Principles of Communications (通信系统原理)

Undergraduate Course

Chapter 4: Linear Systems

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Chapter 4: Linear Systems
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Introduction
Basic Definitions

- The word “system” is defined as “a group or combination of inter-related, inter-dependent, or interacting elements forming a collective entity”.
- In the context of a communications system the interacting elements, such as electronic amplifiers, mixers, detectors, etc., are themselves subsystems made up of components such a resistors, capacitors, and transistors.
- An understanding of how systems behave, and are described, is therefore important to the analysis of electronic communications equipment.

Introduction
Properties of Linear Systems

- Electronic communications equipment is predominantly composed of interconnected linear subsystems.
- What is a linear system?
  - Principle of superposition: *A system is linear if its response to the sum of any two inputs is the sum of its responses of each of the inputs alone.*
  - If $x_i(t)$ are inputs to a system and $y_i(t)$ are the corresponding outputs then superposition can be expressed mathematically as:

$$y(t) = \sum_i y_i(t) \quad \text{when} \quad x(t) = \sum_i x_i(t)$$

  - Proportionality follows directly from linearity:

$$y(t) = my_1(t) \quad \text{when} \quad x(t) = mx_1(t)$$

Introduction
Properties of Linear Systems

- What is a linear system (cont.)?
  - A further property that systems often have is time invariance. This means that the output of a system does not depend on when the input is applied:

$$y(t) = y_1(t-T) \quad \text{when} \quad x(t) = x_1(t-T)$$

  - The majority of communications subsystems obey both the superposition and time invariance principles, and are therefore called time invariant linear systems (TILS).
  - TILS can be defined using a single formula:

If $\quad y_1(t) = S\{x_1(t)\} \;\; and \;\; y_2(t) = S\{x_2(t)\}$

then $\quad S\{ax_1(t-T) + bx_2(t-T)\} = ay_1(t-T) + by_2(t-T)$

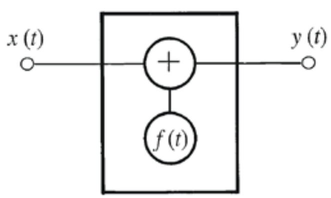
where $S\{\,\}$ represents the functional operation of the system.

Introduction
Properties of Linear Systems

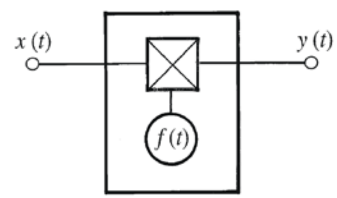
- Linear systems cannot be strictly realized in practice, since any device behaves non-linearly if excited by signals of large enough amplitude.
- However, many non-linear systems are at least approximately linear if driven by small enough signals, because the transfer characteristic of a non-linear (memoryless) system can normally be represented by a polynomial of the form: $y(t) = ax(t) + bx^2(t) + cx^3(t) + \cdots$. For small enough input signals (and providing $a \neq 0$) only the first term is significant and the system behaves linearly.
- Some compelling reasons for studying and using linear systems:
  - Linear systems conserve the shape of sinusoidal signals.
  - The electric and magnetic properties of free space are linear.
  - The electric and magnetic properties of many materials are linear over a large range of field strengths.
  - Many general mathematical techniques are available for describing, analyzing and synthesizing linear systems.

Introduction
Properties of Linear Systems

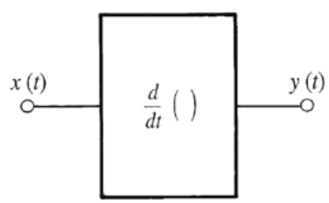
- **[Example 3.1]**
- Demonstrate the linearity or otherwise of the systems represented by the following diagrams.



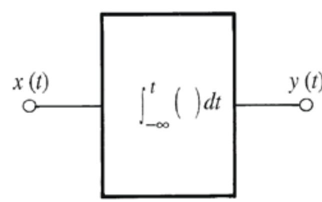
(a) Addition of a function



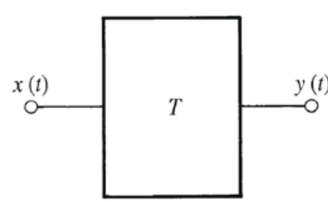
(b) Multiplication by a function



(c) Differentiation



(d) Integration



(e) Time delay (T seconds)

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Time Domain Description of Linear Systems
Basic Definitions

- Consider a linear system with discrete (or sampled) input $x_1, x_2, \cdots, x_N$ and discrete output $y_1, y_2, \cdots, y_M$. Each output is then given by a weighted sum of all the inputs:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & \cdots & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ G_{M1} & G_{M2} & \cdots & \cdots & G_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix}$$

  i.e.

$$y_i = \sum_{j=1}^{N} G_{ij} x_j$$

- If the system is a physical system operating in real time then $G_{ij} = 0$ for all values of x_j occurring after y_i.

Time Domain Description of Linear Systems
Continuous Signals

- If the discrete input and output are replaced with continuous equivalents, i.e. $y_i \rightarrow y(t)$ and $x_j \rightarrow x(\tau)$ then the discrete summation becomes continuous integration.
- For physical systems operating in real time, future values of input do not contribute to current, or past, values of output.
- Furthermore, if input signals are allowed to start at a time arbitrarily distant in the past then

$$y(t) = \int_{-\infty}^{t} G(t,\tau)x(\tau)d\tau$$

- These systems are called causal since only past and current input values affect (or cause) outputs.

Time Domain Description of Linear Systems

Impulse Response

- Replacing the input with a (unit strength) impulse, $\delta(\tau)$, results in the system's impulse response:

$$h(t) = \int_{-\infty}^{t} G(t,\tau)\delta(\tau)d\tau$$

- If the impulse is applied at time $\tau = T$ then, assuming the system is time invariant, the output is:

$$h(t-T) = \int_{-\infty}^{t} G(t,\tau)\delta(\tau - T)d\tau$$

- The sampling property of $\delta(\tau - T)$ under integration means that $G(t,T)$ can be interpreted as the response to an impulse applied at time $\tau = T$, i.e.: $h(t-T) = G(t,T)$.

Time Domain Description of Linear Systems
Convolution

- Replacing T with τ (a change of notation only) gives

$$y(t) = \int_{-\infty}^{t} h(t-\tau)x(\tau)d\tau$$

- If non-causal systems are allowed then

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

- The output of a time invariant linear system is therefore given by the convolution of the system's input with its impulse response, i.e.

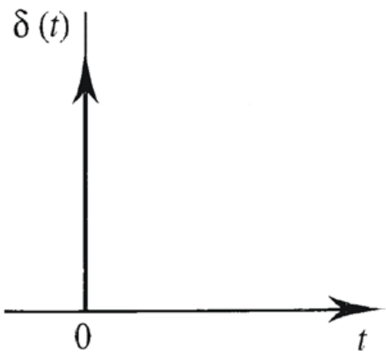
$$y(t) = h(t) * x(t)$$

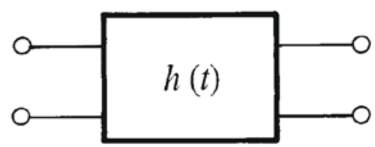
- The commutative property of convolution means that $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$

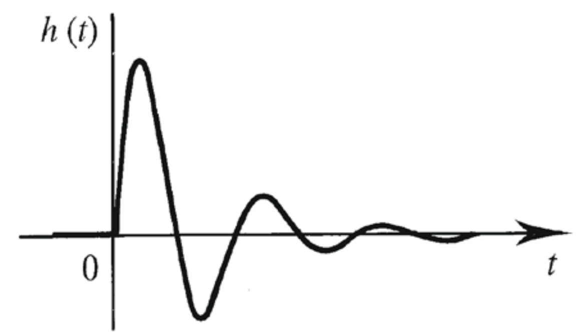
Time Domain Description of Linear Systems
Convolution

- The above expressions are consistent with the definition of an impulse response since

$$y(t) = h(t) * \delta(t) = h(t)$$







Time Domain Description of Linear Systems

Step Response

- If a system is driven with a step signal, $u(t)$ (sometimes called the Heaviside step), defined by

$$u(t) = \begin{cases} 1.0, & t > 0 \\ 0.5, & t = 0 \\ 0, & t < 0 \end{cases}$$

then the output of the system (i.e. its step response) is:

$$q(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

- Since $u(t-\tau) = 0$ for $\tau > t$ and $h(\tau) = 0$ for $\tau < 0$, we have

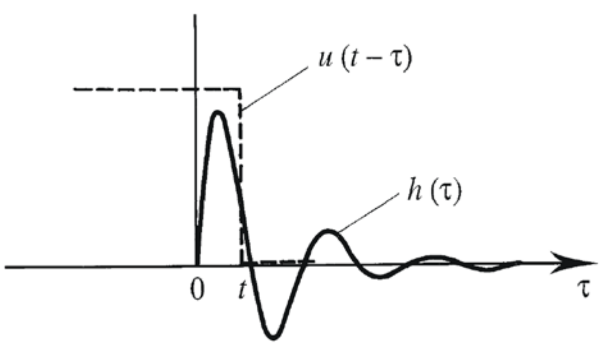
$$q(t) = \int_{0}^{t} h(\tau)u(t-\tau)d\tau$$

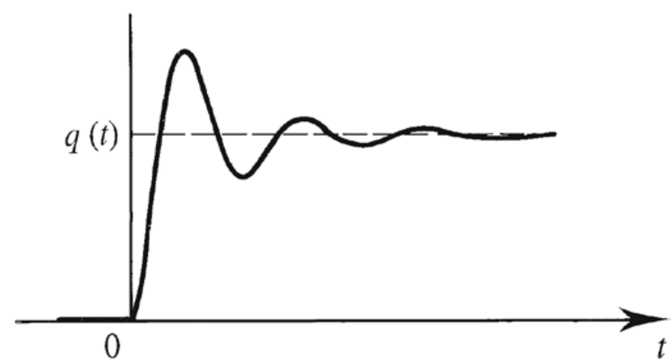
Time Domain Description of Linear Systems
Step Response

- In the region $0 < \tau < t$, $u(t-\tau) = 1.0$, i.e.

$$q(t) = \int_0^t h(\tau)d\tau$$

- The step response is thus the integral of the impulse response.
- Conversely, the impulse response is the derivative of the step response: $h(t) = \frac{d}{dt}q(t)$.





Time Domain Description of Linear Systems
Step Response

- **[Example 3.2]**
- Find and sketch the impulse response of the system which has the step response $\Lambda(t-1)$.

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Frequency Domain Representation
Basic Definitions

- In the time domain the output of a time invariant linear system is the convolution of its input and its impulse response, i.e.: $y(t) = h(t) * x(t)$.
- The equivalent frequency domain expression is found by taking the Fourier transform of both sides:

$$\mathrm{FT}\{y(t)\} = \mathrm{FT}\{h(t) * x(t)\} = \mathrm{FT}\{h(t)\}\mathrm{FT}\{x(t)\}$$

- Let $X(f)$ and $Y(f)$ be respectively the input and output voltage spectra, and $H(f)$ be the frequency response of the system. We have

$$Y(f) = H(f)X(f)$$

Frequency Domain Representation
Basic Definitions

- **[Example 3.3]**
- A linear system with the impulse response $h(t) = 2.0\Pi\left(\frac{t-1}{2}\right)$ is driven by the input signal

$$v_{in}(t) = 3.0\Pi\left(\frac{t-1}{2}\right) - 3.0\Pi\left(\frac{t-3}{2}\right)$$

Find

i. the voltage spectral density of the input signal
ii. the frequency response of the system
iii. the voltage spectral density of the output signal
iv. the (time domain) output signal

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Causality

- Since physical systems should not respond to inputs before the inputs have been applied, realizable systems must be causal, i.e.:

$$h(t) = 0, \qquad \text{for } t < 0$$

- An equivalent way of expressing the above equation is

$$h(t) = u(t)h(t)$$

where $u(t)$ is the Heaviside step function.

- The frequency response must therefore satisfy

$$H(f) = \mathrm{FT}\{u(t)\} * \mathrm{FT}\{h(t)\} = \left[\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right] * H(f)$$

$$= \frac{1}{2}H(f) + \left[\frac{1}{j2\pi f} * H(f)\right]$$

i.e.: $$H(f) = \frac{1}{j\pi f} * H(f)$$

Causality

- **[Example 3.4]**
- Establish which of the following systems are causal and which are not:
  - *i.* $h(t) = \Lambda(t-3)$
  - *ii.* $h(t) = e^{-(t-10)^2}$
  - *iii.* $h(t) = u(t)e^{-t}$
  - *iv.* $H(f) = e^{-f^2}$
  - *v.* $H(f) = \Pi(f)$
  - *vi.* $H(f) = \Lambda(f-3) + \Lambda(f+3)$

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Random Signals and Linear Systems
Basic Definitions

- The effect of a linear system on a deterministic signal is specified completely by
$$y(t) = h(t) * x(t)$$
or alternatively, by
$$Y(f) = H(f)X(f)$$
- Random signals cannot be specified as deterministic functions either in the time domain or in the frequency domain. It follows that the above equations are not useful for information bearing signals or noise.
- In practice, however, the two properties of such signals which must most commonly be specified are their power spectra and their probability density functions.

Random Signals and Linear Systems
Power Spectral Density

- Let $G_x(f)$ and $G_y(f)$ denote the power spectral densities of the input and output signals respectively, we have

$$G_y(f) = |H(f)|^2 G_x(f) \qquad (\mathrm{V}^2/\mathrm{Hz})$$

- If the system input is an energy signal then the power spectral densities are replaced by energy spectral densities

$$E_y(f) = |H(f)|^2 E_x(f) \qquad (\mathrm{V}^2\, s/\mathrm{Hz})$$

Random Signals and Linear Systems
Power Spectral Density

- The equivalent time domain description is obtained by taking the inverse Fourier transform.
- Using the Wiener-Kintchine theorem:

$$R_{yy}(\tau) = R_{hh}(\tau) * R_{xx}(\tau)$$

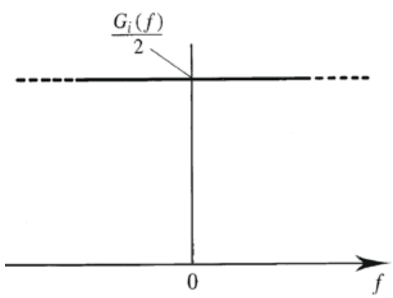
  where R is the correlation function.
- In practice, the frequency domain description is almost always more convenient.
- [Example] Find the output power spectral density for a simple RC filter when it is driven by white noise. What is the total noise power at the filter's output?

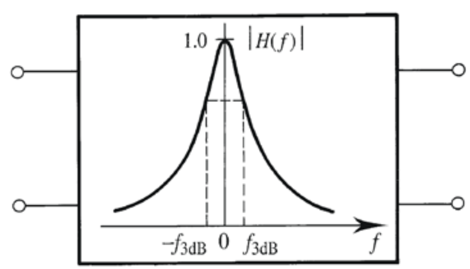
Random Signals and Linear Systems
Noise Power Spectral Density

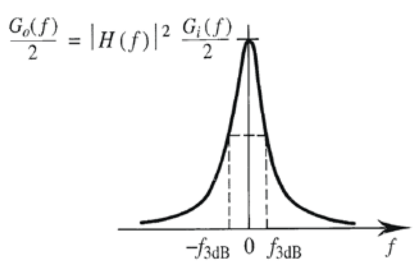
- The problem is shown schematically in the figure.
- The frequency response of the RC filter is given by

$$H(f) = \frac{1}{1 + j(f/f_{3\mathrm{dB}})}$$

where $f_{3\mathrm{dB}}$ is the filter -3 dB, or cut-off, frequency.







Random Signals and Linear Systems

Noise Power Spectral Density

- The power spectral density at the filter output is

$$G_o(f) = |H(f)|^2 G_i(f) = \left|\frac{1}{1 + j(f/f_{3\mathrm{dB}})}\right|^2 G_i(f) = \frac{1}{1 + \left(\frac{f}{f_{3\mathrm{dB}}}\right)^2} G_i(f)$$

- Interpreting $G_i(f)$ and $G_o(f)$ as one-sided, the total noise power, N, at the filter output is:

$$N = \int_0^\infty G_o(f) df = \int_0^\infty \frac{1}{1 + \left(\frac{f}{f_{3\mathrm{dB}}}\right)^2} G_i(f)\ df$$

- Using the change of the variable $u = f/f_{3\mathrm{dB}}$ provides ($G_i(f)$ is a constant)

$$N = G_i f_{3\mathrm{dB}} \int_0^\infty \frac{1}{1 + u^2} du = G_i f_{3\mathrm{dB}} [\tan^{-1} u]_0^\infty = \frac{G_i f_{3\mathrm{dB}} \pi}{2} \quad (\mathrm{V}^2)$$

Random Signals and Linear Systems

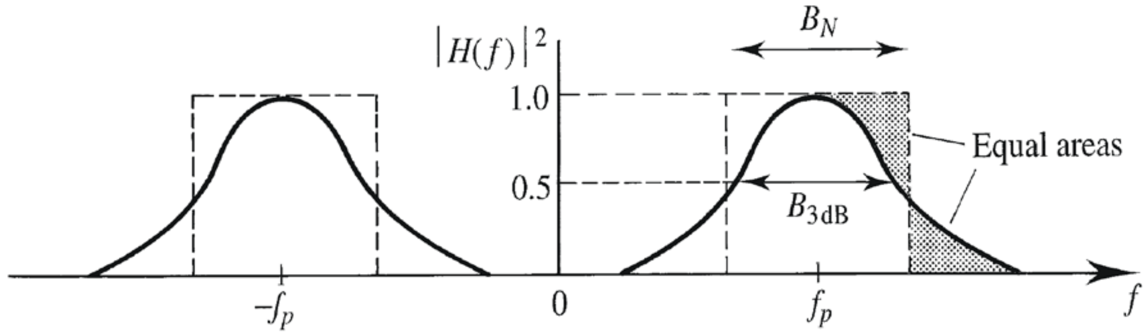
Noise Bandwidth

- The noise bandwidth, B_N, of a filter is defined as that width which a rectangular frequency response would need to have to pass the same noise power as the filter, given identical white noise at the input to both. Mathematically:

$$B_N = \int_0^\infty \frac{|H(f)|^2}{\left|H(f_p)\right|^2} df$$

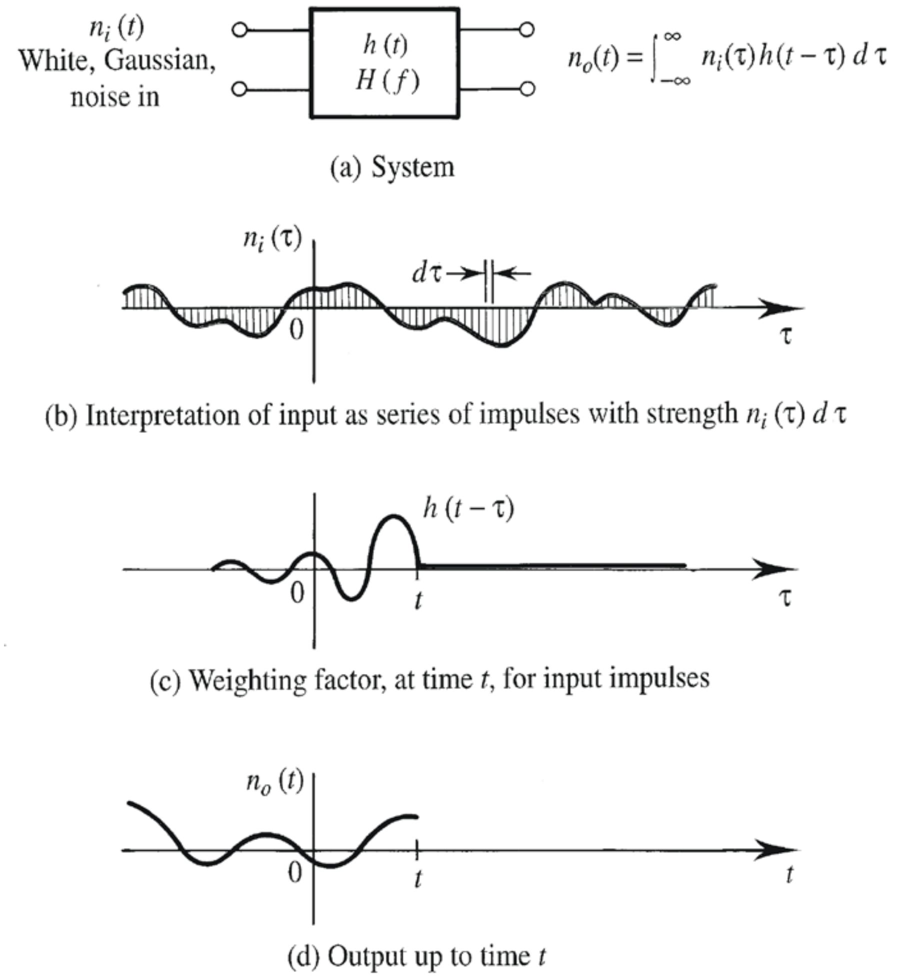
where f_p is the frequency of peak amplitude response.

- In general, noise bandwidth is not equal to the -3 dB bandwidth.



Random Signals and Linear Systems
PDF of Filtered Noise

- Not only the power spectral density of a random signal is changed when it is filtered but, in general, so is its pdf.
- There is no general analytical method of deriving the pdf of the output from the pdf of the input, when systems have non-zero memory.
- An important exception is the pdf of filtered Gaussian noise:
  - Filtered white Gaussian noise is Gaussian.
  - Filtered Gaussian noise is Gaussian.



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Summary

- Linear systems obey the principle of superposition.
- A linear system's time domain output is given by its time domain input convolved with its impulse response.
- The impulse response of a linear system is the time derivative of its step response.
- The output (complex) voltage spectrum of a linear system is the voltage spectrum of its input multiplied by its (complex) frequency response.
- A system's impulse response and frequency response form a Fourier transform pair.
- All physically realizable system are causal.

Summary

- The power spectral density (PSD) of a random signal at the output of a linear system is given by the PSD at its input multiplied by its squared amplitude response.
- The autocorrelation of a random signal at the output of a linear system is the convolution of the input signal's autocorrelation with the autocorrelation of the system's impulse response.
- No general, analytical methods are currently known for predicting the pdf at the output of systems with memory. A useful result of the special case of Gaussian, is that filtered Gaussian noise is Gaussian.

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