Greek Mythology

Chaos [ΧΑΟΣ]

The rude and shapeless mass whose appearance could not be described, for there was no light by which it could be seen. It existed before the formation of the universe; and from it a Supreme Power created the world.

Chaos was presented as one of the oldest of the gods and as one of the deities of the infernal regions.

The Butterfly Effect

- ***The flap of a butterfly's wings in Brazil can set off a tornado in Texas***
- **This is a parable about sensitive dependence on initial conditions**
- **A tiny difference is amplified until two outcomes are totally different**
- **Due to inevitable chaos, long term weather forecasting is impossible**

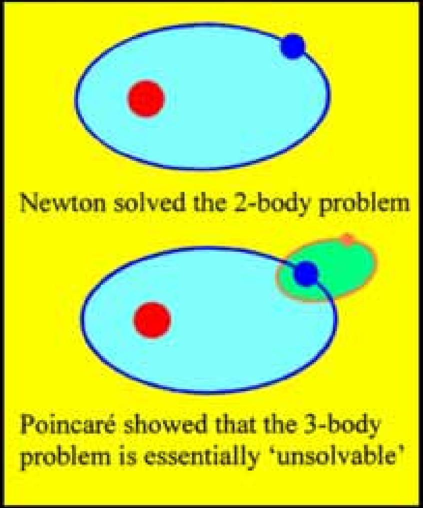
Henri Poincaré: (1890's)

"... it may happen that small differences in the initial conditions will produce very large ones in the final phenomena. A small error in the former produce an enormous error in the latter. Prediction becomes impossible. ..."

Henri Poincare

Birth of Chaos Theory

- In 1887 the King of Sweden offered a prize to the person who could answer the question "Is the solar system stable?"
- Poincare, a French mathematician, won the prize with his work on the three-body problem
- He considered, for example, just the Sun, Earth and Moon orbiting in a plane under their mutual gravitational attractions
- Like the pendulum, this system has some unstable solutions
- Introducing a Poincare section, he saw that homoclinic tangles must occur
- These would then give rise to chaos and unpredictability



Poincare Section

To examine chaos, Poincare used the idea of a section

This cuts across the phase-space orbits

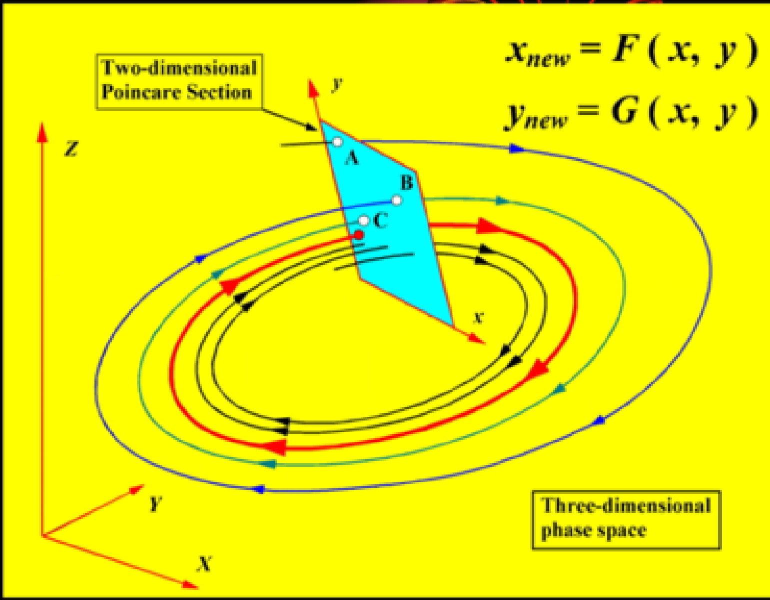
The original system flows in continuous time

On the section, we observe steps in discrete time

The flow is replaced by what is called an iterated map

The dimension of the phase-space is reduced by one

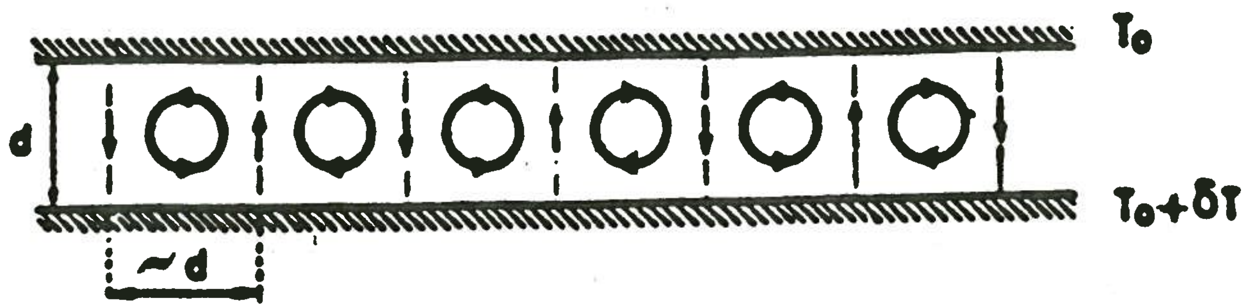
@



Third Order Systems

- Minimum system order to exhibit chaos is third order for autonomous systems.
- A second order forced system can be thought of as a third order autonomous system.

Rayleigh-Benard Convection



Lorenz 1963

* Two plates with fluid between them. The lower plate being hotter than the upper plate.

* As the Rayleigh number was increasa clockwise and counter clockwise rotations were not predictable.

* First physical dissipative system where chaos was found.

The Lorenz Model

* Turbulent convection in Fluids

$$\boxed{\begin{aligned} \dot{x} &= p\,y - p\,x \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned}}$$

where x represents a velocity mode

y, z represent temperature modes

p ~ Prandtle number, r ~ Rayleigh #, b ~ aspect ratio

* $\dot{x} = \dot{y} = \dot{z} = 0$ at three points

$$(0,0,0), \quad \left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$$

and

$$\left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$$

* $$\underset{\sim}{A} = \begin{bmatrix} -p & p & 0 \\ -z+r & -1 & -x \\ y & x & -b \end{bmatrix}$$

Jacobian Matrix

At $(0,0,0)$, the eigen value equation is

$$(\hat{\lambda}+b)\{\hat{\lambda}^2+(p+1)\hat{\lambda}-p(r-1)\}=0$$

All eigen values are real and one of them is positive.

$\Rightarrow$ $\boxed{\text{origin is a "saddle point"}}$.

At $\left(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\right)$, the eigen value equation is

$$\hat{\lambda}^3+(p+b+1)\hat{\lambda}^2+b(p+r)\hat{\lambda}+2pb(r-1)=0$$

The necessary and sufficient condition for the eigen values to be in the open left half-plane is obtained using the Routh-Hurwitz theorem of classical stability theory

$$(p+b+1)\,b(p+r) > 2pb(r-1)$$

When this condition is satisfied, both equilibrium points, away from the origin, are asymptotically stable.

♫ When this condition is not satisfied, then the two equilibrium points, away from the origin, are unstable. Under these circumstances, the system exhibits "very complicated" or turbulent behaviour.

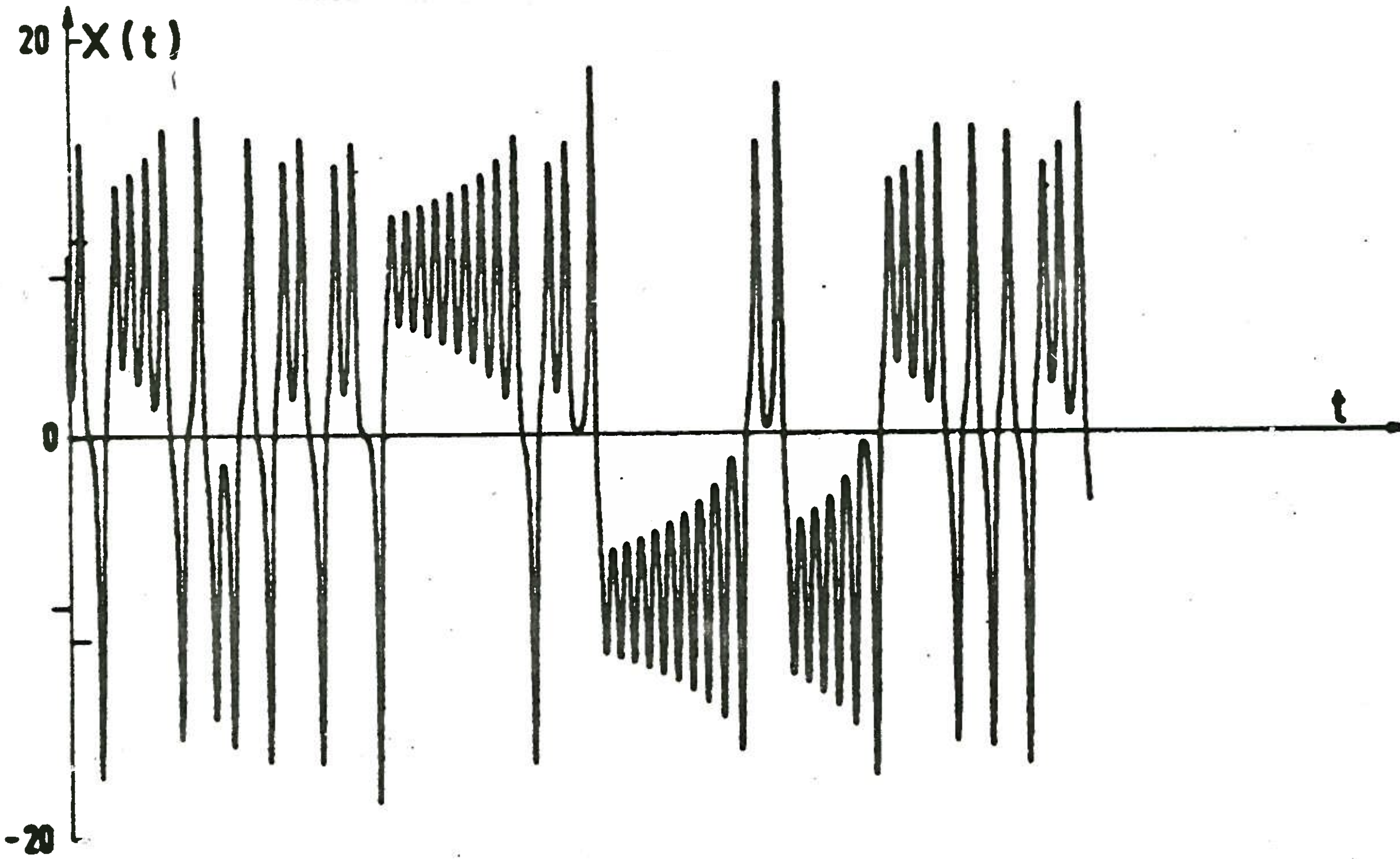
Computer Model

* Let $b = \frac{8}{3}$, $p = 10$, $r = 28$.

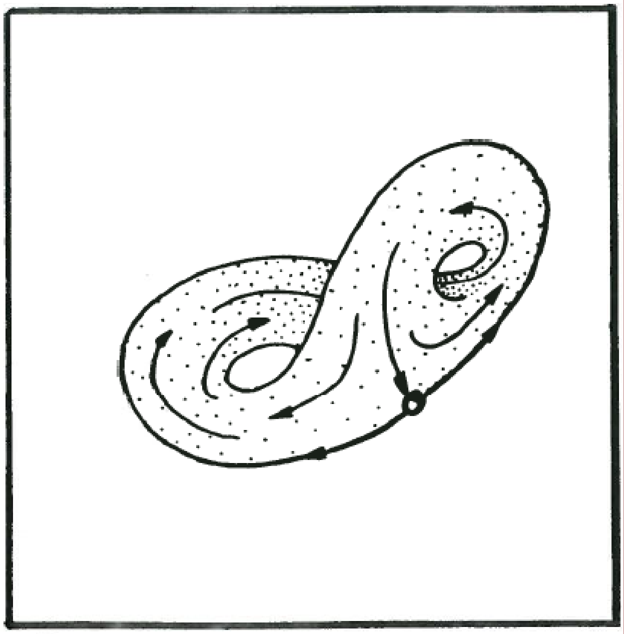
Then both equilibrium points (away from the origin) are unstable.

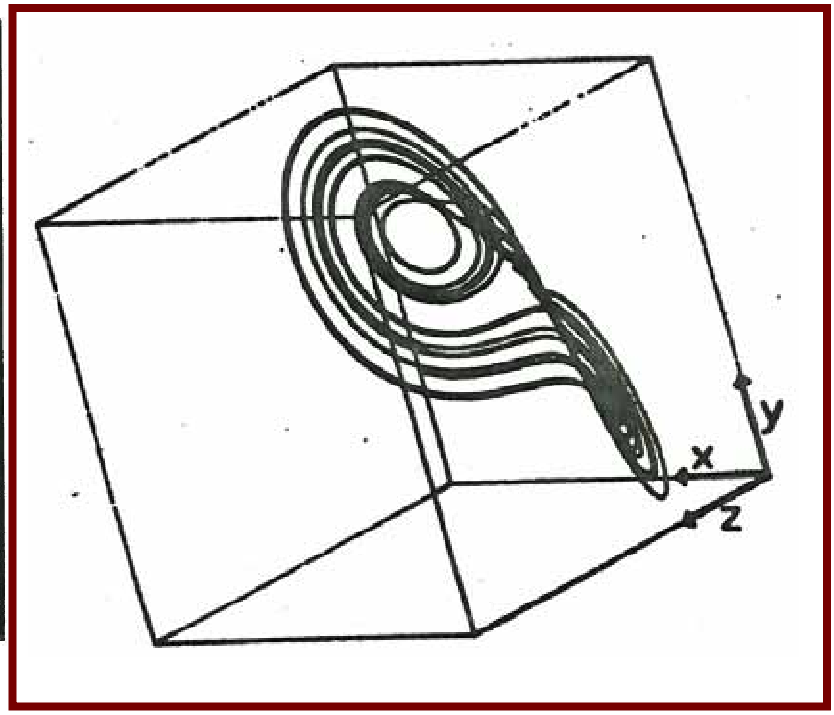
* Solve for x, y, z as variables that change with respect to time.

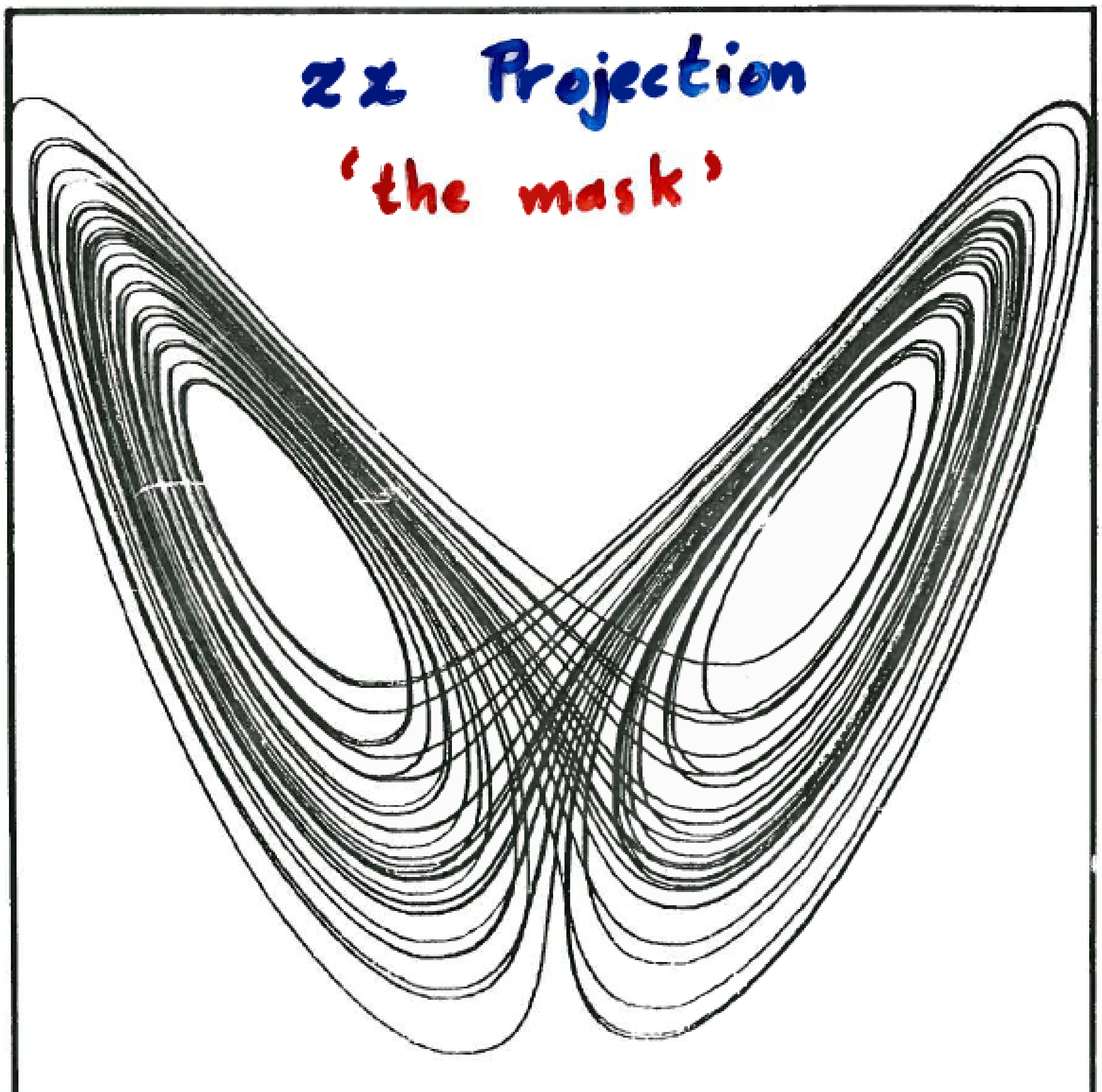
* Estimate state trajectories in the 3-dimensional state space as well as projections on 2-dimensional subspaces.



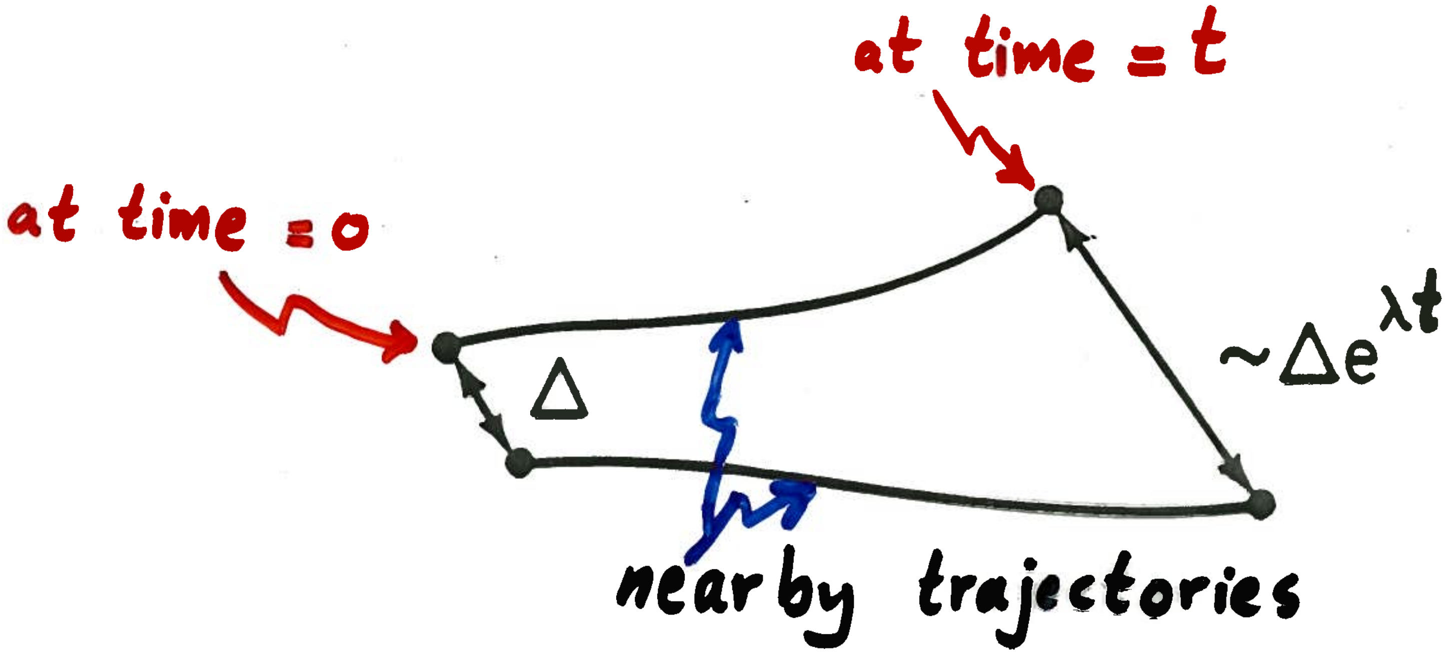
20
X (t)
0
t
-20




y
x
z

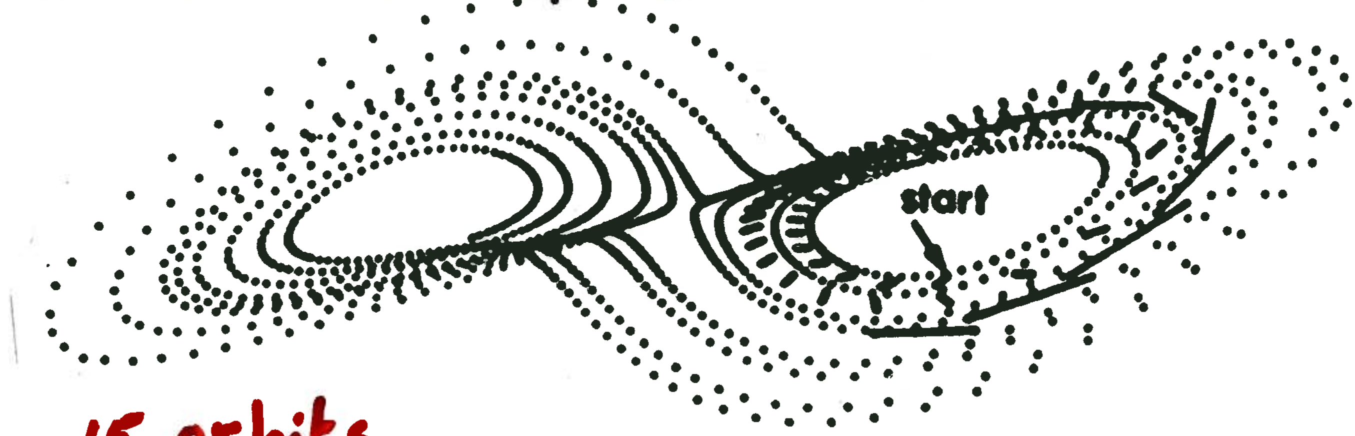

zx Projection
'the mask'

HOW TO DEFINE THE LARGEST LYAPUNOV EXPONENT?

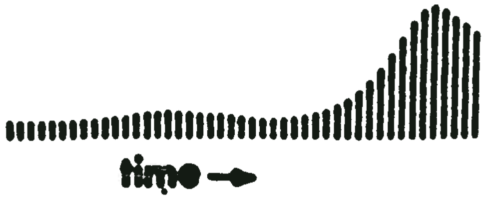


λ : LARGEST LYAPUNOV EXPONENT

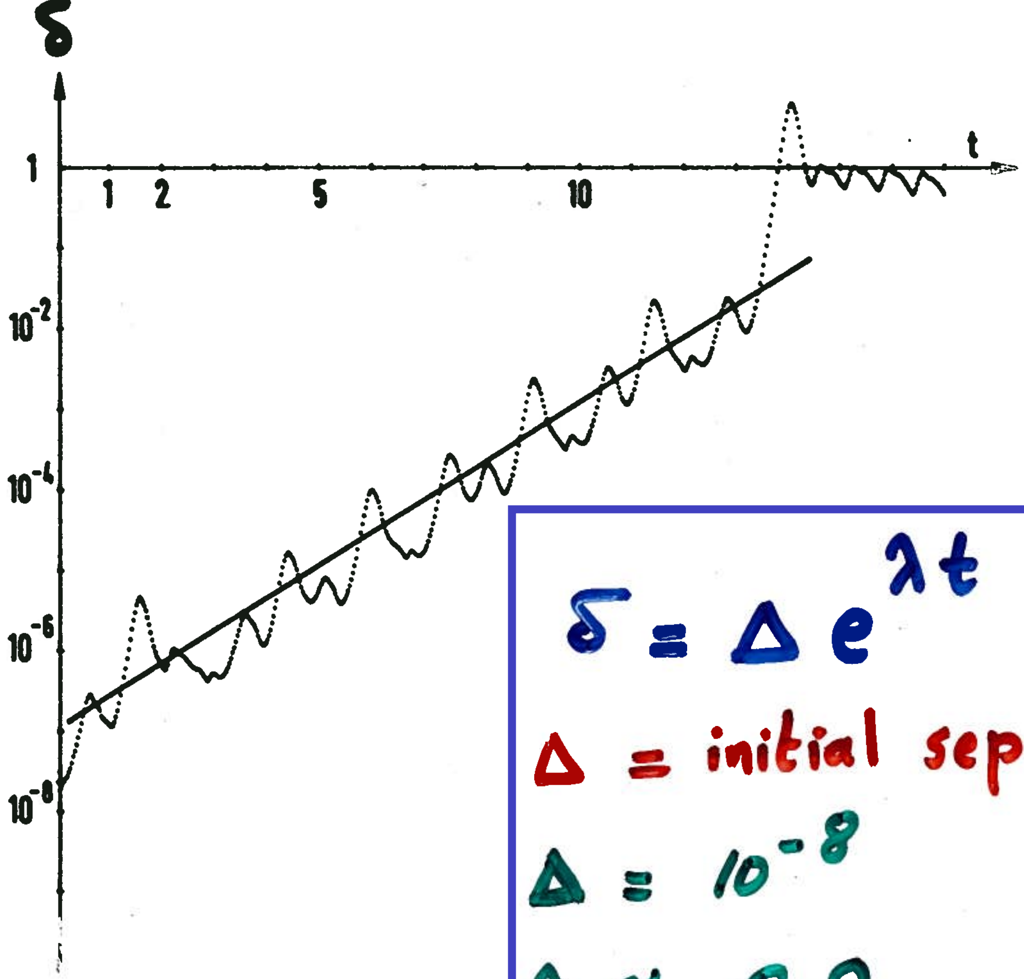
Short-term evolution of separation vector between nearby points :



....15 orbits



magnitude of evolving vector δ.

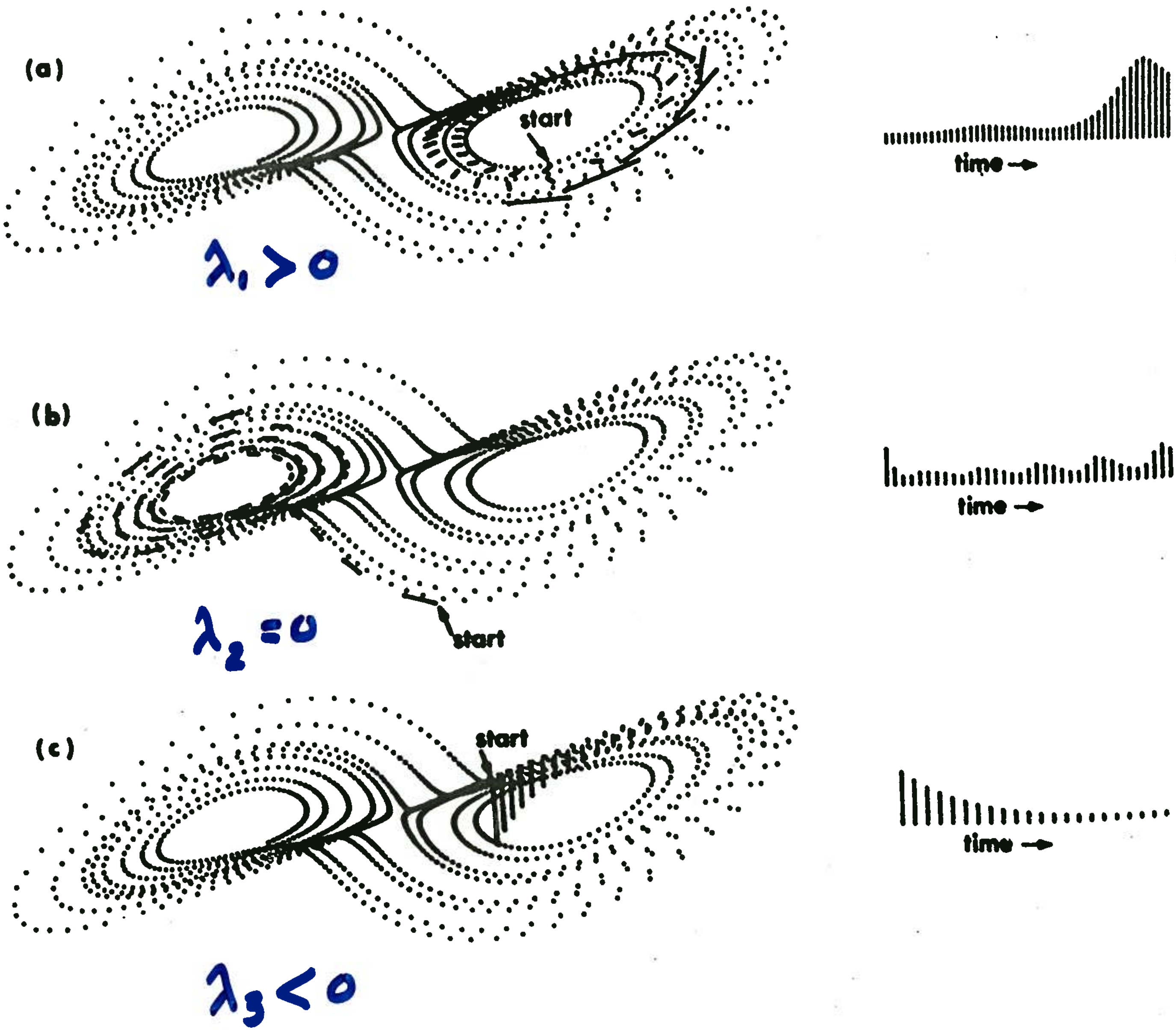


$$\delta = \Delta e^{\lambda t}$$

Δ = initial separation

$\Delta = 10^{-8}$

$\lambda \approx 0.9$


(a)
start
time →
λ1 > 0
(b)
start
time →
λ2 = 0
(c)
start
time →
λ3 < 0

* For an attractor, contraction must outweigh expansion, hence

$$\sum_{i=1}^{n} \lambda_i < 0$$

* For a non-chaotic attracting set $\lambda_i \leq 0$ for all "i".

* For a chaotic attractor, at least one

$$\lambda_i > 0$$

$$\lambda_{i+1} = 0$$

$$\lambda_{i+2} < 0$$

* In a three-dimensional state space, the signs of the Lyapunov exponents of different attractor types are:

$(-,-,-)$; $(0,-,-)$; $(0,0,-)$; $(+,0,-)$

fixed point | limit cycle | quasiperiodic | chaotic attractor

CHAOS

* If $\lambda > 0$ for typical aperiodic orbits on the attractor $\Longrightarrow$ CHAOS.

* $\boxed{\text{Distance scales as } \Delta e^{\lambda t}}$

* Neighboring orbits diverge exponentially when chaotic.

* $\boxed{\text{Nothing is going to infinity.}}$

CHAOS IS CHARACTERIZED BY

SENSITIVE DEPENDENCE

ON THE INITIAL CONDITIONS

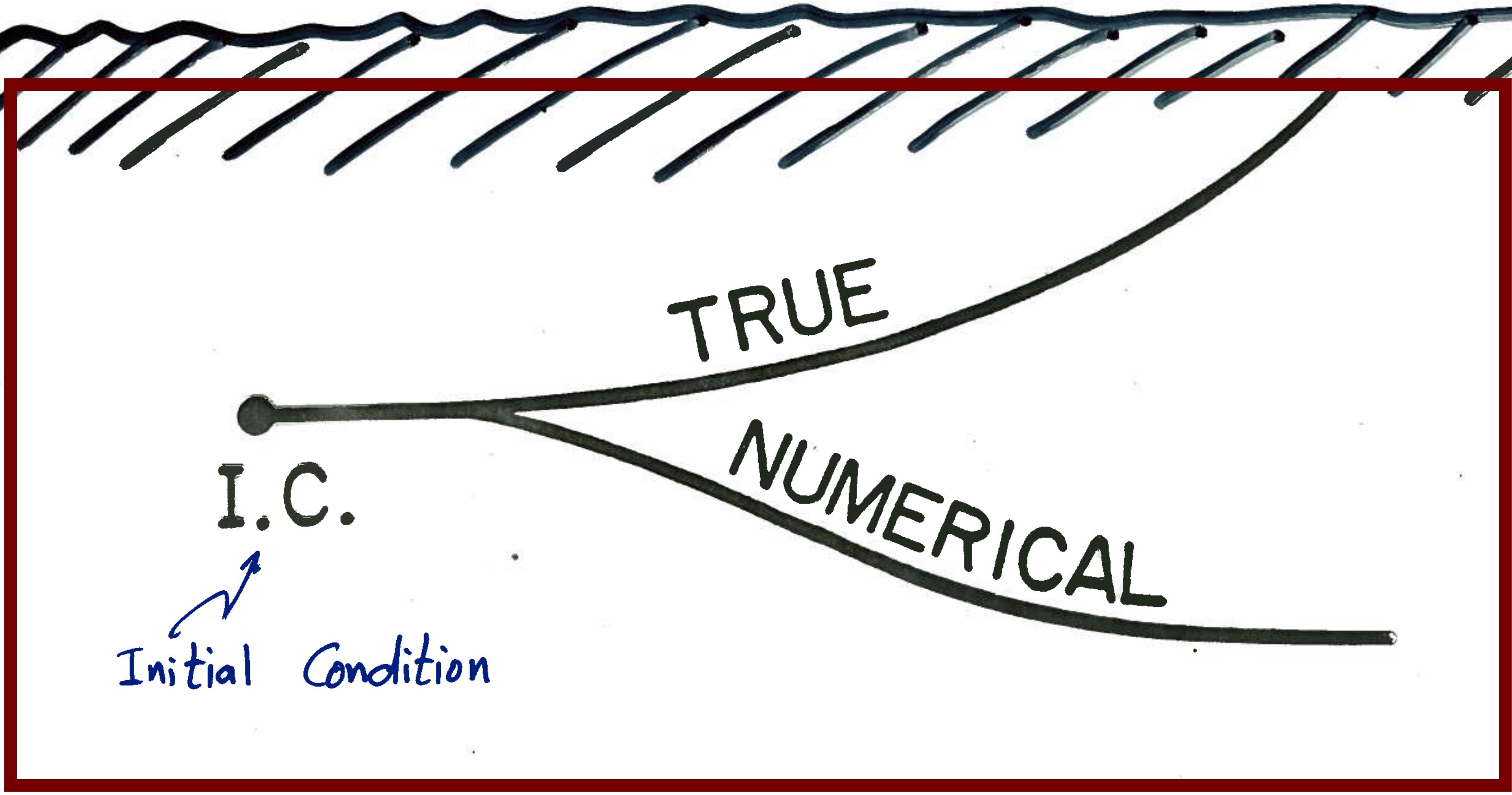
* PROPERTY IS CENTRAL FOR CONNECTING:

DETERMINISTIC EQUATIONS

AND

CHAOTIC SOLUTIONS

SHADOWING OF A CHAOTIC TRAJECTORY



TRUE AND NUMERICAL TRAJECTORIES SEPARATE EXPONENTIALLY FROM EACH OTHER.

Theorem **(Haken [1983])**

Existence of $\lambda = 0$

If, given system $\boxed{\dot{\underset{\sim}{x}} = \underset{\sim}{f}(\underset{\sim}{x}),}$ $\underset{\sim}{x}(0) = \underset{\sim}{x}_0$

i) $\phi_t(\underset{\sim}{x}_0)$ *is bounded for* $t \geq 0$,

ii) $\phi_t(\underset{\sim}{x}_0)$ *does not tend toward an equilibrium point, and*

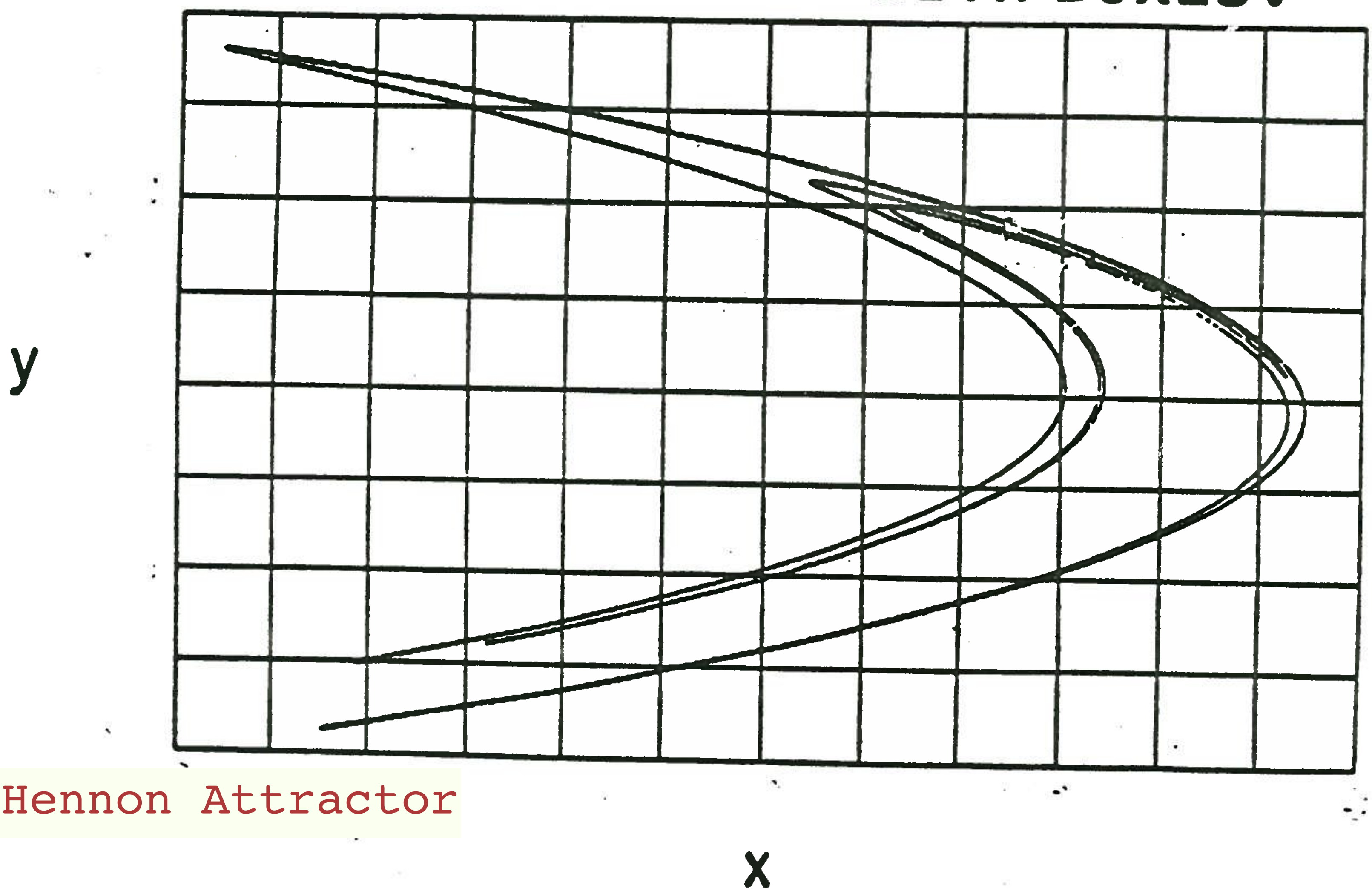
iii) $\underset{\sim}{f}$ *has a finite number of zeros;*

then at least one of the Lyapunov exponents is zero.

$\Phi_t(\underset{\sim}{x}_o)$ is a state trajectory, when it is bounded then it belongs to an attractor (if it does not tend towards an equilibrium point).

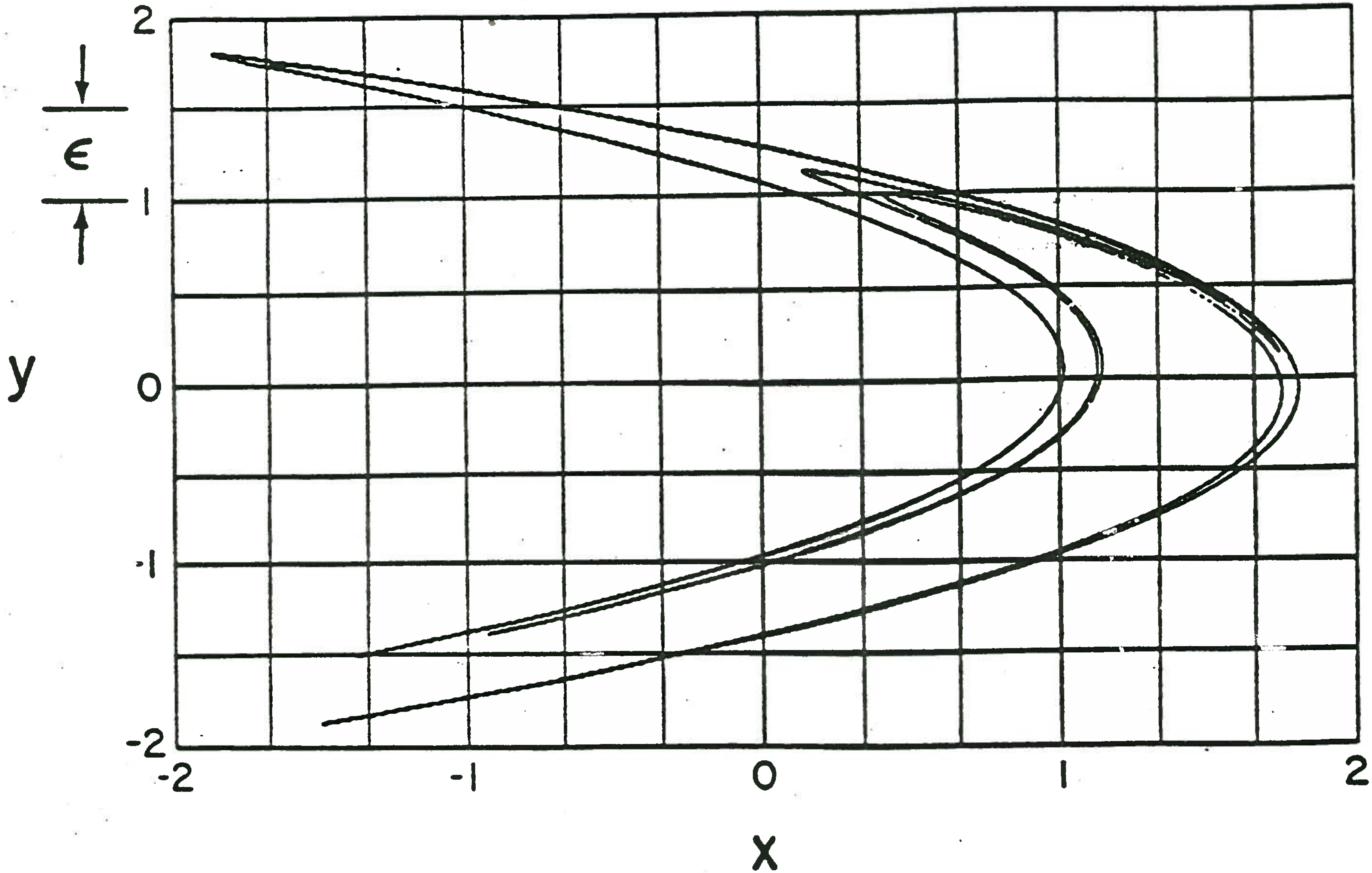
HOW TO DEFINE DIMENSION?

* MANY POSSIBLE WAYS OF DEFINING D.
* CHOOSE THE BOX-COUNTING DIMENSION: COVER THE ATTRACTOR WITH BOXES.



Hennon Attractor

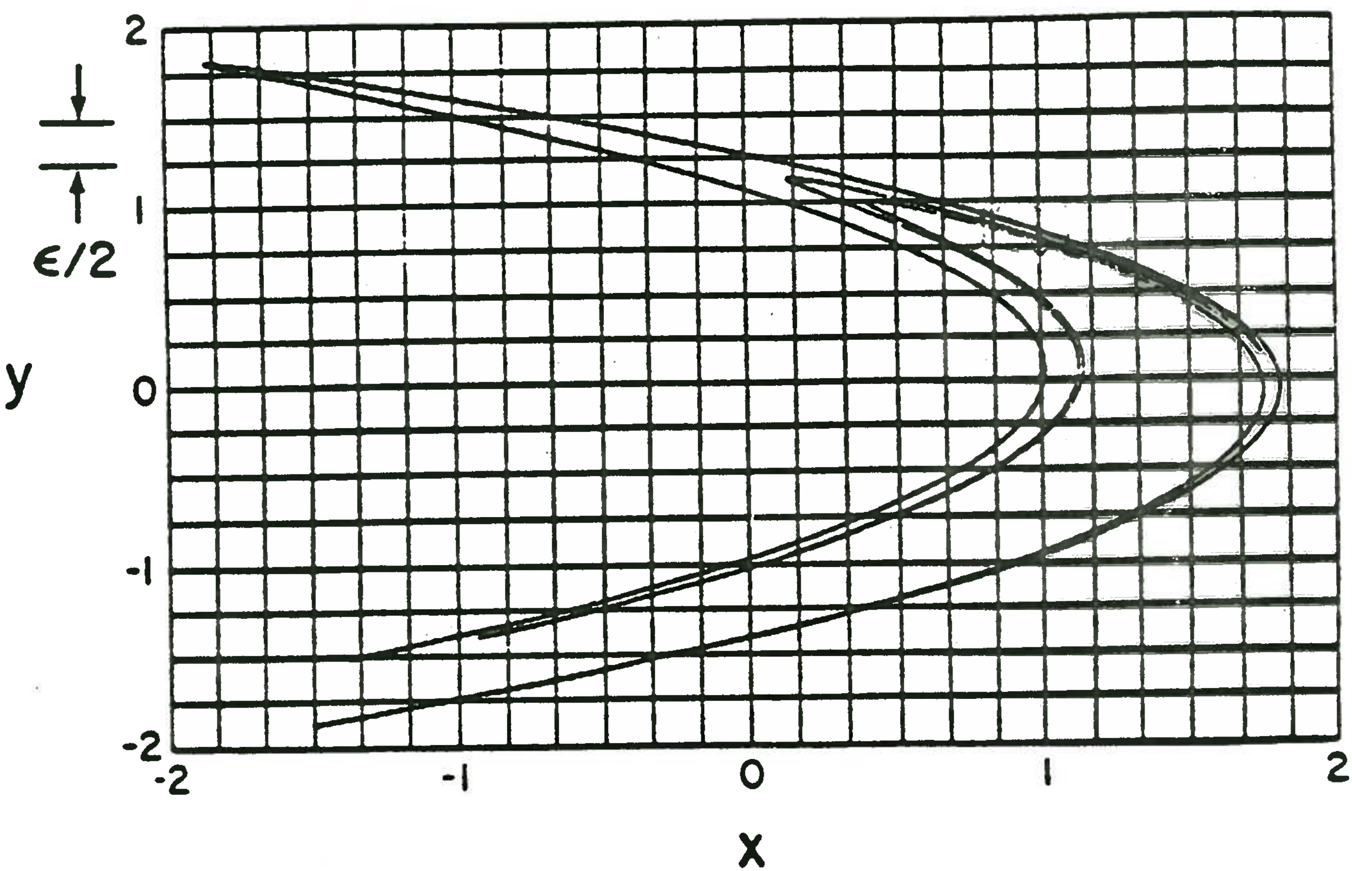
N(ϵ): NUMBER OF BOXES NEEDED TO COVER THE SET



* IN THIS CASE N(ϵ) = 40

WHAT HAPPENS IF ϵ IS REDUCED?

SAY, $\epsilon \rightarrow \frac{\epsilon}{2}$?



Now, $N(\frac{\epsilon}{2}) = 95$

THE HAUSDORFF DIMENSION

OR BOX-COUNTING DIMENSION

$$N(\epsilon) \sim \left(\frac{1}{\epsilon}\right)^D$$

or CAPACITY Dimension D_{cap}

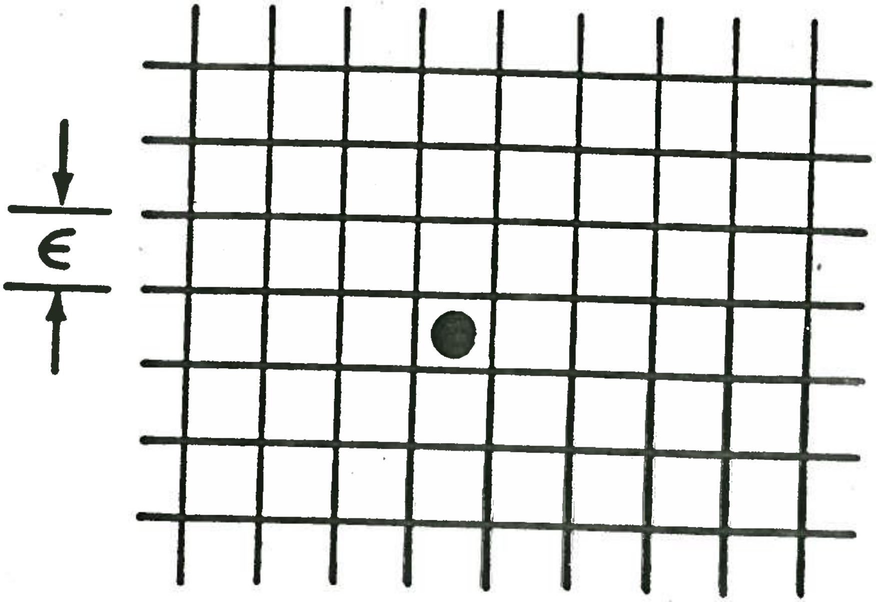
OR

$$D = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \frac{1}{\epsilon}} = 1.25$$

natural logarithm

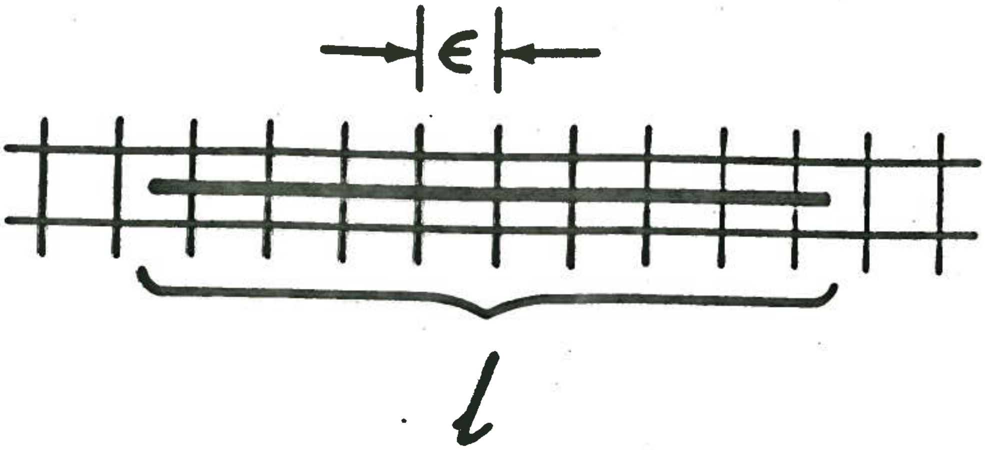
TYPICALLY, $D \neq$ INTEGER.

DIMENSION OF A POINT



$N(\epsilon) = 1 \Rightarrow D = 0$

DIMENSION OF A LINE

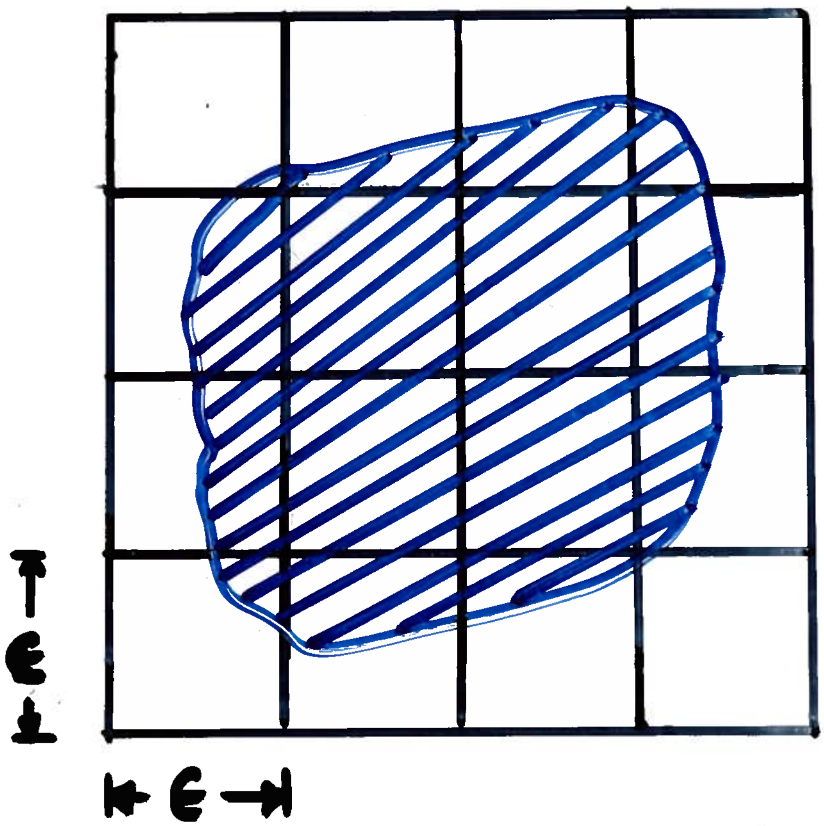


$$N(\epsilon) = \frac{\ell}{\epsilon}$$

$$D = \lim_{\epsilon \to 0} \frac{\ln \frac{\ell}{\epsilon}}{\ln \frac{1}{\epsilon}} = 1$$

$$D = \lim_{\epsilon \to 0} \left[\frac{\ln \ell - \ln \epsilon}{\ln 1 - \ln \epsilon} \right] = \lim_{\epsilon \to 0} \left[-\frac{\ln \ell}{\ln \epsilon} + 1 \right] = 1$$

DIMENSION OF A SURFACE OF AREA S



$$N(\epsilon) = \frac{S}{\epsilon^2}$$

$$D = \lim_{\epsilon \to 0} \frac{\ln S - 2 \ln \epsilon}{\ln 1 - \ln \epsilon} = \lim_{\epsilon \to 0} \left\{ -\frac{\ln S}{\ln \epsilon} + 2 \right\}$$

$$= 2$$

Correlation Dimension

- It is a **probabilistic** covering of $N(\epsilon)$ volume elements of diameter ϵ.
- Assume N_T points of a trajectory have been collected. Let n_i points lie in the ith volume element, then

$$P_i = \lim_{N_T \to \infty} \frac{n_i}{N_T} \; ; \quad i = 1, \dots, N(\epsilon)$$

$$\lim_{\epsilon \to 0} \frac{\ln \sum_{i=1}^{N(\epsilon)} P_i^2}{\ln \epsilon} = D_c$$

♫ $\sum_{i=1}^{N(\epsilon)} P_i^2 = \frac{1}{N_T^2} \{$ the no. of pairs (n_i, n_j) such that $\|n_i - n_j\| < \epsilon \}$ ="coverage"

♫ $D_c \leq D_{cap}$

The Embedding Theorem

Takens [Takens, 1981] proved that one can reconstruct the attractor from a time series of a single component. To do this, the phase space may be reconstructed with the embedding vectors [Packard et al., 1980]

$$\boxed{\vec{x}(t_i) = [x(t_i), x(t_i + l\Delta t), \ldots, x(t_i + (d-1)l\Delta t)]}$$

where

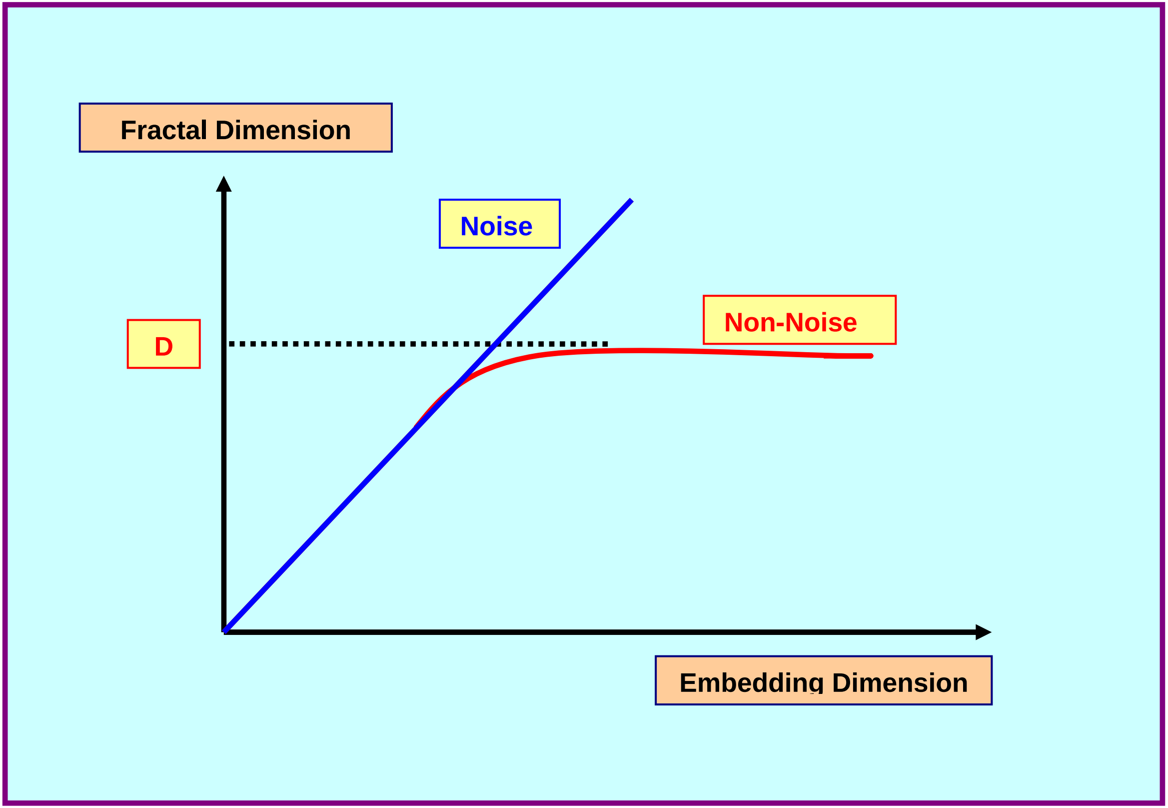
d denotes the embedding dimension,
$\triangle$t is the sample time,
l is an appropriate integer, and
D is the fractal dimension of the compact manifold containing the attractor

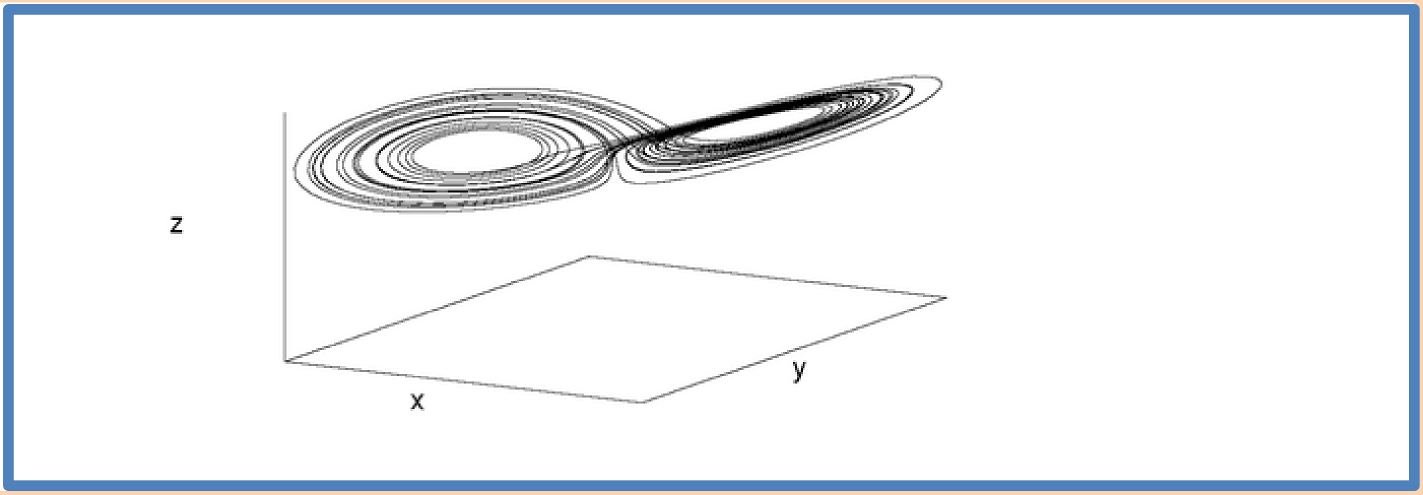
The embedding theorem states that there generically exists an embedding between the reconstructed and original phase space if

$$\boxed{d \geq 2D + 1}$$

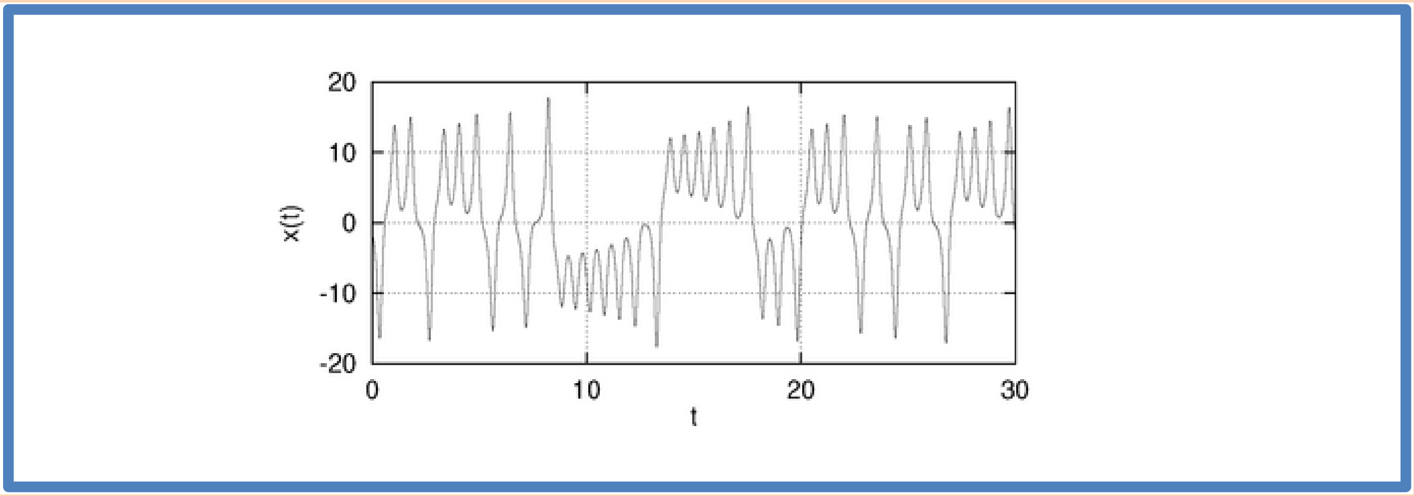
This implies that the dimension and entropy spectra of the reconstructed attractor are the same as those of the ``real'' attractor, since the spectra are invariant under a smooth coordinate change However, it is assumed that there are an infinite number of infinitely precise measurements of the single component.

For infinitely long time series, the sample time and the integer can be chosen almost arbitrarily. In practice however, choosing the optimal length of the time series and the optimal time delay is a difficult problem

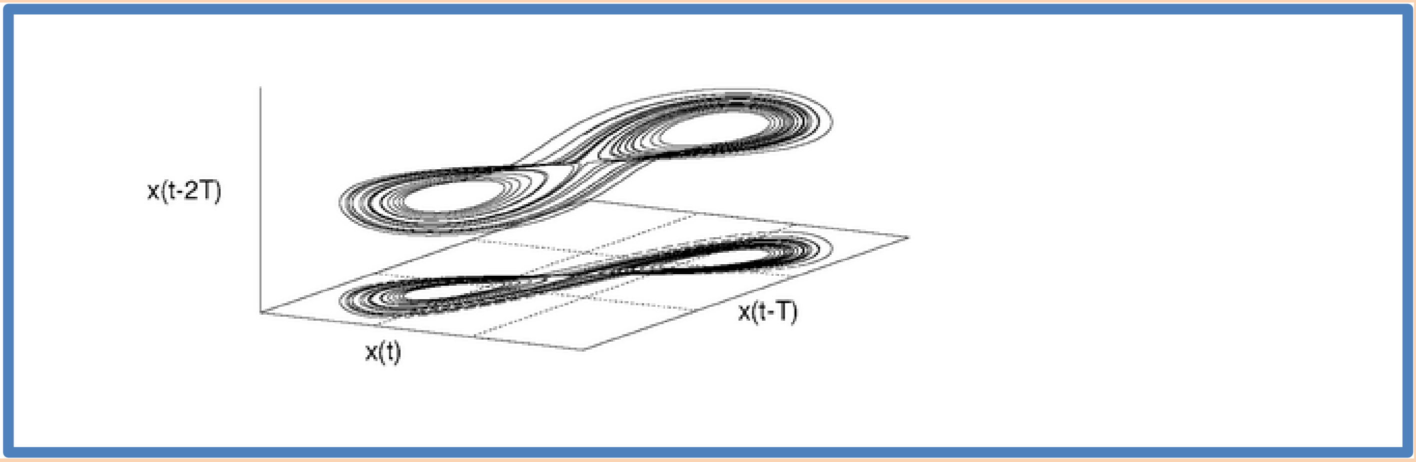

Fractal Dimension
Noise
D
Non-Noise
Embedding Dimension



The Lorenz attractor



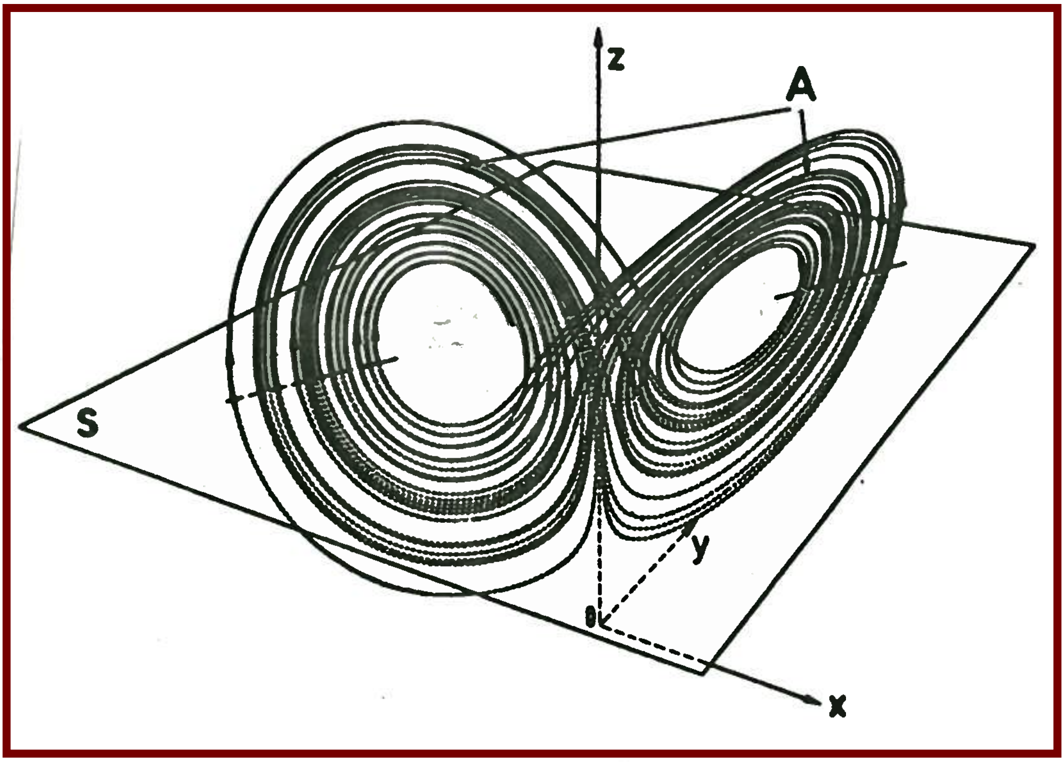
Times series formed by x coordinate

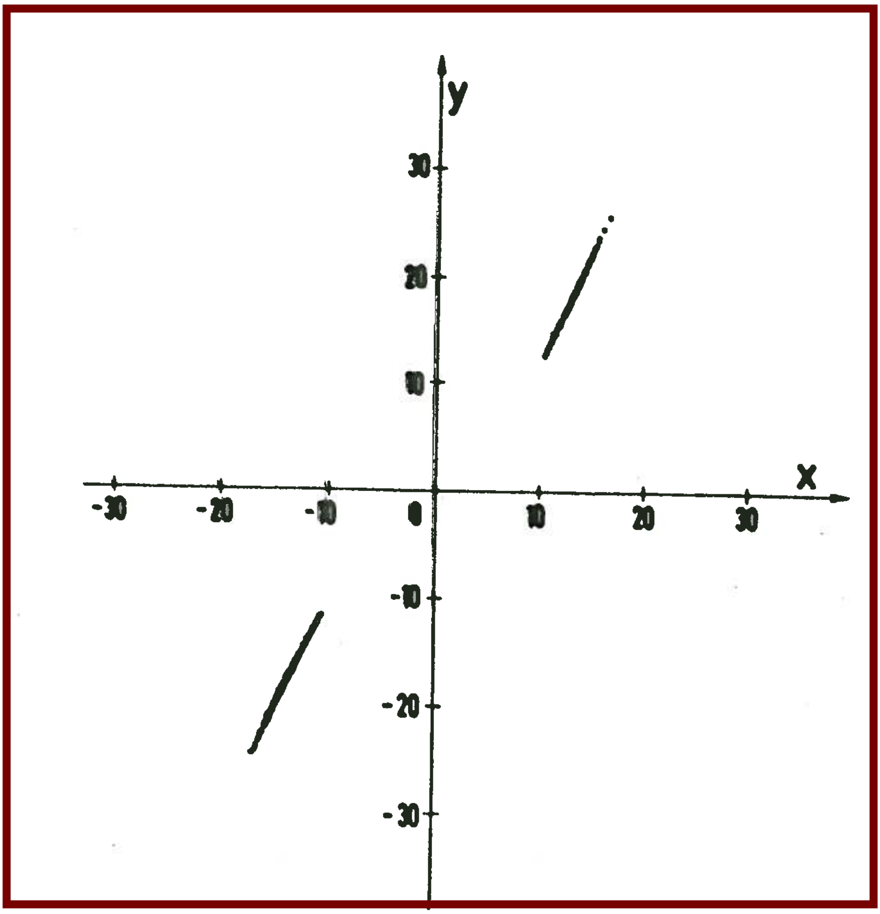


Reconstructed attractor

The figure shows a reconstruction of the fractal attractor for the well-known Lorenz system, whose fractal dimension is slightly larger than 2. The time series shown consists of the *x* coordinate of the system traced as a function of time. In the third image, triples of time series values [$x(t)$, $x(t-\tau)$, $x(t-2\tau)$] are plotted. The topological structure of the Lorenz attractor is preserved by the reconstruction.

Poincaré Section





Poincaré Maps

- In a Poincaré section, the transformation leading from one point to the next is a continuous mapping T of S into itself. Such a transformation is called a Poincaré map.

$$P_{k+1} = T(P_k)$$

$$= T(T(P_{k-1})) = T^2(P_{k-1})$$

$$= \ldots\ldots$$

- The Poincaré section and map have the same kind of topolgical properties as the flow from which they arise.

- The Poincaré section can practically be considered to be a set of points distributed along a curve.

In this case, one defines a coordinate x for each point on the curve, and studies how x varies with time.

The Poincaré map on this one-dimensional graph is called a first return map.

$$\boxed{x_{k+1} = f(x_k)}$$

* The intersection of the first return map with the identity map

$$\boxed{x_{k+1} = x_k}$$

gives the fixed points of the map.

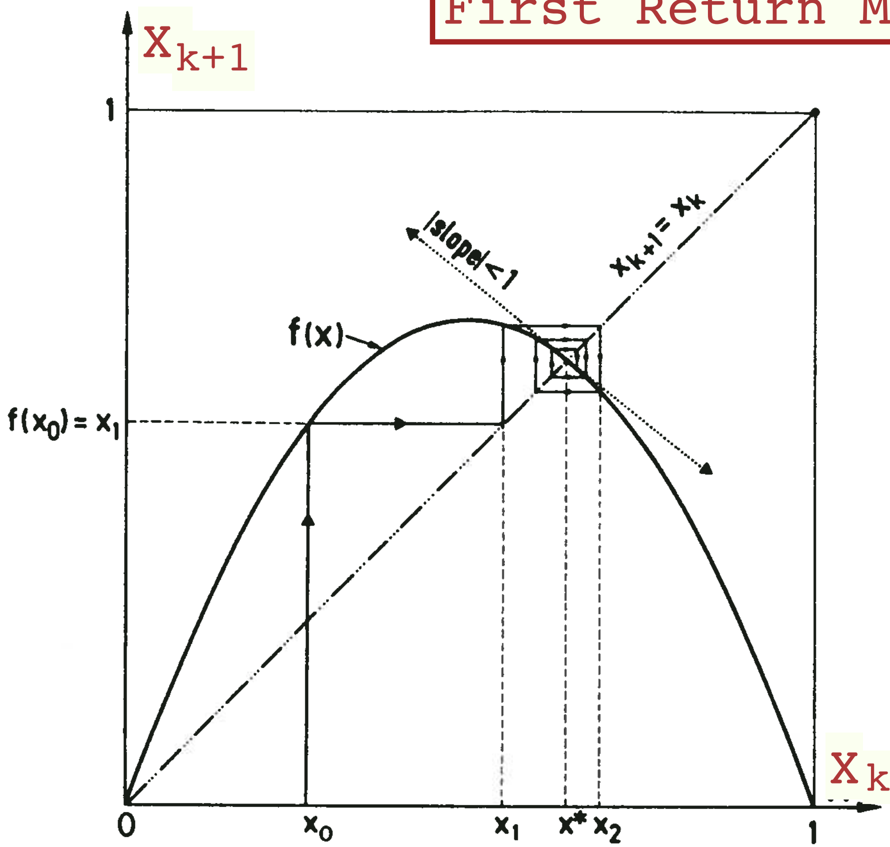
* If the slope of f at the fixed point is of absolute value less than one, the the fixed point is stable, otherwise it is unstable.

Logistic Equation

$$x_{k+1} = \boxed{2.8}\, x_k (1 - x_k)$$

$$x \in [0, 1]$$

First Return Map



Fixed Points
origin & x^*

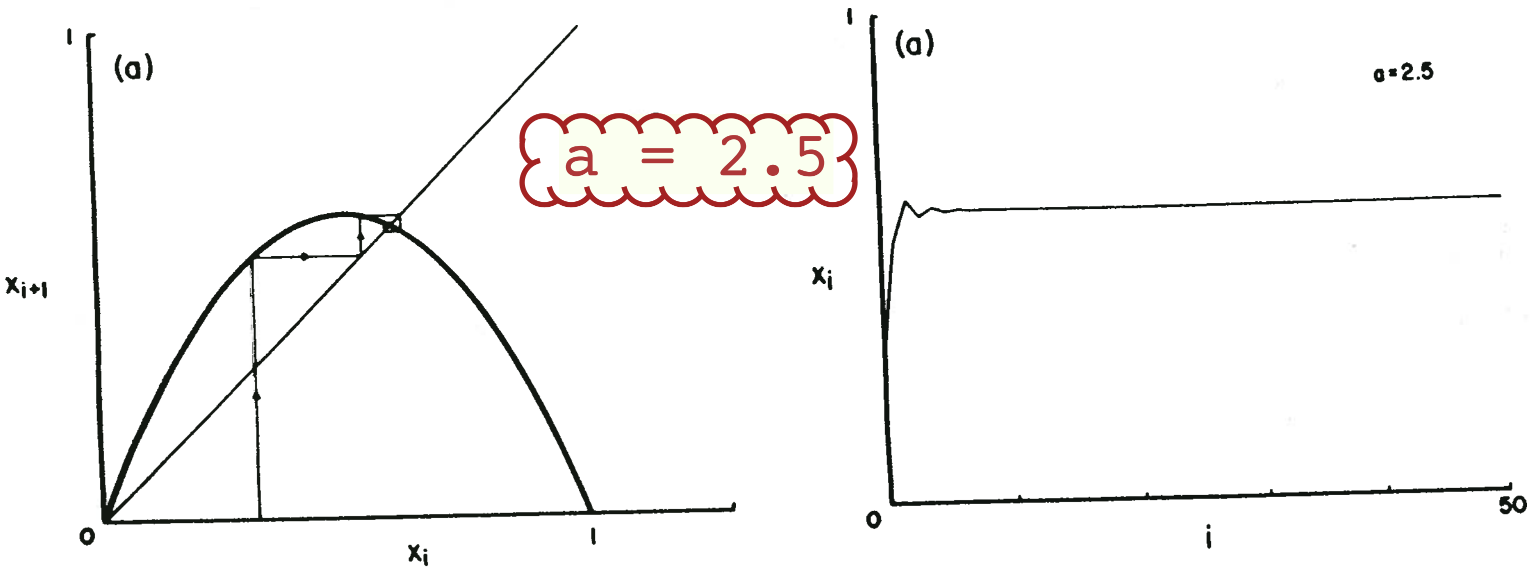
Origin is unstable |slope| > 1

x^* is stable |slope| < 1

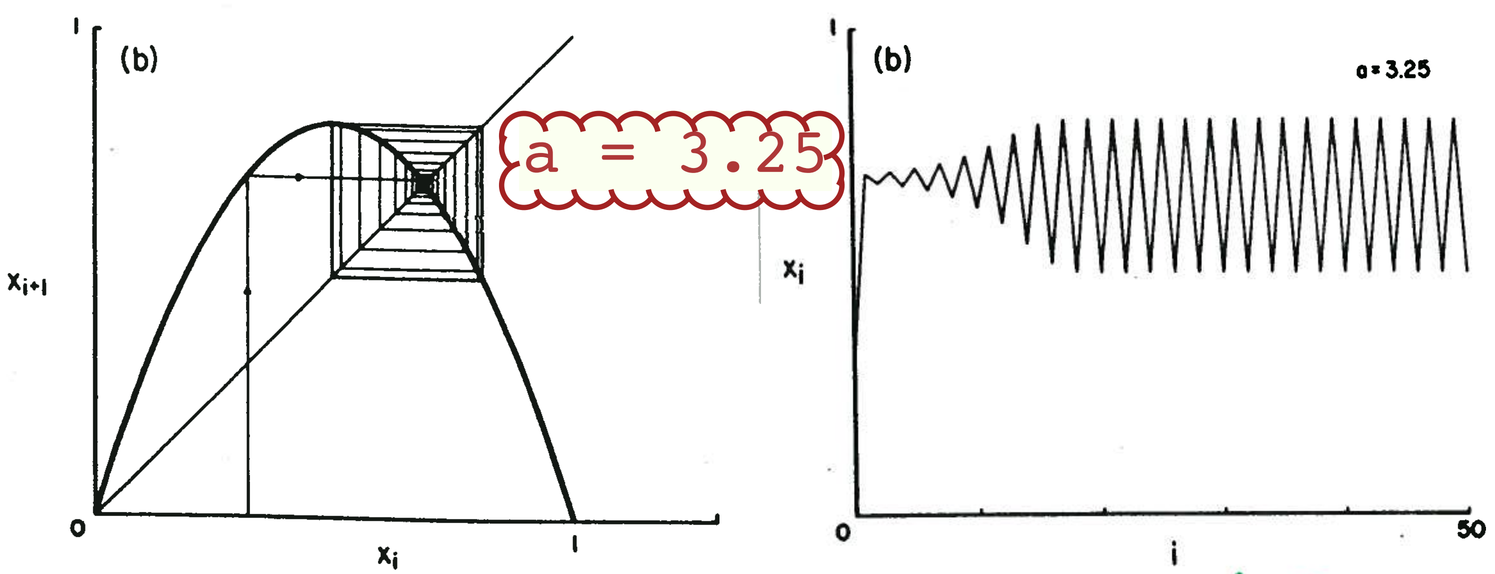
$x_i \triangleq$ value at iteration i

$$x_{i+1} = a \, x_i (1 - x_i)$$

logistic equation

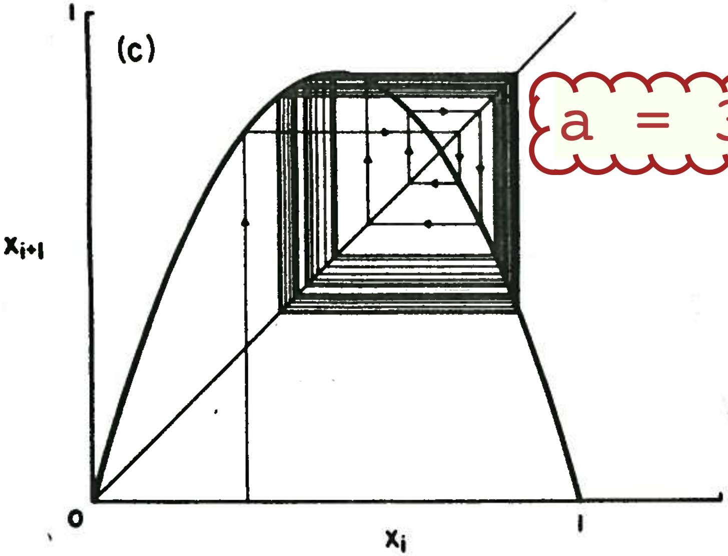


First Return Map

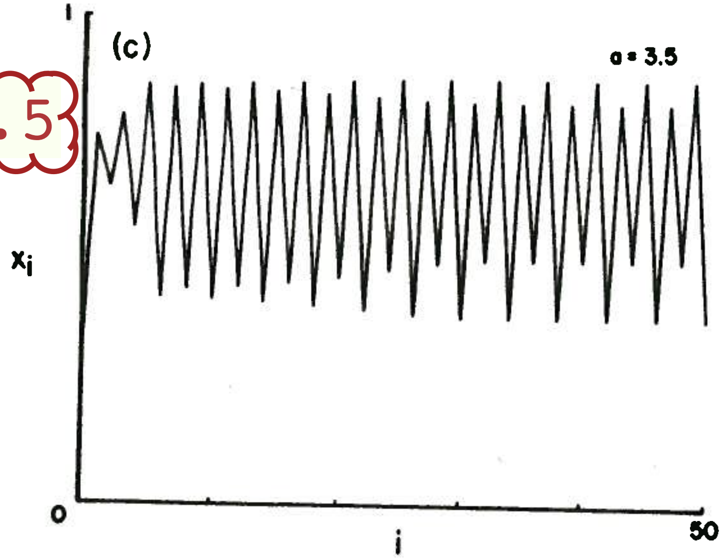


period 2

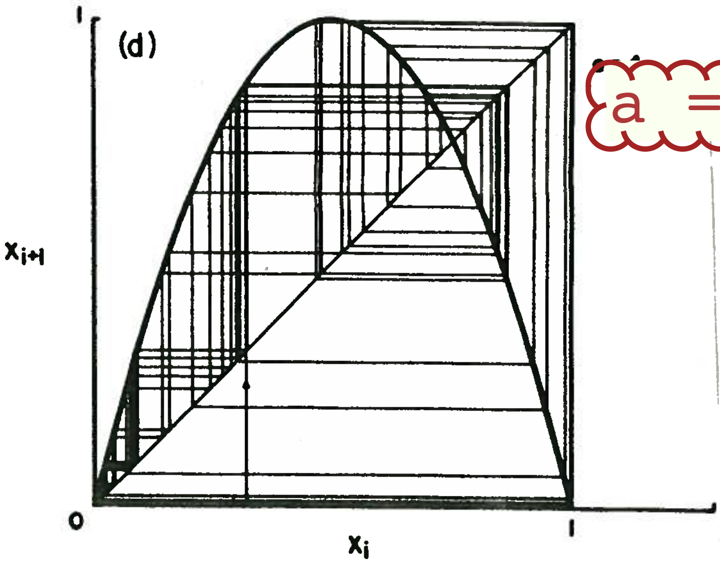
period 4



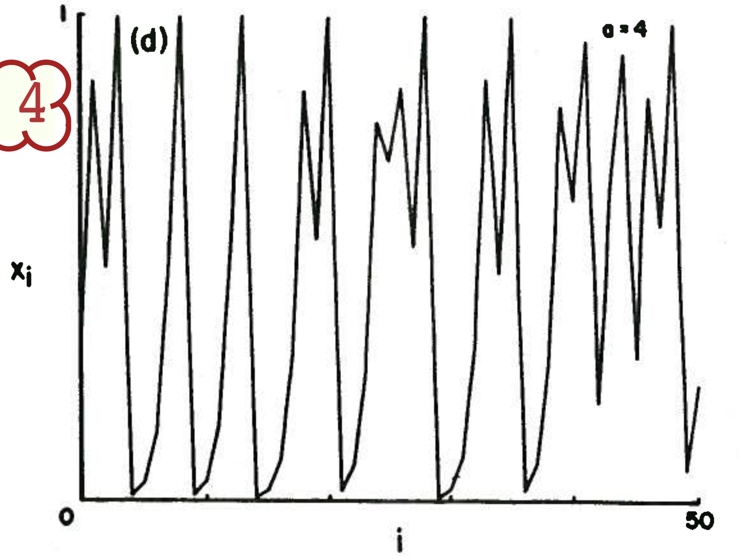
a = 3.5



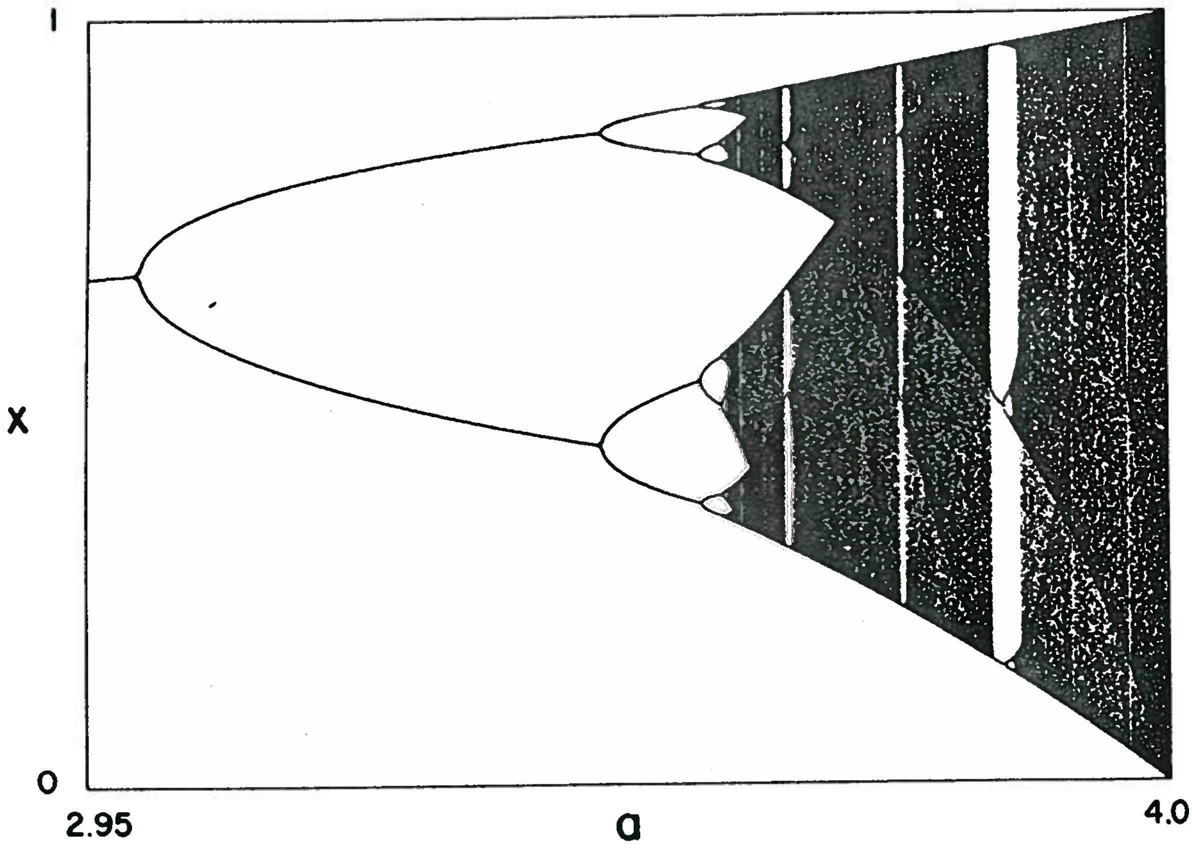
chaos



a = 4

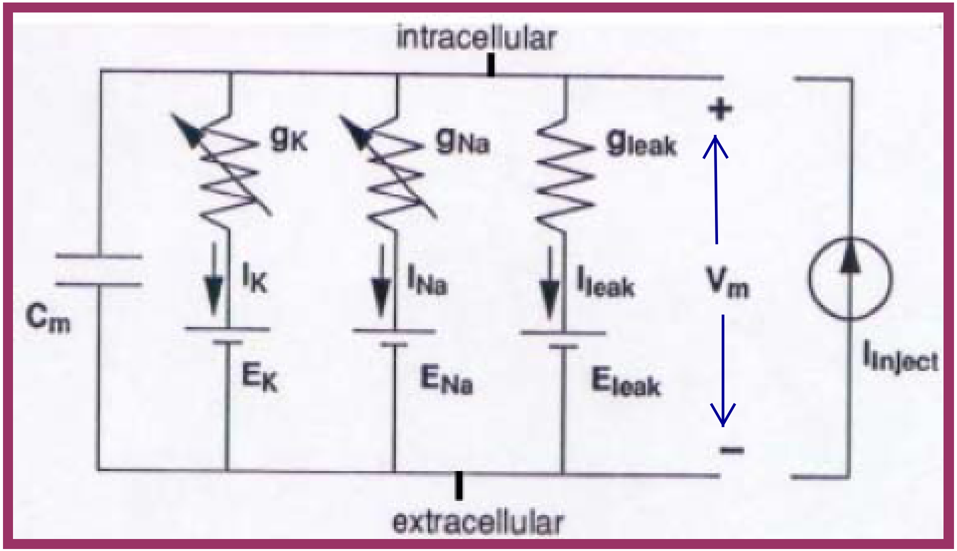


The Bifurcation Diagram



- Steady state results are plotted as x against a
- A Feigenbaum cascade of bifurcations leads to chaos
- All cascades have smaller cascades within them
- The complex fractal pattern shrinks indefinitely

Parallel Conductance Model for a Patch of Cellular Membrane Containing Voltage-Gated Ionic Channels When Stimulated by a Current Injection



Fourth Order Systems

The Hodgkin - Huxley Model

$$I_{inject} = c_m \frac{dV_m}{dt} + g_{Na}[V_m - E_{Na}] + g_K[V_m - E_K] + g_{leak}[V_m - E_{leak}]$$

$$g_{Na} = \bar{g}_{Na}\, m^3 h$$

$$g_K = \bar{g}_K\, n^4$$

resting conductance

resting membrane voltage

$$g_{leak} = \bar{g}_{leak} \quad ; \quad E_{leak} = V_{resting}$$

$$\frac{dm}{dt} = \alpha_m(1-m) - \beta_m m$$

$$\frac{dn}{dt} = \alpha_n(1-n) - \beta_n n$$

$$\frac{dh}{dt} = \alpha_h(1-h) - \beta_h h$$

The four state variables are V_m, n, m, h

where ↑ (for squid axons)

the rate variables

$$\alpha_m(v_m) = \frac{0.1\,[25 - v_m]}{\exp\left[\frac{25 - v_m}{10}\right] - 1}$$

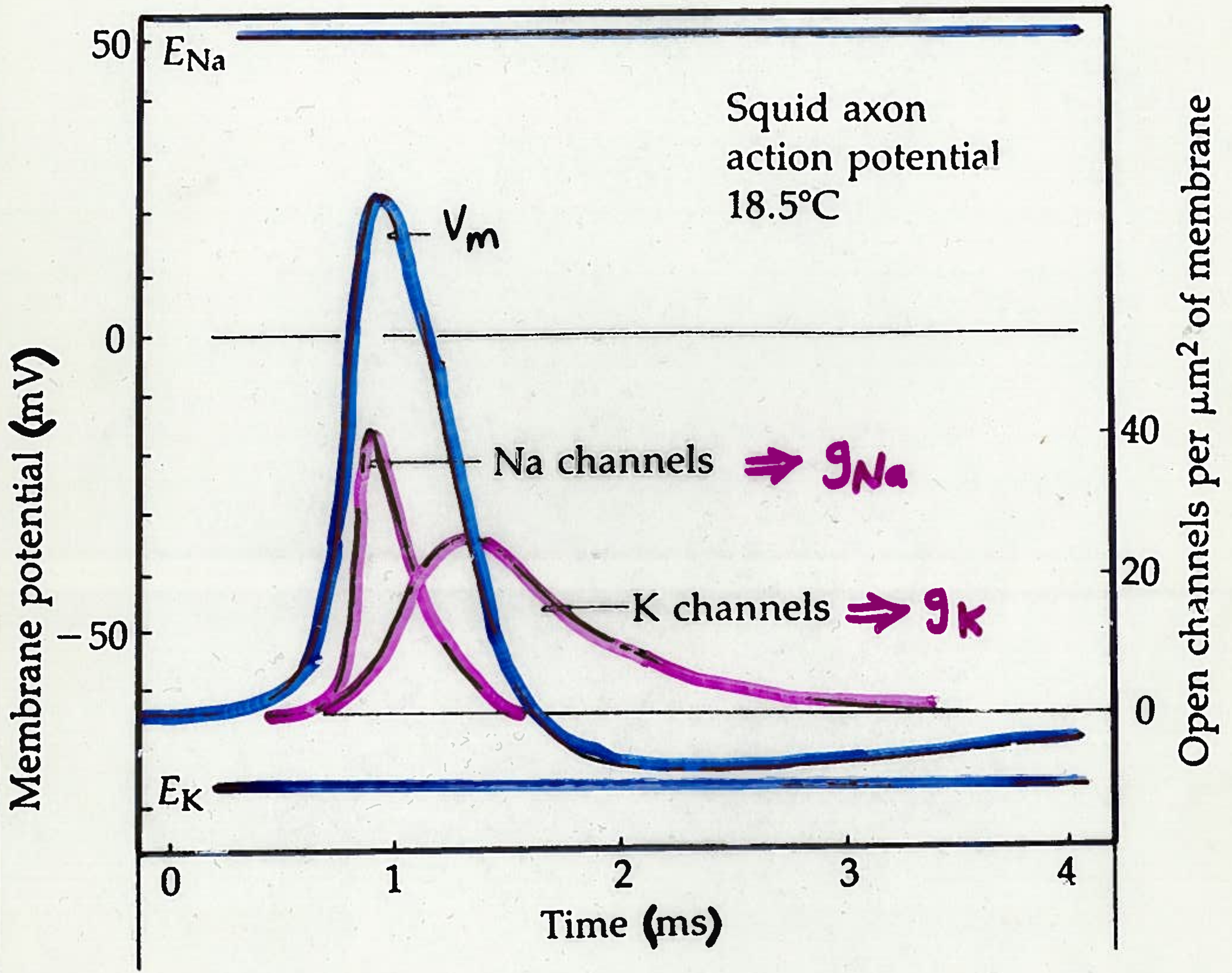
$$\beta_m(v_m) = 4\,\exp\left[-\frac{v_m}{18}\right]$$

$$\alpha_n(v_m) = \frac{0.01\,[10 - v_m]}{\exp\left[\frac{10 - v_m}{10}\right] - 1}$$

$$\beta_n(v_m) = 0.125\,\exp\left[-\frac{v_m}{80}\right]$$

$$\alpha_h(v_m) = 0.07\,\exp\left[-\frac{v_m}{20}\right]$$

$$\beta_h(v_m) = \frac{1}{\exp\left[\frac{30 - v_m}{10}\right] + 1}$$

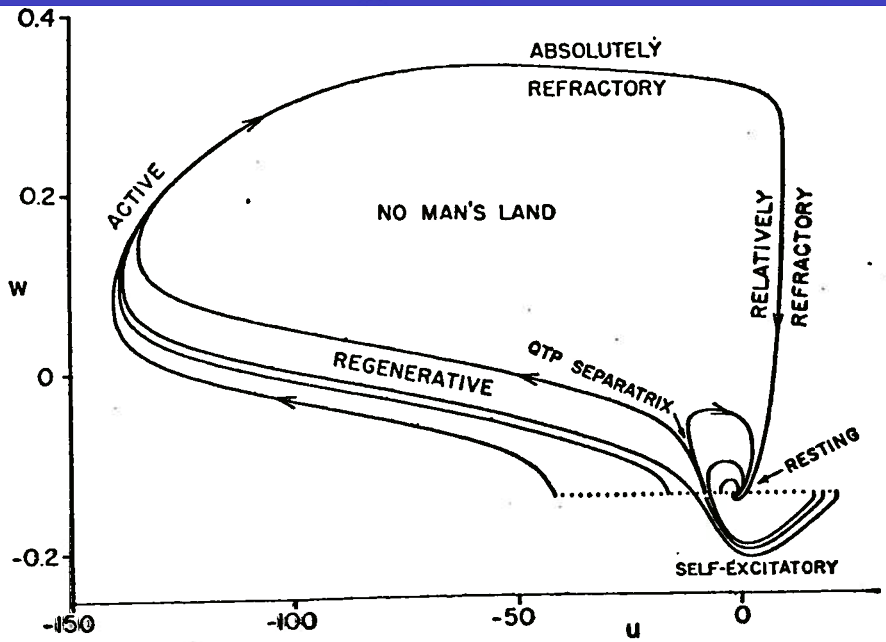


50
0
−50
E_{Na}
E_K
Membrane potential (mV)
Squid axon
action potential
18.5°C
V_m
Na channels ⇒ g_{Na}
K channels ⇒ g_K
40
20
0
Open channels per μm² of membrane
0
1
2
3
4
Time (ms)

Let $u = V - 36m$

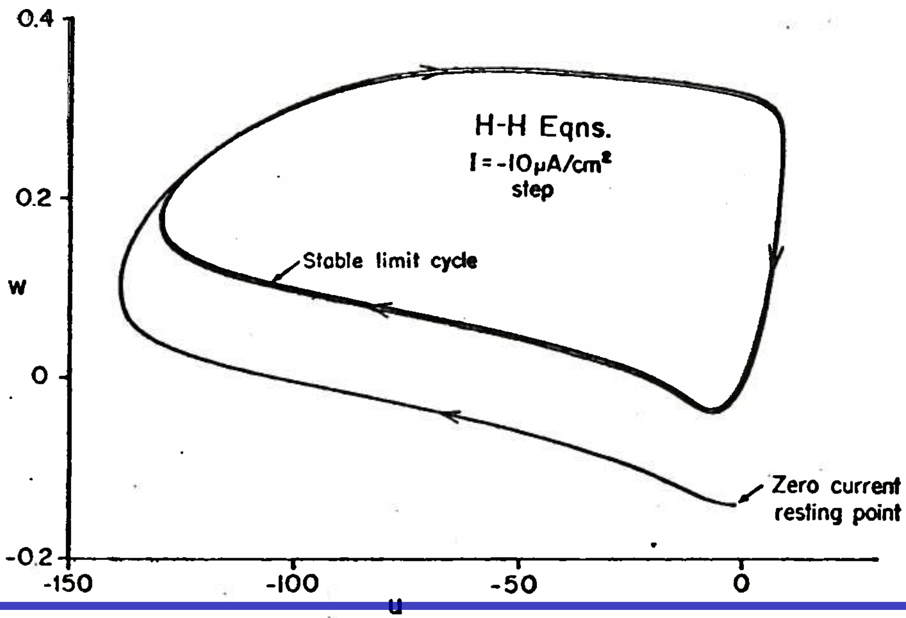
$w = \frac{1}{2}[n - h]$

$I = 0$



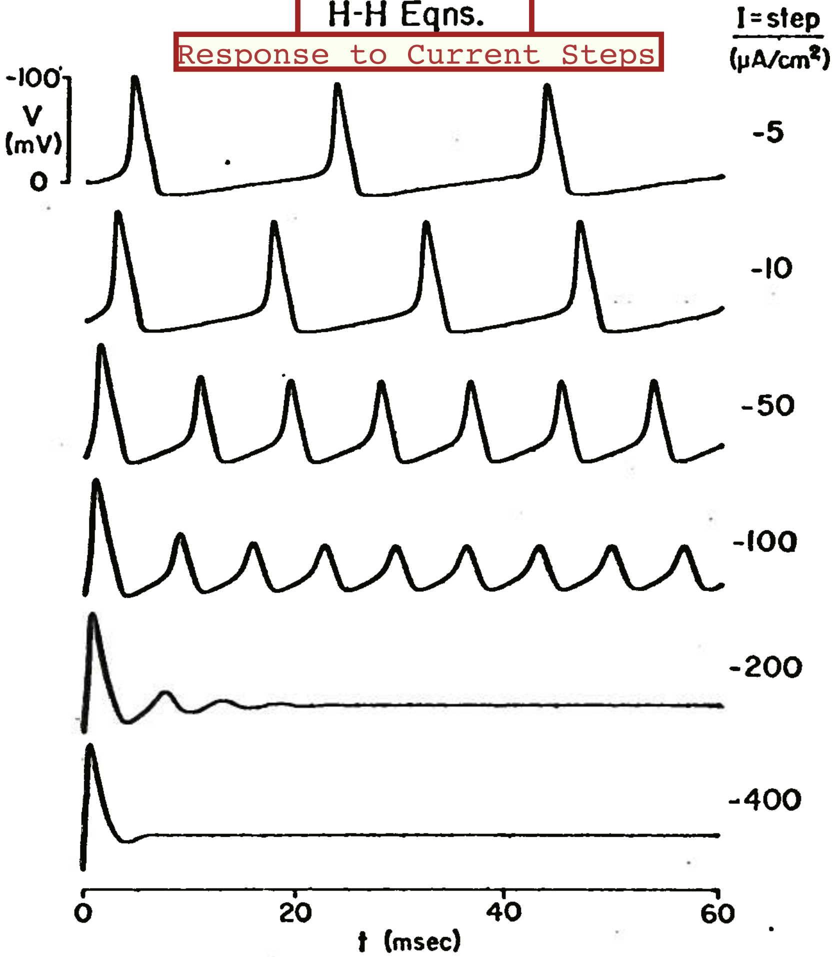
u-w plane portraits of Hodgkin-Huxley are similar to Fitzhugh-Nagumo state portraits

$I = -10\ \mu A/cm^2$

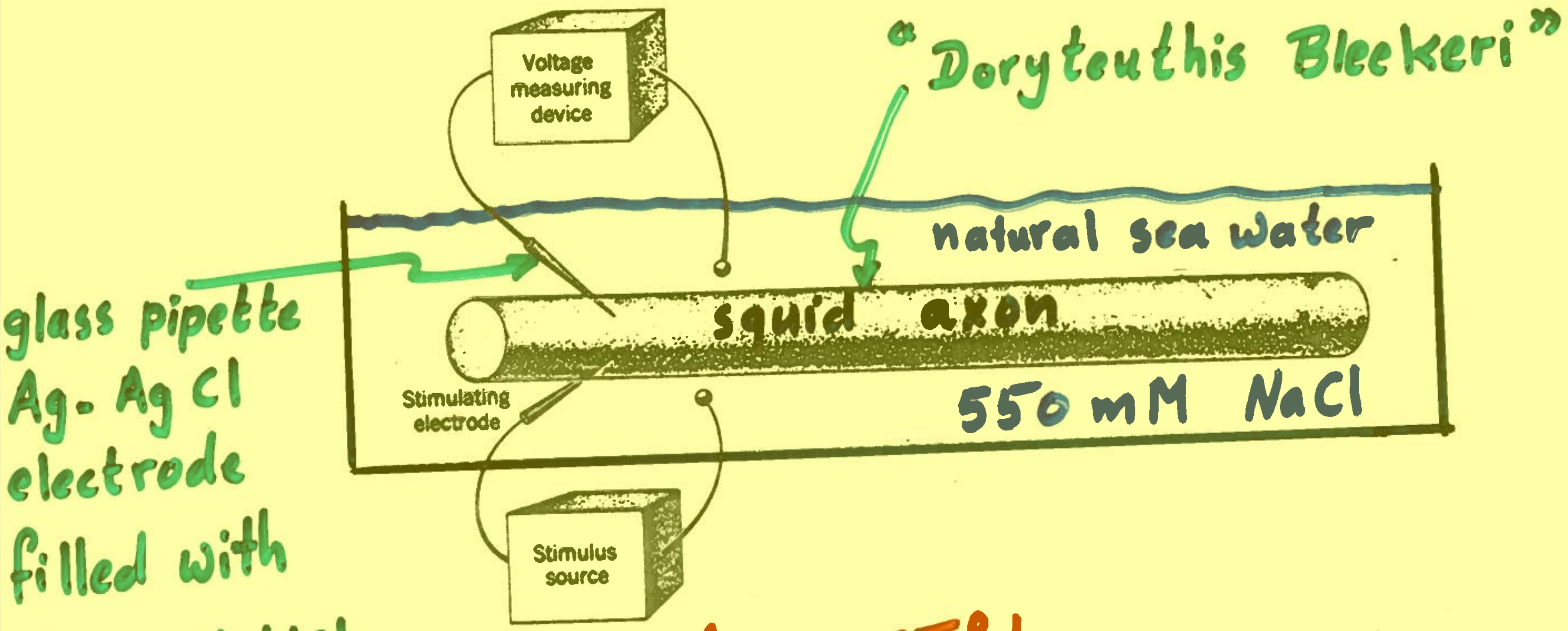


H-H Eqns.

Response to Current Steps



Experimental set up

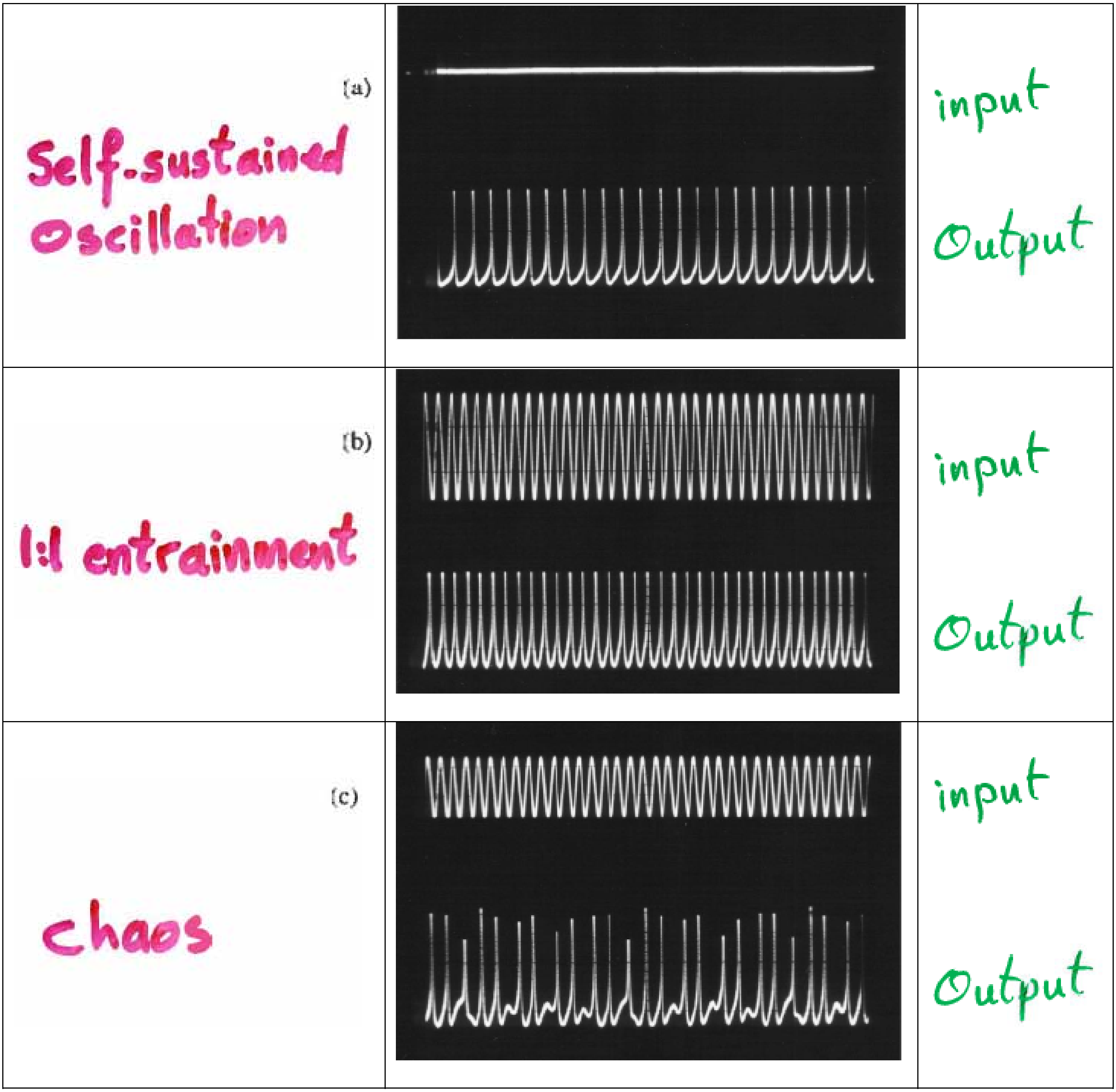


$$I_s = A_s \sin 2\pi f_s t$$

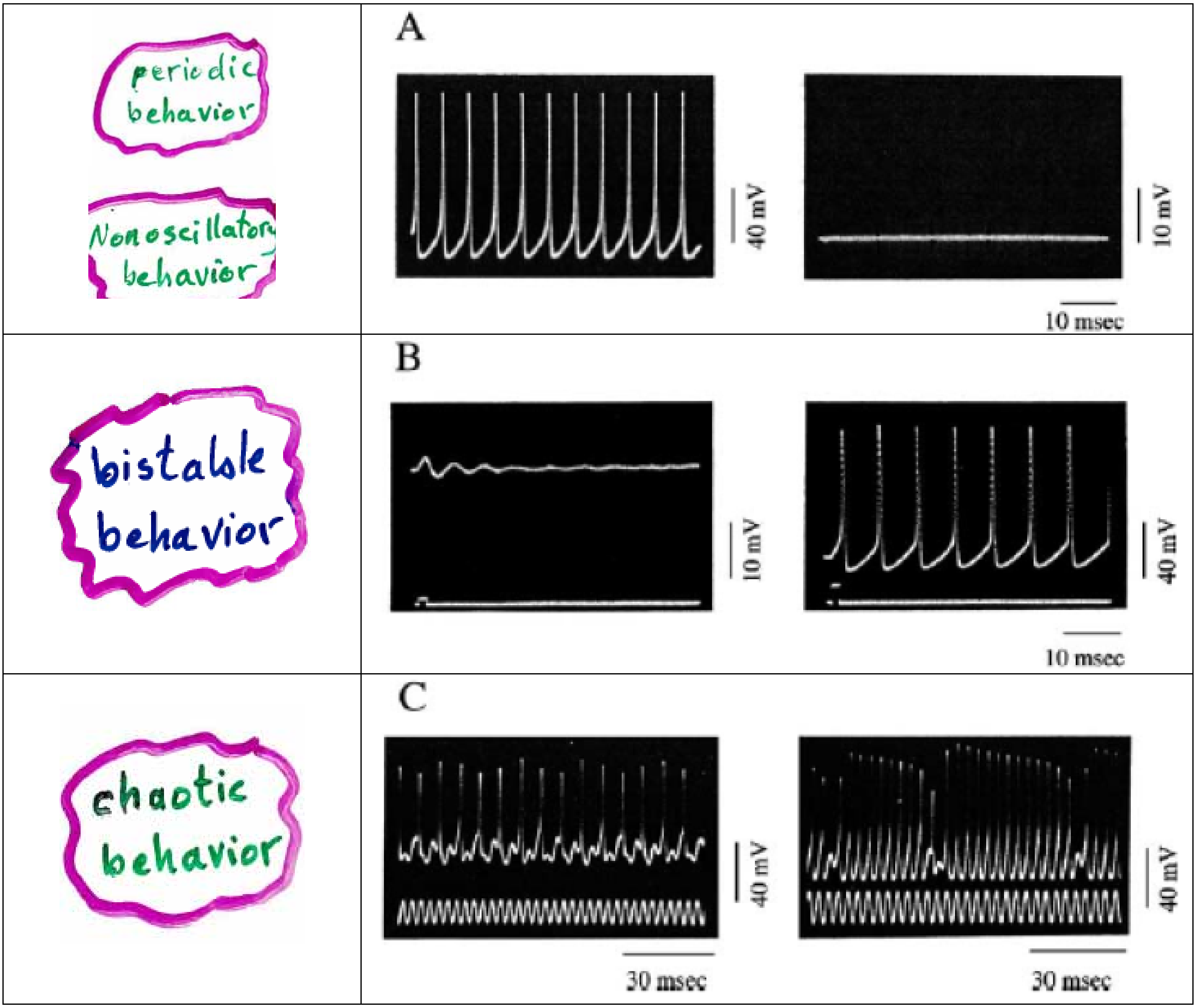
[1] K. Aihara, G. Matsumoto (1982), Temporally coherent organization and instabilities in squid giant axons, J. Theor. Biol., 95: 697 -720.

[2] K. Aihara, G. Matsumoto, Y. Ikegaya (1984), Periodic and nonperiodic responses of a periodically forced Hodgkin -Huxley oscillator, J. Theor. Biol., 109: 249 -269.

[3] K. Aihara, G. Matsumoto, Chaotic oscillations and bifurcations in squid giant axons (1986), in: A.V. Holden (Ed.), Chaos, University Press, Princeton,NJ, pp. 257 -269.

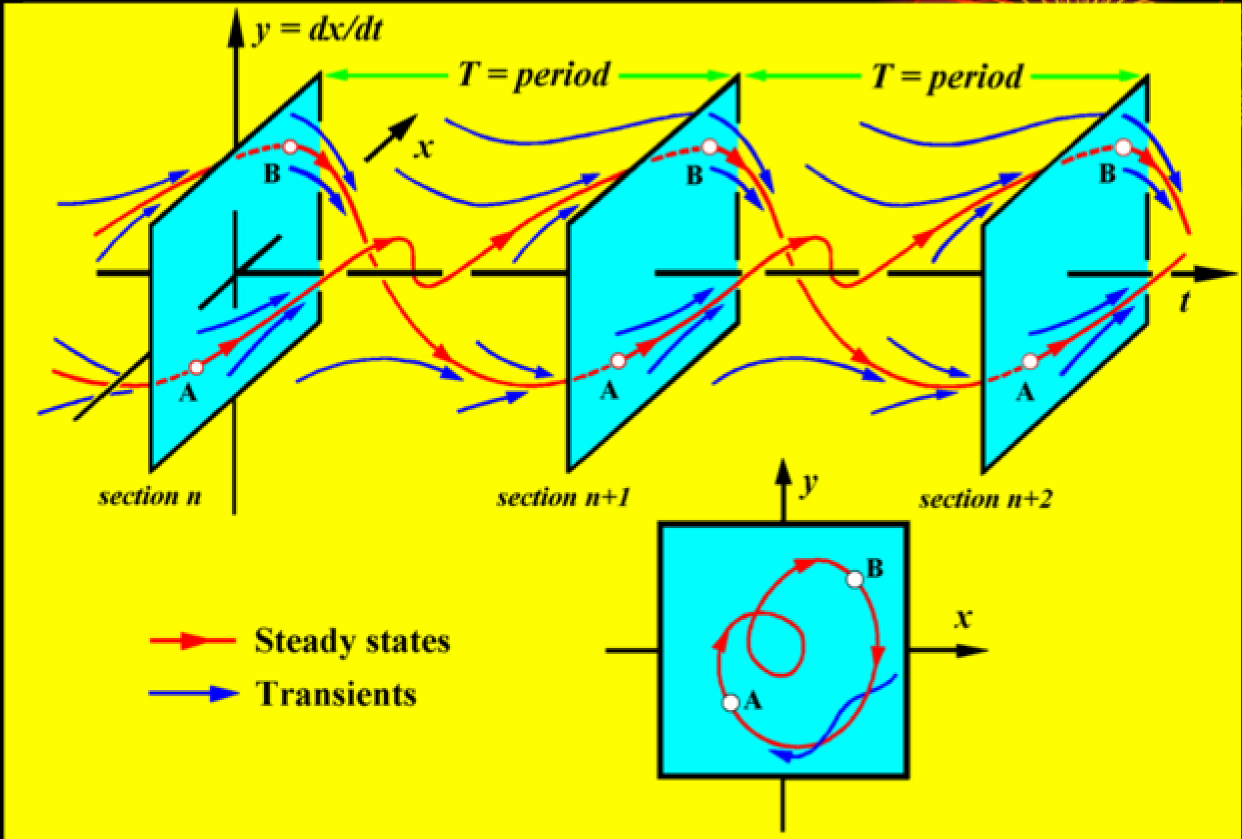


Examples of electrophysiological results with squid giant axons. (a) A self-sustained oscillation with repetitive firing of action potentials, (b) a 1/1-synchronized forced oscillation, and (c) a chaotic forced oscillation. In each figure, the upper and lower wave forms show the stimulating current and the membrane potential, respectively. Giant axon of squid (*Doryteuthis bleekeri*) were used in the experiment. The self-sustained oscillation in (a) was induced by bating the axon in a mixture of natural sea water and 550-mM NaCl The forced oscillations in (b) and (c) were produced by stimulating the self-sustained neural oscillator with sinusoidal currents through an internally platinized platinum wire electrode



Periodic and non-periodic behavior of a squid giant axon. (**A**) Periodic oscillations (left) and membrane potential at rest (right) after exposure of the preparation for 0.25 and 6.25 min to an external solution containing the equivalent of 530 and 550 mM NaCl, respectively. (**B**) Bistable behavior of an axon placed in a 1/3.5 mixture of NSW 550 mM NaCl. Note the switch from subliminal (left) to supraliminal (right) self-sustained oscillations, produced by a stimulating pulse of increasing intensity. (**C**) Chaotic oscillations in response to sinusoidal currents. The values of the natural oscillating frequency and the stimulating frequency (f_n) were 136 and 328 Hz (left) and 228 and 303 Hz (right). In each panel, the upper and lower traces represent the membrane potential and the activating current, respectively.

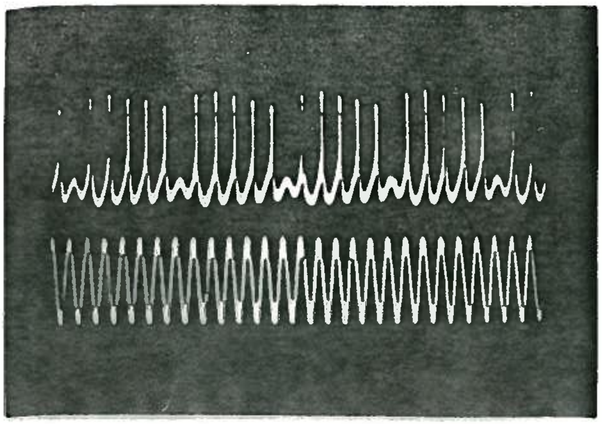
Stroboscopic Section



$\left(\frac{dV}{dt} \text{ vs } V\right) \Longleftrightarrow \left(V_{k+1} \text{ vs } V_k\right)$

$f_i = 181$ Hz

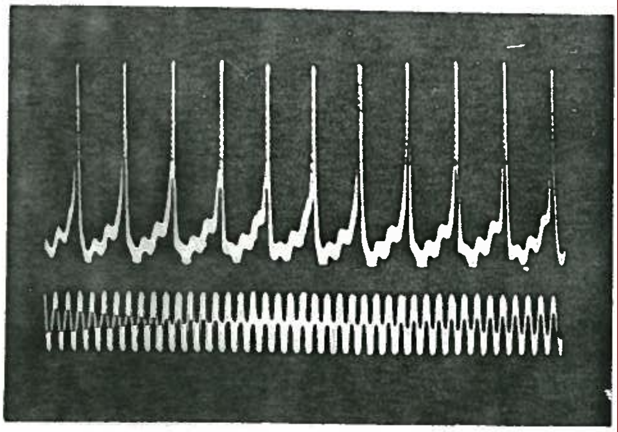
265 Hz
3.27 μA



120 mV
12 μA
30 ms

$f_i = 187$ Hz

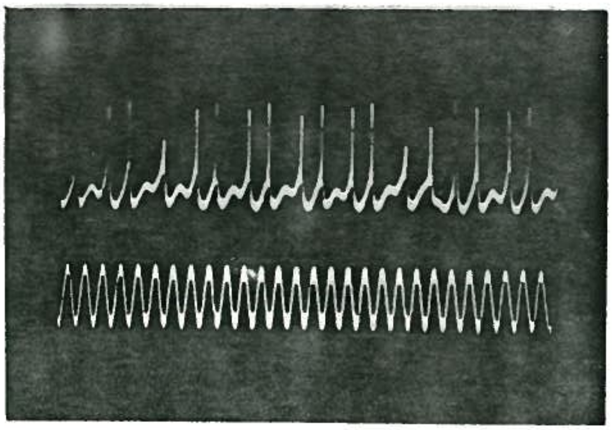
800 Hz
2 μA



60 mV
12 μA
15 ms

$f_i = 179$ Hz

270 Hz
2.37 μA

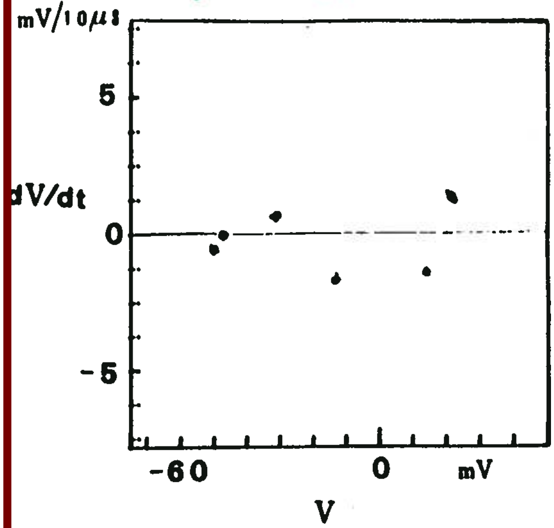


120 mV
12 μA
30 ms

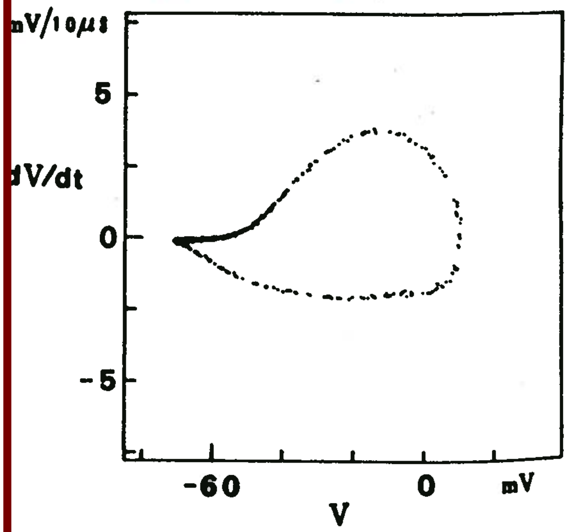
action potentials

stimulus cycles

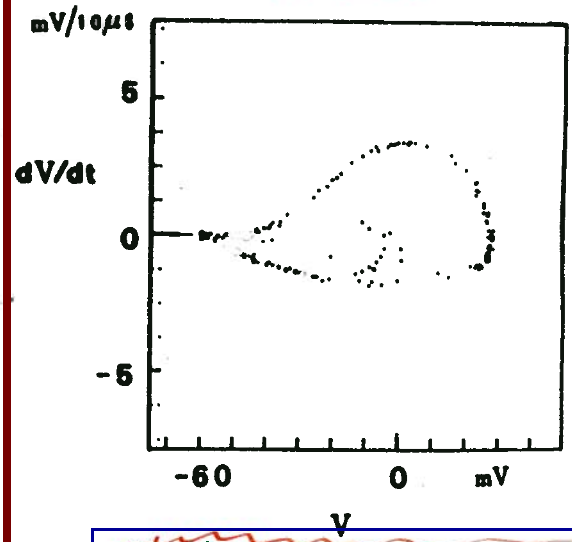
5/6 synchronized



quasi-periodic



chaotic



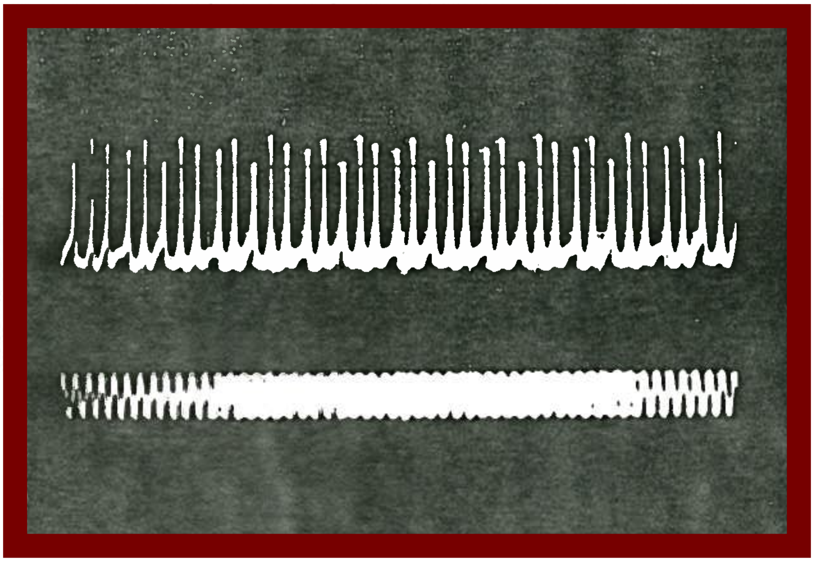
Stroboscopic Plots

Period Doubling Bifurcation

$f_i = 177$ Hz

5/7 sync.

(a)

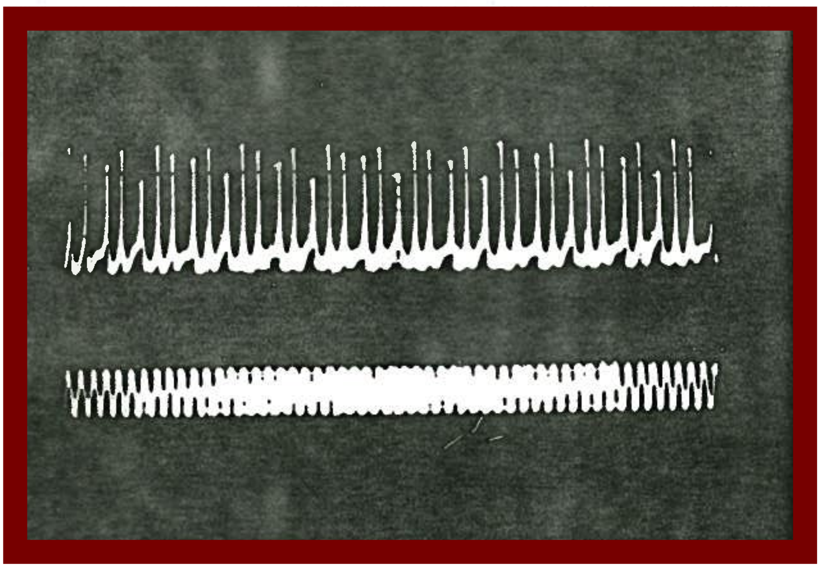


$f_s = 250$ Hz

$A_s = 1.19\ \mu A$

10/14 sync.

(b)

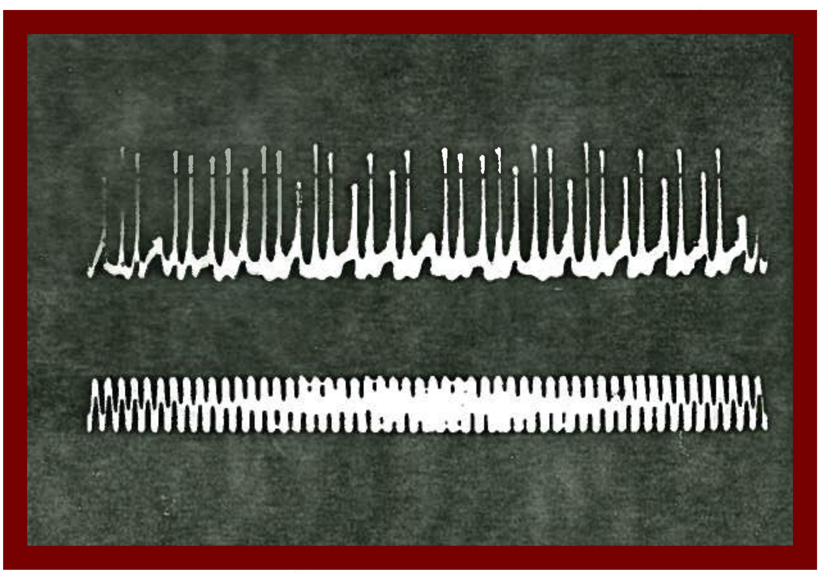


$A_s = 1.33\ \mu A$

120 mV
12 μA
60 ms

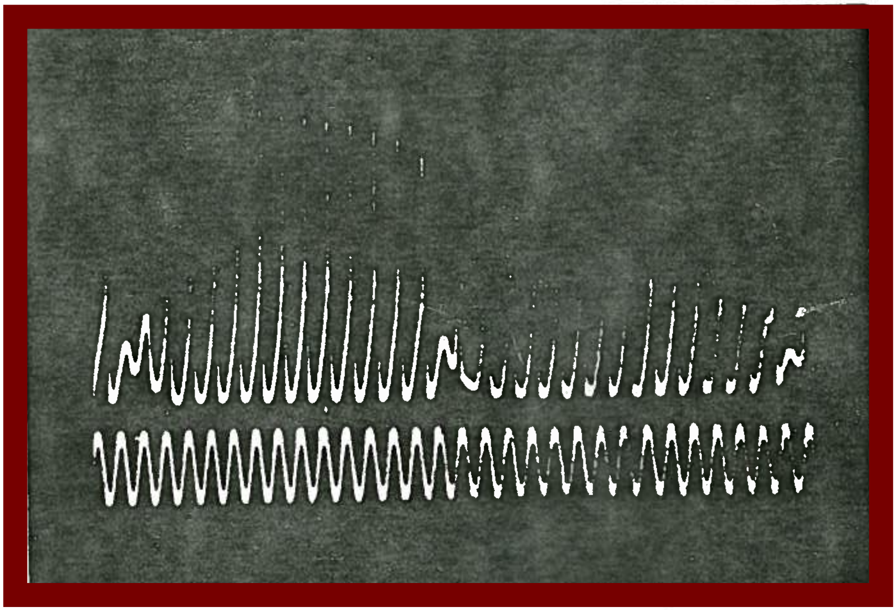
chaotic

(c)



$A_s = 1.63\ \mu A$

Intermittency



$f_i = 228$ Hz

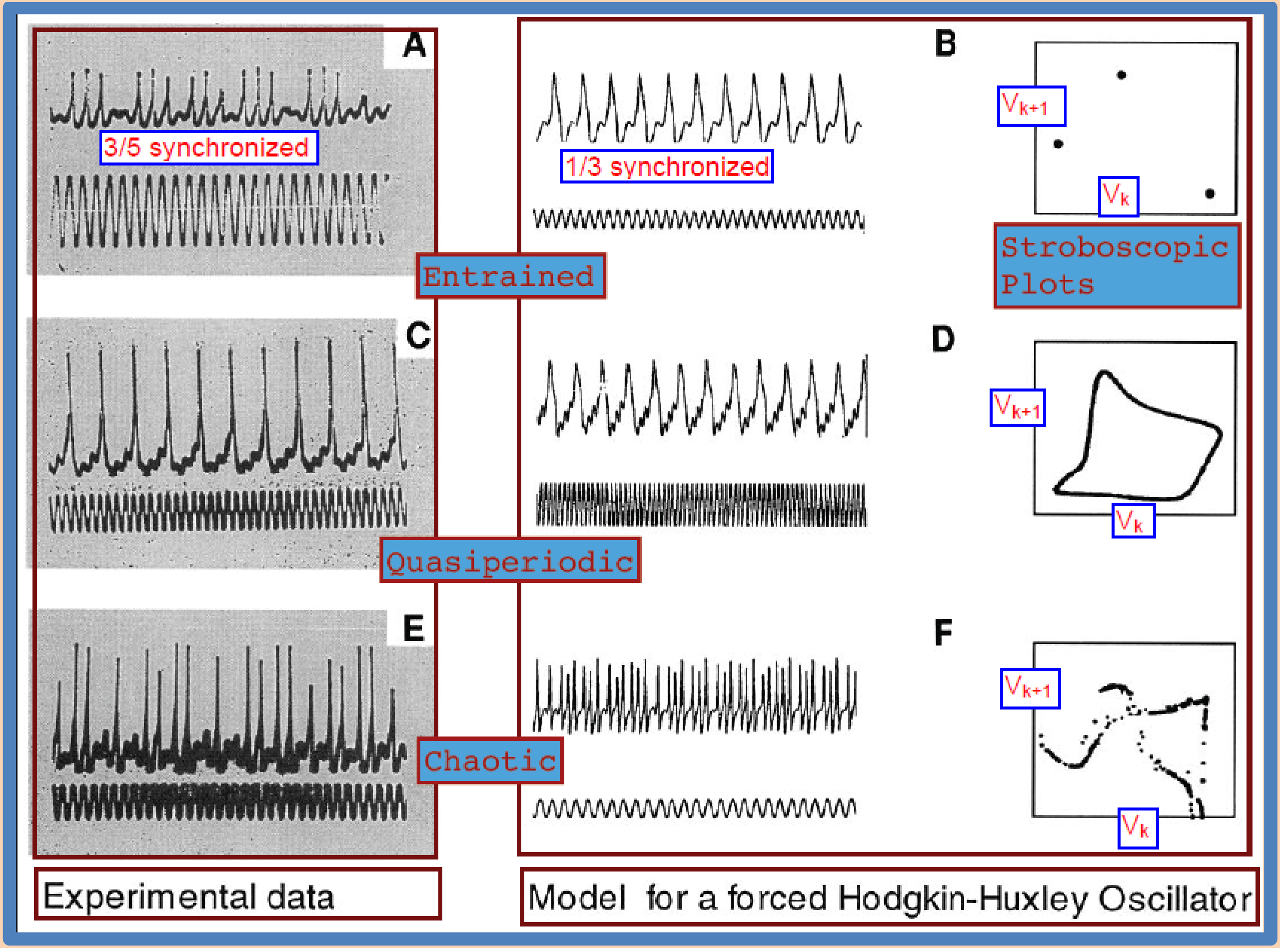
$f_s = 303$ Hz

$A_s = 2\ \mu A$

60 mV

12 μA

30 ms


A
3/5 synchronized
Entrained
C
Quasiperiodic
E
Chaotic
B
1/3 synchronized
V_{k+1}
V_k
Stroboscopic Plots
D
V_{k+1}
V_k
F
V_{k+1}
V_k
Experimental data
Model for a forced Hodgkin-Huxley Oscillator

Motion at large in the State Plane

Energy - Balance Plane

Consider a simple nonlinear conservative system

$$\ddot{x} + f(x) = 0$$

where $f(x)$ is a nonlinear analytic functio

$$\dot{x} = y$$
$$\dot{y} = -f(x)$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-f(x)}{y}$$

$$y\,dy = -f(x)\,dx$$ Integrate both sides

$$\boxed{\frac{y^2}{2} + V(x) = h}$$ = constant of integration

$\Rightarrow$ Law of conservation of energy

$\frac{y^2}{2} = \frac{1}{2}\dot{x}^2$ = Kinetic energy for a unit mass

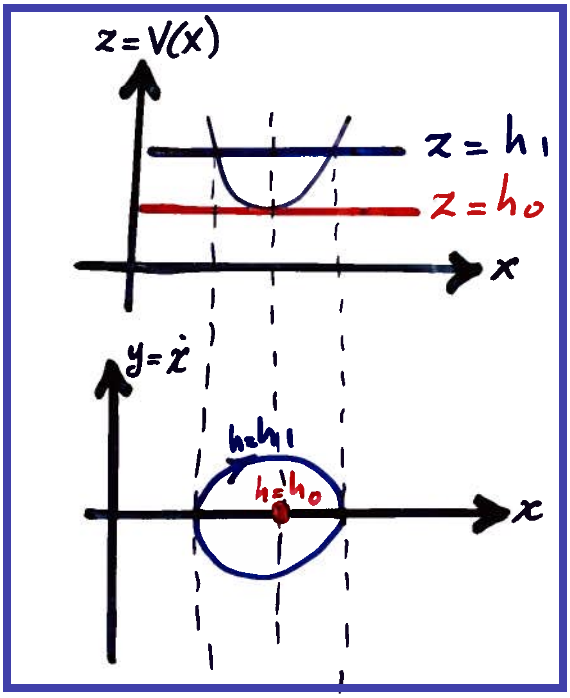
$$\boxed{V(x) = \int f(x)\,dx = \text{Potential energy of the system}}$$

h = energy constant or total energy

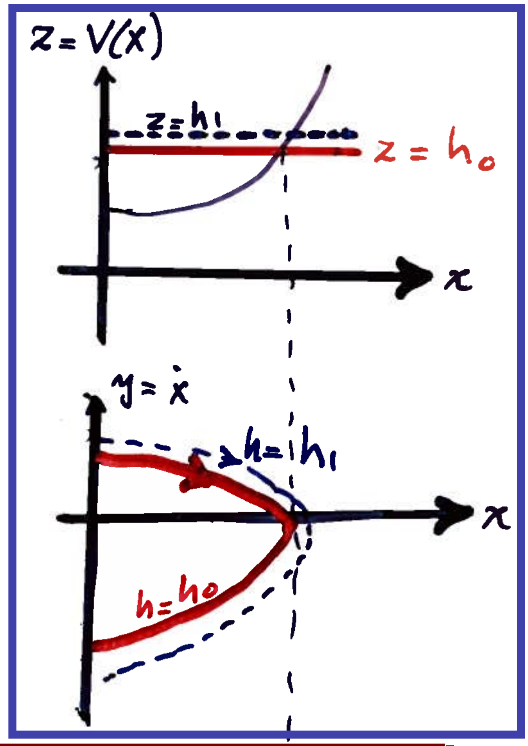
$$\boxed{\frac{y^2}{2} = h - V(x)}$$

For $h < V(x)$, kinetic energy is negative, hence no motion of state $\boxed{y = \dot{x} = \phi(x) = \left[2h - 2V(x)\right]^{\frac{1}{2}}}$

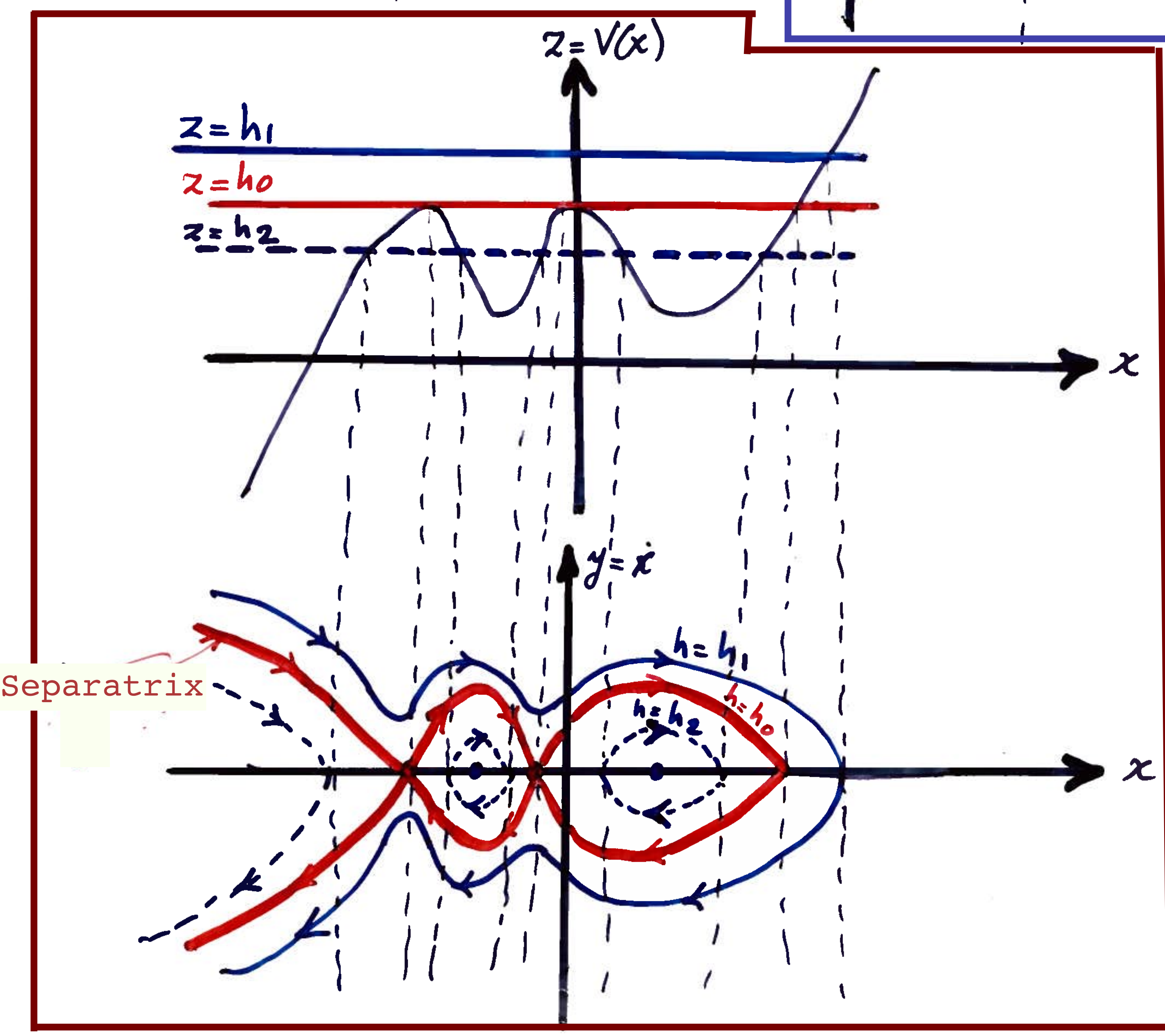
For $h > V(x)$, then for each value of "h" there is a whole curve $\boxed{y = \phi(x)}$ on the xy-plane, called the equi-energy curve. The state point will move along this curve when the total energy of motion is equal to "h".



$z=V(x)$
$z=h_1$
$z=h_0$
x
$y=\dot{x}$
$h=h_1$
$h=h_0$
x



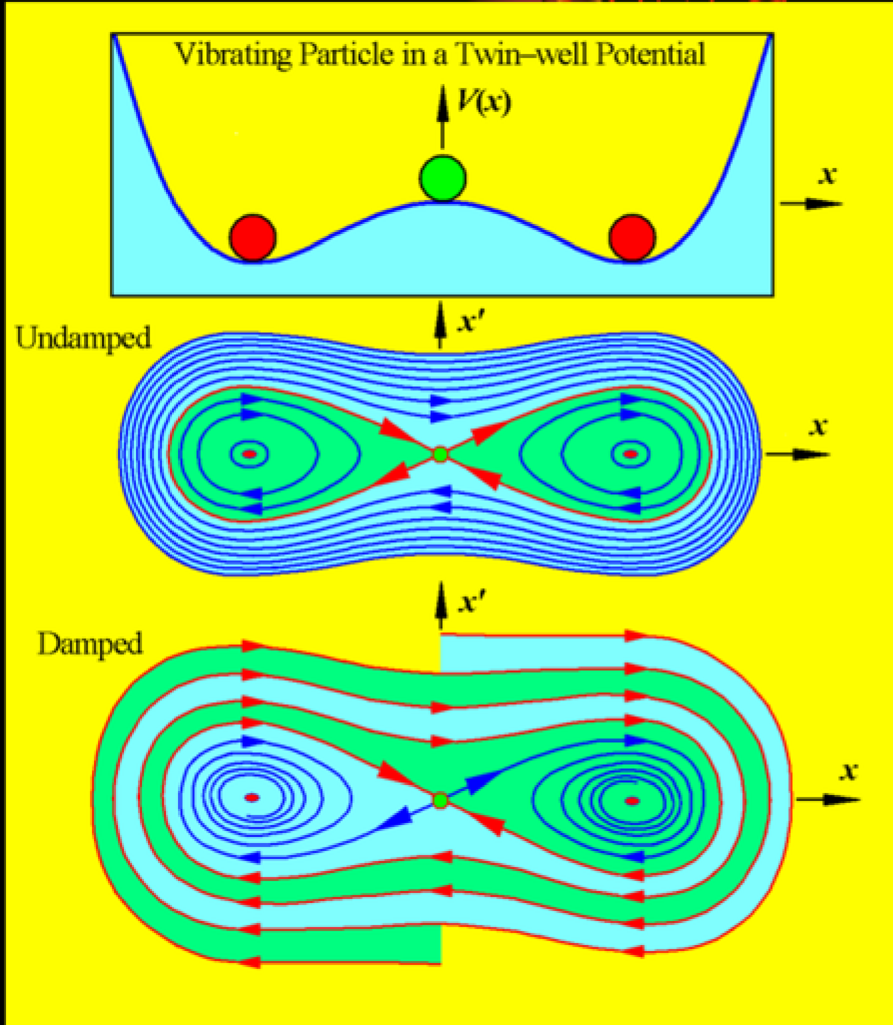
$z=V(x)$
$z=h_1$
$z=h_0$
x
$y=\dot{x}$
$h=h_1$
x
$h=h_0$



$z=V(x)$
$z=h_1$
$z=h_0$
$z=h_2$
x
$y=\dot{x}$
$h=h_1$
Separatrix
$h=h_2$
$h=h_0$
x

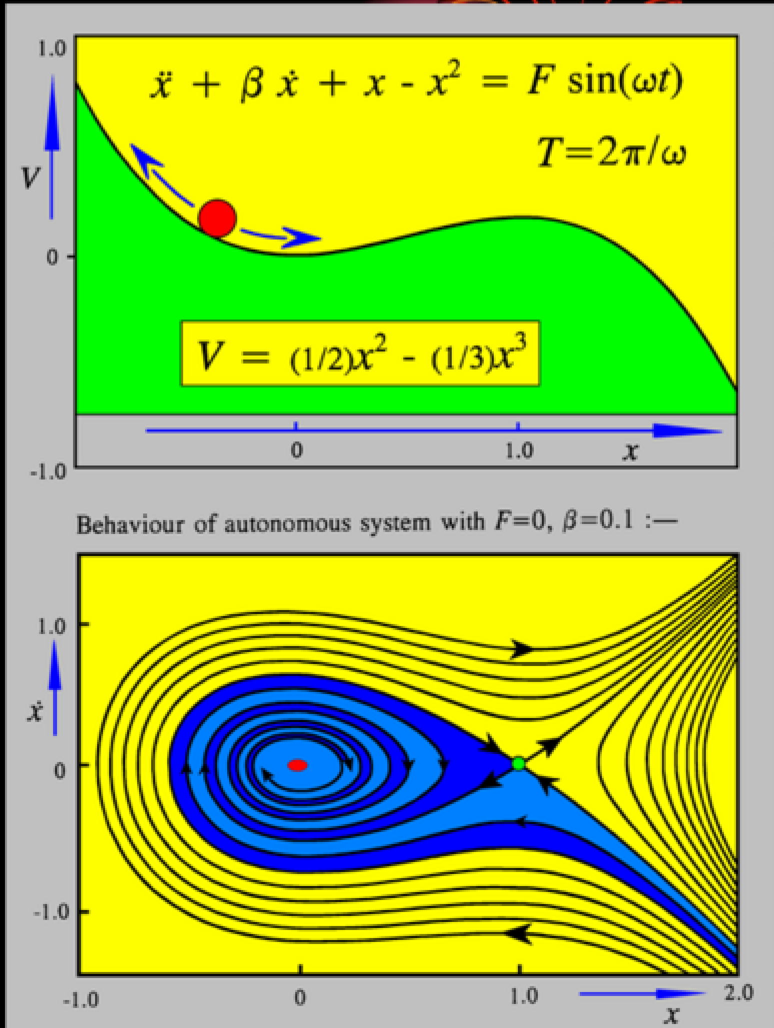
Twin-Well Oscillator

- **Imagine a ball rolling on the top energy surface**
- **The undamped 2D portrait has closed orbits**
- **With damping we have 2 attractors in their basins**



Escape from a Well

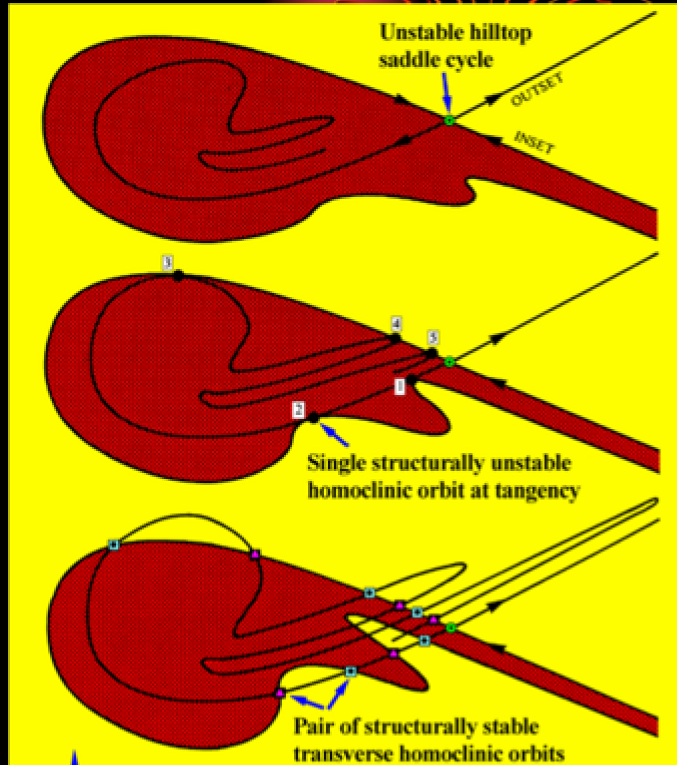
- Escape from a potential well is a recurring problem in science and engineering
- Consider a damped particle in a well excited by a direct periodic driving force
- If the driving is switched off, the 2D phase portrait is as shown in the lower picture
- The safe basin of attraction is shown in Blue.
- Starts in the Yellow area, escape over the hilltop to infinity.



Homoclinic Tangling
(like that foreseen by Poincare in 1887)

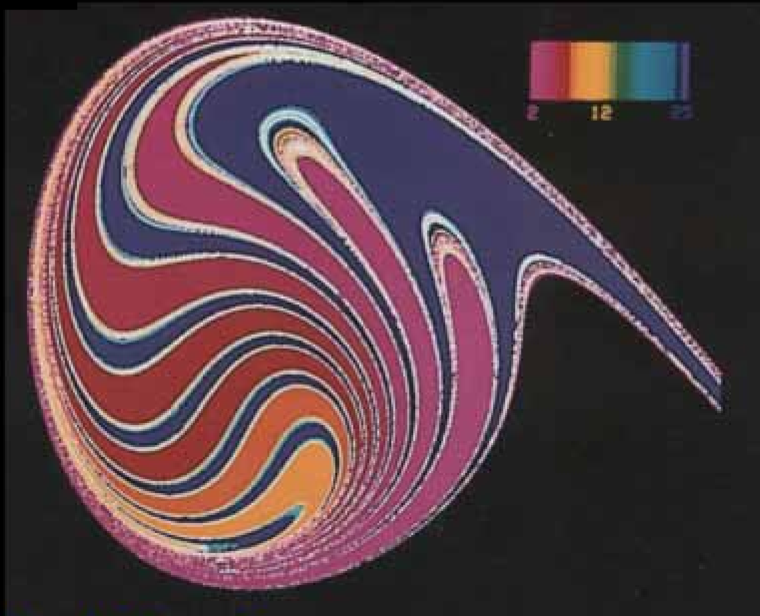
- Once we start driving, the phase portrait is in 3D
- We must view the basin in a stroboscopic section
- The hill-top solution still has an inset and an outset
- Solutions step along these lines [redacted]
- As the driving increases, the inset and outset get tangled
- They intersect one another an infinite number of times
- The boundary of the safe basin becomes fractal

@



Stroboscopic Section =>
Time Sampling =>
Poincare Section

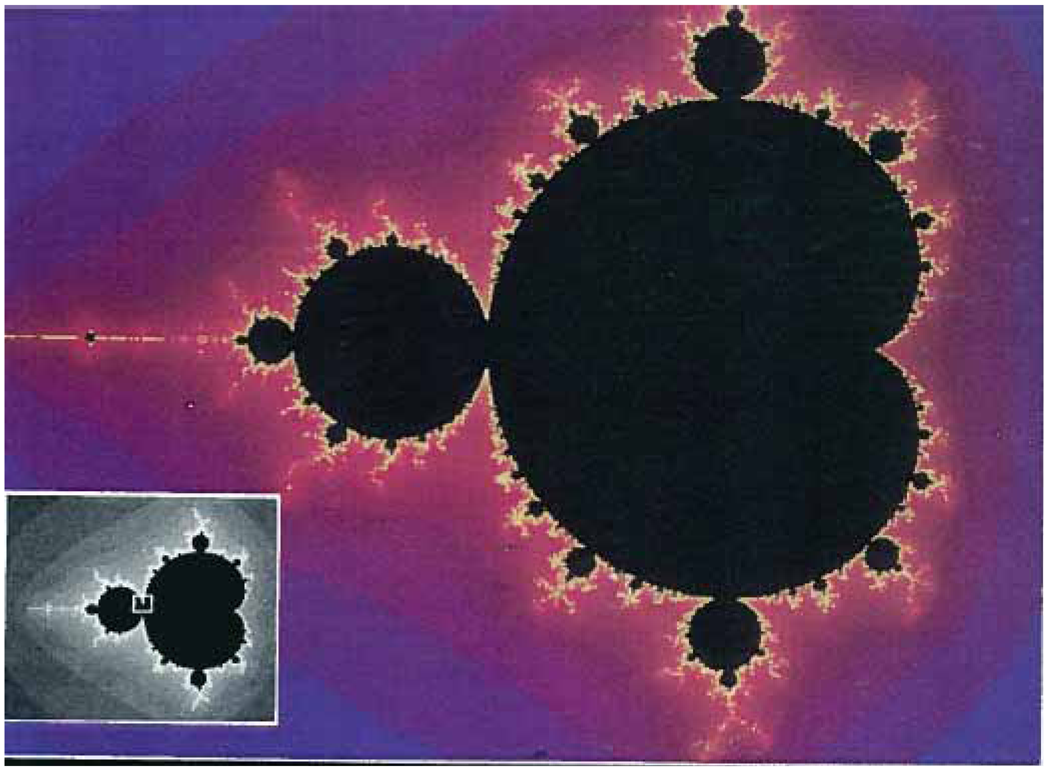
Fractal Basin Erosion

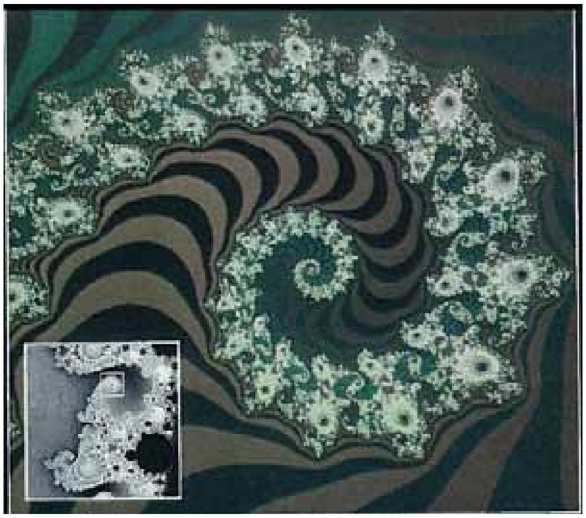
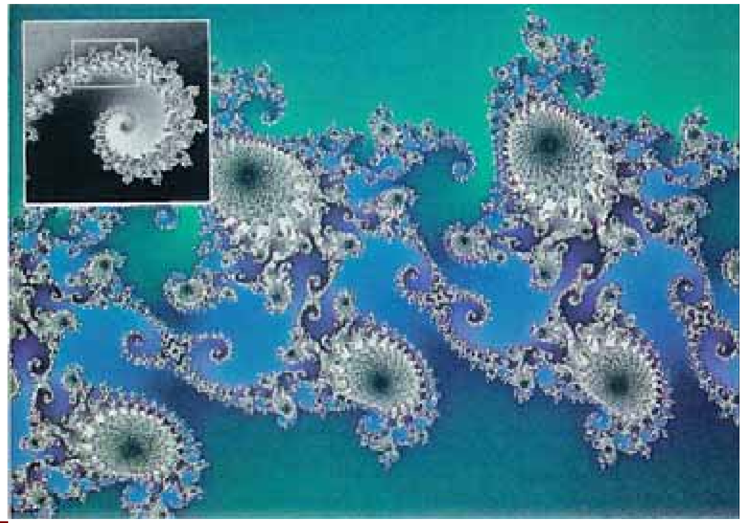
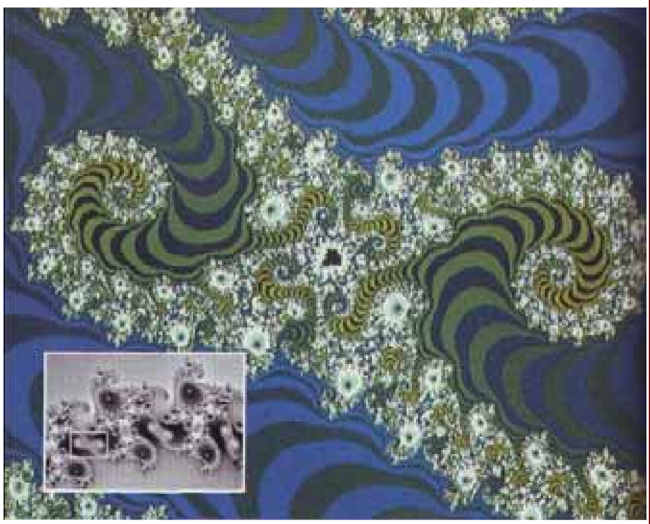
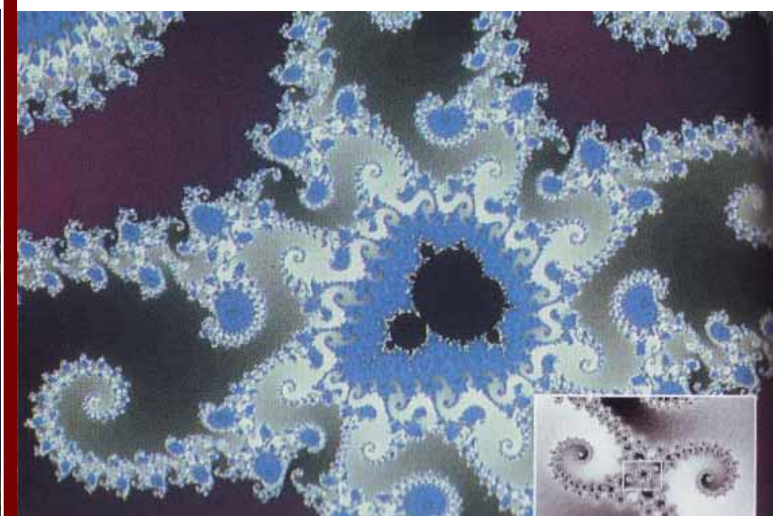


- As the driving increases, fractal fingers created by the homoclinic tangling make a sudden incursion into the safe basin: the integrity of the in-well motions is lost
- Colours show escape time, measured in driving periods

The Mandelbrot Set

A fractal Object



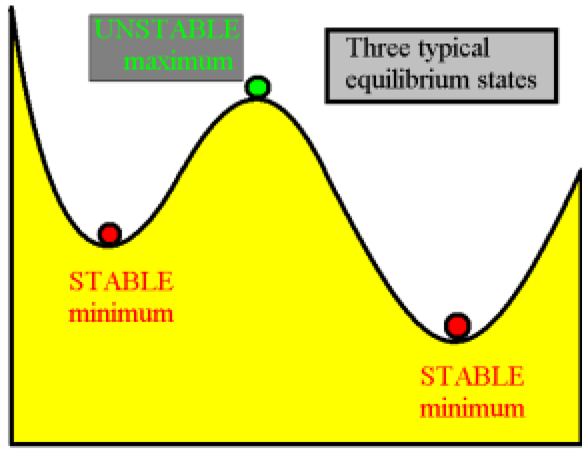





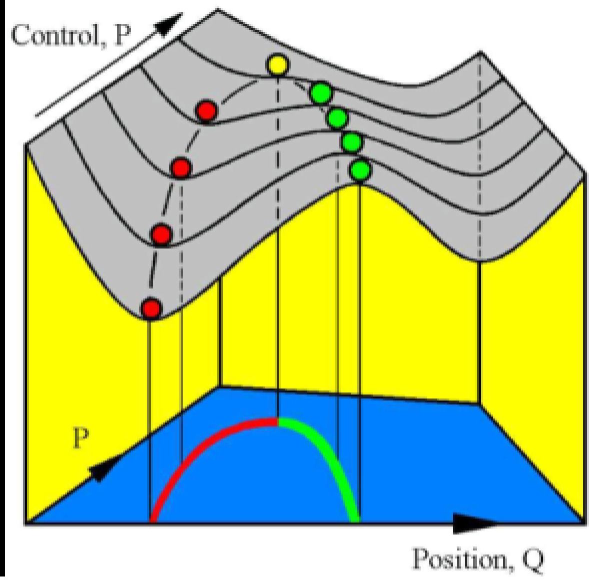

Catastrophe Theory

Typical Events

- A typical energy surface for a rolling ball has only hills and valleys (where energy V is locally quadratic)
- With one control, P, we can typically encounter a fold (where V is cubic)

@





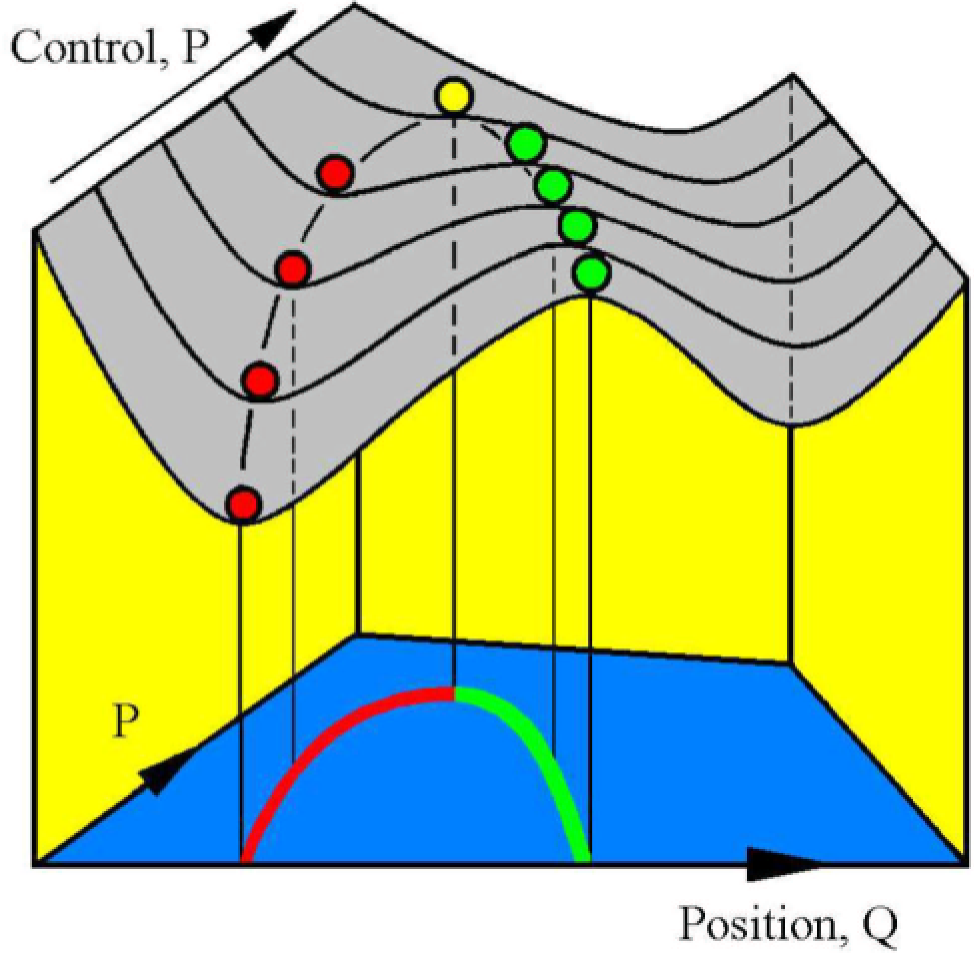
The Fold Catastrophe

Instability at a Fold

This is the most typical way that a system can become unstable

The graph of load (P) against the position (Q) folds back

At the fold, the ball rolls off to a new position



Angry Dogs

Imagine that 2 emotions control how a dog will behave

His rage (P) and his fear (R)

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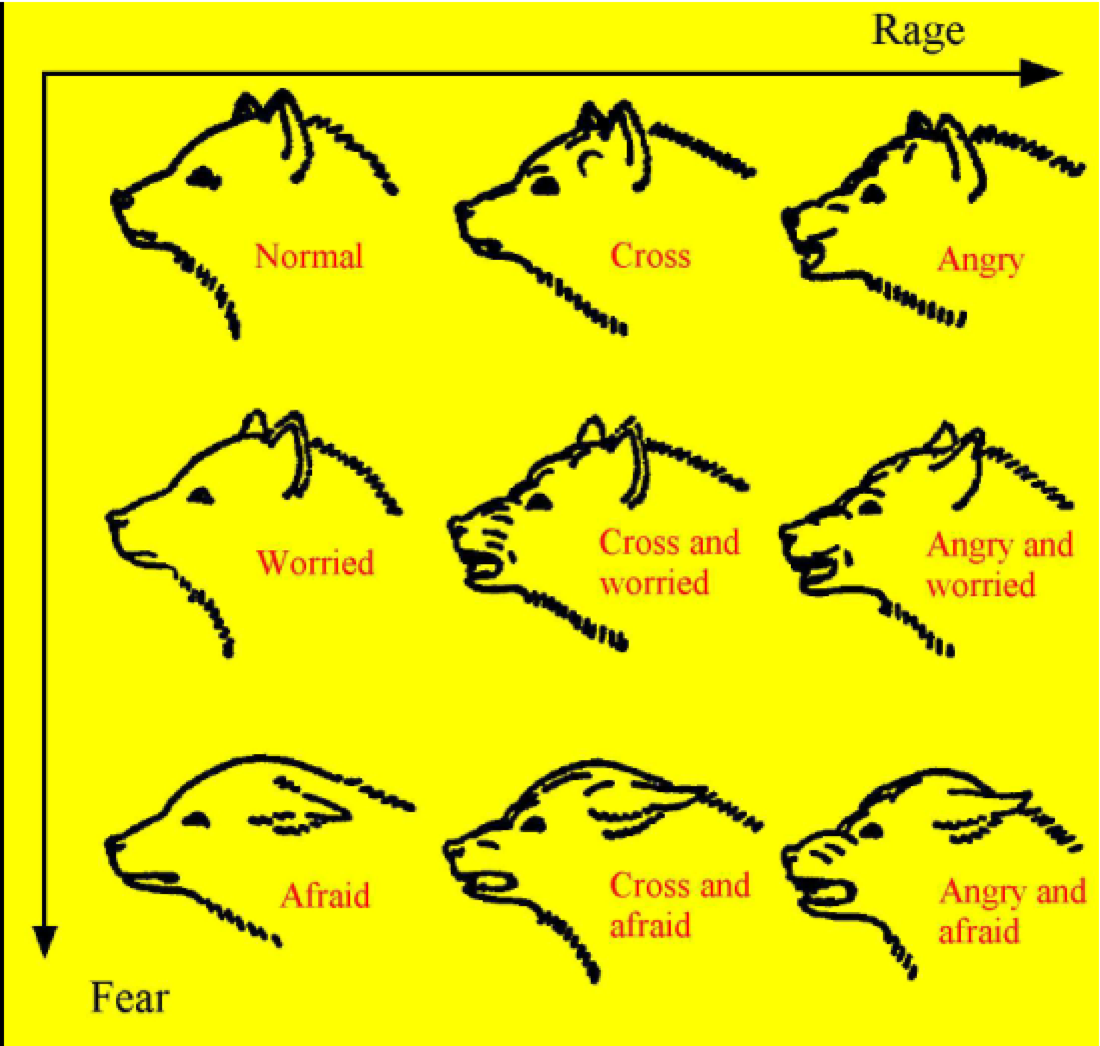
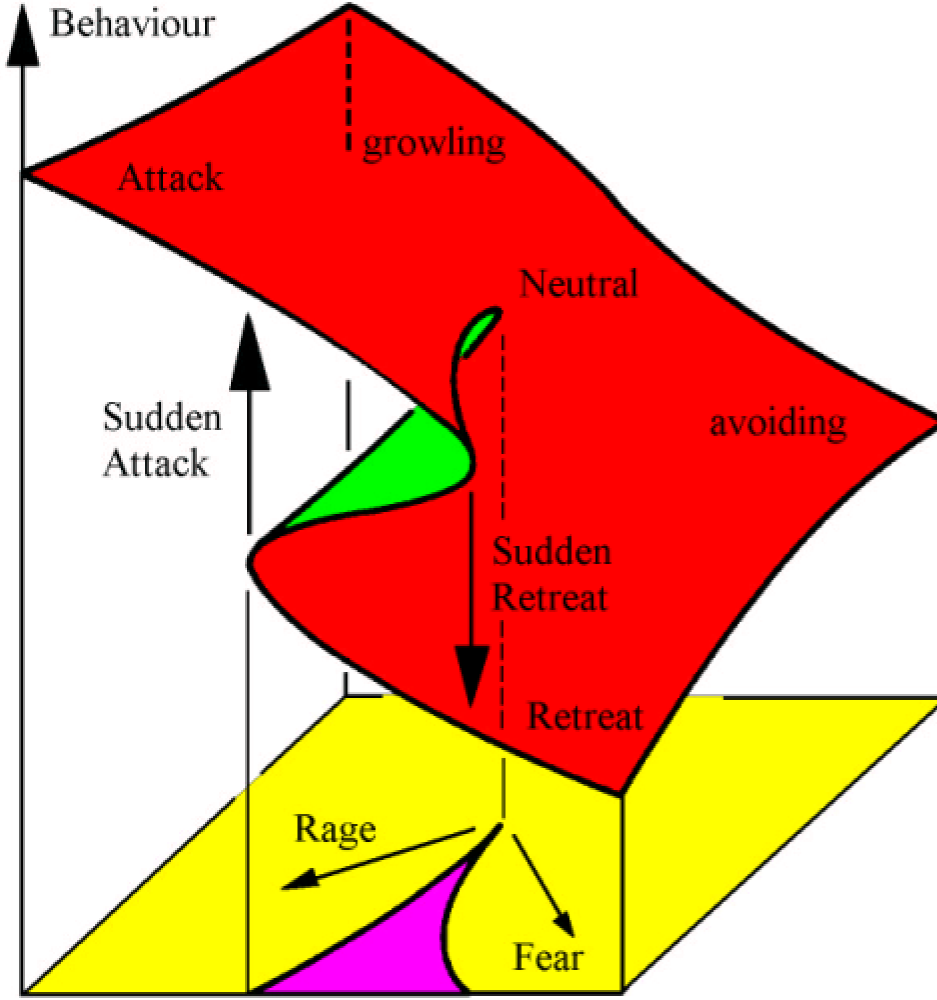


Sudden Attack

Think of a dog as a black box with 2 knobs marked rage and fear

At a prescribed control setting, we examine its behaviour

This 3D surface is another way of looking at a pitchfork or cusp



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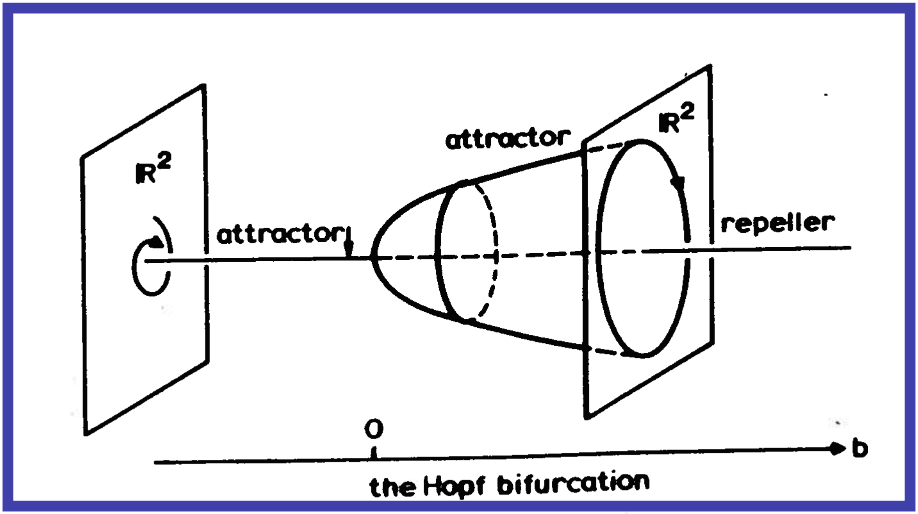
The Hopf bifurcation: An immediate word of warning is necessary, because not all stable bifurcations of oscillators are elementary catastrophes; it depends on whether or not a stably bifurcating Lyapunov function exists. As yet the non-elementary bifurcations are unclassified. The most famous counter example is the Hopf bifurcation, which is the one-dimensional bifurcation exhibited by the parametrised Van der Pol oscillator, with parameter b.

$$\boxed{\ddot{x} + \epsilon(x^2 - b)\dot{x} + x = 0}$$

When $b < 0$ the flow in the phase plane R^2 has only an attractor point at the origin. When $b > 0$ the origin turns into a repeller and an attracting limit cycle appears of radius approximately $2\sqrt{b}$. Thus, the non-wandering set, as b varies, consists of (or more precisely is differentially equivalent to and within ϵ of) a paraboloid and its axis (



The Hopf bifurcation

Van der Pol with large damping: Now consider what happens when the damping ϵ becomes large. Let us replace ϵ by K to indicate its largeness. The phase plane with co-ordinates $x, \dot{x}$ is no longer a good geometrical way to represent the oscillator, because even though x may remain bounded, the velocity $\dot{x}$ becomes very large. Therefore, it is better to use the 'dual' phase plane with co-ordinates $x, \int x$, as follows. We begin with the Van der Pol oscillator

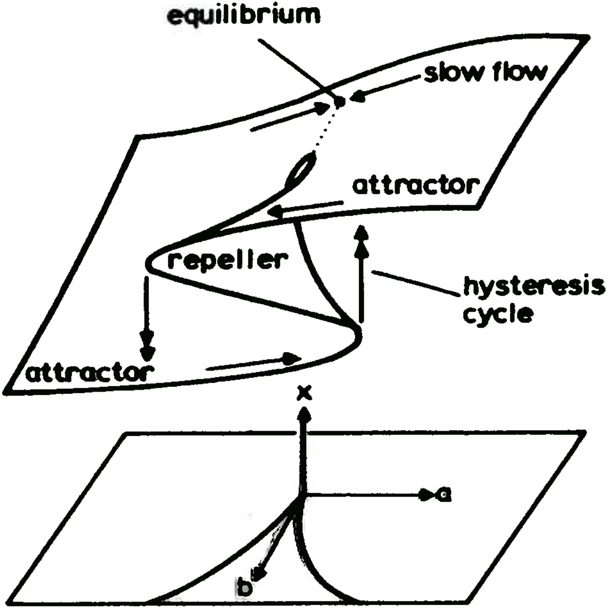
$$\ddot{x} + K(3x^2 - b)\dot{x} + x = 0$$

where K is a large constant, b a parameter and the factor 3 is put in for convenience. Suppose that $x, \dot{x}$ take initial values $x_0, \dot{x}_0$. Let

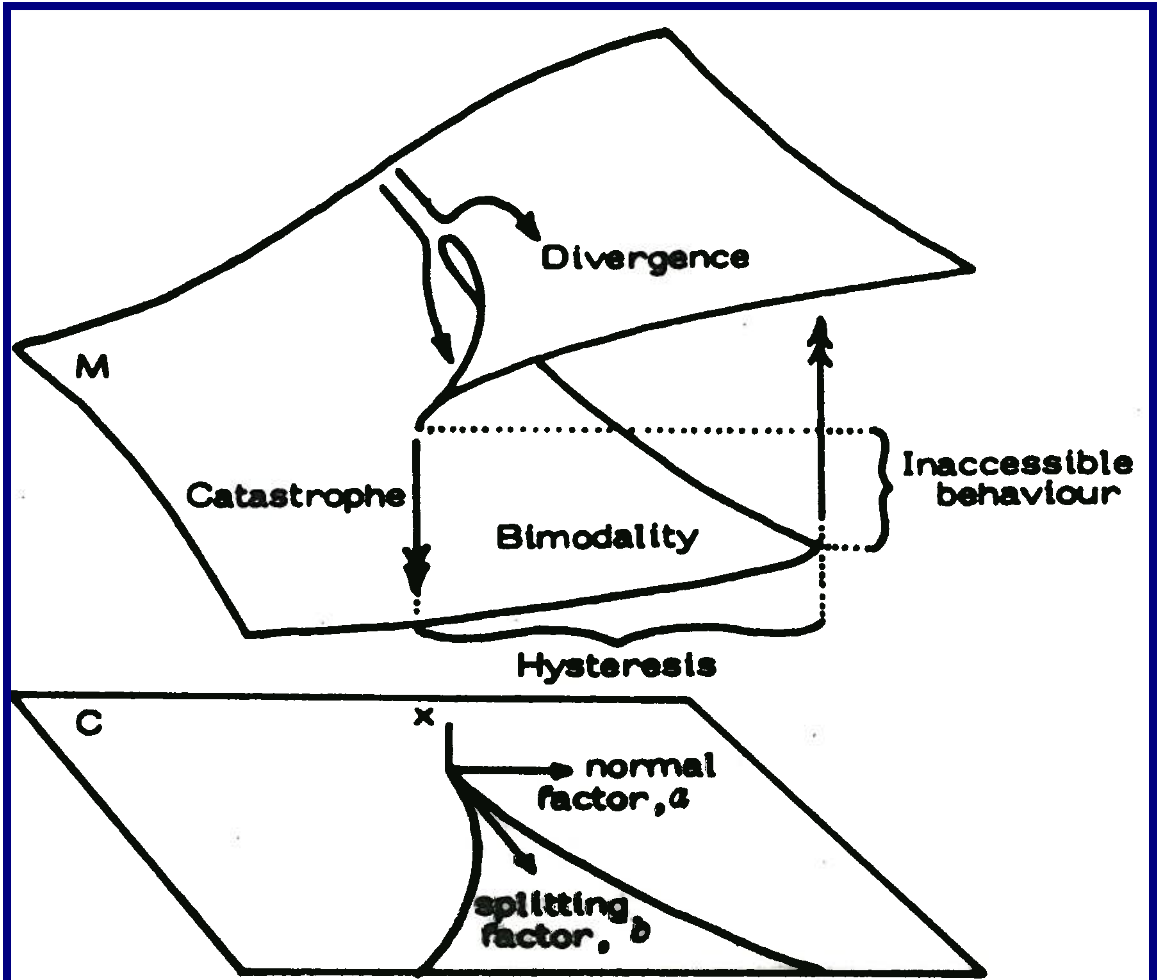
$$a_0 = x_0^3 - bx_0 - \frac{1}{K}\dot{x}_0$$

$$a(t) = a_0 - \frac{1}{K}\int_0^t x(\tau)\,d\tau$$

$$\dot{a} = -\frac{1}{K}x$$

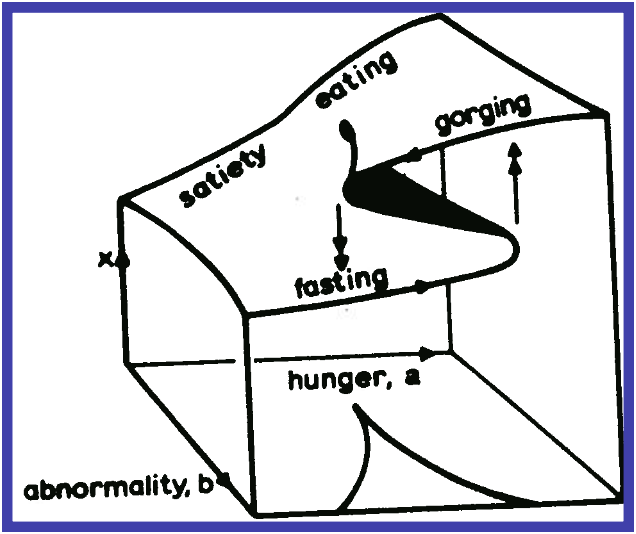


Van der Pol with large damping is a cusp catastrophe with feedback flow

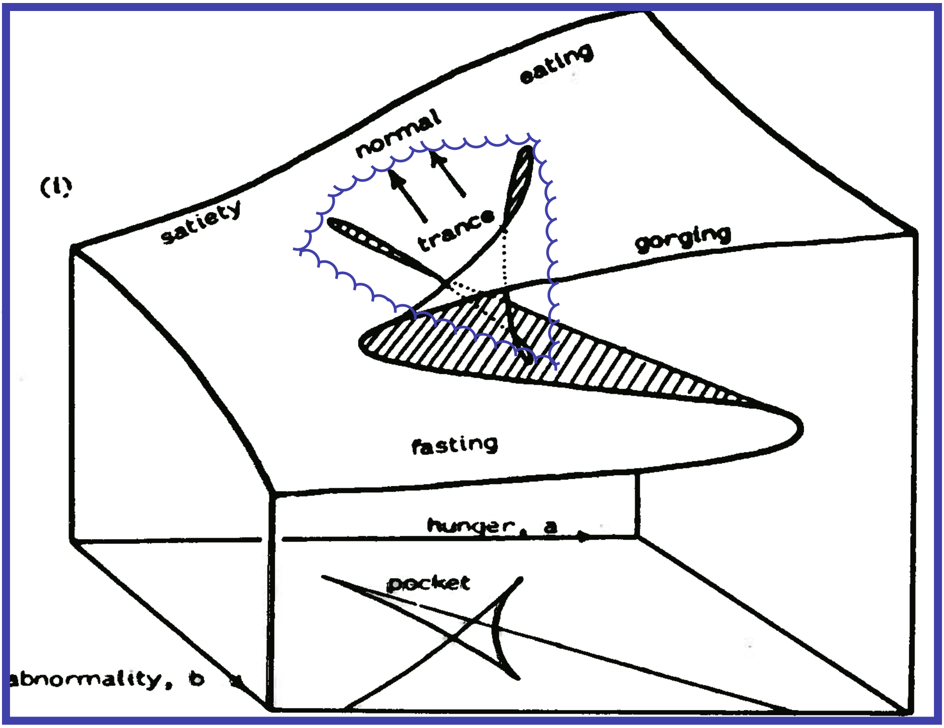


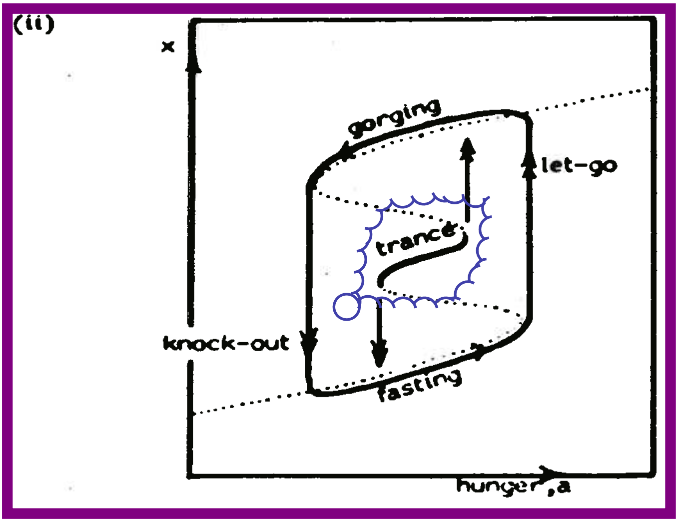
Five characteristic properties of the cusp-catastrophe are bimodality, inaccessibility, catastrophe, hysteresis and divergence (The unstable middle sheet has been removed)

Anorexia



Initial behaviour model for anorexia





The effect of the butterfly factor, $d > 0$

(i) When therapy starts the trance states appear as a new triangular sheet of stable behaviour over the pocket in between the upper and lower sheets. The new sheet opens up a pathway back to normality

(ii) The trance states sit inside the hysteresis cycle; initially they are fragile, and coming out of the trance is a catastrophic jump into either a fasting or a gorging frame of mind