Principles of Communications (通信系统原理)

Undergraduate Course

Chapter 3: Random Signals and Noise

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Chapter 3: Random Signals and Noise

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Chapter 3: Random Signals and Noise

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Introduction

- Deterministic signals cannot convey information.
- Unpredictability or randomness is an essential property for information bearing signals.
- While one type of random signal creates information (i.e. increases knowledge) at a communication receiver, another type of random signal, known as noise, destroys it (i.e. decreases knowledge).
- Interference (signals arising from information sources other than the intended one) can be considered as a type of noise.
- Describing information signals and noise mathematically should use probability theory.

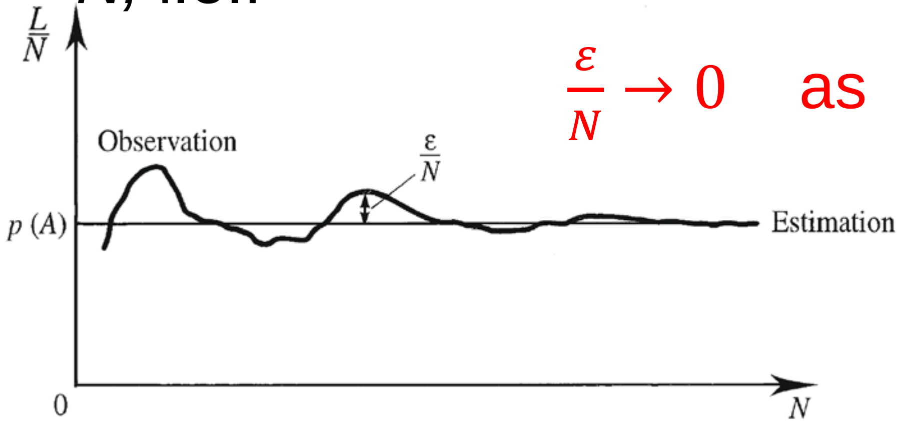
Probability
Basic Definitions

- Consider an experiment with three possible, random, outcomes *A*, *B*, and *C*. If the experiment is repeated *N* times and the outcome *A* occurs *L* times then the probability of outcome *A* is defined by

$$P(A) \triangleq \lim_{N \to \infty} \left\{ \frac{L}{N} \right\}$$

- Let the error be ε. The ratio ε / N tends to zero for large *N*, i.e.:

$$\frac{\varepsilon}{N} \to 0 \quad \text{as} \quad N \to \infty$$



Probability
Basic Definitions

- This experiment can be performed *N* times with one set of apparatus (temporal experiment) or *N* times, simultaneously, with *N* sets of identical apparatus (ensemble experiment).
- If, after *N* trials, the outcome *A* occurs L times and the outcome *B* occurs M times, and *A* and *B* are *mutually exclusive* (i.e. they cannot occur together), then the probability that *A* or *B* occurs is

$$P(A \text{ or } B) = \lim_{N\to\infty}\left\{\frac{L+M}{N}\right\} = \lim_{N\to\infty}\left\{\frac{L}{N}\right\} + \lim_{N\to\infty}\left\{\frac{M}{N}\right\} = P(A) + P(B)$$

- This law of additive probabilities can be used for any number of mutually exclusive events.
- The outcome of the experimental trials is called a random variable. Random variables can be discrete or continuous.

Probability
Basic Definitions

- **[Example 2.1]**
- A dice is thrown once. What is the probability that: (i) the dice shows 3; (ii) the dice shows 6; (iii) the dice shows a number greater than 2; (iv) the dice does not show 5?

Chapter 3: Random Signals and Noise

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Probability Theory

Conditional Probabilities, Joint Probabilities and Bayes's Rule

- The probability of event *A* occurring given that event *B* is known to have occurred, $P(A|B)$, is called the conditional probability of *A* on *B* (or the probability of *A* conditional on *B*).
- The probability of *A* and *B* occurring together, $P(A,B)$, is called the joint probability of *A* and *B*.
- Joint and conditional probabilities are related by:
$$P(A,B) = P(B)P(A|B) = P(A)P(B|A)$$
- The Bayes's rule:
$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

Probability Theory

Conditional Probabilities, Joint Probabilities and Bayes's Rule

- **[Example 2.2]**
- Four cards are dealt off the top of a shuffled pack of 52 playing cards. What is the probability that all the cards will be of the same suit?

Probability Theory
Statistical Independence

- Events A and B are statistically independent if the occurrence of one does not affect the probability of the other occurring, i.e.:

$$P(A|B) = P(A) \quad \text{and} \quad P(B|A) = P(B)$$

- It follows that for statistically independent events

$$P(A,B) = P(A)P(B)$$

- **[Example 2.3]**
- Two cards are dealt one at a time, face up, from a shuffled pack of cards. Show that these two events are not statistically independent.

Probability Theory

Discrete Probability of Errors in a Data Block

- Performance prediction in a digital coding system: What is the probability of having more than a given number of errors in a fixed length codeword?
- Assume that the probability of single bit (binary digit) error is P_e, the number of errors is R', and n is the block length, i.e. we require to determine the probability of having more than R' errors in a block of n digits:

$$P(> R' \text{ errors}) = 1 - P(\leq R' \text{ errors})$$
$$= 1 - [P(0 \text{ errors}) + P(1 \text{ errors}) + \cdots + P(R' \text{errors})]$$

where we assume that the errors are independent.

Probability Theory

Discrete Probability of Errors in a Data Block

- For the case of j errors in n digits, with a probability of error per digit of P_e, we have

$$P(j \text{ errors}) = {}^{n}C_j (P_e)^j (1 - P_e)^{n-j}$$

where the binomial coefficient ${}^{n}C_j$ is given by

$${}^{n}C_j = \frac{n!}{j!\,(n-j)!} = \binom{n}{j}$$

- The probability of having more than R' errors is thus:

$$P(> R' \text{errors}) = 1 - \sum_{j=1}^{R'} P(j \text{ errors})$$

Probability Theory

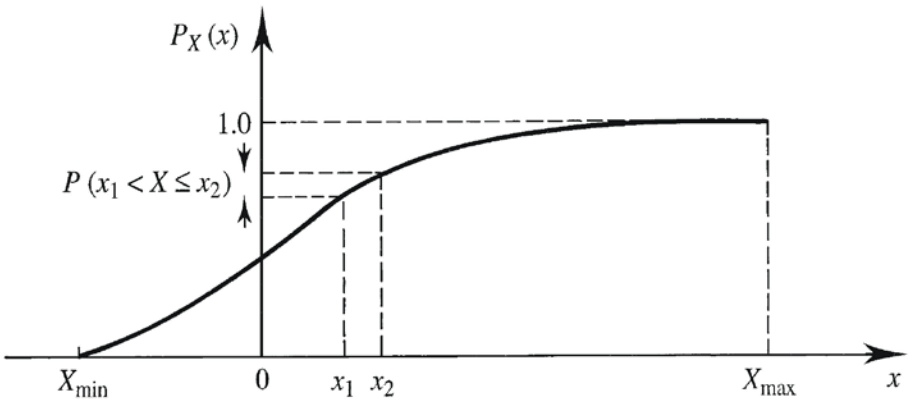
Discrete Probability of Errors in a Data Block

- **[Example 2.4]**
- If the probability of single digit error is 0.01 and hence the probability of correct digit reception is 0.99, calculate the probability of 0 to 2 errors occurring in a 10-digit codeword.

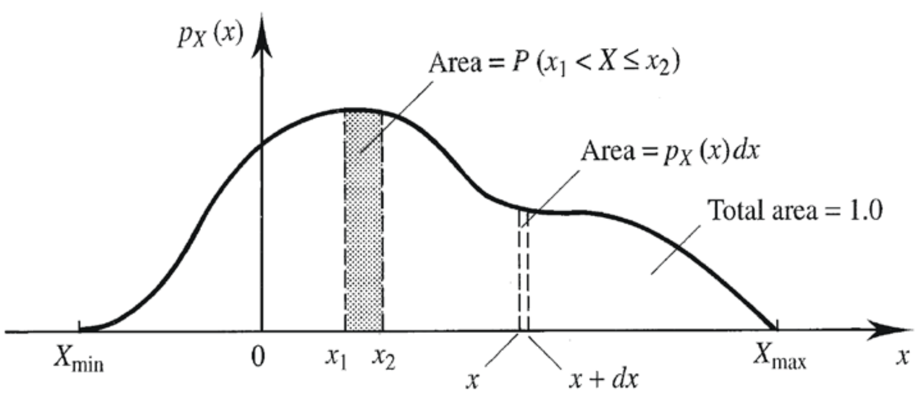
Probability Theory
Cumulative Distributions

- A cumulative distribution (CD, also called a probability distribution) is a curve showing the probability that the value of the random variable X will be less than or equal to some specific value x:

$$P_X(x) = P(X \leq x)$$



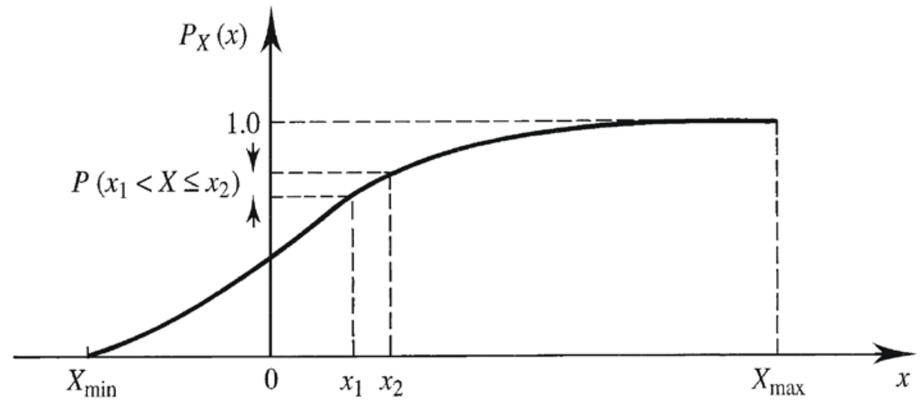
(a) Cumulative distribution of X



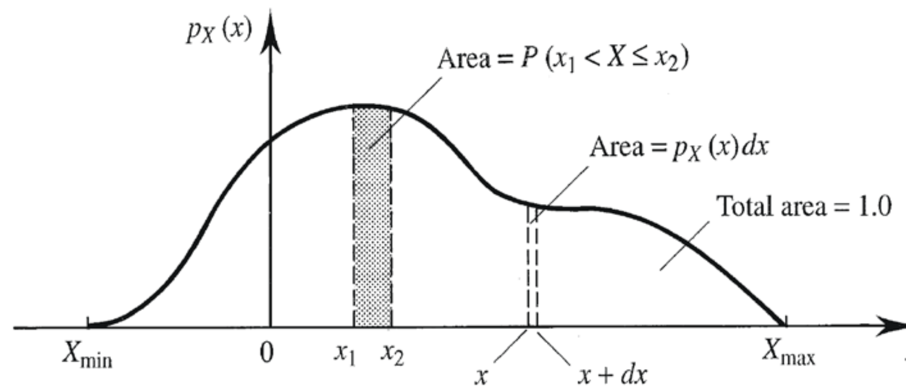
(b) Probability density function of X

Probability Theory
Cumulative Distributions (Properties)

- $0 \leq P_X(x) \leq 1$, for $-\infty \leq x \leq \infty$
- $P_X(-\infty) = 0$
- $P_X(\infty) = 1$
- $P_X(x_2) - P_X(x_1)$
  $= P(x_1 < X \leq x_2)$
- $\frac{dP_X(x)}{dx} \geq 0$
- Exceedance curves are complementary to CDs:
  $P(X > x) = 1 - P_X(x)$



(a) Cumulative distribution of X



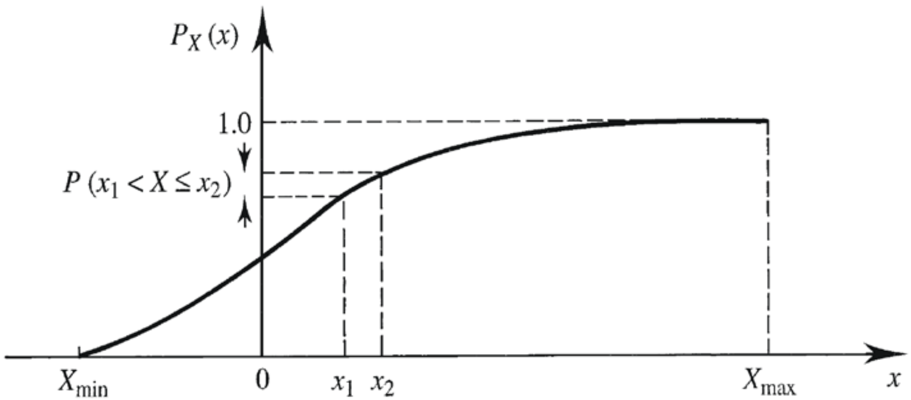
(b) Probability density function of X

Probability Theory
Probability Density Functions

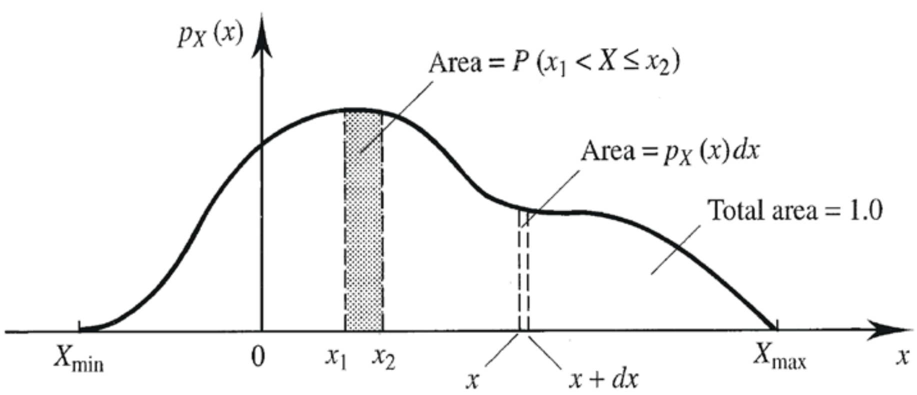
- Probability density functions (pdfs) give the probability that the value of a random variable, X, lies between x and $x + dx$, i.e.:

$$P_X(x + dx) - P_X(x) = \frac{dP_X(x)}{dx} dx$$

- The factor $\frac{dP_X(x)}{dx}$, normally denoted by $p_X(x)$, is defined as the pdf of X.



(a) Cumulative distribution of X



(b) Probability density function of X

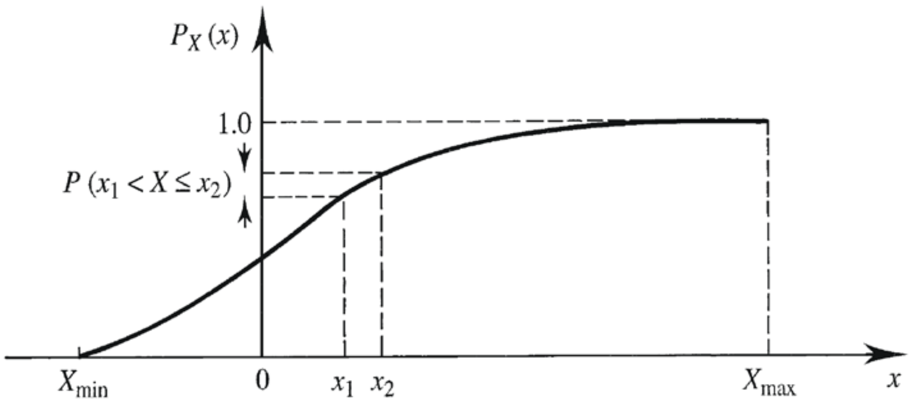
Probability Theory
Probability Density Functions

- Pdfs and CDs are related by

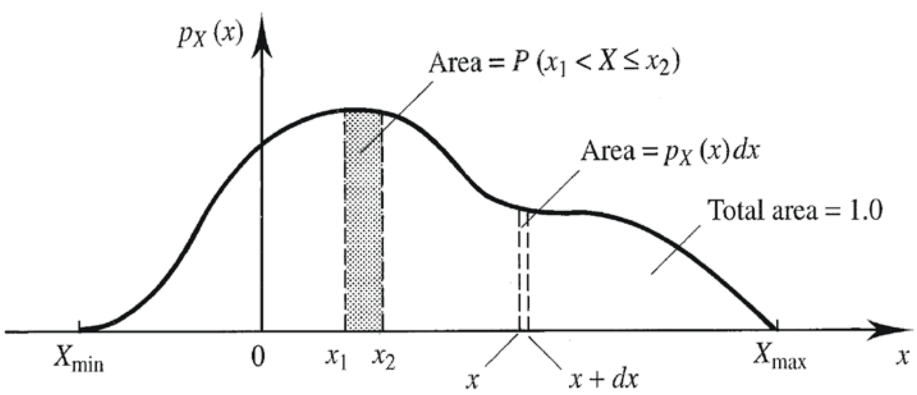
$$p_X(x) = \frac{dP_X(x)}{dx}$$

$$P_X(x) = \int_{-\infty}^{x} p_X(x')dx'$$

- Properties of pdfs
  - $\int_{-\infty}^{\infty} p_X(x)dx = 1$
  - $\int_{x_1}^{x_2} p_X(x)dx = P(x_1 < X \leq x_2)$
- CDs and pdfs can represent continuous, discrete or mixed random variables.



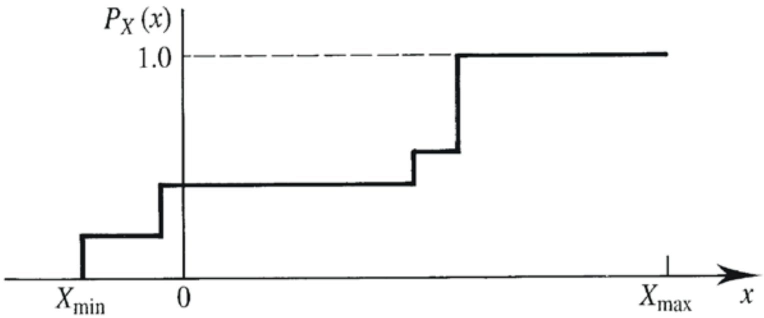
(a) Cumulative distribution of X



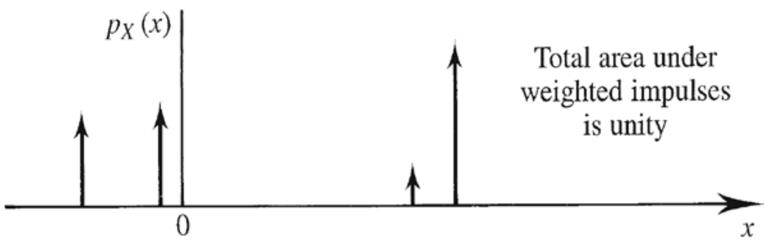
(b) Probability density function of X

Probability Theory

CDFs and PDFs of Discrete Mixed Random Variables

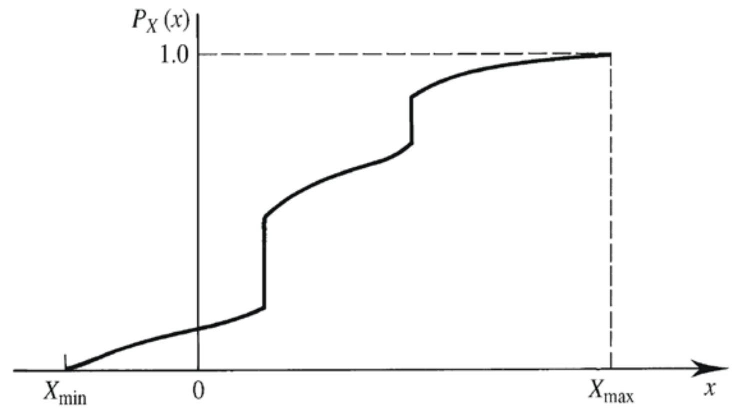


(a) Cumulative distribution of X

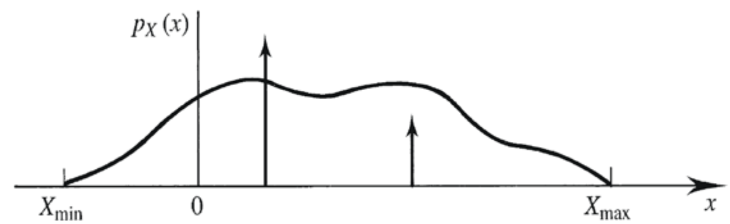


(b) Probability density function of X

Probability descriptions of a discrete random variable (stepped CD and discrete pdf)



(a) Cumulative distribution of X



(b) Probability density function of X

Probability descriptions of a mixed random variable

Probability Theory

CDFs and PDFs of Discrete Mixed Random Variables

- **[Example 2.5]**
- A random voltage has a pdf given by

$$p(V) = ku(V+4)e^{-3(V+4)} + 0.25\delta(V-2)$$

where $u(\quad)$ is the Heaviside step function.

i. Sketch the pdf;
ii. Find the probability that $V = 2$ V;
iii. Find the value of k;
iv. Find and sketch the CD of V.

Probability Theory
Moments

- The first moment of a random variable X is defined by

$$\bar{X} = \int_{-\infty}^{\infty} xp(x)dx$$

where $\bar{X}$ denotes an ensemble mean (or expected value $\mathrm{E}[X]$)

- The second moment is defined as

$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 p(x)dx$$

and represents the mean square of the random variable.

- The square root of the second moment is the root mean square (RMS) value of X.

Probability Theory
Moments

- Higher order moments are defined by

$$\overline{X^n} = \int_{-\infty}^{\infty} x^n p(x) dx$$

- Central moments are the net moments of a random variable taken about its mean.
- The second central moment is defined as

$$\overline{(X - \bar{X})^2} = \int_{-\infty}^{\infty} (x - \bar{X})^2 p(x) dx$$

and is usually called the variance of the random variable.

- The square root of the second central moment is the standard deviation of the random variable (normally denoted by s, but for a Gaussian random variable by σ).

Probability Theory
Moments

- Higher order central moments are defined by

$$\overline{(X-\bar{X})^n} = \int_{-\infty}^{\infty} (X-\bar{X})^n p(x)dx$$

- The 3rd and 4th central moments divided by s^3 and s^4 respectively are called skew and kurtosis. These are a measure of pdf asymmetry and peakiness.
- These higher order moments are of current interest for the analysis of non-stationary signals, such as speech, and they are also appropriate for the analysis of non-Gaussian signals.

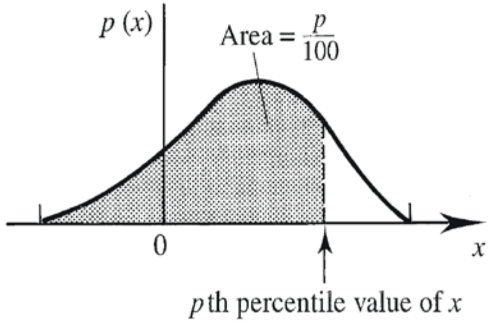
Probability Theory
Percentiles

- The pth percentile is the value of X below which $p\%$ of the total area under the pdf lies, i.e.

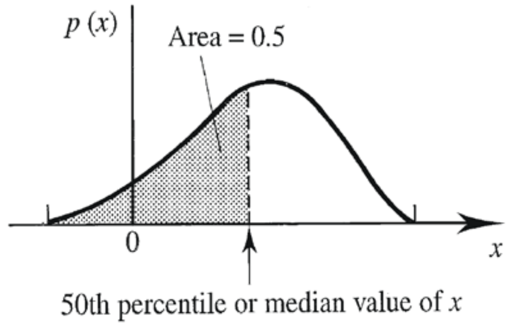
$$\int_{-\infty}^{x} p(x')dx' = \frac{p}{100}$$

where x is the pth percentile.

- In the special case of $p = 50$, the value of x is called the median and the pdf is divided into two equal areas.



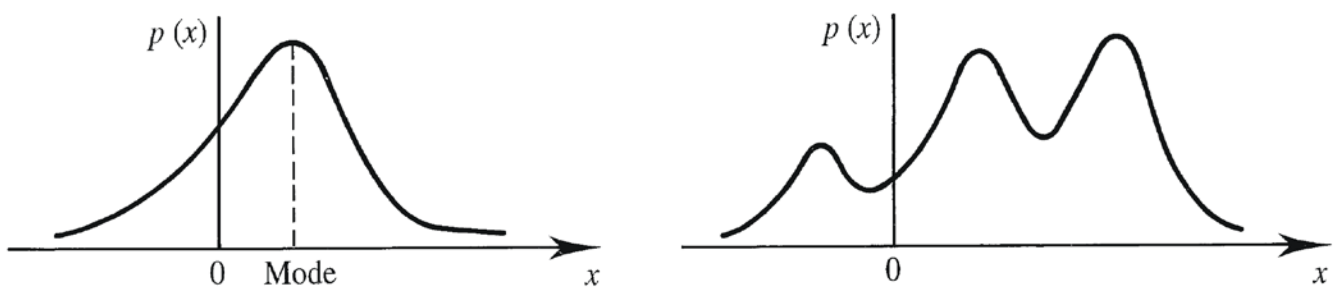
(a) pth percentile value



(b) Median value

Probability Theory
Modes

- The mode of a pdf is the value of x for which $p(x)$ is a maximum.
- For a pdf with a single mode this can be interpreted as the most likely value of X. In general pdfs can be multimodal.
- Moments, percentiles and modes are all examples of *statistics*, i.e. numbers summarizing the behavior of a random variable.
- The difference between probabilistic and statistical models is that statistical models usually give an incomplete description of random variables.



(a) Unimodal distribution (b) Multimodal distribution

Probability Theory
Moments (Interpretation)

- The first moment (mean value) can be interpreted as the DC voltage (or current).
- The second moment (mean square value) can be interpreted as the total power (the total normalized power, or the power dissipated in a 1 Ω load).
- The second central moment (s^2, variance) can be interpreted as the AC power (i.e. the power dissipated in a 1 Ω load by the fluctuating/AC component of voltage).

  $s^2 = \langle [x(t) - \langle x(t) \rangle]^2 \rangle = \langle x^2(t) - 2x(t)\langle x(t) \rangle + \langle x(t) \rangle^2 \rangle$
  $= \langle x^2(t) \rangle - \langle x(t) \rangle^2$

  where the angular brackets, $\langle \quad \rangle$, indicate time average.

Probability Theory
Moments (Interpretation)

- **[Example 2.6]**
- A random voltage has a pdf given by

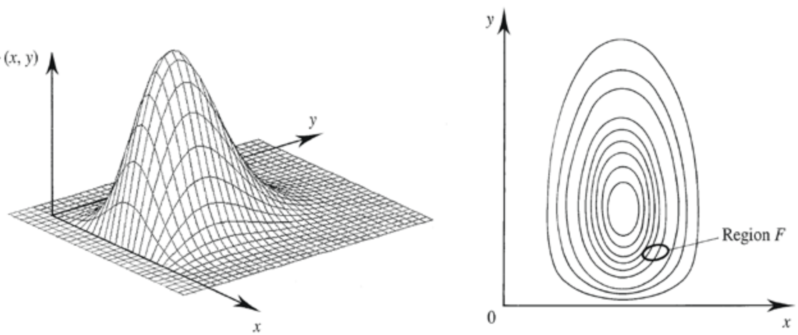
$$p(V) = u(V)3e^{-3V}$$

- Find the DC voltage, the power dissipated in a 1 Ω load and the median value of voltage. What power would be dissipated at the output of an AC coupling capacitor?
  - Standard integral: $\int x^n e^{ax} dx = e^{ax} \left[\frac{x^n}{a} - \frac{nx^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} - \right.$

Probability Theory
Joint and Marginal PDFs

- The joint probability density function, $p_{X,Y}(x,y)$, of random variables X and Y are defined such that $p_{X,Y}(x,y)dxdy$ is the probability that X lies in the range x to $x+dx$ and Y lies in the range y to $y+dy$.
- The volume under the surface representing a bivariate random variable is 1.0, i.e.

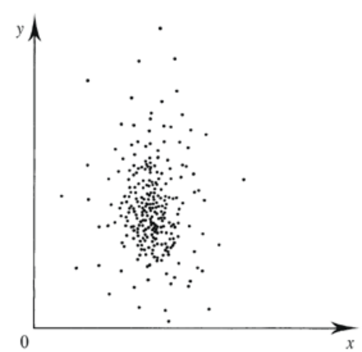
$$\int_X \int_Y p_{X,Y}(x,y)dxdy = 1.0$$



(a) 3-dimensional view of pdf surface (b) Contour plot of pdf surface

- The probability of finding the bivariate random variable in any particular region, F, is

$$P([X,Y] \text{ lies within } F) = \int\int_F p_{X,Y}(x,y)dxdy$$



(c) Scattergram of joint samples

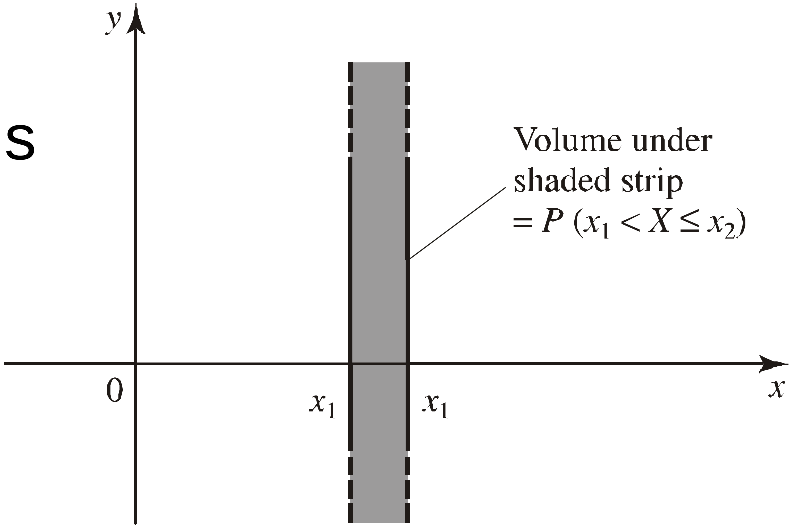
Probability Theory
Joint and Marginal PDFs

- If the joint pdf, $p_{X,Y}(x, y)$, of a bivariate variable is known then the probability that X lies in the range x_1 to x_2 (irrespective of the value of Y) is called a marginal probability of X and is found by integrating over all Y:

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x, y)dy$$

- Similarly, the marginal pdf of Y is

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x, y)dx$$



Probability Theory
Joint and Marginal PDFs

- **[Example 2.7]**
- Two quantized signals have the following discrete joint pdfs

		X			
		1.0	1.5	2.0	2.5
Y	-1.0	0.15	0.08	0.06	0.05
	-0.5	0.10	0.13	0.06	0.05
	0.0	0.04	0.07	0.05	0.05
	0.5	0.01	0.02	0.03	0.05

Find and sketch the marginal pdfs of X and Y

Probability Theory

Joint Moments, Correlation and Covariance

- The joint moments of $p_{X,Y}(x,y)$ are defined by

$$\overline{X^n Y^m} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^m p_{X,Y}(x,y)dxdy$$

- In the special case of $n = m = 1$ the joint moment is called the *correlation* of X and Y.
  - A large positive value of correlation means that when x is high then y, *on average*, will also be high.
  - A large negative value of correlation means that when x is high then y, on average, will be low.
  - A small value of correlation means that x gives little information about the magnitude or sign of y.

Probability Theory

Joint Moments, Correlation and Covariance

- The joint central moments of $p_{X,Y}(x,y)$ are defined by

$$\overline{(X-\bar{X})^n(Y-\bar{Y})^m} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{X})^n(y-\bar{Y})^m p_{X,Y}(x,y)dxdy$$

- In the special case of $n = m = 1$ the joint central moment is called the *covariance* of X and Y. This is because the mean values of X and Y have been subtracted before correlating the resulting zero mean variables.
- Covariance refers to the correlation of the *varying* parts of X and Y.
- If X and Y are already zero mean variables then the correlation and covariance are identical.

Probability Theory

Joint Moments, Correlation and Covariance

- The definitions of joint, and joint central, moments can be extended to more than two random variables.
- The random variables X and Y are uncorrelated if
$$\overline{XY} = \bar{X}\bar{Y}$$
- This implies the *covariance* (not the correlation) is zero:
$$\overline{(X-\bar{X})(Y-\bar{Y})} = 0$$
- Correlation of X and Y can be 0, only if either $\bar{X}$ or $\bar{Y}$ is 0.
- *Statistically independent random variables must be uncorrelated*. The converse, however, is not true.
  - An exception is when the random variables are Gaussian. In this case, uncorrelatedness does imply statistical independence.

Probability Theory

Joint Moments, Correlation and Covariance

- The normalized correlation coefficient, ρ, between two random variables is the correlation between the corresponding standardized variables, standardization in this context implying zero mean and unit standard deviation, i.e.

$$\rho = \overline{\left(\frac{X - \bar{X}}{s_X}\right)\left(\frac{Y - \bar{Y}}{s_Y}\right)}$$

where s_X and s_Y are the standard deviations of X and Y.

- Since ρ can also be interpreted as the covariance of the random variables with normalized standard deviation then $\rho = 0$ can be viewed as the defining property for uncorrelatedness.

Probability Theory

Joint Moments, Correlation and Covariance

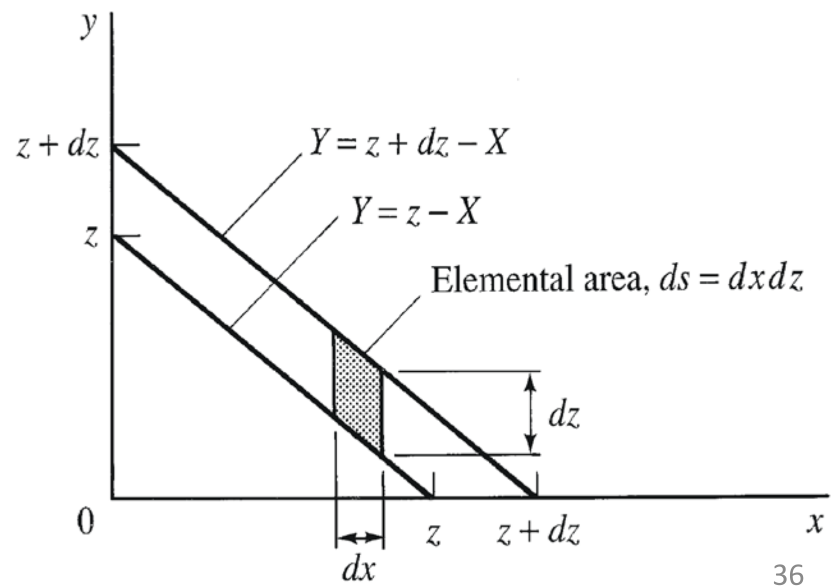
- **[Example 2.8]**
- Find the correlation and covariance of the discrete joint pdf described in Example 2.7, i.e.

		X			
		1.0	1.5	2.0	2.5
Y	-1.0	0.15	0.08	0.06	0.05
	-0.5	0.10	0.13	0.06	0.05
	0.0	0.04	0.07	0.05	0.05
	0.5	0.01	0.02	0.03	0.05

Probability Theory
Central Limit Theorem

- If two independent random variables X and Y are added, and their joint pdf, $p_{X,Y}(x,y)$, is known, what is the pdf, $p_Z(z)$, of their sum, $Z = X + Y$?
- When $Z = z$ (i.e. Z takes on a particular value z) then $Y = z - X$
- When $Z = z + dz$ then

  $Y = z + dz - X$
- They represent straight lines in the X, Y plane.



Probability Theory
Central Limit Theorem

- The probability that Z lies in the range z to $z + dz$ is given by the volume contained under $p_{X,Y}(x, y)$ in the strip between these two lines, i.e.:

$$P(z < Z \leq z + dz) = \int_{strip} p_{X,Y}(x, y) ds \qquad \text{or}$$

$$p_Z(z) dz = \int_{strip} p_{X,Y}(x, z - x) dx dz$$

- Therefore

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X,Y}(x, z - x) dx$$

Probability Theory
Central Limit Theorem

- If X and Y are statistically independent then
$$p_{X,Y}(x, z-x) = p_X(x)p_Y(z-x)$$
- Thus
$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)p_Y(z-x)dx$$
- The pdf of the sum of independent random variables is therefore the convolution of their individual pdfs.
- The multiple convolution of pdfs which arises when many independent random variables are added has a surprising and important consequence.

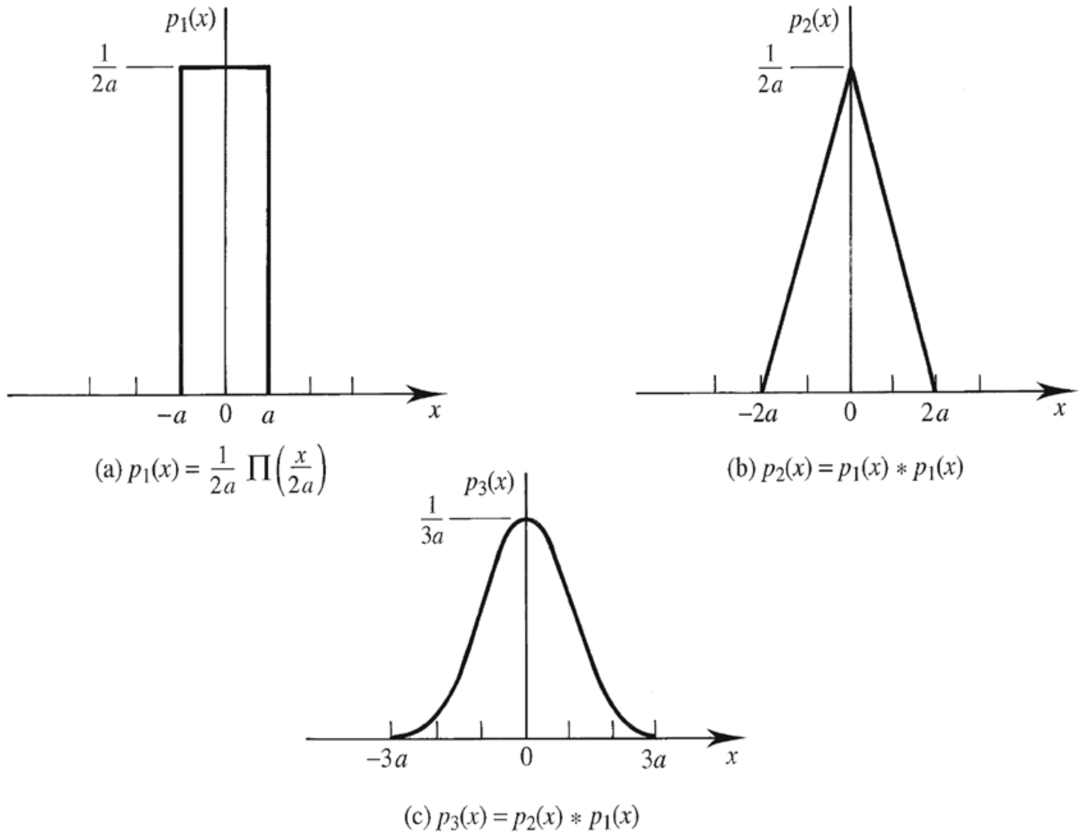
Probability Theory
Central Limit Theorem

- Convolution almost always results in a function smoother than either of the functions being convolved.
- After a few convolutions this repeated smoothing results in a distribution which approximates a Gaussian function:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{X})^2}{2\sigma^2}}$$

- The approximation gets better when the number of convolutions increases.
- The tendency for multiple convolutions to give rise to Gaussian functions is called the central limit theorem and accounts for the ubiquitous nature of Gaussian noise.

Probability Theory
Central Limit Theorem



Multiple self convolution of a rectangular pulse

Probability Theory
Central Limit Theorem

- In the context of statistics the central limit theorem can be stated as follows:
  - *If N statistically independent random variables are added, the sum will have a probability density function which tends to a Gaussian function as N tends to infinity, irrespective of the original random variable pdfs.*
- A second consequence of the central limit theorem is that the pdf of the product of N independent random variables will tend to a log-normal distribution as N tends to infinity, since multiplication of functions corresponds to addition of their logarithms.

Probability Theory
Central Limit Theorem

- If two Gaussian random variables are added their sum will also be a Gaussian random variable. The result holds even if the random variables are correlated.
- The mean of the sum is

$$\bar{Z} = \bar{X} + \bar{Y}$$

- The variance of the sum is

$$\sigma_{X \pm Y}^2 = \sigma_X^2 \pm 2\rho\sigma_X\sigma_Y + \sigma_Y^2$$

- For uncorrelated (and therefore independent) Gaussian random variables the variances are simply added.

Probability Theory
Central Limit Theorem

- **[Example 2.9]**
- X and Y are zero mean Gaussian random currents. When applied individually to 1 Ω resistive loads they dissipate 4.0 W and 1.0 W of power respectively. When both are applied to the load simultaneously the power dissipated is 3.0 W. What is the correlation between X and Y?

- **[Example 2.10]**
- Find the pdf of the sum of two independent random voltages, which have a uniform pdf given by

$$p(V) = \begin{cases} 0.5, & |V| \leq 1.0 \\ 0 & |V| > 1.0 \end{cases}$$

Chapter 3: Random Signals and Noise
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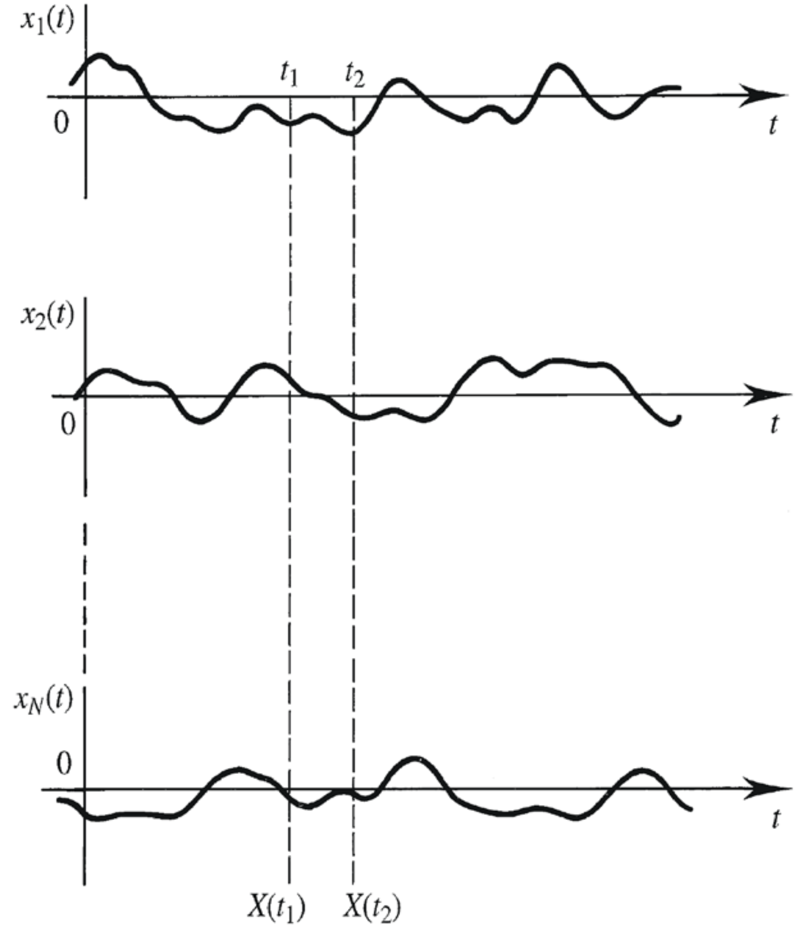
Random Processes
Basic Definitions

- The term random process usually refers to a random variable which is a function of time (or occasionally a function of position) and is strictly defined in terms of an ensemble (i.e. collection) of time functions
- Such an ensemble of functions may, in principle, be generated using many sets (perhaps an infinite number) of identical sources.



Random process, $X(t)$, as ensemble of sample functions, $x_i(t)$

Random Processes
Basic Definitions

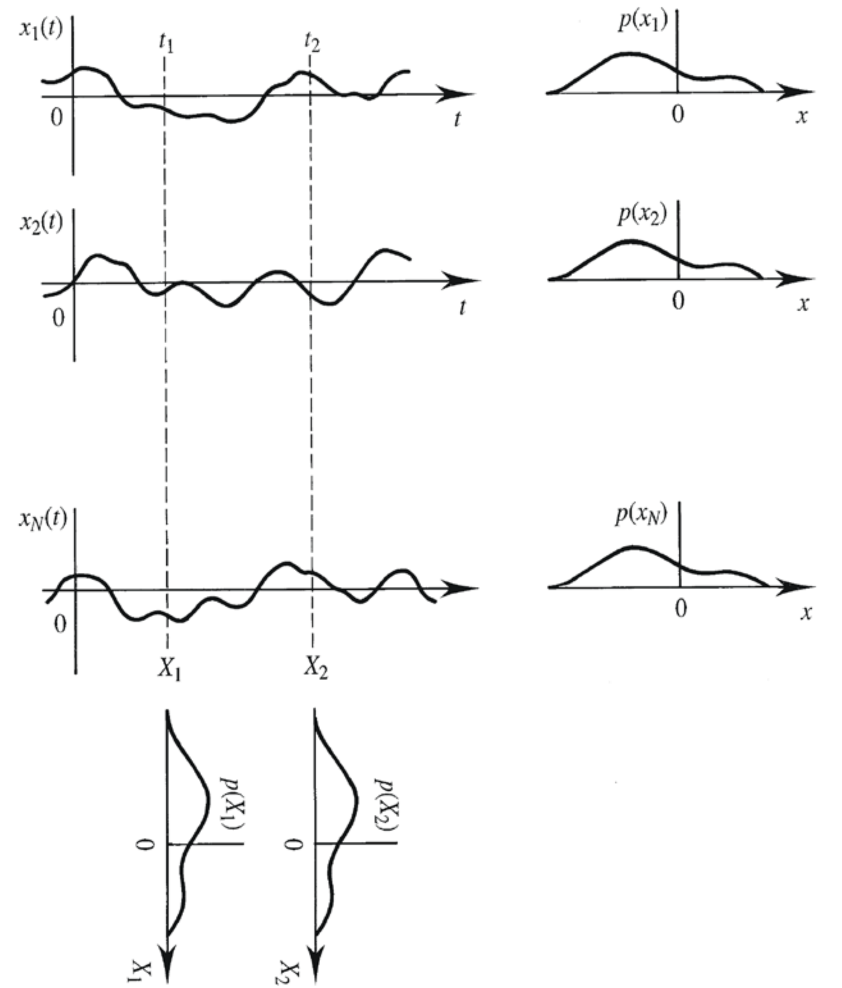
- Some notations:
  - The random process (i.e. the entire ensemble of functions) is denoted by $X(t)$
  - $X(t_1)$ or X_1 denotes an ensemble of samples taken at time t_1 and constitutes a random variable
  - $x_i(t)$ is the ith sample function of the ensemble
- Random processes, like other types of signal, can be classified in a number of different ways, e.g.:
  - Continuous or discrete
  - Analogue or digital (or mixed)
  - Deterministic or non-deterministic
  - Stationary or non-stationary
  - Ergodic or non-ergodic

Random Processes
Stationarity and Ergodicity

- Stationarity relates to the time independence of a random process's statistics.
- There are two definitions:
  - A random process is said to be stationary in the *strict* (sometimes called narrow) sense if all its pdfs (joint, conditional and marginal) are the same for any value of t, i.e. if none of its statistics change with time.
  - A random process is said to be stationary in the *loose* (sometimes called wide) sense if its mean value, $\overline{X(t)}$, is independent of time, t, and the correlation, $\overline{X(t_1)X(t_2)}$, depends only on time difference $\tau = t_2 - t_1$.

Random Processes
Stationarity and Ergodicity

- Ergodicity relates to the equivalence of ensemble and time averages.
- It implies that each sample function, $x_i(t)$, of the ensemble has the same statistical behavior as any set of ensemble values, $X(t_j)$.



Identity of sample function pdfs, $p(x_i)$, and ensemble random variable pdfs, $p(X_i)$, for an ergodic process.

Random Processes
Stationarity and Ergodicity

- For an ergodic process:

$$\langle x_i^n(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x_i^n(t)dt = \int_{-\infty}^{\infty} X_j^n p\big(X_j\big)dx = \overline{X^n(t_j)}$$

  for any i and j.

- Ergodicity is a stronger (more restrictive) condition on a random process than stationarity, i.e.

  Ergodicity $\Rightarrow$ stationarity

  Stationarity $\nRightarrow$ ergodicity

Random Processes
Stationarity and Ergodicity

- **[Example 2.11]**
- Consider a random process as a cosinusoid function with random phases, i.e.:

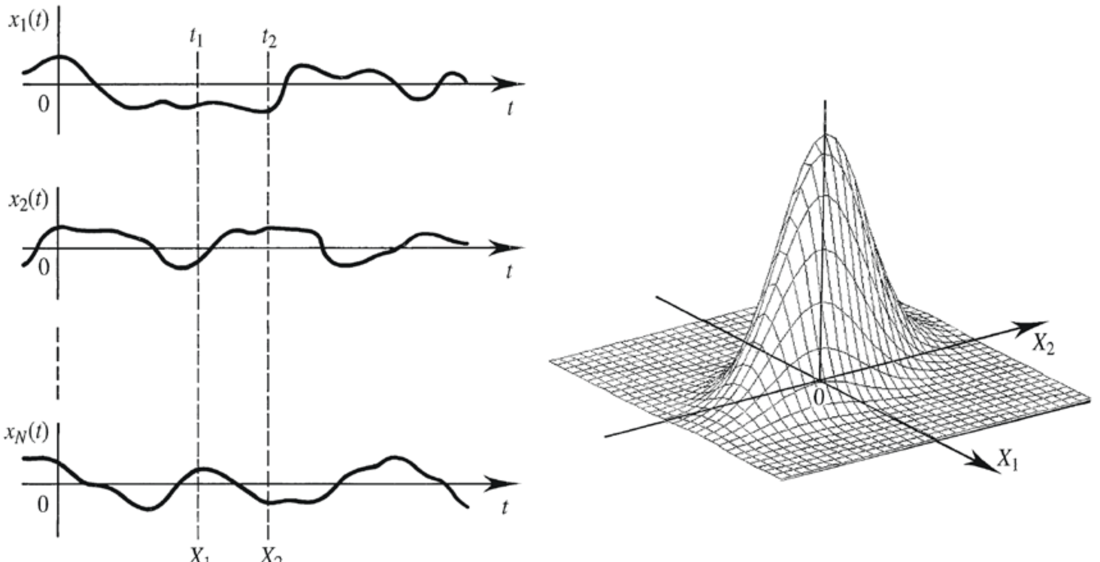
$$X(t) = A\cos(\omega_c t + \theta)$$

where A and ω_c are constants, θ is a uniformly distributed random variable within $(0, 2\pi)$. Is $X(t)$ an ergodic process?

Random Processes

Strict and Loose Sense Gaussian Processes

- A sample function, $x_i(t)$, is said to belong to a Gaussian random process, $X(t)$, in the *strict* sense if the random variables $X_1 = X(t_1)$, $X_2 = X(t_2)$, $\cdots$, $X_N = X(t_N)$ have an N-dimensional joint Gaussian pdf.



(a) Random process, X

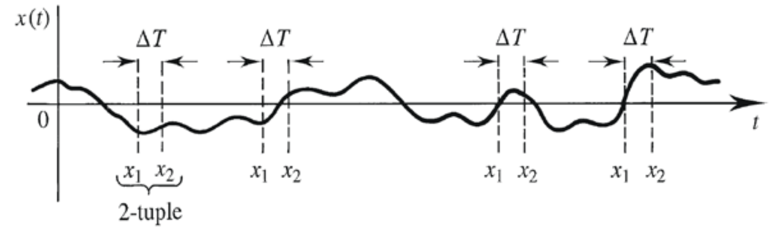
(b) Joint Gaussian pdf of X_1 and X_2 (shown here with non-zero correlation)

Example ($N = 2$) of the joint Gaussian pdf of random variables taken from a strict sense Gaussian process.

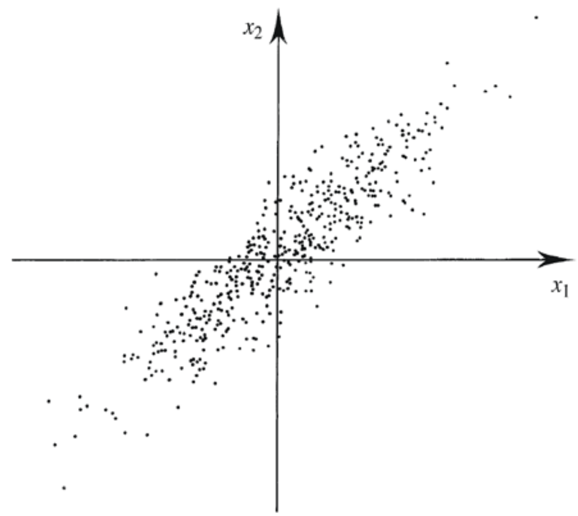
Random Processes

Strict and Loose Sense Gaussian Processes

- For an ergodic process the strict sense Gaussian condition can be defined in terms of a single sample function.
- If the joint pdf of multiple sets of N-tuple samples, taken with fixed time intervals between the samples of each N-tuple, is N-variate Gaussian then the process is Gaussian in the strict sense.



(a) N-tuple ($N = 2$) samples with constant sample separation (ΔT) taken at random times from a sample function, $x(t)$, of the random process $X(t)$



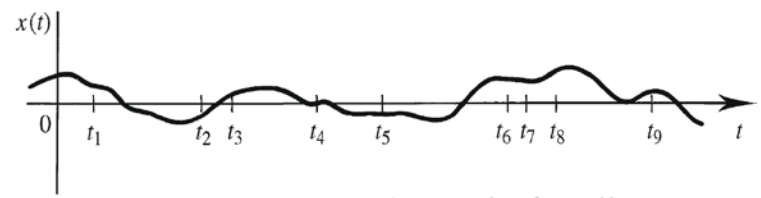
(b) Joint, N-variate ($N = 2$), Gaussian scattergram for $p(x_1, x_2)$

Single sample function definition of ergodic, strict sense Gaussian, random process.

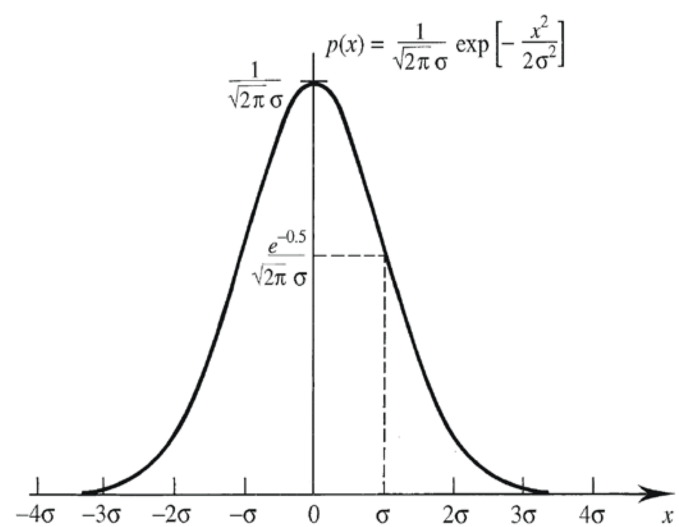
Random Processes

Strict and Loose Sense Gaussian Processes

- A sample function, $x_i(t)$, is said to belong to a Gaussian random process in the *loose* sense if isolated samples taken from $x_i(t)$ come from a Gaussian pdf.
  - All strict sense Gaussian processes are also loose sense Gaussian processes.
  - A strict sense Gaussian process is the most structureless, random or unpredictable statistical process possible. It describes thermal noise which is present in all practical systems.
  - A strict sense N-dimensional Gaussian pdf is specified completely by its first and second order moments, i.e. its means, variances and covariances, as all higher moments of the Gaussian pdf are zero.



(a) Isolated samples $x(t_n)$ taken at random from $x(t)$



(b) Gaussian distribution of samples from (a)

Definition of a loose sense Gaussian process.

Random Processes

Autocorrelation and Power Spectral Density

- A simple pdf is insufficient to fully describe a random signal (i.e. a sample function from a random process) because it contains no information about the signal's rate of change.
- Such information is available in the joint pdfs, $p(X_1, X_2)$, of random variables, $X(t_1)$ and $X(t_2)$, separated by $\tau = t_2 - t_1$.
- These joint pdfs are not usually known in full but partial information about them is often available in the form of the correlation, $\overline{X(t_2)X(t_2 - \tau)}$.
- For ergodic signals the ensemble average taken at any time is equal to the temporal average of any sample function, i.e.

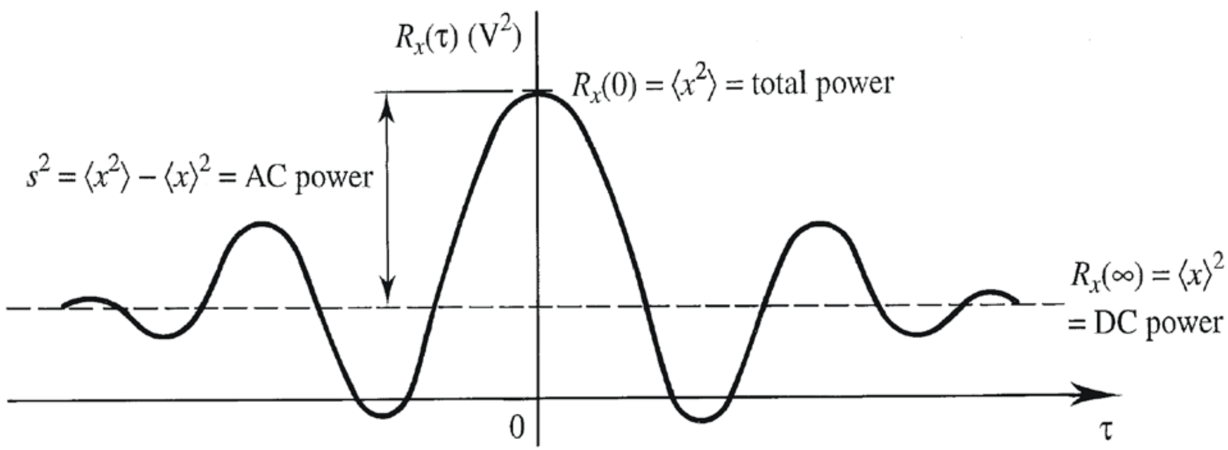
$$\overline{X(t)X(t-\tau)} = \langle x(t)x(t-\tau)\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)dt$$

$$= R_x(\tau) \ \ (V^2)$$

Random Processes

Autocorrelation and Power Spectral Density

- Properties of the autocorrelation function, $R_x(\tau)$, of a sample function, $x(t)$, taken from a real, ergodic, random process:
  - $R_x(\tau)$ is real and has even symmetry: $R_x(-\tau) = R_x(\tau)$.
  - $R_x(\tau)$ has a maximum magnitude at $\tau = 0$ which corresponds to the mean square value (or normalized power in) $x(t)$: $\langle x^2(t)\rangle = R_x(0) > |R_x(\tau)|$ for all $\tau \neq 0$
  - $R_x(\infty)$ is the square mean value (normalized DC power) of $x(t)$: $R_x(\infty) = \langle x(t)\rangle^2$
  - $R_x(0) - R_x(\infty)$ is the variance of $x(t)$: $R_x(0) - R_x(\infty) = \langle x^2(t)\rangle - \langle x(t)\rangle^2 = s^2$
  - The autocorrelation function and two-sided power spectral density of $x(t)$ form a Fourier transform pair (Wiener-Kintchine theorem): $R_x(\tau) \overset{FT}{\Leftrightarrow} G_x(f)$



Random Processes
Autocorrelation and Power Spectral Density

- Properties of power spectral density:
  - $G_x(f)$ is real and has even symmetry about $f = 0$: $G_x(-f) = G_x(f)$.
  - The area under $G_x(f)$ is the mean square value of (or normalized power in) $x(t)$: $\int_{-\infty}^{\infty} G_x(f)df = \langle x^2(t) \rangle$
  - If $x(t)$ has units of V then $G_x(f)$ has units of V^2/Hz.
  - The area under any impulse in $G_x(f)$ occurring at $f = 0$ is the square mean value of (or normalized DC power in) $x(t)$: $\int_{0-}^{0+} G_x(f)df = \langle x(t) \rangle^2$.
  - The area under $G_x(f)$, excluding any impulse function at $f = 0$ is the variance of $x(t)$ or the normalized power in the fluctuating component of $x(t)$, i.e.
$$\int_{-\infty}^{0-} G_x(f)df + \int_{0+}^{\infty} G_x(f)df = \langle x^2(t) \rangle - \langle x(t) \rangle^2$$
  - $G_x(f)$ is positive for all f: $G_x(f) \geq 0$
- White noise is a random signal with particularly extreme spectral and autocorrelation properties. *It has no self-similarity with any time shifted version of itself so its autocorrelation function consists of a single impulse at zero delay, and its power spectral density is flat.*

Random Processes

Autocorrelation and Power Spectral Density

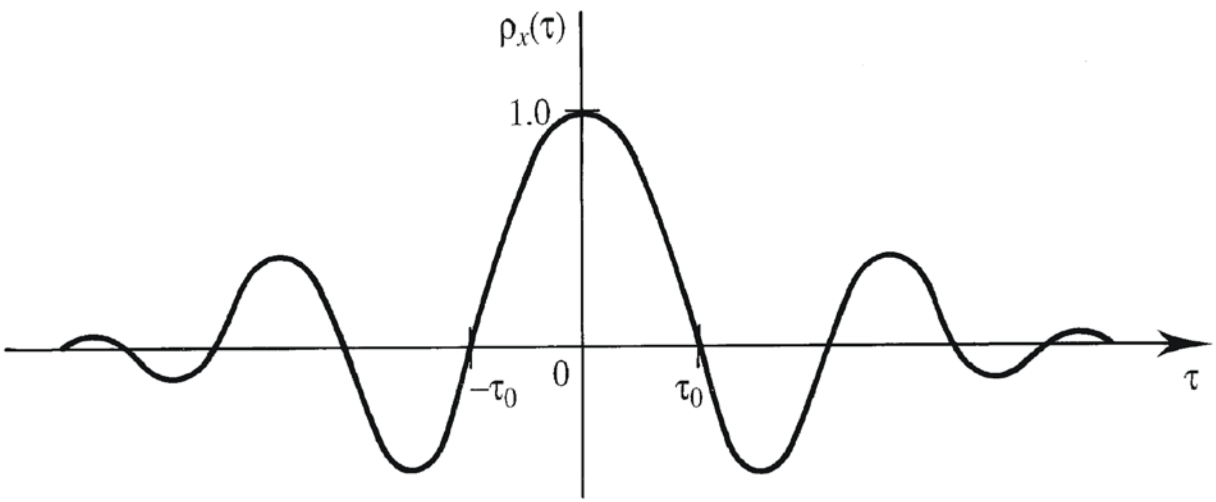
- **[Example 2.12]**
- Find and sketch the autocorrelation function of the stationary random signal whose power spectral density is

$$G_x(f) = 3.0\left[\Lambda\left(\frac{f-5000}{1000}\right) + \Lambda\left(\frac{f+5000}{1000}\right)\right]$$

Random Processes
Signal Memory and Decorrelation Time

- A normalized autocorrelation function can be defined by subtracting any DC value present in $x(t)$, dividing by the resulting RMS value and autocorrelating the result:

$$\rho_x(\tau) = \frac{\langle x(t)x(t-\tau) - \langle x(t)\rangle^2\rangle}{\langle x^2(t)\rangle - \langle x(t)\rangle^2}$$



Random Processes

Signal Memory and Decorrelation Time

- Practical signals have a finite memory, i.e. samples taken close enough together are highly correlated.
- The decorrelation time, τ_0, of a signal provides a quantitative measure of this memory and is defined as the minimum time shift, τ, required to reduce $\rho_x(\tau)$ to some predetermined, or reference, value.
- The reference value depends on the application and/or preference. Popular choices are $\rho_x(\tau_0) = \frac{1}{\sqrt{2}}, 0.5, \frac{1}{e}, \text{and } 0$.
- For a random signal or noise with a white power spectral density, $\tau_0 = 0$. the autocorrelation function of white noise is impulsive. This means that adjacent samples taken from a white noise process are uncorrelated no matter how closely the samples are spaced.

Random Processes
Cross Correlation

- The cross correlation of functions, taken from two real ergodic random processes, is

$$R_{xy}(\tau) = \langle x(t)y(t-\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)y(t-\tau)dt$$

  - $R_{xy}(\tau)$ is real and $R_{xy}(-\tau) = R_{yx}(\tau)$. (In general $R_{xy}(-\tau) \neq R_{xy}(\tau)$).
  - $\left[R_x(0)R_y(0)\right]^{1/2} > |R_{xy}(\tau)|$, and $\frac{\left[R_x(0)+R_y(0)\right]}{2} > |R_{xy}(\tau)|$, for all τ.
  - If $x(t)$ and $y(t)$ have units of V then $R_{xy}(\tau)$ has the units of V^2
  - For statistically independent random processes, $R_{xy}(\tau) = R_{yx}(\tau)$, and if either process has zero mean then $R_{xy}(\tau) = R_{yx}(\tau) = 0$ for all τ.
  - The Fourier transform of $R_{xy}(\tau)$ is often called a cross-power spectral density, $G_{xy}(f)$ since its units are V^2/Hz: $R_{xy}(\tau) \overset{FT}{\Leftrightarrow} G_{xy}(f)$
  - For complex functions: $R_{xy}(\tau) = \langle x(t)y^*(t-\tau)\rangle = \langle x^*(t)y\,(t+\tau)\rangle$

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Summary

- Variables are said to be random if their particular value at specified future times cannot be predicted. Information about their probable future values is often available from a probability model.
- Bayes's rule relates joint, conditional and unconditional probabilities.
- Pdfs are the derivative of the CD.
- Moments, central moments and modes are statistics of random variables.
- Joint pdfs give the probability that two or more random variables will concurrently take particular values between the specified limits.
- A marginal pdf is the pdf of one random variable irrespective of the value of any other random variable.
- The correlation of two random variables is their mean product.
- The covariance is the mean product of their fluctuating (zero mean) components only, being zero for uncorrelated signals.
- Statistically independent random variables are always uncorrelated but the converse is not true.

Summary

- The pdf of the sum of independent random variables is the convolution of their individual pdfs and, for the sum of many independent random variables, this results in a Gaussian pdf.
- If the random variables are independent then the mean of their sum is the sum of their means and the variance of their sum is the sum of their variances.
- Random processes are random variables which change with time.
- Random processes which are ergodic are statistically stationary but the converse is not necessarily true.
- Signal memory is characterized by the signal's autocorrelation function.
- The Wiener-Kintchine theorem identifies the power and energy spectral densities of power and energy signals with the Fourier transform of these signals' autocorrelation functions.
- Cross correlation relates to the similarity between a pair of different functions, one offset from the other by a time shift.

Principles of Communications (通信系统原理)

Undergraduate Course

Chapter 3: Random Signals and Noise

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