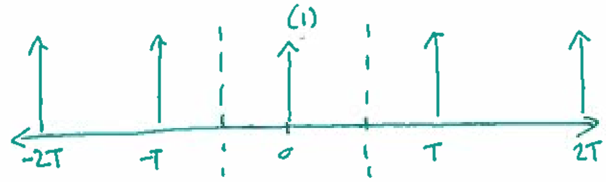
Problem Set 7 - Jiaying Wei

1. Consider a train of unit impulses separated by T time units given by

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

a. sketch it



b. Find the fourier series representation of p(t)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt} \qquad C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-\frac{2\pi}{T}jkt}\, dt \qquad x(t) = \begin{cases} 1 @ kT \\ 0 @ \text{else} \end{cases}$$

If we think about the picking property of δ then it should pick the value of $e^{-\frac{2\pi}{T}jk(0)} = 1$ w/ an area of 1 so the $\int dt = 1$

$$C_k = \frac{1}{T}\left(\int_{-T/2}^{0} 0\, dt + \int_{-0}^{+0} 1 \cdot e^{-\frac{2\pi}{T}jkt}\, dt + \int_{0}^{T/2} 0\, dt\right)$$

$$\boxed{C_k = \frac{1}{T}(0 + 1 + 0) = \frac{1}{T}}$$ (so it appears that there are no dependencys on k)

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} e^{j\frac{2\pi}{T}kt}}$$

c. Let a function x(t) be represented as a Fourier series w/ an infinite # of terms:

ie

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt}$$

Find X(w) in terms of C_k

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$ ← the inverse CTFT

$$X(\omega) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} C_k e^{j\frac{2\pi}{T}kt}\, dt$$

$$C_k = \frac{\omega_0}{2\pi} X(\omega_0 k) \qquad \omega_0 = \frac{2\pi}{T} \text{ (fundamental freq)}$$

$$C_k = \frac{\cancel{2\pi}}{T} \cdot \frac{1}{\cancel{2\pi}} X(\omega) \qquad \omega = \omega_0 k$$

$$\boxed{C_k = \frac{1}{T} X(\omega)}$$

d. Using your answer to the previous parts find $P(\omega)$

$P(\omega)$ is like $X(\omega)$ for $p(t)$

so plug & chug

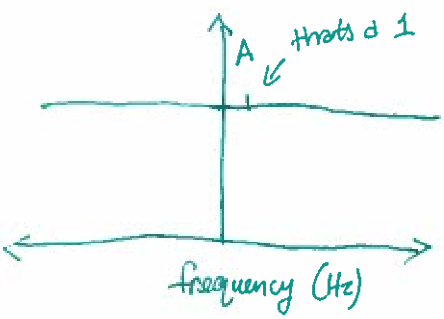
$$C_k = \frac{1}{T} \qquad C_k = \frac{1}{T} X(\omega)$$

$$\frac{1}{T} = \frac{1}{T} X(\omega)$$

$$P(\omega) = X(\omega) = 1$$

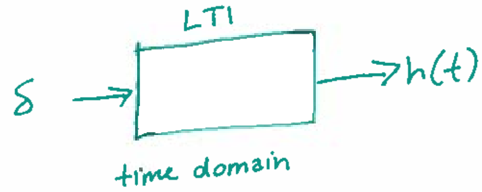
which makes sense because the impulse has a amplitude of 1 @ all frequencies; that's why it can characterize systems

e. sketch $P(\omega)$

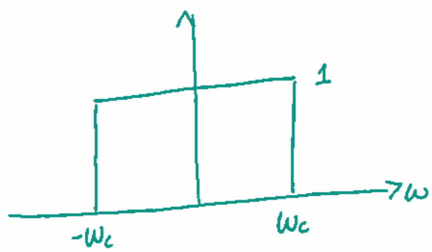


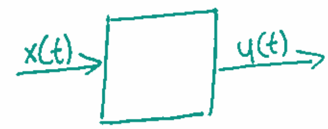
Changing T (the spacing between impulses) will do absolutely nothing in the frequency domain. Since we're only evaluating the ~~CT~~ CTFT over a single period.

2. Consider an LTI system w/ impulse response $h(t)$ input $x(t)$, output $y(t)$



$H(\omega)$ - frequency domain





$y(t) = x(t) * h$

a. Find $h(t)$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega) e^{j\omega t}\, d\omega \qquad H(\omega) = \begin{cases} 0 & @ \text{ else} \\ 1 & @ -\omega_c < \omega < \omega_c \end{cases}$$

~~$Y(\omega) = X(t)H(t)$~~

$Y(\omega) = X(\omega)H(\omega)$

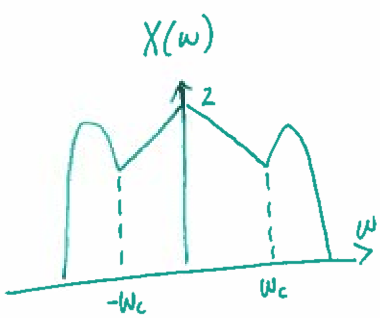
$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\, e^{j\omega t}\, d\omega$$

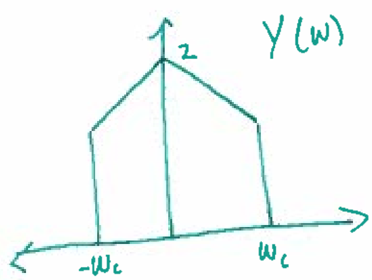
$$h(t) = \frac{1}{2\pi}\left(\frac{1}{jt} e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}\right) = \frac{1}{\pi t}\left(\frac{e^{j\omega_c t}}{2j} - \frac{e^{-j\omega_c t}}{2j}\right) \qquad \theta = \omega_c t$$

$$\boxed{h(t) = \frac{1}{2\pi}\left(\frac{e^{j\omega_c t} - e^{-j\omega_c t}}{jt}\right)} = \boxed{\frac{1}{t\pi}\sin(\omega_c t)}$$

b

sketch $Y(\omega)$





everything between $-\omega_c$ and ω_c was kept anything else was destroyed

c This is an ideal lowpass because anything above ω_c gets destroyed and anthing below is persevered $*1$

And since negative frequencies just get reflected up its the same behavior on the negative end.

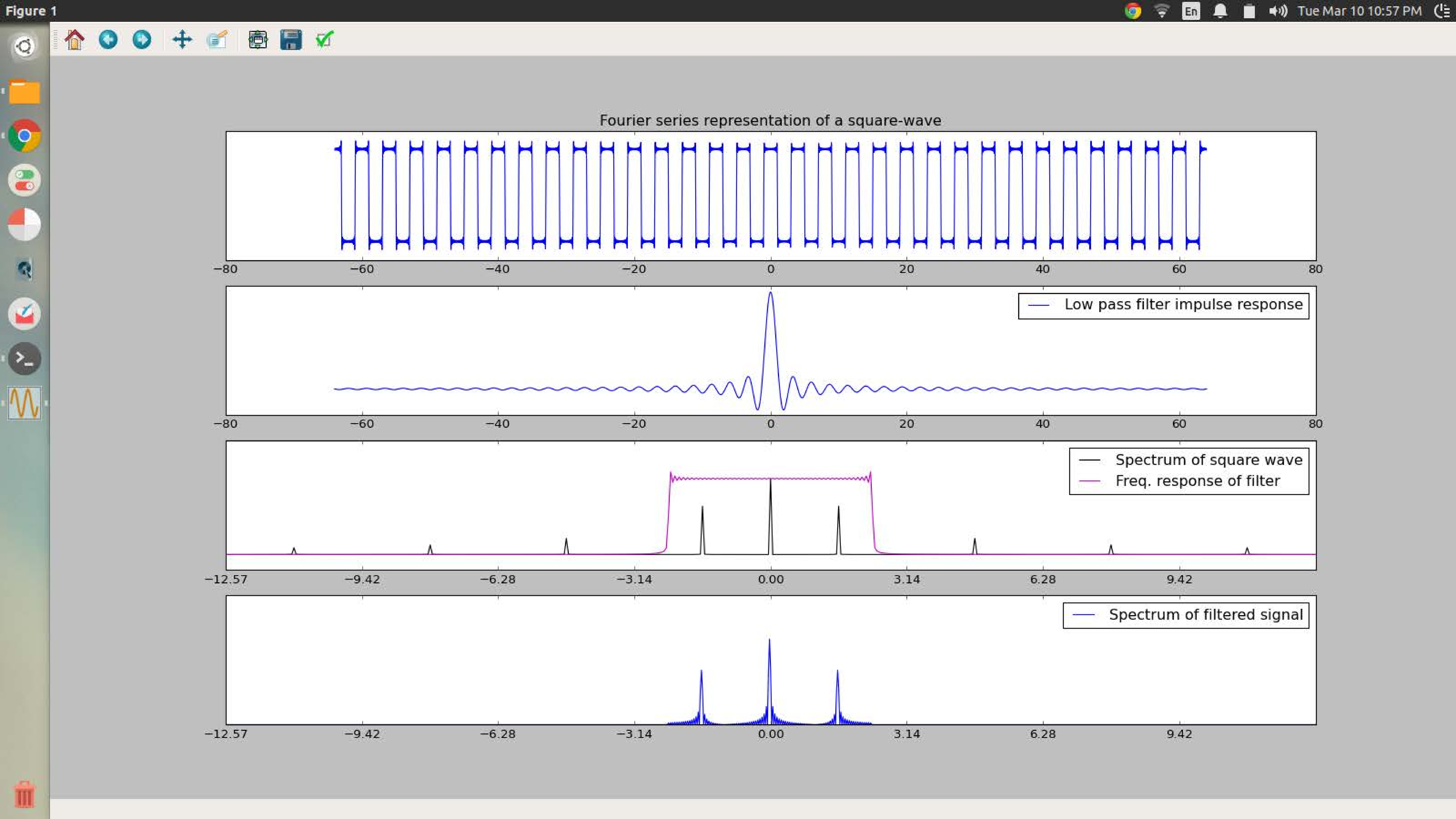


Figure 1
En
Tue Mar 10 10:57 PM
Fourier series representation of a square-wave
−80
−60
−40
−20
0
20
40
60
80
Low pass filter impulse response
−80
−60
−40
−20
0
20
40
60
80
Spectrum of square wave
Freq. response of filter
−12.57
−9.42
−6.28
−3.14
0.00
3.14
6.28
9.42
Spectrum of filtered signal
−12.57
−9.42
−6.28
−3.14
0.00
3.14
6.28
9.42

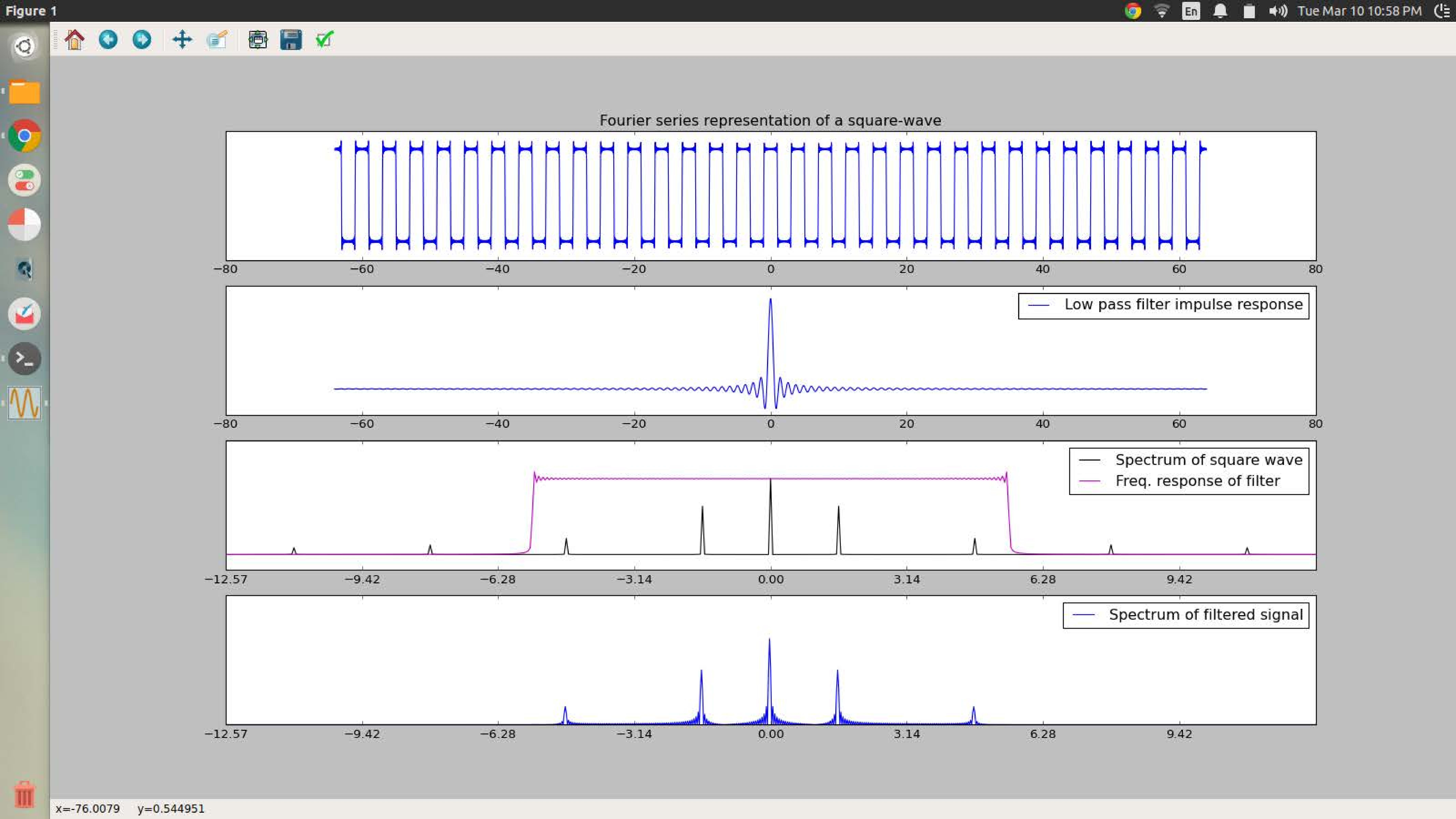
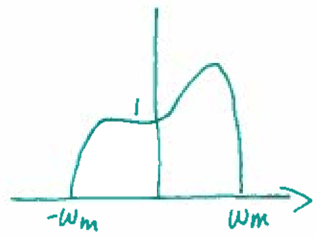


Figure 1
Tue Mar 10 10:58 PM
En
Fourier series representation of a square-wave
−80 −60 −40 −20 0 20 40 60 80
Low pass filter impulse response
−80 −60 −40 −20 0 20 40 60 80
Spectrum of square wave
Freq. response of filter
−12.57 −9.42 −6.28 −3.14 0.00 3.14 6.28 9.42
Spectrum of filtered signal
−12.57 −9.42 −6.28 −3.14 0.00 3.14 6.28 9.42
x=-76.0079 y=0.544951

3. Consider a signal $x(t)$ which is band limited to the range $[-\omega_m, \omega_m]$
(this signal only has frequencies from $-\omega_m$ to ω_m)

$x(t)$ $X(\omega)$



$x(t) \rightarrow \boxed{\quad} \rightarrow y(t)$

where $y(t) = x(t)\cos(\omega_c t)$

$\omega_c >> \omega_M$ ← much greater than

~~a. sketch $Y(\omega)$ $Y(\omega) = X(\omega)H(\omega)$~~

~~$x(t) \rightarrow \boxed{\quad} \rightarrow x(t)\cos(\omega_c t)$~~

~~time domain~~

~~* This system scales $x(t)$ by $\cos(\omega_c t)$~~

~~doesn't this t make it not time invariant~~

~~Scaling in the time domain =~~

a. sketch $Y(\omega)$

$y(t) = x(t)\cos(\omega_c t)$

↑ multiplication in the time domain

From the table

$y(t) \longrightarrow Y(\omega)$

$x(t)h(t) \longrightarrow \frac{1}{2\pi} X * H(\omega)$

$$Y(\omega) = \frac{1}{2\pi} X * H(\omega) \quad \text{where} \quad h(t) = \cos(\omega_c t)$$

From the table again

$\cos(\omega_o t) \rightarrow \pi\delta(\omega - \omega_o) + \pi\delta(\omega + \omega_o)$

so

$$Y(\omega) = \frac{1}{2\pi} X * \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

* Convolving a function w/ an impulse moves the function to the location of the impulse and scales it

So convolving $\frac{1}{2\pi} X$ with $\pi\delta$ @ $-\omega_c$ and $\pi\delta$ @ ω_c should result in: