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DSC: Scheduling Parallel Tasks on an Unbounded Number of Processors[*]

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Abstract

We present a low complexity heuristic named the Dominant Sequence Clustering algorithm (DSC) for scheduling parallel tasks on an unbounded number of completely connected processors. The performance of DSC is comparable or even better on average than other higher complexity algorithms. We assume no task duplication and nonzero communication overhead between processors. Finding the optimum solution for arbitrary directed acyclic task graphs (DAGs) is NP-complete. DSC finds optimal schedules for special classes of DAGs such as fork, join, coarse grain trees and some fine grain trees. It guarantees a performance within a factor of two of the optimum for general coarse grain DAGs. We compare DSC with three higher complexity general scheduling algorithms: the ETF by Hwang, Chow, Anger and Lee [11], Sarkar's clustering algorithm [17] and the MD by Wu and Gajski [19]. We also give a sample of important practical applications where DSC has been found useful.

Index Terms – Clustering, directed acyclic graph, heuristic algorithm, optimality, parallel processing, scheduling, task precedence.

1 Introduction

We study the scheduling problem of directed acyclic weighted task graphs (DAGs) on unlimited processor resources and a completely connected interconnection network. The task and edge weights are deterministic. This problem is important in the development of parallel programming tools and compilers for scalable MIMD architectures [17, 19, 21, 23]. Sarkar [17] has proposed a two step method for scheduling with communication: (1) Schedule on unbounded number of a completely connected architecture. The result of this step will be clusters of tasks, with the constraint that all tasks in a cluster must execute in the same processor. (2) If the number of clusters is larger than the number of processors then merge the clusters

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further to the number of physical processors and also incorporate the network topology in the merging step.

In this paper, we present an efficient algorithm for the first step of Sarkar's approach. Algorithms for the second step are discussed elsewhere [23]. The objective of scheduling is to allocate tasks onto the processors and then order their execution so that task dependence is satisfied and the length of the schedule, known as the parallel time, is minimized. In the presence of communication, the complexity of the above scheduling problem has been found to be much more difficult than the classical scheduling problem where communication is ignored. The general problem is NP-complete and even for simple graphs such as fine grain trees or the concatenation of a fork and a join together the complexity is still NP-complete, Chretienne [3], Papadimitriou and Yannakakis [15] and Sarkar [17]. Only for special classes of DAGs, such as join, fork, and coarse grain tree, special polynomial algorithms are known, Chretienne [4], Anger, Hwang and Chow [2].

There have been two approaches in the literature addressing the general scheduling problem. The first approach considers heuristics for arbitrary DAGs and the second studies optimal algorithms for special classes of DAGs. When task duplication is allowed, Papadimitriou and Yannakakis [15] have proposed an approximate algorithm for a DAG with equal task weights and equal edge weights, which guarantees a performance within 50% of the optimum. This algorithm has a complexity of $O(v^3(v \log v + e))$ where v is the number of tasks and e is the number of edges. Kruatrachue and Lewis [13] have also given an $O(v^4)$ algorithm for a general DAG based on task duplication. One difficulty with allowing task duplication is that duplicated tasks may require duplicated data among processors and thus the space complexity could increase when executing parallel programs on real machines.

Without task duplication, many heuristic scheduling algorithms for arbitrary DAGs have been proposed in the literature, e.g. Kim and Browne [12], Sarkar [17], Wu and Gajski [19]. A detailed comparison of four heuristic algorithms is given in Gerasoulis and Yang [9]. One difficulty with most existing algorithms for general DAGs is their high complexity. As far as we know no scheduling algorithm exists that works well for arbitrary graphs, finds optimal schedules for special DAGs and also has a low complexity. We present one such algorithm in this paper with a complexity of $O((v + e) \log v)$, called the Dominant Sequence Clustering (DSC) algorithm. We compare DSC with ETF algorithm by Hwang, Chow, Anger, and Lee [11] and discuss the similarities and differences with the MD algorithm proposed by Wu and Gajski [19].

The organization is as follows: Section 2 introduces the basic concepts. Section 3 describes an initial design of the DSC algorithm and analyzes its weaknesses. Section 4 presents an improved version of DSC that takes care of the initial weaknesses and analyzes how DSC achieves both low complexity and good performance. Section 5 gives a performance bound for a general DAG. It shows that the performance of DSC is within 50% of the optimum for coarse grain DAGs, and it is optimal for join, fork, coarse grain trees and a class of fine grain trees. Section 6 discusses the related work, presents experimental results and compares the performance of DSC with Sarkar's, ETF and MD algorithms.

2 Preliminaries

A directed acyclic task graph (DAG) is defined by a tuple $G = (V, E, \mathcal{C}, \mathcal{T})$ where V is the set of task nodes and $v = |V|$ is the number of nodes, E is the set of communication edges and $e = |E|$ is the number of edges, $\mathcal{C}$ is the set of edge communication costs and $\mathcal{T}$ is the set of node computation costs. The value $c_{i,j} \in \mathcal{C}$ is the communication cost incurred along the edge $e_{i,j} = (n_i, n_j) \in E$, which is zero if both nodes are mapped in the same processor. The value $\tau_i \in \mathcal{T}$ is the execution time of node $n_i \in V$. $PRED(n_x)$ is the set of immediate predecessors of n_x and $SUCC(n_x)$ is the set of immediate successors of n_x. An example of DAG is shown Fig. 1(a) with 7 tasks $n_1, n_2, \cdots, n_7$. Their execution times are on the right side of the bullets and edge weights are written on the edges.

The task execution model is the compile time *macro-dataflow* model. A task receives all input before starting execution in parallel, executes to completion without interruption, and immediately sends the output to all successor tasks in parallel, see Wu and Gajski [19], Sarkar [17]. Duplication of the same task in separate processors is not allowed.

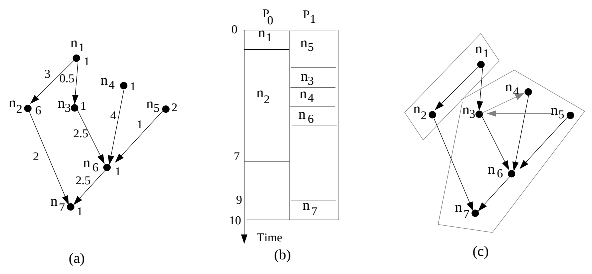

Figure 1: (a) A weighted DAG. (b) A Gantt chart for a schedule. (c) A scheduled DAG.

Given a DAG and an unbounded number of completely connected processors, the scheduling problem consists of two parts: The task to processor assignment, called *clustering* in this paper and the task execution ordering for each processor. A DAG with a given clustering but without the task execution ordering is called a *clustered graph.* A communication edge weight in a clustered graph becomes zero if the start and end nodes of this edge are in the same cluster. Fig. 1(c) shows a clustering of the DAG in (a) excluding the dashed edges for ordering. The tasks n_1 and n_2 constitute one cluster and the rest of the tasks another. Fig. 1(b) shows a Gantt chart of a scheduling in which the processor assignment for each task and the starting and completion times are defined. An equivalent way of representing a schedule is shown in Fig. 1(c), called a *scheduled DAG.* A scheduled DAG contains both clustering and task execution ordering information. The dashed pseudo edges between n_3 and n_4 in Fig. 1(c) represent the execution ordering imposed by the schedule. The task starting times are not explicitly given in a scheduled DAG but can be computed by traversing this DAG.

$CLUST(n_x)$ stands for the cluster of node n_x. If a cluster contains only one task, it is called a *unit* cluster. We distinguish between two types of clusters, the linear and the nonlinear. Two tasks are called *independent* if there are no dependence paths between them. A cluster is called *nonlinear* if there are two independent tasks in the same cluster, otherwise it is called *linear.* In Fig. 1(c), there are two clusters, the cluster that contains n_1 and n_2 is linear and the other cluster is nonlinear because n_3 and n_4 are independent tasks in Fig. 1(a). A schedule imposes an ordering of tasks in nonlinear clusters. Thus the nonlinear clusters of a DAG can be thought of as linear clusters in the scheduled DAG if the execution orders between independent tasks are counted as edges. A linear clustering preserves the parallelism present in a DAG while nonlinear clustering reduces parallelism by sequentializing parallel tasks. A further discussion of this issue can be found in Gerasoulis and Yang [8].

The *critical path* of a clustered graph is the longest path in that graph including both nonzero communication edge cost and task weights in that path. The parallel time in executing a clustered DAG is determined by the critical path of the scheduled DAG and not by the critical path of the clustered DAG. We call the critical path of a scheduled DAG the *dominant sequence*, DS for short, to distinguish it from the critical path of the clustered DAG. For Fig. 1(c), the critical path of the clustered graph is $< n_1, n_2, n_7 >$ and the DS is still that path. If the weight of n_5 is changed from 2 to 6, then the critical path of the clustered graph remains the same $< n_1, n_2, n_7 >$, but the DS changes to $< n_5, n_3, n_4, n_6, n_7 >$. A path that is not a DS is called a SubDS.

Let $tlevel(n_x)$ be the length of the longest path from an entry (top) node to n_x excluding the weight of n_x in a DAG. Symmetrically, let $blevel(n_x)$ be the length of the longest path from n_x to an exit (bottom) node. For example, in Fig. 1(c), $tlevel(n_1) = 0$, $blevel(n_1) = 10$, $tlevel(n_3) = 2, blevel(n_3) = 4$. The following formula can be used to determine the parallel time from a scheduled graph:

$$PT = \max_{n_x \in V}\{tlevel(n_x) + blevel(n_x)\}. \tag{1}$$

2.1 Scheduling as successive clustering refinements

Our approach for solving the scheduling problem with unlimited resources is to consider a scheduling algorithm as performing a sequence of clustering refinement steps. As a matter of fact, most of the existing algorithms can be characterized by using such a framework [9]. The initial step assumes that each node is mapped in a unit cluster. At each step, the algorithm tries to improve on the previous clustering by merging appropriate clusters. A merging operation is performed by zeroing an edge cost connecting two clusters [1].

Sarkar's algorithm

We consider Sarkar's algorithm [17], pp. 123-131, as an example of an edge-zeroing clustering refinement algorithm. This algorithm first sorts the e edges of the DAG in a decreasing order of edge weights and then performs e clustering steps by examining edges from left to right in the sorted list. At each step, it examines one edge and zeroes this edge if the parallel time does not increase. Sarkar's algorithm requires the computation of the parallel time at each step and this problem is also NP_complete. Sarkar uses the following strategy for ordering: Order independent tasks in a cluster by using the highest *blevel* first

[1]Two clusters will not be merged if there is no edge connecting them since this cannot decrease the parallel time.

priority heuristic, where *blevel* is the value computed in the previous step. The new parallel time is then computed by traversing the scheduled DAG in $O(v+e)$ time. Since there are e steps the overall complexity is $O(e(e+v))$.

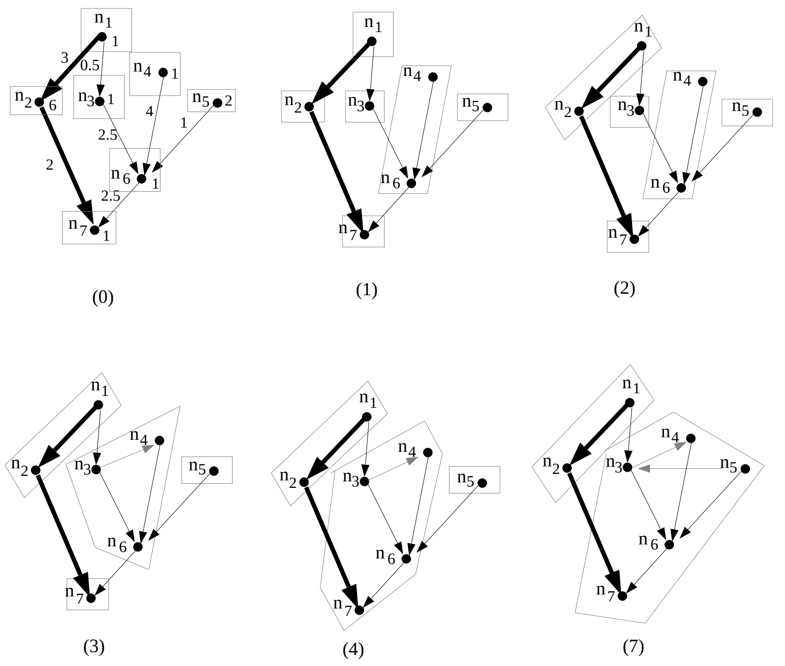

Figure 2: Clustering steps of Sarkar's algorithm for Fig. 1(a).

step i	edge examined	PT if zeroed	zeroing	PT_i
0				13
1	(n_4, n_6)	13	yes	13
2	(n_1, n_2)	10	yes	10
3	(n_3, n_6)	10	yes	10
4	(n_6, n_7)	10	yes	10
5	(n_2, n_7)	11	no	10
6	(n_1, n_3)	11	no	10
7	(n_5, n_6)	10	yes	10

Table 1: Clustering steps of Sarkar's algorithm corresponding to Fig. 2.

Fig. 2 shows the clustering steps of Sarkar's algorithm for the DAG in Fig. 1(a). The sorted edge list with respect to edge weights is $\{(n_4,n_6),(n_1,n_2),(n_3,n_6),(n_6,n_7),(n_2,n_7),(n_1,n_3),(n_5,n_6)\}$. Table 1 traces the execution of this algorithm where PT stands for the parallel time and PT_i is the parallel time for executing the clustered graph at the completion of step i. Initially each task is in a separate cluster as shown in Fig. 2(0) and the thick path indicates the DS whose length is $PT_0 = 13$. At step 1, edge (n_4, n_6) is examined and PT remains 13 if this edge is zeroed. Thus this zeroing is accepted. In step 2, 3 and 4, shown in Fig. 2(2), (3) and (4), all examined edges are zeroed since each zeroing does not increase PT. At step 5, edge (n_2, n_7) is examined and by zeroing it the parallel time increases from 10 to 11. Thus this zeroing is rejected. Similarly, at step 6 zeroing (n_1, n_3) is rejected. At step 7 (n_5, n_6) is zeroed and a pseudo edge from n_5 to n_3 is added because after step 6 $blevel(n_3) = 3$ and $blevel(n_5) = 5$. Finally two clusters are produced with $PT = 10$.

3 An Initial Design of the DSC Algorithm

3.1 Design considerations for the DSC algorithm

As we saw in the previous section, Sarkar's algorithm zeroes the highest communication edge. This edge, however, may not be in a DS and as a result the parallel time may not be reduced at all (see the zeroing of edge (n_4, n_6) in step 1 Fig. 2). In order to reduce the parallel time, we must examine the schedule of a clustered graph to identify a DS and then try to reduce its length. *The main idea behind the DSC algorithm is to perform a sequence of edge zeroing steps with the goal of reducing the length of a DS at each step. The challenge is to implement this idea with low complexity so that it can be used for large task graphs.* Thus we would like to develop an algorithm having the following goals:

G1: The complexity should be low.
G2: The parallel time should be minimized.
G3: The efficiency should be maximized.

There are several difficulties in the implementation of algorithms that satisfy the above goals:

1. The goals G1, G2 and G3 could conflict with each other. For example the maximization of the efficiency conflicts with the minimization in the parallel time. When such conflicts arise, then G1 is given priority over G2 and G3, and G2 over G3. In other words, we are interested in algorithms with low complexity that attains the minimum possible parallel time and the maximum possible efficiency.

2. Let us assume that the DS has been determined. A careful selection of edges to be examined and zeroed must be made to avoid high complexity and at the same time reduce the parallel time. Consider the example in Fig. 2(0). Initially the DS is $< n_1, n_2, n_7 >$. To reduce the length of that DS, we need to zero at least one edge in DS. Hence we need to decide which edges should be zeroed. We could zero either one or both edges. If the edge (n_1, n_2) is zeroed, then the parallel time reduces from 13 to 10. If (n_2, n_7) is zeroed the parallel time reduces to 11. If both edges are zeroed the parallel time reduces to 9.5. Therefore there are many possible ways of edge zeroing and we discuss three approaches:

- **AP1: Multiple DS edge zeroing with maximum *PT* reduction:** This is a greedy approach that will try to get the maximum reduction of the parallel time at each clustering step. However, it is not always true that multiple DS edge zeroing could lead to the maximum PT reduction since after zeroing one DS edge of a path, the other edges in that path may not be in the DS of the new graph.

- **AP2: One DS zeroing of maximum weight edge:** Considering a DS could become a SubDS by zeroing only one edge of this DS, instead of multiple edge zeroing, we could zero only one edge at each step to make smaller reductions in the parallel time but then perform more steps. For example, we could choose one edge to zero at each step, say the largest weighted edge in DS. In Fig. 2(0), the length of current DS $< n_1, n_2, n_7 >$ could be reduced more by zeroing edge (n_1, n_2) instead of (n_2, n_7). Zeroing the largest weighted edge may increase the complexity but may not necessarily lead to a better solution.

- **AP3: One DS edge zeroing with low complexity:** Instead of zeroing the highest edge weight we could allow for more flexibility and chose to zero the one DS edge that leads to a low complexity algorithm.

  Determining a DS for a clustered DAG could take at least $O(v + e)$ time if the computation is not done incrementally. Repeating this computation for all steps will result in at least $O(v^2)$ complexity. Thus it is necessary to use an incremental computation of DS from one step to the next to avoid the traversal of the entire DAG at each step. A proper selection of a DS edge for zeroing simplifies the incremental computation of the DS at the next step.

It is not clear how to implement AP1 or AP2 with a low complexity and also there is no guarantee that AP1 or AP2 will be better than AP3. We will use AP3 to develop our algorithm.

3. Since one of the goals is G3, i.e. reducing the number of unnecessary clusters and increase the efficiency, we also need to allow for zeroing non-DS edges. The questions is when to do SubDS zeroing. One approach is to always zero DS edges until the algorithm stops and then follow up with non-DS zeroing. Another approach, followed in this paper, is to interleave the non-DS zeroing with DS zeroing, which gives more flexibility and results in a lower complexity.

4. Since backtracking could result in high complexity we would like to avoid it as much as possible. For this reason, we need to impose the non-increase in the parallel time constraint from one step to the next:

$$PT_{i-1} \geq PT_i.$$

Sarkar imposes this constraint explicitly in his edge zeroing process, by comparing the parallel time at each step. Here, we will use an implicit constraint to avoid the explicit computation of parallel time in order to reduce the complexity.

In the next subsection, we present an initial version of DSC algorithm and then identify its weaknesses so that we can improve its performance.

3.2 DSC-I: An initial version of DSC

An algorithmic description of DSC-I is given in Fig. 3. An unexamined node is called *free* if all of its predecessors have been examined.

Fig. 4 shows the initial status (step 0) and step i of DSC-I, corresponding to the While loop iteration i

```
1. $EG = \emptyset$. $UEG = V$.
2. Compute $blevel$ for each node and set $tlevel = 0$ for each entry node.
3. Every task is marked unexamined and assumed to constitute one unit cluster.
4. While there is an unexamined node Do
5.       Find a free node $n_f$ with highest priority from $UEG$.
6.       Merge $n_f$ with the cluster of one of its predecessors such that $tlevel(n_f)$
         decreases in a maximum degree. If all zeroings increase $tlevel(n_f)$, $n_f$ remains in a
         unit cluster.
7.       Update the priority values of $n_f$'s successors.
8.       $UEG = UEG - \{n_f\}$; $EG = EG + \{n_f\}$.
9. EndWhile
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Figure 3: The DSC-I algorithm.

at Fig. 3, $1 \leq i \leq 5$, in scheduling a DAG with 5 tasks. The thick paths of each step represent DSs and dashed edges pseudo execution edges. We provide an explanation of the DSC-I algorithm in Fig. 3:

Priority definition and DS identification: (Line 5 in Fig. 3) Based on Equation (1), we define the priority for each task at each step

$$PRIO(n_f) = tlevel(n_f) + blevel(n_f).$$

Thus we can identify a DS node as the one with the highest priority. In Fig. 4(0), the DS nodes are n_1, n_2 and n_5 and they have the highest priority value 14.5.

The topological order for task and edge examination: (Line 5 in Fig. 3) During the execution of DSC-I the graph consists of two parts, the examined part EG and the unexamined part UEG. Initially all nodes are marked *unexamined*, see Fig. 4(0). DSC-I selects only an unexamined free task n_f with the highest priority. Then it examines an incoming edge of this task for zeroing or not zeroing. In Fig. 4, tasks selected at step 1, 2, 3, 4, and 5 are n_1, n_2, n_4, n_3 and n_5 respectively. Notice that the selected task n_f belongs to a DS if there are free DS nodes, otherwise it belongs to a SubDS. In Fig. 4, step 1 selects task n_1 which is in a DS but step 3 selects task n_4 which is not in a DS. The order of such selection is equivalent to *topologically* traversing the graph.

The reason for following a topological order of task examination and zeroing one DS edge, is that we can localize the effect of edge zeroing on the priority values for the rest of the unexamined tasks. In this way, even though zeroing changes the priorities of many nodes, DSC-I only needs to compute the changes for nodes which are immediate successors of the currently examined task n_f, see Line 7 of Fig. 3. For Fig. 4(1), after n_1 is examined, the priority of n_5 does not need to be updated until one of its predecessors is examined at step 2. More explanations are given by Property 3.2.

Edge zeroing criterion: (Line 6 in Fig. 3) The criterion for accepting a zeroing is that the value of $tlevel(n_f)$ of the highest priority free node *does not increase* by such a zeroing. If it does then such zeroing is not accepted and n_f becomes a new cluster in EG. Notice that by reducing $tlevel(n_f)$ all paths

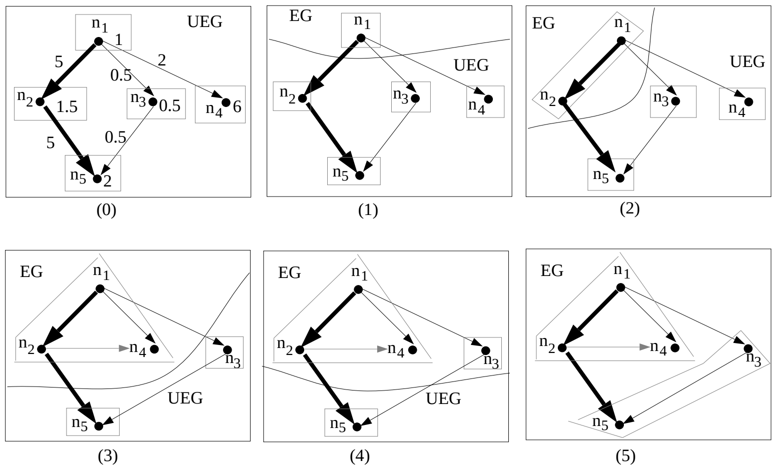


Figure 4: The result of DSC-I after each step in scheduling a DAG.

going through n_f could be compressed and as a result the DS length could be reduced. In Fig. 4(2), n_2 is selected and (n_1, n_2) is zeroed. The zeroing is accepted since $tlevel(n_2)$ reduces from 6 to 1.

Task placement in EG: (Line 6 in Fig. 3) When an edge is zeroed then a free task n_f is merged to the cluster where one of its predecessors resides. Our scheduling heuristic adds a pseudo edge from the *last* task of this cluster to n_f if they are independent. In Fig. 4(3), n_4 is selected and the zeroing of (n_1, n_4) is accepted since $tlevel(n_4)$ reduces from 3 to 2.5. A pseudo edge (n_2, n_4) is added in EG.

The DSC-I satisfies the following properties:

Property 3.1 $PT_{i-1} \geq PT_i$.

Equation (1) implies that reducing the priority value of tasks would lead to the reduction of the parallel time. Thus the constraint that *tlevel* values do not increase implies this property. A formal proof will be given in Section 4.5.

In DSC-I, *tlevel* and *blevel* values are re-used after each step, leading to a reduction of the complexity when determining DS nodes. The following property explains how the topological traversal and the cluster merging rule of DSC-I reduces the complexity.

Property 3.2 *For the DSC-I algorithm, $tlevel(n_x)$ remains unchanged if $n_x \in EG$ and $blevel(n_x)$ remains unchanged if $n_x \in UEG$.*

Proof: If $n_x \in UEG$, then the topological traversal implies that all descendants of n_x are in UEG. Since n_x and its descendants are in separate unit clusters, $blevel(n_x)$ remains unchanged before it is examined. Also for nodes in EG, all clusters in EG can be considered "linear" by counting the pseudo execution edges. When a free node is merged to a "linear" cluster it is always attached to the last node of that "linear" cluster. Thus $tlevel(n_x)$ remains unchanged after n_x has been examined. □

Property 3.3 *The time complexity of DSC-I algorithm is $O(e + v \log v)$.*

Proof: From Property 3.2, the priority of a free node n_f can be easily determined by using

$$tlevel(n_f) = \max_{n_j \in PRED(n_f)} \{tlevel(n_j) + \tau_j + c_{j,f}\}.$$

Once $tlevel(n_j)$ is computed after its examination at some step, where n_j is the immediate predecessor of n_f, this value is propagated to $tlevel(n_f)$. Afterwards $tlevel(n_j)$ remains unchanged and does not affect the value of $tlevel(n_f)$.

At each step of DSC-I, we maintain a priority list FL that contains all free tasks in UEG. This list can be implemented using a balanced search tree data structure. At the beginning, FL is empty and there is no initial overhead in setting up this data structure. The overhead occurs when maintaining the proper order amongst tasks after an insertion or deletion of a task. This operation costs $O(\log |FL|)$ where $|FL| \leq v$. Since each task in a DAG is inserted to and deleted from FL once during the entire execution of DSC-I, the total complexity for maintaining FL is at the most $2v \log v$.

The main computational cost of DSC-I algorithm is spent in the While loop (Line 4 in Fig. 3). The number of steps (iterations) is v. For Line 5, each step costs $O(\log v)$ for finding the head of FL and v steps cost $O(v \log v)$. For Line 6, each step costs $O(|PRED(n_f)|)$ when examining the immediate predecessors of task n_f. For the v steps the cost is $\sum_{n_f \in V} O(|PRED(n_f)|) = O(e)$. For line 7, each step costs $O(|SUCC(n_f)|)$ to update the priority values of the immediate successors of n_f, and similarly the cost for v steps is $O(e)$. When a successor of n_f is found free at Line 7, it is added to FL, and the overall cost is $O(v \log v)$. Thus the total cost for DSC-I is $O(e + v \log v)$. $\square$

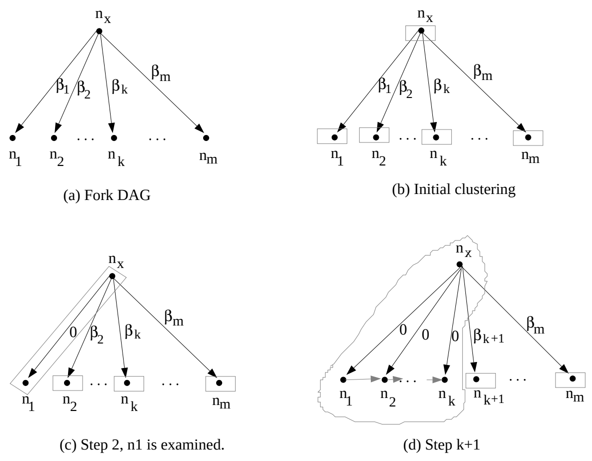
3.3 An Evaluation of DSC-I

In this subsection, we study the performance of DSC-I for some DAGs and propose modifications to improve its performance. Because a DAG is composed of a set of join and fork components, we consider the strengths and weaknesses of DSC-I in scheduling fork and join DAGs. Then we discuss a problem arising when zeroing non-DS edges due to the topological ordering of the traversal.

DSC-I for fork DAGs

Fig. 5 shows the clustering steps of DSC-I for a fork DAG. Without loss of generality, assume that the leaf nodes in (a) are sorted such that $\tau_j + \beta_j \geq \tau_{j+1} + \beta_{j+1}, \quad j = 1 : m - 1$. The steps are described below.

Fig. 5(b) shows the initial clustering where each node is in a unit cluster. $EG = \{\}$, n_x is the only free task in UEG and $PRIO(n_x) = \tau_x + \beta_1 + \tau_1$. At step 1, n_x is selected and it has no incoming edges. It remains in a unit cluster and $EG = \{n_x\}$. After that, $n_1, n_2, \cdots, n_m$ become free and n_1 has the highest priority, $PRIO(n_1) = \tau_x + \beta_1 + \tau_1$, and $tlevel(n_1) = \tau_x + \beta_1$. At step 2 shown in 5(c), n_1 is selected and merged to the cluster of n_x and $tlevel(n_1)$ is reduced, in a maximum degree, to τ_x. At step $k + 1$, n_k is



n_x
β_1 β_2 β_k β_m
n_1 n_2 n_k n_m
(a) Fork DAG
(b) Initial clustering
0 β_2 β_k β_m
(c) Step 2, n1 is examined.
0 0 0 β_{k+1} β_m
n_{k+1}
(d) Step k+1

Figure 5: (c) and (d) are the results of DSC-I after step 2 and $k+1$ for a fork DAG.

selected. The original leftmost scheduled cluster in 5(d) is a "linear" chain $n_x, n_1, \cdots, n_{k-1}$. If attaching n_k to the end of this chain does not increase $tlevel(n_k) = \tau_x + \beta_k$, the zeroing of edge (n_x, n_k) is accepted and the new $tlevel(n_k) = \tau_x + \sum_{j=1}^{k-1} \tau_j$. Thus the condition for accepting or not accepting a zeroing can be expressed as: $\sum_{j=1}^{k-1} \tau_j \leq \beta_k$.

It is easy to verify that DSC-I always zeroes DS edges at each step for a fork DAG and the parallel time strictly decreases monotonically. It turns out the DSC-I algorithm is optimal for this case and a proof will be given in section 5.

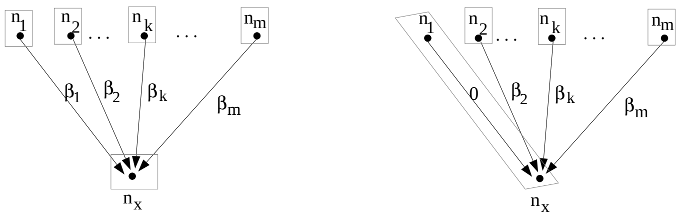
DSC-I for join DAGs

Let us consider the DSC-I algorithm for join DAGs.

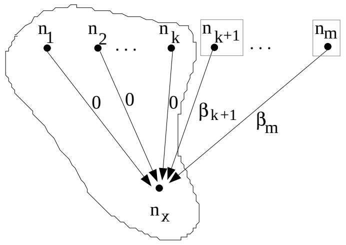
Fig. 6 shows a join DAG, the final clustering and the optimum clustering. Again we assume that $\tau_j + \beta_j \geq \tau_{j+1} + \beta_{j+1}, \quad j = 1 : m-1$. The steps of DSC-I are described below.

Fig. 6(a) shows the initial clustering. Nodes $n_1, n_2, \cdots, n_m$ are free. One DS is $< n_1, n_x >$. Step 1 selects n_1 which is in DS. No incoming edge exists for n_1 and no zeroing is performed. The DS is still $< n_1, n_x >$ after step 1. Now n_x becomes *partially free*. A node is partial free if it is in UEG and at least one of its predecessors has been examined but not all of its predecessors have been examined. Step 2 selects n_2 which is not in DS. No incoming edge exists and n_2 remains in a unit cluster in EG. Step $m+1$ selects n_x which is now in the DS and then (n_1, n_x) is zeroed. The final result of DSC-I is shown in Fig. 6(b) which may not be optimal. The optimal result for a join DAG shown in Fig. 6(c) is symmetric to the previous optimal solution for a fork.

The join example shows that zeroing only one incoming edge of n_x is not sufficient to attain the optimum without backtracking. In general, when a free node is examined, zeroing multiple incoming edges of this free node instead of zeroing one edge could result in a reduction of *tlevel* in a maximum possible degree. As a consequence, the length of DS or SubDS going through this node could be reduced even more substantially.



(a) Join DAG, initial clustering. (b) Final clustering for DSC-I



(c) An optimal clustering.

Figure 6: DSC-I clustering for a join DAG.

To achieve such a greedy goal, a low complexity minimization procedure that zeroes *multiple* incoming edges of the selected free node is needed to be incorporated in the DSC algorithm.

Chretienne [3] has proposed an optimal algorithm for a fork and join, which zeroes multiple edges. The complexity of his algorithm is $O(m \log B)$ where $B = min\{\sum_{i=1}^{m} \tau_i, \beta_1 + \tau_1\} + \tau_x$. Al-Mouhamed [1] has also used the idea of zeroing multiple incoming edges of a task to compute a lower bound for scheduling a DAG, using an $O(m^2)$ algorithm, but no feasible schedules that reach the bound are produced by his algorithm. Since we are interested in lower complexity algorithms we will use a new optimum algorithm with an $O(m \log m)$ complexity.

Dominant Sequence Length Reduction Warranty (DSRW)

We describe another problem with DSC-I. When a DS node n_y is partial free, DSC-I suspends the zeroing of its incoming edges and examines the current non-DS free nodes according to a priority-based topological order. Assume that $tlevel(n_y)$ could be reduced by δ if such a zeroing was not suspended. We should be able to get at least the same reduction for $tlevel(n_y)$ at some future step when DSC-I examines the free node n_y. Then the length of DS going through n_y will also be reduced. However, this is not the case with DSC-I.

Fig. 4(2) shows the result of step 2 of DSC-I after n_2 is merged to $CLUST(n_1)$. The new DS depicted with the thick arrow is $< n_1, n_2, n_5 >$ and it goes through partial free node n_5. If (n_2, n_5) was zeroed at step 3, $tlevel(n_5)$ would have been decreased by $\delta = 5$. Then the length of the current DS $< n_1, n_2, n_5 >$ would also have been reduced by 5. But due to the topological traversal rule, a free task n_4 is selected at step 3 because $PRIO(n_4) = 9 \geq PRIO(n_3) = 4.5$. Then n_4 is merged to $CLUST(n_1)$ since $tlevel(n_4)$ can be reduced from 3 to $\tau_1 + \tau_2 = 2.5$. Such zeroing affects the future compression of DS $< n_1, n_2, n_5 >$. When n_5 is free at step 5, $tlevel(n_5) = \tau_1 + \tau_2 + c_{2,5} = 7.5$ and it is impossible to reduce $tlevel(n_5)$ further by moving it to $CLUST(n_2)$. This is because n_5 will have to be linked after n_4, which makes

$tlevel(n_5) = \tau_1 + \tau_2 + \tau_4 = 8.5.$

To guarantee the effective reduction of $tlevel(n_y)$ of a partial free DS node n_y at the future step, we impose a condition called DSRW. The basic idea is to avoid the edge zeroing operations that affect the reduction of $tlevel(n_y)$. The details are described in Section 4.3.

4 The Final Form of the DSC Algorithm

The main improvements to DSC-I are the minimization procedure for $tlevel(n_f)$, maintaining a partial free list(PFL) and imposing the constraint DSRW. The DSC algorithm is described in Fig. 7.

The clustering sequence of Fig. 7 starts from examining the entry tasks of a DAG. We call it *forward clustering*. There is another way to obtain a schedule for a DAG: first invert this DAG by changing the directions of all precedence edges, obtain a schedule using Fig. 7 for the inverted DAG, then invert this schedule to produce a schedule with the same length for the original DAG. We call it *backward* clustering. These two approaches may have different performance because of different precedence structures. In fact, in Section 5.3 we show that forward clustering cannot obtain the optimum for certain trees but backward clustering can. The complete version of the DSC algorithm performs both forward and backward clustering and chooses the solution with a shorter schedule. We will mainly analyze the properties of DSC with respect to forward clustering. Backward clustering has the same properties except dealing with an inverted graph and it does not increase the order of the algorithm complexity.

4.1 Priority Lists

At each clustering step, we maintain two node priority lists, a partial free list PFL and a free list FL both sorted in a descending order of their task priorities. When two tasks have the same priority we choose the one with the most immediate successors. If there is still a tie, we break it randomly. Function $head(L)$ returns the first node in the sorted list L, which is the task with the highest priority. If $L = \{\}$, $head(L) = NULL$ and the priority value is set to 0.

The $tlevel$ value of a node is propagated to its successors only after this node has been examined. Thus the priority value of a partial free node can be updated using only the $tlevel$ from its examined immediate predecessors. Because only part of predecessors are considered, we define the priority of a partial free task:

$$pPRIO(n_y) = ptlevel(n_y) + blevel(n_y), \quad ptlevel(n_y) = \max_{n_j \in PRED(n_y) \bigcap EG} \{tlevel(n_j) + \tau_j + c_{j,y}\}.$$

In general, $pPRIO(n_y) \leq PRIO(n_y)$ and if a DS goes through an edge (n_j, n_y) where n_j is an examined predecessor of n_y, then we have $pPRIO(n_y) = PRIO(n_j) = PRIO(n_y)$. By maintaining $pPRIO$ instead of $PRIO$ the complexity is reduced considerably. As we will prove later maintaining $pPRIO$ does not adversely affect the performance of DSC since it can still correctly identify the DS at each step.

```
1.   $EG = \emptyset$. $UEG = V$. Add all free entry nodes to $FL$.
2.   Compute $blevel$ for each node and set $tlevel = 0$ for each free node.
3.   WHILE $UEG \neq \emptyset$ DO
4.       $n_x = head(FL)$;/* the free task with the highest priority $PRIO$. */
5.       $n_y = head(PFL)$; /* the partial free task with the highest priority $pPRIO$.*/
6.       IF $(PRIO(n_x) \geq pPRIO(n_y))$ THEN
7.           Call the minimization procedure to reduce $tlevel(n_x)$.
8.           If no zeroing is accepted, $n_x$ remains in a unit cluster.
9.       ELSE
10.          Call the minimization procedure to reduce $tlevel(n_x)$ under constraint DSRW.
11.          If no zeroing is accepted, $n_x$ remains in a unit cluster.
12       ENDIF
13.      Update the priorities of $n_x$'s successors and put $n_x$ into $EG$.
14.  ENDWHILE
```

Figure 7: The DSC Algorithm.

4.2 The minimization procedure for zeroing multiple incoming edges

To reduce $tlevel(n_x)$ in DSC a minimization procedure that zeroes multiple incoming edges of free task n_x is needed. An optimal algorithm for a join DAG has been described in [8] and an optimal solution is shown in Fig. 6(c). The basic procedure is to first sort the nodes such that $\tau_j + \beta_j \geq \tau_{j+1} + \beta_{j+1}, \ \ j = 1 : m-1$, and then zero edges from left to right as long as the parallel time reduces after each zeroing, i.e. *linear searching* of the optimal point. This is equivalent to satisfying the condition $(\sum_{j=1}^{k-1} \tau_j \leq \beta_k)$ for each accepted zeroing. Another optimum algorithm for join is to determine the optimum point k first by using a *binary search* between 1 and m such that $(\sum_{j=1}^{k-1} \tau_j \leq \beta_k)$ and then zero all edges to the left of k.

We modify the optimum binary search join algorithm to minimize $tlevel(n_x)$ when n_x is free. The procedure is shown in Fig. 8. Assume that $PRED(n_x) = \{n_1, n_2, \cdots, n_m\}$. Notice that all predecessors are in EG but now $CLUST(n_1), \cdots, CLUST(n_m)$ may not be unit clusters. We sort the predecessors of n_x such that $tlevel(n_j) + \tau_j + c_{j,x} \geq tlevel(n_{j+1}) + \tau_{j+1} + c_{j+1,x}$. To reduce $tlevel(n_x)$ we must zero the edge (n_1, n_x). A problem arises when the algorithm zeroes an edge of a predecessor n_p which has other children than n_x. When task n_p is moved from $CLUST(n_p)$ to $CLUST(n_1)$, the *tlevel* of the children of n_p will be affected. As a result, the length of paths going through those children will most likely increase. The constraint $PT_{i-1} \geq PT_i$ may no longer hold and maintaining task priorities becomes complicated. To avoid such cases, we will exclude from the minimization procedure predecessors[2] which have children other than n_x unless they are already in $CLUST(n_1)$.

The binary search algorithm in Fig. 8 finds the best stopping point k and those predecessors n_j of n_x $(2 \leq j \leq k)$ that are not in $CLUST(n_1)$ must be extracted from their corresponding clusters and attached to $CLUST(n_1)$. In computing the new $tlevel(n_x)$, those predecessors of n_x in $CLUST(n_1)$ are ordered for execution in an increasing order of their *tlevel* values. The *tlevel* ordering could be performed before the

[2]DSC reschedules all predecessors that have been examined and have n_x as the only child. This is the only backtracking currently allowed in DSC without increasing complexity.

1. Sort the predecessors of n_x such that

$$tlevel(n_j) + \tau_j + c_{j,x} \geq tlevel(n_{j+1}) + \tau_{j+1} + c_{j+1,x}, \quad j = 1 : m-1.$$

2. Let h be the maximum integer from 2 to m such that for $2 \leq t \leq h$ node n_t satisfies the following constraint:

If n_t is not in $CLUST(n_1)$, then n_t does not have any children other than n_x.

3. Find the optimum point k between 1 and h using the binary search algorithm. Zero (n_1, n_x) up to (n_k, n_x) so that $tlevel(n_x)$ is minimum.

Figure 8: The minimization procedure for DSC.

binary searching.

We determine the complexity of this procedure. The *tlevel* ordering of all predecessors is done once at a cost of $O(m \log m)$. The ordering for step 1 at Figure 8 costs $O(m \log m)$. The binary search computes the effect of the ordered predecessors to $tlevel(n_x)$ at a cost of $O(m)$ at each step. The total number of search steps is $O(\log m)$. Thus the total cost of the above algorithm is $O(m \log m)$. If linear search was used instead the total cost would increase to $O(m^2)$ since the number of search steps increases to m.

4.3 Imposing constraint DSRW

As we saw previously when there is no DS going through any free task and there is one DS passing through a partial free node n_y, then zeroing non-DS incoming edges of free nodes could affect the reduction of $tlevel(n_y)$ in the future steps. We impose the following constraint on DSC to avoid such side-effects:

DSRW: Zeroing incoming edges of a free node should not affect the reduction of $ptlevel(n_y)$ if it is reducible by zeroing an incoming DS edge of n_y.

There are two problems that we must address in the implementation of DSRW. First we must detect if $ptlevel(n_y)$ is reducible and second we must make sure that DSRW is satisfied.

1. To detect the reducibility of $ptlevel(n_y)$ we must examine the result of the zeroing of an incoming DS edge of n_y. To find such an incoming DS edge we only examine the result of the zeroing each incoming edge (n_j, n_y) where n_j is a predecessor of n_y and $n_j \in EG$. As we will prove in section 4.5 $ptlevel(n_y) = tlevel(n_y)$. This implies that such partial reducibility is sufficient to guarantee that if the parallel time was reducible by zeroing the DS incoming edges of a partial free DS node n_y, then $tlevel(n_y)$ is reducible when n_y becomes free. Hence the DS can be compressed at that time.

2. After detecting the partial reducibility at step i for node n_y, we implement the constraint DSRW as follows: Assume that n_p is one examined predecessor of n_y and zeroing (n_p, n_y) would reduce $ptlevel(n_y)$,

then no other nodes are allowed to move to $CLUST(n_p)$ until n_y becomes free.

For the example in Fig. 4(2), $n_y = n_5$, $ptlevel(n_5) = \tau_1 + \tau_2 + c_{2,5} = 7.5$, $pPRIO(n_5) = 9.5$, $PRIO(n_3) = 4.5$ and $PRIO(n_4) = 9$. We have that $pPRIO(n_5) > PRIO(n_4)$, which implies DS goes through partial free node n_5 by Theorem 4.1 in section 4.5. And $ptlevel(n_5)$ could be reduced if (n_2, n_5) was zeroed. Then $CLUST(n_2)$ cannot be touched before n_5 becomes free. Thus n_3 and n_4 cannot be moved to $CLUST(n_2)$ and they remain in the unit clusters in EG. When finally n_5 becomes free, (n_2, n_5) is zeroed and PT is reduced from 9.5 to 9.

It should be pointed out that DSRW is a greedy heuristic which attempts to reduce the parallel time as much as possible. It is always possible that this constraint could prevent the generation of an optimal solution. In Section 6, we examine the performance of DSC on scheduling randomly generated DAGs and compare with other algorithms.

4.4 A running trace of DSC

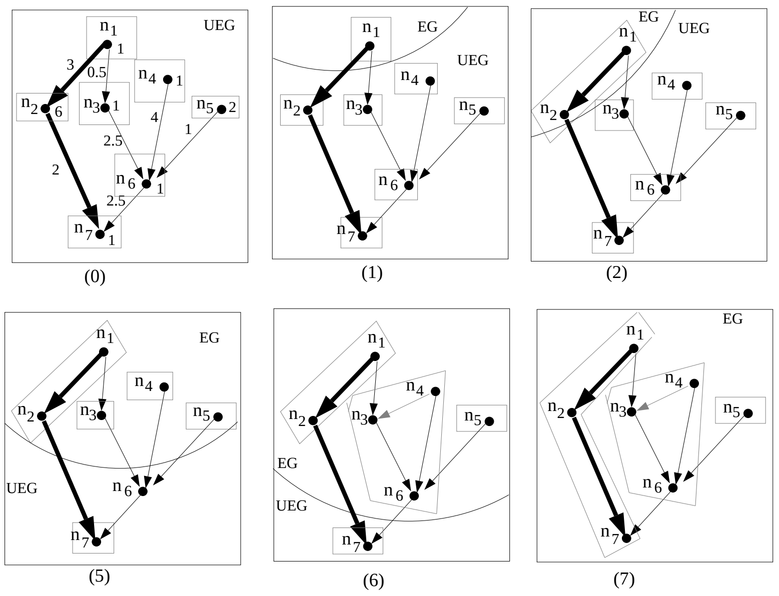


Figure 9: The result of DSC after each step in scheduling the DAG shown in Fig. 1(a).

We demonstrate the DSC steps by using a DAG example shown in Fig. 1(a). The steps 0, 1, 2, 5, 6 and 7 are shown in Fig. 9. The thick paths are the DSs and dashed pseudo edges are the execution order within a cluster. We provide an explanation for each step below. The superscript of a task node in FL or PFL indicates its priority value.

(0) Initially
$UEG = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$ $\quad PT_0 = 13 \quad FL = \{n_1^{0+13}, n_4^{0+9.5}, n_5^{0+7.5}\} \quad PFL = \{\}$.

(1) n_1 is selected, $tlevel(n_1) = 0$, it cannot be reduced so $CLUST(n_1)$ remains a unit cluster. Then
$UEG = \{n_2, n_3, n_4, n_5, n_6, n_7\} \quad PT_1 = 13 \quad FL = \{n_2^{4+9}, n_3^{1.5+8}, n_4^{0+9.5}, n_5^{0+7.5}\} \quad PFL = \{\}$.

(2) n_2 is selected, $n_x = n_2$, $n_y = NULL$, and $tlevel(n_2) = 4$. By zeroing the incoming edge (n_1, n_2) of n_2, $tlevel(n_2)$ reduces to 1. Thus this zeroing is accepted and after that step,
$UEG = \{n_3, n_4, n_5, n_6, n_7\}$ $PT_2 = 10$ $FL = \{n_3^{1.5+8}, n_4^{0+9.5}, n_5^{0+7.5}\}$ $PFL = \{n_7^{9+1}\}$.

(3) n_3 is selected, $n_x = n_3$ with $PRIO(n_3) = 1.5 + 8 = 9.5$ and $n_y = n_7$ with $pPRIO(n_7) = 10$. Notice that by zeroing (n_2, n_7) the $tlevel(n_7)$ reduces to 8.5. Thus we impose the DSRW constraint. Since zeroing (n_1, n_3) affects $tlevel(n_7)$ this zeroing is not accepted because of DSRW. The $tlevel(n_3)$ remains the same and $CLUST(n_3)$ remains a unit cluster in EG.

(4), (5) n_4 and n_5 are selected respectively and their clusters again remain unit clusters. After that,

$UEG = \{n_6, n_7\}$ $PT_5 = 10$ $FL = \{n_6^{5+4.5}\}$ $PFL = \{n_7^{9+1}\}$.

(6) n_6 is selected and its incoming edges (n_3, n_6) and (n_4, n_6) are zeroed by the minimization procedure. Node n_4 is ordered for execution first because $tlevel(n_4) = 0$, which is smaller than $tlevel(n_3) = 1.5$. Then $tlevel(n_6)$ is reduced to 2.5 and

$UEG = \{n_7\}$ $PT_6 = 10$ $FL = \{n_7^{9+1}\}$ $PFL = \{\}$.

(7), n_7 is selected and (n_2, n_7) is zeroed so that $tlevel(n_7)$ is reduced from 9 to 7.

$UEG = \{\}$ $PT_7 = 8$ $FL = \{\}$ $PFL = \{\}$.

Finally three clusters are generated with $PT = 8$.

4.5 DSC Properties

In this subsection, we study several properties of DSC related to the identification of DS, the reduction of the parallel time and the computational complexity. Theorem 4.1 indicates that DSC (Line 6 and 9 in Fig. 7) correctly locates DS nodes at each step even if we use the partial priority $pPRIO$. Theorems 4.2 and 4.3 show how DSC warranties in the reduction of the parallel time. Theorem 4.4 studies the complexity of DSC.

The correctness in locating DS nodes

The goal of DSC is to reduce the length of DSs in a sequence of steps. To do that it must correctly identify unexamined DS nodes. For Lemma 4.1 and 4.2 and Theorem 4.1, we assume that *a DS goes through* UEG, since it is only then that DSC needs to identify and compress DS. If DS does not go through UEG but only through EG, then all DS nodes have been examined and the DS can no longer be compressed because backtracking is not allowed in general.

Since the DSC algorithm examines the nodes topologically the free list FL is always non-empty. By definition all tasks in FL and PFL are in UEG. It is obvious that a DS must go through tasks in either FL or PFL, when it goes through UEG. The interesting question is if a DS also goes through the heads of the priority lists of FL and PFL, since then the algorithm will correctly locate DS nodes for examination. The answer is given in the following lemmas and Theorem 4.1.

Lemma 4.1 *Assume that $n_x = head(FL)$ after step i. If there are DSs going through free nodes in FL, then one DS must go through n_x.*

Proof: At the completion of step i the parallel time is $PT_i = PRIO(n_s)$, where n_s is a DS node. Assume that no DS goes through n_x. Then one DS must go through another non-head free node n_f. This implies that $PRIO(n_f) = PT_i > PRIO(n_x)$, which is a contradiction because n_x has the highest priority in FL. □

Lemma 4.2 *Assume that $n_y = head(PFL)$ after step i. If DSs only go through partial free nodes in PFL, then one DS must go through n_y. Moreover, $pPRIO(n_y) = PRIO(n_y)$.*

Proof: First observe that the starting node of a DS must be an entry node of this DAG. If this node is in UEG, then it must be free which is impossible since the assumption says that DSs only go through partial free nodes in UEG. Thus the starting node must be in EG. As a result a DS must start from a node in EG and go through an examined node n_j to its unexamined partial free successor n_p. Then because they are in the same DS we have that $PRIO(n_p) = PRIO(n_j) = pPRIO(n_p)$. The proof now becomes similar to the previous Lemma. Suppose that no DS goes through n_y. We have that $pPRIO(n_y) \leq PRIO(n_y) < PT_i = PRIO(n_p) = pPRIO(n_p)$ which contradicts the assumption that n_y is the head of PFL.

Next we prove that $pPRIO(n_y) = PRIO(n_y)$. Suppose $pPRIO(n_y) < PRIO(n_y)$. This implies that the DS, where n_y belongs to, does not pass through any examined immediate predecessor of n_y. As a result DS must go through some other nodes in UEG. Thus, there exists an ancestor node n_a of n_y, such that the DS passes through an edge (n_q, n_a) where n_q is the examined predecessor of n_a and n_a is partial free since it is impossible to be free by the assumptions of this Lemma. We have that $pPRIO(n_a) = PRIO(n_a) = PRIO(n_y) > pPRIO(n_y)$ which shows n_y is not the head of PFL. This a contradiction. □

The following theorem shows that the condition $PRIO(n_x) \geq pPRIO(n_y)$ used by DSC algorithm in Line 6 of Fig. 7), correctly identifies DS nodes.

Theorem 4.1 *Assume that $n_x = head(FL)$ and $n_y = head(PFL)$ after step i and that there is a DS going through UEG. If $PRIO(n_x) \geq pPRIO(n_y)$ then a DS goes through n_x. If $PRIO(n_x) < pPRIO(n_y)$, then a DS goes through n_y and there is no DS going through any free node in FL.*

Proof: If $PRIO(n_x) \geq pPRIO(n_y)$, then we will show that a DS goes through n_x. First assume DS goes through FL or both FL and PFL. Then according to Lemma 4.1 it must go through n_x. Next assume that DS goes through PFL only. Then from Lemma 4.2 it must go through n_y, implying that $pPRIO(n_y) = PRIO(n_y) = PT_i > PRIO(n_x)$ which is a contradiction since $PRIO(n_x) \geq pPRIO(n_y)$.

If $PRIO(n_x) < pPRIO(n_y)$, suppose that a DS passes a free node. Then according to Lemma 4.1, $PRIO(n_x) = PT_i \geq PRIO(n_y) \geq pPRIO(n_y)$ which is a contradiction again. Thus the DSs must only go through partial free nodes and one of them must go through n_y by Lemma 4.2. □

The warranty in reducing parallel time

We will show that during DSC clustering steps, the parallel time of the clustered graph monotonically decreases. Moreover, after any clustering step i, if there exists an algorithm that could reduce the parallel time of the clustered graph by zeroing some edge, then DSC will guarantee a reduction at some future step.

Lemma 4.3 *Assume that $n_x = head(FL)$ and $n_y = head(PFL)$ after step i. The parallel time for executing the clustered graph after step i of DSC is:*

$$PT_i = \max\{PRIO(n_x),\ pPRIO(n_y),\ \max_{n_e \in EG}\{PRIO(n_e)\}\}.$$

Proof: There are three cases in the proof. (1) If DS nodes are only within EG, then by definition $PT_i = \max_{n_e \in EG}\{PRIO(n_e)\}$. (2) If a DS goes through a free node, then $PT_i = PRIO(n_x)$ by Lemma 4.1. (3) If there is a DS passing through UEG but this DS only passes through partial free nodes, then $PT_i = PRIO(n_y) = pPRIO(n_y)$ by Lemma 4.2 . □

Theorem 4.2 *For each step i of DSC, $PT_{i-1} \geq PT_i$.*

Proof: For this proof we rename the priority values of $n_x = head(FL)$, and $n_y = head(PFL)$ after step $i-1$ as $PRIO(n_x, i-1)$ and $pPRIO(n_y, i-1)$ respectively. We need to prove, that $PT_{i-1} \geq PT_i$, where

$$PT_{i-1} = \max\{PRIO(n_x, i-1),\ pPRIO(n_y, i-1),\ \max_{n_e \in EG}\{PRIO(n_e, i-1)\}\}$$

$$PT_i = \max\{PRIO(n_x^*, i),\ pPRIO(n_y^*, i),\ \max_{n_e \in EG \bigcup \{n_x\}}\{PRIO(n_e, i)\}\}$$

and n_x^* and n_y^* are the new heads of FL and PFL after step i.

We prove first that $PRIO(n_x^*, i) \leq PT_{i-1}$.

Since n_x^* is in the free list after step $i-1$, it must be in UEG and also it must be either the successor of n_x or independent of n_x. At step i, DSC picks up task n_x to examine its incoming edges for zeroing. We consider the effect of such zeroing on the priority value of n_x^*. Since the minimization procedure does not increase $tlevel(n_x)$, the length of the paths going through n_x decreases or remains unchanged. Thus the priority values of the descendants of n_x could decrease but not increase. The priority values of other nodes in EG remain the same since the minimization procedure excludes those predecessors of n_x that have children other than n_x. Thus if n_x^* is the successor of n_x, then $PRIO(n_x^*, i) \leq PRIO(n_x^*, i-1)$, otherwise $PRIO(n_x^*, i) = PRIO(n_x^*, i-1)$. Since $PRIO(n_x^*, i-1) \leq PT_{i-1}$, then $PRIO(n_x^*, i) \leq PT_{i-1}$.

Similarly we can prove that $PRIO(n_y^*, i) \leq PT_{i-1}$.

Next we prove that $\max_{n_e \in EG \bigcup \{n_x\}}\{PRIO(n_e, i)\}\} \leq PT_{i-1}$. We have $PRIO(n_x, i) \leq PRIO(n_x, i-1)$ from the minimization procedure. We only need to examine the effect of zeroing for n_x on the priorities of nodes in EG. The minimization procedure may increase the *tlevel* values of some predecessors of n_x, say n_p, but it guarantees that the length of the paths going through n_p and n_x do not increase. Thus the new value of $PRIO(n_p)$ satisfies $PRIO(n_p, i) \leq PRIO(n_x, i-1)$. Since $PRIO(n_x, i-1) \leq PT_{i-1}$, then $PRIO(n_p, i) \leq PT_{i-1}$.

The minimization procedure may also increase the *blevel* values of some nodes in the cluster in EG to which n_x is attached. A pseudo edge is added from the last node of that cluster, say n_e, to n_x if n_e and n_x are independent. If there is an increase in the priority value of n_e, then the reason must be that adding this pseudo edge introduces a new path going through n_x with longer length compared with the other existing paths for n_e. Since the minimization procedure has considered the effect of attaching n_x after n_e on $tlevel(n_x)$, the length of paths will be less than or equal to $PRIO(n_x, i-1) \leq PT_{i-1}$. Thus $PRIO(n_e, i) \leq PT_{i-1}$. Similarly we can prove that other nodes in $CLUST(n_e)$ satisfy this inequality. Since the priority values of nodes, say n_o, that are not the predecessors of n_x or not in $CLUST(n_e)$ will not be affected by the minimization procedure, we have $PRIO(n_o, i-1) = PRIO(n_o, i)$. Thus $\max_{n_e \in EG \bigcup \{n_x\}}\{PRIO(n_e, i)\}\} \leq PT_{i-1}$. □

Theorem 4.3 *After step i of DSC, if the current parallel time is reducible by zeroing one incoming edge of a node in UEG, then DSC guarantees that the parallel time will be reduced at some step greater or equal to $i+1$.*

Proof: The assumption that the parallel time reduces by zeroing *one* edge (n_r, n_s), implies that a DS must go through UEG. This implies that the edge (n_r, n_s) belongs to all DSs that go through UEG, otherwise the parallel time is not reducible. There are three cases:

(1) $n_r \in EG$ and n_s is free.
We prove that the node n_s must be the head of FL. If it is not, then another free node in UEG, n_f, must have the same priority and thus belong to a DS. Because all DSs go through (n_r, n_s), n_f must be a successor of n_s. Then n_s and n_f cannot be both free, which is a contradiction.

Also since $PT_i = tlevel(n_s) + blevel(n_s)$ is reducible by zeroing (n_r, n_s), and $blevel(n_s)$ does not change by such a zeroing, then $tlevel(n_s)$ is reducible. Thus during the DSC execution, $n_s = head(FL)$ will be picked up at step $i+1$, and since $tlevel(n_s)$ is reducible the minimization procedure will accept the zeroing of (n_r, n_s) and the parallel time will reduce at that step.

(2) $n_r \in EG$ and n_s is partial free.

We prove that n_s must be the head of PFL. We show it by contradiction. Assume n_f ($n_f \neq n_s$) is the head of PFL. Since all DSs go through (n_r, n_s), no free nodes are in DSs and n_f must be a successor of n_s. Then $pPRIO(n_f) < PRIO(n_f)$. But by Lemma 4.2 $pPRIO(n_f) = pPRIO(n_f)$, which is a contradiction.

Assume that $PT_i = ptlevel(n_s) + blevel(n_s)$ is reducible by $\delta > 0$ when zeroing (n_r, n_s), and $blevel(n_s)$ does not change by such a zeroing. Then $ptlevel(n_s)$ is reducible by at least δ. Thus, during the execution of DSC, the reducibility of $ptlevel(n_s)$ will be detected at step $i+1$. Afterwards non-DS edges are zeroed until n_s becomes free. However, the reducibility of $ptlevel(n_s)$ is not affected by such zeroings because of DSRW. A non-DS node remains a non-DS node and no other nodes are moved to $CLUST(n_r)$. When n_s becomes free, $ptlevel(n_s)$ is still reducible by at least δ, and since other SubDS either decrease or remain unchanged, then the parallel time is also reducible by at least δ. The rest of the proof becomes the same as in the previous case.

(3) $n_r \in UEG$.
Assume that n_r becomes examined at step $j \geq i+1$. At step $i+1$ all DSs must go through (n_r, n_s). If from $i+1$ to step j the parallel time has been reduced then the theorem is true. If the parallel time has not been reduced then at least one DS has not been compressed and all DSs still go through (n_r, n_s) because the minimization procedure guarantees that no other DSs will be created. Thus the parallel time will be reducible at step j by zeroing (n_r, n_s) and the proof becomes the same as in the above cases. $\square$

The following corollary is the direct result of Case 2 in the above Theorem.

Corollary 4.3 *Assume that $n_y \in PFL$, $n_y \in DS$ at step i and that zeroing of an incoming edge of n_y from its scheduled predecessor would have reduced PT by δ. Then DSC guarantees that when n_y becomes free at step j ($j > i$), PT can be reduced by at least δ.*

The complexity

Lemma 4.4 *After a node n_s becomes free or partial free, $blevel(n_s)$ remains unchanged until n_s is examined. Let n_j be a predecessor of n_s, $tlevel(n_j)$ remains unchanged from the step when n_j becomes examined to the step when n_s is examined.*

Proof: Referring to Property 3.2 of DSC-I, DSC is the same as DSC-I except that the minimization procedure changes *tlevel* values of some examined predecessors, say n_h, of the currently-selected free task, say n_x. But n_h does not have any children other than n_x. Thus after a node n_s becomes partial free or free but before it is examined, the *tlevel* value of its examined predecessors will remain unchanged. $\square$

Theorem 4.4 *The time complexity of DSC is $O((v+e)\log v)$ and the space complexity is $O(v+e)$.*

Proof: The difference in the complexity between DSC-I and DSC results from the minimization procedure within the While loop in Fig. 7. In DSC we also maintain PFL but the cost for the v steps is the same $O(v \log v)$ when we use the balanced search trees. Therefore, for Line 4 and 5 in Fig. 7, v steps cost $O(v \log v)$. For Line 7 and 8 (or Line 9 and 10), the minimization procedure costs $O(|PRED(n_x)| \log |PRED(n_x)|)$ at each step. Since $\sum_{n_x \in V} |PRED(n_x)| = e$ and $|PRED(n_x)| < v$, v steps cost $O(e \log v)$. When imposing DSRW, the predecessors of a partial free node are checked and the overall cost is at the most $O(v + e)$.

For Line 13, the *tlevel* values of the successors of n_x are updated. Those successors could be in PFL and the list needs to be rearranged since their $pPRIO$ values could be changed. The cost of adjusting each successor in PFL is $O(\log |PFL|)$ where $|PFL| < v$. The step cost for Line 13 is $O(|SUCC(n_x)| \log v)$. Since $\sum_{n_x \in V} |SUCC(n_x)| = e$, the total cost for maintaining PFL during v steps is $O(e \log v)$.

Also for Line 13 when one successor becomes free, it needs to be added to FL with cost $O(\log |FL|)$ where $|FL| \leq v$. Since there are total v task that could become free during v steps, the total cost in Line 13 spent for FL during v steps is $O(v \log v)$.

Notice that according to Lemma 4.4, after n_x has been moved to EG at Line 13, its *tlevel* value will not affect the priority of tasks in PFL and FL in the rest of the steps. Thus the updating of FL or PFL occurs only once with respect to each task. Therefore the time complexity of DSC is $O((e + v) \log v)$. The space needed for DSC is to store the DAG and FL/PFL. The space complexity is $O(v + e)$. $\square$

5 Performance Bounds and Optimality of DSC

In this section we study the performance characteristics of DS. We give an upper bound for a general DAG and prove the optimality for forks, joins, coarse grain trees and a class of fine grain trees. Since this scheduling problem is NP-complete for a general fine grain tree and for a DAG which is a concatenation of a fork and a join (series parallel DAG) [3, 5], the analysis shows that DSC not only has a low complexity but also attains an optimality degree that a general polynomial algorithm could achieve.

5.1 Performance bounds for general DAGs

A DAG consists of *fork* (F_x) and/or *join* (J_x) structures such as the ones shown in Fig. 5(a) and 6(a). In [8], we define the grain of DAG as follows:

Let

$$g(F_x) = \min_{k=1:m} \{\tau_k\} / \max_{k=1:m} \{c_{x,k}\}, \quad g(J_x) = \min_{k=1:m} \{\tau_k\} / \max_{k=1:m} \{c_{k,x}\}.$$

Then the *granularity* of G is: $g(G) = \min_{n_x \in V} \{g_x\}$ where $g_x = \min\{g(F_x), g(J_x)\}$.

We call a DAG *coarse grain* if $g(G) \geq 1$. For a coarse grain DAG, each task receives or sends a small amount of communication compared to the computation of its neighboring tasks. In [8], we prove the following two theorems:

Theorem 5.1 *For a coarse grain DAG, there exists a linear clustering that attains the optimal solution.*

Theorem 5.2 *Let PT_{opt} be the optimum parallel time and PT_{lc} be the parallel time of a linear clustering, then $PT_{lc} \le (1 + \frac{1}{g(G)})PT_{opt}$. For a coarse grain DAG, $PT_{lc} \le 2 \times PT_{opt}$.*

The following theorem is a performance bound of DSC for a general DAG.

Theorem 5.3 *Let PT_{dsc} be the parallel time by DSC for a DAG G, then $PT_{dsc} \le (1 + \frac{1}{g(G)})PT_{opt}$. For a coarse grain DAG, $PT_{dsc} \le 2 \times PT_{opt}$.*

Proof: In the initial step of DSC, all nodes are in the separate clusters, which is a linear clustering. By Theorem 5.2 and Theorem 4.2 we have that

$$PT_{dsc} \le \ldots \le PT_k \le \ldots \le PT_1 \le PT_0 \le (1 + \frac{1}{g(G)})PT_{opt}.$$

For coarse grain DAGs the statement is obvious. $\square$

5.2 Optimality for join and fork

Theorem 5.4 *DSC derives optimal solutions for fork and join DAGs.*

Proof: For a fork, DSC performs the exact same zeroing sequence as DSC-I. The clustering steps are shown in Fig. 5. After n_x is examined, DSC will examine free nodes $n_1, n_2, \cdots, n_m$ in a decreasing order of their priorities. The priority value for each free node is the length of each path $< n_x, n_1 >, \cdots, < n_x, n_m >$. If we assume that $\beta_k + \tau_k \ge \beta_{k+1} + \tau_{k+1}$ for $1 \le k \le m-1$, then the nodes are sorted as n_1, $n_2, \cdots, n_m$ in the free list FL.

We first determine the optimal time for the fork and then show that DSC achieves the optimum. Assume the optimal parallel time to be PT_{opt}. If $PRIO(n_h) = \tau_x + \tau_h + \beta_h > PT_{opt}$ for some h, then β_i must have been zeroed for $i = 1 : h$, otherwise we have a contradiction. All other edges $i > h$ need not be zeroed because zeroing such edge does not decrease PT but could increase PT. Let the optimal zeroing stopping point be h and assume $\beta_{m+1} = \tau_{m+1} = 0$. Then the optimal PT is: $PT_{opt} = \tau_x + max(\sum_{j=1}^{h} \tau_j, \beta_{h+1} + \tau_{h+1})$.

DSC zeroes edges from left to right as many as possible up to the point k as shown in Fig. 5(d) such that: $\sum_{j=1}^{k-1} \tau_j \le \beta_k$ and $\sum_{j=1}^{k} \tau_j > \beta_{k+1}$. We will prove that $PT_{opt} = PT_{dsc}$ by contradiction. Suppose that $k \ne h$ and $PT_{opt} < PT_{dsc}$. There are two cases:

(1) If $h < k$, then $\sum_{j=1}^{h} \tau_j < \sum_{j=1}^{k} \tau_j \le \beta_k + \tau_k \le \beta_{h+1} + \tau_{h+1}$. Thus $PT_{opt} = \tau_x + \beta_{h+1} + \tau_{h+1} \ge \tau_x + \beta_k + \tau_k \ge \tau_x + max(\sum_{j=1}^{k} \tau_j, \beta_{k+1} + \tau_{k+1}) = PT_{dsc}$.

(2) If $h > k$, then since $\sum_{j=1}^{h} \tau_j \geq \sum_{j=1}^{k+1} \tau_j > \beta_{k+1}+\tau_{k+1} \geq \beta_{h+1}+\tau_{h+1}$, we have that $PT_{opt} = \tau_x+\sum_{j=1}^{h} \tau_j \geq \tau_x + max(\sum_{j=1}^{k} \tau_j, \beta_{k+1} + \tau_{k+1}) = PT_{dsc}$.

There is a contradiction in both cases.

For a join, the DSC uses the minimization procedure to minimize the *tlevel* value of the root and the solution is symmetrical to the optimal result for a fork. □

5.3 Optimality for in/out trees

An *in-tree* is a directed tree in which the root has outgoing degree zero and other nodes have the outgoing degree one. An *out-tree* is a directed tree in which the root has incoming degree zero and other nodes have the incoming degree one.

Scheduling in/out trees is still NP-complete in general as shown by Chretienne [5] and DSC will not give the optimal solution. However, DSC will yield optimal solutions for coarse grain trees and a class of fine grain trees.

Coarse grain trees

Theorem 5.5 *DSC gives an optimal solution for a coarse grain in-tree.*

Proof: Since all paths in an in-tree go through the tree root, say n_x, $PT = tlevel(n_x) + blevel(n_x) = tlevel(n_x) + \tau_x$. We claim $tlevel(n_x)$ is minimized by DSC. We prove it by induction on the depth of the in-tree (d). When $d = 0$, it is trivial. When $d = 1$, it is a join DAG and $tlevel(n_x)$ is minimized. Assume it is true for $d < k$.

When $d = k$, let the predecessors of root n_x be $n_1, \cdots, n_m$. Since each sub-tree rooted with n_i has depth $< k$ and the disjoint subgraphs cannot be clustered together by DSC, DSC will obtain the minimum *tlevel* time for each n_j where $1 \leq j \leq m$ according to the induction hypothesis.

When n_x becomes free, $tlevel(n_x) = \max_{1 \leq j \leq m}\{CT(n_j)+c_{j,x}\}$, $CT(n_j) = tlevel(n_j)+\tau_j$. Without loss of generality, assume that (n_1, n_x) is in a DS and $tlevel(n_1)+\tau_1+c_{1,x}$ has the highest value and $tlevel(n_2)+\tau_2+c_{2,x}$ has the second highest value. DSC will zero (n_1, n_x), and $tlevel(n_x) = \max(CT(n_1), \max_{2 \leq j \leq m}\{CT(n_j)+c_{j,x}\})$. (DSC will not zero any more edges because of the coarse grain condition. DSC might zero (n_2, n_x) when $g(G) = 1$, but $tlevel(n_x)$ does not decrease.)

We need to prove that $tlevel(n_x)$ is the smallest possible. Since the tree is coarse-grain, by Theorem 5.1, linear clustering can be used for deriving the optimal solution. Thus we can assume S^* is an optimal schedule that uses linear clustering. Let $tlevel^*(n_j)$ be the *tlevel* value of n_j in schedule S^*, and $CT^*(n_j) = tlevel^*(n_j) + \tau_j$, $CT^*(n_j) \geq CT(n_j)$ for $1 \leq j \leq m$ from the assumption that $CT(n_j)$ is minimum. Now we will show that $tlevel^*(n_x) \geq tlevel(n_x)$. There are two cases:

(1) If in S^* the zeroed incoming edge of n_x is (n_1, n_x), then

$$tlevel^*(n_x) = \max(CT^*(n_1), \max_{2 \le j \le m}\{CT^*(n_j) + c_{j,x}\}) \ge \max(CT(n_1), \max_{2 \le j \le m}\{CT(n_j) + c_{j,x}\}) = tlevel(n_x).$$

(2) If in S^* the zeroed incoming edge of n_x is not (n_1, n_x), say it is (n_m, n_x). Thus

$$tlevel^*(n_x) = \max(CT^*(n_m), \max_{1 \le j \le m-1}\{CT^*(n_j) + c_{j,x}\}) \ge \max(CT(n_m), \max_{1 \le j \le m-1}\{CT(n_j) + c_{j,x}\}).$$

Because of $g(G) \ge 1$, $CT(n_1) + c_{1,x} \ge \max_{2 \le j \le m}\{CT(n_j) + c_{j,x}\}$, then $tlevel^*(n_x) \ge CT(n_1) + c_{1,x} \ge tlevel(n_x)$. □

DSC solves an in-tree in time $O(v \log v)$ where v is the number of nodes in this tree. It can be verified that the condition $g(G) \ge 1$ for the in-tree optimality of DSC can be relaxed as $g(J_x) \ge 1$ for all join structures J_x in this tree. Under the same or similar condition, Chretienne [4] and Anger, Hwang and Chow [2] developed $O(v)$ optimal tree scheduling algorithms. These two algorithms are only specific to this kind of trees.

For an out-tree, the forward clustering of DSC may not find the optimum directly. But for the inverted graph, the backward clustering can obtain the optimum since an inverted out-tree is an in-tree.

Fine grain trees

Finding optimal solutions for general fine grain trees is NP-complete. However, DSC is able to obtain optimum for a class of fine grain trees. A *single-spawn* out-tree is an out tree such that at most one successor of a non-leaf tree node, say n_x, can spawn successors. Other successors of n_x are leaf nodes. A *single-merge* in-tree is an inverse of a single-spawn out tree. Examples of such trees are shown in Fig. 10.

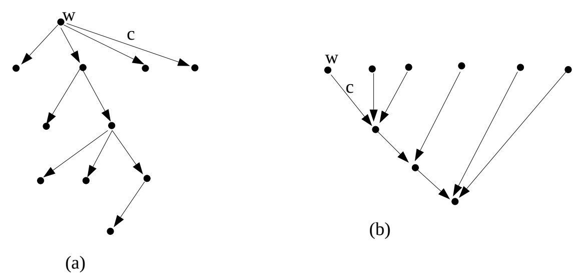


Figure 10: (a) A single-spawn out-tree. (b) A single-merge in-tree.

Theorem 5.6 *Given a single-spawn out-tree or single-merge in-tree with an equal computation weight w for each task and an equal communication weight c for each edge, DSC is optimal for this tree.*

Proof: We will present a proof for an out-tree by induction on the height (h) of this tree. The proof for an in-tree is similar. When $h = 2$, it is a fork and DSC is optimal. Assume DSC obtains the optimum when $h = k$.

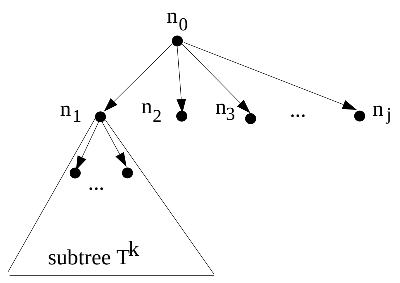


Figure 11: A single-spawn out-tree named T^{k+1} with height $h = k+1$.

When $h = k+1$, we assume without loss of generality that the successors of root n_0 are $n_1, n_2, \cdots, n_j$ and that n_1 spawns successors. First we assume n_0 has more than 1 successors, i.e. $j > 1$. Fig. 11 depicts this tree. We call the entire tree T^{k+1} and the subtree rooted in n_1 T^k. The height of T^k is k and it has q tasks where $q > 1$. We claim that DSC will examine all nodes in the subtree T^k first before examining other successors of n_0. At step 1, n_0 is examined and all successors of n_0 become free. Node n_1 has priority $PRIO(n_1) \geq 3w + 2c$ and other successors of n_0 have priority $2w + c$. Then n_1 is examined at step 2, (n_0, n_1) is zeroed and all successors of n_1 are added to the free list. The priority of n_1's successors $\geq 3w + c$. Thus they will be examined before $n_2, \cdots, n_j$. Recursively, all n_1's descendants will be freed and have priority $\geq 3w + c$. Thus from step 2 to step $q+1$, all q nodes in T^k are examined one by one. After step $q+1$, DSC looks at $n_2, n_3, \cdots, n_j$.

Since from step 2 to $q+1$, DSC clusters T^k only, DSC obtains an optimal clustering solution for this subtree by the induction hypothesis. We call the parallel time for this subtree $PT_{opt}(T^k)$. Then the parallel time after step $q+1$ is: $PT_{dsc}^{q+1} = \max(w + PT_{opt}(T^k), 2w + c)$.

Let PT_{dsc} be the time after the final step of DSC for T^{k+1}. We study the following two cases:

(1) One edge in T^k is not zeroed by DSC. Then $PT_{opt}(T^k) \geq 2w + c$ implying $PT_{dsc}^{q+1} = w + PT_{opt}(T^k)$. Let PT_{dsc} be the time of the final step, since the stepwise parallel time of DSC monotonically decreases, $PT_{dsc}^{q+1} \geq PT_{dsc}$. Also since the optimal parallel time for a graph should be no less than that for its subgraph, $PT_{opt}(T^{k+1}) \geq w + PT_{opt}(T^k)$. Thus $PT_{dsc} \leq PT_{opt}(T^{k+1})$ and DSC is optimal for this case.

(2) All edges in T^k are zeroed by DSC. Then $PT_{opt}(T^k) = qw$ and $PT_{dsc}^{q+1} = \max(w + qw, 2w + c)$. If $w + qw \geq 2w + c$, i.e., $c \leq (q-1)w$ then $PT_{dsc} \leq PT_{dsc}^{q+1} = w + PT_{opt}(T^k) \leq PT_{opt}(T^{k+1})$. Since otherwise $c > (q-1)w$ and $PT_{dsc}^{q+1} = 2w + c$. We claim that all edges in T^k and edge (n_0, n_1) should be zeroed by any optimal clustering for T^{k+1}. If they are not, then $PT_{opt}(T^{k+1}) \geq 3w + c > PT_{dsc}^{q+1} \geq PT_{dsc}$, which is impossible. Since all nodes in T^k and n_0 are in the same cluster, the optimal clustering for the entire out-tree T^{k+1} can be considered as clustering a fork with "leaf-node" n_1 having a weight qw. Because DSC is optimal for a fork, DSC will get the optimum for T^{k+1}.

Finally we examine the case that n_0 only has one successor, i.e. $j = 1$. DSC first zeroes edge (n_0, n_1) then gets the optimum for T^k. Thus $PT_{dsc} = w + PT_{opt}(T^k) = PT_{opt}(T^{k+1})$. □

We do not know of another proof of polynomiality of the above class of fine grain DAGs in the literature. An open question remains if there exist a larger class of fine grain trees which are tractable in polynomial time, say for example the weights are not uniform in the above trees.

6 A Comparison with Other Algorithms and Experimental Results

There are many clustering algorithms for general DAGs, e.g. Sarkar [17], Kim and Browne [12], and Wu and Gajski [19]. A comparison of these algorithms is given in Gerasoulis and Yang [9]. In this section, we compare DSC with the MD algorithm of Wu and Gajski [19] and the ETF by Hwang, Chow, Anger and Lee [11]. We also provide an experimental comparison for ETF, DSC and Sarkar's algorithms.

6.1 The MD algorithm

Wu and Gajski's [19] have proposed two algorithms the MCP and MD. We refer the reader to [19] for the description of both algorithms as well as the terms used in this paragraph. The authors use the notion of *as-soon-as-possible*(ASAP) starting time $T_S(n_i)$ and the *as-late-as-possible*(ALAP) time $T_L(n_i)$ and *latest finishing time*, $T_F(n_i)$. The *relative mobility* of a node is defined by $(T_L(n_i) - T_S(n_i))/w(n_i)$ where $w(n_i)$ is the task weight.

The MCP algorithm uses $T_L(n_i)$ as a node priority. It then selects the free node with the smallest node priority, which is equivalent to selecting the node with the largest $blevel(n_i)$, and schedules it to a processor that allows its *earliest starting time*.

The MD algorithm uses the relative mobility as a node priority and at each step of scheduling, it identifies a task n_p using the smallest relative mobility. It then examines the available processors starting from PE_0 and it schedules n_p to the first processor that satisfies a condition called Fact 1 [19], pp. 336[3]. An intuitive explanation for Fact 1 is that scheduling a task n_p to a processor m should not increase the length of the current critical path (DS).

The complexity of the original algorithm is $O(v^3)$ as shown in [19], pp. 337. The corrected version of MD has a better performance but slightly higher complexity. This is because of the re-computation of T_S and T_F for each scanned processor. For each scheduling step, the complexity of MD is $O(p(v+e))$ where p is the number of scanned processors and $O(v+e)$ for the mobility information and checking Fact 1 for each processor, and since there are v steps the total complexity for the revised MD is $O(pv(v+e))$.

The idea of identifying the important tasks in DSC is the same as in MD, i.e. the smallest relative mobility

[3]In a recent personal communication [20], the authors have made the following corrections to the MD algorithm presented in [19]: (1) For Fact 1 when considering processor m, the condition "for each k" should change to "there exists k", [19], pp. 336. (2) The T_F and T_S computation [19], pp. 336, should assume that task n_p is scheduled on processor m. (3) When n_p is scheduled to processor m, n_p is inserted before the first task in the task sequence of processor m that satisfies the inequality listed in Fact 1.

identifies DS nodes which have the maximum *tlevel* + *blevel*. However, the way to identify DS nodes is different. The DSC uses a priority function with an $O(\log v)$ computing scheme, while MD uses the relative mobility function with a computing cost of $O(v+e)$. Another difference is that when DSC picks a DS task n_p to schedule, it uses the minimization procedure to reduce the *tlevel* of this task and thus decrease the length of DS going through this task. On the other hand, the MD scans the processors from the left to right to find the first processor satisfying Fact 1. Even though Fact 1 guarantees the non-increase of the current critical path, it does not necessarily make the path length shorter.

For a fork or join, MD picks up the DS node at each step and we can show that it produces an optimal solution. For a coarse grain tree, as we saw in our optimality proof, the *tlevel* must be reduced at each step. Since the MD schedules a task to a processor which does not necessarily decrease the *tlevel* at each step, the MD may not produce the optimum in general. A summary of this comparison is given in Table 2.

6.2 The ETF algorithm

ETF [11] is a scheduling algorithm for a bounded number of processors with arbitrary network topologies. At each scheduling step, ETF finds a free task whose starting time is the smallest and then assigns this task to a processor in which the task execution can be started as early-as-possible. If there is a tie then the task with the highest *blevel* is scheduled and this heuristic is called the ETF/CP in [11].

ETF is designed for scheduling on a bounded number of processors. Thus to compare the performance of DSC and ETF we first apply DSC to determine the number of processors(clusters), which we then use as an input to the ETF algorithm.

We discuss the differences and similarities of DSC and ETF as follows. For a node priority DSC uses *tlevel* + *blevel*, and then selects the *largest* node priority. On the other hand, the ETF uses the *earliest-task-first* which is similar to using *tlevel* as node priority and then selecting the *smallest* node priority for scheduling. For the scheduling step DSC and ETF use the same idea, i.e. try to reduce *tlevel* by scheduling to a processor that can start a task *as early as possible*. However, the technique for choosing a processor is different. ETF places a task to the processor that allows the earliest starting time without re-scheduling its predecessors, while DSC uses the minimization procedure that could re-schedule some of the predecessors. It should be mentioned that the MCP [19] algorithm also schedules a task to a processor that allows its earliest starting time as in ETF. However, the node priority for MCP is *blevel* as opposed to *tlevel* used by ETF.

The complexity of ETF is higher than DSC. Since at each step ETF examines all free tasks on all possible processors to find the minimum starting time, the complexity of ETF is $O(pw)$ where p is the number of processors used, w is the maximum size of the free task list. For v tasks, the total complexity is $O(pwv)$. In our case, $p = O(v)$, $w = O(v)$, thus the worst complexity is $O(v^3)$. We have used the balanced searching tree structure for the ETF algorithm in finding the values of a clock variable, NM, [11], pp. 249-250. However, the complexity of ETF for finding the earliest task at each step cannot be reduced since the earliest starting time of a task depends on the location of processors to be assigned. In practice, the average complexity of ETF could be lower than $O(v^3)$. For the Choleski decomposition DAG described in section 6.3, $p = O(\sqrt{v})$ and $w = O(\sqrt{v})$, thus the actual complexity is $O(v^2)$. In section 6.3, we will compare the CPU time spent for DSC and ETF on a SUN 4 workstation.

For a join DAG, ETF does not use a minimization procedure such as DSC, and it may not be optimal. For a fork, ETF may pick up a task with the earliest starting time but this task may not be in a DS and thus ETF may not give the optimum. For a coarse grain in-tree, ETF places a task to the processor of its successor which allows the earliest starting time for this task. We can use a similar approach as in DSC to prove the optimality of ETF for coarse grain in-trees.

A summary of the comparison is described in Table 2. Notice the similarities and differences between MCP and ETF and between MD and DSC. For a detailed comparison of MCP and DSC, see [9].

	MCP [19]	ETF [11]	MD [19]	DSC
Task priority	*blevel*	earliest task first (*tlevel*)	relative mobility	$tlevel + blevel$
DS task first	no	no	yes	yes
Processor selection	processor for earliest starting	processor for earliest starting	first processor satisfying Fact 1	minimization procedure
Complexity	$O(v^2 \log v)$	$O(pv^2)$	$O(pv(v+e))$	$O((v+e)\log v)$
Join/Fork	no	no	optimal	optimal
Coarse grain in-tree	optimal	optimal	no	optimal

Table 2: A comparison of MCP, ETF, MD, and DSC.

6.3 Random DAGs

Due to the NP-completeness of this scheduling problem, heuristic ideas used in DSC cannot always lead to an optimal solution. Thus it is necessary to compare the average performance of different algorithms using randomly generated graphs. Since both the MD and DSC are using the DS to identify the important tasks, we expect a similar performance from both methods. On the other hand, ETF, DSC and Sarkar's are based on different principles and it is of interest to conduct an experimental comparison of these three methods.

We have generated 180 random DAGs as follows: We first randomly generate the number of layers in each DAG. We then randomly place a number of independent tasks in each layer. Next we randomly link the edges between tasks at different layers. Finally, we assign random values to task and edge weights. The following statistic information is important for analyzing the performance of scheduling algorithms:

W: The range of independent tasks in each layer. It approximates the average degree of parallelism.
L: The number of layers.
R/C: The average ratio of task weights over edge weights. It approximates the graph granularity.

The 180 graphs are classified into three groups of 60 graphs each based on their R/C values.

M1: The R/C range is 0.8-1.2. The average weights of computation and communication are close.
M2: The R/C range is 3-10. The graphs are coarse grain.
M3: The R/C range is 0.1-0.3. The graphs are fine grain.

Each group is further classified into 6 subgroups with 10 graphs each, based on the values of W and L. The results of scheduling group M1 , M2 and M3 are summarized in Table 3, 4 and 5. The fifth and sixth columns of the tables show the parallel time improvement ratio of DSC over ETF and Sarkar's algorithm. The improvement ratio of DSC over algorithm A is defined as

$$DSC/A = 1 - \frac{PT_{dsc}}{PT_A}.$$

For group M1, DSC/ETF shows an improvement by 3% and DSC/Sarkar's by 20%. For the coarse grain group M2, the performance differences are insignificant between ETF and DSC and small between Sarkar's and DSC. For the fine grain group M3, the performance is similar to M1 except in the first subgroup where Sarkar's performs the best. This is because for this group the degree of parallelism and the number of layers are small and the granularity is relatively fine. This implies that communication dominates the computation and since Sarkar's algorithm reduces the communication volume by zeroing the largest communication edge at each step, it can get the largest reduction sooner. This is not the case for the other subgroups since then the size of graphs is larger and DSC is given more opportunities (steps) to zero edges.

Layer range	Width Avg/Max	#tasks range	#edge range	DSC/ETF avg	DSC/Sarkar avg
9-11	4/11	44-94	57-206	4.58%	15.73%
9-11	9/20	64-107	118-255	4.49%	17.49%
18-21	5/11	84-121	131-276	4.23%	18.59%
18-20	9/22	158-210	334-552	3.27%	23.28%
18-20	19/41	313-432	691-1249	1.95%	23.40%
36-40	11/22	397-618	900-2374	1.30%	25.93%
Average				3.30%	20.74%

Table 3: DSC vs. ETF and Sarkar's for group M1. The R/C range is between 0.8 and 1.2.

A summary of the experiments for the 180 random DAGs is given in Table 6. We list the percentage of cases when the performance of DSC is better, the same, and worse than that of the other two algorithms. This experiment shows that the average performance of DSC is better than that of Sarkar's and is slightly better than that of ETF.

To see the differences in the complexity between DSC and ETF and demonstrate the practicality of DSC, we consider an important DAG in numerical computing, the Choleski decomposition (CD) DAG, Cosnard et al. [7]. For a matrix size n the degree of parallelism is n, the number of tasks v is about $n^2/2$ and the number of edges e is about n^2. The average R/C is 2. The performance of the two algorithms is given in Table 7. We show the *PT* improvement as well as the total CPU time spent in scheduling this DAG on a SUN4 workstation.

Layer range	Width Avg/Max	#tasks range	#edge range	DSC/ETF avg	DSC/Sarkar avg
9-11	4/11	26-76	40-149	0.26%	3.60%
9-11	8/21	69-101	115-228	0.07%	5.04%
18-21	5/10	93-115	201-331	0.02%	5.56%
19-21	10/22	172-247	383-833	0.04%	6.16%
18-20	20/41	255-441	571-1495	0.02%	6.30%
35-41	11/23	378-504	676-1452	-0.03%	6.67%
Average				0.06%	5.56%

Table 4: DSC vs. ETF and Sarkar's for group M2. The R/C range between 3 and 10. Coarse grain DAGs.

Layer range	Width Avg/Max	#tasks range	#edge range	DSC/ETF avg	DSC/Sarkar avg
9-10	4/11	38-68	34-187	-2.90%	-5.10%
9-11	9/21	54-118	83-257	2.79%	10.58%
18-20	5/12	84-153	154-469	-0.13%	18.43%
19-21	10/22	181-277	441-924	3.00%	25.96%
18-20	20/41	346-546	843-1992	3.78%	33.04%
35-41	11/23	391-474	632-1459	7.62%	33.41%
Average				2.36%	19.39%

Table 5: DSC vs. ETF and Sarkar's for group M2. The R/C range between 0.1 and 0.3. Fine grain DAGs.

	DSC vs. ETF	DSC vs. Sarkar
Avg DSC/A	1.91%	15.23%
#cases, better	56.67%	93.89%
#cases, same	23.33%	0.00%
#cases, worse	20.00%	6.11%

Table 6: A summary of the performance of three algorithms for the 180 DAGs.

To explain why the values of CPU in Table 7 makes sense for different n, we examine the complexity of the algorithms for this DAG. The complexity of DSC is $O((v+e)\log v)$ which is $O(n^2 \log n)$ for this case. When n increases by 2, CPU_{dsc} increases by about 4 times. For ETF, the complexity is $O(pvw)$, for this case $p = w = n$ and $v = n^2/2$, thus the complexity is $O(n^4)$. When n increases by 2, CPU_{etf} increases by about 16.

n	v	e	DSC/ETF	CPU_{dsc}	CPU_{etf}
10	55	90	5.00%	0.06 sec	0.05 sec
20	210	380	4.15%	0.18 sec	0.63 sec
40	820	1560	2.47%	0.70 sec	10.4 sec
80	3240	6320	1.33%	3.01 sec	171.0 sec
160	12880	25440	0.69%	13.1 sec	2879 sec
320	51360	102080	0.35%	56.8 sec	55794sec (15 hrs)

Table 7: DSC vs. ETF for the CD DAG. CPU is the time spent for scheduling on a Sun4 workstation.

7 Conclusions

We have presented a low complexity scheduling algorithm with performance comparable or even better on average to much higher complexity heuristics. The low complexity makes DSC very attractive in practice. DSC can be used in the first step of Sarkar's [17] two step approach to scheduling on a bounded number of processors. We have already incorporated DSC in our programming environment PYRROS [23] that has produced very good results on real architectures such as nCUBE-II and INTEL/i860. DSC is also useful for partitioning and clustering of parallel programs [17, 21]. A particular area that DSC could be useful is scheduling irregular task graphs. Pozo [16] has used DSC to investigate the performance of sparse matrix methods for distributed memory architectures. Wolski and Feo [18] have extended the applicability of DSC to program partitioning for NUMA architectures.

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