Lecture 10.2: hypothesis testing for beta

EXAMPLE 1 (HYPERTENSION, PEDIATRICS)

Newborn blood pressure (y) is thought to be affected by weight (x_1) and age (x_2) when both blood pressure and weight are measured.

$$y = 53.45019 + 0.12558x_1 + 5.88772x_2$$

Table 1: Sample data for infant birth weight, age and blood pressure for 16 infants.

i	Weight (oz) (x_1)	Age (days) (x_2)	SBP (mm HG) (y)
1	135	3	89
2	120	4	90
3	100	3	83
4	105	2	77
5	130	4	92
6	125	5	98
7	125	2	82
8	105	3	85
9	120	5	96
10	90	4	95
11	120	2	80
12	95	3	79
13	120	3	86
14	150	4	97
15	160	3	92
16	125	3	88

1. *F* TEST FOR JOINT HYPOTHESIS

We would like to test the overall hypothesis that the predictors when considered together have significant impact on the outcome.

$$H_0\colon \beta_1 = \cdots = \beta_k = 0 \text{ versus } H_a\colon \text{least one } \beta_j \neq 0.$$

This is the overall hypothesis that at least some of the β_j *'s* are different from zero, but without specifying which one is different.

Similar to the F test for SLR

- Estimate the regression parameters using the method of least squares, and compute Reg SS and Res SS

$$\text{Res SS} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

$$\text{Reg SS} = \text{Total SS} - \text{Res SS}$$

$$\text{Total SS} = \sum_{i=1}^{n} (y_i - \bar{y})^2$$

$$\hat{y}_i = a + \sum_{j=1}^{k} b_j x_{ij}$$

x_{ij} = jth independent variable for the ith subject, $j = 1,\ldots, k$; $i = 1,\ldots, n$

- *F* statistic (**Table 1**):

$$F = \frac{(\text{RegSS})/k}{(\text{ResSS})/(n-k-1)} = \frac{\text{RegMS}}{\text{ResMS}}.$$

- *Null distribution:* Let $e \sim N(0, \sigma^2)$. If H_0 is true, then $F \sim F_{k,n-k-1}$, the centralized F distribution with $(k, n-k-1)$ degrees of freedom.

- The *exact p-value* of the observed F value is given by $p = \Pr\left(F_{k,n-k-1} > F\right)$.

- *Decision rule:* Reject the H_0 at nominal level $\alpha \in (0,1)$ if $F > F_{k,n-k-1,1-\alpha}$ (equivalently, $p < \alpha$), where the critical value $F_{k,n-k-1,1-\alpha}$, is the $1-\alpha$ percentile of $F_{k,n-k-1}$.

Table 2: Analysis of variance (ANOVA)				
Source	DF	Sum of Squares	Mean Squares	F
Regression	k	RegSS	$\text{RegMS} = \dfrac{\text{RegSS}}{k}$	$F = \dfrac{\text{RegMS}}{\text{ResMS}}.$
Residual	$n - k - 1$	ResSS	$\text{ResMS} = \dfrac{\text{ResSS}}{n - k - 1}$	
Total	$n - 1$	TotSS	**The p value** of a realized F is: $p = \Pr\left(F \sim F_{k,n-k-1} > F\right).$	

Solve the following problems using the data from Example 1.

(1) Compute the Least Squares Fit and R^2.

(2) Perform F test (at level $\alpha = 0.05$.) on $H_0: \beta_1 = \beta_2 = 0$ versus H_a: $\beta_1^2 + \beta_2^2 > 0$ (at least one of the β's is not zero).

(1) Multiple Linear Regression Least Squares Fit:

$$y = 53.45019 + 0.12558x_1 + 5.88772x_2.$$

(2) F TEST

$H_0: \beta_1 = \beta_2 = 0$ **versus** $H_a: \beta_1^2 + \beta_2^2 > 0$.

$F = 48.08.$

Since $n = 16, k = 2$ **and** $\alpha = 0.05$, **we have** $F_{k,n-k-1,1-\alpha} = F_{2,13,0.95} = 3.805565.$

Confidently, we reject H_0 **since the realized** $F = 48.08 > F_{2,13,0.95}$ **(the** p**-value** $< 0.0001 < \alpha$**).**

$$R^2 = \frac{\text{SSReg}}{SS\ total} = \frac{591.03564}{670.93750} = 0.8809$$

That is, 88.097% of the total variation in the SBP is accounted for by the linear combination of weight and age.

Table 2.1: ANOVA					
Source	DF	SS	MS	F	P-value
Reg(x_1, x_2)	2	591.03564	295.51782	48.08	<.0001
Residual	13	79.90186	6.14630		
Total	15	670.93750			

2. TESTS FOR A PARTIAL HYPOTHESIS

Often, we want to know whether an *individual predictor* x_j has a significant effect on outcome y after *controlling for the other predictors*. The partial hypothesis on β_j is

$$H_{0j}: \beta_j = 0 \text{ versus } H_{aj}: \beta_j \neq 0.$$

we assume other β is making a contribution under either hypothesis

(1) PARTIAL *T* TEST

- The t statistic:

$$t(x_j|\text{other } x's) = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}.$$

- Null distribution: Let $e \sim N(0, \sigma^2)$. If $H_{0j}: \beta_j = 0$ is true, then $t(x_j|\text{other } x's) \sim t_{n-k-1}$, the centralized t distribution with $n - k - 1$ degrees of freedom.

- The *exact two-tailed p* value is $p = 2 \times \Pr\left(t_{n-k-1} > \left|t(x_j|\text{other } x's)\right|\right)$.

- Decision rule: Reject H_{0j} at nominal level $\alpha \in (0,1)$ if $|t(x_j|\text{other } x's)| > t_{n-k-1,1-\alpha/2}$ (equivalently, $p < \alpha$).

EXAMPLE 1 (HYPERTENSION, PEDIATRICS): PARTIAL *T* TESTS

Perform tests for the partial contributions of weight and age in predicting SBP in infants (using SAS or R output).

Solution: From the output of lm(.) in R, I obtain **Table 3**. The output Pr(> $|t|$) of the lm(.) module is double-tailed p value.

Table 3: Partial t tests				
predictor	$\hat{\beta}_j$	$se(\hat{\beta}_j)$	t value	p value
x_1=weight(oz)	0.1256	0.0343	$3.6575 > t_{13,0.975}$	2.90e-3 <0.05
x_2=age (day)	5.8877	0.6802	$8.6558 > t_{13,0.975}$	9.34e-7 <0.05

Since $n = 16, k = 2$ and $\alpha = 0.05$, we have $t_{n-k-1,1-\alpha/2} = t_{13,0.975} = 2.1604$.

All the realized t values surpass the critical value and their p values are surpassed by α, *suggesting* that both weight and age have highly significant associations with SBP, *even after controlling for* the other variable.

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept) 53.45019    4.53189  11.794 2.57e-08 ***
weight       0.12558    0.03434   3.657   0.0029 ** 
age          5.88772    0.68021   8.656 9.34e-07 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(2) PARTIAL *F* TEST

- Aforesaid partial t test for the effect of one particular predictor adjusts for the contribution of the other predictor.
- To better understand this point, let us *'develop' partial F test* for

$$H_{02}: \beta_2 = 0 \text{ vs. } H_{a2}: \beta_2 \neq 0$$

in full model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$$

after adjusting for the contribution of x_1.

- If H_{02} is false, then we have the *full model* and $\text{RegSS}(x_1, x_2)$.

- If H_{02} is true, then we have the *reduced model* $y = \beta_0 + \beta_1 x_1 + e$ and $\text{RegSS}(x_1)$.

- The extra sum of squares due to x_2 after *adjusting for* x_1 is

$$\text{RegSS}(x_2|x_1) = \text{RegSS}(x_1, x_2) - \text{RegSS}(x_1).$$

- Hence,

$$\text{RegMS}(x_2|x_1) = \frac{\text{RegSS}(x_2|x_1)}{1} = \text{RegSS}(x_2|x_1).$$

- The F test statistic is

$$F(x_2|x_1) = \frac{\text{Reg MS}(x_2|x_1)}{\text{Res MS}(x_1, x_2)}.$$

- Null distribution: Let $e \sim N(0, \sigma^2)$. If H_{02} is true, then $F(x_2|x_1) \sim F_{1,n-3}$.

- The exact p value of a realized $F(x_2|x_1)$ is given by $p = Pr\left(F_{1,n-3} > F(x_2|x_1)\right)$.

- Decision rule: Reject H_{02} when the realized $F(x_2|x_1) > F_{1,n-3,1-\alpha}$ (equivalently, $p < \alpha$).

EXAMPLE 1: PARTIAL F TESTS

Perform partial F tests for the partial contributions of weight and age in predicting SBP in infants (using SAS or R output).

Solution: Let y =SBP, x_1 =weight, x_2 =age. The full model is $\boldsymbol{y} = \boldsymbol{\beta_0} + \boldsymbol{\beta_1 x_1} + \boldsymbol{\beta_2 x_2} + e$.

(1) The hypothesis is $\boldsymbol{H_{02}}: \boldsymbol{\beta_2} = \boldsymbol{0}$ versus $\boldsymbol{H_{a2}}: \boldsymbol{\beta_2} \neq 0$.

If H_{02} is false, then I obtain RegSS(x_1, x_2) =591.03564 and ResMS$(x_1, x_2) = 6.14630$ by fitting the full model (**Table 2.1**). If H_{02} is true, then I obtain RegSS$(x_1) =$ 130.5375 by fitting the *reduced model* $y = \beta_0 + \beta_1 x_1 + e$.

The extra sum of squares due to x_2 after adjusting for x_1 is

$$\begin{aligned}\text{RegSS}(x_2|x_1) &= \text{RegSS}(x_1, x_2) - \text{RegSS}(x_1)\\ &= 591.0356 - 130.5375\\ &= 460.4981.\end{aligned}$$

Hence, $\text{Reg MS}(x_2|x_1) = \text{Reg SS}(x_2|x_1) = 460.4981$, and the realized *F test statistic* is

$$F(x_2|x_1) = \frac{\text{Reg MS}(x_2|x_1)}{\text{Res MS}(x_1, x_2)} = \frac{460.4981}{6.14630} = 74.9228.$$

From **Table 3** I see $t(x_2|x_1) = 8.6558$. It follows that $t^2(x_2|x_1) =$74.9229$= F(x_2|x_1)$.

(2) The hypothesis is $H_{01}: \beta_1 = 0$ versus $H_{a1}: \beta_1 \neq 0$.

- If H_{01} is false, then I obtain RegSS(x_1, x_2) =591.03564 and ResMS$(x_1, x_2) = 6.14630$ by fitting the full model (Table 7.2).

- If H_{01} is true, then I obtain Reg SS$(x_2) = 508.81657$ by fitting the *reduced model* $y = \beta_0 + \beta_1 x_2 + e$.

- The extra sum of squares due to x_1 after adjusting for x_2 is

$$\text{Reg SS}(x_1|x_2) = \text{Reg SS}(x_1, x_2) - \text{Reg SS}(x_2)$$
$$= 591.0356 - 508.81657 = 82.2190.$$

Hence, Reg $\text{MS}(x_1|x_2) = \text{Reg SS}(x_1|x_2) = 82.2190$, and the realized *F test statistic* is

$$F(x_1|x_2) = \frac{\text{Reg MS}(x_1|x_2)}{\text{Res MS}(x_1, x_2)} = \frac{82.2190}{6.14630} = 13.3770.$$

From **Table 3,** I see $t(x_1|x_2) = 3.6575$. It follows that $t^2(x_1|x_2) = 13.3773 = F(x_1|x_2)$.

For $\alpha = 0.05$ and $n = 16$, I see critical value: $F_{1,n-3,0.95} = F_{1,13,0.95} = 4.67$. Since both $F(x_1|x_2)$ and $F(x_2|x_1)$ are larger than 4.67, I reject both H_{01} and H_{02}.

This analysis suggests that both weight and age have highly significant associations with SBP, even after controlling for the other predictor.

- **Note 1:** Assume linearity, $e \sim N(0, \sigma^2)$, homoscedasticity, and independence. If H_{0j} is true, then $t^2(x_j | \text{other } x's) = F(x_j | \text{other } x's) \sim F_{1,n-k-1}$.

- **Note 2:** As illustrated by the Example, $t^2(x_j | \text{other } x's) = F(x_j | \text{other } x's)$ for arbitrary specific data points $(n > k + 1)$, none of the above assumptions are needed.

- **Note 3:** $t^2_{n-k-1,1-\alpha/2} = F_{1,n-k-1,1-\alpha}$ for $\alpha \in (0,1)$.