Notes on the project ParametricAcousticArray

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Contents

1 List of Symbols

Symbols	Descriptions
a	radius of the source
c_0	isentropic speed of sound at ambient values of pressure and density
f_1, f_2	$f_1 = N_1 f_\mathrm{b}$, $f_2 = N_2 f_b$, primary frequencies
f_b	$= \gcd f_2 - f_1$, basic frequency
f_m	$= (f_1 + f_2)/2$, mean primary frequency
k	$= 2\pi f/c_0$ wavenumber
k_m	$= (k_1 + k_2)/2$ wavenumber of the mean primary frequency
l_D	$= (\beta k \epsilon)^{-1}$, shock formation distance of a plane wave
L_x, L_y	the size of the rectangular transducer in x, y direction, and $L_y \geqslant L_x$
N_m	$= (N_1 + N_2)/2$
p	pressure
p_0	ambient pressure
$\bar{p}$	$= (p - p_0)/P_0$, acoustic pressure normalized to P_0
P_0	$\rho_0 c_0 v_0$, acoustic pressure peak amplitude on the source
R_D	$= ka^2/2 = \pi a^2/\lambda$, Rayleigh distance of primary frequency
R_Dm	$= k_\mathrm{m} a^2/2 = \pi a^2/\lambda_\mathrm{m}$, Rayleigh distance of mean primary frequency
t	time
$\mathbf{u}$	$= (u_x, u_y) = \mathbf{r}/L_y = (x, y)/L_y$
v_0	characteristic velocity peak amplitude on the source
x, y, z	dimensional coordinates, z along the direction of propagation
$\mathbf{r}$	(x, y)
r	$\lvert\mathbf{r}\rvert$
α	absorption coefficient, in Neper per meter
β	parameter of nonlinearity
ϵ	$= u_0/c_0$, Mach number
σ	$= z/R_\mathrm{D}$, nondimensional z
σ_D	$= l_\mathrm{D}/R_\mathrm{D}$
σ_Dm	$= l_\mathrm{Dm}/R_\mathrm{Dm}$
χ	$= x/R_\mathrm{Dm}$
ψ	$= y/R_\mathrm{Dm}$
γ	$= L_y/L_x$, the aspect ratio
$\nabla^2_{x,y}$	$= \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$
∇^2_u	$\dfrac{\partial^2}{\partial \mathbf{u}^2} = \dfrac{1}{u}\dfrac{\partial}{\partial u}\left(u\dfrac{\partial}{\partial u}\right)$
$\nabla^2_{\chi,\psi}$	$= \dfrac{\partial^2}{\partial \chi^2} + \dfrac{\partial^2}{\partial \psi^2}$

2 Theoretical Model and Analytical Solutions

2.1 KZK Equation

Nonlinear propagation of finite amplitude sound waves in dissipative and homogenous fluid is decribed by the following KZK equation:

$$\frac{\partial^2 \bar{p}}{\partial \sigma \partial \tau} = \alpha R_\mathrm{D} \frac{\partial^3 \bar{p}}{\partial \tau^3} + \frac{1}{4}\nabla^2_\perp \bar{p} + \frac{1}{2\sigma_\mathrm{D}} \frac{\partial^2 (\bar{p})^2}{\partial \tau^2} \tag{1}$$

If the source is bifrequency, i.e.

$$f_i = N_i f_\mathrm{b}, \quad N_i \in \mathbb{N}^+, \quad i = 1, 2 \tag{2}$$

2.1.1 Rectangular profile

The Eq. (1) is written as

$$\frac{\partial^2 \bar{p}}{\partial \sigma \partial \tau} = \alpha R_{\mathrm{D}} \frac{\partial^3 \bar{p}}{\partial \tau^3} + \frac{1}{\pi \gamma} \nabla_u^2 \bar{p} + \frac{1}{2\sigma_{\mathrm{D}}} \frac{\partial^2 (\bar{p})^2}{\partial \tau^2} \tag{3}$$

The transformations are:

$$\sigma_{\mathrm{m}} = z/R_{\mathrm{Dm}}, \quad T = (1+\sigma_{\mathrm{m}})\bar{p}, \quad \tau_{\mathrm{m}} = \tau - \frac{\pi\gamma}{4} \frac{\xi^2}{N_{\mathrm{m}}(1+\sigma_{\mathrm{m}})}, \quad (\chi, \psi) = \frac{(u_x, u_y)}{1+\sigma_{\mathrm{m}}} \tag{4}$$

Then, the transformed KZK equation is obtained by Eq. (3)

$$\frac{\partial^2 T}{\partial \sigma_{\mathrm{m}} \partial \tau_{\mathrm{m}}} = \frac{\alpha_{\mathrm{m}}}{N_{\mathrm{m}}^2} R_{\mathrm{Dm}} \frac{\partial^3 T}{\partial \tau_{\mathrm{m}}^3} + \frac{N_{\mathrm{m}}}{\pi\gamma(1+\sigma_{\mathrm{m}})^2} \nabla_{\chi,\psi}^2 T + \frac{1}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m}})} \frac{\partial^2 (T^2)}{\partial \tau_{\mathrm{m}}^2} \tag{5}$$

2.2 Spectral Solutions

A solution of Eq. (3) is sought in the form of a Fourier series

$$\bar{p} = \sum_{n=1}^{\infty} (g_n \sin n\tau + h_n \cos n\tau) = \sum_{n=1}^{\infty} \bar{p}_n \tag{6}$$

where g_n and h_n are functions of spatial coordinates.

The solution of Eq. (5) is

$$T = \sum_{n=1}^{\infty} [g_n(\chi, \psi, \sigma_{\mathrm{m}}) \sin n\tau_{\mathrm{m}} + h_n(\chi, \psi, \sigma_{\mathrm{m}}) \cos n\tau_{\mathrm{m}}] = \sum_{n=1}^{\infty} T_n \tag{7}$$

2.3 Coupled PDEs

We thus obtain the following set of coupled partial differential equations for g_n and h_n.

2.3.1 Rectangular Profile

$$\frac{\partial g_n}{\partial \sigma} = -\alpha_n R_{\mathrm{Dm}} g_n + \frac{1}{\pi\gamma n(1+\sigma_{\mathrm{m}})^2} \nabla_{\chi,\psi}^2 h_n + \frac{n}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m}})} \times \left[\frac{1}{2} \sum_{m=1}^{n-1} (g_m g_{n-m} - h_m h_{n-m}) - \sum_{m=n+1}^{\infty} (g_{m-n} g_m + h_{m-n} h_m) \right] \tag{8}$$

$$\frac{\partial h_n}{\partial \sigma} = -\alpha_n R_{\mathrm{Dm}} h_n - \frac{1}{\pi\gamma n(1+\sigma_{\mathrm{m}})^2} \nabla_{\chi,\psi}^2 g_n + \frac{n}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m}})} \times \left[\sum_{m=1}^{n-1} g_m h_{n-m} + \sum_{m=n+1}^{\infty} (h_{m-n} g_m - g_{m-n} h_m) \right] \tag{9}$$

2.4 Boundary Conditions

The boundary condition for g_n and h_n at $\sigma = \sigma_{\mathrm{m}} = 0$ is

$$g_n = \frac{1}{\pi} \int_{-\pi}^{\pi} c \sin n\tau_{\mathrm{m}} \, \mathrm{d}\tau_{\mathrm{m}}$$

$$h_n = \frac{1}{\pi} \int_{-\pi}^{\pi} c \cos n\tau_{\mathrm{m}} \, \mathrm{d}\tau_{\mathrm{m}}$$

2.4.1 Circular Profile

The boundary condition at $\sigma = \sigma_{\mathrm{m}} = 0$ is

$$\bar{p}(\sigma_{\mathrm{m}} = 0, \boldsymbol{\xi}, \tau) = c(\boldsymbol{\xi}, \tau) \tag{10}$$

Suppose it is

$$\bar{p}(\sigma_{\mathrm{m}} = 0, \boldsymbol{\xi}, \tau) = c_1(\boldsymbol{\xi}) \sin N_1\tau + c_2(\boldsymbol{\xi}) \sin N_2\tau \tag{11}$$

2.4.2 Rectangular Profile

The boundary condition at $\sigma = \sigma_\mathrm{m} = 0$ is

$$\bar{p}(\sigma_\mathrm{m} = 0, \mathbf{u}, \tau) = c(\mathbf{u}, \tau) \tag{12}$$

Suppose it is

$$\bar{p}(\mathbf{u}, \sigma_\mathrm{m} = 0, \tau) = c_1(\mathbf{u}) \sin N_1\tau + c_2(\mathbf{u}) \sin N_2\tau \tag{13}$$

$$\begin{aligned} &= c_1(\mathbf{u})\left[\cos\left(\frac{\pi\gamma N_1}{4N_\mathrm{m}}\xi^2\right)\sin N_1\tau_\mathrm{m} + \sin\left(\frac{\pi\gamma N_1}{4N_\mathrm{m}}\xi^2\right)\cos N_1\tau_\mathrm{m}\right] \\ &+ c_2(\mathbf{u})\left[\cos\left(\frac{\pi\gamma N_1}{4N_\mathrm{m}}\xi^2\right)\sin N_2\tau_\mathrm{m} + \sin\left(\frac{\pi\gamma N_1}{4N_\mathrm{m}}\xi^2\right)\cos N_2\tau_\mathrm{m}\right] \end{aligned} \tag{14}$$

Then

$$g_{N_i}(\chi, \psi, \sigma_\mathrm{m} = 0) = c_i(\chi, \psi) \cos\left[\frac{\pi\gamma N_i}{4N_\mathrm{m}}(\chi^2 + \psi^2)\right], \quad i = 1, 2 \tag{15}$$

$$h_{N_i}(\chi, \psi, \sigma_\mathrm{m} = 0) = c_i(\chi, \psi) \sin\left[\frac{\pi\gamma N_i}{4N_\mathrm{m}}(\chi^2 + \psi^2)\right], \quad i = 1, 2 \tag{16}$$

$$g_n(\chi, \psi, \sigma_\mathrm{m} = 0) = h_n(\chi, \psi, \sigma_\mathrm{m} = 0) = 0, \quad n \neq N_1, N_2 \tag{17}$$

2.5 Sound Power of Waves

The overall power of a wave is found by integrating the intensity across the entire field. Within the parabolic approximation the linear plane wave impedance relation is valid [1], and the nondimensional power of a wave is given by (Eq. (2.6) in [2])

$$\overline{\mathcal{P}}(\sigma_\mathrm{m}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \bar{p}^2(\xi, \sigma_\mathrm{m}, \tau)\xi \,\mathrm{d}\xi \,\mathrm{d}\tau$$

The nondimensional power of an individual harmonic component is given by (Eq (2.7) in [2])

$$\overline{\mathcal{P}}_n(\sigma_\mathrm{m}) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \bar{p}_n^2(\xi, \sigma_\mathrm{m}, \tau)\xi \,\mathrm{d}\xi \,\mathrm{d}\tau \tag{18}$$

Transform the coordinates into $(\zeta, \sigma_\mathrm{m})$, Eq. (18) can be written into

$$\begin{aligned} \overline{\mathcal{P}}_n(\sigma_\mathrm{m}) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} [g_n(\zeta, \sigma_\mathrm{m}) \sin n\tau_\mathrm{m} + h_n(\zeta, \sigma_\mathrm{m}) \cos n\tau_\mathrm{m}]^2 \,\mathrm{d}\zeta \,\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} [(g_n(\zeta, \sigma_\mathrm{m}) \cos\varphi_n + h_n(\zeta, \sigma_\mathrm{m}) \sin\varphi_n) \sin n\tau \\ &\qquad + (-g_n(\zeta, \sigma_\mathrm{m}) \sin\varphi_n + h_n(\zeta, \sigma_\mathrm{m}) \cos\varphi_n) \cos n\tau]^2 \,\zeta \,\mathrm{d}\zeta \,\mathrm{d}\tau \end{aligned} \tag{19}$$

$$\begin{aligned} &= \frac{1}{2}\int_0^{\infty} \Big[(g_n(\zeta, \sigma_\mathrm{m}) \cos\varphi_n + h_n(\zeta, \sigma_\mathrm{m}) \sin\varphi_n)^2 \\ &\qquad + (-g_n(\zeta, \sigma_\mathrm{m}) \sin\varphi_n + h_n(\zeta, \sigma_\mathrm{m}) \cos\varphi_n)^2\Big] \,\zeta \,\mathrm{d}\zeta \end{aligned} \tag{20}$$

where

$$\varphi_n = \frac{n}{N_\mathrm{m}}(1 + \sigma_\mathrm{m})\zeta^2$$

3 Numerical Methods

3.1 Discretization

3.1.1 Rectangular Profile

$$\sigma_i = i\Delta\sigma, \quad i = 0, 1, \cdots, I-1, I, \quad \sigma_0 = 0, \quad \sigma_I = \sigma_{\max}, \quad \Delta\sigma = \frac{\sigma_{\max}}{I}$$

$$\chi_j = j\Delta\chi, \quad j = 0, 1, \cdots, J-1, J, \quad \chi_0 = 0, \quad \chi_J = \chi_{\max}, \quad \Delta\chi = \frac{\chi_{\max}}{J}$$

$$\psi_k = k\Delta\psi, \quad k = 0, 1, \cdots, K-1, K, \quad \psi_0 = 0, \quad \psi_K = \psi_{\max}, \quad \Delta\psi = \frac{\psi_{\max}}{K}$$

For $\partial g_n/\partial\sigma$ or $\partial h_n/\partial\sigma$, we adopt Backward Implicit Finite Difference

$$\left.\frac{\partial g_n(\chi,\psi,\sigma_{\mathrm{m}})}{\partial\sigma_{\mathrm{m}}}\right|_{i,j,k} = \frac{g_n^{i,j,k} - g_n^{i-1,j,k}}{\Delta\sigma_{\mathrm{m}}}, \quad i = 1, 2, \cdots, I \tag{21}$$

where

$$\begin{aligned} g_n^{i,j,k} &\approx g_n(\chi_j, \psi_k, \sigma_{\mathrm{m},i}) \\ h_n^{i,j,k} &\approx h_n(\chi_j, \psi_k, \sigma_{\mathrm{m},i}) \end{aligned} \tag{22}$$

For $\nabla^2_{\chi,\psi} g_n$ or $\nabla^2_{\chi,\psi} h_n$, 4 cases are considered:
(1) $j = 1, 2, \cdots, J-1$, and $k = 1, 2, \cdots, K-1$

$$\nabla^2_{\chi,\psi}\big|_{i,j,k} = \frac{g_n^{i,j-1,k} - 2g_n^{i,j,k} + g_n^{i,j+1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,j,k-1} - 2g_n^{i,j,k} + g_n^{i,j,k+1}}{(\Delta\psi)^2} \tag{23}$$

(2) $j = 1, 2, \cdots, J-1$, and $k = 0$

$$\nabla^2_{\chi,\psi}\big|_{i,j,0} = \frac{g_n^{i,j-1,0} - 2g_n^{i,j,0} + g_n^{i,j+1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,j,0} + 2g_n^{i,j,1}}{(\Delta\psi)^2} \tag{24}$$

(3) $j = 0$, and $k = 1, 2, \cdots, K-1$

$$\nabla^2_{\chi,\psi}\big|_{i,0,k} = \frac{-2g_n^{i,0,k} + 2g_n^{i,1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,0,k-1} - 2g_n^{i,0,k} + g_n^{i,0,k+1}}{(\Delta\psi)^2} \tag{25}$$

(4) $j = 0$, and $k = 0$

$$\nabla^2_{\chi,\psi}\big|_{i,0,0} = \frac{-2g_n^{i,0,0} + 2g_n^{i,1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,0,0} + 2g_n^{i,0,1}}{(\Delta\psi)^2} \tag{26}$$

where the symmetric conditions are used

$$\begin{aligned} \left.\frac{\partial g_n}{\partial\chi}\right|_{\chi=0} &= \left.\frac{\partial h_n}{\partial\chi}\right|_{\chi=0} = 0, \quad n = 1, 2, \cdots, N \\ \left.\frac{\partial g_n}{\partial\psi}\right|_{\psi=0} &= \left.\frac{\partial h_n}{\partial\psi}\right|_{\psi=0} = 0, \quad n = 1, 2, \cdots, N \end{aligned} \tag{27}$$

Discrete Eq. (8) and Eq. (9), we have
(1) $j = 1, 2, \cdots, J-1$, and $k = 1, 2, \cdots, K-1$

$$\begin{aligned} g_n^{i,j,k} - g_n^{i-1,j,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,j,k} + \mathcal{G}_n^{i-1,j,k} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\ \times\left[\frac{h_n^{i,j-1,k} - 2h_n^{i,j,k} + h_n^{i,j+1,k}}{(\Delta\chi)^2} + \frac{h_n^{i,j,k-1} - 2h_n^{i,j,k} + h_n^{i,j,k+1}}{(\Delta\psi)^2}\right] \end{aligned} \tag{28}$$

$$\begin{aligned} h_n^{i,j,k} - h_n^{i-1,j,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,j,k} + \mathcal{H}_n^{i-1,j,k} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\ \times\left[\frac{g_n^{i,j-1,k} - 2g_n^{i,j,k} + g_n^{i,j+1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,j,k-1} - 2g_n^{i,j,k} + g_n^{i,j,k+1}}{(\Delta\psi)^2}\right] \end{aligned} \tag{29}$$

(2) $j = 1, 2, \cdots, J-1$, and $k = 0$

$$
\begin{aligned}
g_n^{i,j,0} - g_n^{i-1,j,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,j,0} + \mathcal{G}_n^{i-1,j,0} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\
\times \left[\frac{h_n^{i,j-1,0} - 2h_n^{i,j,0} + h_n^{i,j+1,0}}{(\Delta\chi)^2} + \frac{-2h_n^{i,j,0} + 2h_n^{i,j,1}}{(\Delta\psi)^2} \right]
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
h_n^{i,j,0} - h_n^{i-1,j,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,j,0} + \mathcal{H}_n^{i-1,j,0} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\
\times \left[\frac{g_n^{i,j-1,0} - 2g_n^{i,j,0} + g_n^{i,j+1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,j,0} + 2g_n^{i,j,1}}{(\Delta\psi)^2} \right]
\end{aligned} \tag{31}
$$

(3) $j = 0$, and $k = 1, 2, \cdots, K-1$

$$
\begin{aligned}
g_n^{i,0,k} - g_n^{i-1,0,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,0,k} + \mathcal{G}_n^{i-1,0,k} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\
\times \left[\frac{-2h_n^{i,0,k} + 2h_n^{i,1,k}}{(\Delta\chi)^2} + \frac{h_n^{i,0,k-1} - 2h_n^{i,0,k} + h_n^{i,0,k+1}}{(\Delta\psi)^2} \right]
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
h_n^{i,0,k} - h_n^{i-1,0,k} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,0,k} + \mathcal{H}_n^{i-1,0,k} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\
\times \left[\frac{-2g_n^{i,0,k} + 2g_n^{i,1,k}}{(\Delta\chi)^2} + \frac{g_n^{i,0,k-1} - 2g_n^{i,0,k} + g_n^{i,0,k+1}}{(\Delta\psi)^2} \right]
\end{aligned} \tag{33}
$$

(4) $j = 0$, and $k = 0$

$$
\begin{aligned}
g_n^{i,0,0} - g_n^{i-1,0,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} g_n^{i,0,0} + \mathcal{G}_n^{i-1,0,0} + \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\
\times \left[\frac{-2h_n^{i,0,0} + 2h_n^{i,1,0}}{(\Delta\chi)^2} + \frac{-2h_n^{i,0,0} + 2h_n^{i,0,1}}{(\Delta\psi)^2} \right]
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
h_n^{i,0,0} - h_n^{i-1,0,0} = -\Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}} h_n^{i,0,0} + \mathcal{H}_n^{i-1,0,0} - \frac{\Delta\sigma_{\mathrm{m}}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2} \\
\times \left[\frac{-2g_n^{i,0,0} + 2g_n^{i,1,0}}{(\Delta\chi)^2} + \frac{-2g_n^{i,0,0} + 2g_n^{i,0,1}}{(\Delta\psi)^2} \right]
\end{aligned} \tag{35}
$$

where

$$
\mathcal{G}_n^{i,j,k} = \frac{n\Delta\sigma_{\mathrm{m}}}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m},i})} \left[\frac{1}{2}\sum_{m=1}^{n-1} (g_m^{i,j,k} g_{n-m}^{i,j,k} - h_m^{i,j,k} h_{n-m}^{i,j,k}) - \sum_{m=n+1}^{N} (g_m^{i,j,k} g_{m-n}^{i,j,k} + h_m^{i,j,k} h_{m-n}^{i,j,k}) \right] \tag{36}
$$

$$
\mathcal{H}_n^{i,j,k} = \frac{n\Delta\sigma_{\mathrm{m}}}{2N_{\mathrm{m}}\sigma_{\mathrm{Dm}}(1+\sigma_{\mathrm{m},i})} \left[\sum_{m=1}^{n-1} g_m^{i,j,k} h_{n-m}^{i,j,k} + \sum_{m=n+1}^{N} (g_m^{i,j,k} h_{m-n}^{i,j,k} - h_m^{i,j,k} g_{m-n}^{i,j,k}) \right] \tag{37}
$$

3.2 Matrix Form

3.2.1 Rectangular Profile

$$
\begin{bmatrix} \mathbf{E}_n & -\mathbf{A}_n^i \\ \mathbf{A}_n^i & \mathbf{E}_n \end{bmatrix} \begin{bmatrix} \mathbf{G}_n^i \\ \mathbf{H}_n^i \end{bmatrix} = \begin{bmatrix} \mathbf{G}_n^{i-1} \\ \mathbf{H}_n^{i-1} \end{bmatrix} + \begin{bmatrix} \mathcal{G}_n^{i-1} \\ \mathcal{H}_n^{i-1} \end{bmatrix} \tag{38}
$$

where

$$
\mathbf{E}_n = (1 + \Delta\sigma_{\mathrm{m}}\alpha_n R_{\mathrm{Dm}})\mathbf{I}_{JK\times JK}
$$

$$
\mathbf{G}_n^i = \begin{bmatrix} \mathbf{g}_n^{i,0,\cdot} \\ \mathbf{g}_n^{i,1,\cdot} \\ \vdots \\ \mathbf{g}_n^{i,J-1,\cdot} \end{bmatrix}, \quad \mathbf{g}_n^{i,j,\cdot} = \begin{bmatrix} \mathbf{g}_n^{i,j,0} \\ \mathbf{g}_n^{i,j,1} \\ \vdots \\ \mathbf{g}_n^{i,j,K-1} \end{bmatrix}, \quad \mathbf{H}_n^i = \begin{bmatrix} \mathbf{h}_n^{i,0,\cdot} \\ \mathbf{h}_n^{i,1,\cdot} \\ \vdots \\ \mathbf{h}_n^{i,J-1,\cdot} \end{bmatrix}, \quad \mathbf{h}_n^{i,j,\cdot} = \begin{bmatrix} \mathbf{h}_n^{i,j,0} \\ \mathbf{h}_n^{i,j,1} \\ \vdots \\ \mathbf{h}_n^{i,j,K-1} \end{bmatrix}
$$

$$\mathcal{G}_n^i = \begin{bmatrix} \mathcal{G}_n^{i,0,\cdot} \\ \mathcal{G}_n^{i,1,\cdot} \\ \vdots \\ \mathcal{G}_n^{i,J-1,\cdot} \end{bmatrix}, \quad \mathcal{G}_n^{i,j,\cdot} = \begin{bmatrix} \mathcal{G}_n^{i,j,0} \\ \mathcal{G}_n^{i,j,1} \\ \vdots \\ \mathcal{G}_n^{i,j,K-1} \end{bmatrix}, \quad \mathcal{H}_n^i = \begin{bmatrix} \mathcal{H}_n^{i,0,\cdot} \\ \mathcal{H}_n^{i,1,\cdot} \\ \vdots \\ \mathcal{H}_n^{i,J-1,\cdot} \end{bmatrix}, \quad \mathcal{H}_n^{i,j,\cdot} = \begin{bmatrix} \mathcal{H}_n^{i,j,0} \\ \mathcal{H}_n^{i,j,1} \\ \vdots \\ \mathcal{H}_n^{i,j,K-1} \end{bmatrix}$$

$$\mathbf{A}_n^i = \frac{\Delta\sigma_\mathrm{m}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2(\Delta\chi)^2}\mathbf{A}_{n,\chi} + \frac{\Delta\sigma_\mathrm{m}}{\pi\gamma n(1+\sigma_{\mathrm{m},i})^2(\Delta\psi)^2}\mathbf{A}_{n,\psi}$$

$$\mathbf{A}_{n,\chi} = \begin{bmatrix} \mathbf{B} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{B} \end{bmatrix}_{JK\times JK}, \quad \mathbf{B} = \begin{bmatrix} -2 & 2 & & & & 0 \\ 1 & -2 & 1 & & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ 0 & & & & 1 & -2 \end{bmatrix}_{K\times K}$$

$$\mathbf{A}_{n,\psi} = \begin{bmatrix} -2\mathbf{I}_{K\times K} & 2\mathbf{I}_{K\times K} & & & & \mathbf{0}_{K\times K} \\ \mathbf{I}_{K\times K} & -2\mathbf{I}_{K\times K} & \mathbf{I}_{K\times K} & & & \\ & & & \ddots & & \\ & & & \mathbf{I}_{K\times K} & -2\mathbf{I}_{K\times K} & \mathbf{I}_{K\times K} \\ \mathbf{0}_{K\times K} & & & & \mathbf{I}_{K\times K} & -2\mathbf{I}_{K\times K} \end{bmatrix}_{JK\times JK}$$

4 Test Data

4.1 Circular Profile

4.1.1 Kamakura 1989

Fig. 6

$$\Delta\sigma = 3\times 10^{-4}, \quad \sigma_\mathrm{max} = 10, \quad \Delta\xi = 0.03, \quad \xi_\mathrm{max} = 6, \quad T = 23.7°\mathrm{C}, \quad h_\mathrm{r} = 65.2\%$$

$$M = 18, \quad f_1 = 25\mathrm{kHz}, \quad f_2 = 30\mathrm{kHz}, \quad P_0(25\mathrm{kHz}) = 109.5\mathrm{dB}, \quad P_0(30\mathrm{kHz}) = 108.5\mathrm{dB}$$

$$a = 0.21\mathrm{m}$$

References

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