LU factorization and its communication avoiding version

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October 17, 2023

Plan

LU factorization

Block LU factorization

Communication avoiding LU factorization

Norms and other notations

$$\begin{aligned}
\|A\|_F &= \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2} \\
\|A\|_2 &= \sigma_{max}(A) \\
\|A\|_\infty &= \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \\
\|A\|_1 &= \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|
\end{aligned}$$

Inequalities $|x| \le |y|$ and $|A| \le |B|$ hold componentwise.

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Algebra of the LU factorization

LU factorization

Compute the factorization PA = LU

Example

Given the matrix

$$A = \begin{pmatrix} 3 & 1 & 3 \\ 6 & 7 & 3 \\ 9 & 12 & 3 \end{pmatrix}$$

Let

$$M_1 = \begin{pmatrix} 1 & & \\ -2 & 1 & \\ -3 & & 1 \end{pmatrix}, \quad M_1 A = \begin{pmatrix} 3 & 1 & 3 \\ 0 & 5 & -3 \\ 0 & 9 & -6 \end{pmatrix}$$

Algebra of the LU factorization

- In general

$$\begin{aligned} A^{(k+1)} &= M_k A^{(k)} := \begin{pmatrix} I_{k-1} & & & & \\ & 1 & & & \\ & -m_{k+1,k} & 1 & & \\ & \dots & & \ddots & \\ & -m_{n,k} & & & 1 \end{pmatrix} A^{(k)}, \text{where} \\ M_k &= I - m_k e_k^T, \quad M_k^{-1} = I + m_k e_k^T \end{aligned}$$

where e_k is the k-th unit vector, $m_k = (0, \dots, 0, 1, m_{k+1,k}, \dots, m_{n,k})^T$, $e_i^T m_k = 0, \forall i \leq k$

- The factorization can be written as

$$M_{n-1} \dots M_1 A = A^{(n)} = U$$

Algebra of the LU factorization

- We obtain

$$
\begin{aligned}
A &= M_1^{-1} \dots M_{n-1}^{-1} U \\
&= (I + m_1 e_1^T) \dots (I + m_{n-1} e_{n-1}^T) U \\
&= \left(I + \sum_{i=1}^{n-1} m_i e_i^T \right) U \\
&= \begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ m_{n1} & m_{n2} & \dots & 1 \end{pmatrix} U = LU
\end{aligned}
$$

The need for pivoting

- For stability, avoid division by small diagonal elements
- For example

$$A = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 1 & 3 \\ 6 & 2 & 3 \end{pmatrix} \tag{1}$$

has an LU factorization if we permute the rows of matrix A

$$PA = \begin{pmatrix} 6 & 2 & 3 \\ 0 & 3 & 3 \\ 3 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ 0.5 & & 1 \end{pmatrix} \cdot \begin{pmatrix} 6 & 2 & 3 \\ & 3 & 3 \\ & & 1.5 \end{pmatrix} \tag{2}$$

- Partial pivoting allows to bound the multipliers $m_{ik} \leq 1$ and hence $|L| \leq 1$
- Factorization referred to as LU with partial pivoting (LUPP) or also Gaussian elimination with partial pivoting GEPP

Wilkinson's backward error stability result

Growth factor g_W defined as

$$g_W = \frac{\max_{i,j,k} |a_{ij}^k|}{\max_{i,j} |a_{ij}|}$$

Note that

$$|u_{ij}| = |a_{ij}^i| \leq g_W \max_{i,j} |a_{ij}|$$

Theorem (Wilkinson's backward error stability result, see also [N.J.Higham, 2002] for more details)

Let $A \in \mathbb{R}^{n \times n}$ and let $\hat{x}$ be the computed solution of $Ax = b$ obtained by using GEPP. Then

$$(A + \Delta A)\hat{x} = b, \qquad \|\Delta A\|_\infty \leq n^2 \gamma_{3n} g_W(n) \|A\|_\infty,$$

where $\gamma_n = nu/(1 - nu)$, u is machine precision and assuming $nu < 1$.

The growth factor

- The LU factorization is backward stable if the growth factor is small (grows linearly with n).
- For partial pivoting, the growth factor $g(n) \leq 2^{n-1}$, and this bound is attainable.
- In practice it is on the order of $n^{2/3} - n^{1/2}$

Exponential growth factor for Wilkinson matrix

$$A = diag(\pm 1) \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \ldots & 0 & 1 \\ -1 & -1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 1 \\ -1 & -1 & \cdots & -1 & 1 & 1 \\ -1 & -1 & \cdots & -1 & -1 & 1 \end{bmatrix}$$

Experimental results for special matrices

Several errror bounds for GEPP, the normwise backward error η and the componentwise backward error w $(r = b - Ax)$.

$$\eta = \frac{||r||_1}{||A||_1 \; ||x||_1 + ||b||_1},$$
$$w = \max_i \frac{|r_i|}{(|A| \; |x| + |b|)_i}.$$

matrix	cond(A,2)	g_W	$\|\|L\|\|_1$	cond$(U,1)$	$\frac{\|\|PA-LU\|\|_F}{\|\|A\|\|_F}$	η	w_b
hadamard	1.0E+0	4.1E+3	4.1E+3	5.3E+5	0.0E+0	3.3E-16	4.6E-15
randsvd	6.7E+7	4.7E+0	9.9E+2	1.4E+10	5.6E-15	3.4E-16	2.0E-15
chebvand	3.8E+19	2.0E+2	2.2E+3	4.8E+22	5.1E-14	3.3E-17	2.6E-16
frank	1.7E+20	1.0E+0	2.0E+0	1.9E+30	2.2E-18	4.9E-27	1.2E-23
hilb	8.0E+21	1.0E+0	3.1E+3	2.2E+22	2.2E-16	5.5E-19	2.0E-17

- Two reasons considered to be important for the average case stability [Trefethen and Schreiber, 90]:
  - the multipliers in L are small
  - the correction introduced at each elimination step is of rank 1

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Block formulation of the LU factorization

Partitioning of matrix A of size $n \times n$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is of size $b \times b$, A_{21} is of size $(m-b) \times b$, A_{12} is of size $b \times (n-b)$ and A_{22} is of size $(m-b) \times (n-b)$.

Block LU algebra

The first iteration computes the factorization:

$$P_1^T A = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} L_{11} & \\ L_{21} & I_{n-b} \end{bmatrix} \cdot \begin{bmatrix} U_{11} & U_{12} \\ & A^1 \end{bmatrix}$$

The algorithm continues recursively on the trailing matrix A^1.

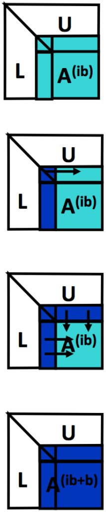
Block LU factorization - the algorithm

1. Compute the LU factorization with partial pivoting of the first block column
$$P_1 \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix} U_{11}$$
2. Pivot by applying the permutation matrix P_1^T on the entire matrix,
$$\bar{A} = P_1^T A.$$
3. Solve the triangular system
$$L_{11} U_{12} = \bar{A}_{12}$$
4. Update the trailing matrix,
$$A^1 = \bar{A}_{22} - L_{21} U_{12}$$
5. Compute recursively the block LU factorization of A^1.

LU Factorization as in ScaLAPACK

LU factorization on a P = Pr x Pc grid of processors
For ib = 1 to n-1 step b
$A(ib) = A(ib:n, ib:n)$

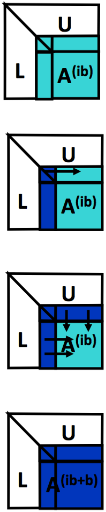
1. Compute panel factorization
   - find pivot in each column, swap rows
2. Apply all row permutations
   - broadcast pivot information along the rows
   - swap rows at left and right
3. Compute block row of U
   - broadcast right diagonal block of L of current panel
4. Update trailing matrix
   - broadcast right block column of L
   - broadcast down block row of U



Cost of LU Factorization in ScaLAPACK

LU factorization on a P = Pr x Pc grid of processors
For ib = 1 to n-1 step b
$A(ib) = A(ib:n, ib:n)$

1. Compute panel factorization
   - $\#messages = O(n \log_2 P_r)$
2. Apply all row permutations
   - $\#messages = O(n/b(\log_2 P_r + \log_2 P_c))$
3. Compute block row of U
   - $\#messages = O(n/b \log_2 P_c)$
4. Update trailing matrix
   - $\#messages = O(n/b(\log_2 P_r + \log_2 P_c)$



Cost of parallel block LU

Consider that we have a $\sqrt{P} \times \sqrt{P}$ grid, block size b

$$\gamma \cdot \left(\frac{2/3n^3}{P} + \frac{n^2 b}{\sqrt{P}} \right) + \beta \cdot \frac{n^2 \log P}{\sqrt{P}} + \alpha \cdot \left(1.5 n \log P + \frac{3.5 n}{b} \log P \right).$$

Plan

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The LU factorization of a tall skinny matrix

First try the obvious generalization of TSQR.

$$
\begin{aligned}
W &= \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \Pi_{00} L_{00} U_{00} \\ \Pi_{10} L_{10} U_{10} \\ \Pi_{20} L_{20} U_{20} \\ \Pi_{30} L_{30} U_{30} \end{pmatrix} \\
&= \begin{pmatrix} \Pi_{00} & & & \\ & \Pi_{10} & & \\ & & \Pi_{20} & \\ & & & \Pi_{30} \end{pmatrix} \begin{pmatrix} L_{00} & & & \\ & L_{10} & & \\ & & L_{20} & \\ & & & L_{30} \end{pmatrix} \left(\begin{array}{c} U_{00} \\ U_{10} \\ \hline U_{20} \\ U_{30} \end{array}\right) = \Pi_0 L_0 \left(\begin{array}{c} U_{00} \\ U_{10} \\ \hline U_{20} \\ U_{30} \end{array}\right) \\
\left(\begin{array}{c} U_{00} \\ U_{10} \\ \hline U_{20} \\ U_{30} \end{array}\right) &= \left(\begin{array}{c} \Pi_{01} L_{01} U_{01} \\ \hline \Pi_{11} L_{11} U_{11} \end{array}\right) = \left(\begin{array}{c|c} \Pi_{01} & \\ \hline & \Pi_{11} \end{array}\right) \left(\begin{array}{c|c} L_{01} & \\ \hline & L_{11} \end{array}\right) \left(\begin{array}{c} U_{01} \\ \hline U_{11} \end{array}\right) = \Pi_1 L_1 \left(\begin{array}{c} U_{01} \\ \hline U_{11} \end{array}\right) \\
\left(\begin{array}{c} U_{01} \\ \hline U_{11} \end{array}\right) &= \Pi_{02} \cdot L_{02} \cdot U_{02}
\end{aligned}
$$

The final factorization is:

$$
W = \Pi_0 L_0 \Pi_1 L_1 \Pi_{02} L_{02} U_{02}
$$

Obvious generalization of TSQR to LU

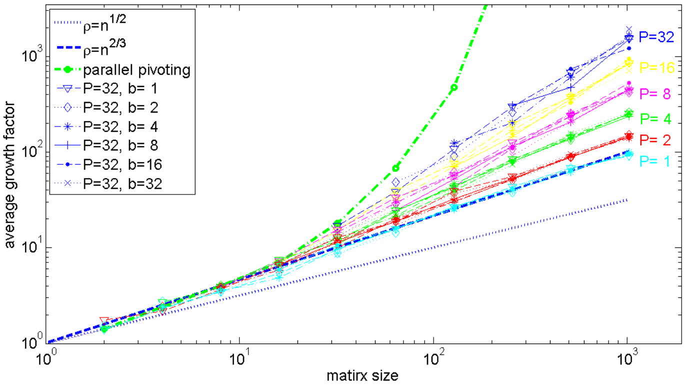
- Block parallel pivoting:

$$\begin{tikzcd}
W_0 \arrow[r] & U_{00} \arrow[dr] & & \\
& & U_{01} \arrow[ddr] & \\
W_1 \arrow[r] & U_{10} \arrow[ur] & & \\
& & & U_{02} \\
W_2 \arrow[r] & U_{20} \arrow[dr] & & \\
& & U_{11} \arrow[uur] & \\
W_3 \arrow[r] & U_{30} \arrow[ur] & &
\end{tikzcd}$$

- Block pairwise pivoting:
  - uses a flat tree and is optimal in the sequential case
  - introduced by Barron and Swinnerton-Dyer, 1960: block LU factorization used to solve a system with 100 equations on EDSAC 2 computer using an auxiliary magnetic-tape
  - used in PLASMA for multicore architectures and FLAME for out-of-core algorithms and for multicore architectures

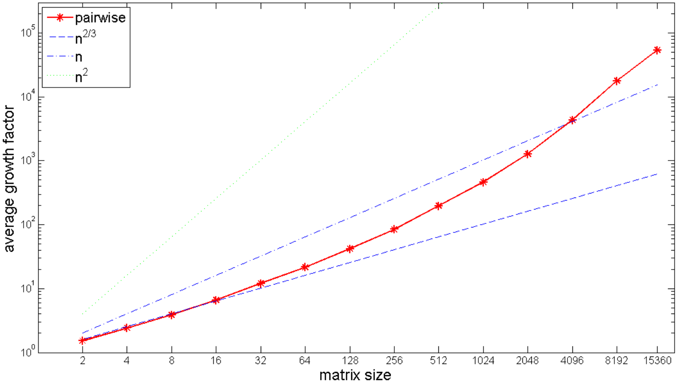
$$\begin{tikzcd}
W_0 \arrow[r] & U_{00} \arrow[r] & U_{01} \arrow[r] & U_{02} \arrow[r] & U_{03} \\
W_1 \arrow[urr] & & & & \\
W_2 \arrow[uurrr] & & & & \\
W_3 \arrow[uuurrrr] & & & &
\end{tikzcd}$$

Block parallel pivoting



- Unstable for large number of processors P
- When P=number rows, it corresponds to parallel pivoting, known to be unstable (Trefethen and Schreiber, 90)

Block pairwise pivoting



- Results shown for random matrices
- Will become unstable for large matrices

Tournament pivoting - the overall idea

- At each iteration of a block algorithm

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{where} W = \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix}$$

  where A_{11} is of size $b \times b$, A_{21} is of size $(m-b) \times b$, A_{12} is of size $b \times (n-b)$ and A_{22} is of size $(m-b) \times (n-b)$.

  - Preprocess W to find at low communication cost good pivots for the LU factorization of W, return a permutation matrix P
  - Permute the pivots to top, ie compute PA
  - Compute LU with no pivoting of W, update trailing matrix, obtain

$$PA = \begin{pmatrix} L_{11} & \\ L_{21} & I_{n-b} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ & A_{22} - L_{21} U_{12} \end{pmatrix}$$

- For details see [Grigori et al., 2011]

Tournament pivoting for a tall skinny matrix

1. Compute GEPP factorization of each W_i, find permutation Π_0

$$W = A(:, 1:b) \quad = \quad \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} \Pi_{00} L_{00} U_{00} \\ \Pi_{10} L_{10} U_{10} \\ \Pi_{20} L_{20} U_{20} \\ \Pi_{30} L_{30} U_{30} \end{pmatrix}, \quad \begin{array}{l} \text{Pick b pivot rows, form } A_{00} \\ \text{Same for } A_{10} \\ \text{Same for } A_{20} \\ \text{Same for } A_{30} \end{array}$$

2. Perform $\log_2 P$ times GEPP factorization of selected $2b \times b$ rows, find permutations Π_1, Π_2

$$\begin{pmatrix} A_{00} \\ A_{10} \\ A_{20} \\ A_{30} \end{pmatrix} = \begin{pmatrix} \Pi_{01} L_{01} U_{01} \\ \Pi_{11} L_{11} U_{11} \end{pmatrix}, \quad \begin{array}{l} \text{Pick b pivot rows, form } A_{01} \\ \text{Same for } A_{11} \end{array}$$

$$\begin{pmatrix} A_{01} \\ A_{11} \end{pmatrix} = \left(\Pi_{02} L_{02} U_{02} \right), \quad \begin{array}{l} \text{Pick b pivot rows that will} \\ \text{be used as pivots to factor } W \end{array}$$

3. Perform LU factorization with no pivoting of the permuted matrix

$$\Pi_2^T \Pi_1^T \Pi_0^T W = LU$$

Tournament pivoting

P_0: $W_0 = \begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 2 & 0 \\ 1 & 2 \end{pmatrix} = \Pi_0 L_0 U_0$, $\quad \Pi_0^T W_0 = \begin{pmatrix} 2 & 4 \\ 2 & 0 \end{pmatrix}$

P_1: $W_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 4 & 1 \\ 1 & 0 \end{pmatrix} = \Pi_1 L_1 U_1$, $\quad \Pi_1^T W_1 = \begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$

P_2: $W_2 = \begin{pmatrix} 0 & 1 \\ 1 & 4 \\ 0 & 0 \\ 0 & 2 \end{pmatrix} = \Pi_2 L_2 U_2$, $\quad \Pi_2^T W_2 = \begin{pmatrix} 1 & 4 \\ 0 & 2 \end{pmatrix}$

P_3: $W_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \\ 4 & 2 \end{pmatrix} = \Pi_3 L_3 U_3$, $\quad \Pi_3^T W_3 = \begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}$

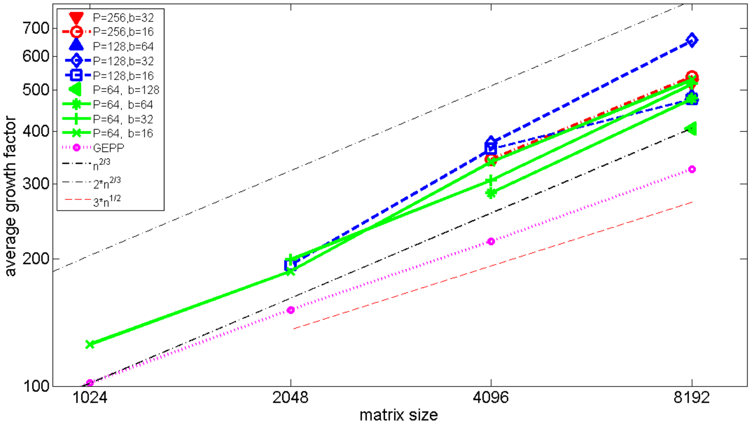
$\overline{W}_0 = \begin{pmatrix} 2 & 4 \\ 2 & 0 \\ 4 & 1 \\ 2 & 0 \end{pmatrix} = \overline{\Pi}_0 \overline{L}_0 \overline{U}_0$, $\quad \overline{\Pi}_0^T \overline{W}_0 = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$

$\overline{W}_2 = \begin{pmatrix} 1 & 4 \\ 0 & 2 \\ 4 & 2 \\ 0 & 2 \end{pmatrix} = \overline{\Pi}_2 \overline{L}_2 \overline{U}_2$, $\quad \overline{\Pi}_2^T \overline{W}_2 = \begin{pmatrix} 4 & 2 \\ 1 & 4 \end{pmatrix}$

$\underline{W}_0 = \begin{pmatrix} 4 & 1 \\ 2 & 4 \\ 4 & 2 \\ 1 & 4 \end{pmatrix} = \underline{\Pi}_0 \underline{L}_0 \underline{U}_0$, $\quad \underline{\Pi}_0^T \underline{W}_0 = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$

Good pivots for factorizing W

Growth factor for binary tree based CALU



- Random matrices from a normal distribution
- Same behaviour for all matrices in our test, and $|L| \leq 4.2$

Our "proof of stability" for CALU

- CALU as stable as GEPP in following sense:
  In exact arithmetic, CALU process on a matrix A is equivalent to GEPP process on a larger matrix G whose entries are blocks of A and zeros.
- Example of one step of tournament pivoting:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix},$$

$$\begin{tikzcd} A_{11} \arrow[rr] & & \bar{A}_{11} \\ A_{21} \arrow[r] & A_{21} \arrow[ur] & \\ A_{31} \arrow[ur] & & \end{tikzcd}$$

$$G = \begin{pmatrix} \bar{A}_{11} & & \bar{A}_{12} \\ A_{21} & A_{21} & \\ & -A_{31} & A_{32} \end{pmatrix}$$

- Proof possible by using original rows of A during tournament pivoting (not the computed rows of U).

Outline of "proof of stability"

- After the factorization of first panel by CALU, A_{32}^s (the Schur complement of A_{32}) is not bounded as in GEPP,

$$\begin{pmatrix} \Pi_{11} & \Pi_{12} & \\ \Pi_{21} & \Pi_{22} & \\ & & I \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \\ A_{31} & A_{32} \end{pmatrix} = \begin{pmatrix} \bar{L}_{11} & & \\ \bar{L}_{21} & I_b & \\ \bar{L}_{31} & & I_{m-2b} \end{pmatrix} \cdot \begin{pmatrix} \bar{U}_{11} & \bar{U}_{12} \\ & \bar{A}_{22}^s \\ & A_{32}^s \end{pmatrix}$$

- but A_{32}^s can be obtained by GEPP on larger matrix G formed from blocks of A

$$G = \begin{pmatrix} \bar{A}_{11} & & \bar{A}_{12} \\ A_{21} & A_{21} & \\ & -A_{31} & A_{32} \end{pmatrix} = \begin{pmatrix} \bar{L}_{11} & & \\ A_{21}\bar{U}_{11}^{-1} & L_{21} & \\ & -L_{31} & I_{m-2b} \end{pmatrix} \begin{pmatrix} \bar{U}_{11} & & \bar{U}_{12} \\ & U_{21} & -L_{21}^{-1}A_{21}\bar{U}_{11}^{-1}\bar{U}_{12} \\ & & A_{32}^s \end{pmatrix}$$

- GEPP on G does not permute and

$$\begin{aligned} L_{31}L_{21}^{-1}A_{21}\bar{U}_{11}^{-1}\bar{U}_{12} + A_{32}^s &= L_{31}U_{21}\bar{U}_{11}^{-1}\bar{U}_{12} + A_{32}^s = A_{31}\bar{U}_{11}^{-1}\bar{U}_{12} + A_{32}^s \\ &= \bar{L}_{31}\bar{U}_{12} + A_{32}^s = A_{32} \end{aligned}$$

Growth factor in exact arithmetic

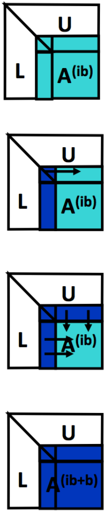
- Matrix of size m-by-n, reduction tree of height $H = \log_2(P)$
- In practice growth factor for GEPP and CALU is on the order of $n^{2/3} - -n^{1/2}$

	matrix of size $m \times (b+1)$		
	TSLU(b,H)		GEPP
	upper bound	attained	upper bound
$\|L\|$	2^{bH}	$2^{(b-2)H-(b-1)}$	1
g_W	$2^{b(H+1)}$	2^b	2^b
	matrix of size $m \times n$		
	CALU(b,H)		GEPP
	upper bound	attained	upper bound
$\|L\|$	2^{bH}	$2^{(b-2)H-(b-1)}$	1
g_W	$2^{n(H+1)-1}$	2^{n-1}	2^{n-1}

Cost of LU Factorization in ScaLAPACK

LU factorization on a P = Pr x Pc grid of processors
For ib = 1 to n-1 step b
$A(ib) = A(ib:n, ib:n)$

1. Compute panel factorization
   - $\#messages = O(n/b \log_2 P_r)$
2. Apply all row permutations
   - $\#messages = O(n/b(\log_2 P_r + \log_2 P_c))$
3. Compute block row of U
   - $\#messages = O(n/b \log_2 P_c)$
4. Update trailing matrix
   - $\#messages = O(n/b(\log_2 P_r + \log_2 P_c)$



CALU based on TSLU

Cost of CALU vs ScaLAPACK's PDGETRF

- $n \times n$ matrix on $\sqrt{P} \times \sqrt{P}$ processor grid, block size b
- Flops: $(2/3)n^3/P + 3/2n^2 b \log_2 P/\sqrt{P}$ vs $(2/3)n^3/P + n^2 b/P^{1/2}$
- Bandwidth: $n^2 \log_2 P/\sqrt{P}$ vs same
- Latency: $3n \log_2 P/b$ vs $1.5n \log_2 P$

Close to optimal (modulo log P factors)

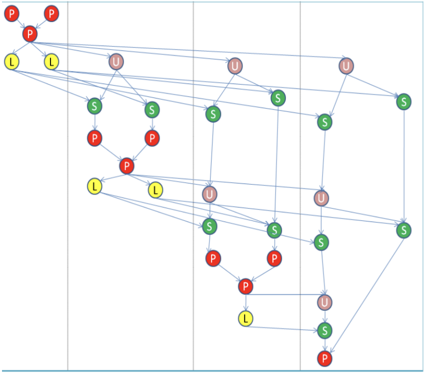
- Assume $O(n^2/P)$ memory/processor, $O(n^3)$ algorithm
- Choose b near $n/\sqrt{P}$ (its upper bound)
- Bandwidth lower bound: $\Omega(n^2/\sqrt{P})$ – just $\log_2 P$ smaller
- Latency lower bound: $\Omega(\sqrt{P})$ – just polylog(P) smaller

Performance vs ScaLAPACK

- Parallel TSLU (LU on tall-skinny matrix)
  - IBM Power 5
    Up to 4.37x faster (16 procs, $1M \times 150$)
  - Cray XT4
    Up to 5.52x faster (8 procs, $1M \times 150$)
- Parallel CALU (LU on general matrices)
  - Intel Xeon (two socket, quad core)
    Up to 2.3x faster (8 cores, $10^6 \times 500$)
  - IBM Power 5
    Up to 2.29x faster (64 procs, 1000×1000)
  - Cray XT4
    Up to 1.81x faster (64 procs, 1000×1000)
- Details in SuperComputing'08 (LG, Demmel, Xiang), IPDPS'10 (S. Donfack, LG).

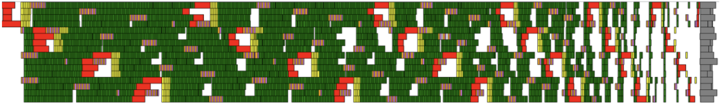
CALU and its task dependency graph

- The matrix is partitioned into blocks of size $T \times b$
- The computation of each block is associated with a task.

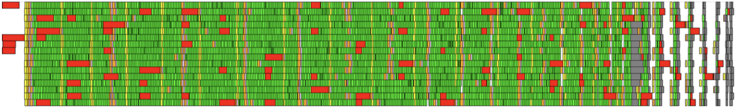


Scheduling CALU's Task Dependency Graph

- **Static scheduling**
  - \+ Good locality of data
  - \- Ignores noise



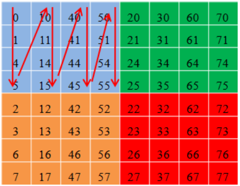
- **Dynamic scheduling**
  - \+ Keeps cores busy
  - \- Poor usage of data locality
  - \- Can have large dequeue overhead



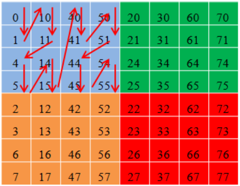
Lightweight scheduling

- Emerging complexities of multi- and mani-core processors suggest a need for self-adaptive strategies
  - One example is work stealing
- Goal
  - Design a tunable strategy that is able to provide a good trade-off between load balance, data locality, and dequeue overhead
  - Provide performance consistency
- Approach: combine static and dynamic scheduling
  - Shown to be efficient for regular mesh computation [B. Gropp and V. Kale]

Possible data layouts



Block cyclic layout (BCL)



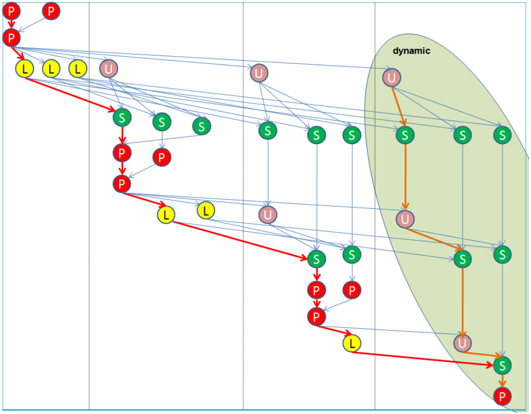
Two level block layout (2l-BL)

Data layout/scheduling	Static	Dynamic	Static(%dynamic)
Column Major Layout (CM)		x	
Block Cyclic Layout (BCL)	x	x	x
2-level Block Cyclic Layout (2l-BL)	x	x	x

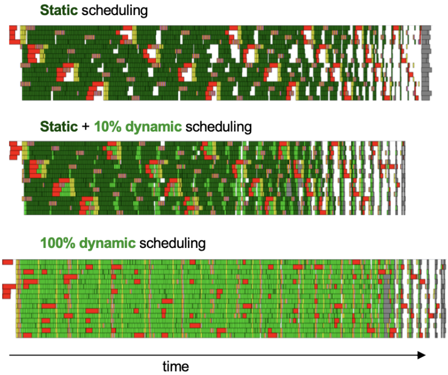
Lightweight scheduling

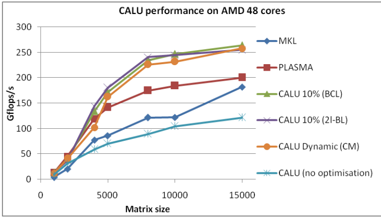
- A self-adaptive strategy to provide
  - A good trade-off between load balance, data locality, and dequeue overhead.
  - Performance consistency
  - Shown to be efficient for regular mesh computation [B. Gropp and V. Kale]

- Combined static/dynamic scheduling
  - A thread executes in priority its statically assigned tasks
  - When no task ready, it picks a ready task from the dynamic part
  - The size of the dynamic part is guided by a performance model



Best performance of CALU on multicore architectures





- Reported performance for PLASMA uses LU with block pairwise pivoting

Acknowledgement

- Some of the examples taken from [Golub and Van Loan, 1996]

References (1)

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