Principles of Communications (通信系统原理)

Undergraduate Course

Chapter 8: Decision Theory

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Chapter 8: Decision Theory

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Introduction

- The detection of baseband equiprobable binary symbols with Gaussian noise chooses midway between those voltages which are to be detected.
- In the more general case of arbitrary noise distributions and unequal symbol probabilities, two major decision criteria, Bayes and Neyman-Pearson, are normally applied.
- The *Bayes decision criterion* is used extensively in binary communications while the *Neyman-Pearson criterion* is used more in radar applications.
- The principal difference between them is that the Bayes decision rule assumes known a priori source statistics for the occurrence of digital ones and zeros, while the Neyman-Pearson criterion makes no such assumption.

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A Priori, Conditional and Posteriori Probabilities

- There are four probabilities and two probability density functions associated with symbol transmission and reception:
  - $P(0)$ - a priori probability of transmitting the digit 0.
  - $P(1)$ - a priori probability of transmitting the digit 1.
  - $p(v|0)$ - conditional probability density function of detecting voltage v given that a digital 0 was transmitted.
  - $p(v|1)$ - conditional probability density function of detecting voltage v given that a digital 1 was transmitted.
  - $P(0|v)$ - a posteriori probability that a digital 0 was transmitted given that voltage v was detected.
  - $P(1|v)$ - a posteriori probability that a digital 1 was transmitted given that voltage v was detected.

A Priori, Conditional and Posteriori Probabilities

- The terms a priori and a posteriori imply reasoning from cause to effect and reasoning from effect to cause respectively.
- The cause of a communication event is the symbol transmitted and the effect is the voltage detected.
- The a priori probabilities (in communications applications) are usually known in advance.
- The conditional probabilities are dependent on the data (1 or 0) transmitted.
- The a posteriori probabilities can only be established after many events (i.e. symbol transmissions and receptions) have been completed.

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Symbol Transmission Matrix
Basics

- When communications systems operate in the presence of noise, there is the possibility of transmitted symbols being sufficiently corrupted to be interpreted, at the receiver, in error.
- If the characteristics of the noise are known, or many observations of symbol transmissions and receptions are made, the probability of each transmitted symbol being interpreted as other symbols can be found.
- These transition probabilities are denoted by $P(i|j)$, where i represents the symbol received and j represents the symbol transmitted.
- $P(i|j)$ is the probability that a symbol will be interpreted as i given that j was transmitted.
- The symbol transition probabilities can be arranged as the elements of a matrix to describe the end-to-end properties of a communications channel.

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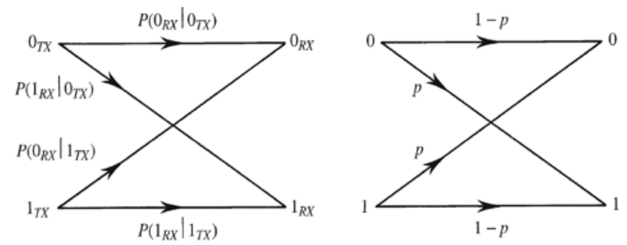
Symbol Transmission Matrix

Binary Symmetric Channel

- In a binary channel four types of communication events can occur:
  - 0 transmitted and 0 received
  - 0 transmitted and 1 received
  - 1 transmitted and 1 received
  - 1 transmitted and 0 received



(a) Schematic of binary channel (b) Schematic of binary symmetric channel

- Denoting transmitted symbols with subscript TX and received symbols with subscript RX, the symbol transition matrix is:

$$\begin{bmatrix} P(0_{RX}|0_{TX}) & P(0_{RX}|1_{TX}) \\ P(1_{RX}|0_{TX}) & P(1_{RX}|1_{TX}) \end{bmatrix}$$

- If the probability, p, of a transmitted 0 being received in error as a 1 is equal to the probability of a transmitted 1 being received as a 0, the binary channel is said to be symmetric and the transition matrix is

$$\begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

Symbol Transmission Matrix
Binary Symmetric Channel

- The conditional probabilities in the matrix must sum vertically to unity:

$$P(0_{RX}|0_{TX}) + P(1_{RX}|0_{TX}) = 1$$

- For the binary symmetric case:

$$P(0_{RX}|0_{TX}) = P(1_{RX}|1_{TX}) = 1 - p$$

- The unconditional probability of receiving a 0 is therefore:

$$P_{0_{RX}} = P(0_{TX})P(0_{RX}|0_{TX}) + P(1_{TX})P(0_{RX}|1_{TX})$$

which can be rewritten for the binary symmetric channel as

$$P_{0_{RX}} = P(0)(1 - p) + P(1)p$$

- Similarly

$$P_{1_{RX}} = P(1)(1 - p) + P(0)p$$

- The above equations relate the observed probabilities of received symbols to the a priori probabilities of transmitted symbols and the error probabilities of the channel.

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Symbol Transmission Matrix
Multi-symbol Transmission

- For an M-symbol communication alphabet the 2×2 transition matrix of the binary channel must be extended to an $M \times M$ matrix.
- [**Example 7.1**] Consider a source with $M = 6$ symbols, $A\ B\ C\ D\ E\ F$, and a 6-ary receiver which can distinguish these symbols. The transition matrix, $\mathbf{T}$, for this system contains 6×6 transition probabilities

$$\mathbf{T} = \begin{bmatrix} 1/2 & 0/1 & 1/24 & 0/1 & 0/1 & 1/8 \\ 1/4 & 1/2 & 1/6 & 1/3 & 1/4 & 1/12 \\ 1/8 & 1/8 & 1/4 & 1/6 & 0/1 & 1/12 \\ 0/1 & 1/3 & 1/4 & 1/3 & 1/4 & 1/4 \\ 1/8 & 0/1 & 1/6 & 1/12 & 1/2 & 1/8 \\ 0/1 & 1/24 & 1/8 & 1/12 & 0/1 & 1/3 \end{bmatrix}$$

Symbol Transmission Matrix
Multi-symbol Transmission

- The a priori transmission probabilities of the six symbols are

$$P(A) = 0.1 \quad P(B) = 0.15 \quad P(C) = 0.4$$
$$P(D) = 0.05 \quad P(E) = 0.2 \quad P(F) = 0.1$$

- Find the probability of error
   i. when only the symbol D is transmitted
   ii. when a random string of symbols is transmitted
   iii. For the latter case, find the probability of receiving the symbol C.

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Bayes Decision Criterion
Basics

- Bayes decision criterion is the most widely applied decision rule in communications systems, and operates so as to minimize the average *cost* (in terms of errors or lost information) of making decisions.
- Decision Cost: In binary transmission there are two ways to lose information:
  - Information is lost when a transmitted digital 1 is received in error as a digital 0
  - Information is lost when a transmitted digital 0 is received in error as a digital 1
- The cost in the sense of lost information due to mechanisms 1 and 2 is denoted here by C_0 and C_1 respectively.
- There is no cost (i.e. no information is lost) when correct decisions are made.

Bayes Decision Criterion
Expected Conditional Decision Costs

- The expected conditional cost, $C(0|v)$, incurred when a detected voltage v is interpreted by a decision circuit as a digital 0 is given by

$$C(0|v) = C_0 P(1|v)$$

where C_0 is the cost if the decision is in error and $P(1|v)$ is the (a posteriori) probability that the decision is in error.

- By symmetry, the expected conditional cost incurred when v is interpreted as a digital 1 is given by

$$C(1|v) = C_1 P(0|v)$$

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Bayes Decision Criterion
Optimum Decision Rule

- A rational decision rule to adopt is to interpret each detected voltage, v, as either a 0 or a 1, so as to minimize the expected conditional cost, i.e.

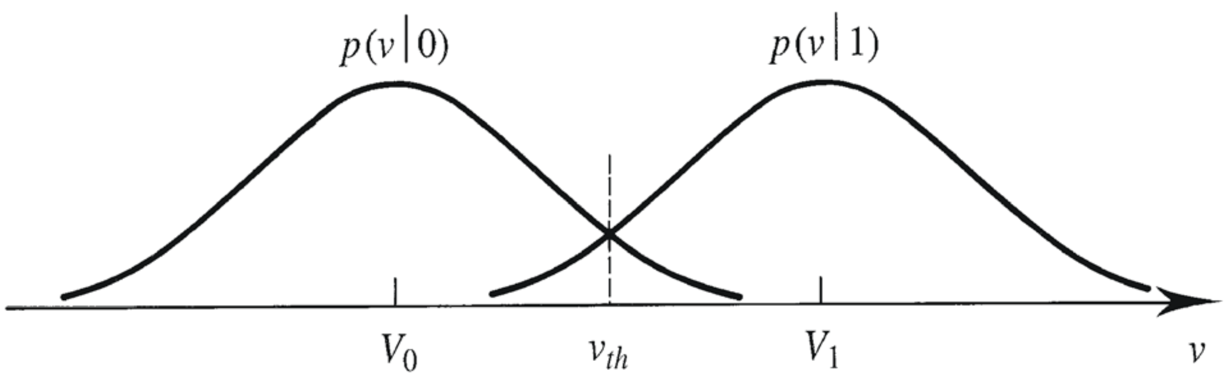
$$\begin{cases} 1 & \text{if } C(1|v) < C(0|v) \\ 0 & \text{if } C(1|v) > C(0|v) \end{cases} \quad \text{or equivalently} \quad \begin{cases} 1 & \text{if } \dfrac{P(0|v)}{P(1|v)} < \dfrac{C_0}{C_1} \\ 0 & \text{if } \dfrac{P(0|v)}{P(1|v)} > \dfrac{C_0}{C_1} \end{cases}$$

- If $C_0 = C_1$, then the above formula represents a maximum a posteriori (MAP) probability decision criterion.
- This is one form of Bayes decision criterion, and uses a posteriori probabilities, however, which are not usually known.

Bayes Decision Criterion
Optimum Decision Rule

- It can be transformed to a more useful form by using Bayes theorem: $P(0|v)=\frac{p(v|0)P(0)}{p(v)}$ and $P(1|v)=\frac{p(v|1)P(1)}{p(v)}$.
- Since $\frac{P(0|v)}{P(1|v)}=\frac{p(v|0)P(0)}{p(v|1)P(1)}$, we have

$$\begin{cases} 1 & \text{if } \dfrac{p(v|0)P(0)}{p(v|1)P(1)} < \dfrac{C_0}{C_1} \\ 0 & \text{if } \dfrac{p(v|0)P(0)}{p(v|1)P(1)} > \dfrac{C_0}{C_1} \end{cases}$$



PDFs for binary transmissions V_0 or V_1

Bayes Decision Criterion
Optimum Decision Rule

- Rearranging the above rule gives Bayes criterion in a form using conditional probability density functions and the usually known a priori source probabilities:

$$\begin{cases} 1 & \text{if } \dfrac{p(v|0)}{p(v|1)} < \dfrac{C_0 P(1)}{C_1 P(0)} \\ 0 & \text{if } \dfrac{p(v|0)}{p(v|1)} > \dfrac{C_0 P(1)}{C_1 P(0)} \end{cases}$$

- $\frac{p(v|0)}{p(v|1)}$ is called the *likelihood ratio* and $\frac{C_0 P(1)}{C_1 P(0)}$ is a likelihood threshold, denoted by L_{th}.
- If $C_0 = C_1$ and $P(0) = P(1)$, then $L_{th} = 1$ and the above rule is called a maximum likelihood decision criterion.

Bayes Decision Criterion
Comparison of Receiver Types

Receiver	A priori probabilities known	Decision costs known	Assumptions	Decision criterion
Bayes	Yes	Yes	None	$\begin{cases} 1 & \text{if } \dfrac{p(v\|0)}{p(v\|1)} < \dfrac{C_0 P(1)}{C_1 P(0)} \\ 0 & \text{if } \dfrac{p(v\|0)}{p(v\|1)} > \dfrac{C_0 P(1)}{C_1 P(0)} \end{cases}$
MAP	Yes	No	$C_0 = C_1$	$\begin{cases} 1 & \text{if } \dfrac{p(v\|0)}{p(v\|1)} < \dfrac{P(1)}{P(0)} \\ 0 & \text{if } \dfrac{p(v\|0)}{p(v\|1)} > \dfrac{P(1)}{P(0)} \end{cases}$
Maximum likelihood	No	No	$C_0 P(1) = C_1 P(0)$	$\begin{cases} 1 & \text{if } \dfrac{p(v\|0)}{p(v\|1)} < 1 \\ 0 & \text{if } \dfrac{p(v\|0)}{p(v\|1)} > 1 \end{cases}$

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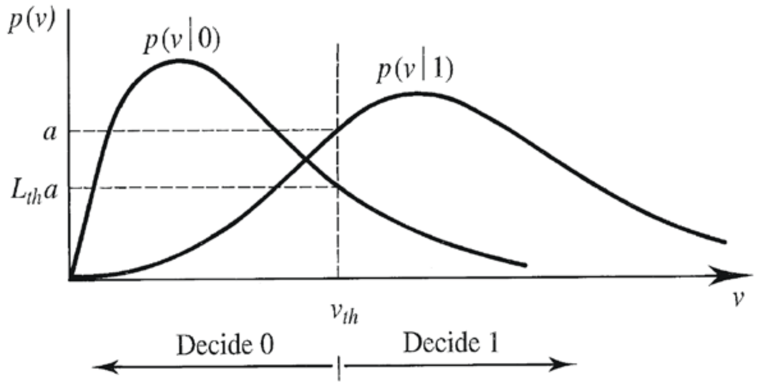
Bayes Decision Criterion
Optimum Decision Threshold Voltage

- Bayes decision criterion represents a general solution to setting the optimum reference or threshold voltage, v_{th}, in a receiver decision circuit.
- The threshold voltage which minimizes the expected conditional cost of each decision is the value of v which satisfies:

$$\frac{p(v|0)}{p(v|1)} = \frac{C_0 P(1)}{C_1 P(0)} = L_{th}$$



Optimum decision threshold for envelope detected OOK signal

If $L_{th} = 1.0$, such as would be the case for statistically independent, equiprobable symbols with equal error costs, then the voltage threshold would occur at the intersection of the two conditional pdfs.

Bayes Decision Criterion
Average Unconditional Decision Cost

- As well as minimizing the cost of each decision, Bayes criterion also minimizes the average cost of decisions.
- The average cost, $\bar{C}$, of making decision is given by

$$\bar{C} = C_1 P(0) P(1_{RX}|0_{TX}) + C_0 P(1) P(0_{RX}|1_{TX})$$

- Since

$$P(1_{RX}|0_{TX}) = \int_{v_{th}}^{\infty} p(v|0)\,dv$$

$$P(0_{RX}|1_{TX}) = \int_{-\infty}^{v_{th}} p(v|1)\,dv$$

we have

$$\bar{C} = C_1 P(0) \int_{v_{th}}^{\infty} p(v|0)\,dv + C_0 P(1) \int_{-\infty}^{v_{th}} p(v|1)\,dv$$

Bayes Decision Criterion
Average Unconditional Decision Cost

- Bayes decision criterion sets the decision threshold v_{th} so as to minimize $\bar{C}$.
- Using $\frac{d}{dx}\int_c^x f(t)dt = f(x)$ and $\frac{d}{dx}\int_x^c f(t)dt = -f(x)$, we have $\frac{d\bar{C}}{dv_{th}} = C_1 P(0)[-p(v_{th}|0)] + C_0 P(1)[p(v_{th}|1)]$
- Setting it to zero gives
$$C_1 P(0) p(v_{th}|0) = C_0 P(1) p(v_{th}|1)$$
- Equivalently
$$\frac{p(v_{th}|1)}{p(v_{th}|0)} = \frac{C_1}{C_0}\frac{P(0)}{P(1)}$$
- The solution v_{th} gives Bayes criterion (i.e. optimum) decision voltage.

Bayes Decision Criterion
Exercises

- **[Example 7.2] Binary transmission**
- Consider a binary transmission system subject to additive Gaussian noise which has a mean value of 1.0 V and a standard deviation of 2.5 V. A digital 1 is represented by a rectangular pulse with amplitude 4.0 V and a digital 0 is represented by a rectangular pulse with amplitude -4.0 V. The costs due to each type of error are identical (i.e. $C_0 = C_1$) but the a priori probabilities of symbol transmission are different. $P(1)$ being twice $P(0)$, as the symbols are not statistically independent. Find the optimum decision threshold voltage and the resulting probability of symbol error.

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Neyman-Pearson Decision Criterion

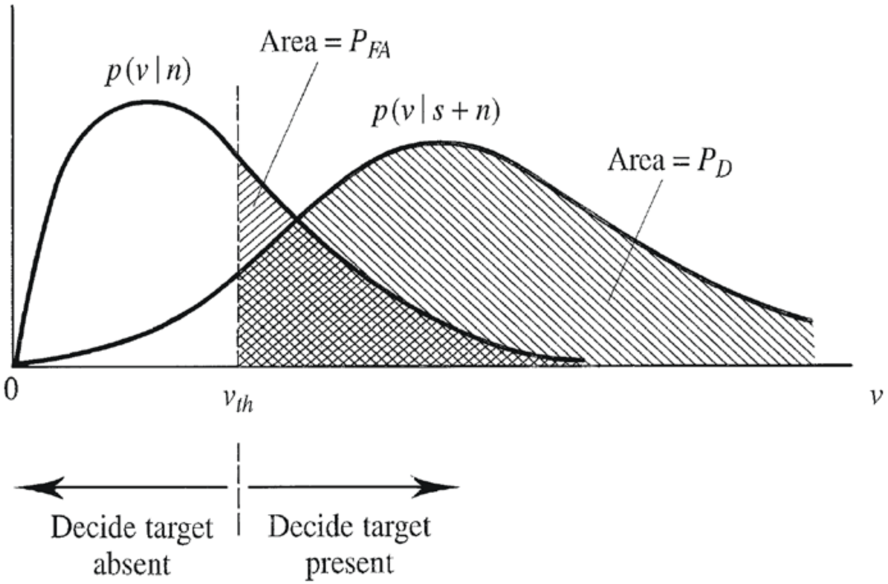
- The Neyman-Pearson decision criterion requires only a posteriori probabilities.
- Unlike Bayes decision rule it does not require a priori source probability information.
- This criterion is particularly appropriate to pulse detection in Gaussian noise, as occurs in radar applications, where the source statistics (i.e. probabilities of presence, and absence, of a target) are unknown.
- It also works well when $C_0 \gg C_1$ or, in radar terms, when the information cost of erroneously deciding a target is present is much less than the information cost of erroneously deciding a target is absent.
- In this context the important probabilities are those for *target detection*, P_D, and *target false alarm*, P_{FA}.

Neyman-Pearson Decision Criterion

- The two probabilities are

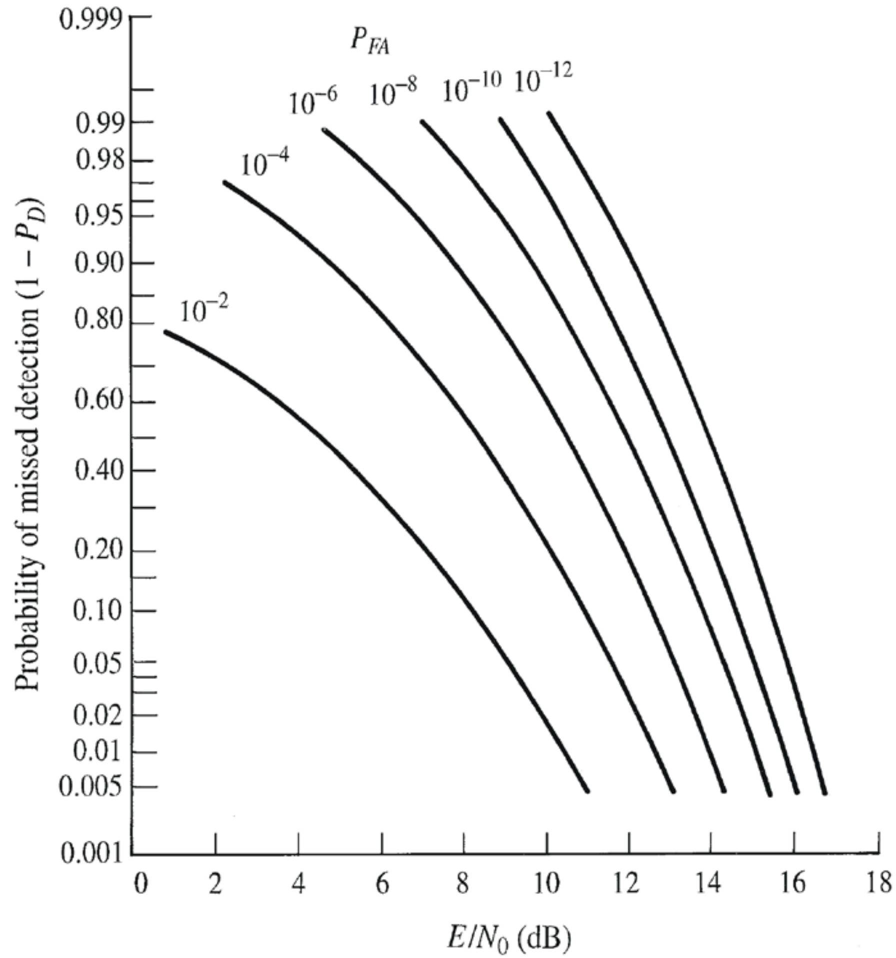
$$P_D = \int_{v_{th}}^{\infty} p(v|s+n)dv \quad \text{and} \quad P_{FA} = \int_{v_{th}}^{\infty} p(v|n)dv$$

where s denotes signal (arising due to reflections from a target) and n denotes noise.



Neyman-Pearson Decision Criterion

- In the Neyman-Pearson detector the threshold voltage, v_{th}, is chosen to give an acceptable value of P_{FA}.
- The detection performance is dependent on both the choice of P_{FA} and the ratio of received pulse energy to noise power spectral density, E/N_0.



Pulse detection in Gaussian noise using Neyman-Pearson criterion

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Summary

- Transition probabilities can be assembled into a matrix to describe the end to end error properties of a communications channel.
- Bayes decision criterion is the criterion most often used in digital communications receivers. It is optimum in the sense that it minimizes the cost of making decisions.
- Bayes criterion allows the optimum threshold voltage(s), which delineate(s) decision regions, to be found.
- Maximum a posteriori probability (MAP) and maximum likelihood detectors are special cases of a Bayes criterion detector, appropriate when decision costs, or decision costs and a priori probabilities, are unknown.
- The Neyman-Pearson decision criterion is normally used in radar applications. It has the advantage over Bayes criterion that a priori symbol probabilities, whilst known to be very different, need not be quantified.

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