Durham University

Electrical Engineering II

ENGI2191

Complex Numbers and Phasors

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E108

Lecture Objectives:

Aim of this lecture:

The aim of this lecture is to understand the concepts of phasor transform and analyse the electric circuits in phasor domain.

Intended Learning Outcomes:

At the completion of the lecture and associated problems you should be able to:

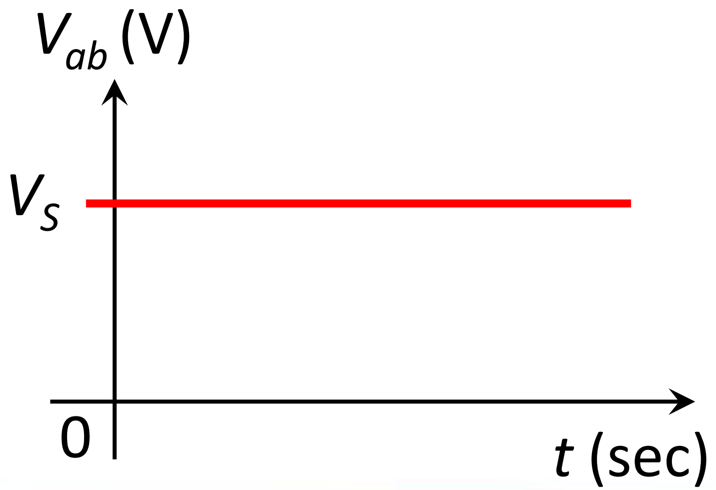
- Understand the fundamental concepts of complex numbers
- Identify the performance of the basic ideal circuit elements in phasor domain
- Analyse the electric circuits in phasor domain

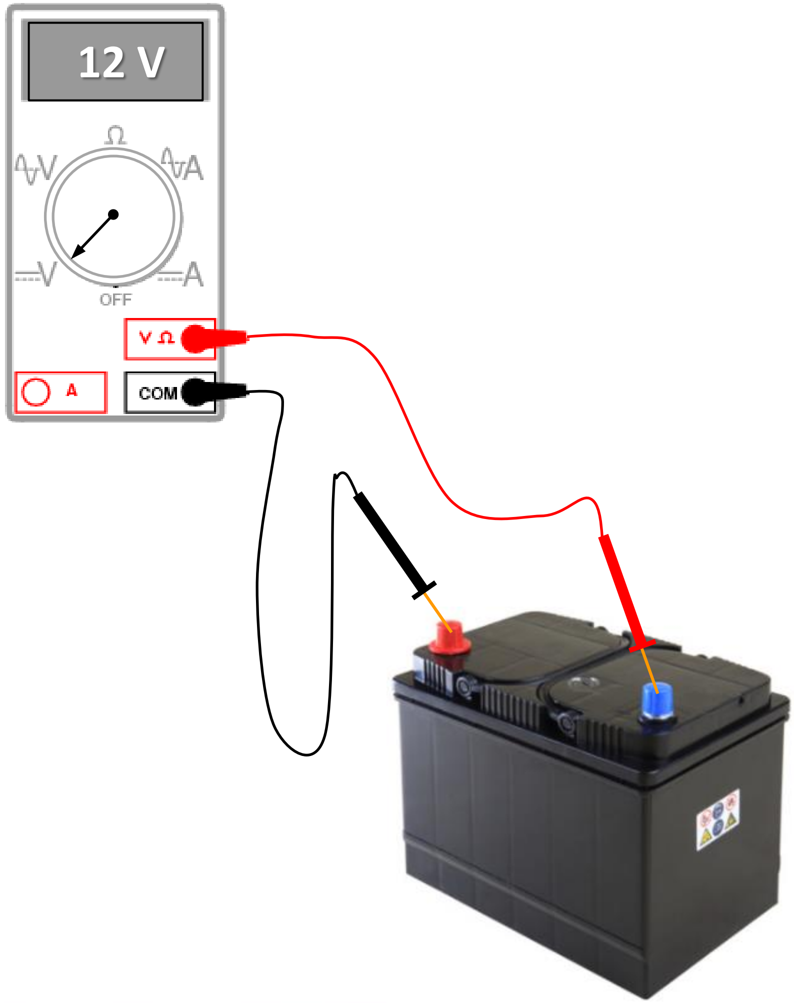
Lecture Objectives:

Battery, an example of Direct Current (DC) electric source.

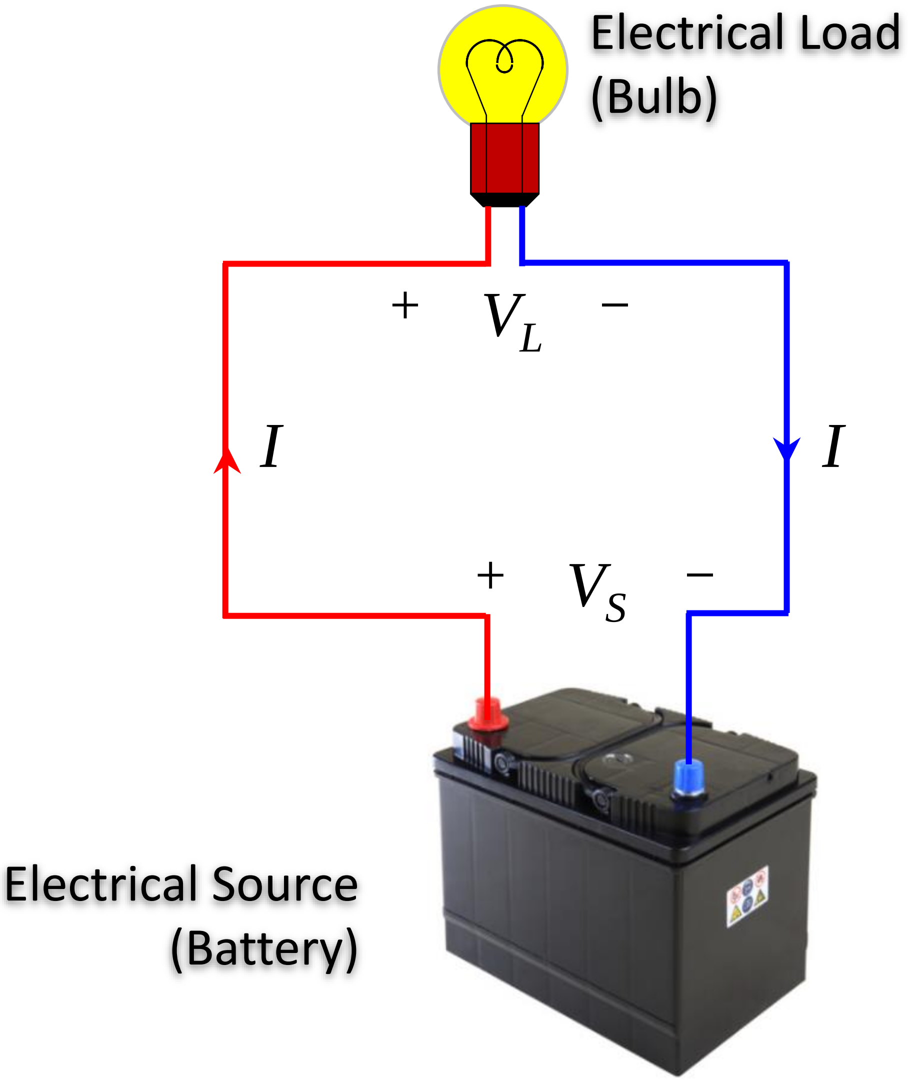
Car battery

Phone battery

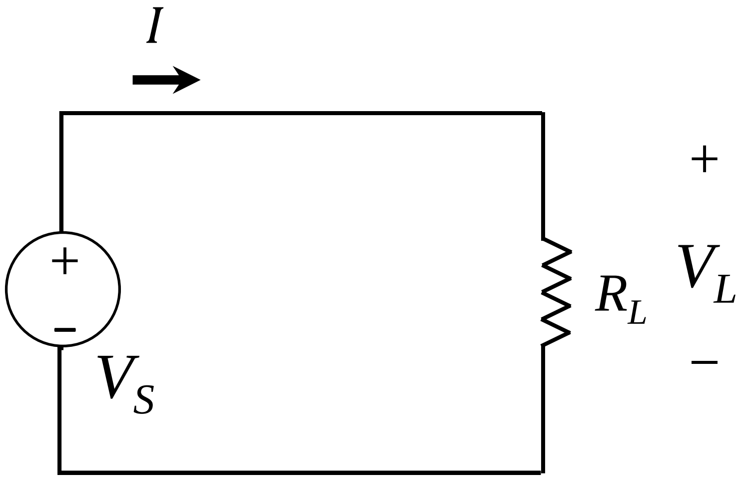




Lecture Objectives:



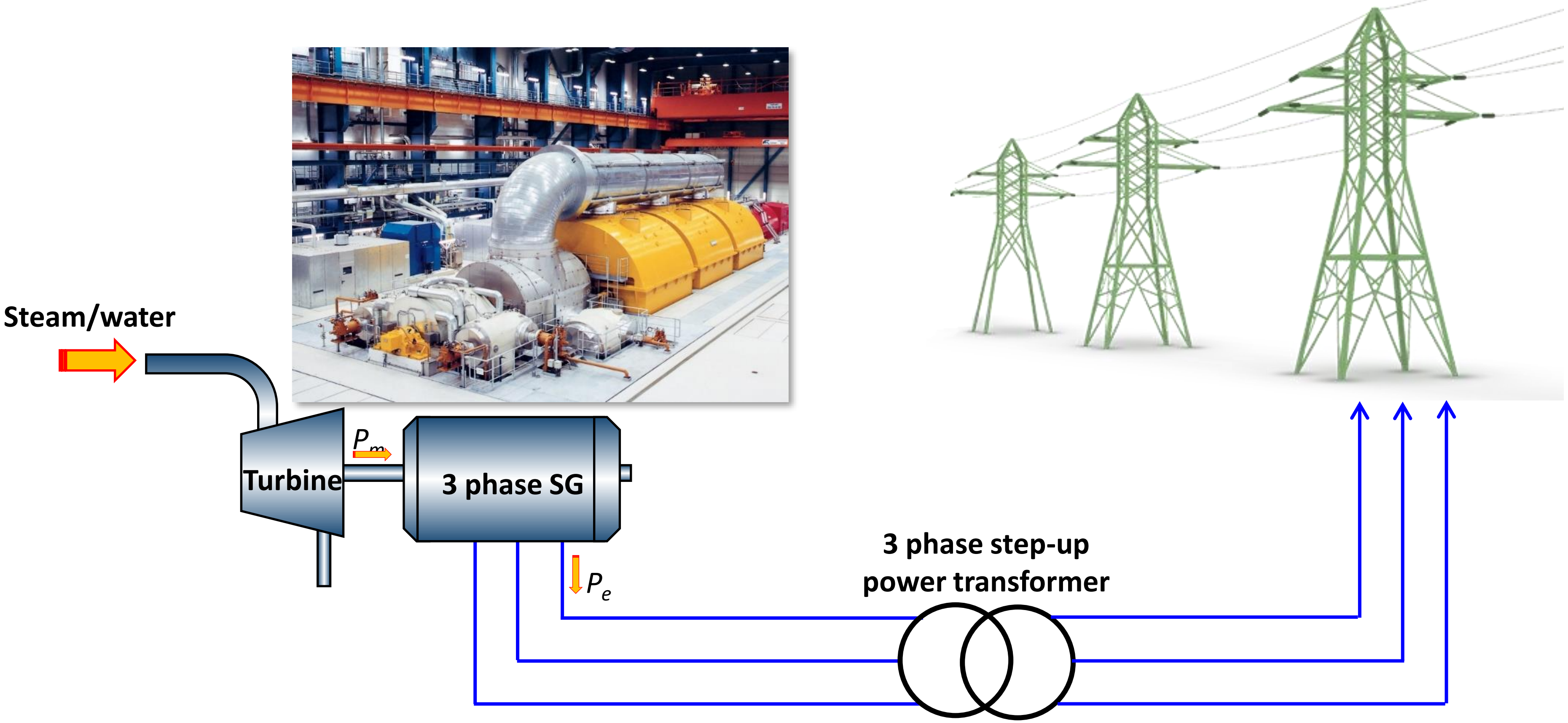
Equivalent circuit of this electric system.



$$I = \frac{V_S}{R_L}$$

Lecture Objectives:

- In power systems, electricity is generated as Alternating Current (AC).
- AC electric networks can be analysed in phasor domain for which we use “Complex Numbers”.

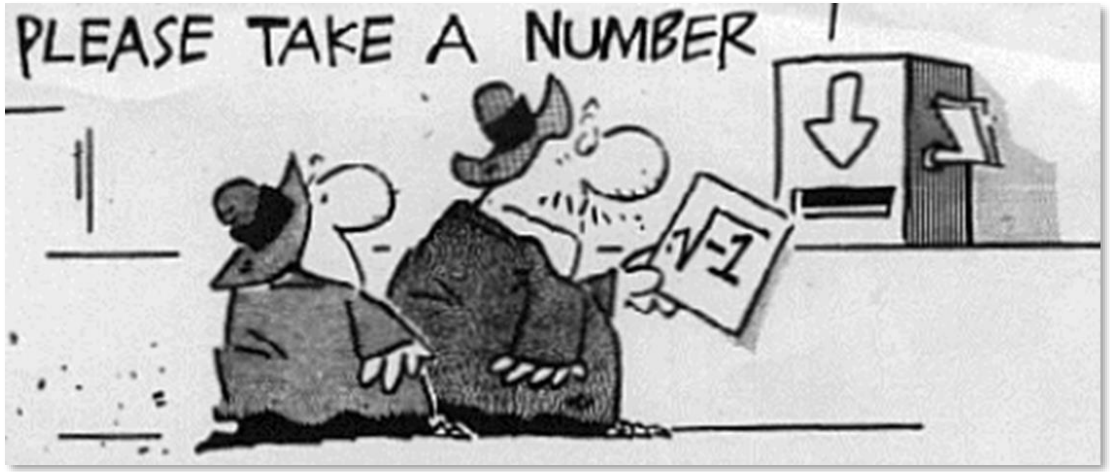


An introduction to complex numbers

$$(Z = a + jb)$$

Imaginary and complex numbers:

Most of the math calculations use "Real Numbers": $2, 0, -8, \sqrt{5}, \pi, e$



$$x^2 + 4 = 0 \Rightarrow x^2 = -4 \Rightarrow x = \sqrt{-4}$$

$$\sqrt{4-5} = \sqrt{-1} \Rightarrow$$ **No real solution !**

In the real number system, we can't take the square root of negatives, therefore the "Complex Number" system was created.

Imaginary and complex numbers:

Imaginary numbers were invented, so that negative numbers would have square roots and certain equations would have solutions.

These numbers were devised using an imaginary unit named i.

$$i = \sqrt{-1} \qquad i^2 = -1$$

In electrical engineering, a lower case i represents a time-dependent (ac) current, so imaginary unit is shown by symbol j. So in this module we define:

$$j = \sqrt{-1} \qquad j^2 = -1$$

$$x = \sqrt{-4} = \sqrt{-1}\sqrt{4} = j2$$

Imaginary and complex numbers:

A complex number is a number of the form:

$$z = a + jb$$

Imaginary part

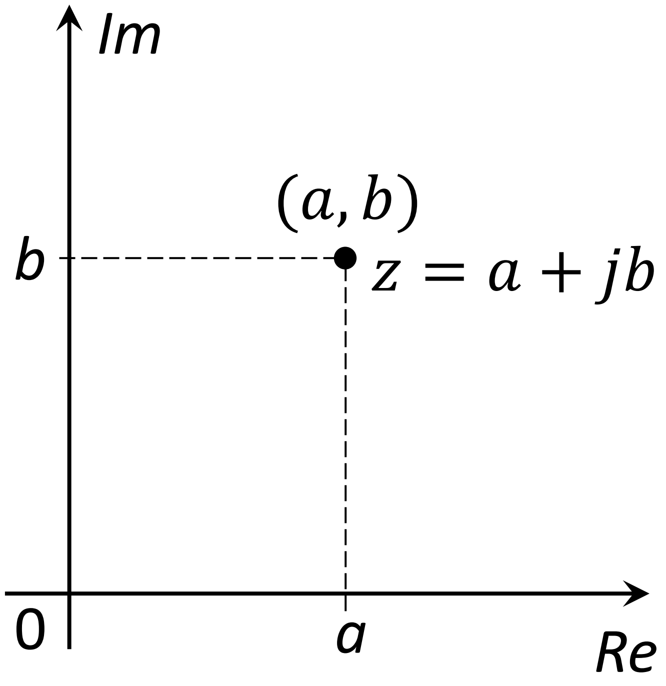
Imaginary unit

Real part

Complex number

Cartesian (rectangular) form of a complex number.

Geometric plot of complex numbers:

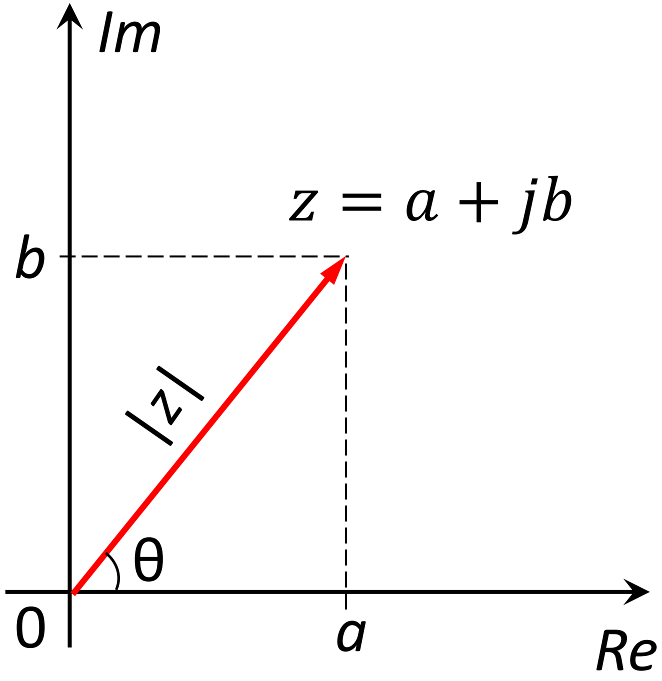


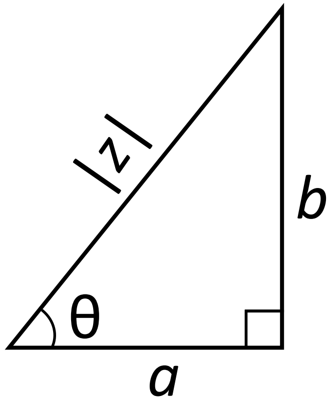
$$a = Re(z)$$

$$b = Im(z)$$

Imaginary and complex numbers:

A complex number can be visualised in a two-dimensional number line, known as an *Argand diagram*, or the *complex plane*:





A complex number can be expressed in terms of a magnitude or length, and an angle as:

$$|z| = \sqrt{a^2 + b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

$$z = |z|\angle\theta$$

Polar form of a complex number

Imaginary and complex numbers:

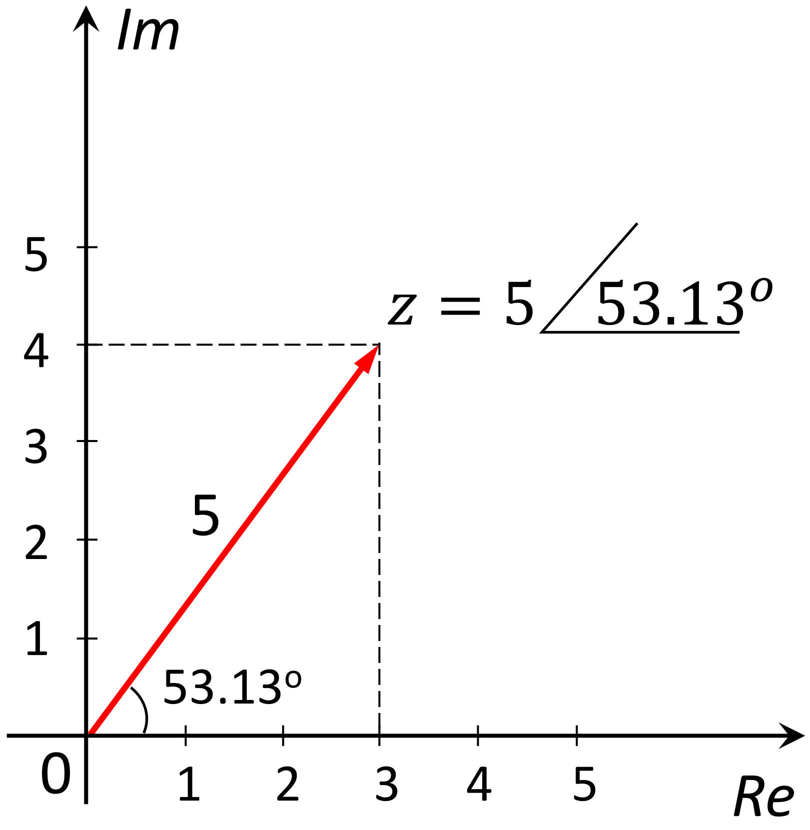
Example:

Z= 3+j4

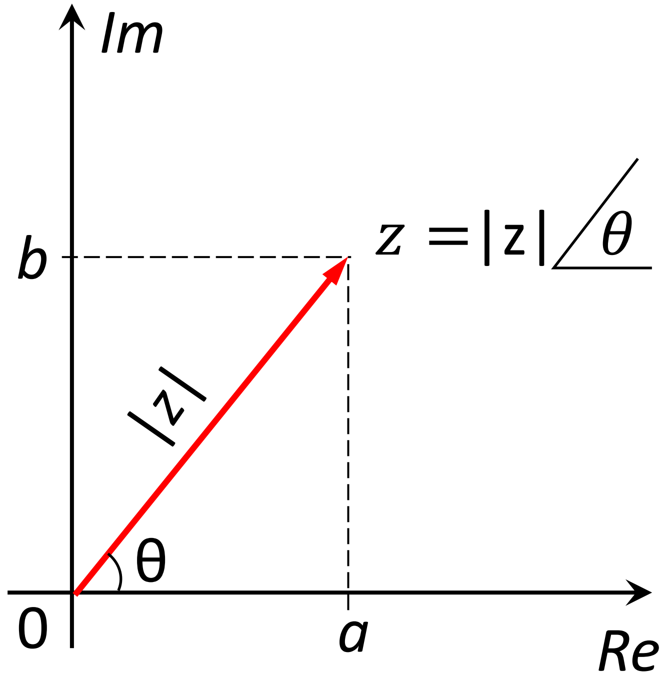
$$|z| = \sqrt{a^2 + b^2} = \sqrt{(3)^2+(4)^2} = \sqrt{25} = 5$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{4}{3} = 53.13^o$$

$$z = |z|\angle\theta = 5\angle 53.13^o$$



Imaginary and complex numbers:



$$a = |z| \; \cos\theta \qquad b = |z| \; \sin\theta$$

$$z = |z| \; \cos\theta + j\,|z| \sin\theta$$

Imaginary and complex numbers:

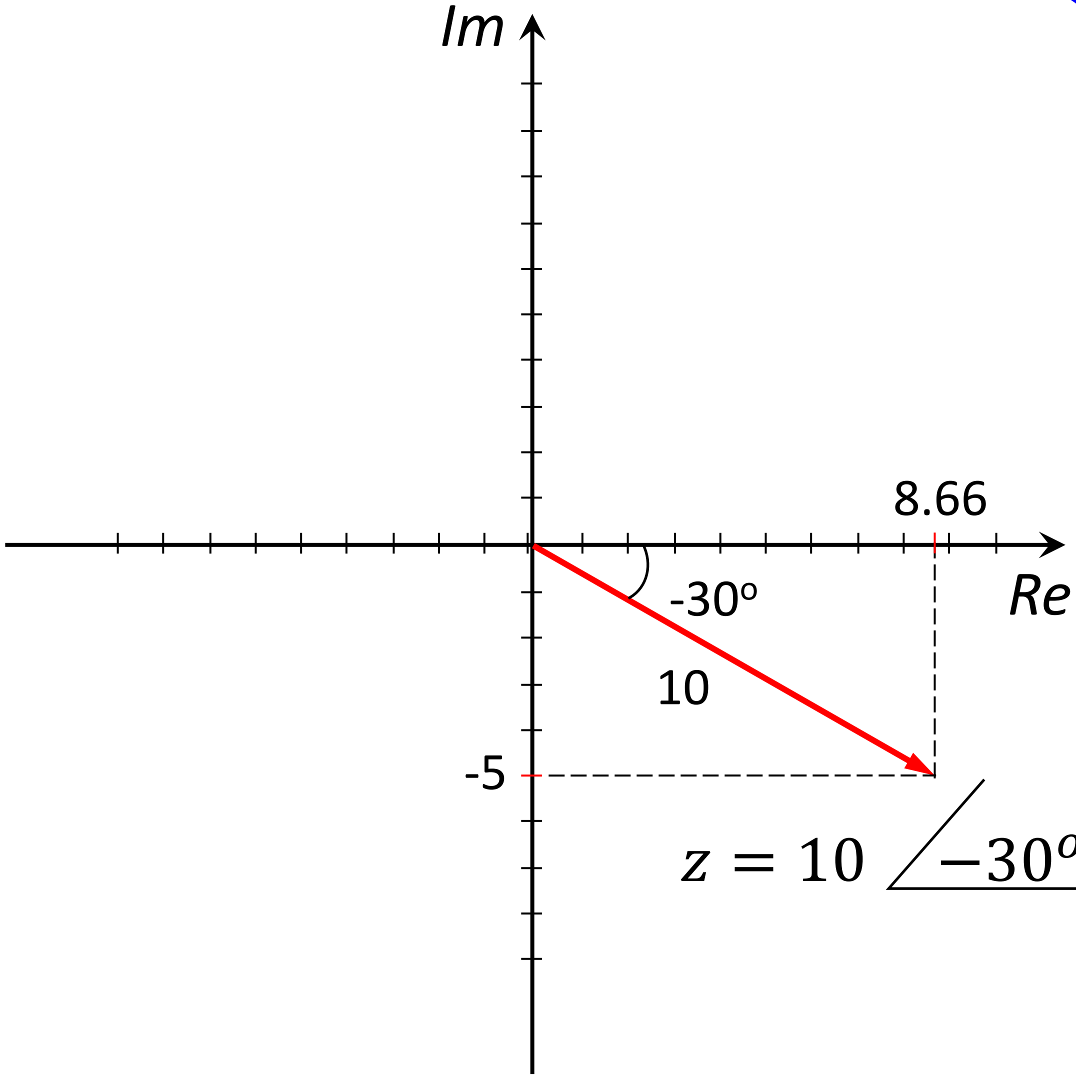
Example:

$$z = 10 \angle -30^o$$

$$a = |z| \cos\theta = 10 \cos(-30) = 8.66$$

$$b = |z| \sin\theta = 10 \sin(-30) = -5$$

$$z = 8.66 - j5$$



$$z = 10 \angle -30^o$$

Operations on Complex Numbers:

Addition of Complex Numbers

Add real parts, add imaginary parts.

$$z_1 = a_1 + j\,b_1 \qquad z_2 = a_2 + j\,b_2$$

$$z_1 + z_2 = (a_1 + j\,b_1) + (a_2 + j\,b_2)$$

$$= (a_1 + a_2) + j\,(b_1 + b_2)$$

Subtraction of Complex Numbers

Subtract real parts, subtract imaginary parts.

$$z_1 = a_1 + j\,b_1 \qquad z_2 = a_2 + j\,b_2$$

$$z_1 - z_2 = (a_1 + j\,b_1) - (a_2 + j\,b_2)$$

$$= (a_1 - a_2) + j\,(b_1 - b_2)$$

Imaginary and complex numbers:

Example:

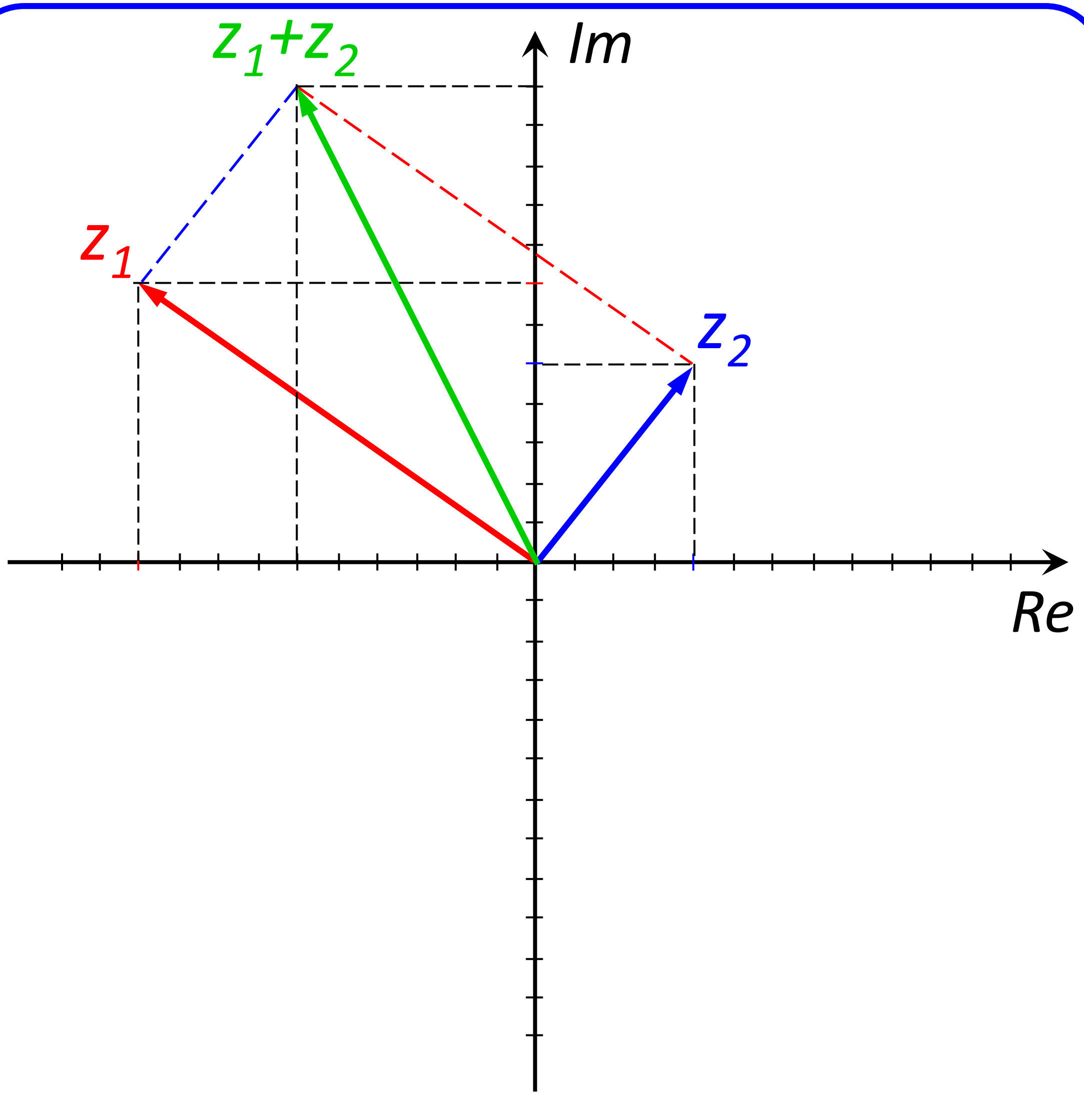
z_1= -10+j 7 z_2= 4+j 5

z_1+z_2= ?

z_1+z_2= (-10+j 7)+(4+j 5)

=(-10+4)+j (7+5)

z_1+z_2=-6+j 12



Imaginary and complex numbers:

Example:

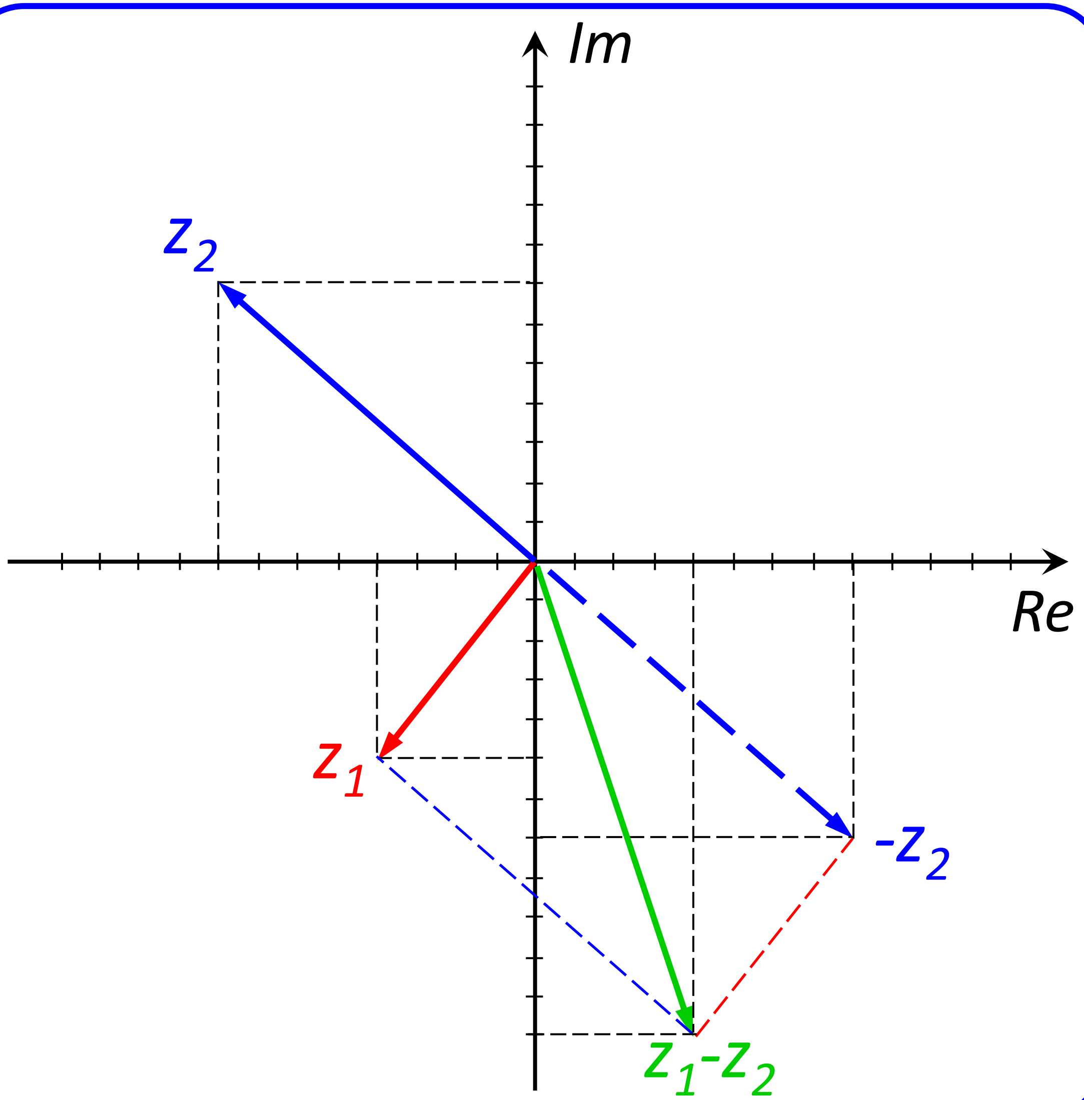
$z_1 = -4 - j\,5 \qquad z_2 = -8 + j\,7$

$z_1 - z_2 = ?$

$z_1 - z_2 = (-4 - j\,5) - (-8 + j\,7)$

$= (-4 - (-8)) + j\,(-5 - 7)$

$z_1 - z_2 = 4 - j\,12$



Imaginary and complex numbers:

Multiplication of Complex Numbers

$z_1 = a_1 + j\,b_1 \qquad z_2 = a_2 + j\,b_2$

$z_1 z_2 = ?$

Expand brackets as usual

Hint: $j^2=-1$

$$z_1 z_2 = (a_1 + j\,b_1)(a_2 + j\,b_2)$$

$$= a_1 a_2 + j a_1 b_2 + j b_1 a_2 - b_1 b_2$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + b_1 a_2)$$

$$z_1 = |z_1| \angle \theta_1 \qquad z_2 = |z_2| \angle \theta_2$$

$$z_1 z_2 = |z_1||z_2| \angle \theta_1 + \theta_2$$

Imaginary and complex numbers:

Example:

z_1= -3+j 4 z_2= 8+j 6

z_1z_2= (-3+j 4) (8+j 6)

=(-3) × 8 + j(-3) × 6 + j4 × 8 – 4 × 6

z_1z_2=(-24 - 24) + j(-18 + 32)

z_1z_2= -48 + j14

$$z_1 = 5\angle -53.13^o \qquad z_2 = 10\angle 36.86^o$$

$$z_1z_2 = (5 \times 10)\angle(-53.13 + 36.86)$$

$$z_1z_2 = 50\angle -16.27^o$$

$$z_1z_2 = 50\cos(-16.27) + j\,50\sin(-16.27)$$

$$z_1z_2 = -48 + j14$$

Imaginary and complex numbers:

Division of Complex Numbers

$$z_1 = |z_1| \angle \theta_1 \qquad z_2 = |z_2| \angle \theta_2$$

$$\frac{z_1}{z_2} = \frac{|z_1| \angle \theta_1}{|z_2| \angle \theta_2}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \angle \theta_1 - \theta_2$$

Example:

$$z_1 = 36 \angle 30^o \qquad z_2 = 3 \angle -45^o$$

$$\frac{z_1}{z_2} = \frac{36 \angle 30^o}{3 \angle -45^o} = 12 \angle (30 - (-45))$$

$$\frac{z_1}{z_2} = 12 \angle 75^o$$

Imaginary and complex numbers:

Complex Conjugate:

The complex conjugate of a complex number *z*, is denoted *z**, and is defined as:

$$z = a + j\,b \qquad z^* = a - j\,b$$

$$z = |z| \angle \theta \qquad z^* = |z| \angle -\theta$$

Example:

$$z = 36\angle 30^o \qquad z^* = 36\angle -30^o$$

$$z = -\sqrt{10} - j\,5 \qquad z^* = -\sqrt{10} + j\,5$$

Prove the following equations:

$$z\,z^* = (a^2+b^2) \qquad z\,z^* = |z|^2$$

Imaginary and complex numbers:

Division of Complex Numbers

$z_1 = a_1 + j\,b_1 \qquad z_2 = a_2 + j\,b_2$

$$\frac{z_1}{z_2} = \left(\frac{a_1 + j\,b_1}{a_2 + j\,b_2}\right) \times \left(\frac{a_2 - j\,b_2}{a_2 - j\,b_2}\right)$$

Drill:

$z_1 = 3 - j \qquad z_2 = 4 - j\,2$

Calculate $\dfrac{z_1}{z_2}$

Answer:

$$\frac{z_1}{z_2} = \frac{7 + j}{10}$$

Module description:

Reading list:

- DeCarlo Lin, “*Linear Circuit Analysis*”, Oxford University Press, Second Edition, 2003
- W H Hayt, J E Kemmerly, S M Durbin, “*Engineering Circuit Analysis*”, McGraw-Hill, 9^{th} Edition, 2019