The Principles of Acoustics

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Sound Waves in Ideal Fluids

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Introduction

Acoustic waves

Acoustic waves constitute one kind of **pressure fluctuation** that can exist in a **compressible fluid**.

- mechanical waves
- media: **elastic continuum**, e.g. gases, liquids, solids

Ideal Fluid: lossless, no dissipitive effects

- **inviscid**: No **viscous force**
- No **heat conduction**

Continuum Hypothesis

- **fluid particles**(**elements**)
  - infinitesimal volume of the fluid large enough to contain millions of molecules $\rightarrow$ contious medium
  - small enough $\rightarrow$ all acoustic variables are uniform throughout

Two Descrption Methods

Lagrangian Method

- Follow **the individual fluid particle** located at $\mathbf{R}_0 = (a, b, c)$ at time $t = t_0$.
- The location of **this** particle $\mathbf{R} = (X, Y, Z)$ at time t is:

$$\begin{cases} X = X(a, b, c, t) \\ Y = Y(a, b, c, t) \\ Z = Z(a, b, c, t) \end{cases}$$

Eulerian Method

- Don't care about the specific fluid particles.
- The **field representation** is used.
- The **physical quantities** f at the location $\mathbf{r} = (x, y, z)$ at time t: $f(x, y, z, t)$.
  - 3 **state variables**: pressure (p), density (ρ), and temperature (T)
  - 3 **velocity components**: $\mathbf{v} = (v_x, v_y, v_z)$
- **6 (differential)**[a] **equations** are needed to solve the above 6 physical quantites
  - 5 equations: **conservation theorems**
  - 1 equation: **state equation**

[a] Why **differential equations**? How about **integral equations?**

- **particle displacement**: $\Delta \mathbf{R} \equiv \mathbf{R} - \mathbf{R}_0 = \boldsymbol{\xi} = (\xi, \eta, \zeta)$
  - the location of the particle:

$$\begin{cases} X = a + \xi(a, b, c, t) \\ Y = b + \eta(a, b, c, t) \\ Z = c + \zeta(a, b, c, t) \end{cases}$$

- the **volume** of the fluid particle: $\Delta V(a, b, c, t)$
  - $\Delta V_0 \equiv \Delta V(a, b, c, t_0) = \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$
  - $\Delta V \equiv \Delta V(a, b, c, t) = \mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z$

$$\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z = |\, J \,|\, \mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c$$

$$\left(\begin{bmatrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{bmatrix} = \begin{bmatrix} \partial X/\partial a & \partial X/\partial b & \partial X/\partial c \\ \partial Y/\partial a & \partial Y/\partial b & \partial Y/\partial c \\ \partial Z/\partial a & \partial Z/\partial b & \partial Z/\partial c \end{bmatrix} \begin{bmatrix} \mathrm{d}a \\ \mathrm{d}b \\ \mathrm{d}c \end{bmatrix} \right)$$

  - **Jacobbian determinant** $(X_a \equiv \partial X/\partial a\,, \xi_a \equiv \partial \xi/\partial a)$:

$$|J| = \begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = \begin{vmatrix} 1+\xi_a & \xi_b & \xi_c \\ \eta_a & 1+\eta_b & \eta_c \\ \zeta_a & \zeta_b & 1+\zeta_c \end{vmatrix}$$

Lagrangian Method

- **equilibrium density**: $\rho_0 \equiv \rho(a, b, c, t_0)$
- **instantaneous density**: $\rho \equiv \rho(X, Y, Z, t)$
- **conservation of mass**: $\rho\, \mathrm{d}X \mathrm{d}Y \mathrm{d}Z = \rho_0\, \mathrm{d}a \mathrm{d}b \mathrm{d}c$

The Equation of Continuity

$$\rho|J| = \rho_0$$

where $|J|$ is defined in:

- 3D: $|J| = \begin{vmatrix} X_a & X_b & X_c \\ Y_a & Y_b & Y_c \\ Z_a & Z_b & Z_c \end{vmatrix} = \begin{vmatrix} 1+\xi_a & \xi_b & \xi_c \\ \eta_a & 1+\eta_b & \eta_c \\ \zeta_a & \zeta_b & 1+\zeta_c \end{vmatrix}$
- 2D: $|J| = \begin{vmatrix} X_a & X_b \\ Y_a & Y_b \end{vmatrix} = \begin{vmatrix} 1+\xi_a & \xi_b \\ \eta_a & 1+\eta_b \end{vmatrix}$
- 1D: $|J| = X_a = 1 + \xi_a$

- **mass flux (density)** is the rate of mass flow per unit area: $\mathbf{j}_\mathrm{m} = \rho\mathbf{v}$
- The **velocity** at $\mathbf{r}$ and time t: $\mathbf{v} = \mathbf{v}(\mathbf{r}, t) = (v_x, v_y, v_z)$, $v = ||\mathbf{v}||$
- The **volume** of a bulk of fluid: V, **surface** of V: $\mathbf{S} = S\mathbf{n}$
- The **principle of mass conservation**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho\,\mathrm{d}V = -\oint\!\!\oint_S \mathbf{j}_\mathrm{m}\cdot\mathrm{d}\mathbf{S}$$

- The Gauss theorem:

$$\oint\!\!\oint_S \mathbf{f}\cdot\mathrm{d}\mathbf{S} = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{f}\,\mathrm{d}V$$

Exact Continuity Equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j}_\mathrm{m} = 0$$

Import Remark

Remember that **ALL** the physical quantities are the function of $\mathbf{r}$ and t **IN DEFUALT**.

1. Suppose there exists a net **volume velocity** output of q units per unit volume per unit time (the unit is s^{-1}, i.e. per second), please show that the exact continuity equation is:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j}_{\mathrm{m}} = \rho q$$

Eulerian Method

- The **velocity field** at loction $\mathbf{r}$ and time t: $\mathbf{v}(x, y, z, t)$
- The particle at $\mathbf{r}$ at time t travels to $\mathbf{r} + \Delta\mathbf{r}$ at time $t + \Delta t$, and the veloctiy of this particle is: $\mathbf{v}(\mathbf{r} + \Delta\mathbf{r}, t + \Delta t)$
- The **acceleration** of this particle:

$$\mathbf{a} \equiv \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

- The Taylor expansion:

$$\mathbf{v}(\mathbf{r}+\Delta\mathbf{r}, t + \Delta t) = \mathbf{v}(x, y, z, t) + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial t}\,\mathrm{d}t + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial x} v_x\,\mathrm{d}t + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial y} v_y\,\mathrm{d}t + \frac{\partial \mathbf{v}(x, y, z, t)}{\partial z} v_z\,\mathrm{d}t\,, \quad \left(v_x\,\mathrm{d}t = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \mathrm{d}x\right)$$

$$\Longrightarrow \boxed{\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}} = \frac{\partial \mathbf{v}}{\partial t} + v_x\frac{\partial \mathbf{v}}{\partial x} + v_y\frac{\partial \mathbf{v}}{\partial y} + v_z\frac{\partial \mathbf{v}}{\partial z} = \boxed{\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)\mathbf{v}}$$

- A particle travels from $\mathbf{r} = (x, y, z)$ at time t to $\mathbf{r} + \Delta\mathbf{r} = (x + \Delta x, y + \Delta y, z + \Delta z)$ at time $t + \Delta t$. According to Taylor expansion, we have

$$\begin{aligned}\mathbf{v}(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) &= \mathbf{v}(x, y, z, t) \\ &\quad + \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{(x,y,z,t)} \Delta t + \left.\frac{\partial \mathbf{v}}{\partial x}\right|_{(x,y,z,t)} \Delta x + \left.\frac{\partial \mathbf{v}}{\partial y}\right|_{(x,y,z,t)} \Delta y + \left.\frac{\partial \mathbf{v}}{\partial z}\right|_{(x,y,z,z)} \Delta z\end{aligned}$$

$$\Longrightarrow \quad \frac{\mathbf{v}(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - \mathbf{v}(x, y, z, t)}{\Delta t}$$
$$= \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{(x,y,z,t)} \frac{\Delta t}{\Delta t} + \left.\frac{\partial \mathbf{v}}{\partial x}\right|_{(x,y,z,t)} \frac{\Delta x}{\Delta t} + \left.\frac{\partial \mathbf{v}}{\partial y}\right|_{(x,y,z,t)} \frac{\Delta y}{\Delta t} + \left.\frac{\partial \mathbf{v}}{\partial z}\right|_{(x,y,z,t)} \frac{\Delta z}{\Delta t}$$

$$\begin{aligned}\Longrightarrow \quad \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{(x,y,z,t)} &= \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{(x,y,z,t)} + \left.\frac{\partial \mathbf{v}}{\partial x}\right|_{(x,y,z,t)} v_x(x, y, z, t) \\ &\quad + \left.\frac{\partial \mathbf{v}}{\partial y}\right|_{(x,y,z,t)} v_y(x, y, z, t) + \left.\frac{\partial \mathbf{v}}{\partial z}\right|_{(x,y,z,t)} v_z(x, y, z, t)\end{aligned}$$

$$\Longrightarrow \quad \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{(x,y,z,t)} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{(x,y,z,t)} + \left[(v_x, v_y, v_z)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right]\mathbf{v}\Big|_{(x,y,z,t)}$$

$$\Longrightarrow \quad \left.\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right|_{(x,y,z,t)} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{(x,y,z,t)} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}|_{(x,y,z,t)}$$

where $\mathbf{v}(x, y, z, t) = (v_x(x, y, z, t), v_y(x, y, z, t), v_z(x, y, z, t))$

- The **volume** of a bulk of fluid: ΔV
- The **surface** of ΔV: $\Delta\mathbf{S} = \Delta S\mathbf{n}$
- The **total pressure** when $\Delta V \to 0$:

$$-\oiint_{\Delta S} P\,\mathrm{d}\mathbf{S} = -\iiint_{\Delta V} \boldsymbol{\nabla} P\,\mathrm{d}V \xrightarrow{\Delta V \to 0} -\boldsymbol{\nabla} P \Delta V$$

- Newton's seconda law

$$(\rho \Delta V)\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} P \Delta V$$

Eulerian Equation

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} P$$

or

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P$$

1. Suppose there exists a **body force density** $\rho\mathbf{g}$ (which has the physical dimensions of force per unit mass), please show that the Eulerian equation is:

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} P + \rho\mathbf{g}$$

Definition (Body Force)

- A **body force** is a force that acts throught the volume of a body.
  - ✓: gravity, electric forces, magnetic forces, Centrifugal force
  - ×: shear forces, normal forces
- The **body force density**, $\rho\mathbf{g}$ is defined so that the volume integral of it gives the **body force**:

$$\mathbf{G}_{\text{body}} = \iiint_V \rho\mathbf{g}(\mathbf{r})\,\mathrm{d}\,V$$

  - where $\mathbf{g}$ is the **external body force density field** acting on the system

Equation of State

For fluid media, the equation of state must relate **3 physical quantities** descrbing the thermodynamic behavior of the fuid: such as $\{\rho, P, T\}$, $\{\rho, P, s\}$ or $\{\rho, P, V\}$.

$$f(\rho, P, T) = 0, \quad f(\rho, P, s) = 0 \quad \text{or} \quad f(\rho, P, V) = 0$$

where s is the **entropy per mass** of the fluid.

Equation of State for a Perfect Gas

$$P = \rho \mathcal{R} T_{\mathrm{K}}$$

where T_{K} in kelvins (K) is the **absolute temperature**, and $\mathcal{R}$ is the **specific gas constant** and depends on the **universal gas constant** and the **molecular weight**. For air, $\mathcal{R} \approx 287 \mathrm{J} \cdot \mathrm{kg}^{-1} \mathrm{K}^{-1}$.

Equation of State

- The acoustic processes are nearly **isentropic** (**adiabatic and reversible**)
- The **entropy** of the fluid remains nearly constant: $\dfrac{\mathrm{d}s(P,\rho)}{\mathrm{d}t} = 0 \Longrightarrow P = P(\rho)$
- Definition of **sound speed**: $c^2 \equiv \dfrac{\partial P}{\partial \rho} = c^2(\rho, s)$
  - $c_s^2 = \dfrac{\mathrm{d}P}{\mathrm{d}\rho} = c_s^2(\rho)$
- Definition of **compressibility** and **bulk modulus**:

$$\beta_s = \frac{1}{\kappa_s} \equiv -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_s = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial P}\right)_s = \frac{1}{\rho c_s^2}$$

Equation of State

General Fluids Adiabat

$$P = P(\rho) = P(\delta) = P_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \rho_0^n \left(\frac{\partial^n P}{\partial \rho^n}\right)_{s,0} \delta^n$$
$$= P_0 + \frac{1}{1!} A\delta + \frac{1}{2!} B\delta^2 + \frac{1}{3!} C\delta^3 + \cdots$$

where $\delta = (\rho - \rho_0)/\rho_0$, $A = \rho_0 \left(\frac{\partial P}{\partial \rho}\right)_{s,0}$, $B = \rho_0^2 \left(\frac{\partial^2 P}{\partial \rho^2}\right)_{s,0}$, $C = \rho_0^3 \left(\frac{\partial^3 P}{\partial \rho^3}\right)_{s,0}$.

Perfect Gas Adiabat

$$P = P_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$
$$= P_0 + \frac{1}{1!} \rho_0 \gamma \delta + \frac{1}{2!} \rho_0^2 \gamma(\gamma - 1)\delta^2 + \cdots$$

where γ is the **ratio of specific heats** (or **ratio of heat capacities**). For air, $\gamma \approx 1.40$.

- Under mechanical disturbances, the quantities are changed as:

$$P_0 \to P, \quad T_0 \to T, \quad \rho_0 \to \rho$$

  - **pressure disturbance (excess pressure)**: $p = \Delta P (= \mathrm{d}P) = P - P_0$
  - **density disturbance (excess density)**: $\rho' = \Delta\rho (= \mathrm{d}\rho) = \rho - \rho_0$
  - **temperature disturbance (excess density)**: $\tau = \Delta T (= \mathrm{d}T) = T - T_0$
  - **velocity disturbance (excess density)**: $\mathbf{v}' = \Delta\mathbf{v} (= \mathrm{d}\mathbf{v}) = \mathbf{v} - \mathbf{v}_0$
    - $\mathbf{v}' = \mathbf{v}$ if $\mathbf{v}_0 = 0$
- **(scalar) pressure field**: $p(\mathbf{r}, t)$
- **(vector) velocity field**: $\mathbf{v}(\mathbf{r}, t)$
- **(scalar) velocity potential field**: $\mathbf{v} = -\boldsymbol{\nabla}\Phi$ or $\mathbf{v} = \boldsymbol{\nabla}\Phi$
  - the particle velocity must be **irrotational**, i.e. $\boldsymbol{\nabla} \times \mathbf{v} = 0$

Sound Pressure with Its Units

Sound (Acoustic) Pressure

$$p \equiv P - P_0$$

- **effective sound pressure** (only for periodical or quasi-periodical acoustic waves:

$$p_e(x) = \sqrt{\frac{1}{T}\int_0^T p^2(\mathbf{r}, t)\,\mathrm{d}t}$$

- Units for sound pressure
  - $1\text{Pa} = 1\text{N/m}^2 = 10\text{Dyne/cm}^2$
  - $1\text{bar} = 100\text{kPa} = 10^5\text{Pa}$

Assumptions for Linear Wave Equation

- Assumptions for the fluids to be investigated:
  - **staionery**: $\mathbf{v}_0 = 0$
  - **homogeneity of initial conditions**: $P_0(\mathbf{r}) = \text{const.}, \rho_0(\mathbf{r}) = \text{const.}$
  - **lossless**: adiabat, i.e. $s(t) = \text{const.}$
- Linear assumptions:

$$p \ll P_0, |\boldsymbol{\xi}| \ll \lambda, \rho' \ll \rho_0, |\mathbf{v}| \ll c_0$$

  - $c_s^2 \approx c_0^2 = \left(\dfrac{\mathrm{d}P}{\mathrm{d}\rho}\right)_{s,0}$
  - $\kappa_s \approx \kappa_{s,0} = \rho_0 c_0^2, \beta_s \approx \beta_{s,0} = \dfrac{1}{\rho_0 c_0^2}$

- **exact continuity equation** $\rightarrow$ **linear continuity equation**:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \rightarrow \frac{\partial \rho'}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v} \approx 0$$

- **Eulerian equation** $\rightarrow$ linear:

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} P \rightarrow \rho_0 \frac{\partial \mathbf{v}}{\partial t} \approx -\boldsymbol{\nabla} p$$

- **equation of state** $\rightarrow$ linear:

$$P = P_0 + \frac{1}{1!}\rho_0 \left(\frac{\partial P}{\partial \rho}\right)_{s,0} \delta + \frac{1}{2!}\rho_0^2 \left(\frac{\partial^2 P}{\partial \rho^2}\right)_{s,0} \delta^2 + \cdots$$

$$\rightarrow p \approx \rho' c_0^2$$

- Eliminate ρ',* we have

$$\begin{cases} \beta_{s,0}\dfrac{\partial p}{\partial t} \approx -\boldsymbol{\nabla}\cdot\mathbf{v} \\ \dfrac{\partial \mathbf{v}}{\partial t} \approx -\dfrac{1}{\rho_0}\boldsymbol{\nabla} p \end{cases}, \quad \left(\beta_{s,0} = \frac{1}{\kappa_{s,0}} = \frac{1}{\rho_0 c_0^2}\right)$$

- Eliminate $\mathbf{v}$,† we have

Linear Wave Equation (for Sound Pressure)

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p \approx 0$$

*Why ρ'? Because the variation of the density is hard to detect in practice.

†Why $\mathbf{v}$? Because the velocity is a vector, which is harder to handle than a scalar like sound pressure p.

Useful Equations

Calculate $\mathbf{v}$ by p

$$\mathbf{v} = -\frac{1}{\rho_0}\int \boldsymbol{\nabla} p \,\mathrm{d}t$$

Proof: by Eulerian euqation directly.

Linear Wave Equation (for Velocity)

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{v} \approx 0$$

Proof: Eliminate p with $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{v} + \nabla^2\mathbf{v} = \nabla^2\mathbf{v}$.

Linear Wave Equation (for Velocity Potential)

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\Phi \approx 0, \quad \left(p \approx \rho_0\frac{\partial\Phi}{\partial t}\right)$$

where $\mathbf{v} = -\boldsymbol{\nabla}\Phi$.

Definition (Nondimensionalized Variables)

$$\tau = \omega t, \quad \xi = kx, \quad \nu = \frac{v}{c_0}, \quad q = \frac{p}{\kappa_{s,0}}, \quad \delta = \frac{\rho'}{\rho_0}, \quad \chi^2 + 1 = \left(\frac{c}{c_0}\right)^2 \text{[a]}$$

[a] Why don't we define this as $\chi^2 = \left(\frac{c-c_0}{c_0}\right)^2$? Because c is always no less than c_0.

1. Suppose there exist q and $\mathbf{g}$, show that

Linear Inhomogenous Wave Equation (for Sound Pressure)

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p \approx \rho_0\left(\boldsymbol{\nabla}\cdot\mathbf{g} - \frac{\partial q}{\partial t}\right)$$

2. Suppose the $\mathbf{g}$ has **potential**, i.e. $\mathbf{g} = -\boldsymbol{\nabla}\Psi$, show that

Linear Wave Equation (for Velocity Potential)

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\Phi \approx -q - \frac{1}{c_0^2}\frac{\partial \Psi}{\partial t}$$

3. Textbook B: 4-4

4. Suppose the potential is defined as:

$$\mathbf{v} = \boldsymbol{\nabla}\Phi, \quad \mathbf{g} = \boldsymbol{\nabla}\Psi.$$

Show that

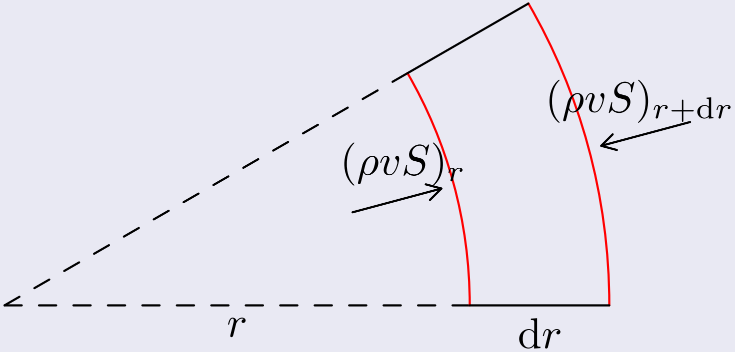
$$p \approx -\rho_0\left(\frac{\partial \Phi}{\partial t} - \Psi\right)$$

and

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\Phi \approx q - \frac{1}{c_0^2}\frac{\partial \Psi}{\partial t}$$

Description

- 1. the **shape of the wave front** remains the same
- 2. the **direction of propagating of the wave** remains the same



A Special Wave Equation

- **continuity equation:**

$$\frac{\partial}{\partial t}(\rho S\,\mathrm{d}r) = (\rho v S)_r - (\rho v S)_{r+\mathrm{d}r} = -\frac{\partial(\rho v S)}{\partial r}\,\mathrm{d}r \quad \rightarrow \quad \boxed{S\frac{\partial \rho}{\partial t} \approx -\rho_0 \frac{\partial (vS)}{\partial r}}$$

- **Eulerian equation:**

$$\rho\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\partial P}{\partial r} \quad \rightarrow \quad \boxed{\rho_0\frac{\partial v}{\partial t} \approx -\frac{\partial p}{\partial r}}$$

- **equation of state:**

$$P = P(\rho) \quad \rightarrow \quad \boxed{p \approx c_0^2 \rho'}$$

- **linear wave equation:**

$$\boxed{\left[\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r}\frac{\mathrm{d}(\ln S)}{\mathrm{d}r}\right] - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} \approx 0}$$

Planar Waves

- 1D wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p = 0$$

$$\frac{\partial^2 p}{\partial \xi \partial \eta} = 0, (\xi = c_0 t - x, \eta = c_0 t + x)$$

$$\Longrightarrow p = f(\xi) + g(\eta) = f(c_0 t - x) + g(c_0 t + x)$$

- Proof

$$\xi = \xi(x,t), \quad \eta = \eta(x,t)$$

$$\frac{\partial}{\partial \xi} = \left(\frac{\partial \xi}{\partial t}\right)^{-1}\frac{\partial}{\partial t} + \left(\frac{\partial \xi}{\partial x}\right)^{-1}\frac{\partial}{\partial x} = \frac{1}{c_0}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial \eta} = \left(\frac{\partial \eta}{\partial t}\right)^{-1}\frac{\partial}{\partial t} + \left(\frac{\partial \eta}{\partial x}\right)^{-1}\frac{\partial}{\partial x} = \frac{1}{c_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$$

$$\begin{aligned}\frac{\partial^2}{\partial \xi \partial \eta} &= \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) \\ &= \left(\frac{1}{c_0}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) = \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\end{aligned}$$

Planar Waves

- 1D wave equation

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p = 0$$

$$\Longrightarrow p = f(c_0\,t - x) + g(c_0\,t + x)$$

- **propagating wave** in x direction: $p = p(c_0\,t - x)$
- (3D wave equation) propagating wave in **unit vector** $\mathbf{n} = (n_x, n_y, n_z)$ direction

$$p = p(c_0\,t - \mathbf{n}\cdot\mathbf{r})$$

- **wave front**: a curved surface where the physical quantites are the same

$$\zeta = c_0\,t - \mathbf{n}\cdot\mathbf{r} = \text{const.}$$

$$\mathrm{d}\zeta = 0 \Longrightarrow c_0 = \frac{\partial}{\partial t}(\mathbf{n}\cdot\mathbf{r})$$

- velocity

$$\mathbf{v} = -\frac{1}{\rho_0}\int \boldsymbol{\nabla} p\,\mathrm{d}t = \frac{p}{\rho_0\,c_0}\mathbf{n}, \quad \left(\boldsymbol{\nabla} p(\zeta) = -\mathbf{n}\frac{\mathrm{d}p}{\mathrm{d}\zeta}, \int p(\zeta)\,\mathrm{d}t = \frac{1}{c_0}\int p(\zeta)\,\mathrm{d}\zeta\right)$$

- the **angular frequency**: ω

$$p(\mathbf{r}, t) = p(\mathbf{r})\mathrm{e}^{\mathrm{j}\omega t}, \quad (\mathrm{j}^2 = -1)$$

Linear Harmonic Wave Equation (Helmholtz Equation)

$$(\nabla^2 + k^2)\,p(\mathbf{r}) = 0$$

the **wave number**: $k = \omega / c_0$

- 1D wave equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2\right)p(x) = 0, \Longrightarrow p(x) = p_\mathrm{a}\mathrm{e}^{\mathrm{j}kx} \quad \text{or} \quad p_\mathrm{a}\mathrm{e}^{-\mathrm{j}kx}$$

- propagating wave with frequency ω

$$p_\pm(x, t) = p_\mathrm{a}\mathrm{e}^{\mathrm{j}(\omega t \mp kx)}$$

$$p(\mathbf{r}, t) = p_\mathrm{a}\mathrm{e}^{\mathrm{j}(\omega t - \mathbf{k}\cdot\mathbf{r})}, \quad (\mathbf{k} = k\mathbf{n})$$

- particle velocity

$$v(\mathbf{r}, t) = \mathbf{n}v_\mathrm{a}\mathrm{e}^{\mathrm{j}(\omega t - \mathbf{k}\cdot\mathbf{r})}, \quad \left(v_\mathrm{a} = \frac{p_\mathrm{a}}{\rho_0 c_0}\right)$$

Sound Speed – the propagating speed of sound waves

$$c \equiv \sqrt{\frac{\mathrm{d}P}{\mathrm{d}\rho}} = \sqrt{\frac{\kappa_s}{\rho}} = \sqrt{\frac{1}{\rho\beta_s}}, \quad c_0 = c|_{\rho=\rho_0}$$

- **incompressible fluids/solids**: $\kappa_s \to \infty, \quad \beta_s \to 0, \quad c_0 \to 0$
- **rarefied gas**: $\kappa_s \to 0, \quad \beta_s \to \infty, \quad c_0 \to 0$
- **Adiabatic and Ideal Gas**:

$$P\rho^{-\gamma} = \text{const.} \quad \text{or} \quad PV^{\gamma} = \text{const.}, \quad (\gamma_{\text{air}} = 1.4)$$

$$c = \sqrt{\frac{\gamma P}{\rho}}, \quad c_0 = \sqrt{\frac{\gamma P_0}{\rho_0}}$$

- **Clapeyron Euation for Ideal Gas**:

$$PV = \frac{M}{\mu}RT, \quad = P = \frac{\rho}{\mu}RT$$

$$c_0 = \sqrt{\frac{\gamma}{\mu}RT_0} = c_0|_{T_\mathrm{C}=0^\circ\mathrm{C}}\sqrt{1+\frac{T_\mathrm{C}}{273}} \approx 331.6 + 0.6\,T_\mathrm{C}, \quad (T_\mathrm{C} = T - 273)$$

- **Clapeyron Euation for Ideal Gas**:

$$PV = \frac{M}{\mu} RT, \quad = P = \frac{\rho}{\mu} RT$$

  - P: the **pressure** of the gas
  - V: the **volumne** of the gas
  - M: the **total mass** of the gas (in kilograms)
  - μ: the **molar mass** of the gas (in kilograms per mole)
  - R: the ideal, or universal, **gas constant**, equal to the product of the Boltzmann constant and the Avogadro constant
  - T: the **absolute temperature** of the gass (in Kelvins)
  - ρ: the **density** of the gass

- Calculate c_0 by T_C, the **temperature** in Celsius

$$c_0 = \sqrt{\frac{\gamma}{\mu} RT_0} = c_0|_{T_\mathrm{C}=0^\circ\mathrm{C}} \sqrt{1 + \frac{T_\mathrm{C}}{273}} \approx 331.6 + 0.6\, T_\mathrm{C}, \quad (T_\mathrm{C} = T - 273)$$

- Check for linear assumption in air: $p_\mathrm{a} = 0.1\,\mathrm{Pa}, \quad T_\mathrm{C} = 20^\circ\mathrm{C}, \quad \rho_0 = 1.21\,\mathrm{kg/m^3}$

$$c_0 \approx 343.6\,\mathrm{m/s}, \quad v_\mathrm{a} = \frac{p_\mathrm{a}}{\rho_0 c_0} \approx 2.4 \times 10^{-4} \mathrm{m/s} \ll c_0$$

Acoustic impedance, specific acoustic impedance, and characteristic impendace

- **Acoustic Impedance**: $Z_\mathrm{a} = \dfrac{p}{U}$
  - lumped parameter system
- **Specific Acoustic Impedance**: $Z_\mathrm{s} = \dfrac{p}{v}$
  - distributed parameter system
  - real part: **specific acoustic resistance**
  - imaginary part: **specific acoustic reactance**
- plane wave
  - positve direction: $Z_\mathrm{s} = \rho_0 c_0$
  - negative direction: $Z_\mathrm{s} = -\rho_0 c_0$
- **characteristic (specific acoustic) impedance**

$$z_0 = \rho_0 c_0$$

  - represents the acoustic properties of the media

Sound Energy

- consider the fluid particle $\Delta V = \Delta V(\mathbf{r}, t)$
  - $t = 0 \Longrightarrow \Delta V_0 = \Delta V(\mathbf{r}, 0), \quad P_0, \quad \rho_0$
- **kinetic energy**:

$$\Delta E_\mathrm{k} = \frac{1}{2}(\rho_0 \Delta V_0) v^2, \quad (v = |\mathbf{v}|)$$

- **potential energy**: the work done by sound pressure p

$$\Delta E_\mathrm{p} = -\int_{\Delta V_0}^{\Delta V} p \,\mathrm{d}(\Delta V) \approx \Delta V_0 \beta_{s,0} \int_0^p p \,\mathrm{d}p = \frac{1}{2}\beta_{s,0} p^2 \Delta V_0$$

$$\left(\mathrm{d}(\rho \Delta V) = 0 \to \frac{\mathrm{d}(\Delta V)}{\Delta V_0} \approx -\frac{\mathrm{d}\rho'}{\rho_0} \xrightarrow{p = c_0^2 \rho'} \mathrm{d}(\Delta V) \approx -\Delta V_0 \beta_{s,0} \,\mathrm{d}p\right)$$

- **total energy**:

$$\Delta E = \Delta E_\mathrm{k} + \Delta E_\mathrm{p} = \frac{1}{2}\left(\rho_0 v^2 + \beta_{s,0} p^2\right) \Delta V_0$$

Sound Energy Density

- the sound energy per volume (in Pascals)

$$\varepsilon = \frac{\Delta E}{\Delta V_0} = \frac{\Delta E_{\mathrm{k}} + \Delta E_{\mathrm{p}}}{\Delta V_0} = \frac{1}{2}\left(\rho_0 v^2 + \beta_{s,0} p^2\right)$$

- plane wave: $\Delta E_{\mathrm{k}} = \Delta E_{\mathrm{p}}$

$$v = \frac{p}{\rho_0 c_0} \rightarrow \varepsilon = \beta_{s,0} p^2 = \frac{p^2}{\rho_0 c_0^2}$$

- **average** of the sound energy density for plane harmonic waves

$$\bar{\varepsilon} = \frac{\overline{p^2}}{\rho_0 c_0^2} = \frac{p_{\mathrm{a}}^2}{2\rho_0 c_0^2} = \frac{p_{\mathrm{e}}^2}{\rho_0 c_0^2}, \quad \left(\overline{p^2} = \frac{1}{2}|p|^2 = p_{\mathrm{e}}^2 = \frac{1}{2} p_{\mathrm{a}}^2\right)$$

Sound Power, Sound Intensity

- **sound power** of arbitrary curved surface $\mathbf{S}$ (W)

$$W = \iint_S \mathbf{v} \cdot (p\,\mathrm{d}\mathbf{S}) = \iint_S p\mathbf{v} \cdot \mathrm{d}\mathbf{S} = \iint_S \mathbf{I} \cdot \mathrm{d}\mathbf{S}\,,$$

- **sound energy flux (density)**: the rate of energy transfer per unit area ($\mathrm{W} \cdot \mathrm{m}^{-2} = \mathrm{J} \cdot \mathrm{m}^{-2} \cdot \mathrm{s}^{-1}$)

$$\mathbf{I} \equiv p\mathbf{v}$$

- **sound intensity**: time average of sound energy flux

$$\bar{\mathbf{I}} = \overline{p\mathbf{v}}$$

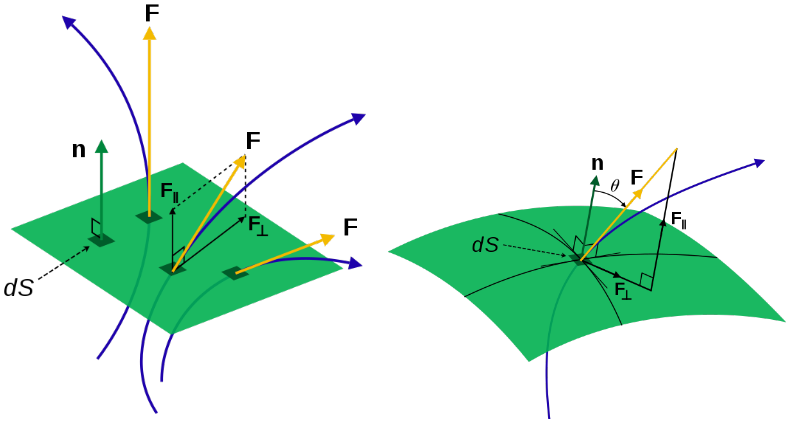
- plane sound waves:

$$\bar{\mathbf{I}} = \frac{1}{\rho_0 c_0} \overline{p^2}\mathbf{n} = \bar{\varepsilon} c_0 \mathbf{n}$$

- harmonic plane sound waves:

$$\bar{\mathbf{I}} = \frac{1}{2} \mathrm{Re}(p^* \mathbf{v}) = \frac{1}{\rho_0 c_0} |p|^2 \mathbf{n} = (p_\mathrm{e} v_\mathrm{e})\mathbf{n}$$

- **Flux**: describes the quantity which passes through a surface



Flux*

- **Flux**: describes the quantity which passes through a surface
- **Flux as flow rate per unit area**: $\mathbf{j}$
  - transport phenomena: heat transfer, mass transfer, and fluid dynamics
  - dimensions: [quantity]·[time]$^{-1}$·[area]$^{-1}$
  - vector
  - also called **flux density**
  - q: physical quantity that flows

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \iint_S \mathbf{j} \cdot \mathrm{d}\mathbf{S}$$

- **Flux as a surface integral**: Φ_F
  - electromagnetism, calculus
  - dimensions: [quantity]·[time]$^{-1}$
  - scalar

$$\Phi_F = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Sound Pressure Level, Sound Intensity Level

- **Sound Pressure Level (SPL)**: a logarithmic measure of the **effective pressure** of a sound relative to a reference value

$$\mathrm{SPL} \equiv \ln\left(\frac{p_\mathrm{e}}{p_\mathrm{ref}}\right)\mathrm{Np} \equiv 2\lg\left(\frac{p_\mathrm{e}}{p_\mathrm{ref}}\right)\mathrm{B} \equiv 20\lg\left(\frac{p_\mathrm{e}}{p_\mathrm{ref}}\right)\mathrm{dB}$$

  - p_e: the **effective sound pressure** or root mean square sound pressure
- $p_\mathrm{ref} = 20\mu\mathrm{Pa}$ in air, which is often considered as the **threshold of human hearing at** 1kHz (roughly the sound of a mosquito flying 3 m away
- **sound intensity level**

$$\mathrm{SIL} = 10\lg\left(\frac{I}{I_\mathrm{ref}}\right)\mathrm{dB}, \quad I_\mathrm{ref} = 10 \times 10^{-12}\mathrm{W}\cdot\mathrm{m}^2$$

- relation between the SPL and SIL:

$$\mathrm{SIL} = \mathrm{SPL} + 10\lg\frac{400}{\rho_0 c_0}$$

- the boundary conditions s.t. Snell's law:

$$\begin{cases} p_{\mathrm{ia}} + p_{\mathrm{ra}} = p_{\mathrm{ta}} \\ \dfrac{p_{\mathrm{ia}} - p_{\mathrm{r}}\,a}{z_{\mathrm{n}1}} = \dfrac{p_{\mathrm{t}}\,a}{z_{\mathrm{n}2}} \end{cases} \Longrightarrow \begin{cases} 1 + r_{\mathrm{p}} = t_{\mathrm{p}} \\ \dfrac{1 - r_{\mathrm{p}}}{z_{\mathrm{n}1}} = \dfrac{t_{\mathrm{p}}}{z_{\mathrm{n}2}} \end{cases}, \left(\begin{array}{c} z_{\mathrm{n}1} \equiv z_{\mathrm{n}}\,i = -z_{\mathrm{n}}\,r \\ z_{\mathrm{n}2} \equiv z_{\mathrm{n}}\,t \end{array} \right)$$

- the ratio of density: $m \equiv \rho_2/\rho_1$, refractive index: $n = k_2/k_1 = c_1/c_2$
- reflection and transmission coeff.:

$$r_{\mathrm{p}} \equiv \frac{p_{\mathrm{r}}\,a}{p_{\mathrm{i}}\,a} = \frac{z_{\mathrm{n}2} - z_{\mathrm{n}1}}{z_{\mathrm{n}2} - z_{\mathrm{n}1}} = \frac{R_2 \cos\theta_{\mathrm{i}} - R_1 \cos\theta_{\mathrm{t}}}{R_2 \cos\theta_{\mathrm{i}} + R_1 \cos\theta_{\mathrm{t}}} = \frac{m\cos\theta_{\mathrm{i}} - \sqrt{n^2 - \sin^2\theta_{\mathrm{i}}}}{m\cos\theta_{\mathrm{i}} + \sqrt{n^2 - \sin^2\theta_{\mathrm{i}}}}$$

$$t_{\mathrm{p}} \equiv \frac{p_{\mathrm{t}}\,a}{p_{\mathrm{i}}\,a} = \frac{2z_{\mathrm{n}2}}{z_{\mathrm{n}2} - z_{\mathrm{n}1}} = \frac{2R_2 \cos\theta_{\mathrm{i}}}{R_2 \cos\theta_{\mathrm{i}} + R_1 \cos\theta_{\mathrm{t}}} = \frac{2m\cos\theta_{\mathrm{i}}}{m\cos\theta_{\mathrm{i}} + \sqrt{n^2 - \sin^2\theta_{\mathrm{i}}}}$$

Sound Waves in Ducts Filled with Fluids

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September 24, 2018

Radiation of Sound

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Green Functions with Infinitely Large Rigid Boundary

$$G(\mathbf{r};\mathbf{r}_0) = g(|\mathbf{r}-\mathbf{r}_0|) + g(|\mathbf{r}-\mathbf{r}_0'|), \quad \begin{cases} \mathbf{r}_0 = (x_0, y_0, z_0) \\ \mathbf{r}_0' = (x_0, y_0, -z_0) \end{cases}$$

- only the sound field on the surface xOz is needed to compute for symmetry
- location of the field point P: $\mathbf{r} = r(\sin\theta, 0, \cos\theta)$
- volume velocity of an element on the piston: $\mathrm{d}\,Q_0$
- location of a point on the piston $\mathrm{d}\,Q_0$: $\mathbf{r}_0 = \rho(\cos\varphi, \sin\varphi, 0)$
- distance between P and $\mathrm{d}\,Q_0$: $h = |\mathbf{r} - \mathbf{r}_0| \approx r - \rho\sin\theta\cos\varphi, (r \gg a)$

$$\Phi(\mathbf{r}) = \iint_{S_{\mathrm{pis}}} 2g(\mathbf{r};\mathbf{r}_0)\,u_{\mathrm{a}}(\mathbf{r}_0)\,\mathrm{d}S_0 \xlongequal{u_{\mathrm{a}}=\mathrm{const}} u_{\mathrm{a}} \iint_{S_{\mathrm{pis}}} \frac{\mathrm{e}^{-\mathrm{j}kh}}{2\pi h}\,\mathrm{d}S_0$$

Sound Field at Far Field (Radiation from a Baffled Circular Plane Piston)

$$\Phi(\mathbf{r}, t) \approx 2Q(t)g(r)D(\theta)$$

where $\quad Q(t) = Q_\mathrm{a}\mathrm{e}^{\mathrm{j}\omega t}, \quad Q_\mathrm{a} = u_\mathrm{a}\left(\pi a^2\right), \quad g(r) \equiv \dfrac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r}, \quad D(\theta) \equiv \dfrac{2\mathrm{J}_1(ka\sin\theta)}{ka\sin\theta}$

- the property of Bessel functions: $\left[\frac{2\mathrm{J}_1(x)}{x}\right]_{x=0} = 1$

Viscous Absorption of Sound

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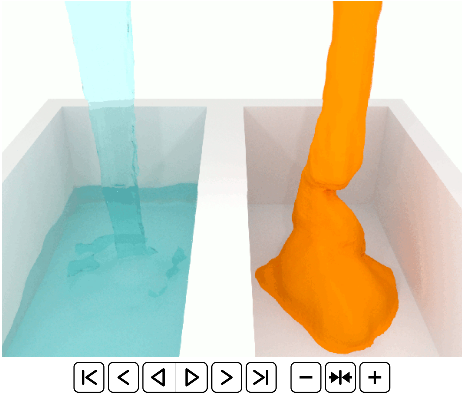 ‡

[‡] Available at: https://www.wikiwand.com/en/Viscosity

Definition (Viscosity)

- Definition 1. **the first and second viscosity** μ_1, μ_2
- Definition 2. **the shear viscosity** η, and **the bulk viscosity** ζ
- the relations:

$$\begin{cases} \mu_1 = \eta \\ \mu_2 = \zeta - \frac{2}{3}\eta \end{cases}, \quad \begin{cases} \eta = \mu_1 \\ \zeta = \frac{2}{3}\mu_1 + \mu_2 \end{cases}$$

Conservation of Momentum

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V \rho v_i \,\mathrm{d}V = \iiint_V \rho g_i \,\mathrm{d}V - \oiint_S \rho v_i v_k \,\mathrm{d}S_k + \oiint_S T_{ik}\,\mathrm{d}S_k$$

- **Cartesian momentum density component**: ρv_i
- **force density (force per volume)**: ρg_i
- **stress tensor**: T_{ik}

- by Gauss theorem:

$$\frac{\partial}{\partial t}(\rho v_i) = \rho g_i - \frac{\partial}{\partial x_k}(\rho v_i v_k) + \frac{\partial}{\partial x_k}(T_{ik})$$

Conservation of Momentum Equation

- by using the Continuity Equation, we have

$$\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} \equiv \rho\frac{\partial v_i}{\partial t} + \rho v_{i,k} v_k = \rho g_i + \frac{\partial}{\partial x_k}(T_{ik})$$

Strain and Stress

- **strain rate tensor (deformation velocity tensor)**

$$d_{ij} \equiv \frac{1}{2}(v_{i,j} + v_{j,i})$$

- **stress tensor**

$$T_{ij} = \mu_1 \Theta \delta_{ij} + 2\mu_2 d_{ij}, \quad (\Theta \equiv d_{kk} = \boldsymbol{\nabla} \cdot \mathbf{v})$$

Navier-Stokes Equation

General Navier-Stokes Equation

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} - \boldsymbol{\nabla}P + \mu_1\nabla^2\mathbf{v} + (\mu_1+\mu_2)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{v}$$

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} - \boldsymbol{\nabla}P + \eta\nabla^2\mathbf{v} + (\zeta+\frac{\eta}{3})\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{v}$$

$$\left(\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} = \rho g_i - \frac{\partial P}{\partial x_i} + \mu_1\frac{\partial^2 v_i}{\partial x_k\partial x_k} + (\mu_1+\mu_2)\frac{\partial^2 v_k}{\partial x_i\partial x_k}\right)$$

Classical Navier-Stokes Equation

- with the relation: $\mu_2 + \frac{2}{3}\mu_1 = 0$

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} - \boldsymbol{\nabla}P + \mu_1\nabla^2\mathbf{v} + \frac{\mu_1}{3}\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{v}$$

- with the relation: $\zeta = 0$

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\mathbf{g} - \boldsymbol{\nabla}P + \eta\nabla^2\mathbf{v} + \frac{\eta}{3}\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{v}$$

Nonlinear Acoustics

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Theorem (Gauss Theorem in Tensor Fields)

$$\iiint_V \frac{\partial \mathbf{F}_i}{\partial x_i}\,\mathrm{d}\,V = \oiint_S \mathbf{F}_i\,\mathrm{d}S_i \left(= \oiint_S \mathbf{F}_i n_i\,\mathrm{d}S\right)$$

Appendix: Complex Numbers with Their Computations

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Notations

Convention Notations

Type	Notation
complex number	X, Y, Z
real number	$x = \mathrm{Re}(X), y = \mathrm{Re}(Y), z = \mathrm{Re}(Z)$
real number	a, b, c, d
exponential	$Z_1 = \rho_1 \mathrm{e}^{\mathrm{j}\varphi_1}, Z_2 = \rho_2 \mathrm{e}^{\mathrm{j}\varphi_2}$

- relations

$$z = \frac{1}{2}(Z + Z^*) = \mathrm{Re}(Z)$$

Operations

- **addition**

$$z = ax \pm by \Longrightarrow Z = aX \pm bY$$

- **multiplication**

$$z = xy \Longrightarrow Z = X\,\mathrm{Re}(Y) = \mathrm{Re}(X)\,Y$$

Proof

$$\begin{aligned}
z &= \mathrm{Re}(X) \times \mathrm{Re}(Y) \\
&= \frac{1}{2}(X + X^*) \times \frac{1}{2}(Y + Y^*) \\
&= \frac{1}{2}\,\mathrm{Re}(XY + X^* Y) \\
\therefore Z &= \frac{1}{2}(XY + XY^*) \\
&= \mathrm{Re}(X)\,Y = \mathrm{Re}(X)\,Y^* \\
&= X\,\mathrm{Re}(Y) = X^*\,\mathrm{Re}(Y)
\end{aligned}$$

- **integral and derivative**

$$\begin{cases} \int \mathrm{Re}[X(t)]\mathrm{d}t & = \mathrm{Re}\left(\int X(t)\,\mathrm{d}t\right) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}[X(t)] & = \mathrm{Re}\left[\frac{\mathrm{d}}{\mathrm{d}t}X(t)\right] \end{cases} \Longrightarrow \begin{cases} a\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx + d\int x\,\mathrm{d}t & = 0 \\ a\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} + b\frac{\mathrm{d}X}{\mathrm{d}t} + cX + d\int X\,\mathrm{d}t & = 0 \end{cases}$$

- Show that

$$Z_1 + Z_2 = \sqrt{\rho_1^2 + \rho_2^2 + 2\rho_1\rho_2\cos(\varphi_1 - \varphi_2)}\mathrm{e}^{\mathrm{j}\varphi}, \tan\varphi = \frac{\rho_1\sin\varphi_1 + \rho_2\sin\varphi_2}{\rho_1\cos\varphi_1 + \rho_2\cos\varphi_2}$$

- Show that

$$\mathrm{e}^{-\mathrm{j}\varphi} + \mathrm{e}^{\mathrm{j}\varphi} = 2\cos\varphi, \quad \mathrm{e}^{-\mathrm{j}\varphi} - \mathrm{e}^{\mathrm{j}\varphi} = -2\mathrm{j}\sin\varphi$$

- Show that

$$1 + \mathrm{e}^{\mathrm{j}\varphi} = \mathrm{e}^{\mathrm{j}\frac{\varphi}{2}} 2\cos\frac{\varphi}{2}, \quad 1 - \mathrm{e}^{\mathrm{j}\varphi} = \mathrm{e}^{\mathrm{j}\frac{\varphi}{2}} \times (-2\mathrm{j})\sin\frac{\varphi}{2}$$

- Show that

$$\sum_{n=1}^{N} \mathrm{e}^{\mathrm{j}n\varphi} = \mathrm{e}^{\mathrm{j}\frac{N+1}{2}\varphi} \frac{\sin\left(\frac{n-1}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}$$

$$\sum_{n=1}^{N} \cos n\varphi = \cos\left(\frac{N+1}{2}\varphi\right)\frac{\sin\left(\frac{n-1}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}, \sum_{n=1}^{N} \sin n\varphi = \sin\left(\frac{N+1}{2}\varphi\right)\frac{\sin\left(\frac{n-1}{2}\varphi\right)}{\sin\left(\frac{1}{2}\varphi\right)}$$

- Show that

$$\frac{1 + Z_1}{1 - Z_1} = Z_2 \Longrightarrow Z_1 = \frac{Z_2 - 1}{Z_2 + 1}$$

Appendix: Tensors and Einstein Summation Convention

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Notations

Convention Notations

Type	Rank	Vector Notation	Index Notation
scalar	0	α, β, Φ	α, β, Φ
vector	1	$\mathbf{u}, \mathbf{v}, \mathbf{w}$	u_i, v_i, w_i
tensor	2	$\mathbf{A}, \mathbf{B}, \mathbf{C}$	A_{ij}, B_{ij}, C_{ij}
tensor	3	$\mathbf{T}$	T_{ijk}

Definition (Tensors)

Tensor (2-order) $\mathbf{A}$ is defined by two vectors $\mathbf{u}, \mathbf{v}$ as

$$\mathbf{A} = \mathbf{u}\mathbf{v} = \sum_{j=1}^{3}\sum_{i=1}^{3} u_i v_j \mathbf{e}_i \mathbf{e}_j \equiv \sum_{j=1}^{3}\sum_{i=1}^{3} A_{ij} \mathbf{e}_i \mathbf{e}_j$$

Free Indices

1. appear **only once** within each additive term in an expression. i is a **free index**:

$$a_i = \epsilon_{ikl} b_k c_l + D_{ik} e_k$$

2. imply 3 **ditinct equations**.

$$a_i = b_i + c_i \Longrightarrow \begin{cases} a_1 = b_1 + c_1 \\ a_2 = b_2 + c_2 \\ a_3 = b_3 + c_3 \end{cases}$$

3. The same letter must be used for the free index in **every** additive term.
4. The number of free indices can be larger than 1, and equals the **rank** of the term.
5. The first free index in a term corresponds to the **row**, and the second corresponds to the **column**

$$u_i = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Dummy Indices

1. appear **twice** within an additive term of an expression. j and k are both **dummy indices**:

$$a_i = \epsilon_{ikl} b_k c_l + D_{ik} e_k$$

2. imply a **summation** over the range of the index:

$$a_{kk} \equiv a_{11} + a_{22} + a_{33}$$

$$a_i = \sum_{l=1}^{3} \sum_{=1}^{3} \epsilon_{ikl} b_k c_l + \sum_{k=1}^{3} D_{ik} e_k$$

3. **local** to an individual additive term. Thus, it can be renamed, just like the x or t in integrals:

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 f(t)\,\mathrm{d}t$$

Definition (Kronecker Delta)

A rank-2 symmetric tensor defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad \text{or} \qquad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition (Alternating Unit Tensor or Levi-Cavita Symbol)

A rank-3 **antisymmetric** tensor

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123, 231, 312 \\ 0 & \text{if} \quad (i-j)(j-k)(k-i) = 0 \\ -1 & \text{if} \quad ijk = 132, 213, 321 \end{cases}$$

Important Indentity

$$\epsilon_{ikl}\epsilon_{imn} = \delta_{km}\delta_{ln} - \delta_{kn}\delta_{lm}$$

Commutation and Association in Vector and Index Notation

- In general, **vector notation** DO NOT have commutative or associative properties

$$\mathbf{u} \times \mathbf{v} \neq \mathbf{v} \times \mathbf{u}$$

- All of the terms in index notation are scalars, and only **multiplication/division and addition/subtraction operations** $(+, -, \times, \div)$ are defined.
- commutative and associative properties hold in index notation:

$$a_i b_j = b_j a_i, \quad (a_i b_j) c_k = a_i (b_j c_k)$$

- In general, **calculus operators** $(\int, \mathrm{d}, \partial)$ are not commutative

Vector Operations using Index Notation

- multiplication of a vctor by a scalar

$$\alpha \mathbf{u} = \mathbf{v} \Longrightarrow \alpha u_i = v_i$$

- scalar product of two vectors (a.k.a. dot or inner product)

$$\mathbf{u} \cdot \mathbf{v} = \alpha \Longrightarrow u_k v_k = \alpha$$

- scalar product of two tensors (a.k.a. dot or inner product)

$$\mathbf{A} : \mathbf{B} = \alpha \Longrightarrow A_{kl} B_{lk} = \alpha$$

- tensor product of two vectors (a.k.a dyadic product)

$$\mathbf{u}\mathbf{v} = \mathbf{A} \Longrightarrow u_i v_j = A_{ij}$$

- tensor product of two tensors

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{C} \Longrightarrow A_{ik} B_{kj} = C_{ij}$$

  - NOTE this is NOT an inner product

Vector Operations using Index Notation

- vector product of a tensor and a vector

$$\mathbf{u} \cdot \mathbf{A} = \mathbf{v} \Longrightarrow u_k A_{ki} = v_i$$

$$\mathbf{A} \cdot \mathbf{u} = \mathbf{v} \Longrightarrow A_{ik} u_k = v_i$$

  - NOTE $\mathbf{u} \cdot \mathbf{A} \neq \mathbf{A} \cdot \mathbf{u}$
- cross product of two vectors

$$\mathbf{u} \times \mathbf{v} = \mathbf{w} \Longrightarrow \epsilon_{ikl} u_k v_l = w_i$$

- contraction or trace of a tensor

$$\mathrm{Tr}\, \mathbf{A} = \alpha \Longrightarrow A_{kk} = \alpha$$

Calculus Operations using Index Notation

- NOTE the spatial Cartesian coordinates are renamed as follows

$$(x, y, z) \rightarrow (x_1, x_2, x_3)$$

- temporal derivative of a scalar field $\Phi(x_1, x_2, x_3, t)$

$$\frac{\partial \Phi}{\partial t} \equiv \partial_0 \Phi \quad \text{or} \quad \partial_t \Phi$$

- gradient of a scalar field $\Phi(x_1, x_2, x_3, t)$

$$\frac{\partial \Phi}{\partial x_i} \equiv \partial_i \Phi$$

- gradient of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\boldsymbol{\nabla} \mathbf{u} = \frac{\partial u_j}{\partial x_i} \equiv \partial_i u_j = \begin{bmatrix} \partial_1 u_1 & \partial_1 u_2 & \partial_1 u_3 \\ \partial_2 u_1 & \partial_2 u_2 & \partial_2 u_3 \\ \partial_3 u_1 & \partial_3 u_2 & \partial_3 u_3 \end{bmatrix}$$

  - the first index is the row index
  - the gradient **increases by one the rank** of the expression on which it operates

- divergence of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \partial_k u_k = \alpha$$

  - the divergence **decreases by one the rank** of the expression on which it operates
  - it is not possible to take the divergence of a scalar

- curl of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\boldsymbol{\nabla} \times \mathbf{u} = \epsilon_{ikl} \partial_k u_l = v_i$$

  - the curl **does not change the rank** of the expression on which it operates
  - it is not possible to take the curl of a scalar

- Laplacian of a vector field $\mathbf{u}(x_1, x_2, x_3, t)$

$$\nabla^2 \mathbf{u} \equiv \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \mathbf{u}) = \partial_k \partial_k u_i = v_i$$

1. Show that
$$\boldsymbol{\nabla} \cdot \mathbf{u} \neq \mathbf{u} \cdot \boldsymbol{\nabla}$$
2. Show that
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$$
3. Show that
$$\mathbf{u} \times \mathbf{v} \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{w} \cdot \mathbf{v})\mathbf{u}$$
4. Show that
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\Phi) = \mathbf{0}$$
5. Show that
$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) - \nabla^2\mathbf{u}$$
6. Show that
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$
7. Show that
$$\boldsymbol{\nabla} \cdot (\mathbf{u}\mathbf{v}) = (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{v} + (\boldsymbol{\nabla} \cdot \mathbf{u})\mathbf{v}$$

Decomposition of a Tensor into Symmetric and Antisymmertric Parts

Definition (Symmetric and Antisymmetric Tensors)

- a tensor is **symmetric** if it is equal to its transpose

$$A_{ij} = A_{ji}$$

- a tensor is **antisymmetric** if it is equal to the negative of its transpose

$$A_{ij} = -A_{ij}$$

Theorem (Decomposition of a Tensor into Symmetric and Antisymmetric Parts)

- any arbitray tensor $\mathbf{A}$ may be composed into sum of a symmetric tensor and an antisymmetric tensor

$$A_{ij} = A_{(ij)} + A_{[ij]}$$

where $$A_{(ij)} = \frac{1}{2}(A_{ij} + A_{ji}), \quad A_{[ij]} = \frac{1}{2}(A_{ij} - A_{ji})$$

- the antisymmetric component can be calculated as

$$A_{[ij]} = \frac{1}{2}\epsilon_{ijk}\epsilon_{klm}A_{lm}$$

- the scalar product of any symmetric and antisymmetric tensor is zeros

$$\mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A} = 0 \quad \text{if} \quad A_{ij} = A_{ji} \quad \text{and} \quad B_{ij} = -B_{ji}$$