Problem Set 8 - Jiaying Wei

1. Consider a signal $x(t)$ which is bandwidth limited to ω_m

$x(t)$

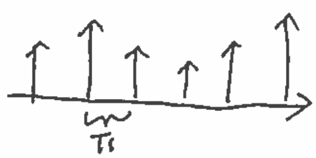
$X(\omega)$

$p(t)$ is an impulse train w/ impulses seperated by T_s

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$

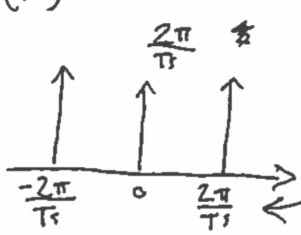
$$x_p(t) = x(t)p(t)$$

a) Sketch $x_p(t)$



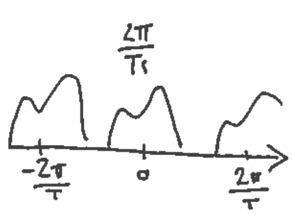
← impulse train w/ shape of $x(t)$ like discrete sampling

b) Sketch $P(\omega)$



frequency is one over period

c) Sketch $X_p(\omega)$

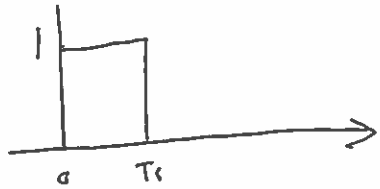


d) To ensure that you have all the information in $X_p(\omega)$ to be able to recover $X(\omega)$ you need to sample $\frac{2\pi}{T_s}$ of the original otherwise they will overlap and you can't unbake the cake.

e) You bandpass it

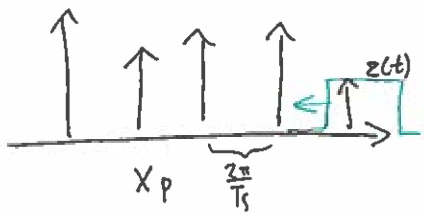
and then inverse CTFT it

f) Consider $z(t)$

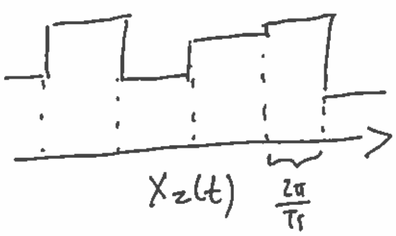


* $X_z(t)$ is a zero order hold reconstruction of the sampled signal $x_p(t)$

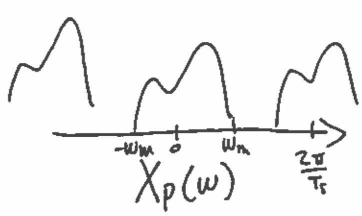
g) Sketch $x_z(t) = x_p * z(t)$

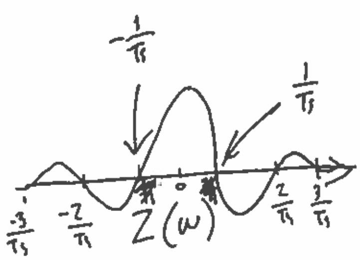


As $z(t)$ travels ← : convolving w/ $x_p(t)$ it takes on the value of the impulse its covering

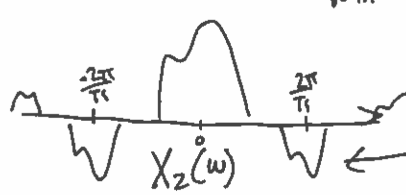


h) Sketch $X_z(\omega)$ which should be ~~x~~ = to $X_p(\omega)Z(\omega)$



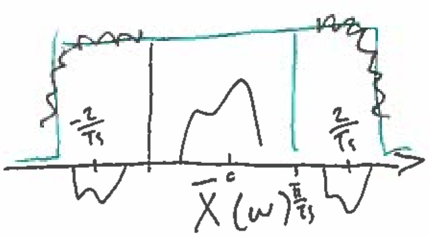


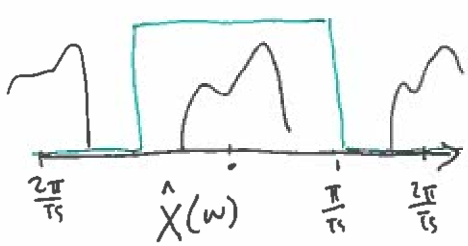
assuming that $w_m < \frac{1}{T_s}$

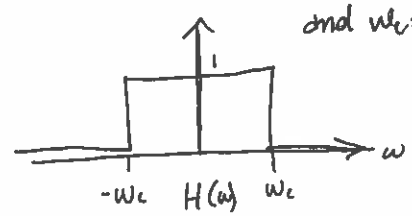


← not necessarily upside down

i) Sketch ~~Xz~~ $\bar{X}(\omega) = X_z(\omega)H(\omega)$ and $\hat{X}(\omega) = X_p(\omega)H(\omega)$ where $H(\omega)$ is







and $w_c = \frac{\pi}{T_s}$

j) ↑ this ones more distorted b/c

↑ this ones a little less distorted

k) What is the ratio of $\bar{X}(w_M)$ to $\hat{X}(w_M)$

$w_M = \frac{\pi}{T_s}$

The ratio is 1:1

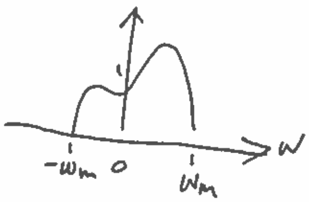
$\bar{X}(w_m) = 0$

$\hat{X}(w_m) = 0$

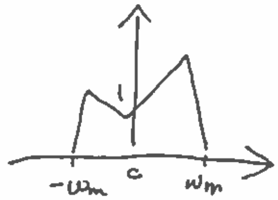
2. Consider $y(t) = x_1(t)\cos(\omega_1 t) + x_2(t)\cos(\omega_2 t)$

$\underbrace{X_1(\omega) = 0 \quad X_2(\omega) = 0}_{\text{if } |\omega| > \omega_m}$

$X_1(\omega)$



$X_2(\omega)$



assume

$\omega_1 \gg \omega_m$

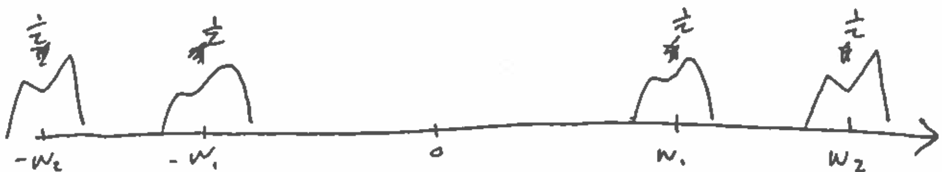
$\omega_2 \gg \omega_m$

$\omega_1 + 2\omega_m < \omega_2$ ← they dont overlap

$\omega_2 > \omega_1$

a. Sketch $Y(\omega)$

$$Y(\omega) = X_1(\omega)\left(\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)\right)$$
$$+ X_2(\omega)\left(\pi\delta(\omega-\omega_2) + \pi\delta(\omega+\omega_2)\right)$$

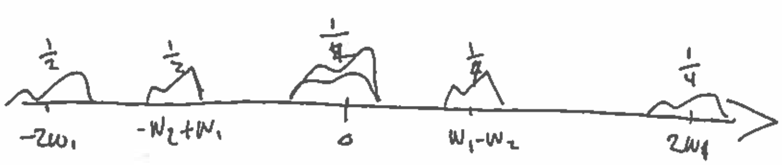


$Y(\omega)$

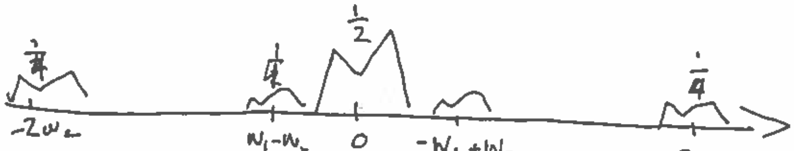
b. Sketch the Forier Transforms of $y(t)\cos(\omega_1 t)$ and $y(t)\cos(\omega_2 t)$

$$FT\{y(t)\cos(\omega_1 t)\} = \begin{pmatrix} X_1(\omega)\left(\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)\right) \\ + X_2(\omega)\left(\pi\delta(\omega-\omega_2) + \pi\delta(\omega+\omega_2)\right)\end{pmatrix}\left(\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)\right)$$

$$= X_1(\omega)\left(\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)\right)^2$$
$$+ X_2(\omega)\left(\pi\delta(\omega-\omega_2) + \pi\delta(\omega+\omega_2)\right)\left(\pi\delta(\omega-\omega_1) + \pi\delta(\omega+\omega_1)\right)$$

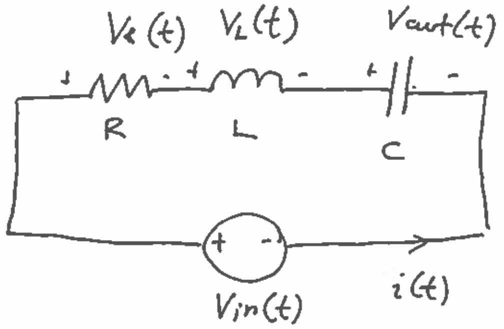


$FT\{y(t)\cos(\omega_1 t)\}$



c. We can recover $x_1(t)$ by bandpassing $FT\{y(t)\cos(\omega_1 t)\}$ and multiplying by 2. We can recover $x_2(t)$ by bandpassing $FT\{y(t)\cos(\omega_2 t)\}$ and multiplying by 2.

3. RLC System



$$i(t) = C\frac{d}{dt}V_{out}(t)$$

$$V_L(t) = L\frac{d}{dt}i(t)$$

a. Write a differential equation relating Vout and Vin(t)

$$V_R(t) + V_L(t) + \underline{V_{out}(t)} = \underline{V_{in}(t)}$$

$$iR + L\frac{d}{dt}i + V_{out}(t) = V_{in}(t)$$

$$C\frac{d}{dt}V_{out}(t)R + L\frac{d}{dt}C\frac{d}{dt}V_{out}(t) + V_{out} = V_{in}(t)$$

$$LC\frac{d^2}{dt}V_{out}(t) + RC\frac{d}{dt}V_{out}(t) + V_{out} = V_{in}(t)$$

b Find $H(\omega)$ the transfer function that relates $\frac{\text{input (fq)}}{\text{output (fq)}}$

$$LCj^2\omega^2 V_{out}(\omega) + RCj\omega V_{out}(\omega) + V_{out}(\omega) = V_{in}(\omega)$$

$$V_{out}(\omega)\left(LCj^2\omega^2 + RCj\omega + 1\right) = V_{in}(\omega)$$

$$H(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{1}{LCj^2\omega^2 + RCj\omega + 1} = \frac{1}{-LC\omega^2 + RCj\omega + 1}$$

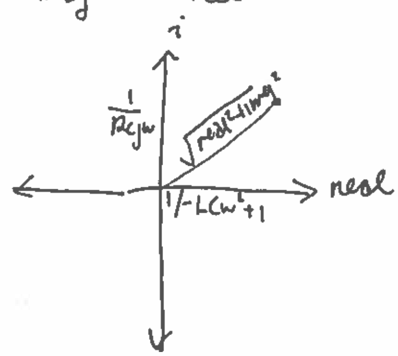
$$= \frac{1}{\underbrace{RCj\omega}_{imag}\ \underbrace{-LC\omega^2 + 1}_{real}}$$

c. Find the magnitude of $H(\omega)$

$$|H(\omega)| = \frac{1}{\sqrt{(RC\cancel{j}\omega)^2 + (-LC\omega^2+1)^2}}$$

$$\cancel{= \frac{1}{\sqrt{-R^2C^2\omega^2}}}$$

* You don't include the j in there because it's the axis kinda like if $-LC\omega^2+1$ real had a real attached to it



a. As a function of R L C find the value of ω that maximizes $|H(\omega)|$

$$|H(\omega)| = \frac{1}{\sqrt{(RC\cancel{j}\omega)^2 + (-LC\omega^2+1)^2}} = \left((RC\cancel{j}\omega)^2 + (-LC\omega^2+1)^2\right)^{-\frac{1}{2}}$$

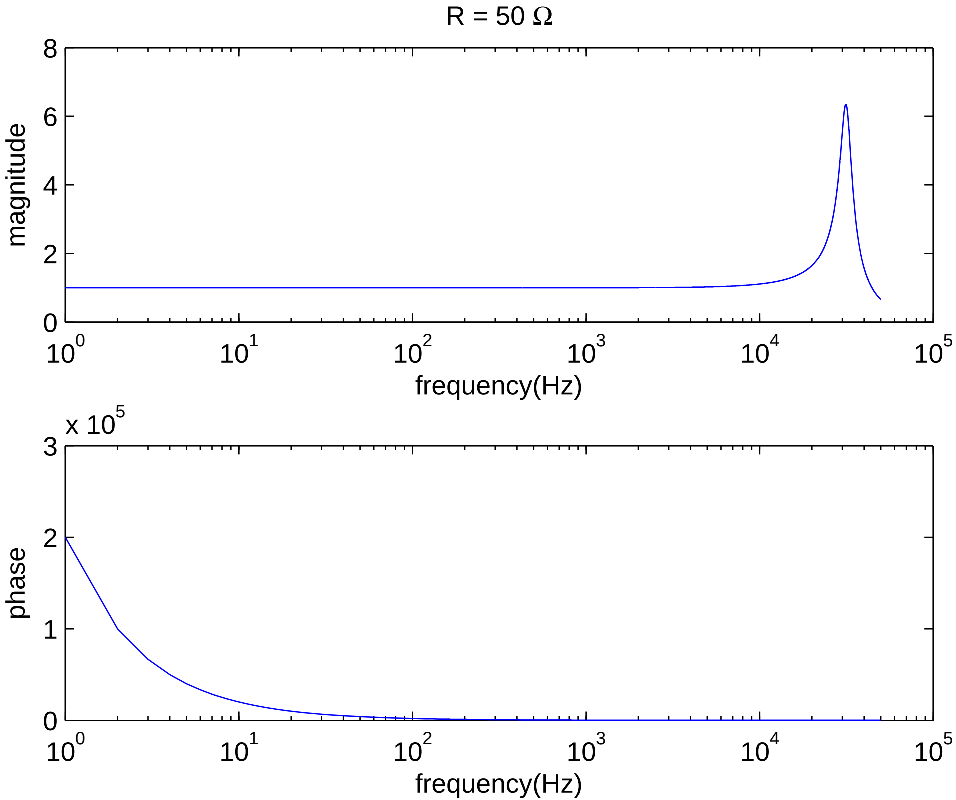
$$\frac{d}{d\omega}H(\omega) = \cancel{\frac{-1}{2}\left(2RC_j\omega - 2\cdot 2LC\omega\right)^{-\frac{3}{2}}} \quad 2(-LC\omega^2+1)\cdot -2LC\omega$$

$$= -\frac{1}{2}\left((RC\cancel{j}\omega)^2 + (-LC\omega^2+1)^2\right)^{-\frac{3}{2}}\left(2RC\cancel{j}\omega - 4LC\right)$$

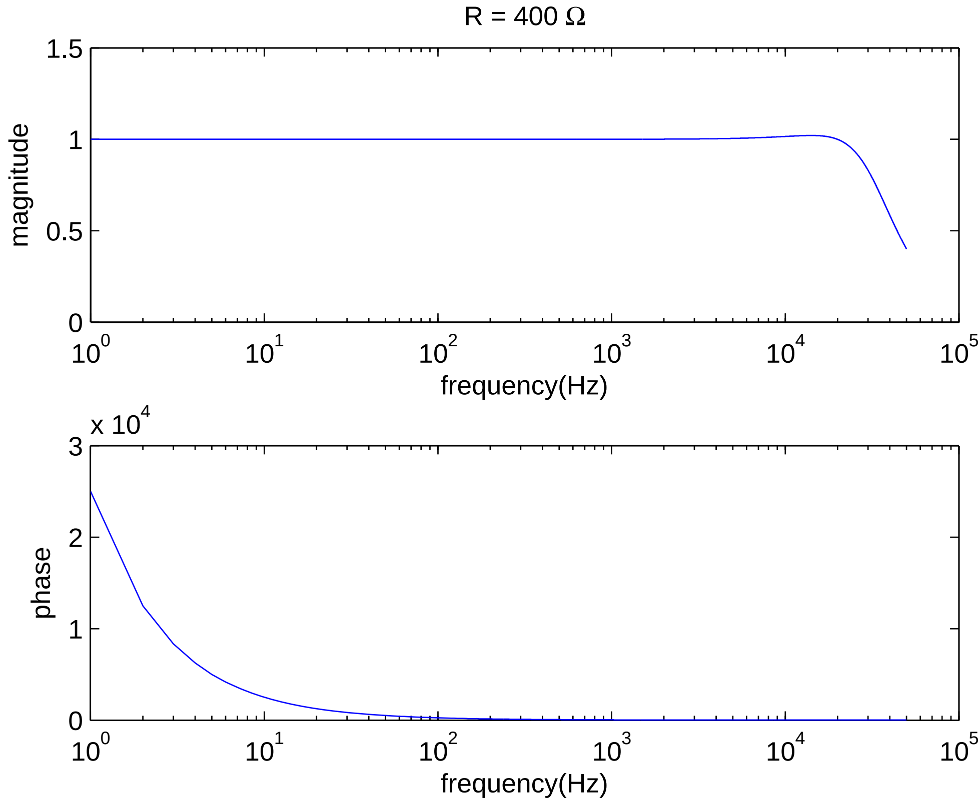
$$= -\frac{1}{2}\left((RC\omega)^2 + (-LC\omega^2+1)^2\right)^{-\frac{3}{2}}\left(2RC\omega + -4LC\omega(-LC\omega^2+1)\right)$$

* So using the calculus
  - ↳ when ~~theres a~~ the 1st derivative $=0$ theres a min/max
- ↳ then using Wolfram Alpha to find the roots

$\frac{d}{d\omega}H(\omega) = 0$ when $\omega = \frac{\sqrt{2L-R}}{\sqrt{2}\sqrt{C}L}$ when $\sqrt{C}L \neq 0$ or $\omega = -\frac{\sqrt{2L-R}}{\sqrt{2}\sqrt{C}L}$ when $\sqrt{C}L \neq 0$ or $\omega = 0$ when $CL \neq 0$



R = 50 Ω
magnitude
8
6
4
2
0
10^0
10^1
10^2
10^3
10^4
10^5
frequency(Hz)
x 10^5
phase
3
2
1
0
10^0
10^1
10^2
10^3
10^4
10^5
frequency(Hz)



R = 400 Ω
magnitude
1.5
1
0.5
0
10^0
10^1
10^2
10^3
10^4
10^5
frequency(Hz)
x 10^4
3
2
1
0
phase
10^0
10^1
10^2
10^3
10^4
10^5
frequency(Hz)