Notes on the project ParametricAcousticArray

Jiaxin Zhong

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1 List of Symbols

Symbols	Descriptions
a	radius of the source
c_0	isentropic speed of sound at ambient values of pressure and density
f_1, f_2	$f_1 = N_1 f_\mathrm{b}$, $f_2 = N_2 f_b$, primary frequencies
f_b	$= \gcd f_2 - f_1$, basic frequency
f_m	$= (f_1 + f_2)/2$, mean primary frequency
k	$= 2\pi f/c_0$ wavenumber
k_m	$= (k_1 + k_2)/2$ wavenumber of the mean primary frequency
l_D	$= (\beta k \epsilon)^{-1}$, shock formation distance of a plane wave
L_x, L_y	the size of the rectangular transducer in x, y direction, and $L_y \geqslant L_x$
N_m	$= (N_1 + N_2)/2$
p	pressure
p_0	ambient pressure
$\bar{p}$	$= (p - p_0)/P_0$, acoustic pressure normalized to P_0
P_0	$\rho_0 c_0 v_0$, acoustic pressure peak amplitude on the source
R_D	$= ka^2/2 = \pi a^2/\lambda$, Rayleigh distance of primary frequency
R_Dm	$= k_\mathrm{m} a^2/2 = \pi a^2/\lambda_\mathrm{m}$, Rayleigh distance of mean primary frequency
t	time
$\mathbf{u}$	$= (u_x, u_y) = \mathbf{r}/L_y = (x, y)/L_y$
v_0	characteristic velocity peak amplitude on the source
x, y, z	dimensional coordinates, z along the direction of propagation
$\mathbf{r}$	(x, y)
r	$\lvert\mathbf{r}\rvert$
α	absorption coefficient, in Neper per meter
β	parameter of nonlinearity
ϵ	$= u_0/c_0$, Mach number
σ	$= z/R_\mathrm{D}$, nondimensional z
σ_D	$= l_\mathrm{D}/R_\mathrm{D}$
σ_Dm	$= l_\mathrm{Dm}/R_\mathrm{Dm}$
χ	$= x/R_\mathrm{Dm}$
ψ	$= y/R_\mathrm{Dm}$
γ	$= L_y/L_x$, the aspect ratio
$\nabla^2_{x,y}$	$= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$
∇^2_u	$\dfrac{\partial^2}{\partial \mathbf{u}^2} = \dfrac{1}{u}\dfrac{\partial}{\partial u}\left(u\dfrac{\partial}{\partial u}\right)$
$\nabla^2_{\chi,\psi}$	$= \dfrac{\partial^2}{\partial \chi^2} + \dfrac{\partial^2}{\partial \psi^2}$

2 Theoretical Model and Analytical Solutions

2.1 KZK Equation

Nonlinear propagation of finite amplitude sound waves in dissipative and homogenous fluid is decribed by the following KZK equation:

$$\frac{\partial^2 \bar{p}}{\partial \sigma \partial \tau} = \alpha R_\mathrm{D} \frac{\partial^3 \bar{p}}{\partial \tau^3} + \frac{1}{4}\nabla^2_\perp \bar{p} + \frac{1}{2\sigma_\mathrm{D}} \frac{\partial^2 (\bar{p})^2}{\partial \tau^2} \tag{1}$$

If the source is bifrequency, i.e.

$$f_i = N_i f_\mathrm{b}, \quad N_i \in \mathbb{N}^+, \quad i = 1, 2 \tag{2}$$

2.1.1 Rectangular profile

The Eq. (1) is written as

$$\frac{\partial^2 \bar{p}}{\partial \sigma \partial \tau} = \alpha R_{\mathrm{D}} \frac{\partial^3 \bar{p}}{\partial \tau^3} + \frac{1}{\pi \gamma} \nabla_u^2 \bar{p} + \frac{1}{2\sigma_{\mathrm{D}}} \frac{\partial^2 (\bar{p})^2}{\partial \tau^2} \tag{3}$$

The transformations are:

$$\sigma_{\mathrm{m}} = z/R_{\mathrm{Dm}}, \quad T = (1+\sigma_{\mathrm{m}})\bar{p}, \quad \tau_{\mathrm{m}} = \tau - \frac{\pi\gamma}{4} \frac{\xi^2}{N_{\mathrm{m}}(1+\sigma_{\mathrm{m}})}, \quad (\chi, \psi) = \frac{(u_x, u_y)}{1+\sigma_{\mathrm{m}}} \tag{4}$$

Then, the transformed KZK equation is obtained by Eq. (3)

$$\frac{\partial^2 T}{\partial \sigma_{\mathrm{m}} \partial \tau_{\mathrm{m}}} = \frac{\alpha_{\mathrm{m}}}{N_{\mathrm{m}}^2} R_{\mathrm{Dm}} \frac{\partial^3 T}{\partial \tau_{\mathrm{m}}^3} + \frac{N_{\mathrm{m}}}{\pi\gamma(1+\sigma_{\mathrm{m}})^2} \nabla_{\chi,\psi}^2 T + \frac{1}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m}})} \frac{\partial^2 (T^2)}{\partial \tau_{\mathrm{m}}^2} \tag{5}$$

2.2 Spectral Solutions

A solution of Eq. (3) is sought in the form of a Fourier series

$$\bar{p} = \sum_{n=1}^{\infty} (g_n \sin n\tau + h_n \cos n\tau) = \sum_{n=1}^{\infty} \bar{p}_n \tag{6}$$

where g_n and h_n are functions of spatial coordinates.

The solution of Eq. (5) is

$$T = \sum_{n=1}^{\infty} [g_n(\chi, \psi, \sigma_{\mathrm{m}}) \sin n\tau_{\mathrm{m}} + h_n(\chi, \psi, \sigma_{\mathrm{m}}) \cos n\tau_{\mathrm{m}}] = \sum_{n=1}^{\infty} T_n \tag{7}$$

2.3 Coupled PDEs

We thus obtain the following set of coupled partial differential equations for g_n and h_n.

2.3.1 Rectangular Profile

$$\begin{aligned} \frac{\partial g_n}{\partial \sigma} &= -\alpha_n R_{\mathrm{Dm}} g_n + \frac{1}{\pi\gamma n(1+\sigma_{\mathrm{m}})^2} \nabla_{\chi,\psi}^2 h_n + \frac{n}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m}})} \\ &\quad \times \left[\frac{1}{2} \sum_{m=1}^{n-1} (g_m g_{n-m} - h_m h_{n-m}) - \sum_{m=n+1}^{\infty} (g_{m-n} g_m + h_{m-n} h_m) \right] \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{\partial h_n}{\partial \sigma} &= -\alpha_n R_{\mathrm{Dm}} h_n - \frac{1}{\pi\gamma n(1+\sigma_{\mathrm{m}})^2} \nabla_{\chi,\psi}^2 g_n + \frac{n}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m}})} \\ &\quad \times \left[\sum_{m=1}^{n-1} g_m h_{n-m} + \sum_{m=n+1}^{\infty} (h_{m-n} g_m - g_{m-n} h_m) \right] \end{aligned} \tag{9}$$

2.4 Boundary Conditions

The boundary condition for g_n and h_n at $\sigma = \sigma_{\mathrm{m}} = 0$ is

$$g_n = \frac{1}{\pi} \int_{-\pi}^{\pi} c \sin n\tau_{\mathrm{m}} \, \mathrm{d}\tau_{\mathrm{m}}$$

$$h_n = \frac{1}{\pi} \int_{-\pi}^{\pi} c \cos n\tau_{\mathrm{m}} \, \mathrm{d}\tau_{\mathrm{m}}$$

2.4.1 Circular Profile

The boundary condition at $\sigma = \sigma_{\mathrm{m}} = 0$ is

$$\bar{p}(\sigma_{\mathrm{m}} = 0, \boldsymbol{\xi}, \tau) = c(\boldsymbol{\xi}, \tau) \tag{10}$$

Suppose it is

$$\bar{p}(\sigma_{\mathrm{m}} = 0, \boldsymbol{\xi}, \tau) = c_1(\boldsymbol{\xi}) \sin N_1 \tau + c_2(\boldsymbol{\xi}) \sin N_2 \tau \tag{11}$$

2.4.2 Rectangular Profile

The boundary condition at $\sigma = \sigma_{\mathrm{m}} = 0$ is

$$\bar{p}(\sigma_{\mathrm{m}} = 0, \mathbf{u}, \tau) = c(\mathbf{u}, \tau) \tag{12}$$

Suppose it is

$$\begin{aligned}\bar{p}(\mathbf{u}, \sigma_{\mathrm{m}} = 0, \tau) &= c_1(\mathbf{u}) \sin N_1 \tau + c_2(\mathbf{u}) \sin N_2 \tau \qquad (13)\\ &= c_1(\mathbf{u}) \left[\cos\left(\frac{\pi\gamma N_1}{4N_{\mathrm{m}}}\xi^2\right) \sin N_1 \tau_{\mathrm{m}} + \sin\left(\frac{\pi\gamma N_1}{4N_{\mathrm{m}}}\xi^2\right) \cos N_1 \tau_{\mathrm{m}} \right] \\ &+ c_2(\mathbf{u}) \left[\cos\left(\frac{\pi\gamma N_1}{4N_{\mathrm{m}}}\xi^2\right) \sin N_2 \tau_{\mathrm{m}} + \sin\left(\frac{\pi\gamma N_1}{4N_{\mathrm{m}}}\xi^2\right) \cos N_2 \tau_{\mathrm{m}} \right] \qquad (14)\end{aligned}$$

Then

$$g_{N_i}(\chi, \psi, \sigma_{\mathrm{m}} = 0) = c_i(\chi, \psi) \cos\left[\frac{\pi\gamma N_i}{4N_{\mathrm{m}}}(\chi^2 + \psi^2)\right], \quad i = 1, 2 \tag{15}$$

$$h_{N_i}(\chi, \psi, \sigma_{\mathrm{m}} = 0) = c_i(\chi, \psi) \sin\left[\frac{\pi\gamma N_i}{4N_{\mathrm{m}}}(\chi^2 + \psi^2)\right], \quad i = 1, 2 \tag{16}$$

$$g_n(\chi, \psi, \sigma_{\mathrm{m}} = 0) = h_n(\chi, \psi, \sigma_{\mathrm{m}} = 0) = 0, \quad n \neq N_1, N_2 \tag{17}$$

3 Numerical Methods

3.1 Discretization

3.1.1 Rectangular Profile

$$\sigma_i = i\Delta\sigma, \quad i = 0, 1, \cdots, I-1, I, \quad \sigma_0 = 0, \quad \sigma_I = \sigma_{\max}, \quad \Delta\sigma = \frac{\sigma_{\max}}{I}$$

$$\chi_j = j\Delta\chi, \quad j = 0, 1, \cdots, J-1, J, \quad \chi_0 = 0, \quad \chi_J = \chi_{\max}, \quad \Delta\chi = \frac{\chi_{\max}}{J}$$

$$\psi_k = k\Delta\psi, \quad k = 0, 1, \cdots, K-1, K, \quad \psi_0 = 0, \quad \psi_K = \psi_{\max}, \quad \Delta\psi = \frac{\psi_{\max}}{K}$$

For $\partial g_n / \partial \sigma$ or $\partial h_n / \partial \sigma$, we adopt Backward Implicit Finite Difference

$$\left.\frac{\partial g_n(\chi, \psi, \sigma_{\mathrm{m}})}{\partial \sigma_{\mathrm{m}}}\right|_{i,j,k} = \frac{g_n^{i,j,k} - g_n^{i-1,j,k}}{\Delta\sigma_{\mathrm{m}}}, \quad i = 1, 2, \cdots, I \tag{18}$$

where

$$\begin{aligned} g_n^{i,j,k} &\approx g_n(\chi_j, \psi_k, \sigma_{\mathrm{m},i}) \\ h_n^{i,j,k} &\approx h_n(\chi_j, \psi_k, \sigma_{\mathrm{m},i}) \end{aligned} \tag{19}$$

For $\nabla^2_{\chi,\psi} g_n$ or $\nabla^2_{\chi,\psi} h_n$, 4 cases are considered:
(1) $j = 1, 2, \cdots, J-1$, and $k = 1, 2, \cdots, K-1$

$$\nabla^2_{\chi,\psi}\big|_{i,j,k} = \frac{g_n^{i,j-1,k} - 2g_n^{i,j,k} + g_n^{i,j+1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,j,k-1} - 2g_n^{i,j,k} + g_n^{i,j,k+1}}{(\Delta\psi)^2} \tag{20}$$

(2) $j = 1, 2, \cdots, J-1$, and $k = 0$

$$\nabla^2_{\chi,\psi}\big|_{i,j,0} = \frac{g_n^{i,j-1,0} - 2g_n^{i,j,0} + g_n^{i,j+1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,j,0} + 2g_n^{i,j,1}}{(\Delta\psi)^2} \tag{21}$$

(3) $j = 0$, and $k = 1, 2, \cdots, K-1$

$$\nabla^2_{\chi,\psi}\big|_{i,0,k} = \frac{-2g_n^{i,0,k} + 2g_n^{i,1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,0,k-1} - 2g_n^{i,0,k} + g_n^{i,0,k+1}}{(\Delta\psi)^2} \tag{22}$$

(4) $j=0$, and $k=0$

$$\nabla^2_{\chi,\psi}\big|_{i,0,0} = \frac{-2g_n^{i,0,0}+2g_n^{i,1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,0,0}+2g_n^{i,0,1}}{(\Delta\psi)^2} \tag{23}$$

where the symmetric conditions are used

$$\begin{aligned} \left.\frac{\partial g_n}{\partial \chi}\right|_{\chi=0} &= \left.\frac{\partial h_n}{\partial \chi}\right|_{\chi=0} = 0, \quad n=1,2,\cdots,N \\ \left.\frac{\partial g_n}{\partial \psi}\right|_{\psi=0} &= \left.\frac{\partial h_n}{\partial \psi}\right|_{\psi=0} = 0, \quad n=1,2,\cdots,N \end{aligned} \tag{24}$$

Discrete Eq. (8) and Eq. (9), we have

(1) $j=1,2,\cdots,J-1$, and $k=1,2,\cdots,K-1$

$$g_n^{i,j,k} - g_n^{i-1,j,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,j,k} + \mathcal{G}_n^{i-1,j,k} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{h_n^{i,j-1,k}-2h_n^{i,j,k}+h_n^{i,j+1,k}}{(\Delta\chi)^2} + \frac{h_n^{i,j,k-1}-2h_n^{i,j,k}+h_n^{i,j,k+1}}{(\Delta\psi)^2}\right] \tag{25}$$

$$h_n^{i,j,k} - h_n^{i-1,j,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,j,k} + \mathcal{H}_n^{i-1,j,k} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{g_n^{i,j-1,k}-2g_n^{i,j,k}+g_n^{i,j+1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,j,k-1}-2g_n^{i,j,k}+g_n^{i,j,k+1}}{(\Delta\psi)^2}\right] \tag{26}$$

(2) $j=1,2,\cdots,J-1$, and $k=0$

$$g_n^{i,j,0} - g_n^{i-1,j,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,j,0} + \mathcal{G}_n^{i-1,j,0} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{h_n^{i,j-1,0}-2h_n^{i,j,0}+h_n^{i,j+1,0}}{(\Delta\chi)^2} + \frac{-2h_n^{i,j,0}+2h_n^{i,j,1}}{(\Delta\psi)^2}\right] \tag{27}$$

$$h_n^{i,j,0} - h_n^{i-1,j,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,j,0} + \mathcal{H}_n^{i-1,j,0} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{g_n^{i,j-1,0}-2g_n^{i,j,0}+g_n^{i,j+1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,j,0}+2g_n^{i,j,1}}{(\Delta\psi)^2}\right] \tag{28}$$

(3) $j=0$, and $k=1,2,\cdots,K-1$

$$g_n^{i,0,k} - g_n^{i-1,0,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,0,k} + \mathcal{G}_n^{i-1,0,k} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{-2h_n^{i,0,k}+2h_n^{i,1,k}}{(\Delta\chi)^2} + \frac{h_n^{i,0,k-1}-2h_n^{i,0,k}+h_n^{i,0,k+1}}{(\Delta\psi)^2}\right] \tag{29}$$

$$h_n^{i,0,k} - h_n^{i-1,0,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,0,k} + \mathcal{H}_n^{i-1,0,k} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{-2g_n^{i,0,k}+2g_n^{i,1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,0,k-1}-2g_n^{i,0,k}+g_n^{i,0,k+1}}{(\Delta\psi)^2}\right] \tag{30}$$

(4) $j=0$, and $k=0$

$$g_n^{i,0,0} - g_n^{i-1,0,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,0,0} + \mathcal{G}_n^{i-1,0,0} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{-2h_n^{i,0,0}+2h_n^{i,1,0}}{(\Delta\chi)^2} + \frac{-2h_n^{i,0,0}+2h_n^{i,0,1}}{(\Delta\psi)^2}\right] \tag{31}$$

$$h_n^{i,0,0} - h_n^{i-1,0,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,0,0} + \mathcal{H}_n^{i-1,0,0} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \times \left[\frac{-2g_n^{i,0,0}+2g_n^{i,1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,0,0}+2g_n^{i,0,1}}{(\Delta\psi)^2}\right] \tag{32}$$

where

$$\mathcal{G}_n^{i,j,k} = \frac{n\Delta\sigma_{\mathrm{m}}}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m},i})}\left[\frac{1}{2}\sum_{m=1}^{n-1}(g_m^{i,j,k}g_{n-m}^{i,j,k} - h_m^{i,j,k}h_{n-m}^{i,j,k}) - \sum_{n+1}^{N}(g_m^{i,j,k}g_{m-n}^{i,j,k} + h_m^{i,j,k}h_{m-n}^{i,j,k})\right] \tag{33}$$

$$\mathcal{H}_n^{i,j,k} = \frac{n\Delta\sigma_{\mathrm{m}}}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m},i})}\left[\sum_{m=1}^{n-1} g_m^{i,j,k}h_{n-m}^{i,j,k} + \sum_{n+1}^{N}(g_m^{i,j,k}h_{m-n}^{i,j,k} - h_m^{i,j,k}g_{m-n}^{i,j,k})\right] \tag{34}$$

4 Legacy

4.1 Matrix Form

$$I_{2(J-1)}\begin{bmatrix} G_n^i - G_n^{i-1} \\ H_n^i - H_n^{i-1}\end{bmatrix} = \begin{bmatrix} B_{(J-1)\times(J-1)} & A_{(J-1)\times(J-1)} \\ -A_{(J-1)\times(J-1)} & B_{(J-1)\times(J-1)}\end{bmatrix}\begin{bmatrix} G_n^i \\ H_n^i\end{bmatrix} + \begin{bmatrix} \mathcal{H}_n^i \\ \mathcal{G}_n^i\end{bmatrix} + \begin{bmatrix} S_n^{i-1} \\ T_n^{i-1}\end{bmatrix} \tag{35}$$

$$(i = 1, 2, \cdots, I-1, I)$$

It can be written as

$$\left(I_{2(J-1)} - \begin{bmatrix} B & A \\ -A & B\end{bmatrix}\right)\begin{bmatrix} G_n^i \\ H_n^i\end{bmatrix} = I_{2(J-1)}\begin{bmatrix} G_n^{i-1} \\ H_n^{i-1}\end{bmatrix} + \begin{bmatrix} \mathcal{H}_n^i \\ \mathcal{G}_n^i\end{bmatrix} + \begin{bmatrix} S_n^{i-1} \\ T_n^{i-1}\end{bmatrix} \tag{36}$$

where

$$G_n^i = \begin{bmatrix} g_n^{i,1} \\ g_n^{i,2} \\ \vdots \\ g_n^{i,J-1}\end{bmatrix}, \quad G_n^{i-1} = \begin{bmatrix} g_n^{i-1,1} \\ g_n^{i-1,2} \\ \vdots \\ g_n^{i-1,J-1}\end{bmatrix}, \quad H_n^i = \begin{bmatrix} h_n^{i,1} \\ h_n^{i,2} \\ \vdots \\ h_n^{i,J-1}\end{bmatrix}, \quad H_n^{i-1} = \begin{bmatrix} h_n^{i-1,1} \\ h_n^{i-1,2} \\ \vdots \\ h_n^{i-1,J-1}\end{bmatrix} \tag{37}$$

$$\mathcal{G}_n^i = \begin{bmatrix} a_1 g_n^{i,0} \\ O_{(J-3)\times 1} \\ b_{J-1}g_n^{i,J}\end{bmatrix}, \quad \mathcal{H}_n^i = \begin{bmatrix} a_1 h_n^{i,0} \\ O_{(J-3)\times 1} \\ b_{J-1}h_n^{i,J}\end{bmatrix}, \quad h_n^{i,J} = h_n(\sigma_i, \xi_J) = h_n(\sigma_i, \xi_{\max}) = 0 \tag{38}$$

$$A = \begin{bmatrix} q & b_1 & & & & \\ a_2 & q & b_2 & & & \\ & & & \vdots & & \\ & & & a_{J-2} & q & b_{J-2} \\ & & & & a_{J-1} & q \end{bmatrix}, \quad \begin{cases} a_j = (1 - 0.5/j)\dfrac{\Delta\sigma}{4n(\Delta\xi)^2} \\ b_j = (1 + 0.5/j)\dfrac{\Delta\sigma}{4n(\Delta\xi)^2} \\ q = -\dfrac{\Delta\sigma}{2n(\Delta\xi)^2} \end{cases} \quad j = 1, 2, \cdots, J-1 \tag{39}$$

$$B = -\Delta\sigma\alpha_n R_{\mathrm{D}} I_{J-1} \tag{40}$$

$$S_n^{i-1} = \begin{bmatrix} S_n^{i-1,1} \\ S_n^{i-1,2} \\ \vdots \\ S_n^{i-1,J-1}\end{bmatrix}, \quad T_n^{i-1} = \begin{bmatrix} T_n^{i-1,1} \\ T_n^{i-1,2} \\ \vdots \\ T_n^{i-1,J-1}\end{bmatrix} \tag{41}$$

$$S_n^{i-1,j} = \Delta\sigma n\frac{R_{\mathrm{D}}}{2l_{\mathrm{D}}}\left[\frac{1}{2}\sum_{m=1}^{n-1}(g_m^{i-1,j}g_{n-m}^{i-1,j} - h_m^{i-1,j}h_{n-m}^{i-1,j}) - \sum_{m=n+1}^{M}(g_{m-n}^{i-1,j}g_m^{i-1,j} + h_{m-n}^{i-1,j}h_m^{i-1,j})\right] \tag{42}$$

$$T_n^{i-1,j} = \Delta\sigma n\frac{R_{\mathrm{D}}}{2l_{\mathrm{D}}}\left[\frac{1}{2}\sum_{m=1}^{n-1}(h_m^{i-1,j}g_{n-m}^{i-1,j} + g_m^{i-1,j}h_{n-m}^{i-1,j}) + \sum_{m=n+1}^{M}(h_{m-n}^{i-1,j}g_m^{i-1,j} - g_{m-n}^{i-1,j}h_m^{i-1,j})\right] \tag{43}$$

$$g_n^{i,0} = (1 - \Delta\sigma\alpha_n R_{\mathrm{D}})g_n^{i-1,0} + \frac{\Delta\sigma}{n(\Delta\xi)^2}(h_n^{i-1,1} - h_n^{i-1,0}) + S_n^{i-1,0} \tag{44}$$

$$h_n^{i,0} = (1 - \Delta\sigma\alpha_n R_{\mathrm{D}})h_n^{i-1,0} - \frac{\Delta\sigma}{n(\Delta\xi)^2}(g_n^{i-1,1} - g_n^{i-1,0}) + T_n^{i-1,0} \tag{45}$$

$$(i = 1, 2, \cdots, I)$$

5 Crank-Nicolson Method

$$ \tag{46} $$

5.1 Data Set: Kamakura 1989

Fig. 6

$$\Delta\sigma = 3 \times 10^{-4}, \quad \sigma_{\max} = 10, \quad \Delta\xi = 0.03, \quad \xi_{\max} = 6, \quad T = 23.7°\mathrm{C}, \quad h_{\mathrm{r}} = 65.2\%$$

$$M = 18, \quad f_1 = 25\mathrm{kHz}, \quad f_2 = 30\mathrm{kHz}, \quad P_0(25\mathrm{kHz}) = 109.5\mathrm{dB}, \quad P_0(30\mathrm{kHz}) = 108.5\mathrm{dB}$$

$$a = 0.21\mathrm{m}$$