

Computing energy release rates using the Virtual Crack Closure Technique with the Finite Element Method: analytical discussion

Luca Di Stasio^{1,2,a}, Janis Varna^{2,b} and Zoubir Ayadi^{1,c}

¹SI2M, IJL, EEIGM, Université de Lorraine, 6 Rue Bastien Lepage, F-54010 Nancy, France

²Division of Polymer Engineering, Luleå University of Technology, SE-97187 Luleå, Sweden

E-mail: ^aluca.di-stasio@univ-lorraine.fr, ^bjanis.varna@ltu.se, ^czoubir.ayadi@univ-lorraine.fr

Abstract. The effect of crack tip orientation and elements' size in its vicinity on mode splitting in the Virtual Crack Closure Technique is analyzed by means of analytical derivations.

- (i) The total energy release rate is shown to have no direct dependence on the debond angular size, but only an indirect one through the FEM solution of the crack displacement field in the crack tip neighborhood. It thus can be computed using unrotated forces and displacements, i.e. aligned with the global and not with the local reference.
- (ii) Mode I and mode II energy release rate are expressed as a function of the crack displacement and it is shown as in [5] that crack tip forces depend linearly on both components of the crack displacements. However, what in [5] is an assumption used to reverse-engineer the relationship, here the dependency is fully derived in terms of the underlying FEM discretization. It is furthermore shown that crack tip forces do not only depend on crack displacements, but also on the resultant of forces of the 4 elements connected to the crack tip, which represent the influence of the rest of the domain.
- (iii) A new vectorial formulation of the VCCT is proposed, which can be applied to elements of different order and in the presence of quarter-tip singularity (based on the work in [6]). It provides a general framework for the analysis of curved cracks analyzed with FEM. It furthermore expresses directly the dependence on FEM discretization and solution. By implementing the equations provided alongside the classic FEM, it can provide a native formulation which does not rely on the extraction of internal or reaction forces in the post-processing phase.
- (iv) Drawing upon the asymptotic expression of the displacement derived in [7] and presented in [8], it is shown that mode I and mode II for a curved crack behaves as $A \log(\delta) + B$, where δ is the angular discretization at the crack tip. The result is confirmed by the numerical results.

List of acronyms

VCCT	Virtual Crack Closure Technique
BEM	Boundary Element Method
FEM	Finite Element Method

List of symbols

G_I	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate
G_{II}	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate
G_{TOT}	$\left[\frac{J}{m^2}\right]$	Total energy release rate
$G_{I,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate in $r - \theta$ reference frame
$G_{II,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate in $r - \theta$ reference frame
$G_{TOT,r\theta}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate in $r - \theta$ reference frame
$\tilde{G}_{I,xy}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate of equivalent crack in $x - y$ reference frame
$\tilde{G}_{II,xy}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate of equivalent crack in $x - y$ reference frame
$\tilde{G}_{TOT,xy}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate of equivalent crack in $x - y$ reference frame
R_f	$[\mu m]$	Fiber radius
a	$[\mu m]$	Debond size
Δa	$[\mu m]$	Debond increment
$\Delta \theta$	$[rad]$	Half debond angular size
δ	$[rad]$	Angular size of element at the interface close to the crack tip
$u_{x,[A-Z]}$	$[\mu m]$	Displacement along x of a point labeled with a letter in [A-Z]
$u_{y,[A-Z]}$	$[\mu m]$	Displacement along y of a point labeled with a letter in [A-Z]
u_x	$[\mu m]$	Displacement along x -direction
u_y	$[\mu m]$	Displacement along y -direction
u_r	$[\mu m]$	Displacement along r -direction
u_θ	$[\mu m]$	Displacement along θ -direction
$F_{x,[A-Z]}$	$[\mu m]$	Force along x at a point labeled with a letter in [A-Z]
$F_{y,[A-Z]}$	$[\mu m]$	Force along y at a point labeled with a letter in [A-Z]
F_x	$[\mu m]$	Force along x -direction
F_y	$[\mu m]$	Force along y -direction
F_r	$[\mu m]$	Force along r -direction
F_θ	$[\mu m]$	Force along θ -direction
$\underline{\underline{R}}$	$[-]$	Rotation matrix

1. FEM formulation with quadrilateral elements

$$\Pi(u_i) = \int_V \sigma_{ij} \varepsilon_{ij} dV + \int_V \rho \ddot{u}_i u_i dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \quad (1)$$

$$\Pi(u_i) = \int_V \sigma_{ij} \varepsilon_{ij} dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \quad (2)$$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (3)$$

$$\underline{\underline{\varepsilon}}(x, y) = \underline{\underline{\tilde{B}}} \cdot \underline{u}(x, y) \quad (4)$$

$$\underline{\underline{\tilde{B}}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (5)$$

$$\underline{\underline{\sigma}} = \underline{\underline{D}} \cdot \underline{\underline{\varepsilon}} \quad (6)$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \quad \underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \quad (7)$$

$$\underline{\underline{D}} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{bmatrix} \quad \text{with } G = \frac{E}{2(1+\nu)} \text{ for an isotropic material} \quad (8)$$

$$\begin{aligned} E_1 &= \frac{E}{1-\nu^2} & E_2 &= \nu E_1 & \text{for plane stress} \\ E_1 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & E_2 &= \frac{\nu E_1}{1-\nu} & \text{for plane strain} \end{aligned} \quad (9)$$

$$\begin{aligned} \Pi(\underline{u}) &= \frac{1}{2} \int_V \underline{\underline{\varepsilon}}^T \underline{\underline{D}} \cdot \underline{\underline{\varepsilon}} dV - \int_V \underline{F}^T \underline{u} dV - \int_S \underline{f}^T \underline{u} dS = \\ &= \frac{1}{2} \int_V \underline{u}^T \underline{\underline{\tilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\tilde{B}}} \cdot \underline{u} dV - \int_V \underline{F}^T \underline{u} dV - \int_S \underline{f}^T \underline{u} dS \end{aligned} \quad (10)$$

$$\delta \Pi(\delta \underline{u}) = 0 \quad (11)$$

$$\begin{aligned}
\delta\Pi(\delta\underline{u}) &= \Pi(\underline{u} + \delta\underline{u}) - \Pi(\underline{u}) = \\
&= \frac{1}{2} \int_V (\underline{u} + \delta\underline{u})^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot (\underline{u} + \delta\underline{u}) dV - \int_V \underline{F}^T (\underline{u} + \delta\underline{u}) dV - \int_S \underline{f}^T (\underline{u} + \delta\underline{u}) dS + \\
&- \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \int_V \underline{F}^T \underline{u} dV + \int_S \underline{f}^T \underline{u} dS = \\
&= \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \frac{1}{2} \int_V \delta\underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \delta\underline{u} dV \xrightarrow{\approx 0} + \\
&+ \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \delta\underline{u} dV + \frac{1}{2} \int_V \delta\underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \\
&- \int_V \underline{F}^T \underline{u} dV - \int_V \underline{F}^T \delta\underline{u} dV + \\
&- \int_S \underline{f}^T \underline{u} dS - \int_S \underline{f}^T \delta\underline{u} dS + \\
&- \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \int_V \underline{F}^T \underline{u} dV + \int_S \underline{f}^T \underline{u} dS = \\
&= \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \delta\underline{u} dV - \int_V \underline{F}^T \delta\underline{u} dV - \int_S \underline{f}^T \delta\underline{u} dS
\end{aligned} \tag{12}$$

$$\int_V \delta\underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV - \int_V \delta\underline{u}^T \underline{F} dV - \int_S \delta\underline{u}^T \underline{f} dS = 0 \tag{13}$$

$$\underline{u} = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} \quad \underline{u}_N = \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{2,x} \\ u_{2,y} \\ u_{3,x} \\ u_{3,y} \\ u_{4,x} \\ u_{4,y} \\ u_{3,x} \\ u_{3,y} \\ u_{4,x} \\ u_{4,y} \end{bmatrix} \quad \text{or} \quad \underline{u}_N = \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{2,x} \\ u_{2,y} \\ u_{3,x} \\ u_{3,y} \\ u_{4,x} \\ u_{4,y} \\ u_{5,x} \\ u_{5,y} \\ u_{6,x} \\ u_{6,y} \\ u_{7,x} \\ u_{7,y} \\ u_{8,x} \\ u_{8,y} \end{bmatrix} \tag{14}$$

$$\underline{u} = \underline{\underline{N}} \cdot \underline{u}_N \tag{15}$$

$$\underline{\underline{N}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{16}$$

$$\underline{\underline{N}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \tag{17}$$

$$\begin{cases} N_1 = N_1(\xi, \eta) \\ N_2 = N_2(\xi, \eta) \\ N_3 = N_3(\xi, \eta) \\ N_4 = N_4(\xi, \eta) \end{cases} \quad \text{with} \quad \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases} \quad \text{for isoparametric elements} \quad (18)$$

$$\underline{\underline{B}} = \underline{\underline{\tilde{B}}} \cdot \underline{\underline{N}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \quad (19)$$

$$\begin{aligned} \underline{\underline{B}} &= \underline{\underline{\tilde{B}}} \cdot \underline{\underline{N}} = \\ &= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_5}{\partial x} & 0 & \frac{\partial N_6}{\partial x} & 0 & \frac{\partial N_7}{\partial x} & 0 & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} & 0 & \frac{\partial N_5}{\partial y} & 0 & \frac{\partial N_6}{\partial y} & 0 & \frac{\partial N_7}{\partial y} & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial y} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial y} & \frac{\partial N_6}{\partial x} & \frac{\partial N_7}{\partial y} & \frac{\partial N_7}{\partial x} & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{bmatrix} \end{aligned} \quad (20)$$

$$\delta \underline{u} = \delta (\underline{\underline{N}} \cdot \underline{u}_N) = \underline{\underline{N}} \delta \underline{u}_N \quad (21)$$

$$\int_V \delta \underline{u}_N^T \underline{\underline{N}}^T \underline{\underline{\tilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\tilde{B}}} \cdot \underline{\underline{N}} \cdot \underline{u}_N dV - \int_V \delta \underline{u}_N^T \underline{\underline{N}}^T \underline{F} dV - \int_S \delta \underline{u}_N^T \underline{\underline{N}}^T \underline{f} dS = 0 \quad (22)$$

$$\delta \underline{u}_N^T \left(\int_V \underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}} dV \cdot \underline{u}_N - \int_V \underline{\underline{N}}^T \underline{F} dV - \int_S \underline{\underline{N}}^T \underline{f} dS \right) = 0 \quad (23)$$

$$\underline{k} \cdot \underline{u}_N = \underline{F}_N \quad \underline{k} = \int_V (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) dV \quad \underline{F}_N = \int_V \underline{\underline{N}}^T \underline{F} dV + \int_S \underline{\underline{N}}^T \underline{f} dS \quad (24)$$

$$\begin{cases} N_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{cases} \quad \begin{cases} N_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \\ N_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\ N_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\ N_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \\ N_5(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1-\eta) \\ N_6(\xi, \eta) = \frac{1}{2}(1+\xi)(1-\eta^2) \\ N_7(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1+\eta) \\ N_8(\xi, \eta) = \frac{1}{2}(1-\xi)(1-\eta^2) \end{cases} \quad (25)$$

$$\begin{cases} \frac{\partial N_1(\xi, \eta)}{\partial \xi} = -\frac{1}{4}(1-\eta) \\ \frac{\partial N_2(\xi, \eta)}{\partial \xi} = \frac{1}{4}(1-\eta) \\ \frac{\partial N_3(\xi, \eta)}{\partial \xi} = \frac{1}{4}(1+\eta) \\ \frac{\partial N_4(\xi, \eta)}{\partial \xi} = -\frac{1}{4}(1+\eta) \\ \frac{\partial N_1(\xi, \eta)}{\partial \eta} = -\frac{1}{4}(1-\xi) \\ \frac{\partial N_2(\xi, \eta)}{\partial \eta} = -\frac{1}{4}(1+\xi) \\ \frac{\partial N_3(\xi, \eta)}{\partial \eta} = \frac{1}{4}(1+\xi) \\ \frac{\partial N_4(\xi, \eta)}{\partial \eta} = \frac{1}{4}(1-\xi) \end{cases} \quad (26)$$

$$\left\{ \begin{array}{l} \frac{\partial N_1(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 - \eta) (2\xi + \eta) \\ \frac{\partial N_2(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 - \eta) (2\xi - \eta) \\ \frac{\partial N_3(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 + \eta) (2\xi + \eta) \\ \frac{\partial N_4(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 + \eta) (2\xi - \eta) \\ \frac{\partial N_5(\xi, \eta)}{\partial \xi} = -\xi (1 - \eta) \\ \frac{\partial N_6(\xi, \eta)}{\partial \xi} = \frac{1}{2} (1 - \eta^2) \\ \frac{\partial N_7(\xi, \eta)}{\partial \xi} = -\xi (1 + \eta) \\ \frac{\partial N_8(\xi, \eta)}{\partial \xi} = -\frac{1}{2} (1 - \eta^2) \\ \frac{\partial N_1(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 - \xi) (2\eta + \xi) \\ \frac{\partial N_2(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 + \xi) (2\eta - \xi) \\ \frac{\partial N_3(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 + \xi) (2\eta + \xi) \\ \frac{\partial N_4(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 - \xi) (2\eta - \xi) \\ \frac{\partial N_5(\xi, \eta)}{\partial \eta} = -\frac{1}{2} (1 - \xi^2) \\ \frac{\partial N_6(\xi, \eta)}{\partial \eta} = -\eta (1 + \xi) \\ \frac{\partial N_7(\xi, \eta)}{\partial \eta} = \frac{1}{2} (1 - \xi^2) \\ \frac{\partial N_8(\xi, \eta)}{\partial \eta} = -\eta (1 - \xi) \end{array} \right. \quad (27)$$

$$\underline{p} = \underline{\underline{N}} \cdot \underline{p}_N \quad (28)$$

$$\underline{p} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \quad \text{or} \quad \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \\ x_5 \\ y_5 \\ x_6 \\ y_6 \\ x_7 \\ y_7 \\ x_8 \\ y_8 \end{bmatrix} \quad (29)$$

$$\begin{aligned} x &= x(\xi, \eta) = N_1(\xi, \eta) x_1 + N_2(\xi, \eta) x_2 + N_3(\xi, \eta) x_3 + N_4(\xi, \eta) x_4 \\ y &= y(\xi, \eta) = N_1(\xi, \eta) y_1 + N_2(\xi, \eta) y_2 + N_3(\xi, \eta) y_3 + N_4(\xi, \eta) y_4 \end{aligned} \quad (30)$$

$$\begin{aligned} x &= x(\xi, \eta) = N_1(\xi, \eta) x_1 + N_2(\xi, \eta) x_2 + N_3(\xi, \eta) x_3 + N_4(\xi, \eta) x_4 + \\ &\quad + N_5(\xi, \eta) x_5 + N_6(\xi, \eta) x_6 + N_7(\xi, \eta) x_7 + N_8(\xi, \eta) x_8 \\ y &= y(\xi, \eta) = N_1(\xi, \eta) y_1 + N_2(\xi, \eta) y_2 + N_3(\xi, \eta) y_3 + N_4(\xi, \eta) y_4 + \\ &\quad + N_5(\xi, \eta) y_5 + N_6(\xi, \eta) y_6 + N_7(\xi, \eta) y_7 + N_8(\xi, \eta) y_8 \end{aligned} \quad (31)$$

$$\begin{aligned}
g &= \det \left(\underline{\underline{g}} \right) = \\
&= \left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \eta} \right) - \left(\frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} \right)
\end{aligned} \tag{41}$$

$$dV(x, y) = \sqrt{g} dV(\xi, \eta) \tag{42}$$

$$\begin{aligned}
\underline{\underline{k}}_e &= \int_{V_e(x, y)} (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) dV_e(x, y) = \int_{V_e(\xi, \eta)} (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) \sqrt{g} dV_e(\xi, \eta) \\
&= \int_{-1}^1 \int_{-1}^1 (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) \sqrt{g} d\xi d\eta \approx \sum_{i=1}^N \sum_{j=1}^N w_i w_j (\underline{\underline{B}}^T(\xi_i, \eta_j) \cdot \underline{\underline{D}} \cdot \underline{\underline{B}}(\xi_i, \eta_j) \sqrt{g})
\end{aligned} \tag{43}$$

$$k_e = \begin{bmatrix} k_{e|11} & k_{e|12} & k_{e|13} & k_{e|14} & k_{e|15} & k_{e|16} & k_{e|17} & k_{e|18} \\ k_{e|12} & k_{e|22} & k_{e|23} & k_{e|24} & k_{e|25} & k_{e|26} & k_{e|27} & k_{e|28} \\ k_{e|13} & k_{e|23} & k_{e|33} & k_{e|34} & k_{e|35} & k_{e|36} & k_{e|37} & k_{e|38} \\ k_{e|14} & k_{e|24} & k_{e|34} & k_{e|44} & k_{e|45} & k_{e|46} & k_{e|47} & k_{e|48} \\ k_{e|15} & k_{e|25} & k_{e|35} & k_{e|45} & k_{e|55} & k_{e|56} & k_{e|57} & k_{e|58} \\ k_{e|16} & k_{e|26} & k_{e|36} & k_{e|46} & k_{e|56} & k_{e|66} & k_{e|67} & k_{e|68} \\ k_{e|17} & k_{e|27} & k_{e|37} & k_{e|47} & k_{e|57} & k_{e|67} & k_{e|77} & k_{e|78} \\ k_{e|18} & k_{e|28} & k_{e|38} & k_{e|48} & k_{e|58} & k_{e|68} & k_{e|78} & k_{e|88} \end{bmatrix} \tag{44}$$

2. VCCT for first order quadrilateral elements

2.1. Definition of crack tip reference frame



Figure 1. Schematic representation of the discretized crack tip geometry for 1st order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad (45)$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \quad (46)$$

2.2. Calculation of displacements and reaction forces

$$u_x = u_{x,M} - u_{x,F} \quad u_y = u_{y,M} - u_{y,F} \quad (47)$$

$$u_r = \cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y \quad u_\theta = -\sin(\Delta\theta) u_x + \cos(\Delta\theta) u_y \quad (48)$$

$$F_r = \cos(\Delta\theta) F_{x,CT} + \sin(\Delta\theta) F_{y,CT} \quad F_\theta = -\sin(\Delta\theta) F_{x,CT} + \cos(\Delta\theta) F_{y,CT} \quad (49)$$

$$\left\{ \begin{array}{l}
(k_{e,M|11} + k_{e,M|33}) u_{x,MCT} + (k_{e,M|12} + k_{e,M|34}) u_{y,MCT} + \\
+ k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + (k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + k_{M|31} u_{x,NCOI} + k_{M|32} u_{y,NCOI} = 0 \\
\\
(k_{e,M|21} + k_{e,M|43}) u_{x,MCT} + (k_{e,M|22} + k_{e,M|44}) u_{y,MCT} + \\
+ k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + k_{M|41} u_{x,NCOI} + k_{M|42} u_{y,NCOI} = 0 \\
\\
(k_{e,F|77} + k_{e,F|55}) u_{x,FCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,FCT} + \\
+ k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i} + k_{F|57} u_{x,NCOI} + k_{F|58} u_{y,NCOI} = 0 \\
\\
(k_{e,F|87} + k_{e,F|65}) u_{x,FCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,FCT} + \\
+ k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + k_{F|67} u_{x,NCOI} + k_{F|68} u_{y,NCOI} = 0 \\
\\
u_{x,FCT} - u_{x,MCT} - u_{x,DUMMY} = 0 \\
u_{y,FCT} - u_{y,MCT} - u_{y,DUMMY} = 0 \\
\\
u_{x,DUMMY} = 0 \\
u_{y,DUMMY} = 0
\end{array} \right. \quad (50)$$

$$\left\{ \begin{array}{l}
(k_{e,M|11} + k_{e,M|33}) u_{x,MCT} + (k_{e,M|12} + k_{e,M|34}) u_{y,MCT} + \\
+ k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + (k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + k_{M|31} u_{x,NCOI} + k_{M|32} u_{y,NCOI} = 0 \\
\\
(k_{e,M|21} + k_{e,M|43}) u_{x,MCT} + (k_{e,M|22} + k_{e,M|44}) u_{y,MCT} + \\
+ k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + k_{M|41} u_{x,NCOI} + k_{M|42} u_{y,NCOI} = 0 \\
\\
(k_{e,F|77} + k_{e,F|55}) u_{x,MCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,MCT} + \\
+ k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i} + k_{F|57} u_{x,NCOI} + k_{F|58} u_{y,NCOI} = 0 \\
\\
(k_{e,F|87} + k_{e,F|65}) u_{x,MCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,MCT} + \\
+ k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + k_{F|67} u_{x,NCOI} + k_{F|68} u_{y,NCOI} = 0 \\
\\
u_{x,FCT} = u_{x,MCT} \\
u_{y,FCT} = u_{y,MCT} \\
\\
R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{array} \right. \quad (51)$$

$$\left\{ \begin{aligned}
& (k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}) u_{x,MCT} + (k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}) u_{y,MCT} + \\
& + k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + \\
& + (k_{M|31} + k_{F|57}) u_{x,NCOI} + (k_{M|32} + k_{F|58}) u_{y,NCOI} + \\
& + (k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2} + \\
& + \sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i} = 0 \\
\\
& (k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}) u_{x,MCT} + (k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}) u_{y,MCT} + \\
& + k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + \\
& + (k_{M|41} + k_{F|67}) u_{x,NCOI} + (k_{M|42} + k_{F|68}) u_{y,NCOI} + \\
& + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
& + \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} = 0 \\
\\
& u_{x,FCT} = u_{x,MCT} \\
& u_{y,FCT} = u_{y,MCT} \\
\\
& R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
& R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{aligned} \right. \tag{52}$$

$$\left\{ \begin{aligned}
& u_{y,MCT} = -\frac{k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} u_{x,MCT} + \\
& -\frac{k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\
& -\frac{(k_{M|31} + k_{F|57}) u_{x,NCOI} + (k_{M|32} + k_{F|58}) u_{y,NCOI}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\
& -\frac{(k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \\
\\
& \left[(k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}) + \frac{k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} (k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}) \right] u_{x,MCT} + \\
& + \left(k_{e,M|23} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|13} \right) u_{x,M} + \\
& + \left(k_{e,M|24} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|14} \right) u_{y,M} + \\
& + \left(k_{e,F|85} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|75} \right) u_{x,F} + \\
& + \left(k_{e,F|86} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|76} \right) u_{y,F} + \\
& + \left[(k_{M|41} + k_{F|67}) - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} (k_{M|31} + k_{F|57}) \right] u_{x,NCOI} + \\
& + \left[(k_{M|42} + k_{F|68}) - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} (k_{M|32} + k_{F|58}) \right] u_{y,NCOI} + \\
& + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
& -\frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} [(k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8}] + \\
& -\frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} [(k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2}] \\
& + \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} = 0 \\
\\
& u_{x,FCT} = u_{x,MCT} \\
& u_{y,FCT} = u_{y,MCT} \\
\\
& R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
& R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{aligned} \right. \tag{53}$$

$$\left\{ \begin{aligned}
& u_{y,MCT} = -\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} u_{x,MCT} + \\
& -\frac{k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
& -\frac{(k_{M|31}+k_{F|57}) u_{x,NCOI} + (k_{M|32}+k_{F|58}) u_{y,NCOI}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
& -\frac{(k_{M|17}+k_{M|35}) u_{N,MC|7} + (k_{M|18}+k_{M|36}) u_{N,MC|8} + (k_{F|71}+k_{F|53}) u_{N,FC|1} + (k_{F|72}+k_{F|54}) u_{N,FC|2}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} \\
& \left[(k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}) + \frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}) \right] u_{x,MCT} + \\
& + \left(k_{e,M|23} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} k_{e,M|13} \right) u_x + \\
& + \left(k_{e,M|24} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} k_{e,M|14} \right) u_y + \\
& + \left(k_{e,M|23} + k_{e,F|85} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{e,M|13} + k_{e,M|75}) \right) \overset{\text{approx}}{\approx} 0 + \\
& + \left(k_{e,M|24} + k_{e,F|86} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{e,M|14} + k_{e,M|76}) \right) \overset{\text{approx}}{\approx} 0 + \\
& + \left[(k_{M|41} + k_{F|67}) - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{M|31} + k_{F|57}) \right] u_{x,NCOI} + \\
& + \left[(k_{M|42} + k_{F|68}) - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{M|32} + k_{F|58}) \right] u_{y,NCOI} + \\
& + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
& -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} [(k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8}] + \\
& -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} [(k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2}] \\
& + \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} = 0 \\
& u_{x,FCT} = u_{x,MCT} \\
& u_{y,FCT} = u_{y,MCT} \\
& F_{x,CT} = R_{x,FCT} = \\
& = (k_{e,F|77} + k_{e,F|55}) u_{x,FCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,FCT} + \\
& + k_{e,F|75} \overset{\text{approx}}{\approx} 0 + k_{e,F|76} \overset{\text{approx}}{\approx} 0 + \\
& + \sum_{i=1}^4 k_{e,F|7i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|5i} u_{N,FB|i} \\
& F_{y,CT} = R_{y,FCT} = \\
& = (k_{e,F|87} + k_{e,F|65}) u_{x,FCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,FCT} + \\
& + k_{e,F|85} \overset{\text{approx}}{\approx} 0 + k_{e,F|86} \overset{\text{approx}}{\approx} 0 + \\
& + \sum_{i=1}^4 k_{e,F|8i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|6i} u_{N,FB|i}
\end{aligned} \right. \tag{54}$$

$$\left\{ \begin{aligned}
F_{x,CT} &= (k_{e,F|77} + k_{e,F|55}) u_{x,MCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,MCT} + \\
& + \sum_{i=1}^4 k_{e,F|7i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|5i} u_{N,FB|i} \\
F_{y,CT} &= (k_{e,F|87} + k_{e,F|65}) u_{x,MCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,MCT} + \\
& + \sum_{i=1}^4 k_{e,F|8i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|6i} u_{N,FB|i}
\end{aligned} \right. \tag{55}$$

$$\left\{ \begin{aligned}
F_{x,CT} &= K_{xx} u_x + K_{xy} u_y + \\
& + \sum_{i=1}^4 K_{FC,x|i} u_{N,FC|i} + \sum_{i=1, i \neq (3,4,5,6)}^8 K_{FB,x|i} u_{N,FB|i} + \\
& + \sum_{i=5}^8 K_{FC,x|i} u_{N,MC|i} + \sum_{i=7}^8 K_{MB,x|i} u_{N,FB|i} \\
F_{y,CT} &= K_{yx} u_x + K_{yy} u_y + \\
& + \sum_{i=1}^4 K_{FC,y|i} u_{N,FC|i} + \sum_{i=1, i \neq (3,4,5,6)}^8 K_{FB,y|i} u_{N,FB|i} + \\
& + \sum_{i=5}^8 K_{FC,y|i} u_{N,MC|i} + \sum_{i=7}^8 K_{MB,y|i} u_{N,FB|i}
\end{aligned} \right. \tag{56}$$

$$\begin{cases} F_{x,CT} &= K_{xx}u_x + K_{xy}u_y + \tilde{F}_x \\ F_{y,CT} &= K_{yx}u_x + K_{yy}u_y + \tilde{F}_y \end{cases} \quad (57)$$

2.3. Calculation of energy release rates

$$\begin{aligned} G_{I,r\theta} &= \frac{1}{2} \frac{F_r u_r}{R_f \delta} = \\ &= \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y) = \\ &= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) \end{aligned} \quad (58)$$

$$\begin{aligned} G_{II,r\theta} &= \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} = \\ &= \frac{1}{2R_f \delta} (-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y) (-\sin(\Delta\theta) u_x + \cos(\Delta\theta) u_y) = \\ &= \frac{1}{2R_f \delta} (\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y) \end{aligned} \quad (59)$$

$$\begin{aligned} G_{TOT,r\theta} &= G_{I,r\theta} + G_{II,r\theta} = \\ &= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) + \\ &\quad + \frac{1}{2R_f \delta} (\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y) = \\ &= \frac{1}{2R_f \delta} \left(\begin{array}{c} \xrightarrow{1} \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) F_x u_x \end{array} \right) + \\ &\quad + \frac{1}{2R_f \delta} \left(\begin{array}{c} \xrightarrow{0} \\ ((F_x u_y + F_y u_x) - (F_x u_y + F_y u_x)) \cos(\Delta\theta) \sin(\Delta\theta) \end{array} \right) + \\ &\quad + \frac{1}{2R_f \delta} \left(\begin{array}{c} \xrightarrow{1} \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) F_y u_y \end{array} \right) = \\ &= \frac{1}{2} \frac{F_x u_x}{R_f \delta} + \frac{1}{2} \frac{F_y u_y}{R_f \delta} = \\ &= \tilde{G}_{I,xy} + \tilde{G}_{II,xy} = \tilde{G}_{TOT,xy} \end{aligned} \quad (60)$$

$$\begin{aligned} G_{I,r\theta} &= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) = \\ &= \cos^2(\Delta\theta) \frac{F_x u_x}{2R_f \delta} + \left(\frac{F_x u_y}{2R_f \delta} + \frac{F_y u_x}{2R_f \delta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) \frac{F_y u_y}{2R_f \delta} = \\ &= \cos^2(\Delta\theta) \tilde{G}_{I,xy} + \left(\tilde{G}_{I,xy} \frac{u_y}{u_x} + \tilde{G}_{II,xy} \frac{u_x}{u_y} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) \tilde{G}_{II,xy} \end{aligned} \quad (61)$$

$$\begin{aligned}
G_{II,r\theta} &= \frac{1}{2R_f\delta} \left(\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y \right) = \\
&= \sin^2(\Delta\theta) \frac{F_x u_x}{2R_f\delta} - \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) \frac{F_y u_y}{2R_f\delta} = \\
&= \sin^2(\Delta\theta) \tilde{G}_{I,xy} - \left(\tilde{G}_{I,xy} \frac{u_y}{u_x} + \tilde{G}_{II,xy} \frac{u_x}{u_y} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) \tilde{G}_{II,xy}
\end{aligned} \tag{62}$$

2.4. Sensitivity analysis of the FEM solution

$$\begin{cases} F_{x,CT} &= K_{xx} u_x + K_{xy} u_y + \tilde{F}_x \\ F_{y,CT} &= K_{yx} u_x + K_{yy} u_y + \tilde{F}_y \end{cases} \tag{63}$$

$$\begin{aligned}
G_{I,r\theta} &\sim \frac{1}{2R_f\delta} \cos^2(\Delta\theta) \left(K_{xx} u_x^2 + K_{xy} u_y u_x + \tilde{F}_x u_x \right) + \\
&+ \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) \left(K_{xx} u_x u_y + K_{xy} u_y^2 + \tilde{F}_x u_y \right) + \\
&+ \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) \left(K_{yx} u_x^2 + K_{yy} u_y u_x + \tilde{F}_y u_x \right) + \\
&+ \frac{1}{2R_f\delta} \sin^2(\Delta\theta) \left(K_{yx} u_y u_x + K_{yy} u_y^2 + \tilde{F}_y \right)
\end{aligned} \tag{64}$$

$$\begin{aligned}
G_{II,r\theta} &\sim \frac{1}{2R_f\delta} \sin^2(\Delta\theta) k_x u_x^2(\Delta\theta) + \\
&- \frac{1}{2R_f\delta} (k_x + k_y) u_x(\Delta\theta) u_y(\Delta\theta) \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta} \cos^2(\Delta\theta) k_y u_y^2(\Delta\theta)
\end{aligned} \tag{65}$$

$$G_{TOT,r\theta} \sim \frac{1}{2R_f\delta} (k_x u_x^2(\Delta\theta) + k_y u_y^2(\Delta\theta)) \tag{66}$$

$$\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial \Delta\theta} &\sim \frac{1}{R_f\delta} \cos^2(\Delta\theta) k_x u_x(\Delta\theta) \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + \\
&+ \frac{1}{2R_f\delta} (k_x + k_y) \left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} u_y(\Delta\theta) + u_x(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \frac{1}{R_f\delta} \sin^2(\Delta\theta) k_y u_y(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \\
&+ \frac{1}{2R_f\delta} (k_y u_y^2(\Delta\theta) - k_x u_x^2(\Delta\theta)) \sin(2\Delta\theta) + \\
&+ \frac{1}{2R_f\delta} (k_x u_x(\Delta\theta) u_y(\Delta\theta) + k_y u_y(\Delta\theta) u_x(\Delta\theta)) \cos(2\Delta\theta)
\end{aligned} \tag{67}$$

$$\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f \delta} \sin^2(\Delta\theta) k_x u_x(\Delta\theta) \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + \\
& - \frac{1}{2R_f \delta} (k_x + k_y) \left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} u_y(\Delta\theta) + u_x(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{R_f \delta} \cos^2(\Delta\theta) k_y u_y(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \\
& + \frac{1}{2R_f \delta} (k_x u_x^2(\Delta\theta) - k_y u_y^2(\Delta\theta)) \sin(2\Delta\theta) + \\
& - \frac{1}{2R_f \delta} (k_x u_x(\Delta\theta) u_y(\Delta\theta) + k_y u_y(\Delta\theta) u_x(\Delta\theta)) \cos(2\Delta\theta)
\end{aligned} \tag{68}$$

$$\frac{\partial G_{TOT,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f \delta} \left(k_x u_x(\Delta\theta) \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + k_y u_y(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} \right) \tag{69}$$

2.5. Discretization error

$$u_r = \cos(\Delta\theta - \delta) u_x + \sin(\Delta\theta - \delta) u_y \quad u_\theta = -\sin(\Delta\theta - \delta) u_x + \cos(\Delta\theta - \delta) u_y \tag{70}$$

$$F_r = \cos(\Delta\theta) F_{x,CT} + \sin(\Delta\theta) F_{y,CT} \quad F_\theta = -\sin(\Delta\theta) F_{x,CT} + \cos(\Delta\theta) F_{y,CT} \tag{71}$$

$$\begin{aligned}
\tilde{G}_{I,r\theta} &= \frac{1}{2} \frac{F_r u_r}{R_f \delta} = \\
&= \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\cos(\Delta\theta - \delta) u_x + \sin(\Delta\theta - \delta) u_y) = \\
&= \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y) \cos(\delta) + \\
&\quad + \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\sin(\Delta\theta) u_x - \cos(\Delta\theta) u_y) \sin(\delta) = \\
&= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) \cos(\delta) + \\
&\quad + \frac{1}{2R_f \delta} (-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x) \sin(\delta) = \\
&= G_{I,r\theta} \cos(\delta) + \\
&\quad + \frac{1}{2R_f \delta} (-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x) \sin(\delta) = \\
&= G_{I,r\theta} \cos(\delta) - \frac{1}{2R_f \delta} F_r u_\theta \sin \delta
\end{aligned} \tag{72}$$

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_{I,r\theta} &= \lim_{\delta \rightarrow 0} G_{I,r\theta} \cos(\delta) + \\
&+ \lim_{\delta \rightarrow 0} \frac{1}{2R_f \delta} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \sin(\delta) = \\
&= G_{I,r\theta} + \\
&+ \frac{1}{2R_f} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \lim_{\delta \rightarrow 0} \frac{\sin(\delta)}{\delta} = \\
&= G_{I,r\theta} + \frac{1}{2R_f} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \\
&= G_{I,r\theta} - \frac{1}{2R_f} F_r u_\theta
\end{aligned} \tag{73}$$

$$\begin{aligned}
\tilde{G}_{II,r\theta} &= \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} = \\
&= \frac{1}{2R_f \delta} \left(-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y \right) \left(-\sin(\Delta\theta - \delta) u_x + \cos(\Delta\theta - \delta) u_y \right) = \\
&= \frac{1}{2R_f \delta} \left(\sin^2(\Delta\theta) \cos(\delta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) \cos(\delta) + \cos^2(\Delta\theta) \cos(\delta) F_y u_y \right) + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) \sin(\delta) + \cos^2(\Delta\theta) \sin(\delta) F_y u_x - \sin^2(\Delta\theta) \sin(\delta) F_x u_y \right) = \\
&= \frac{1}{2R_f \delta} \left(\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y \right) \cos(\delta) + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= G_{II,r\theta} \cos(\delta) + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= G_{II,r\theta} \cos(\delta) + \frac{1}{2R_f \delta} F_\theta u_r \sin \delta
\end{aligned} \tag{74}$$

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_{II,r\theta} &= \lim_{\delta \rightarrow 0} G_{II,r\theta} \cos(\delta) + \\
&+ \lim_{\delta \rightarrow 0} \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= G_{II,r\theta} + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \lim_{\delta \rightarrow 0} \frac{\sin(\delta)}{\delta} = \\
&= G_{II,r\theta} + \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) = \\
&= G_{II,r\theta} + \frac{1}{2R_f} F_\theta u_r
\end{aligned} \tag{75}$$

$$\begin{aligned}
\tilde{G}_{TOT,r\theta} &= \tilde{G}_{I,r\theta} + \tilde{G}_{II,r\theta} = \\
&= (G_{I,r\theta} + G_{II,r\theta}) \cos(\delta) + \\
&+ \frac{1}{2R_f\delta} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \sin(\delta) + \\
&+ \frac{1}{2R_f\delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= (G_{I,r\theta} + G_{II,r\theta}) \cos(\delta) + \frac{1}{2R_f\delta} (F_y u_x - F_x u_y) \sin(\delta) = \\
&= G_{TOT,r\theta} \cos(\delta) + \frac{1}{2R_f\delta} (F_y u_x - F_x u_y) \sin(\delta) = \\
&= G_{TOT,r\theta} \cos(\delta) + \frac{1}{2R_f\delta} (F_\theta u_r - F_r u_\theta) \sin(\delta)
\end{aligned} \tag{76}$$

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_{TOT,r\theta} &= \lim_{\delta \rightarrow 0} G_{TOT,r\theta} \cos(\delta) + \lim_{\delta \rightarrow 0} \frac{1}{2R_f\delta} (F_y u_x - F_x u_y) \sin(\delta) = \\
&= G_{TOT,r\theta} + \frac{1}{2R_f} (F_y u_x - F_x u_y) \lim_{\delta \rightarrow 0} \frac{\sin(\delta)}{\delta} - 0 \\
&= G_{TOT,r\theta} + \frac{1}{2R_f} (F_y u_x - F_x u_y) = \\
&= G_{TOT,r\theta} + \frac{1}{2R_f} (F_\theta u_r - F_r u_\theta)
\end{aligned} \tag{77}$$

2.6. Contact region

$$u_r = 0 \tag{78}$$

$$\cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y = 0 \tag{79}$$

$$u_y = -\frac{u_x}{\tan(\Delta\theta)} \tag{80}$$

$$\begin{aligned}
u_\theta &= -\sin(\Delta\theta) u_x - \frac{\cos^2(\Delta\theta)}{\sin(\Delta\theta)} u_x = \\
&= -\frac{u_x}{\sin(\Delta\theta)}
\end{aligned} \tag{81}$$

$$F_r = \cos(\Delta\theta) F_{x,CT} + \sin(\Delta\theta) F_{y,CT} \quad F_\theta = -\sin(\Delta\theta) F_{x,CT} + \cos(\Delta\theta) F_{y,CT} \tag{82}$$

$$G_{I,r\theta} = \frac{1}{2} \frac{F_r u_r}{R_f \delta} = 0 \tag{83}$$

$$\begin{aligned}
G_{II,r\theta} &= \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} = \\
&= \frac{1}{2R_f \delta} (-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y) \left(-\frac{u_x}{\sin(\Delta\theta)} \right) = \\
&= \frac{1}{2R_f \delta} \left(F_x u_x - \frac{F_y u_x}{\tan(\Delta\theta)} \right) \\
&= \frac{1}{2R_f \delta} \left(F_x - \frac{F_y}{\tan(\Delta\theta)} \right) u_x
\end{aligned} \tag{84}$$

3. VCCT for second order quadrilateral elements

3.1. Definition of crack tip reference frame



Figure 2. Schematic representation of the discretized crack tip geometry for 2^{nd} order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad (85)$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \quad (86)$$

3.2. Calculation of displacements and reaction forces

$$\begin{aligned} u_{x,1} &= u_{x,M1} - u_{x,F1} & u_{y,1} &= u_{y,M1} - u_{y,F1} \\ u_{x,2} &= u_{x,M2} - u_{x,F2} & u_{y,2} &= u_{y,M2} - u_{y,F2} \end{aligned} \quad (87)$$

$$\begin{aligned} u_{r,1} &= \cos(\Delta\theta) u_{x,1} + \sin(\Delta\theta) u_{y,1} & u_{\theta,1} &= -\sin(\Delta\theta) u_{x,1} + \cos(\Delta\theta) u_{y,1} \\ u_{r,2} &= \cos(\Delta\theta) u_{x,2} + \sin(\Delta\theta) u_{y,2} & u_{\theta,2} &= -\sin(\Delta\theta) u_{x,2} + \cos(\Delta\theta) u_{y,2} \end{aligned} \quad (88)$$

$$\begin{aligned}
F_{r,1} &= \cos(\Delta\theta) F_{x,1} + \sin(\Delta\theta) F_{y,1} & F_{\theta,1} &= -\sin(\Delta\theta) F_{x,1} + \cos(\Delta\theta) F_{y,1} \\
F_{r,2} &= \cos(\Delta\theta) F_{x,2} + \sin(\Delta\theta) F_{y,2} & F_{\theta,2} &= -\sin(\Delta\theta) F_{x,2} + \cos(\Delta\theta) F_{y,2}
\end{aligned} \tag{89}$$

3.3. Calculation of energy release rates

$$\begin{aligned}
G_{I,r\theta} &= \frac{1}{2R_f\delta} (F_{r,1}u_{r,1} + F_{r,2}u_{r,2}) = \\
&= \frac{1}{2R_f\delta} (\cos(\Delta\theta) F_{x,1} + \sin(\Delta\theta) F_{y,1}) (\cos(\Delta\theta) u_{x,1} + \sin(\Delta\theta) u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (\cos(\Delta\theta) F_{x,2} + \sin(\Delta\theta) F_{y,2}) (\cos(\Delta\theta) u_{x,2} + \sin(\Delta\theta) u_{y,2}) + \\
&= \frac{1}{2R_f\delta} (\cos^2(\Delta\theta) F_{x,1}u_{x,1} + (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,1}u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (\cos^2(\Delta\theta) F_{x,2}u_{x,2} + (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,2}u_{y,2})
\end{aligned} \tag{90}$$

$$\begin{aligned}
G_{II,r\theta} &= \frac{1}{2R_f\delta} (F_{\theta,1}u_{\theta,1} + F_{\theta,2}u_{\theta,2}) = \\
&= \frac{1}{2R_f\delta} (-\sin(\Delta\theta) F_{x,1} + \cos(\Delta\theta) F_{y,1}) (-\sin(\Delta\theta) u_{x,1} + \cos(\Delta\theta) u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (-\sin(\Delta\theta) F_{x,2} + \cos(\Delta\theta) F_{y,2}) (-\sin(\Delta\theta) u_{x,2} + \cos(\Delta\theta) u_{y,2}) = \\
&= \frac{1}{2R_f\delta} (\sin^2(\Delta\theta) F_{x,1}u_{x,1} - (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,1}u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (\sin^2(\Delta\theta) F_{x,2}u_{x,2} - (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,2}u_{y,2})
\end{aligned} \tag{91}$$

$$\begin{aligned}
G_{TOT,r\theta} &= G_{I,r\theta} + G_{II,r\theta} = \\
&= \frac{1}{2R_f\delta} \left(\cos^2(\Delta\theta) F_{x,1}u_{x,1} + (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,1}u_{y,1} \right) + \\
&\quad + \frac{1}{2R_f\delta} \left(\cos^2(\Delta\theta) F_{x,2}u_{x,2} + (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,2}u_{y,2} \right) + \\
&\quad + \frac{1}{2R_f\delta} \left(\sin^2(\Delta\theta) F_{x,1}u_{x,1} - (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,1}u_{y,1} \right) + \\
&\quad + \frac{1}{2R_f\delta} \left(\sin^2(\Delta\theta) F_{x,2}u_{x,2} - (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,2}u_{y,2} \right) = \\
&= \frac{1}{2R_f\delta} \cos^2(\Delta\theta) (F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}) + \\
&\quad + \frac{1}{2R_f\delta} ((F_{x,1}u_{y,1} + F_{x,1}u_{y,1}) + (F_{y,2}u_{x,2} + F_{y,2}u_{x,2})) \cos(\Delta\theta) \sin(\Delta\theta) + \\
&\quad + \frac{1}{2R_f\delta} \sin^2(\Delta\theta) (F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}) + \\
&\quad - \frac{1}{2R_f\delta} ((F_{x,1}u_{y,1} + F_{x,1}u_{y,1}) + (F_{y,2}u_{x,2} + F_{y,2}u_{x,2})) \cos(\Delta\theta) \sin(\Delta\theta) = \\
&= \frac{1}{2R_f\delta} \left(\begin{array}{c} \xrightarrow{1} \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) (F_{x,1}u_{x,1} + F_{x,2}u_{x,2}) \end{array} \right) + \\
&\quad + \frac{1}{2R_f\delta} \begin{array}{c} \xrightarrow{0} \\ ((F_{x,1}u_{y,1} + F_{x,1}u_{y,1}) - (F_{x,1}u_{y,1} + F_{x,1}u_{y,1})) \cos(\Delta\theta) \sin(\Delta\theta) + \\ \xrightarrow{0} \end{array} \\
&\quad + \frac{1}{2R_f\delta} \begin{array}{c} \xrightarrow{0} \\ ((F_{y,2}u_{x,2} + F_{y,2}u_{x,2}) - (F_{y,2}u_{x,2} + F_{y,2}u_{x,2})) \cos(\Delta\theta) \sin(\Delta\theta) + \\ \xrightarrow{1} \end{array} \\
&\quad + \frac{1}{2R_f\delta} \left(\begin{array}{c} \xrightarrow{1} \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) (F_{y,1}u_{y,1} + F_{y,2}u_{y,2}) \end{array} \right) = \\
&= \frac{1}{2} \frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{R_f\delta} + \frac{1}{2} \frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{R_f\delta} = \\
&= \tilde{G}_{I,xy} + \tilde{G}_{II,xy} = \tilde{G}_{TOT,xy}
\end{aligned}$$

$$\begin{aligned}
G_{I,r\theta} &= \cos^2(\Delta\theta) \frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{2R_f\delta} + \\
&+ \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \sin^2(\Delta\theta) \frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{2R_f\delta} = \\
&= \cos^2(\Delta\theta) \tilde{G}_{I,xy} + \sin^2(\Delta\theta) \tilde{G}_{II,xy} + \\
&+ \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta)
\end{aligned} \tag{93}$$

$$\begin{aligned}
G_{II,r\theta} &= \sin^2(\Delta\theta) \frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{2R_f\delta} + \\
&- \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \cos^2(\Delta\theta) \frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{2R_f\delta} = \\
&= \sin^2(\Delta\theta) \tilde{G}_{I,xy} + \cos^2(\Delta\theta) \tilde{G}_{II,xy} + \\
&- \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta)
\end{aligned} \tag{94}$$

3.4. Sensitivity analysis of the FEM solution

$$\begin{aligned}
F_{x,1} &\sim k_{x,1}u_{x,1} & F_{y,1} &\sim k_{y,1}u_{y,1} \\
F_{x,2} &\sim k_{x,2}u_{x,2} & F_{y,2} &\sim k_{y,2}u_{y,2}
\end{aligned} \tag{95}$$

$$\begin{aligned}
G_{I,r\theta} &\sim \frac{1}{2R_f\delta} \cos^2(\Delta\theta) (k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2) + \\
&+ \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) (k_{x,1}u_{x,1}u_{y,1} + k_{y,1}u_{y,1}u_{x,1} + k_{x,2}u_{x,2}u_{y,2} + k_{y,2}u_{y,2}u_{x,2}) + \\
&+ \frac{1}{2R_f\delta} \sin^2(\Delta\theta) (k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2)
\end{aligned} \tag{96}$$

$$\begin{aligned}
G_{II,r\theta} &\sim \frac{1}{2R_f\delta} \sin^2(\Delta\theta) (k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2) + \\
&- \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) (k_{x,1}u_{x,1}u_{y,1} + k_{y,1}u_{y,1}u_{x,1} + k_{x,2}u_{x,2}u_{y,2} + k_{y,2}u_{y,2}u_{x,2}) + \\
&+ \frac{1}{2R_f\delta} \cos^2(\Delta\theta) (k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2)
\end{aligned} \tag{97}$$

$$G_{TOT,r\theta} \sim \frac{1}{2R_f\delta} ((k_{x,1}u_{x,1}^2(\Delta\theta) + k_{x,2}u_{x,2}^2(\Delta\theta)) + (k_{y,1}u_{y,1}^2(\Delta\theta) + k_{y,2}u_{y,2}^2(\Delta\theta))) \tag{98}$$

$$\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta} \cos^2(\Delta\theta) \left(k_{x,1} u_{x,1}(\Delta\theta) \frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} + k_{x,2} u_{x,2}(\Delta\theta) \frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& + \frac{1}{2R_f\delta} (k_{x,1} + k_{y,1}) \left(\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} u_{y,1}(\Delta\theta) + u_{x,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{2R_f\delta} (k_{x,2} + k_{y,2}) \left(\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} u_{y,2}(\Delta\theta) + u_{x,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{R_f\delta} \sin^2(\Delta\theta) \left(k_{y,1} u_{y,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} + k_{y,2} u_{y,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& + \frac{1}{2R_f\delta} ((k_{y,1} u_{y,1}^2 + k_{y,2} u_{y,2}^2) - (k_{x,1} u_{x,1}^2 + k_{x,2} u_{x,2}^2)) \sin(2\Delta\theta) + \\
& + \frac{1}{2R_f\delta} ((k_{x,1} + k_{y,1}) u_{x,1} u_{y,1} + (k_{x,2} + k_{y,2}) u_{x,2} u_{y,2}) \cos(2\Delta\theta)
\end{aligned} \tag{99}$$

$$\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta} \sin^2(\Delta\theta) \left(k_{x,1} u_{x,1}(\Delta\theta) \frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} + k_{x,2} u_{x,2}(\Delta\theta) \frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& - \frac{1}{2R_f\delta} (k_{x,1} + k_{y,1}) \left(\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} u_{y,1}(\Delta\theta) + u_{x,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& - \frac{1}{2R_f\delta} (k_{x,2} + k_{y,2}) \left(\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} u_{y,2}(\Delta\theta) + u_{x,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{R_f\delta} \cos^2(\Delta\theta) \left(k_{y,1} u_{y,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} + k_{y,2} u_{y,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& + \frac{1}{2R_f\delta} ((k_{x,1} u_{x,1}^2 + k_{x,2} u_{x,2}^2) - (k_{y,1} u_{y,1}^2 + k_{y,2} u_{y,2}^2)) \sin(2\Delta\theta) + \\
& - \frac{1}{2R_f\delta} ((k_{x,1} + k_{y,1}) u_{x,1} u_{y,1} + (k_{x,2} + k_{y,2}) u_{x,2} u_{y,2}) \cos(2\Delta\theta)
\end{aligned} \tag{100}$$

$$G_{TOT,r\theta} \sim \frac{1}{R_f\delta} \left(\left(k_{x,1} u_{x,1} \frac{\partial u_{x,1}}{\partial \Delta\theta} + k_{x,2} u_{x,2} \frac{\partial u_{x,2}}{\partial \Delta\theta} \right) + \left(k_{y,1} u_{y,1} \frac{\partial u_{y,1}}{\partial \Delta\theta} + k_{y,2} u_{y,2} \frac{\partial u_{y,2}}{\partial \Delta\theta} \right) \right) \tag{101}$$

4. A vectorial formulation of the VCCT

4.1. Vectorial formulation

m is the order of the element's shape functions, and $p, q = 1$ represents the crack tip

$$\underline{\underline{R}}_{\Delta\theta} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad \underline{\underline{R}}_{\Delta\theta}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad (102)$$

$$\underline{\underline{P}}_{\delta}(p) = \begin{bmatrix} \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \\ -\sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \end{bmatrix} \quad (103)$$

$$\underline{\underline{P}}_{\delta}^{-1} = \underline{\underline{P}}_{\delta}^T = \begin{bmatrix} \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & -\sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \\ \sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \end{bmatrix} \quad (104)$$

$$\underline{\underline{Q}}_{\delta}(q) = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & \sin\left(\frac{q-1}{m}\delta\right) \\ -\sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \quad (105)$$

$$\underline{\underline{Q}}_{\delta}^{-1} = \underline{\underline{Q}}_{\delta}^T = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & -\sin\left(\frac{q-1}{m}\delta\right) \\ \sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \quad (106)$$

$$\frac{\partial \underline{\underline{R}}_{\Delta\theta}}{\partial \Delta\theta} = \underline{\underline{D}} \cdot \underline{\underline{R}}_{\Delta\theta} \quad (107)$$

$$\frac{\partial \underline{\underline{P}}_{\delta}}{\partial \delta} = \left(1 + \frac{1-p}{m}\right) \underline{\underline{D}} \cdot \underline{\underline{P}}_{\delta} \quad (108)$$

$$\frac{\partial \underline{\underline{Q}}_{\delta}}{\partial \delta} = \frac{q-1}{m} \underline{\underline{D}} \cdot \underline{\underline{Q}}_{\delta} \quad (109)$$

$$\underline{\underline{D}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (110)$$

$$\underline{\underline{F}}_{xy} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \underline{\underline{F}}_{r\theta} = \begin{bmatrix} F_r \\ F_{\theta} \end{bmatrix} \quad (111)$$

$$\underline{\underline{u}}_{xy} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \underline{\underline{u}}_{r\theta} = \begin{bmatrix} u_r \\ u_{\theta} \end{bmatrix} \quad (112)$$

$$\underline{\underline{G}} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} \quad (113)$$

$$\underline{\underline{F}}_{r\theta} = \underline{\underline{Q}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{\underline{F}}_{xy} \quad \underline{\underline{u}}_{r\theta} = \underline{\underline{P}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{\underline{u}}_{xy} \quad (114)$$

$$\underline{\underline{F}}_{xy} = \underline{\underline{K}}_{xy} \underline{\underline{u}}_{xy} + \tilde{\underline{\underline{F}}}_{xy} = \underline{\underline{K}}_{xy} \underline{\underline{u}}_{xy} + \tilde{\underline{\underline{K}}}_N \underline{\underline{u}}_N \quad (115)$$

$$\begin{aligned}
G_{TOT} &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \underline{u}_{r\theta,p}^T \underline{T}_{pq}^T \underline{F}_{r\theta,q} = \\
&= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{F}_{r\theta,q} \underline{u}_{r\theta,p}^T \underline{T}_{pq}^T \right) = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\begin{bmatrix} t_{pq|11} F_{r,q} u_{r,p} & t_{pq|12} F_{r,q} u_{\theta,p} \\ t_{pq|21} F_{\theta,q} u_{r,p} & t_{pq|22} F_{\theta,q} u_{\theta,p} \end{bmatrix} \right) = \\
&= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \underline{F}_{xy,q} \underline{u}_{xy,p}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right)
\end{aligned} \tag{116}$$

$$\begin{aligned}
\underline{T}_{pq} &= \begin{cases} \underline{I} & \text{for } p = q < 2 \\ \underline{0} & \text{otherwise} \end{cases} & \text{for } 1^{st} \text{ order quadrilateral elements} \\
&= \begin{cases} \underline{I} & \text{for } p = q < 3 \\ \underline{0} & \text{otherwise} \end{cases} & \text{for } 2^{nd} \text{ order quadrilateral elements} \\
&= \begin{cases} \underline{I} & \text{for } p = q < 4 \\ \underline{0} & \text{otherwise} \end{cases} & \text{for } 3^{rd} \text{ order quadrilateral elements} \\
&= \begin{cases} \begin{pmatrix} (14 - \frac{33\pi}{8}) \underline{I} & \text{for } p = 1, q = 1 \\ (-52 + \frac{33\pi}{2}) \underline{I} & \text{for } p = 1, q = 2 \\ (17 - \frac{21\pi}{4}) \underline{I} & \text{for } p = 2, q = 1 \\ (-\frac{7}{2} + \frac{21\pi}{16}) \underline{I} & \text{for } p = 2, q = 2 \end{pmatrix} & \text{for } 2^{nd} \text{ order quarter-point quadrilateral elements} \\ \begin{pmatrix} (8 - \frac{21\pi}{8}) \underline{I} & \text{for } p = 1, q = 3 \\ (-32 + \frac{21\pi}{2}) \underline{I} & \text{for } p = 2, q = 3 \\ \underline{0} & \text{otherwise} \end{pmatrix} & \end{cases} \\
&= \begin{cases} \begin{pmatrix} (-11187 + \frac{7155\pi}{2}) \underline{I} & \text{for } p = 1, q = 1 \\ (38556 - \frac{24543\pi}{2}) \underline{I} & \text{for } p = 1, q = 2 \\ (-53055 + \frac{33777\pi}{2}) \underline{I} & \text{for } p = 1, q = 3 \\ (\frac{11396}{3} - \frac{9575\pi}{8}) \underline{I} & \text{for } p = 2, q = 1 \\ (-12936 + \frac{33003\pi}{8}) \underline{I} & \text{for } p = 2, q = 2 \\ (17988 - \frac{45837\pi}{8}) \underline{I} & \text{for } p = 2, q = 3 \\ (-\frac{8453}{3} + \frac{3595\pi}{4}) \underline{I} & \text{for } p = 3, q = 1 \\ (9804 - \frac{12411\pi}{4}) \underline{I} & \text{for } p = 3, q = 2 \\ (-13587 + \frac{17289\pi}{4}) \underline{I} & \text{for } p = 3, q = 3 \\ (6948 - \frac{17685\pi}{8}) \underline{I} & \text{for } p = 1, q = 4 \\ (-23976 + \frac{60993\pi}{8}) \underline{I} & \text{for } p = 2, q = 4 \\ (33372 - \frac{84807\pi}{8}) \underline{I} & \text{for } p = 3, q = 4 \\ \underline{0} & \text{otherwise} \end{pmatrix} & \text{for } 3^{rd} \text{ order quarter-point quadrilateral elements} \end{cases}
\end{aligned} \tag{117}$$

$$\begin{aligned}
G_{TOT} &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \left(\underline{K}_{xy} \underline{u}_{xy} + \tilde{\underline{F}}_{xy} \right) \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right) = \\
&= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \tilde{\underline{F}}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right)
\end{aligned} \tag{118}$$

$$\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \underline{F}_{xy,q} \underline{u}_{xy,p}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right) \tag{119}$$

$$\begin{aligned}
\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \tilde{\underline{F}}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right) = \\
&= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{Q}_{\delta} \underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{P}_{\delta}^T \underline{T}_{pq}^T \right)
\end{aligned} \tag{120}$$

4.2. Sensitivity analysis

$$\begin{aligned}
\frac{\partial \underline{G}}{\partial \delta} &= -\frac{1}{2R_f\delta^2} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) - \frac{1}{2R_f\delta^2} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \underline{D}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \underline{D}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \frac{\partial \underline{u}_N}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) = \\
&= \frac{1}{\delta} \underline{G} + \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\frac{1}{2R_f\delta} \underline{R}_{\Delta\theta} \left(\underline{K}_{xy} \underline{u}_{xy} + \tilde{\underline{K}}_N \underline{u}_N \right) \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \underline{D}^T \right) + \\
&+ \frac{1}{R_f\delta} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \frac{\partial \underline{u}_N}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right)
\end{aligned} \tag{121}$$

Following Comninou

$$u(\delta) \sim \sqrt{\delta} (\sin, \cos) (\epsilon \log \delta) \quad \epsilon = \frac{1}{2\pi} \log \left(\frac{1-\beta}{1+\beta} \right) \tag{122}$$

$$\underline{u}_{xy}, \underline{u}_N \sim u(\delta) \sim \sqrt{\delta} (\sin, \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} 0 \tag{123}$$

$$\underline{u}_{xy} \underline{u}_{xy}^T, \underline{u}_N \underline{u}_{xy}^T \sim u^2(\delta) \sim \delta (\sin^2, \cos^2, \sin \cdot \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} 0 \tag{124}$$

$$\frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T, \frac{\partial \underline{u}_N}{\partial \delta} \underline{u}_{xy}^T \sim -\frac{1}{2} (\sin^2, \cos^2, \sin \cdot \cos) (\epsilon \log \delta) + (-\sin^2, \cos^2, \pm \sin \cdot \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} \text{finite} \tag{125}$$

$$\underline{G} \sim \frac{1}{\delta} \underline{u}_{xy} \underline{u}_{xy}^T \sim \frac{1}{\delta} u^2(\delta) \sim (\sin^2, \cos^2, \sin \cdot \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} \text{finite} \tag{126}$$

$$\lim_{\delta \rightarrow 0} \frac{\partial G}{\partial \delta} \sim \frac{1}{\delta} \left(\overrightarrow{F(\delta)}^0 + \underline{C} \right) \quad (127)$$

$$\lim_{\delta \rightarrow 0} \frac{\partial G}{\partial \delta} \sim \frac{1}{\delta} \xrightarrow{f \, d\delta} \lim_{\delta \rightarrow 0} G \sim A \log(\delta) + B \quad (128)$$

$$\begin{aligned} \frac{\partial G}{\partial \Delta \theta} = & \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{D}} \underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} u_{xy} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{D}} \underline{\underline{R}}_{\Delta \theta} \underline{\underline{\tilde{K}}}_N u_N u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \\ & + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} u_{xy} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{D}}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{\tilde{K}}}_N u_N u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{D}}^T \underline{\underline{R}}_{\delta}^T \right) + \\ & + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} \frac{\partial u_{xy}}{\partial \Delta \theta} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{\tilde{K}}}_N \frac{\partial u_N}{\partial \Delta \theta} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \\ & + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} u_{xy} \frac{\partial u_{xy}^T}{\partial \Delta \theta} \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{\tilde{K}}}_N u_N \frac{\partial u_{xy}^T}{\partial \Delta \theta} \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) \end{aligned} \quad (129)$$

References

- [1] Griffiths R. 1994 Stiffness matrix of the four-node quadrilateral element in closed form *Int. J. Numer. Meth. Eng.* **57**(2) 109–143
- [2] Krueger R. 2004 Virtual crack closure technique: History, approach, and applications *Appl. Mech. Rev.* **57**(2) 109–143
- [3] ABAQUS 2016 ABAQUS 2016 Analysis User's Manual *Online Documentation Help: Dassault Systèmes*
- [4] Rice J. R. 1968 A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks *J. Appl. Mech.* **35** 379–386
- [5] Valvo, P. S. 2012 A Revised Virtual Crack Closure Technique for Physically Consistent Fracture Mode Partitioning *Int. J. Fract.* **173**(1) 1–20
- [6] Raju, I.S. 1987 Calculation of Strain-Energy Release Rates With Higher Order and Singular Finite Elements *Eng. Fract. Mech.* **28**(3) 251–274
- [7] Williams, M. L. 1959 The Stresses Around a Fault or Crack in Dissimilar Media *B. Seismol. Soc. Am.* **49**(2)
- [8] Comninou, M. 1990 An overview of interface cracks *Eng. Fract. Mech.* **37** 197–208