

The Principles of Acoustics

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September 19, 2018



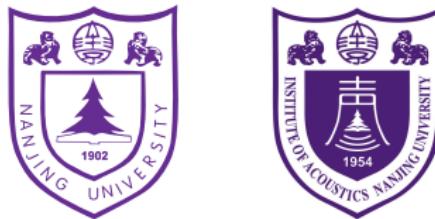
- **Textbook A:** 《声学原理》—程建春著
- **Textbook B:** 《声学基础》第二版—杜功焕等著
- **Lecture Notes** — JIAJIN ZHONG
- 《声学基础》课程课件—王新龙著
- 《理论声学》课程课件—王新龙著
- 网易博客博文: <http://xlwangnu.blog.163.com/> — 王新龙著
- 《数学物理方法》第二版—吴崇试等著
- *Fundamentals of Acoustics - 4ed* — LAWRENCE E. KINSLER, etc.
- *Mathematical Methods in the Physical Sciences - 3ed* — MARY L. BOAS, etc.

Sound Waves in Ideal Fluids

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Introduction

Acoustic waves

Acoustic waves constitute one kind of **pressure fluctuation** that can exist in a **compressible fluid**.

- mechanical waves
- media: **elastic continuum**, e.g. gases, liquids, solids

Ideal Fluid: lossless, no dissipative effects

- **inviscid**: No **viscous force**
- No **heat conduction**

Continuum Hypothesis

- **fluid particles(elements)**
 - infinitesimal volume of the fluid large enough to contain millions of molecules → continuous medium
 - small enough → all acoustic variables are uniform throughout

Two Description Methods

Lagrangian Method

- Follow **the individual fluid particle** located at $\mathbf{R}_0 = (a, b, c)$ at time $t = t_0$.
- The location of **this** particle $\mathbf{R} = (X, Y, Z)$ at time t is:

$$\begin{cases} X = X(a, b, c, t) \\ Y = Y(a, b, c, t) \\ Z = Z(a, b, c, t) \end{cases}$$

Eulerian Method

- Don't care about the specific fluid particles.
- The **field representation** is used.
- The **physical quantities** f at the location $\mathbf{r} = (x, y, z)$ at time t : $f(x, y, z, t)$.
 - 3 **state variables**: pressure (p), density (ρ), and temperature (T)
 - 3 **velocity components**: $\mathbf{v} = (v_x, v_y, v_z)$
- 6 (differential)^a equations** are needed to solve the above 6 physical quantities
 - 5 equations: **conservation theorems**
 - 1 equation: **state equation**

^aWhy **differential equations**? How about **integral equations**?

Lagrangian Method

- **particle displacement:** $\Delta \mathbf{R} \equiv \mathbf{R} - \mathbf{R}_0 = \boldsymbol{\xi} = (\xi, \eta, \zeta)$

- the location of the particle:

$$\begin{cases} X = a + \xi(a, b, c, t) \\ Y = b + \eta(a, b, c, t) \\ Z = c + \zeta(a, b, c, t) \end{cases}$$

- the **volume** of the fluid particle: $\Delta V(a, b, c, t)$

- $\Delta V_0 \equiv \Delta V(a, b, c, t_0) = da db dc$
- $\Delta V \equiv \Delta V(a, b, c, t) = dX dY dZ$

$$dX dY dZ = |J| da db dc$$

$$\left(\begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} = \begin{bmatrix} \partial X / \partial a & \partial X / \partial b & \partial X / \partial c \\ \partial Y / \partial a & \partial Y / \partial b & \partial Y / \partial c \\ \partial Z / \partial a & \partial Z / \partial b & \partial Z / \partial c \end{bmatrix} \begin{bmatrix} da \\ db \\ dc \end{bmatrix} \right)$$

- **Jacobian determinant** ($X_a \equiv \partial X / \partial a$, $\xi_a \equiv \partial \xi / \partial a$):

$$|J| = \begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = \begin{vmatrix} 1 + \xi_a & \xi_b & \xi_c \\ \eta_a & 1 + \eta_b & \eta_c \\ \zeta_a & \zeta_b & 1 + \zeta_c \end{vmatrix}$$

- **equilibrium density:** $\rho_0 \equiv \rho(a, b, c, t_0)$
- **instantaneous density:** $\rho \equiv \rho(X, Y, Z, t)$
- **conservation of mass:** $\rho dX dY dZ = \rho_0 da db dc$

The Equation of Continuity

$$\rho|J| = \rho_0$$

where $|J|$ is defined in:

- 3D: $|J| = \begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = \begin{vmatrix} 1 + \xi_a & \xi_b & \xi_c \\ \eta_a & 1 + \eta_b & \eta_c \\ \zeta_a & \zeta_b & 1 + \zeta_c \end{vmatrix}$
- 2D: $|J| = \begin{vmatrix} X_a & X_b \\ Y_a & Y_b \end{vmatrix} = \begin{vmatrix} 1 + \xi_a & \xi_b \\ \eta_a & 1 + \eta_b \end{vmatrix}$
- 1D: $|J| = X_a = 1 + \xi_a$

Eulerian Method

The Equation of Continuity

- **mass flux** is the rate of mass flow per unit area: $\mathbf{j}_m = \rho \mathbf{v}$
- The **velocity** at \mathbf{r} and time t : $\mathbf{v} = \mathbf{v}(\mathbf{r}, t) = (v_x, v_y, v_z)$, $v = ||\mathbf{v}||$
- The **volume** of a bulk of fluid: V , **surface** of V : $\mathbf{S} = S\mathbf{n}$
- The **principle of mass conservation**:

$$\frac{d}{dt} \iiint_V \rho dV = - \oint_S \mathbf{j}_m \cdot d\mathbf{S}$$

- The Gauss theorem:

$$\oint_S \mathbf{f} \cdot d\mathbf{S} = \iiint_V \nabla \cdot \mathbf{f} dV$$

Exact Continuity Equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}_m = 0$$

Important Remark

Remember that **ALL** the physical quantities are the function of \mathbf{r} and t **IN DEFUALT.**

Eulerian Method

Homework

- ① Suppose there exists a net **volume velocity** output of q units per unit volume per unit time (the unit is s^{-1} , i.e. per second), please show that the exact continuity equation is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}_m = \rho q$$

Eulerian Method

- The **velocity field** at location \mathbf{r} and time t : $\mathbf{v}(x, y, z, t)$
- The particle at \mathbf{r} at time t travels to $\mathbf{r} + \Delta\mathbf{r}$ at time $t + \Delta t$, and the velocity of this particle is: $\mathbf{v}(\mathbf{r} + \Delta\mathbf{r}, t + \Delta t)$
- The **acceleration** of this particle:

$$\mathbf{a} \equiv \frac{d\mathbf{v}}{dt}$$

- The Taylor expansion:

$$\begin{aligned}\mathbf{v}(\mathbf{r} + \Delta\mathbf{r}, t + \Delta t) &= \mathbf{v}(x, y, z, t) + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial t} dt + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial x} v_x dt \\ &\quad + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial y} v_y dt + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial z} v_z dt, \quad \left(v_x dt = \frac{dx}{dt} dt = dx \right)\end{aligned}$$

$$\implies \frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v}$$

- A particle travels from $\mathbf{r} = (x, y, z)$ at time t to $\mathbf{r} + \Delta\mathbf{r} = (x + \Delta x, y + \Delta y, z + \Delta z)$ at time $t + \Delta t$. According to Taylor expansion, we have

$$\mathbf{v}(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) = \mathbf{v}(x, y, z, t)$$

$$+ \left. \frac{\partial \mathbf{v}}{\partial t} \right|_{(x,y,z,t)} \Delta t + \left. \frac{\partial \mathbf{v}}{\partial x} \right|_{(x,y,z,t)} \Delta x + \left. \frac{\partial \mathbf{v}}{\partial y} \right|_{(x,y,z,t)} \Delta y + \left. \frac{\partial \mathbf{v}}{\partial z} \right|_{(x,y,z,z)} \Delta z$$

$$\begin{aligned} &\implies \frac{\mathbf{v}(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - \mathbf{v}(x, y, z, t)}{\Delta t} \\ &= \left. \frac{\partial \mathbf{v}}{\partial t} \right|_{(x,y,z,t)} \frac{\Delta t}{\Delta t} + \left. \frac{\partial \mathbf{v}}{\partial x} \right|_{(x,y,z,t)} \frac{\Delta x}{\Delta t} + \left. \frac{\partial \mathbf{v}}{\partial y} \right|_{(x,y,z,t)} \frac{\Delta y}{\Delta t} + \left. \frac{\partial \mathbf{v}}{\partial z} \right|_{(x,y,z,t)} \frac{\Delta z}{\Delta t} \end{aligned}$$

$$\begin{aligned} &\implies \left. \frac{d\mathbf{v}}{dt} \right|_{(x,y,z,t)} = \left. \frac{\partial \mathbf{v}}{\partial t} \right|_{(x,y,z,t)} + \left. \frac{\partial \mathbf{v}}{\partial x} \right|_{(x,y,z,t)} v_x(x, y, z, t) \\ &\quad + \left. \frac{\partial \mathbf{v}}{\partial y} \right|_{(x,y,z,t)} v_y(x, y, z, t) + \left. \frac{\partial \mathbf{v}}{\partial z} \right|_{(x,y,z,t)} v_z(x, y, z, t) \end{aligned}$$

$$\begin{aligned}\implies \frac{d\mathbf{v}}{dt} \Big|_{(x,y,z,t)} &= \frac{\partial \mathbf{v}}{\partial t} \Big|_{(x,y,z,t)} + \left[(v_x, v_y, v_z) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \right] \mathbf{v} \Big|_{(x,y,z,t)} \\ \implies \frac{d\mathbf{v}}{dt} \Big|_{(x,y,z,t)} &= \frac{\partial \mathbf{v}}{\partial t} \Big|_{(x,y,z,t)} + (\mathbf{v} \cdot \nabla) \mathbf{v} \Big|_{(x,y,z,t)}\end{aligned}$$

where

$$\mathbf{v}(x, y, z, t) = (v_x(x, y, z, t), v_y(x, y, z, t), v_z(x, y, z, t))$$

Eulerian Method

Eulerian Equation

- The **volume** of a bulk of fluid: ΔV
- The **surface** of ΔV : $\Delta \mathbf{S} = \Delta S \mathbf{n}$
- The **total pressure** when $\Delta V \rightarrow 0$:

$$-\iint_{\Delta S} P d\mathbf{S} = -\iiint_{\Delta V} \nabla P dV \xrightarrow{\Delta V \rightarrow 0} -\nabla P \Delta V$$

- Newton's seconda law

$$(\rho \Delta V) \frac{d\mathbf{v}}{dt} = -\nabla P \Delta V$$

Eulerian Equation

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P$$

or

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P$$

- ① Suppose there exists a **body force density** $\rho\mathbf{g}$ (which has the physical dimensions of force per unit mass), please show that the Eulerian equation is:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P + \rho\mathbf{g}$$

Definition (Body Force)

- A **body force** is a force that acts throughout the volume of a body.
 - ✓: gravity, electric forces, magnetic forces, Centrifugal force
 - ×: shear forces, normal forces
- The **body force density**, $\rho\mathbf{g}$ is defined so that the volume integral of it gives the **body force**:

$$\mathbf{G}_{\text{body}} = \iiint_V \rho\mathbf{g}(\mathbf{r}) dV$$

- where \mathbf{g} is the **external body force density field** acting on the system

Equation of State

Equation of State

For fluid media, the equation of state must relate **3 physical quantities** describing the thermodynamic behavior of the fluid: such as $\{\rho, P, T\}$, $\{\rho, P, s\}$ or $\{\rho, P, V\}$.

$$f(\rho, P, T) = 0, \quad f(\rho, P, s) = 0 \quad \text{or} \quad f(\rho, P, V) = 0$$

where s is the **entropy per mass** of the fluid.

Equation of State for a Perfect Gas

$$P = \rho \mathcal{R} T_K$$

where T_K in kelvins (K) is the **absolute temperature**, and \mathcal{R} is the **specific gas constant** and depends on the **universal gas constant** and the **molecular weight**. For air, $\mathcal{R} \approx 287 \text{ J} \cdot \text{kg}^{-1} \text{K}^{-1}$.

Equation of State

- The acoustic processes are nearly **isentropic (adiabatic and reversible)**
- The **entropy** of the fluid remains nearly constant: $\frac{ds(P, \rho)}{dt} = 0 \implies P = P(\rho)$
- Definition of **sound speed**: $c^2 \equiv \frac{\partial P}{\partial \rho} = c^2(\rho, s)$
 - $c_s^2 = \frac{dP}{d\rho} = c_s^2(\rho)$
- Definition of **compressibility** and **bulk modulus**:

$$\beta_s = \frac{1}{\kappa_s} \equiv -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_s = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P} \right)_s = \frac{1}{\rho c_s^2}$$

Equation of State

General Fluids Adiabat

$$\begin{aligned} P = P(\rho) = P(\delta) &= P_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \rho_0^n \left(\frac{\partial^n P}{\partial \rho^n} \right)_{s,0} \delta^n \\ &= P_0 + \frac{1}{1!} A \rho_0 \delta + \frac{1}{2!} \rho_0^2 B \delta^2 + \frac{1}{3!} \rho_0^3 C \delta^3 + \dots \end{aligned}$$

where $\delta = (\rho - \rho_0)/\rho_0$, $A = \left(\frac{\partial P}{\partial \rho} \right)_{s,0}$, $B = \left(\frac{\partial^2 P}{\partial \rho^2} \right)_{s,0}$, $C = \left(\frac{\partial^3 P}{\partial \rho^3} \right)_{s,0}$.

Perfect Gas Adiabat

$$\begin{aligned} P &= P_0 \left(\frac{\rho}{\rho_0} \right)^\gamma \\ &= P_0 + \frac{1}{1!} \rho_0 \gamma \delta + \frac{1}{2!} \rho_0^2 \gamma (\gamma - 1) \delta^2 + \dots \end{aligned}$$

where γ is the **ratio of specific heats** (or **ratio of heat capacities**). For air, $\gamma \approx 1.40$.

Disturbance Quantities

- Under mechanical disturbances, the quantities are changed as:

$$P_0 \rightarrow P, \quad T_0 \rightarrow T, \quad \rho_0 \rightarrow \rho$$

- pressure disturbance:** $p = \Delta P (= dP) = P - P_0$
- density disturbance:** $\rho' = \Delta \rho (= d\rho) = \rho - \rho_0$
- temperature disturbance:** $\tau = \Delta T (= dT) = T - T_0$
- velocity disturbance:** $\mathbf{v}' = \Delta \mathbf{v} (= d\mathbf{v}) = \mathbf{v} - \mathbf{v}_0$
 - $\mathbf{v}' = \mathbf{v}$ if $\mathbf{v}_0 = 0$
- (scalar) pressure field:** $p(\mathbf{r}, t)$
- (vector) velocity field:** $\mathbf{v}(\mathbf{r}, t)$
- (scalar) velocity potential field:** $\mathbf{v} = -\nabla \Phi$ or $\mathbf{v} = \nabla \Phi$
 - the particle velocity must be **irrotational**, i.e. $\nabla \times \mathbf{v} = 0$

Sound (Acoustic) Pressure

$$p \equiv P - P_0$$

- **effective sound pressure** (only for periodical or quasi-periodical acoustic waves):

$$p_e(x) = \sqrt{\frac{1}{T} \int_0^T p^2(\mathbf{r}, t) dt}$$

- Units for sound pressure

- $1\text{Pa} = 1\text{N/m}^2 = 10\text{Dyne/cm}^2$
- $1\text{bar} = 100\text{kPa} = 10^5\text{Pa}$

Assumptions for Linear Wave Equation

- Assumptions for the fluids to be investigated:
 - **stationery:** $\mathbf{v}_0 = 0$
 - **homogeneity of initial conditions:** $P_0(\mathbf{r}) = \text{const.}$, $\rho_0(\mathbf{r}) = \text{const.}$
 - **lossless:** adiabat, i.e. $s(t) = \text{const.}$
- Linear assumptions:

$$p \ll P_0, |\xi| \ll \lambda, \rho' \ll \rho_0, |\mathbf{v}| \ll c_0$$

- $c_s^2 \approx c_0^2 = \left(\frac{dP}{d\rho} \right)_{s,0}$
- $\kappa_s \approx \kappa_{s,0} = \rho_0 c_0^2, \beta_s \approx \beta_{s,0} = \frac{1}{\rho_0 c_0^2}$

- **exact continuity equation** → **linear continuity equation**:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \rightarrow \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v} \approx 0$$

- **Eulerian equation** → linear:

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla P \rightarrow \rho_0 \frac{\partial \mathbf{v}}{\partial t} \approx -\nabla p$$

- **equation of state** → linear:

$$P = P_0 + \frac{1}{1!} \rho_0 \left(\frac{\partial P}{\partial \rho} \right)_{s,0} \delta + \frac{1}{2!} \rho_0^2 \left(\frac{\partial^2 P}{\partial \rho^2} \right)_{s,0} \delta^2 + \dots$$
$$\rightarrow p \approx \rho' c_0^2$$

Linearization Due to Small Variations

- Eliminate $\rho', *$ we have

$$\begin{cases} \beta_{s,0} \frac{\partial p}{\partial t} \approx -\nabla \cdot \mathbf{v} \\ \frac{\partial \mathbf{v}}{\partial t} \approx -\frac{1}{\rho_0} \nabla p \end{cases}, \quad \left(\beta_{s,0} = \frac{1}{\kappa_{s,0}} = \frac{1}{\rho_0 c_0^2} \right)$$

- Eliminate $\mathbf{v}, ^\dagger$ we have

Linear Wave Equation (for Sound Pressure)

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p \approx 0$$

*Why ρ' ? Because the variation of the density is hard to detect in practice.

†Why \mathbf{v} ? Because the velocity is a vector, which is harder to handle than a scalar like sound pressure p .

Useful Equations

Calculate \mathbf{v} by p

$$\mathbf{v} = -\frac{1}{\rho_0} \int \nabla p \, dt$$

Proof: by Eulerian euqation directly.

Linear Wave Equation (for Velocity)

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{v} \approx 0$$

Proof: Eliminate p with $\nabla(\nabla \cdot \mathbf{v}) = \nabla \times \nabla \times \mathbf{v} + \nabla^2 \mathbf{v} = \nabla^2 \mathbf{v}$.

Linear Wave Equation (for Velocity Potential)

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \Phi \approx 0, \quad \left(p \approx \rho_0 \frac{\partial \Phi}{\partial t} \right)$$

where $\mathbf{v} = -\nabla \Phi$.

*Nondimensionalization

Definition (Nondimensionalized Variables)

$$\tau = \omega t, \quad \xi = kx, \quad \nu = \frac{v}{c_0}, \quad q = \frac{p}{\kappa_{s,0}}, \quad \delta = \frac{\rho'}{\rho_0}, \quad \chi^2 + 1 = \left(\frac{c}{c_0} \right)^2 {}^a$$

^aWhy don't we define this as $\chi^2 = \left(\frac{c-c_0}{c_0} \right)^2$? Because c is always no less than c_0 .

Linear Wave Equation

Homework

- ① Suppose there exist q and \mathbf{g} , show that

Linear Inhomogenous Wave Equation (for Sound Pressure)

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p \approx \rho_0 \left(\nabla \cdot \mathbf{g} - \frac{\partial q}{\partial t} \right)$$

- ② Suppose the \mathbf{g} has **potential**, i.e. $\mathbf{g} = -\nabla\Psi$, show that

Linear Wave Equation (for Velocity Potential)

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \Phi \approx -q - \frac{1}{c_0^2} \frac{\partial \Psi}{\partial t}$$

- ③ Textbook B: 4-4

Linear Wave Equation

Homework

- ④ Suppose the potential is defined as:

$$\mathbf{v} = \nabla\Phi, \quad \mathbf{g} = \nabla\Psi.$$

Show that

$$p \approx -\rho_0 \left(\frac{\partial \Phi}{\partial t} - \Psi \right)$$

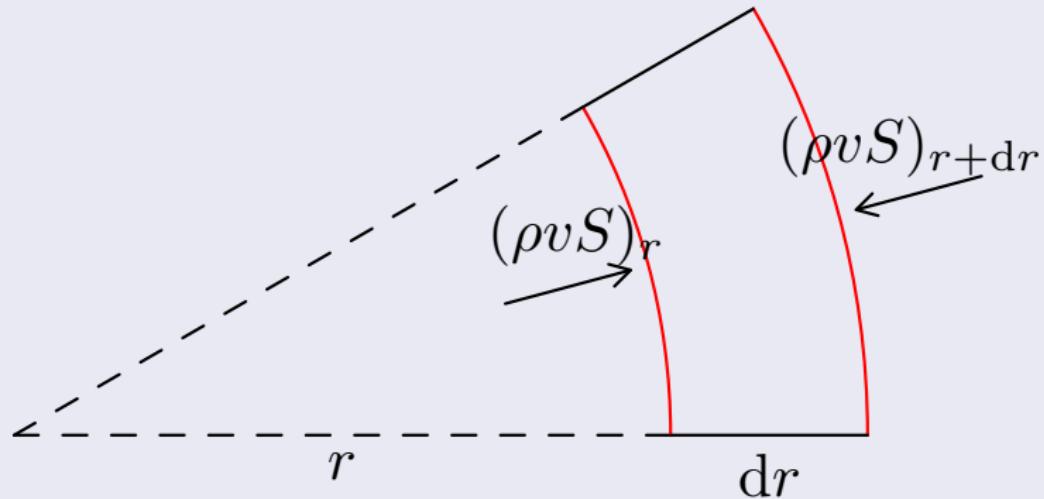
and

$$\left(\nabla^2 - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) \Phi \approx q - \frac{1}{c_0^2} \frac{\partial \Psi}{\partial t}$$

A Special Wave Equation

Description

- 1. the **shape of the wave front** remains the same
- 2. the **direction of propagating of the wave** remains the same



A Special Wave Equation

- **continuity equation:**

$$\rho \frac{dv}{dt} = -\frac{\partial P}{\partial r} \rightarrow \rho_0 \frac{\partial v}{\partial t} \approx -\frac{\partial p}{\partial r}$$

- **Eulerian equation:**

$$\frac{\partial}{\partial t}(\rho S dr) = (\rho v S)_r - (\rho v S)_{r+dr} = -\frac{\partial(\rho v S)}{\partial r} dr \rightarrow S \frac{\partial \rho}{\partial t} \approx -\rho_0 \frac{\partial(vS)}{\partial r}$$

- **equation of state:**

$$P = P(\rho) \rightarrow p \approx c_0^2 \rho'$$

- **linear wave equation:**

$$\left[\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r} \frac{d(\ln S)}{dr} \right] - \frac{1}{c_0^2} \frac{\partial^2 p}{\partial t^2} \approx 0$$

Planar Wave

- 1D wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \right) p = 0$$

$$\frac{\partial^2 p}{\partial \xi \partial \eta} = 0, (\xi = c_0 t - x, \eta = c_0 t + x)$$

$$\implies p = f(\xi) + g(\eta) = f(c_0 t - x) + g(c_0 t + x)$$

- **propagating wave** in x direction: $p = p(c_0 t - x)$

- (3D wave equation) propagating wave in **unit vector** $\mathbf{n} = (n_x, n_y, n_z)$ direction

$$p = p(c_0 t - \mathbf{n} \cdot \mathbf{r})$$

- **wave front:** a curved surface where the physical quantities are the same

$$\zeta = c_0 t \mp \mathbf{n} \cdot \mathbf{r} = \text{const.}$$

$$d\zeta = 0 \implies c_0 = \frac{\partial}{\partial t}(\mathbf{n} \cdot \mathbf{r})$$

- velocity

$$\mathbf{v} = -\frac{1}{\rho_0} \int \nabla p dt = \frac{p}{\rho_0 c_0} \mathbf{n}, \quad \left(\nabla p(\zeta) = \mathbf{n} \frac{dp}{d\zeta}, \int p(\zeta) dt = \frac{1}{c_0} \int p(\zeta) d\zeta \right)$$

Harmonic Plane Wave

- the **angular frequency**: ω

$$p(\mathbf{r}, t) = p(\mathbf{r})e^{j\omega t}$$

Linear Harmonic Plane Wave (Helmholtz) Equation

$$(\nabla^2 + k^2)p(\mathbf{r}) = 0$$

the **wave number**: $k = \omega/c_0$

- 1D wave equation

$$\left(\frac{d^2}{dx^2} + k^2 \right) p(x) = 0, \implies p(x) = p_a e^{jkx} \quad \text{or} \quad p_a e^{-jkx}$$

- propagating wave with frequency ω

$$p_{\pm}(x, t) = p_a e^{j(\omega t \mp kx)}$$

$$p(\mathbf{r}, t) = p_a e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}, \quad (\mathbf{k} = k\mathbf{n})$$

- particle velocity

$$v(\mathbf{r}, t) = \mathbf{n} v_a e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}, \quad \left(v_a = \frac{p_a}{\rho_0 c_0} \right)$$

Sound Speed – the propagating speed of sound waves

$$c \equiv \sqrt{\frac{dP}{d\rho}} = \sqrt{\frac{\kappa_s}{\rho}} = \sqrt{\frac{1}{\rho\beta_s}}, \quad c_0 = c|_{\rho=\rho_0}$$

- **incompressible fluids/solids:** $\kappa_s \rightarrow \infty$, $\beta_s \rightarrow 0$, $c_0 \rightarrow 0$
- **rarefied gas:** $\kappa_s \rightarrow 0$, $\beta_s \rightarrow \infty$, $c_0 \rightarrow 0$
- **Adiabatic and Ideal Gas:**

$$P\rho^{-\gamma} = \text{const.} \quad \text{or} \quad PV^\gamma = \text{const.}, \quad (\gamma_{\text{air}} = 1.4)$$

$$c = \sqrt{\frac{\gamma P}{\rho}}, \quad c_0 = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

- **Clapeyron Equation for Ideal Gas:**

$$PV = \frac{M}{\mu}RT, \quad = P = \frac{\rho}{\mu}RT$$

$$c_0 = \sqrt{\frac{\gamma}{\mu}RT_0} = c_0|_{T_C=0^\circ\text{C}} \sqrt{1 + \frac{T_C}{273}} \approx 331.6 + 0.6T_C, \quad (T_C = T - 273)$$

Sound Speed – the propagating speed of sound waves

- Clapeyron Equation for Ideal Gas:

$$PV = \frac{M}{\mu}RT, \quad P = \frac{\rho}{\mu}RT$$

- P : the **pressure** of the gas
 - V : the **volumne** of the gas
 - M : the **total mass** of the gas (in kilograms)
 - μ : the **molar mass** of the gas (in kilograms per mole)
 - R : the ideal, or universal, **gas constant**, equal to the product of the Boltzmann constant and the Avogadro constant
 - T : the **absolute temperature** of the gass (in Kelvins)
 - ρ : the **density** of the gass
- Calculate c_0 by T_C , the **temperature** in Celsius

$$c_0 = \sqrt{\frac{\gamma}{\mu}RT_0} = c_0|_{T_C=0^\circ\text{C}} \sqrt{1 + \frac{T_C}{273}} \approx 331.6 + 0.6T_C, \quad (T_C = T - 273)$$

- Check for linear assumption in air: $p_a = 0.1 \text{ Pa}$, $T_C = 20^\circ\text{C}$, $\rho_0 = 1.21 \text{ kg/m}^3$

$$c_0 \approx 343.6 \text{ m/s}, \quad v_a = \frac{p_a}{\rho_0 c_0} \approx 2.4 \times 10^{-4} \text{ m/s} \ll c_0$$

Acoustic impedance, specific acoustic impedance, and characteristic impedance

- **Acoustic Impedance:** $Z_a = \frac{p}{U}$
 - lumped parameter system
- **Specific Acoustic Impedance:** $Z_s = \frac{p}{v}$
 - distributed parameter system
 - real part: **specific acoustic resistance**
 - imaginary part: **specific acoustic reactance**
- plane wave
 - positive direction: $Z_s = \rho_0 c_0$
 - negative direction: $Z_s = -\rho_0 c_0$
- **characteristic (specific acoustic) impedance**

$$z_0 = \rho_0 c_0$$

- represents the acoustic properties of the media

Sound Energy

- consider the fluid particle $\Delta V = \Delta V(\mathbf{r}, t)$

- $t = 0 \implies \Delta V_0 = \Delta V(\mathbf{r}, 0), P_0, \rho_0$

- **kinetic energy:**

$$\Delta E_k = \frac{1}{2}(\rho_0 \Delta V_0) v^2, \quad (v = |\mathbf{v}|)$$

- **potential energy:** the work done by sound pressure p

$$\Delta E_p = - \int_{\Delta V_0}^{\Delta V} p d(\Delta V) \approx \Delta V_0 \beta_{s,0} \int_0^p p dp = \frac{1}{2} \beta_{s,0} p^2 \Delta V_0$$

$$\left(\frac{d(\Delta V)}{\Delta V_0} \approx -\frac{d\rho'}{\rho_0} \xrightarrow{p=c_0^2 \rho'} d(\Delta V) \approx -\Delta V_0 \beta_{s,0} dp \right)$$

- **total energy:**

$$\Delta E = \Delta E_k + \Delta E_p = \frac{1}{2} (\rho_0 v^2 + \beta_{s,0} p^2) \Delta V_0$$

Sound Energy Density

- the sound energy per volume (in Pascals)

$$\varepsilon = \frac{\Delta E}{\Delta V_0} = \frac{\Delta E_k}{\Delta E_p} \Delta V_0 = \frac{1}{2} (\rho_0 v^2 + \beta_{s,0} p^2)$$

- plane wave: $\Delta E_k = \Delta E_p$

$$v = \frac{p}{\rho_0 c_0} \rightarrow \varepsilon = \beta_{s,0} p^2 = \frac{p^2}{\rho_c c_0^2}$$

- average** of the sound energy density for plane harmonic waves

$$\bar{\varepsilon} = \frac{\overline{p^2}}{\rho_0 c_0^2} = \frac{p_a^2}{2\rho_0 c_0^2} = \frac{p_e^2}{\rho_0 c_0^2}, \quad \left(\overline{p^2} = \frac{1}{2} |p|^2 = p_e^2 = \frac{1}{2} p_a^2 \right)$$

Oblique Incidence

reflection and transmission coefficients

- the boundary conditions s.t. Snell's law:

$$\begin{cases} p_{ia} + p_{ra} = p_{ta} \\ \frac{p_{ia} - p_{ra}}{z_{n1}} = \frac{p_{ta}}{z_{n2}} \end{cases} \implies \begin{cases} 1 + r_p = t_p \\ \frac{1 - r_p}{z_{n1}} = \frac{t_p}{z_{n2}} \end{cases}, \begin{pmatrix} z_{n1} \equiv z_{ni} = -z_{nr} \\ z_{n2} \equiv z_{nt} \end{pmatrix}$$

- the ratio of density: $m \equiv \rho_2/\rho_1$, refractive index: $n = k_2/k_1 = c_1/c_2$
- reflection and transmission coeff.:

$$r_p \equiv \frac{p_{ra}}{p_{ia}} = \frac{z_{n2} - z_{n1}}{z_{n2} + z_{n1}} = \frac{R_2 \cos \theta_i - R_1 \cos \theta_t}{R_2 \cos \theta_i + R_1 \cos \theta_t} = \frac{m \cos \theta_i - \sqrt{n^2 - \sin^2 \theta_i}}{m \cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}}$$

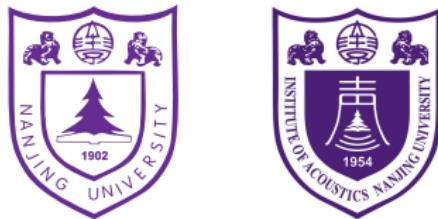
$$t_p \equiv \frac{p_{ta}}{p_{ia}} = \frac{2z_{n2}}{z_{n2} + z_{n1}} = \frac{2R_2 \cos \theta_i}{R_2 \cos \theta_i + R_1 \cos \theta_t} = \frac{2m \cos \theta_i}{m \cos \theta_i + \sqrt{n^2 - \sin^2 \theta_i}}$$

Sound Waves in Ducts Filled with Fluids

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September 19, 2018

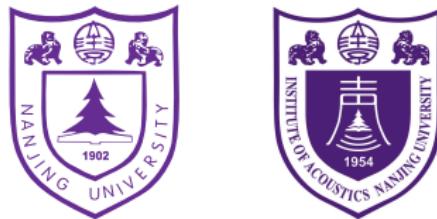


Radiation of Sound

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September 19, 2018



Green Functions with Infinitely Large Rigid Boundary

$$G(\mathbf{r}; \mathbf{r}_0) = g(|\mathbf{r} - \mathbf{r}_0|) + g(|\mathbf{r} - \mathbf{r}'_0|), \quad \begin{cases} \mathbf{r}_0 = (x_0, y_0, z_0) \\ \mathbf{r}'_0 = (x_0, y_0, -z_0) \end{cases}$$

Radiation from a Baffled Circular Plane Piston

- only the sound field on the surface xOz is needed to compute for symmetry
- location of the field point P : $\mathbf{r} = r(\sin \theta, 0, \cos \theta)$
- volume velocity of an element on the piston: dQ_0
- location of a point on the piston dQ_0 : $\mathbf{r}_0 = \rho(\cos \varphi, \sin \varphi, 0)$
- distance between P and dQ_0 : $h = |\mathbf{r} - \mathbf{r}_0| \approx r - \rho \sin \theta \cos \varphi, (r \gg a)$

$$\Phi(\mathbf{r}) = \iint_{S_{\text{pis}}} 2g(\mathbf{r}; \mathbf{r}_0) u_a(\mathbf{r}_0) dS_0 \xrightarrow{u_a = \text{const}} u_a \iint_{S_{\text{pis}}} \frac{e^{-jkh}}{2\pi h} dS_0$$

Radiation from a Baffled Circular Plane Piston

Sound Field at Far Field (Radiation from a Baffled Circular Plane Piston)

$$\Phi(\mathbf{r}, t) \approx 2Q(t)g(r)D(\theta)$$

where $Q(t) = Q_a e^{j\omega t}$, $Q_a = u_a (\pi a^2)$, $g(r) \equiv \frac{e^{-jr}}{4\pi r}$, $D(\theta) \equiv \frac{2J_1(ka \sin \theta)}{ka \sin \theta}$

Radiation from a Baffled Circular Plane Piston

Radiation Directivity (Pattern)

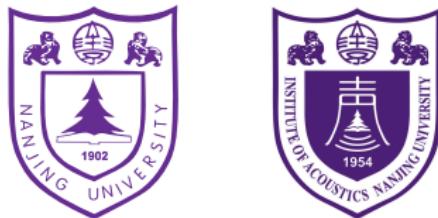
- the property of Bessel functions: $\left[\frac{2J_1(x)}{x} \right]_{x=0} = 1$

Viscous Absorption of Sound

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‡

[‡] Available at: <https://www.wikiwand.com/en/Viscosity>

Definition (Viscosity)

- Definition 1. **the first and second viscosity** μ_1, μ_2
- Definition 2. **the shear viscosity** η , and **the bulk viscosity** ζ
- the relations:

$$\begin{cases} \mu_1 = \eta \\ \mu_2 = \zeta - \frac{2}{3}\eta \end{cases}, \quad \begin{cases} \eta = \mu_1 \\ \zeta = \frac{2}{3}\mu_1 + \mu_2 \end{cases}$$

Conservation of Momentum

$$\frac{d}{dt} \iiint_V \rho v_i dV = \iiint_V \rho g_i dV - \oint_S \rho v_i v_k dS_k + \oint_S T_{ik} dS_k$$

- **Cartesian momentum density component:** ρv_i
- **force density (force per volume):** ρg_i
- **stress tensor:** T_{ik}
- by Gauss theorem:

$$\frac{\partial}{\partial t}(\rho v_i) = \rho g_i - \frac{\partial}{\partial x_k}(\rho v_i v_k) + \frac{\partial}{\partial x_k}(T_{ik})$$

Conservation of Momentum Equation

- by using the Continuity Equation, we have

$$\rho \frac{dv_i}{dt} \equiv \rho \frac{\partial v_i}{\partial t} + \rho v_{i,k} v_k = \rho g_i + \frac{\partial}{\partial x_k}(T_{ik})$$

Strain and Stress

- **strain rate tensor (deformation velocity tensor)**

$$d_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i})$$

- **stress tensor**

$$T_{ij} = \mu_1 \Theta \delta_{ij} + 2\mu_2 d_{ij}, \quad (\Theta \equiv d_{kk} = \nabla \cdot \mathbf{v})$$

Navier-Stokes Equation

General Navier-Stokes Equation

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla P + \mu_1 \nabla^2 \mathbf{v} + (\mu_1 + \mu_2) \nabla \nabla \cdot \mathbf{v}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla P + \eta \nabla^2 \mathbf{v} + (\zeta + \frac{\eta}{3}) \nabla \nabla \cdot \mathbf{v}$$

$$\left(\rho \frac{dv_i}{dt} = \rho g_i - \frac{\partial P}{\partial x_i} + \mu_1 \frac{\partial^2 v_i}{\partial x_k \partial x_k} + (\mu_1 + \mu_2) \frac{\partial^2 v_k}{\partial x_i \partial x_k} \right)$$

Classical Navier-Stokes Equation

- with the relation: $\mu_2 + \frac{2}{3}\mu_1 = 0$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla P + \mu_1 \nabla^2 \mathbf{v} + \frac{\mu_1}{3} \nabla \nabla \cdot \mathbf{v}$$

- with the relation: $\zeta = 0$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{g} - \nabla P + \eta \nabla^2 \mathbf{v} + \frac{\eta}{3} \nabla \nabla \cdot \mathbf{v}$$

Nonlinear Acoustics

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Theorem (Gauss Theorem in Tensor Fields)

$$\iiint_V \frac{\partial \mathbf{F}_i}{\partial x_i} dV = \iint_S \mathbf{F}_i dS_i \left(= \iint_S \mathbf{F}_i n_i dS \right)$$

Appendix: Complex Numbers with Their Computations

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September 19, 2018



Convention Notations

Type	Notation
complex number	X, Y, Z
real number	$x = \operatorname{Re}(X), y = \operatorname{Re}(Y), z = \operatorname{Re}(Z)$
real number	a, b, c, d
exponential	$Z_1 = \rho_1 e^{j\varphi_1}, Z_2 = \rho_2 e^{j\varphi_2}$

- relations

$$z = \frac{1}{2}(Z + Z^*) = \operatorname{Re}(Z)$$

Operations

- **addition**

$$z = ax \pm by \implies Z = aX \pm bY$$

- **multiplication**

$$z = xy \implies Z = X \operatorname{Re}(Y) = \operatorname{Re}(X)Y$$

Proof

$$\begin{aligned} z &= \operatorname{Re}(X) \times \operatorname{Re}(Y) \\ &= \frac{1}{2}(X + X^*) \times \frac{1}{2}(Y + Y^*) \\ &= \frac{1}{2} \operatorname{Re}(XY + X^*Y) \\ \therefore Z &= \frac{1}{2}(XY + XY^*) \\ &= \operatorname{Re}(X)Y = \operatorname{Re}(X)Y^* \\ &= X \operatorname{Re}(Y) = X^* \operatorname{Re}(Y) \end{aligned}$$

- **integral and derivative**

$$\begin{cases} \int \operatorname{Re}[X(t)] dt = \operatorname{Re}\left(\int X(t) dt\right) \\ \frac{d}{dt} \operatorname{Re}[X(t)] = \operatorname{Re}\left[\frac{d}{dt} X(t)\right] \end{cases} \implies \begin{cases} a \frac{d^2x}{dt^2} + b \frac{dx}{dt} + cx + d \int x dt = 0 \\ a \frac{d^2X}{dt^2} + b \frac{dX}{dt} + cX + d \int X dt = 0 \end{cases}$$

Exercises

- Show that

$$Z_1 + Z_2 = \sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2 \cos(\varphi_1 - \varphi_2)} e^{j\varphi}, \tan \varphi = \frac{\rho_1 \sin \varphi_1 + \rho_2 \sin \varphi_2}{\rho_1 \cos \varphi_1 + \rho_2 \cos \varphi_2}$$

- Show that

$$e^{-j\varphi} + e^{j\varphi} = 2 \cos \varphi, \quad e^{-j\varphi} - e^{j\varphi} = -2j \sin \varphi$$

- Show that

$$1 + e^{j\varphi} = e^{j\frac{\varphi}{2}} 2 \cos \frac{\varphi}{2}, \quad 1 - e^{j\varphi} = e^{j\frac{\varphi}{2}} \times (-2j) \sin \frac{\varphi}{2}$$

- Show that

$$\sum_{n=1}^N e^{jn\varphi} = e^{j\frac{N+1}{2}\varphi} \frac{\sin\left(\frac{n-1}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}$$

$$\sum_{n=1}^N \cos n\varphi = \cos\left(\frac{N+1}{2}\varphi\right) \frac{\sin\left(\frac{n-1}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}, \sum_{n=1}^N \sin n\varphi = \sin\left(\frac{N+1}{2}\varphi\right) \frac{\sin\left(\frac{n-1}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}$$

- Show that

$$\frac{1 + Z_1}{1 - Z_1} = Z_2 \implies Z_1 = \frac{Z_2 - 1}{Z_2 + 1}$$

Appendix: Tensors and Einstein Summation Convention

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September 19, 2018



Notations

Convention Notations

Type	Rank	Vector Notation	Index Notation
scalar	0	α, β, Φ	α, β, Φ
vector	1	$\mathbf{u}, \mathbf{v}, \mathbf{w}$	u_i, v_i, w_i
tensor	2	$\mathbf{A}, \mathbf{B}, \mathbf{C}$	A_{ij}, B_{ij}, C_{ij}
tensor	3	\mathbf{T}	T_{ijk}

Definition (Tensors)

Tensor (2-order) \mathbf{A} is defined by two vectors \mathbf{u}, \mathbf{v} as

$$\mathbf{A} = \mathbf{u}\mathbf{v} = \sum_{j=1}^3 \sum_{i=1}^3 u_i v_j \mathbf{e}_i \mathbf{e}_j \equiv \sum_{j=1}^3 \sum_{i=1}^3 A_{ij} \mathbf{e}_i \mathbf{e}_j$$

Free Indices

- ① appear **only once** within each additive term in an expression. i is a **free index**:

$$a_i = \epsilon_{ikl} b_k c_l + D_{ik} e_k$$

- ② imply 3 **distinct equations**.

$$a_i = b_i + c_i \implies \begin{cases} a_1 = b_1 + c_1 \\ a_2 = b_2 + c_2 \\ a_3 = b_3 + c_3 \end{cases}$$

- ③ The same letter must be used for the free index in **every** additive term.
- ④ The number of free indices can be larger than 1, and equals the **rank** of the term.
- ⑤ The first free index in a term corresponds to the **row**, and the second corresponds to the **column**

$$u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Dummy Indices

- ① appear **twice** within an additive term of an expression. j and k are both **dummy indices**:

$$a_i = \epsilon_{ikl} b_k c_l + D_{ik} e_k$$

- ② imply a **summation** over the range of the index:

$$a_{kk} \equiv a_{11} + a_{22} + a_{33}$$

$$a_i = \sum_{l=1}^3 \sum_{=1}^3 \epsilon_{ikl} b_k c_l + \sum_{k=1}^3 D_{ik} e_k$$

- ③ **local** to an individual additive term. Thus, it can be renamed, just like the x or t in integrals:

$$\int_0^1 f(x) dx = \int_0^1 f(t) dt$$

Special Functions

Definition (Kronecker Delta)

A rank-2 symmetric tensor defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{or} \quad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition (Alternating Unit Tensor or Levi-Cavita Symbol)

A rank-3 **antisymmetric** tensor

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \\ 0 & \text{if } (i-j)(j-k)(k-i) = 0 \\ -1 & \text{if } ijk = 132, 213, 321 \end{cases}$$

Important Identity

$$\epsilon_{ikl}\epsilon_{imn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$$

- In general, **vector notation** DO NOT have commutative or associative properties

$$\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$$

- All of the terms in index notation are scalars, and only **multiplication/division and addition/subtraction operations** ($+, -, \times, \div$) are defined.
- commutative and associative properties hold in index notation:

$$a_i b_j = b_j a_i, \quad (a_i b_j) c_k = a_i (b_j c_k)$$

- In general, **calculus operators** (\int, d, ∂) are not commutative

Vector Operations using Index Notation

- multiplication of a vector by a scalar

$$\alpha \mathbf{u} = \mathbf{v} \implies \alpha u_i = v_i$$

- scalar product of two vectors (a.k.a. dot or inner product)

$$\mathbf{u} \cdot \mathbf{v} = \alpha \implies u_k v_k = \alpha$$

- scalar product of two tensors (a.k.a. dot or inner product)

$$\mathbf{A} : \mathbf{B} = \alpha \implies A_{kl} B_{lk} = \alpha$$

- tensor product of two vectors (a.k.a dyadic product)

$$\mathbf{u}\mathbf{v} = \mathbf{A} \implies u_i v_j = A_{ij}$$

- tensor product of two tensors

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \implies A_{ik} B_{kj} = C_{ij}$$

- NOTE this is NOT an inner product

Vector Operations using Index Notation

- vector product of a tensor and a vector

$$\mathbf{u} \cdot \mathbf{A} = \mathbf{v} \implies u_k A_{ki} = v_i$$

$$\mathbf{A} \cdot \mathbf{u} = \mathbf{v} \implies A_{ik} u_k = v_i$$

- NOTE $\mathbf{u} \cdot \mathbf{A} \neq \mathbf{A} \cdot \mathbf{u}$
- cross product of two vectors

$$\mathbf{u} \times \mathbf{v} = \mathbf{w} \implies \epsilon_{ikl} u_k v_l = w_i$$

- contraction or trace of a tensor

$$\text{Tr } \mathbf{A} = \alpha \implies A_{kk} = \alpha$$

Calculus Operations using Index Notation

- NOTE the spatial Cartesian coordinates are renamed as follows

$$(x, y, z) \rightarrow (x_1, x_2, x_3)$$

- temporal derivative of a scalar field $\Phi(x_1, x_2, x_3, t)$

$$\frac{\partial \Phi}{\partial t} \equiv \partial_0 \Phi \quad \text{or} \quad \partial_t \Phi$$

- gradient of a scalar field $\Phi(x_1, x_2, x_3, t)$

$$\frac{\partial \Phi}{\partial x_i} \equiv \partial_i \Phi$$

- gradient of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\nabla \mathbf{u} = \frac{\partial u_j}{\partial x_i} \equiv \partial_i u_j = \begin{bmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{bmatrix}$$

- the first index is the row index
- the gradient **increases by one the rank** of the expression on which it operates

Calculus Operations using Index Notation

- divergence of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\nabla \cdot \mathbf{u} = \partial_k u_k = \alpha$$

- the divergence **decreases by one the rank** of the expression on which it operates
 - it is not possible to take the divergence of a scalar
- curl of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\nabla \times \mathbf{u} = \epsilon_{ikl} \partial_k u_l = v_i$$

- the curl **does not change the rank** of the expression on which it operates
 - it is not possible to take the curl of a scalar
- Laplacian of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\nabla^2 \mathbf{u} \equiv \nabla \cdot (\nabla \mathbf{u}) = \partial_k \partial_k u_i = v_i$$

Calculus Operations using Index Notation

Exercises

- ① Show that

$$\nabla \cdot \mathbf{u} \neq \mathbf{u} \cdot \nabla$$

- ② Show that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$

- ③ Show that

$$\mathbf{u} \times \mathbf{v} \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$$

- ④ Show that

$$\nabla \times (\nabla \Phi) = \mathbf{0}$$

- ⑤ Show that

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}$$

- ⑥ Show that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

- ⑦ Show that

$$\nabla \cdot (\mathbf{u}\mathbf{v}) = (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla \cdot \mathbf{u})\mathbf{v}$$

Definition (Symmetric and Antisymmetric Tensors)

- a tensor is **symmetric** if it is equal to its transpose

$$A_{ij} = A_{ji}$$

- a tensor is **antisymmetric** if it is equal to the negative of its transpose

$$A_{ij} = -A_{ji}$$

Theorem (Decomposition of a Tensor into Symmetric and Antisymmetric Parts)

- any arbitrary tensor \mathbf{A} may be composed into sum of a symmetric tensor and an antisymmetric tensor

$$A_{ij} = A_{(ij)} + A_{[ij]}$$

where

$$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}), \quad A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$$

- the antisymmetric component can be calculated as

$$A_{[ij]} = \frac{1}{2} \epsilon_{ijk} \epsilon_{klm} A_{lm}$$

- the scalar product of any symmetric and antisymmetric tensor is zeros

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = 0 \quad \text{if} \quad A_{ij} = A_{ji} \quad \text{and} \quad B_{ij} = -B_{ji}$$