

Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

Luca Di Stasio^a, Janis Varna^a

^a*Luleå University of Technology, University Campus, SE-97187 Luleå, Sweden*

Abstract

Keywords: Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

1. Introduction

Main ref [1]

2. Derivation of constitutive relations

2.1. Reference frames

5 **Local reference frame of k -th layer:** index 1 is the in-plane longitudinal or fiber or 0° -direction; index 2 is the in-plane transverse or 90° -direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0° direction; index y is the in-plane transverse direction; index
10 z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \quad (1)$$

V_a is equal to the product of total crack surface S_C and average crack opening

15 u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \quad (2)$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C . It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\begin{aligned} \rho_D &= \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta \theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \mathcal{W}}{L_{lam} \mathcal{W} t_{90^\circ}} R_f \Delta \theta = \frac{1}{n 2L} \frac{1}{k 2L} R_f \Delta \theta = \\ &= \frac{1}{nk 4L^2} R_f \Delta \theta = \frac{V_f}{nk \pi R_f^2} R_f \Delta \theta = \frac{V_f}{nk R_f} \frac{\Delta \theta}{\pi} \end{aligned} \quad (3)$$

20 2.3. Vakulenko-Kachanov tensor

In the local reference frame of k -th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \quad (4)$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \quad (5)$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_i n_j + u_j n_i) dS \quad (6)$$

25 Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$\begin{aligned}
\beta_{11} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_1 n_1 + u_1 n_1) dS = \frac{1}{V_k} \int_{S_C} \cancel{u_1} \vec{n}_1^0 dS = 0 \\
\beta_{22} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_2 + u_2 n_2) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_3 + u_3 n_3) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\
\beta_{12} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\cancel{u_1} \vec{n}_2^0 + u_2 \cancel{n_1})^0 dS = 0 \\
\beta_{13} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\cancel{u_1} \vec{n}_3^0 + u_3 \cancel{n_1})^0 dS = 0 \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_3 + u_3 n_2) dS \\
\beta_{21} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 \cancel{n_1}^0 + \cancel{u_1} \vec{n}_2^0) dS = 0 \\
\beta_{31} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 \cancel{n_1}^0 + \cancel{u_1} \vec{n}_3^0) dS = 0 \\
\beta_{32} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_2 + u_2 n_3) dS = \beta_{23}
\end{aligned} \tag{7}$$

Only 3 independent components of the tensor β_{ij} remain: β_{22} , β_{33} and β_{23} . Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f (-n_2^m) dS \right] \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f (-n_3^m) dS \right] \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} (u_2 n_3 + u_3 n_2) dS = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} (u_2^m n_3^m + u_3^m n_2^m) dS + \int_{S_C^f} (u_2^f n_3^f + u_3^f n_2^f) dS \right] = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f (-n_3^m) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f (-n_2^m) dS \right]
\end{aligned} \tag{8}$$

30 The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta \theta \tag{9}$$

With Eq. 9, we can recast Eq. 8 as

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \left[n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam} w t_{90^\circ}} \left[n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90^\circ}} \left[\int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] \\
\beta_{33} &= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_3^m d\theta \right] \\
\beta_{23} &= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_2^m d\theta \right]
\end{aligned} \tag{10}$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$\begin{aligned}
u_2^m - u_2^f &= (u_r^m - u_r^f) \cos(\theta) - (u_\theta^m - u_\theta^f) \sin(\theta) = \\
&= COD(\theta) \cos(\theta) - CSD(\theta) \sin(\theta) \\
u_3^m - u_3^f &= (u_r^m - u_r^f) \sin(\theta) + (u_\theta^m - u_\theta^f) \cos(\theta) = \\
&= COD(\theta) \sin(\theta) + CSD(\theta) \cos(\theta)
\end{aligned} \tag{11}$$

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ :

$$\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned} \tag{12}$$

Thus, Eq. 10 becomes

$$\begin{aligned}
\beta_{22} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (\cos^2(\theta) - \cos(\theta) \sin(\theta)) - CSD(\theta) (\sin(\theta) \cos(\theta) - \sin^2(\theta))] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (1 + \cos(2\theta) - \sin(2\theta)) + CSD(\theta) (1 - \cos(2\theta) - \sin(2\theta))] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{33} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (\sin(\theta) \cos(\theta) + \sin^2(\theta)) + CSD(\theta) (\cos^2(\theta) + \cos(\theta) \sin(\theta))] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (1 + \sin(2\theta) - \cos(2\theta)) + CSD(\theta) (1 + \sin(2\theta) + \cos(2\theta))] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{23} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) (2 \sin(\theta) \cos(\theta) + \cos^2(\theta) - \sin^2(\theta)) + \\
&\quad - \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) (\sin^2(\theta) - \cos^2(\theta) + 2 \cos(\theta) \sin(\theta)) d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta \\
&\hspace{15em} (13)
\end{aligned}$$

2.4. Analytical modeling of $COD(\theta)$ and $CSD(\theta)$

40 The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of θ , the angular coordinate along the crack which varies between 0 and $\Delta\theta$. Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be expressed as the sum of their average value and a term, respectively $\delta COD(\theta)$ and $\delta CSD(\theta)$, that represents the variation of the function from its average:

45

$$\begin{aligned}
COD(\theta) &= COD_{avg} + \delta COD(\theta) \\
CSD(\theta) &= CSD_{avg} + \delta CSD(\theta).
\end{aligned} \tag{14}$$

By defining $\Delta\Psi$

$$\Delta\Psi = \min(\Delta\theta, \Delta\Phi), \tag{15}$$

we introduce at this point an approximation and assume that the functions $\delta COD(\theta)$ and $\delta CSD(\theta)$ can be expressed as the product of the maximum value of the displacement and a function, respectively $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$:

$$\begin{aligned}
COD(\theta) &= COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max} f\left(\theta - \frac{\Delta\Psi}{2}\right) \\
CSD(\theta) &= CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max} g\left(\theta - \frac{\Delta\theta}{2}\right),
\end{aligned} \tag{16}$$

50 where $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ are assumed to be odd functions over their respective integration domain $[0, \Delta\Psi]$ and $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = 0 \quad \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = 0. \tag{17}$$

We assume the two functions $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ to be two odd polynomials of degree $2n - 1$:

$$f\left(\theta - \frac{\Delta\Psi}{2}\right) = \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} \quad g\left(\theta - \frac{\Delta\theta}{2}\right) = \sum_{k=0}^{n-1} b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1}, \tag{18}$$

which satisfy by construction the conditions expressed in Equation 17. The
55 coefficients a_{2k+1} and b_{2k+1} are determined by imposing that

$$\begin{aligned}
COD(\Delta\Psi) &= 0 \\
CSD(\Delta\theta) &= 0.
\end{aligned} \tag{19}$$

The explicit construction of the polynomials $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ for $n = 1, 2, 3$ (or degree $2n - 1 = 1, 3, 5$) is reported in Appendix A.

$$\begin{aligned}
\beta_{22} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{33} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{23} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta
\end{aligned} \tag{20}$$

$$\begin{aligned}
\frac{1}{\Delta\theta} \int_0^{\Delta\Psi} COD(\theta) d\theta &= \\
&= \frac{1}{\Delta\theta} \int_0^{\Delta\Psi} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2} \right)^{2k+1} \right) d\theta = \\
&= \frac{1}{\Delta\Psi} \left[COD_{avg} \theta + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \left(\theta - \frac{\Delta\Psi}{2} \right)^{2(k+1)} \right] \Big|_0^{\Delta\Psi} = \\
&= COD_{avg} + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \frac{\Delta\Psi^{2(k+1)}}{\Delta\theta}
\end{aligned} \tag{21}$$

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) d\theta = CSD_{avg} + CSD_{max} \sum_{k=0}^{n-1} \frac{b_{2k+1}}{2(k+1)} \Delta\theta^{2k+1} \tag{22}$$

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\min(\Delta\theta, \Delta\Phi)} COD(\theta) \sin(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \right) \sin(2\theta) d\theta = \\
& = -\frac{1}{2\Delta\theta} COD_{avg} [\cos(2\theta)] \Big|_0^{\min(\Delta\theta, \Delta\Phi)} + \\
& + \frac{1}{\Delta\theta} \left[COD_{max} \sum_{i=0}^{n-1} \left(-\frac{1}{2} \right)^{2i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\theta \right) \left(\sum_{k=0}^i a_{2k+1} ((n-1) - 2(k+1))! \theta^{2(k+1)} \right) \right] \Big|_0^{\min(\Delta\theta, \Delta\Phi)} = \\
& = \frac{1}{2\Delta\theta} COD_{avg} (1 - \cos(2 \min(\Delta\theta, \Delta\Phi))) + \\
& + \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{n-1} \left(-\frac{1}{2} \right)^{2i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2 \min(\Delta\theta, \Delta\Phi) \right) \left(\sum_{k=0}^i a_{2k+1} ((n-1) - 2(k+1))! \min(\Delta\theta, \Delta\Phi)^{2(k+1)} \right)
\end{aligned} \tag{23}$$

References

- [1] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), *Comprehensive Composite Materials II*, Elsevier, Oxford, 2018, pp. 192 – 220. doi:<https://doi.org/10.1016/B978-0-12-803581-8.09910-0>.

Appendix A. Explicit expressions for $f(\theta)$ and $g(\theta)$

In the following, recall that

$$\Delta\Psi = \min(\Delta\theta, \Delta\Phi). \tag{A.1}$$

$\mathbf{n} = \mathbf{1}$

$$\begin{aligned}
f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^0 a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) \\
g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^0 b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1 \left(\theta - \frac{\Delta\theta}{2}\right)
\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\int_0^{\Delta\Psi} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta &= \int_0^{\Delta\Psi} a_1\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \left[\frac{a_1}{2}\theta^2 - a_1\frac{\Delta\Psi}{2}\theta\right]_0^{\Delta\Psi} = 0 \quad \forall a_1 \\
\int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta &= \int_0^{\Delta\theta} b_1\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \left[\frac{b_1}{2}\theta^2 - b_1\frac{\Delta\theta}{2}\theta\right]_0^{\Delta\theta} = 0 \quad \forall b_1
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
COD_{avg} + COD_{max}a_1\left(\Delta\Psi - \frac{\Delta\Psi}{2}\right) &= 0 \rightarrow a_1 = -\frac{2}{\Delta\Psi} \frac{COD_{avg}}{COD_{max}} \\
CSD_{avg} + CSD_{max}b_1\left(\Delta\theta - \frac{\Delta\theta}{2}\right) &= 0 \rightarrow b_1 = -\frac{2}{\Delta\theta} \frac{CSD_{avg}}{CSD_{max}}
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\sum_{k=0}^0 \frac{a_{2k+1}}{2(k+1)} \frac{\min(\Delta\theta, \Delta\Phi)^{2(k+1)}}{\Delta\theta} &= \frac{a_1}{2} \frac{\min(\Delta\theta, \Delta\Phi)^2}{\Delta\theta} \\
\sum_{k=0}^0 \frac{b_{2k+1}}{2(k+1)} \frac{\Delta\theta^{2(k+1)}}{\Delta\theta} &= \frac{b_1}{2} \Delta\theta
\end{aligned} \tag{A.5}$$

n = 2

$$\begin{aligned}
f(\theta) &= \sum_{k=0}^1 a_{2k+1}\theta^{2k+1} = a_1\theta + a_3\theta^3 \\
g(\theta) &= \sum_{k=0}^1 b_{2k+1}\theta^{2k+1} = b_1\theta + b_3\theta^3
\end{aligned} \tag{A.6}$$

n = 3

$$\begin{aligned}
f(\theta) &= \sum_{k=0}^2 a_{2k+1}\theta^{2k+1} = a_1\theta + a_3\theta^3 + a_5\theta^5 \\
g(\theta) &= \sum_{k=0}^2 b_{2k+1}\theta^{2k+1} = b_1\theta + b_3\theta^3 + b_5\theta^5
\end{aligned} \tag{A.7}$$