Introduction to randomization and sketching techniques

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Plan

Some background

Random sketching

Randomization for least-squares problem

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Singular value decomposition

For any given $A \in \mathbb{R}^{m \times n}$, $m \geq n$ its singular value decomposition is

$$A = U \Sigma V^T = \begin{pmatrix} U_1 & U_2 & U_3 \end{pmatrix} \cdot \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 & V_2 \end{pmatrix}^T$$

where for a given k,

- $U \in \mathbb{R}^{m \times m}$ is orthogonal matrix, the left singular vectors of A ,
  U_1 is $m \times k$, U_2 is $m \times n - k$, U_3 is $m \times m - n$
- $\Sigma \in \mathbb{R}^{m \times n}$, its diagonal is formed by $\sigma_1(A) \geq \ldots \geq \sigma_n(A) \geq 0$
  Σ_1 is $k \times k$, Σ_2 is $n - k \times n - k$
- $V \in \mathbb{R}^{n \times n}$ is orthogonal matrix, the right singular vectors of A,
  V_1 is $n \times k$, V_2 is $n \times n - k$

Eigenvalues of a symmetric matrix

Symmetric Schur Decomposition

if $A \in \mathbb{R}^{n \times n}$ is symmetric, there exists a real orthogonal Q such that

$$\begin{aligned} Q^T A Q &= diag(\lambda_1, \ldots, \lambda_n), \text{ for } k = 1, \ldots, n \\ AQ(:,k) &= \lambda_k Q(:,k) \end{aligned}$$

Courant-Fischer Minimax Theorem

If $A \in \mathbb{R}^{n \times n}$ is symmetric, then

$$\lambda_k(A) = \max_{dim(S)=k} \min_{0 \neq y \in S} \frac{y^T A y}{y^T y}, \text{ for } k = 1, \ldots, n$$

Properties of SVD

Given $A = U\Sigma V^T$, we have

- $A^T A = V\Sigma^T \Sigma V^T$,
  the right singular vectors of A are a set of orthonormal eigenvectors of $A^T A$.
- $AA^T = U\Sigma^T \Sigma U^T$,
  the left singular vectors of A are a set of orthonormal eigenvectors of AA^T.
- The non-negative singular values of A are the square roots of the non-negative eigenvalues of $A^T A$ and AA^T.
- If $\sigma_k \neq 0$ and $\sigma_{k+1}, \ldots, \sigma_n = 0$, then
  $Range(A) = span(U_1)$, $Null(A) = span(V_2)$,
  $Range(A^T) = span(V_1)$, $Null(A) = span(U_2\ U_3)$.

Norms and condition number

$$
\begin{aligned}
||A||_2 &= \sigma_{max}(A) = \sigma_1(A) \\
||A||_F &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2} = \sqrt{\sigma_1^2(A) + \dots \sigma_n^2(A)} \\
||A||_* &= \sigma_1(A) + \dots \sigma_n(A) \\
\kappa(A) &= \frac{\sigma_{max}(A)}{\sigma_{min}(A)} = \sqrt{\|A^T A\|_2 \|(A^T A)^{-1}\|_2}
\end{aligned}
$$

Some properties:

$$
\begin{aligned}
\max_{i,j}|A(i,j)| &\leq ||A||_2 \leq \sqrt{mn}\max_{i,j}|A(i,j)| \\
||A||_2 &\leq ||A||_F \leq \sqrt{min(m,n)}||A||_2
\end{aligned}
$$

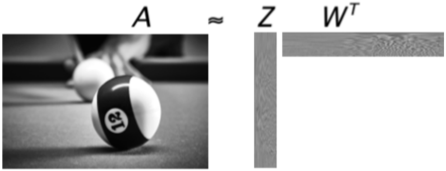
Orthogonal Invariance: If $Q \in \mathbb{R}^{m\times m}$ and $Z \in \mathbb{R}^{n\times n}$ are orthogonal, then

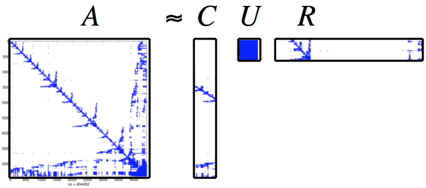
$$||QAZ||_F = ||A||_F$$

$$||QAZ||_2 = ||A||_2$$

Low rank matrix approximation

- Problem: given $A \in \mathbb{R}^{m \times n}$, compute rank-k approximation ZW^T, where Z is $m \times k$ and W^T is $k \times n$.





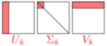
- Problem with diverse applications
  - from scientific computing: fast solvers for integral equations, H-matrices
  - to data analytics: principal component analysis, image processing, ...

$$Ax \rightarrow ZW^T x$$

$$Flops \quad 2mn \rightarrow 2(m+n)k$$

Low rank matrix approximation

- Best rank-k approximation $A_{opt,k} = U_k \Sigma_k V_k^T$ is rank-k truncated SVD of A [Eckart and Young, 1936]



$$\min_{rank(A_k)\leq k} ||A - A_k||_2 = ||A - A_{opt,k}||_2 = \sigma_{k+1}(A) \quad (1)$$

$$\min_{rank(A_k)\leq k} ||A - A_k||_F = ||A - A_{opt,k}||_F = \sqrt{\sum_{j=k+1}^{n} \sigma_j^2(A)} \quad (2)$$

Image, size 1190 × 1920

Rank-10 approximation, SVD

Rank-50 approximation, SVD

- Image source: `https://pixabay.com/photos/billiards-ball-play-number-half-4345870/`

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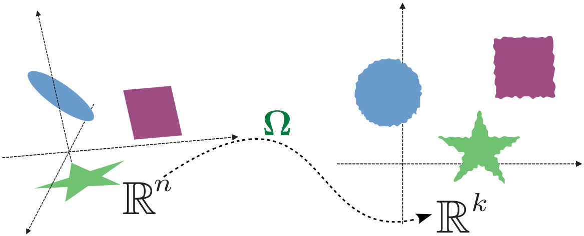
Random sketching



Sketching: embedding of a high dimensional subspace into a low dimensional one, while preserving some geometry, with high probability

Applications: least squares problems, low rank matrix approximation, data compression, column subset selection, orthogonalization of set of vectors, Krylov subspace methods, ...

References: [Johnson and Lindenstrauss, 1984, Dasgupta and Gupta, 2003], [Martinsson and Tropp, 2020]

Image courtesy of O. Balabanov

RandBLAS and RandLAPACK

Ongoing effort to define standards similar to BLAS/LAPACK, organized as

- drivers: few and simple
- computational routines: building blocks for the drivers

RandBLAS - data-oblivious sketching routines

- generate a sketching operator
- apply a sketching operator to a matrix

RandLAPACK: linear algebra problems solved through randomization, e.g.

- least squares
- low rank approximation
- linear solvers
- advanced sketching: leverage scores, sketching operators with tensor product structures

RandBLAS and RandLAPACK

Randomized Numerical Linear Algebra: A Perspective on the Field With an Eye to Software, R. Murray et al, describes:

- basic sketching: dense and sparse sketching operators
- least squares and optimization
- low rank approximation
- full rank matrix decompositions
- kernel methods as arising in machine learning models
- linear solvers and trace estimation
- advanced sketching: leverage scores, sketching operators with tensor product structures

References on RandNLA

Many references available, as:

- Sketching As a Tool for Numerical Linear Algebra [Woodruff, 2014]
- Finding Structure with Randomness: Probabilistic Algorithms for Constructing Approximate Matrix Decompositions [Halko et al., 2011]
- Randomized Numerical Linear Algebra: Foundations and Algorithms [Martinsson and Tropp, 2020]

ε-subspace embedding property

For a given subspace $\mathcal{V} \subset \mathbb{R}^m$ and $\varepsilon \in (0, 1)$, a sketching matrix $\Omega \in \mathbb{R}^{l \times m}$ is an ε-embedding for $\mathcal{V}$ if for all $x_i, x_j \in \mathcal{V}$, we have

$$|\langle \Omega x_i, \Omega x_j \rangle - \langle x_i, x_j \rangle| \leq \epsilon \|x_i\|_2 \|x_j\|_2 \tag{3}$$

- If $x_i = x_j$ we obtain $\|\Omega x_i\|_2^2 = (1 \pm \epsilon)\|x_i\|_2^2$.
- It can also be expressed as: given all vectors $x_i, x_j \in V$ are rescaled to be unit vectors, then for all $x_i, x_j \in V$ we require to hold:

$$\|\Omega(x_i + x_j)\|_2^2 = (1 \pm \epsilon)\|x_i + x_j\|_2^2 \tag{4}$$

Proof that we obtain relation (3):

$$\begin{aligned}
\langle \Omega x_i, \Omega x_j \rangle &= \left(\|\Omega(x_i + x_j)\|_2^2 - \|\Omega x_i\|_2^2 - \|\Omega x_j\|_2^2 \right) / 2 \\
&= \left((1 \pm \epsilon)\|x_i + x_j\|_2^2 - (1 \pm \epsilon)\|x_i\|_2^2 - (1 \pm \epsilon)\|x_j\|_2^2 \right) / 2 \\
&= \langle x_i, x_j \rangle \pm O(\epsilon)
\end{aligned}$$

ε-subspace embedding property

Let W be a matrix whose columns form a basis for $\mathcal{V}$. For simplicity, we refer to an ε-subspace embedding for $\mathcal{V}$ as an ε-embedding for W.

Corollary 1 (Corollary 2.2 in [Balabanov and Grigori, 2022])

If $\Omega \in \mathbb{R}^{l \times m}$ is an ε-embedding for W, then the singular values of W are bounded by

$$(1+\varepsilon)^{-1/2}\sigma_{min}(\Omega W) \ \leq \sigma_{min}(W) \leq \sigma_{max}(W) \leq (1-\varepsilon)^{-1/2}\sigma_{max}(\Omega W).$$

Proof.

Let $a \in \mathbb{R}^{\dim(\mathcal{V})}$ be an arbitrary vector and $x = Wa$. By definition of Ω,

$$(1+\varepsilon)^{-1}\|\Omega x\|^2 \leq \|x\|^2 \leq (1-\varepsilon)^{-1}\|\Omega x\|^2, \text{ which implies that}$$

$$(1+\varepsilon)^{-1/2}\|\Omega W a\| \leq \|Wa\| \leq (1-\varepsilon)^{-1/2}\|\Omega W a\|.$$

The statement of proposition then follows by using definitions of the minimal and the maximal singular values of a matrix. □

Oblivious subspace embedding

Aim: construct Ω such that for any n-dimensional subspace $\mathcal{V} \subset \mathbb{R}^m$

$$\mathbb{P}(\Omega \text{ is } \varepsilon\text{-embedding for } \mathcal{V}) \geq 1 - \delta$$

Definition: oblivious subspace embedding

A random matrix $\Omega \in \mathbb{R}^{l \times m}$ is an oblivious subspace embedding with parameters OSE(n, ϵ, δ) if with probability at least $1 - \delta$ for any n-dimensional subspace $\mathcal{V} \subset \mathbb{R}^m$, for all $x_i, x_j \in \mathcal{V}$, we have

$$|\langle \Omega x_i, \Omega x_j \rangle - \langle x_i, x_j \rangle| \leq \epsilon \|x_i\|_2 \|x_j\|_2 \tag{5}$$

Oblivious subspace embedding (contd)

With high probability, OSEs have bounded norm.

Corollary 2 (Corollary 2.4 in [Balabanov and Grigori, 2022])

If $\Omega \in \mathbb{R}^{l \times m}$ is a $(\varepsilon, \delta/m, 1)$ oblivious ℓ_2-subspace embedding, then with probability at least $1-\delta$, we have

$$\|\Omega\|_F \leq \sqrt{(1+\varepsilon)m}.$$

Proof.

It directly follows from the definition of oblivious embedding, (5), and the union bound argument that Ω is an ε-embedding for each canonical (Euclidean) basis vector. This implies that the ℓ_2-norms of the columns of Ω are bounded from above by $\sqrt{1+\varepsilon}$. The statement of the corollary then follows immediately. $\square$

Random sketching matrices

- $\Omega \in \mathbb{R}^{l \times m}$ whose entries are independent standard normal random variables, multiplied by $1/\sqrt{l}$
  - Ω is OSE(n, ϵ, δ) with $l = \mathcal{O}(\epsilon^{-2}(n + \log \frac{1}{\delta}))$
  - Cost of computing ΩW, $W \in \mathbb{R}^{m \times n}$: $2mnl$ flops
  - Relies on BLAS3 operations when W is dense
- Easy to parallelize, $\Omega W = \sum_{i=1}^{P} \Omega_i W_i$

$$\Omega W = \begin{pmatrix} \Omega_1 & \dots & \Omega_P \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_P \end{pmatrix} = \sum_{i=1}^{P} \Omega_i W_i$$

  - Each processor i owns a block $\Omega_i \in \mathbb{R}^{l \times m/P}$ and a block $W_i \in \mathbb{R}^{m/P \times n}$
  - Each processor computes $\Omega_i W_i \in \mathbb{R}^{l \times n}$
  - Sum-Reduce among all processors to compute $\Omega W = \sum_{i=1}^{P} \Omega_i W_i$
  - Cost of the algorithm

$$(2mnl/P)\gamma + \log_2 P\alpha + ln \log_2 P\beta$$

Fast Johnson-Lindenstrauss transform

Find sparse or structured Ω such that computing ΩW is cheap, e.g. a subsampled random Hadamard transform (SRHT).
Given $m = 2^q, l < m$, the SRHT ensemble embedding $\mathbb{R}^m$ into $\mathbb{R}^l$ is defined as

$$\Omega = \sqrt{\frac{m}{l}} \cdot P \cdot H \cdot D, \text{ where} \tag{6}$$

- $D \in \mathbb{R}^{m \times m}$ is diagonal matrix of uniformly random signs, random variables uniformly distributed on ± 1
- $H \in \mathbb{R}^{m \times m}$ is the normalized Walsh-Hadamard transform
- $P \in \mathbb{R}^{l \times m}$ formed by subset of l rows of the identity, chosen uniformly at random (draws l rows at random from HD).

References: Sarlos'06, Ailon and Chazelle'06, Liberty, Rokhlin, Tygert and Woolfe'06.

Fast Johnson-Lindenstrauss transform (contd)

Definition of Normalized Walsh–Hadamard Matrix
For given $m = 2^q$, $H_m \in \mathbb{R}^{m \times m}$ is the non-normalized Walsh-Hadamard transform defined recursively as,

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \;\; H_m = \begin{pmatrix} H_{m/2} & H_{m/2} \\ H_{m/2} & -H_{m/2} \end{pmatrix}. \tag{7}$$

The normalized Walsh-Hadamard transform is $H = m^{-1/2} H_m$.

Cost of matrix vector multiplication:
For $w \in \mathbb{R}^m$ and $\Omega \in \mathbb{R}^{l \times m}$, computing Ωw costs $2m \log_2 m$ flops.

Random sketching matrices - SRHT

- $\Omega \in \mathbb{R}^{l \times m}$ is a fast Johnson-Lindenstrauss transform, e.g. a subsampled randomized Hadamard transform (SRHT)[1]

$$\Omega = \sqrt{\frac{m}{l}} \cdot R \cdot H \cdot D, \text{ where} \tag{8}$$

$D \in \mathbb{R}^{m \times m}$ is diagonal with independent random signs, $H \in \mathbb{R}^{m \times m}$ is normalized Walsh-Hadamard matrix, $R \in \mathbb{R}^{l \times m}$ draws l rows uniformly at random from HD.

  - Ω is OSE(n, ϵ, δ) with $l = \mathcal{O}(\epsilon^{-2} \left(n + \ln \frac{m}{\delta}\right) \ln \frac{n}{\delta})$
  - Cost of computing ΩW, $W \in \mathbb{R}^{m \times n}$ on P processors:

$$\frac{2mn \log_2 m}{P} \gamma + \log_2 P \alpha + \frac{mn}{P} \log_2 P \beta$$

1. Ailon and Chazelle'06, Liberty, Rokhlin, Tygert and Woolfe'06, Sarlos'06.

Block SRHT for parallelization on P processors

- Ω as in (9) is OSE(n, ϵ, δ) with $l = \mathcal{O}(\epsilon^{-2} \left(n + \ln \frac{m}{\delta}\right) \ln \frac{n}{\delta})$

$$\Omega = [\Omega_1 \quad \Omega_2 \quad \dots \quad \Omega_P] = \sqrt{\frac{m}{Pl}} \cdot [D_{L1} \quad \dots \quad D_{LP}] \begin{bmatrix} RH & & \\ & \ddots & \\ & & RH \end{bmatrix} \begin{bmatrix} D_{R1} & & \\ & \ddots & \\ & & D_{RP} \end{bmatrix}, \tag{9}$$

where $\Omega_i = \sqrt{\frac{m}{Pl}} D_{Li} RHD_{Ri}$, $D_{Li}, D_{Ri} \in \mathbb{R}^{m/P \times m/P}$ are diagonal with independent random signs, $H \in \mathbb{R}^{m/P \times m/P}$ is normalized Walsh-Hadamard matrix, $R \in \mathbb{R}^{l \times m/P}$ is uniform sampling matrix.

- Parallelize as [Balabanov et al., 2022]:

$$\Omega W = \sqrt{\frac{m}{Pl}} \sum_{i=1}^{P} D_{Li} RHD_{Ri} W_i$$

Parallelization of block SRHT

Considering each processor i owns a block $W_i \in \mathbb{R}^{m/P \times n}$, parallelize as:

$$\begin{aligned}
\Omega W &= \begin{pmatrix} \Omega_1 & \dots & \Omega_P \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_P \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{m}{Pl}} D_{L1} RHD_{R1} & \dots & \sqrt{\frac{m}{Pl}} D_{LP} RHD_{RP} W_P \end{pmatrix} \begin{pmatrix} W_1 \\ \vdots \\ W_P \end{pmatrix} \\
&= \sum_{i=1}^{P} \sqrt{\frac{m}{Pl}} D_{Li} RHD_{Ri} W_i
\end{aligned}$$

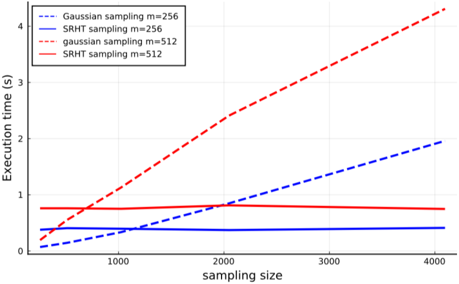
- Root processor broadcasts seed of R
- Each processor i draws R, D_{Li}, D_{Ri}
- Each processor computes $\Omega_i W_i = \sqrt{\frac{m}{Pl}} D_{Li} RHD_{Ri} W_i$, $\Omega_i W_i \in \mathbb{R}^{l \times n}$
- Sum-Reduce among all processors to compute $\Omega W = \sum_{i=1}^{P} \Omega_i W_i$

Cost of the algorithm (some lower order terms ignored)

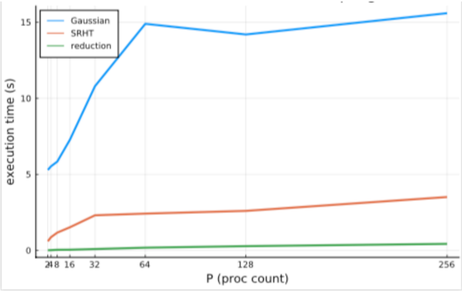
$$2mn \log_2 m/P\gamma + \log_2 P\alpha + ln \log_2 P\beta$$

Performance of Gaussian vs block SRHT

$W \in \mathbb{R}^{n \times m}, n = 64000, P = 1$



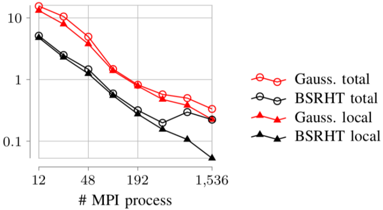
$W \in \mathbb{R}^{nP \times m}, n = 65536, m = 256, k = 4096$



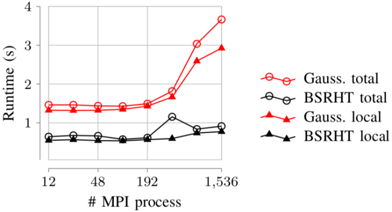
- Results obtained in Julia on nodes formed by 2 Cascade Lake Intel Xeon 5218, 16 cores each, 2.4GHz/core

Performance of Gaussian vs SRHT

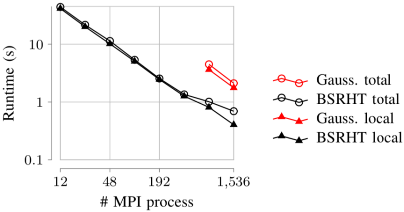
$W \in \mathbb{R}^{m \times n}, m = 10^7, n = 200, l = 2000$



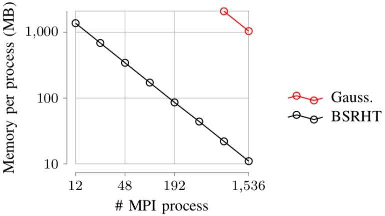
$W \in \mathbb{R}^{m \times n}, m = 10^5 \times P, n = 200, l = 2000$



$W \in \mathbb{R}^{m \times n}, m = 10^8, n = 200, l = 2000$



Memory per processor

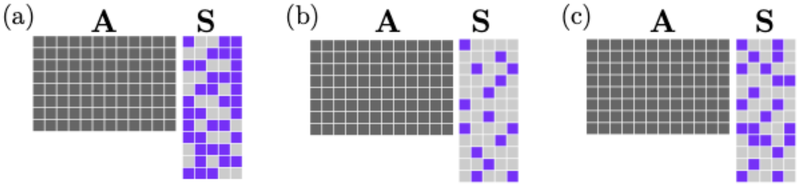


Machine: Intel Skylake 2.7GHz (AVX512), 48 cores per node

Sparse sketching operators

Can be grouped into three categories:

- Short-axis-sparse sketching operators (SASOs)
- Long-axis-sparse sketching operators (LASOs)
- Iid-sparse sketching operators



(a) SASO with 3 nnz per row, (b) LASO with 3 nnz per column, and (c) an iid-sparse sketching operator with iid non-zero entries.

For a more detailed description, see section 2.4 of [Murray et al., 2023], figure from that book, S is the sketching matrix.

Short-axis-sparse sketching operators

Suppose the sketching matrix $S \in \mathbb{R}^{m \times l}$, with $l \ll m$. The short-axis vectors of a SASO should be independent of one another.
Select the locations of k nonzero elements as:

- sample k indices uniformly from $[m]$ without replacement, once for each column, or
- divide $[m]$ into k contiguous subsets of equal size, and then for each column select one index (independently and uniformly) from each of the k index sets.

Plan

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Random sketching

Randomization for least-squares problem

Solving least squares problems

Given $W \in \mathbb{R}^{m\times n}$ full-rank and $b \in \mathbb{R}^m$, with $n \ll m$, solve

$$y := arg \min_{x\in\mathbb{R}^n} \|Wx - b\|_2$$

The unique solution is

$$y = W^+ b, \qquad W^+ = (W^T W)^{-1} W^T$$

Solve by using the QR factorization of W (see lecture on dense QR)

$$W = QR = \begin{pmatrix} \tilde{Q} & \bar{Q} \end{pmatrix} \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix} \tag{10}$$

We obtain

$$\begin{aligned} ||r||_2^2 &= ||b - Wx||_2^2 = ||b - \begin{pmatrix} \tilde{Q} & \bar{Q} \end{pmatrix} \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix} x||_2^2 \\ &= || \begin{pmatrix} \tilde{Q}^T \\ \bar{Q}^T \end{pmatrix} b - \begin{pmatrix} \tilde{R} \\ 0 \end{pmatrix} x||_2^2 = || \begin{pmatrix} \tilde{Q}^T b - \tilde{R}x \\ \bar{Q}^T b \end{pmatrix} ||_2^2 \\ &= ||\tilde{Q}^T b - \tilde{R}x||_2^2 + ||\bar{Q}^T b||_2^2 \end{aligned}$$

Solve $\tilde{R}x = \tilde{Q}^T b$ to minimize $||r||_2$.

Least squares problems

Solve by using the normal equations,

$$W^T W x = W^T b$$

1. with direct methods
   multiply $W^T W$ and compute the Cholesky factorization of the result
2. or with iterative methods without computing explicitely $W^T W$
   use a Krylov subspace solver and at each iteration multiply $W^T W$ with a vector

Randomized least squares - sketch and solve

Solve by using randomization, with $\Omega \in \mathbb{R}^{l \times m}$ being OSE$(n+1, \epsilon, \delta)$ for $\mathcal{V} = range([W, b])$

$$y_s := arg \min_{x \in \mathbb{R}^n} \|\Omega(Wx - b)\|_2$$

or $y_s = (\Omega W)^\dagger (\Omega b)$ We obtain with probability $1 - \delta$:

$$\|Wy_s - b\|_2^2 \leq (1 + \varepsilon)\|Wy - b\|_2^2$$

Proof omitted, but can be found in [Sarlos, 2006]. It can be easily proven that:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \|\Omega(Wx - b)\|_2 = \|\Omega(Wy_s - b)\|_2 &\leq \|\Omega(Wy - b)\|_2 \\ &\leq (1 + \varepsilon)\|Wy - b\|_2 \end{aligned}$$

Randomized least squares - sketch and precondition

Sketch the least squares problem to obtain a preconditioner, and solve a preconditioned least squares problem [Rokhlin and Tygert, 2008], that is

- Compute $S = \Omega W$, $S \in \mathbb{R}^{l \times n}$
- Compute the (rank revealing) QR factorization $S = Q_s R_s \Pi$, $P_s = R_s \Pi$, where Π is a permutation matrix, $Q_s \in \mathbb{R}^{l \times n}$ has orthonormal columns
- Use P_s as a preconditioner and solve $\mathit{arg}\min_{z \in \mathbb{R}^n} \|WP_s^{-1}z - b\|_2$, with WP_s^{-1} being a well conditioned matrix
- Compute the solution to $\mathit{arg}\min_{x \in \mathbb{R}^n} \|Wx - b\|_2$ as $x = P_s^{-1}z$

A more detailed discussion on rank revealing QR will be provided in a future lecture.

Sketch and precondition (contd)

The following results hold:

$$\kappa(WP_S^{-1}) = \frac{\sigma_{max}(WP_s^{-1})}{\sigma_{min}(WP_s^{-1})} \leq \frac{1+\varepsilon}{1-\varepsilon}$$

Sketch of the proof from [Rokhlin and Tygert, 2008]. Consider the following SVD decompositions of $W \in \mathbb{R}^{m\times n}$ and $(\Omega U) \in \mathbb{R}^{l\times n}$:

$$\begin{aligned} W &= U\Sigma V^T, U \in \mathbb{R}^{m\times n}, \Sigma \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{n\times n} \\ \Omega U &= \tilde{U}\tilde{\Sigma}\tilde{V}^T, \tilde{U} \in \mathbb{R}^{m\times n}, \tilde{\Sigma} \in \mathbb{R}^{n\times n}, \tilde{V} \in \mathbb{R}^{n\times n} \end{aligned}$$

It can be shown that $\tilde{U} = Q_s\tilde{Q}$, where $\tilde{Q} \in \mathbb{R}^{n\times n}$ is orthogonal. We obtain:

$$\begin{aligned} P_s &= W\tilde{\Sigma}\tilde{V}^T\Sigma V^T \\ WP_s^{-1} &= U\tilde{V}\tilde{\Sigma}^{-1}\tilde{Q}^T \\ \|(WP_s^{-1})^T(WP_s^{-1})\|_2 &= \|\tilde{\Sigma}^{-1}\|_2 \\ \|((WP_s^{-1})^T(WP_s^{-1}))^{-1}\|_2 &= \|\tilde{\Sigma}\|_2 \end{aligned}$$

This allows to deduce that $\kappa(WP_s^{-1}) = \kappa(\Omega U)$

Sketch and precondition (contd)

$$\kappa(WP_s^{-1}) = \kappa(\Omega U)$$

and since U is orthonormal, Corollary 1 can be applied, from which the bound is obtained.

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