*9.2.2 Pivotal Quantities*

We noted in the example about the *uniform*(0,) that the coverage probability of the interval {aY,bY} did not depend on , while that of {Y+c,Y+d} did. This happened since the former probability could be expressed in terms of Y/, whose distribution does not depend on θ, a quantity known as a *pivotal quantity*.

|  |
| --- |
| *Definition 9.2.6* A r.v. Q(**X**,) = Q(X1,…,Xn,) is a *pivotal quantity* if the distribution of Q(**X**,) is independent of all parameters. That is, if **X**∼F(x|), then Q(**X**,) has the same distribution for all values of . |

Comment:

Note that even if the pdf of the random variable **X** depends both on x and the parameter , the density function of the pivotal quantity Q(**X**,) does not depend on the parameter .

*Example 9.2.7* (Location-scale pivots)

|  |  |  |
| --- | --- | --- |
| Form of pdf | Type of pdf | Pivotal quantity |
| f(x – μ) | Location | - μ |
| (1/σ)f(x/σ) | Scale | /σ |
| (1/σ)f((x-μ)/σ) | Location-scale | ( - μ)/S |

Comments:

* There are lots of pivots in location and scale cases.
* If X1,…,Xn iid n(μ,σ2) then the t statistic (-μ)/(S/) is a pivot because the t distribution does not depend μ and σ2.
* How do we find pivots? *Differences* are pivotal for location problems and *ratios* (or products) are pivotal for scale problems.

*Example 9.2.8* (Gamma pivot)

X1,…,Xn iid exponential(λ) ∼ (1/λ)e-(x/λ), λ > 0.

T = ΣXi ∼ gamma(n, λ) and is sufficient for λ.

The gamma pdf:

f(t|n, λ) = 

Note that t and λ appear together as t/λ.

This pdf belongs to the scale family.

Let Q(T,λ) = 2T/λ. We use the transformation:

t = λq/2 and dt = λdq/2

Inserting into the pdf:

g(q|n) =  = gamma(n,2)

Q(T, λ) ∼ gamma(n, 2)

which does not depend on λ and is a pivot and ∼ χ2(2n) .

(Remember: gamma(p/2, 2) is χ2(p).)

If the pdf of a statistic T, f(t|), can be expressed in the form

f(t|) = g(Q(t,))

for some function g and some monotone function Q (monotone in t for each ). Then it can be shown that Q(t,) is a pivot.

Once we have a pivotal quantity, Q(**X**, ), how can we construct a confidence interval?

For a specified α we can determine constants a and b which do not depend on θ to satisfy

Pθ(a ≤ Q(**X**,) ≤ b) ≥ 1 – α.

Then for each 0 ∈Θ,

A(0) = {**x**: a≤ Q(**X**,0)≤b}

is the *acceptance region* for a level α test of H0:  = 0.

We use the test inversion method to construct a 1 - α *confidence set*:

C(**x**) = {0: a≤ Q(**X**, 0)≤b}

*Example 9.2.9* (Continuation of Example 9.2.8)

We have shown that Q(T,λ) = 2T/λ ∼ .

If we choose two constants a and b to satisfy

P(𝑎≤ ≤b) = 1 – α,

then

Pλ Pλ(𝑎≤ Q(T,λ) ≤b) = P(𝑎≤ ≤b) = 1 – α

Inverting the set (given λ we accept H0 for t in the interval)

A(λ) = {t: 𝑎≤2t/λ ≤b}

gives (given t we believe that λ is in the interval)

C(t) = {λ: 2t/b ≤ λ≤2t/𝑎}

*Example 9.2.10* (Normal Pivotal interval)

If X1,…,Xn are iid n(μ, σ2) it follows that ( - μ)/(σ/) is a pivot.

If σ2 is known we can use this pivot to calculate a confidence interval for μ.

For any constant a,

P= P(-𝑎 ≤ Z ≤ 𝑎)

Inverting this we have



Similarly, if σ is unknown we can use S and the Student t distribution.

Suppose we want a confidence interval for σ.

Since (n-1)S2/σ2 ∼, (n-1)S2/σ2 is a pivotal quantity.

Consider constants 𝑎 and *b* such that

P= P(𝑎≤ ≤ *b*) = 1 – α

Inverting this gives the 1 – α confidence interval



or, equivalently,



Comments:

* Usually one selects the constants 𝑎 and *b* such that one has equal probabilities in the tails. This is, however, not obvious to do, since the distribution is not symmetric. (See Section 9.3).
* We have constructed intervals for μ and σ separately. An alternative would be to find a confidence region for (μ, σ) instead (Exercise 9.14).

Exercises: 9.12, 9.13

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| p | | 3 | | 2 | | 1 | | 0 | |
| 0.235 | | 0.0130 | | 0.1267 | | 0.4126 | | **0.4477** | |
| 0.236 | | 0.0131 | | 0.1277 | | 0.4133 | | **0.4459** | |
| 0.237 | | 0.0133 | | 0.1286 | | 0.4139 | | **0.4442** | |
| **0.238** | | 0.0135 | | 0.1295 | | 0.4146 | | **0.4425** | |
| 0.239 | | 0.0137 | | 0.1304 | | **0.4152** | | **0.4407** | |
| 0.240 | | 0.0138 | | 0.1313 | | **0.4159** | | **0.4390** | |
| 0.241 | | 0.0140 | | 0.1323 | | **0.4165** | | **0.4372** | |
| 0.242 | | 0.0142 | | 0.1332 | | **0.4171** | | **0.4355** | |
| 0.243 | | 0.0143 | | 0.1341 | | **0.4178** | | **0.4338** | |
| 0.244 | | 0.0145 | | 0.1350 | | **0.4184** | | **0.4321** | |
| 0.245 | | 0.0147 | | 0.1360 | | **0.4190** | | **0.4304** | |
| 0.246 | | 0.0149 | | 0.1369 | | **0.4196** | | **0.4287** | |
| 0.247 | | 0.0151 | | 0.1378 | | **0.4202** | | **0.4270** | |
| 0.248 | | 0.0153 | | 0.1388 | | **0.4207** | | **0.4253** | |
| 0.249 | | 0.0154 | | 0.1397 | | **0.4213** | | **0.4236** | |
| 0.250 | | 0.0156 | | 0.1406 | | **0.4219** | | **0.4219** | |
| 0.251 | | 0.0158 | | 0.1416 | | **0.4224** | | **0.4202** | |
| 0.252 | | 0.0160 | | 0.1425 | | **0.4230** | | **0.4185** | |
| 0.253 | | 0.0162 | | 0.1434 | | **0.4235** | | **0.4168** | |
| 0.254 | | 0.0164 | | 0.1444 | | **0.4241** | | **0.4152** | |
| 0.255 | | 0.0166 | | 0.1453 | | **0.4246** | | **0.4135** | |
| ----- | | ----- | | ----- | | **-----** | | **-----** | |
| 0.273 | | 0.0203 | | 0.1625 | | **0.4329** | | **0.3842** | |
| 0.274 | | 0.0206 | | 0.1635 | | **0.4333** | | **0.3827** | |
| 0.275 | | 0.0208 | | 0.1645 | | **0.4336** | | **0.3811** | |
| 0.276 | | 0.0210 | | 0.1655 | | **0.4340** | | **0.3795** | |
| 0.277 | | 0.0213 | | 0.1664 | | **0.4344** | | **0.3779** | |
| 0.278 | | 0.0215 | | 0.1674 | | **0.4348** | | **0.3764** | |
| 0.279 | | 0.0217 | | 0.1684 | | **0.4351** | | **0.3748** | |
| 0.280 | | 0.0220 | | 0.1693 | | **0.4355** | | **0.3732** | |
| 0.281 | | 0.0222 | | 0.1703 | | **0.4358** | | **0.3717** | |
| 0.282 | | 0.0224 | | 0.1713 | | **0.4361** | | **0.3701** | |
| 0.283 | | 0.0227 | | 0.1723 | | **0.4365** | | **0.3686** | |
| 0.284 | | 0.0229 | | 0.1732 | | **0.4368** | | **0.3671** | |
| 0.285 | | 0.0231 | | 0.1742 | | **0.4371** | | **0.3655** | |
| 0.286 | | 0.0234 | | 0.1752 | | **0.4374** | | **0.3640** | |
| 0.287 | | 0.0236 | | 0.1762 | | **0.4377** | | **0.3625** | |
| 0.288 | | 0.0239 | | 0.1772 | | **0.4380** | | **0.3609** | |
| 0.289 | | 0.0241 | | 0.1782 | | **0.4383** | | **0.3594** | |
| 0.290 | | 0.0244 | | 0.1791 | | **0.4386** | | **0.3579** | |
| 0.291 | | 0.0246 | | 0.1801 | | **0.4388** | | **0.3564** | |
| 0.292 | | 0.0249 | | 0.1811 | | **0.4391** | | **0.3549** | |
| 0.293 | | 0.0252 | | 0.1821 | | **0.4394** | | **0.3534** | |
| 0.294 | | 0.0254 | | 0.1831 | | **0.4396** | | **0.3519** | |
| 0.298 | | 0.0265 | | 0.1870 | | **0.4406** | | **0.3459** | |
| 0.299 | | 0.0267 | | 0.1880 | | **0.4408** | | **0.3445** | |
| 0.300 | | 0.0270 | | 0.1890 | | **0.4410** | | **0.3430** | |
| 0.301 | | 0.0273 | | 0.1900 | | **0.4412** | | **0.3415** | |
| 0.302 | | 0.0275 | | 0.1910 | | **0.4414** | | **0.3401** | |
| 0.303 | | 0.0278 | | 0.1920 | | **0.4416** | | **0.3386** | |
| 0.304 | | 0.0281 | | 0.1930 | | **0.4418** | | **0.3372** | |
| **0.305** | | 0.0284 | | 0.1940 | | **0.4420** | | 0.3357 | |
| 0.306 | | 0.0287 | | 0.1950 | | **0.4421** | | 0.3343 | |
| 0.307 | | 0.0289 | | 0.1959 | | **0.4423** | | 0.3328 | |
| 0.308 | | 0.0292 | | 0.1969 | | **0.4425** | | 0.3314 | |
| 0.309 | | 0.0295 | | 0.1979 | | **0.4426** | | 0.3299 | |
| ----- | | ----- | | --- | | ----- | | ----- | |
| 0.361 | | 0.0470 | | 0.2498 | | **0.4422** | | 0.2609 | |
| **0.362** | | 0.0474 | | 0.2508 | | **0.4420** | | 0.2597 | |
| 0.363 | | 0.0478 | | 0.2518 | | **0.4419** | | **0.2585** | |
| 0.364 | | 0.0482 | | 0.2528 | | **0.4417** | | **0.2573** | |
| 0.365 | | 0.0486 | | 0.2538 | | **0.4415** | | **0.2560** | |
| **0.366** | | 0.0490 | | 0.2548 | | **0.4413** | | **0.2548** | |
| 0.367 | | 0.0494 | | **0.2558** | | **0.4412** | | 0.2536 | |
| 0.368 | | 0.0498 | | **0.2568** | | **0.4410** | | 0.2524 | |
| 0.369 | | 0.0502 | | **0.2578** | | **0.4408** | | 0.2512 | |
| 0.370 | | 0.0507 | | **0.2587** | | **0.4406** | | 0.2500 | |
| ----- | | ----- | ----- | | ----- | | ----- | |
| 0.630 | | 0.2500 | **0.4406** | | **0.2587** | | 0.0507 | |
| 0.631 | | 0.2512 | **0.4408** | | **0.2578** | | 0.0502 | |
| 0.632 | | 0.2524 | **0.4410** | | **0.2568** | | 0.0498 | |
| 0.633 | | 0.2536 | **0.4412** | | **0.2558** | | 0.0494 | |
| **0.634** | | 0.2548 | **0.4413** | | **0.2548** | | 0.0490 | |
| 0.635 | | **0.2560** | **0.4415** | | 0.2538 | | 0.0486 | |
| 0.636 | | **0.2573** | **0.4417** | | 0.2528 | | 0.0482 | |
| 0.637 | | **0.2585** | **0.4419** | | 0.2518 | | 0.0478 | |
| **0.638** | | **0.2597** | **0.4420** | | 0.2508 | | 0.0474 | |
| 0.639 | | 0.2609 | **0.4422** | | 0.2498 | | 0.0470 | |
| 0.640 | | 0.2621 | **0.4424** | | 0.2488 | | 0.0467 | |
| ----- | | ----- | ----- | | ----- | | ----- | |
| 0.693 | | 0.3328 | **0.4423** | | 0.1959 | | 0.0289 | | |
| 0.694 | | 0.3343 | **0.4421** | | 0.1950 | | 0.0287 | | |
| **0.695** | | 0.3357 | **0.4420** | | 0.1940 | | 0.0284 | | |
| 0.696 | | **0.3372** | **0.4418** | | 0.1930 | | 0.0281 | | |
| 0.697 | | **0.3386** | **0.4416** | | 0.1920 | | 0.0278 | | |
|  | |  |  | |  | |  | | |
| 0.760 | | **0.4390** | **0.4159** | | | 0.1313 | 0.0138 | | |
| 0.761 | | **0.4407** | **0.4152** | | | 0.1304 | 0.0137 | | |
| **0.762** | | **0.4425** | **0.4146** | | | 0.1295 | 0.0135 | | |
| 0.763 | | **0.4442** | 0.4139 | | | 0.1286 | 0.0133 | | |

Table 9.2.2 *Acceptance region and confidence set for Sterne´s construction, X ∼ binomial(3,p) and 1 - α = .442*

|  |  |  |  |
| --- | --- | --- | --- |
| p | Accept. region | x | Confidence set |
| [.000,.238] | {0} |  |  |
| (.238,.305) | {0,1} | 0 | [.000,.305)(.362,.366) |
| [.305,.362] | {1} |  |  |
| (.362,.366) | {0,1} | 1 | (.238,.634) |
| [.366,.634] | {1,2} |  |  |
| (.634,.638) | {2,3} | 2 | [.366,.762) |
| [.638,.695] | {2} |  |  |
| (.695,.762) | {2,3} | 3 | (.634,.638)(.695,1.00) |
| [.762,1.00] | {3} |  |  |

*9.2.3 Pivoting the CDF*

We have seen that a pivot, Q, leads to a confidence set of the form

C(**x**) = {0 : a ≤ Q(**x**,0) ≤ b}.

If, for every **x**, Q is a monotone function of , then the confidence set C(**x**) is guaranteed to be an interval.

The examples so far, using location and scale transformations, resulted in monotone Q functions, and hence, confidence intervals.

Now assume our CI construction for a parameter  is based on a statistic T with cfd FT(t|).

Assume T to be a continuous variable.

FT(T|) is a random variable that is uniform(0,1) and is a pivotal quantity.

If α1 + α2 = α, then an α-level acceptance region of the hypothesis H0: = 0 is

A(0) = {t: α1 ≤ FT(t|0) ≤ 1 - α2},

with associated confidence set

C(t) = {: α1 ≤ FT(t|) ≤ 1 - α2}.

To guarantee that the confidence set is an interval, we need to have FT(t|) to be monotone in , either increasing or decreasing.

**Theorem 9.2.12 (Pivoting a continuous cdf)**

Let T be a statistic with continuous cdf FT(t|).

Let α1 + α2 = α with 0 < α < 1 be fixed values. Suppose that for each t ∊ 𝓣, the functions L(t) and U(t) can be defined as follows.

1. If FT(t|) is a decreasing function of  for each t, define L(t) and U(t) by

FT(t| U(t)) = α1, FT(t|L(t)) = 1 - α2.

1. If FT(t|) is an increasing function of  for each t, define L(t) and U(t) by

FT(t| U(t)) = 1 - α2, FT(t|L(t)) = α1.

Then the random interval [L(T), U(T)] is a 1 – α confidence interval for .

**Proof:** Part i. Assume we have an acceptance region

{t: α1 ≤ FT(t|0) ≤ 1 - α2}.

Since FT(t|0) is a decreasing function of  for each t, and 1-α2 > α1, L(t) < U(t) and the values L(t) and U(t) are unique. Also, (show a picture!)

FT(t|)< α1 ⟺  > U(t)

FT(t|)> 1 - α2 ⟺  < L(t)

and hence

{: α1 ≤ FT(t|) ≤ 1 - α2} = {: c≤≤ U(t)}.

*Example 9.2.13* (Location exponential interval)

Let X1,…,Xn be iid from f(x|μ) = e-(x-μ)I[μ,∞)(x).

Then Y = min{ X1,…,Xn} is sufficient for μ with pdf

fY(y| μ) = ne-n(y-μ) I[μ,∞)(y).

Here we can find the cdf to be

FY(y| μ) = 1 – e-n(y-μ).

Since this is a decreasing function of μ for each y, we use part i. of the theorem:

FY(y| μU(y) ) = 1 - = α/2

FY(y| μL(y) ) = 1 - = 1 - α/2

We obtain

μU(y) = y + (1/n)log(1 – α/2)

μL(y) = y + (1/n)log(α/2)

Hence the random interval

C(Y) = 

is a 1 – α confidence interval for μ.

**Theorem 9.2.14 (Pivoting a discrete cdf)**

Let T be a discrete statistic with cdf FT(t|) = P(T≤t|).

Let α1 + α2 = α with 0 < α < 1 be fixed values. Suppose that for each t∊𝓣, L(t) and U(t) can be defined as follows.

1. If FT(t|) is a decreasing function of  for each t, define L(t) and U(t) by

P(T≤t|U(t)) = α1, P(T≥t|L(t)) = α2.

1. If FT(t|) is an increasing function of  for each t, define L(t) and U(t) by

P(T≥t|U(t)) = α1, P(T≤t|L(t)) = α2.

Then the random interval [L(T), U(T)] is a 1 – α confidence interval for .

*Example 9.2.15* (Poisson interval estimator)

Let X1,…,Xn be iid from Poisson(λ).

Let Y = ΣXi.

Y is sufficient for λ and Y ∼ Poisson(nλ).

Let α1 = α2 = α/2.

If Y = y0 is observed we should solve for λ in the equations:

= α/2 and = α/2.

Link between Poisson and gamma distributions gives

=  = P(Y≤y0|λ) = P

Thus we take

λ = 

Similarly we have

=  = P(Y≥y0|λ) = P

Thus

