Technical University of Denmark

Time Series Analysis

02417

Assignment 2

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Contents

1 Q 2.1 Stability **1**
- 1.1 Is the process stationary and invertible? . . . 1
- 1.2 Investigate analytically the second order moment representation of the process. . . . 2
- 1.3 Simulate 10 realisations with 200 observations from the process and plot them (Preferably in the same plot). . . . 3
- 1.4 Estimate the ACF for each realisation and plot those (Again, preferably in the same plot). Comment on the results . . . 4
- 1.5 Repeat for the PACF of the same realisations. . . . 5
- 1.6 Calculate the variance of each of the realisations . . . 5
- 1.7 Discuss and compare the analytical and numerical results. . . . 5

2 Q 2.2 Predicting the the number of sales of apartments **7**
- 2.1 Plot the observations along with the two predictions and their 95% prediction intervals. . . . 7

3 Q 2.3 **9**
- 3.1 For both values of ϕ_2 you should calculate the roots of $\phi(z^{-1}) = 0$ 9
- 3.2 For each process, make a histogram plot of the estimates of parameter ϕ_2 and indicate the 95% quantiles. . . . 10
- 3.3 How do different values of ϕ_2 affect the variance/distribution of the estimated ϕ_2? . . . 10
- 3.4 How do different values of σ affect the variance/distribution of the estimated ϕ_2? . . . 10
- 3.5 Plot all the estimated pairs of parameters (ϕ_1; ϕ_2) for the four variations. Comment on what you see and compare with the true values. . . . 11

4 Appendix **13**
- 4.1 Exercise 2.2.1 . . . 13
- 4.2 Exercise 2.2.2 . . . 15
- 4.3 Exercise 2.2.3 . . . 16
- 4.4 More (for 2.2.1) calculations in Maple . . . 20

1 Q 2.1 Stability

Given the process:

$$X_t - 0.8X_{t-1} = \epsilon_t + 0.8\epsilon_{t-1} - 0.5\epsilon_{t-2} \quad (1)$$

Where ϵ_t is white noise with standard deviation $\sigma = 0.4$

1.1 Is the process stationary and invertible?

The AR model is always invertibel and the MA model is invertibel if the roots are within the unit-circle in the z-domain. Determine the of the AR part is stationary by finding the roots:

$$(Z^0 - 0.8Z^{-1}) = (1 - 0.8Z^{-1}) = 0 \quad (2)$$

Then we get that

$$z = \frac{4}{5} = 0.8 \quad (3)$$

Hence we can now conclude that **this process is stationary** since the roots of $\phi(z^{-1})$ lies within the unit circle hence $|0.8| \leq 1$.
By looking at the MA part of the model and find the roots $\theta(z^{-1})$=0 to determine if the process is invertibel:

$$(Z^0 + 0.8Z^{-1} - 0.5Z^{-2}) = 0 \quad (4)$$

The roots are $z_1 = 0.412$, $z_2 = -1.212$. (can be found by solving second order equation) Thus not both of the roots are within the unit circle, leads to **this process is not invertibel.**

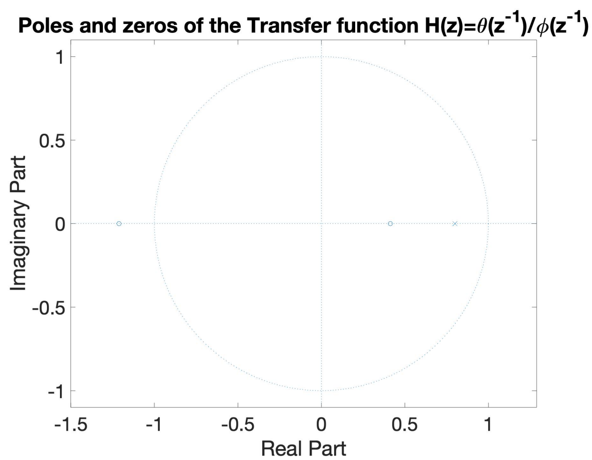

Figure 1: Showing the roots of ARMA process. From this plot we can illustrate again, that the MA has two roots and only one of them is inside the unit circle, AR model has one root and it is within the unitcircle - hence the system is stable but not invertibel

1.2 Investigate analytically the second order moment representation of the process.

The second order moment representation consists of **mean** and **autocovariance**. As we know, white noise has mean $\mu_\epsilon = 0$.

$$X_t - 0.8X_{t-1} = \epsilon_t + 0.8\epsilon_{t-1} - 0.5\epsilon_{t-2} \tag{5}$$

Instead of writing the numbers we substitute the coefficients with θ and ϕ

$$X_t = \phi_1 X_{t-1} + \theta_0\epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} \tag{6}$$

Finding the mean / the expected value of the system X_t

$$E[X_t] = \phi_1 E[X_{t-1}] + \theta_0 E[\epsilon_t] + \theta_1 E[\epsilon_{t-1}] + \theta_2 E[\epsilon_{t-2}] \tag{7}$$

since $\mu_\epsilon = 0$ (white noise) then we have:

$$E[X_t] = \theta_1 E[X_{t-1}] \tag{8}$$

Then equation (8) is only valid if $E[X_t] = E[X_{t-1}] = 0$. **Hence our theoretical mean is** $\mu = 0$

The auto covariance and autocorrelation is found: *(based on (5.96) and (5.97) in textbook)*

$$\gamma_{\varepsilon Y}(k) = \operatorname{Cov}\left[\varepsilon_t, Y_{t+k}\right] = \operatorname{E}\left[Y_t\varepsilon_{t-k}\right] \begin{cases} = 0 & \text{for } k < 0 \\ \neq 0 & \text{for } k \geq 0 \end{cases} \tag{9}$$

$$\gamma_{\varepsilon Y}(k) + \phi_1\gamma_{\varepsilon Y}(k-1) + \cdots + \phi_p\gamma_{\varepsilon Y}(k-p) = \theta_k\sigma_\varepsilon^2, \quad k = 0, 1, \ldots \tag{10}$$

then we have for our system that

For k=0

$$\gamma_{\varepsilon Y}(0) = \theta_0\sigma_\varepsilon^2 \tag{11}$$

For k=1

$$\gamma_{\varepsilon Y}(1) = \theta_1\sigma_\varepsilon^2 - \phi_1\gamma_{\varepsilon Y}(0) = \theta_1\sigma_\varepsilon^2 - \phi_1\theta_0\sigma_\varepsilon^2 \tag{12}$$

For k=2

$$\gamma_{\varepsilon Y}(2) = \theta_2\sigma_\varepsilon^2 - \phi_1\gamma_{\varepsilon Y}(1) \tag{13}$$

For $k \geq 3$

$$\gamma_{\varepsilon Y}(k) = -\phi_1\gamma_{\varepsilon Y}(k-1) \tag{14}$$

Then we define the auto-covariance by solving these equations:

$$\gamma(0) + \phi_1\gamma(1) = \gamma_{\varepsilon Y}(0) + \theta_1\gamma_{\varepsilon Y}(1) + \theta_2\gamma_{\varepsilon Y}(2) \tag{15}$$

$$\gamma(1) + \phi_1\gamma(0) = \theta_1\gamma_{\varepsilon Y}(0) + \theta_2\gamma_{\varepsilon Y}(1) \tag{16}$$

$$\gamma(2) + \phi_1\gamma(1) = \theta_2\gamma_{\varepsilon Y}(0) \tag{17}$$

For $k \geq 3$

$$\gamma(k) + \phi_1 \gamma(k-1) = 0 \tag{18}$$

	k=0	k=1	k=2	For $k \geq 3$
$\gamma_{\varepsilon Y}(k)$	0.16	0.256	0.1248	$\gamma_{\varepsilon Y}(k) = -\phi_1 \gamma_{\varepsilon Y}(k-1)$
$\gamma(k)$	0.84	0.672	0.4576	$\gamma(k) = -\phi_1 \gamma(k-1)$
$\rho(k) = \frac{\gamma(k)}{\gamma(0)}$	1	0.8	0.5447619048	$\rho(k) = \frac{-\phi_1 \gamma(k-1)}{0.84}$

Theoretical mean is $\mu = 0$

1.3 Simulate 10 realisations with 200 observations from the process and plot them (Preferably in the same plot).

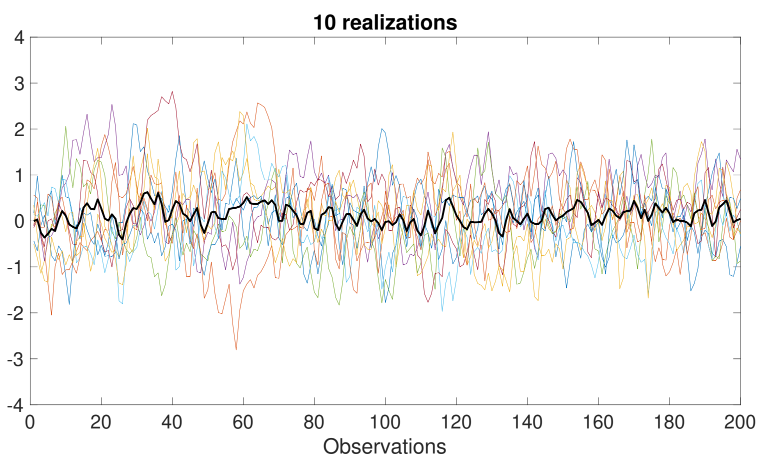

Figure 2: 10 realization shown in one plot, the black line is the mean of 10 realizations.

1.4 Estimate the ACF for each realisation and plot those (Again, preferably in the same plot). Comment on the results

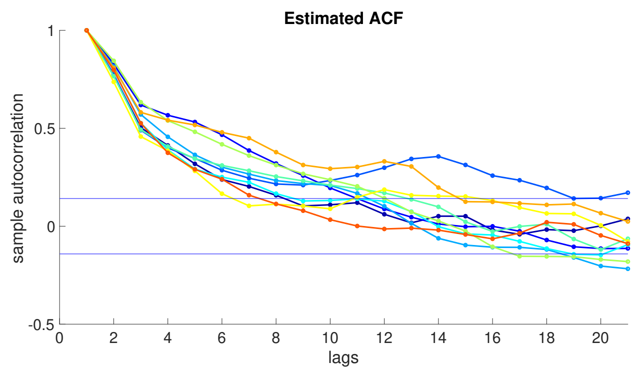

Figure 3: Estimated ACF of 10 realizations. The blue horizontal lines indicates the confidence interval for considering values for being zero.

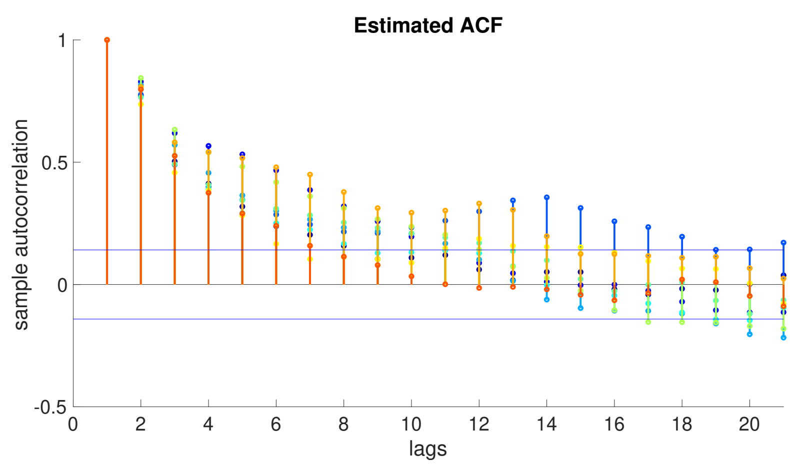

Figure 4: Estimated ACF of 10 realizations (same plot as figure 3 with different plot method).

From the plots of ACF for all 10 observations, it is clear that the values of ACF converges towards zero which reflects the system's stationarity. They are not equally fast converging for example the orange line is converging faster compared to most of the other observations. **This is clear that it is a AR(1) model since it has exponentially decay in the ACF plot**

1.5 Repeat for the PACF of the same realisations.

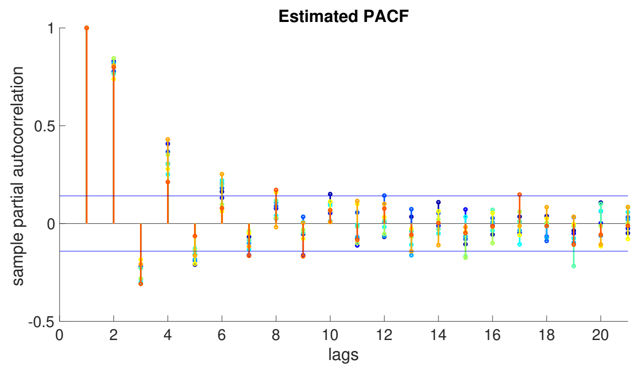

Figure 5: Estimated PACF of 10 realizations

Generally the there are 4 peaks that are significantly above zero. At lag 5, some of the realizations are within the confidence bound. From this plot it is also clear that the model is ARMA since both the ACF and PACF are exponential decaying values with oscillations.

1.6 Calculate the variance of each of the realisations

R1	0.44901
R2	0.69851
R3	0.47741
R4	0.59505
R5	0.61279
R6	0.5126
R7	0.69427
R8	0.41399
R9	0.70449
R10	0.67212

The variance for each realization illustrated in the table above, where it is clear that the variance is larger than $\gamma(0)$

1.7 Discuss and compare the analytical and numerical results.

The theoretical variance is calculated as the auto covariance at lag zero; $\gamma(0) = 0.84$, where most of the realizations (shown in table above) are slight below the theoretical value. This could indicate that the noise can have a affect on the realizations. The auto correlation values $\rho(0)$, $\rho(1)$ and $\rho(2)$ seems to fit the

ACF very well (see figure (3 and table on page 3) As we have plotted the mean line (based on 10 realizations) in figure 2 we can see that the oscillating around zero, and the mean of the mean-line is 0.1134, which is close to the true mean ($\mu = 0$).

2 Q 2.2 Predicting the the number of sales of apartments

This part of exercise illustrates the sales of apartments quarterly. The data extents from 2014 to 2018.

Following model is given for modelling the sales:

$$\left(1-1.04B+0.2B^2\right)\left(1-0.86B^4\right)(Y_t-\mu)=\left(1-0.42B^4\right)\epsilon_t \qquad (19)$$

Where $\sigma_\varepsilon^2 = 36963$ and $\mu = 2070$ We are asked to predict two steps ahead, equivalent to 2019Q1 and 2019Q2.

2.1 Plot the observations along with the two predictions and their 95% prediction intervals.

We can identify that (19) can be rewritten into pure AR model, this means we have to divide the coefficients $(1-0.42B^4)$. This is not favorable, since we do not possess the data from future. Therefore by ultilizing the geometric series (4.51) denoted by:

$$\frac{1}{1-x}=\sum_{i=0}^{\infty}x^i \quad \text{for} \quad |x|<1 \qquad (20)$$

We can now define our model as:

$$\left(1-1.04B+0.2B^2\right)\left(1-0.86B^4\right)\frac{1}{1-0.42B^4}(Y_t-\mu)=\epsilon_t \qquad (21)$$

By using the series we obtain:

$$\frac{1}{1-(0.42B^4)}=\sum_{i=0}^{\infty}(0.42B^4)^i \qquad (22)$$

$$\sum_{i=0}^{5}(0.42B^4)^i=1+0.42B^4+0.176B^8+0.074B^{12}+0.0311B^{16}+0.013B^{20} \qquad (23)$$

Note that these constants are rounded for better visualization, for the actual computation all of the decimals are retains during the calculations. For this experiment, we only have 20 data-points, therefore no need to go to ∞

$$\left(1-1.04B+0.2B^2\right)\left(1-0.86B^4\right)\sum_{i=0}^{5}(0.42B^4)^i(Y_t-\mu)=\epsilon_t \qquad (24)$$

We can consider the model as zero-mean $(Y_t-\mu)$ and pure AR.

$$\left(1-1.04B+0.2B^2\right)\left(1-0.86B^4\right)\sum_{i=0}^{5}(0.42B^4)^iZ_t=\epsilon_t \qquad (25)$$

The one and two step predictions are performed based on equation (5.155):

$$\begin{aligned}\widehat{Y}_{t+k|t} = & -\varphi_1 \mathrm{E}\left[Y_{t+k-1}|Y_t, Y_{t-1}, \ldots\right] - \cdots \\ & - \varphi_{p+d} \mathrm{E}\left[Y_{t+k-p-d}|Y_t, Y_{t-1}, \ldots\right] \\ & + \mathrm{E}\left[\varepsilon_{t+k}|Y_t, Y_{t-1}, \ldots\right] + \theta_1 \mathrm{E}\left[\varepsilon_{t+k-1}|Y_t, Y_{t-1}, \ldots\right] + \cdots \\ & + \theta_q \mathrm{E}\left[\varepsilon_{t+k-q}|Y_t, Y_{t-1}, \ldots\right]\end{aligned} \tag{26}$$

The variance of one-step prediction is only dependent on the variance from the noise

$$\sigma_1^2 = \sigma_\epsilon^2 = 36963 \tag{27}$$

The variance of second step prediction is:

$$\sigma_2^2 = (1 + (\psi_1)^2)\sigma_\epsilon^2 = 76942 \tag{28}$$

Where the Confidence interval with following expression:

$$\widehat{Y}_{t+k|t} \pm u_{\alpha/2}\sigma_k = \widehat{Y}_{t+k|t} \pm u_{\alpha/2}\sigma_\varepsilon \sqrt{1 + \psi_1^2 + \cdots + \psi_{k-1}^2} \tag{29}$$

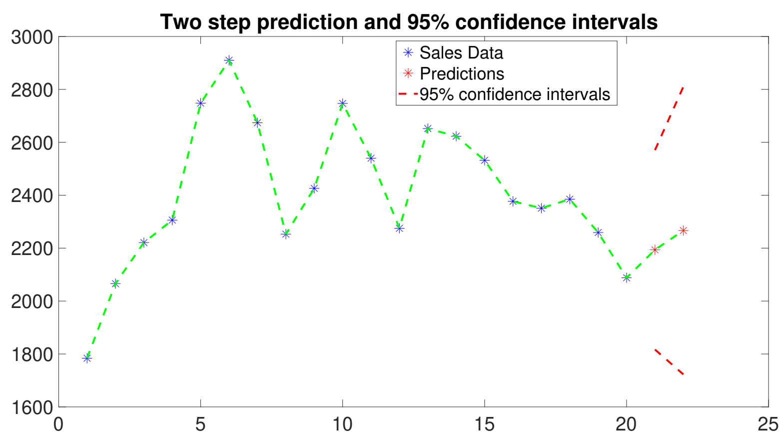

Figure 6: The plot with 2019Q1 and 2019Q2 together with the 95% confidence interval.

The following table shows the results from the calculations in this exercise.

Prediction values:

	Sales	**Lower bound (CI)**	**Upper bound (CI)**
2019Q1	2193.7	1816.8	2570.5
2019Q2	2266.6	1722.9	2810.2

It can be seen in figure 6 that the confidence is larger when prediction step increases which is true, since the variance term also increases with step size.

3 Q 2.3

$$\phi(B)X_t = X_t - 1.5X_{t-1} + \phi_2 X_{t-2} = \epsilon_t \tag{30}$$

3.1 For both values of ϕ_2 you should calculate the roots of $\phi(z^{-1}) = 0$

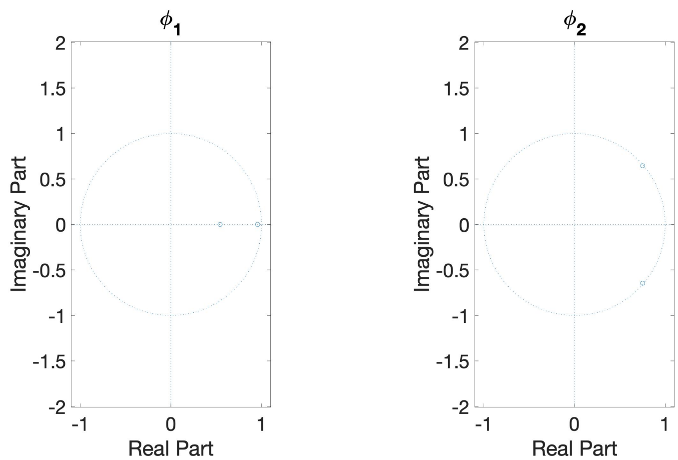

Figure 7: $\phi_2 = 0.52$ has two roots in the unit circle, and $\phi_2 = 0.98$ has both roots close to the edge of the unit circle.

For $\phi_2 = 0.52$ the roots are

$$z_1 = 0.9562$$
$$z_2 = 0.5438$$

For $\phi_2 = 0.98$ we have imaginary parts in the roots which are

$$z_1 = 0.7500 + 0.6461 \cdot i$$
$$z_2 = 0.7500 - 0.6461 \cdot i$$

And absolute values of the imaginary roots are:

$$|z_1| = |z_2| = 0.9899$$

which is corresponding to the distance from the roots and to the center of the unit-circle indicating that both choices of ϕ_2 will make the system stable.

3.2 For each process, make a histogram plot of the estimates of parameter ϕ_2 and indicate the 95% quantiles.

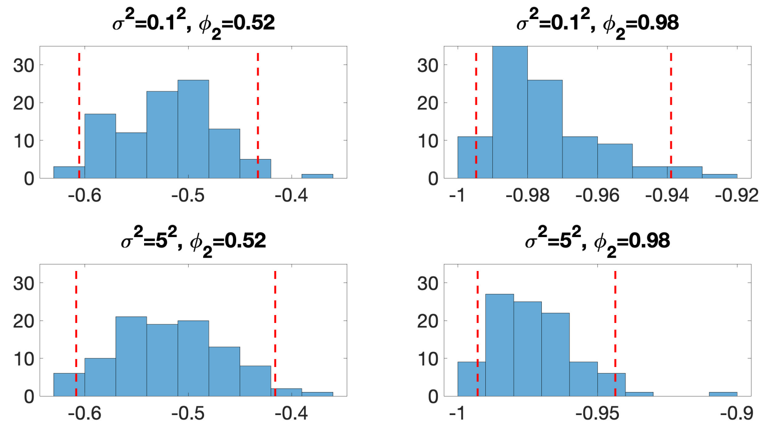

Figure 8: Histogram of estimated parameter and 95% quantiles

3.3 How do different values of ϕ_2 affect the variance/distribution of the estimated ϕ_2?

When choosing small ϕ_2, the distribution is becoming wider, this can be seen in the difference column, e.g $\phi_2 = 0.52$ has distribution widths of 0.17 and 0.19 whereas $\phi_2 = 0.98$ remarkable lower (around 0.05).

Overview of the quantiles:

	2.5%	97.5%	**Difference**
$\sigma^2 = 0.1^2$, $\phi_2 = 0.52$	-0.60535	-0.43254	0.17281
$\sigma^2 = 0.1^2$,$\phi_2 = 0.98$	-0.99475	-0.93891	0.055839
$\sigma^2 = 5^2$, $\phi_2 = 0.52$	-0.60825	-0.41583	0.19243
$\sigma^2 = 5^2$, $\phi_2 = 0.98$	-0.99291	-0.9436	0.049309

The last column (difference) shows the width of the 95% quantiles

3.4 How do different values of σ affect the variance/distribution of the estimated ϕ_2?

From the overview table, it does not show a clear difference between different choices of σ. The distribution is wider / variance is higher for $\phi_2 = 0.52\sigma = 5$ compared to $\phi_2 = 0.52\sigma = 0.1$, but on the other hand this does not apply for $\phi_2 = 0.98$

3.5 Plot all the estimated pairs of parameters (ϕ_1; ϕ_2) for the four variations. Comment on what you see and compare with the true values.

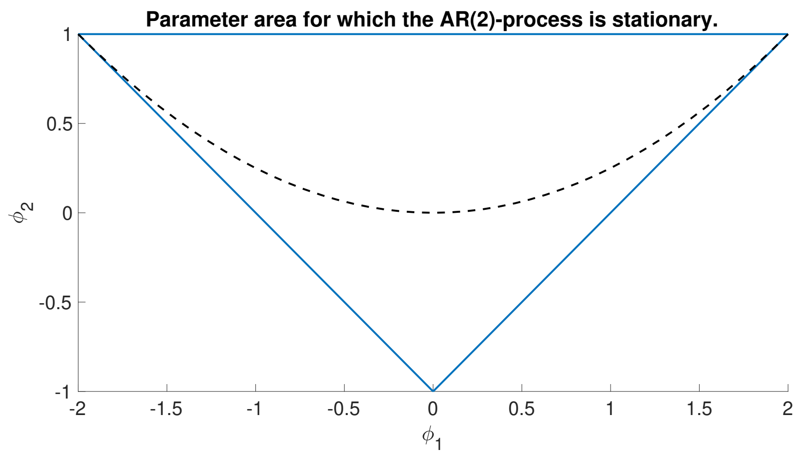

Figure 9: Showing the area of stationarity, values above the black dotted line is complex roots. This blue triangle is based on solving the second order polynomial (characteristic equation). The black dotted line indicates when the discriminant is larger/less than than zero.

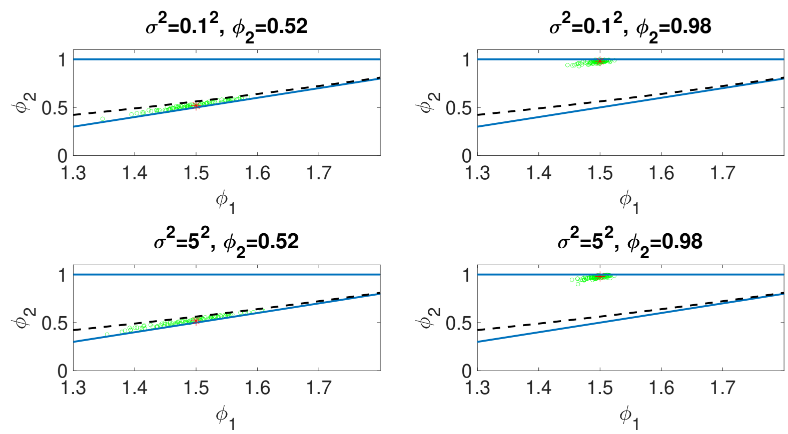

Figure 10: Estimated parameters and the triangle showed in figure 9.

When looking at figure 10, it is clear that when selecting $\phi_2 = 0.52$ we will only have real roots and when $\phi_2 = 0.98$ the roots are complex. All of the estimated values stays within the stationary area. The true values are marked with red *. For $\phi_2 = 0.52$ the estimated values are evenly distributed (the red star is in the center of the estimated values). For $\phi_2 = 0.98$ the star is slightly shifted to the right side (most of the estimated values are less than the true value- this shows same pattern as the histograms, that most of the estimated values is slightly shifted to one side).

4 Appendix

4.1 Exercise 2.2.1

Listing 1: 2.2.1

```
%% Verify the roots of the system:
close all;
p1=[1 0.8 -0.5];
r1=roots(p1);
p2=[1 -0.8];
r2=roots(p2);
figure('WindowState','Maximized')
zplane(r1,r2)
title('Poles and zeros of the Transfer function H(z)=\
    theta(z^-^1)/\phi(z^-^1)')
set(gca, 'fontsize',30)
%% simulate 10 with 200 observations
mdl = arima('AR',0.8,'MA',[0.4124102936
    -0.3401517651],'Constant',0,'Variance',0.4^(2)) ;
Y = simulate(mdl,200,'NumPaths',10) ;
%% plot 10 realizations with 200 observations
figure('WindowState','Maximized')
plot(Y)
hold on
plot(mean(Y,2), 'LineWidth',3, 'color', 'black')
xlabel('Observations'); title('10 realizations')
set(gca, 'fontsize',30); hold off
hgexport(gcf,'10_realizations.eps')
%% Estimated ACF
CM = jet(10);
figure('WindowState','Maximized')
hold on
for k=1:10
[acf,lags,bounds(:,k)] = autocorr(Y(:,k),'NumLags',20)
    ;
plot(acf,'-o','color',CM(k,:), 'LineWidth',3)
end
hold on
line([0 25], [mean(bounds(1,:)) mean(bounds(1,:))],'
    color', 'blue')
line([0 25], [mean(bounds(2,:)) mean(bounds(2,:))],'
    color', 'blue')
xlim([0 21]); xlabel('lags'); ylabel('sample
    autocorrelation');
title('Estimated ACF')
```

```
set(gca, 'fontsize',30);
hgexport(gcf,'ACF_2_1.eps')
%%
CM = jet(10);
figure('WindowState','Maximized')
hold on
for k=1:10
[acf,lags,bounds(:,k)] = autocorr(Y(:,k),'NumLags',20)
    ;
stem(acf,'color',CM(k,:), 'LineWidth',3)
end
hold on
line([0 25], [mean(bounds(1,:)) mean(bounds(1,:))],'
    color', 'blue')
line([0 25], [mean(bounds(2,:)) mean(bounds(2,:))],'
    color', 'blue')
xlim([0 21]); xlabel('lags'); ylabel('sample
    autocorrelation');
title('Estimated ACF')
set(gca, 'fontsize',30);
hgexport(gcf,'ACF_2_1_.eps')
%% Estimated PACF
figure('WindowState','Maximized')
hold on
for k=1:10
[pacf,lags,bounds_par(:,k)] = parcorr(Y(:,k),'NumLags'
    ,20);
plot(pacf,'-o','color',CM(k,:), 'LineWidth',3)
end
hold on
line([0 25], [mean(bounds_par(1,:)) mean(bounds_par
    (1,:))],'color', 'blue')
line([0 25], [mean(bounds_par(2,:)) mean(bounds_par
    (2,:))],'color', 'blue')
xlim([0 21]); xlabel('lags'); ylabel('sample partial
    autocorrelation');
title('Estimated PACF')
set(gca, 'fontsize',30);
hgexport(gcf,'PACF_2_1.eps')
%% Calculate the variance of each of the realisations.
estimated_variance = var(Y);
columnLabels = {'R1','R2','R3','R4','R5','R6','R7','R8
    ','R9','R10'};
matrix2latex(estimated_variance', 'Variance.tex', '
    rowLabels', columnLabels)
```

4.2 Exercise 2.2.2

Listing 2: 2.2.2

```
C = textscan(fopen('A2_sales.txt'), '%s %f','delimiter
    ',' ');
AttributeNames = {'Quater','Sales'};
Data = cell2struct(C,AttributeNames,2);
% Predict the values of Yt corresponding to t = 2019Q1
     and 2019Q2
%(Two steps ahead), together with 95% prediction
    intervals
% for the predictions. !!! Remember plot together.
% k - step predictions (ARMA)
%% define coefficients
% p=[];
p(1)=-1.04;
p(2)=0.2;
p(4)=-0.44;
p(5)=0.4576;
p(6)=-0.088;
p(8)=-0.1848;
p(9)=0.192192;
p(10)=-0.03696;
p(12)=-0.077616;
p(13)=0.08072064;
p(14)=-0.0155232;
p(16)=-0.03259872;
p(17)=0.03390266880;
p(18)=-0.006519744;
p(20)=-0.01369146240;
p(21)=0.01423912089;
%% first estimate
mm=2070;  %the given mean
Z_t=Data.Sales-mm; % susbtract mean
ttt=(-p(1:20))*flip(Z_t);
first_estimate = ttt+mm; % add the mean back
first_estimate
%% second estimate
second_estimate=(-p(1)*(first_estimate-mm)+(-p(2:21))*
    flip(Z_t))+mm;
second_estimate
%% trying to do with arma2ar
ar=arma2ar([1.04 -0.2 0 0.86 -0.8944 0.172],[0 0 0
    -0.42]);
%% first estimate
```

```
mm=2070;  %the given mean
Z_t=Data.Sales-mm; % susbtract mean
ttt=(ar(1:20))*flip(Z_t);
%tt= flip(Z_t);
%test =dot((-ar(1:20)),flip(Z_t) )+mm;
first_estimate = ttt+mm; % add the mean back
first_estimate
%% second estimate
second_estimate=ar(1)*(first_estimate-mm)+(ar(2:21))*
    flip(Z_t)+mm;
second_estimate
%%
first_conf=first_estimate+norminv([0.025 0.975])*sqrt
    (36963);
second_conf=second_estimate+norminv([0.025 0.975])*
    sqrt((1+ (ar(1))^2 )*36963);
%% stem(Data.Sales)
figure('WindowState','Maximized')
plot([Data.Sales], 'b*','MarkerSize',15); hold on
plot([21 22], [first_estimate second_estimate],'r*','
    MarkerSize',15)
plot([21 22], [first_conf(1) second_conf(1)],'--r','
    LineWidth',3)
plot([21 22], [first_conf(2) second_conf(2)],'--r','
    LineWidth',3)
plot([Data.Sales; first_estimate; second_estimate],'--
    g','LineWidth',3)
legend('Sales Data','Predictions','95% confidence
    intervals')
title('Two step prediction and 95% confidence
    intervals ')
set(gca, 'fontsize',30);
pause(5);
hgexport(gcf,'Two_step_predictions.eps')
```

4.3 Exercise 2.2.3

Listing 3: 2.2.3

```
p3=[1 -1.5 0.52];
r3=roots(p3);
p4=[1 -1.5 0.98];
r4=roots(p4);
figure('WindowState','Maximized')
subplot(1,2,1)
```

```
zplane(r3)
title('\phi_1')
set(gca, 'fontsize',30)
subplot(1,2,2)
zplane(r4)
title('\phi_2')
set(gca, 'fontsize',30)
%% histograms
model = arima(2,0,0);
%% different values of sigma and phi
rng(1); % seed
model1=arima('constant', 0, 'AR', {1.5 -0.52},...
    'Variance',0.1^(2));
Y_model1 = simulate(model1,300,'NumPaths',100) ;

model2=arima('constant', 0, 'AR', {1.5 -0.98},...
    'Variance',0.1^(2));
Y_model2 = simulate(model2,300,'NumPaths',100) ;

model3=arima('constant', 0, 'AR', {1.5 -0.52},...
    'Variance',5^(2));
Y_model3 = simulate(model3,300,'NumPaths',100) ;

model4=arima('constant', 0, 'AR', {1.5 -0.98},...
    'Variance',5^(2));
Y_model4 = simulate(model4,300,'NumPaths',100);
%% Find parameters
Pmdl1=[] ; Pmdl2=[] ; Pmdl3=[] ; Pmdl4=[] ;
Pmdl1_1=[]; Pmdl2_1=[]; Pmdl3_1=[]; Pmdl4_1=[];
for p_num=1:100
% Model 1
[Paramdl1,~] = estimate(model,Y_model1(:,p_num),'
   Display','off') ;
Pmdl1(p_num)=cell2mat(Paramdl1.AR(1,2));
Pmdl1_1(p_num)=cell2mat(Paramdl1.AR(1,1));
% Model 2
[Paramdl2,~] = estimate(model,Y_model2(:,p_num),'
   Display','off') ;
Pmdl2(p_num)=cell2mat(Paramdl2.AR(1,2));
Pmdl2_1(p_num)=cell2mat(Paramdl2.AR(1,1));
% Model 3~
[Paramdl3,~] = estimate(model,Y_model3(:,p_num),'
   Display','off') ;
Pmdl3(p_num)=cell2mat(Paramdl3.AR(1,2));
Pmdl3_1(p_num)=cell2mat(Paramdl3.AR(1,1));

```

```
% Model 4
[Paramdl4,~] = estimate(model,Y_model4(:,p_num),'
    Display','off') ;
Pmdl4(p_num)=cell2mat(Paramdl4.AR(1,2));
Pmdl4_1(p_num)=cell2mat(Paramdl4.AR(1,1));
end
%% Find quantile locations
Q1= quantile( Pmdl1 , [0.0250  0.975]) ;
Q2= quantile( Pmdl2 , [0.0250  0.975]) ;
Q3= quantile( Pmdl3 , [0.0250  0.975]) ;
Q4= quantile( Pmdl4 , [0.0250  0.975]) ;
%%
figure('WindowState','Maximized')
subplot(2,2,1)
histogram(Pmdl1); hold on
plot([Q1(1) Q1(1)], [0 35],'--r','LineWidth',3)
plot([Q1(2) Q1(2)], [0 35],'--r','LineWidth',3)
title('\sigma^2=0.1^2, \phi_2=0.52')
ylim([0 35])
set(gca, 'fontsize',30);
subplot(2,2,2)
histogram(Pmdl2); hold on
plot([Q2(1) Q2(1)], [0 35],'--r','LineWidth',3)
plot([Q2(2) Q2(2)], [0 35],'--r','LineWidth',3)
title('\sigma^2=0.1^2, \phi_2=0.98')
ylim([0 35])
set(gca, 'fontsize',30);
subplot(2,2,3)
histogram(Pmdl3); hold on
plot([Q3(1) Q3(1)], [0 35],'--r','LineWidth',3)
plot([Q3(2) Q3(2)], [0 35],'--r','LineWidth',3)
title('\sigma^2=5^2, \phi_2=0.52')
ylim([0 35])

set(gca, 'fontsize',30);
subplot(2,2,4)
histogram(Pmdl4); hold on
plot([Q4(1) Q4(1)], [0 35],'--r','LineWidth',3)
plot([Q4(2) Q4(2)], [0 35],'--r','LineWidth',3)
title('\sigma^2=5^2, \phi_2=0.98')
ylim([0 35])
set(gca, 'fontsize',30);
pause(5);
hgexport(gcf,'Histograms.eps')

%%
```

```
tableQ=[[Q1 abs(Q1(1)-Q1(2))]; [Q2 abs(Q2(1)-Q2(2))];
    [Q3 abs(Q3(1)-Q3(2))]; [Q4 abs(Q4(1)-Q4(2))]]';
row_2 = {'$\sigma^2=0.1^2$, $\phi_2=0.52$',...
    '$\sigma^2=0.1^2$,$ \phi_2=0.98$',...
    '$\sigma^2=5^2$, $\phi_2=0.52$',...
    '$\sigma^2=5^2$, $\phi_2=0.98$'};
col_2 = {'$2.5\%$','$97.5\%$','Difference'};
matrix2latex(tableQ', 'Qtabl.tex', 'rowLabels', row_2,
    'columnLabels',col_2)
%%
figure('WindowState','Maximized')
subplot(2,2,1)
plot(Pmdl1_1,-Pmdl1,'go') ; hold on
line([-2 0], [1 -1],'LineWidth',3);
line([0 2], [-1 1],'LineWidth',3)
line([-2 2], [1 1],'LineWidth',3)
plot(1.5,0.52,'r*','MarkerSize',15)
fplot(@(x) 0.25*(x)^2,[-2 2],'k--','LineWidth',3)
xlabel('\phi_1'); ylabel('\phi_2')
xlim([1.3 1.8])
ylim([0 1.1])
title('\sigma^2=0.1^2, \phi_2=0.52')
set(gca, 'fontsize',30);
subplot(2,2,2)
plot(Pmdl2_1,-Pmdl2,'go') ; hold on
line([-2 0], [1 -1],'LineWidth',3);
line([0 2], [-1 1],'LineWidth',3)
line([-2 2], [1 1],'LineWidth',3)
plot(1.5,0.98,'r*','MarkerSize',15)
fplot(@(x) 0.25*(x)^2,[-2 2],'k--','LineWidth',3)
xlabel('\phi_1'); ylabel('\phi_2')
xlim([1.3 1.8])
ylim([0 1.1])
title('\sigma^2=0.1^2, \phi_2=0.98')
set(gca, 'fontsize',30);
subplot(2,2,3)
plot(Pmdl3_1,-Pmdl3,'go') ; hold on
line([-2 0], [1 -1],'LineWidth',3);
line([0 2], [-1 1],'LineWidth',3)
line([-2 2], [1 1],'LineWidth',3)
plot(1.5,0.52,'r*','MarkerSize',15)
fplot(@(x) 0.25*(x)^2,[-2 2],'k--','LineWidth',3)
xlabel('\phi_1'); ylabel('\phi_2')
xlim([1.3 1.8])
ylim([0 1.1])
title('\sigma^2=5^2, \phi_2=0.52')
```

```
set(gca, 'fontsize',30);
subplot(2,2,4)
plot(Pmdl4_1, -Pmdl4,'go') ; hold on
plot(1.5,0.98,'r*','MarkerSize',15)
line([-2 0], [1 -1],'LineWidth',3);
line([0 2], [-1 1],'LineWidth',3)
line([-2 2], [1 1],'LineWidth',3)
fplot(@(x) 0.25*(x)^2,[-2 2],'k--','LineWidth',3)
xlabel('\phi_1'); ylabel('\phi_2')
xlim([1.3 1.8])
ylim([0 1.1])
%xt = @(t) 0.25*t^2;
%fplot(xt,[-2 2])
title('\sigma^2=5^2, \phi_2=0.98')
set(gca, 'fontsize',30);
pause(5);
hgexport(gcf,'All_estimates.eps')

%%
figure('WindowState','maximized')
line([-2 0], [1 -1],'LineWidth',3);  hold on
line([0 2], [-1 1],'LineWidth',3)
line([-2 2], [1 1],'LineWidth',3)
fplot(@(x) 0.25*(x)^2,[-2 2],'k--','LineWidth',3)
title('Parameter area for which the AR(2)-process is
    stationary.')
xlabel('\phi_1'); ylabel('\phi_2')
set(gca, 'fontsize',30);
pause(5);
hgexport(gcf,'param.eps')
```

4.4 More (for 2.2.1) calculations in Maple

```
> sigma:=0.4;
```
$$\sigma := 0.4 \tag{1.1}$$

```
> theta1:=0.8; theta2:=-0.5
```
$$\theta 1 := 0.8$$
$$\theta 2 := -0.5 \tag{1.2}$$

```
> epsilon1:=-0.8
```
$$\epsilon 1 := -0.8 \tag{1.3}$$

```
> gam0:=sigma^2
```
$$gam0 := 0.16 \tag{1.4}$$

```
> gam1:=theta1*sigma^2-epsilon1*gam0
```
$$gam1 := 0.256 \tag{1.5}$$

```
> gam2:=theta2*sigma^2-epsilon1*gam1
```
$$gam2 := 0.1248 \tag{1.6}$$

```
> lig1:=gamma0 + epsilon1*gamma1= gam0 +theta1*gam1+theta2*gam2
```
$$lig1 := \gamma 0 - 0.8\ \gamma 1 = 0.30240 \tag{1.7}$$

```
> lig2:=gamma1 + epsilon1*gamma0=theta1*gam0+theta2*gam1
```
$$lig2 := -0.8\ \gamma 0 + \gamma 1 = 0. \tag{1.8}$$

```
> lig3:=gamma2+epsilon1*gamma1=theta2*gam0
```
$$lig3 := -0.8\ \gamma 1 + \gamma 2 = -0.080 \tag{1.9}$$

```
> solve({lig1,lig2,lig3},{gamma1, gamma0,gamma2})
```
$$\{\gamma 0 = 0.8400000000, \gamma 1 = 0.6720000000, \gamma 2 = 0.4576000000\} \tag{1.10}$$

```
> gamma0:=0.8400000000
```
$$\gamma 0 := 0.8400000000 \tag{1.11}$$

```
> gamma1:=0.6720000000
```
$$\gamma 1 := 0.6720000000 \tag{1.12}$$

```
> gamma2:=0.4576000000
```
$$\gamma 2 := 0.4576000000 \tag{1.13}$$

```
> gamma1/gamma0
```
$$0.8000000000 \tag{1.14}$$

```
> gamma2/gamma0
```
$$0.5447619048 \tag{1.15}$$

2.2.2

$$\left(1 - 0.86\, B^4 - 1.04\, B + 0.8944\, B^5\right) Y_t = \left(1 - 0.42\, B^4\right) \epsilon_t$$

```
> k1:=(1-1.04*B+0.2*B^2)*(1-0.86*B^4); expand(%)
```

$$k1 := \left(1 - 1.04\, B + 0.2\, B^2\right)\left(1 - 0.86\, B^4\right)$$

$$1 - 0.86\, B^4 - 1.04\, B + 0.8944\, B^5 + 0.2\, B^2 - 0.172\, B^6 \tag{1.1}$$

```
> x:=(0.42*B^4)
```

$$x := 0.42\, B^4 \tag{1.2}$$

```
> k2:=sum(x^(i), i=0..5)
```

$$k2 := 1 + 0.4200000000\, B^4 + 0.1764000000\, B^8 + 0.07408800000\, B^{12} + 0.03111696000\, B^{16} + 0.01306912320\, B^{20} \tag{1.3}$$

```
> 0.42^5
```

$$0.0130691232 \tag{1.4}$$

```
> k1*k2: expand(%)
```

$$-0.002247889190\, B^{26} - 1.04\, B + 0.01423912089\, B^{21} + 0.01168902379\, B^{25} - 0.002738292480\, B^{22} + 1 - 0.1848000000\, B^8 - 0.01552320000\, B^{14} - 0.006519744000\, B^{18} + 0.03390266880\, B^{17} - 0.03696000000\, B^{10} + 0.1921920000\, B^9 + 0.08072064000\, B^{13} - 0.01123944595\, B^{24} - 0.08800000000\, B^6 + 0.4576000000\, B^5 + 0.2\, B^2 - 0.4400000000\, B^4 - 0.01369146240\, B^{20} - 0.03259872000\, B^{16} - 0.07761600000\, B^{12} \tag{1.5}$$

$$-0.44\, B^4 - 1.04\, B + 1 + 0.01423912089\, B^{21} - 0.01369146240\, B^{20} - 0.03259872\, B^{16} - 0.077616\, B^{12} - 0.1848\, B^8 - 0.01123944595\, B^{24} + 0.192192\, B^9 + 0.08072064\, B^{13} + 0.03390266880\, B^{17} - 0.03696\, B^{10} - 0.0155232\, B^{14} - 0.006519744\, B^{18} + 0.4576\, B^5 - 0.088\, B^6 + 0.2\, B^2$$

$$-0.44\, B^4 - 1.04\, B + 1 - 0.01369146240\, B^{20} - 0.03259872\, B^{16} - 0.077616\, B^{12} - 0.1848\, B^8 - 0.01123944595\, B^{24} + 0.192192\, B^9 + 0.08072064\, B^{13} + 0.03390266880\, B^{17} - 0.03696\, B^{10} - 0.0155232\, B^{14} - 0.006519744\, B^{18} + 0.4576\, B^5 - 0.088\, B^6 + 0.2\, B^2 \tag{1.6}$$

$$-0.44\, Y_{t-4} - 1.04\, Y_{t-1} + Y_t + 0.01423912089\, Y_{t-21} - 0.01369146240\, Y_{t-20} - 0.03259872\, Y_{t-16} - 0.077616\, Y_{t-12} - 0.1848\, Y_{t-8} + 0.192192\, Y_{t-9} + 0.08072064\, Y_{t-13} + 0.03390266880\, Y_{t-17} - 0.03696 Y_{t-10} - 0.0155232\, Y_{t-14} - 0.006519744\, Y_{t-18} + 0.4576\, Y_{t-5} - 0.088\, Y_{t-6} + 0.2\, Y_{t-2} = \epsilon_t$$