Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

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Abstract

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1. Introduction

Main ref [1]

2. Derivation of constitutive relations

2.1. Reference frames

5 **Local reference frame of k-th layer:** index 1 is the in-plane longitudinal or fiber or 0°-direction; index 2 is the in-plane transverse or 90°-direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0°direction; index y is the in-plane transverse direction; index 10 z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \tag{1}$$

V_a is equal to the product of total crack surface S_C and average crack opening u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \tag{2}$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90°}} = \frac{n_D \not{w}}{L_{lam} \not{w} t_{90°}} R_f \Delta\theta = \frac{1}{n2L} \frac{1}{k2L} R_f \Delta\theta =$$
$$= \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nk R_f} \frac{\Delta\theta}{\pi} \tag{3}$$

2.3. Vakulenko-Kachanov tensor

In the local reference frame of k-th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \tag{4}$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \tag{5}$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \tag{6}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

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$$\beta_{11} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_1 + u_1 n_1 \right) dS = \frac{1}{V_k} \int_{S_C} \cancel{u_1 n_1}^0 dS = 0$$

$$\beta_{22} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_2 + u_2 n_2 \right) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS$$

$$\beta_{33} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_3 + u_3 n_3 \right) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS$$

$$\beta_{12} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancel{u_1 n_2}^0 + \cancel{u_2 n_1}^0 \right) dS = 0$$

$$\beta_{13} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancel{u_1 n_3}^0 + \cancel{u_3 n_1}^0 \right) dS = 0 \tag{7}$$

$$\beta_{23} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_3 + u_3 n_2 \right) dS$$

$$\beta_{21} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancel{u_2 n_1}^0 + \cancel{u_1 n_2}^0 \right) dS = 0$$

$$\beta_{31} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancel{u_3 n_1}^0 + \cancel{u_1 n_3}^0 \right) dS = 0$$

$$\beta_{32} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_2 + u_2 n_3 \right) dS = \beta_{23}$$

Only 3 independent components of the tensor β_{ij} remain: β_{22}, β_{33} and β_{23}. Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$\beta_{22} = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f \left(-n_2^m \right) dS \right]$$

$$\beta_{33} = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f \left(-n_3^m \right) dS \right]$$

$$\beta_{23} = \frac{1}{V_k} \int_{S_C} \left(u_2 n_3 + u_3 n_2 \right) dS =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} \left(u_2^m n_3^m + u_3^m n_2^m \right) dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f \right) dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f \left(-n_3^m \right) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f \left(-n_2^m \right) dS \right]$$

$$\text{(8)}$$

30   The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta\theta \qquad (9)$$

With Eq. 9, we can recast Eq. 8 as

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$$\beta_{22} = \frac{1}{V_k} \left[n_D R_f w \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] =$$

$$= \frac{1}{L_{lam} w t_{90°}} \left[n_D R_f w \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] =$$

$$= \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90°}} \left[\int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] =$$

$$= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right]$$

$$\beta_{33} = \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_3^m - u_3^f \right) n_3^m d\theta \right]$$

$$\beta_{23} = \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_3^m - u_3^f \right) n_2^m d\theta \right] \tag{10}$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$u_2^m - u_2^f = \left(u_r^m - u_r^f \right) \cos\left(\theta\right) - \left(u_\theta^m - u_\theta^f \right) \sin\left(\theta\right) =$$

$$= COD\left(\theta\right) \cos\left(\theta\right) - CSD\left(\theta\right) \sin\left(\theta\right)$$

$$u_3^m - u_3^f = \left(u_r^m - u_r^f \right) \sin\left(\theta\right) + \left(u_\theta^m - u_\theta^f \right) \cos\left(\theta\right) =$$

$$= COD\left(\theta\right) \sin\left(\theta\right) + CSD\left(\theta\right) \cos\left(\theta\right) \tag{11}$$

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ:

$$n_2^m = \cos\left(\theta\right) - \sin\left(\theta\right)$$

$$n_3^m = \sin\left(\theta\right) + \cos\left(\theta\right) \tag{12}$$

Thus, Eq. 10 becomes

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$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(\cos^2\left(\theta\right) - \cos\left(\theta\right)\sin\left(\theta\right)\right) - CSD\left(\theta\right)\left(\sin\left(\theta\right)\cos\left(\theta\right) - \sin^2\left(\theta\right)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(1 + \cos\left(2\theta\right) - \sin\left(2\theta\right)\right) + CSD\left(\theta\right)\left(1 - \cos\left(2\theta\right) - \sin\left(2\theta\right)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\cos\left(2\theta\right)\right] d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(\sin\left(\theta\right)\cos\left(\theta\right) + \sin^2\left(\theta\right)\right) + CSD\left(\theta\right)\left(\cos^2\left(\theta\right) + \cos\left(\theta\right)\sin\left(\theta\right)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(1 + \sin\left(2\theta\right) - \cos\left(2\theta\right)\right) + CSD\left(\theta\right)\left(1 + \sin\left(2\theta\right) + \cos\left(2\theta\right)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\cos\left(2\theta\right)\right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right)\left(2\sin\left(\theta\right)\cos\left(\theta\right) + \cos^2\left(\theta\right) - \sin^2\left(\theta\right)\right) +$$

$$- \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD\left(\theta\right)\left(\sin^2\left(\theta\right) - \cos^2\left(\theta\right) + 2\cos\left(\theta\right)\sin\left(\theta\right)\right) d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2\left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\sin\left(2\theta\right)\right] d\theta$$

$$(13)$$

2.4. Analytical modeling of $COD\left(\theta\right)$ and $CSD\left(\theta\right)$

The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of θ, the angular coordinate along the crack which varies between 0 and $\Delta\theta$. Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be expressed as the sum of their average value and a term, respectively $\delta COD\left(\theta\right)$ and $\delta CSD\left(\theta\right)$, that represents the variation of the function from its average:

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$$COD\left(\theta\right) = COD_{avg} + \delta COD\left(\theta\right)$$
$$CSD\left(\theta\right) = CSD_{avg} + \delta CSD\left(\theta\right).$$
$$(14)$$

By defining $\Delta\Psi$

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right), \tag{15}$$

we introduce at this point an approximation and assume that the functions $\delta COD\left(\theta\right)$ and $\delta CSD\left(\theta\right)$ can be expressed as the product of the maximum value of the displacement and a function, respectively $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$:

$$COD\left(\theta\right) = COD_{avg} + \delta COD\left(\theta\right) = COD_{avg} + COD_{max}f\left(\theta - \frac{\Delta\Psi}{2}\right)$$
$$CSD\left(\theta\right) = CSD_{avg} + \delta CSD\left(\theta\right) = CSD_{avg} + CSD_{max}g\left(\theta - \frac{\Delta\theta}{2}\right),$$
$$(16)$$

where $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ are assumed to be odd functions over their respective integration domain $[0, \Delta\Psi]$ and $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = 0 \quad \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = 0. \tag{17}$$

We assume the two functions $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ to be two odd polynomials of degree $2n - 1$:

$$f\left(\theta - \frac{\Delta\Psi}{2}\right) = \sum_{k=0}^{n-1} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} \quad g\left(\theta - \frac{\Delta\theta}{2}\right) = \sum_{k=0}^{n-1} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1},$$
$$(18)$$

which satisfy by construction the conditions expressed in Equation 17. The coefficients a_{2k+1} and b_{2k+1} are determined by imposing that

$$COD\left(\Delta\Psi\right) = 0$$
$$CSD\left(\Delta\theta\right) = 0. \tag{19}$$

The explicit construction of the polynomials $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ for $n = 1, 2, 3$ (or degree $2n - 1 = 1, 3, 5$) is reported in Appendix A.

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta$$

$$(20)$$

$$\frac{1}{\Delta\theta} \int_0^{\Delta\Psi} COD(\theta)\, d\theta =$$

$$= \frac{1}{\Delta\theta} \int_0^{\Delta\Psi} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} \right) d\theta =$$

$$= \frac{1}{\Delta\theta} \left[COD_{avg}\theta + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2(k+1)} \right] \Bigg|_0^{\Delta\Psi} =$$

$$= \frac{\Delta\Psi}{\Delta\theta} COD_{avg} + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)\Delta\theta} \left(\left(\frac{\Delta\Psi}{2}\right)^{2(k+1)} - \left(-\frac{\Delta\Psi}{2}\right)^{2(k+1)} \right) =$$

$$= \frac{\Delta\Psi}{\Delta\theta} COD_{avg}$$

$$(21)$$

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta)\, d\theta = CSD_{avg} \tag{22}$$

$$\frac{1}{\Delta\theta}\int_0^{\min(\Delta\theta,\Delta\Phi)} COD\left(\theta\right)\sin\left(2\theta\right)d\theta =$$

$$=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\theta^{2k+1}\right)\sin\left(2\theta\right)d\theta =$$

$$=-\frac{1}{2\Delta\theta}COD_{avg}\left[\cos\left(2\theta\right)\right]\Big|_0^{\min(\Delta\theta,\Delta\Phi)}+$$

$$+\frac{1}{\Delta\theta}\left[COD_{max}\sum_{i=0}^{n-1}\left(-\frac{1}{2}\right)^{2i+1}\sin(\frac{1+mod\left(i,2\right)}{2}\pi-2\theta)\left(\sum_{k=0}^{i}a_{2k+1}\left(\left(n-1\right)-2\left(k+1\right)\right)!\theta^{2(k+1)}\right)\right]\Bigg|_0^{\min(\Delta\theta,\Delta\Phi)}=$$

$$=\frac{1}{2\Delta\theta}COD_{avg}\left(1-\cos\left(2\min\left(\Delta\theta,\Delta\Phi\right)\right)\right)+$$

$$+\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{n-1}\left(-\frac{1}{2}\right)^{2i+1}\sin(\frac{1+mod\left(i,2\right)}{2}\pi-2\min\left(\Delta\theta,\Delta\Phi\right))\left(\sum_{k=0}^{i}a_{2k+1}\left(\left(n-1\right)-2\left(k+1\right)\right)!\min\left(\Delta\theta,\Delta\Phi\right)^{2(k+1)}\right)$$

$$(23)$$

References

[1] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), Comprehensive Composite Materials II, Elsevier, Oxford, 2018, pp. 192 – 220. `doi:https://doi.org/10.1016/B978-0-12-803581-8.09910-0`.

Appendix A. Explicit expressions for $f\left(\theta\right)$ and $g\left(\theta\right)$

In the following, recall that

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right). \qquad (A.1)$$

n = 1

$$f\left(\theta-\frac{\Delta\Psi}{2}\right)=\sum_{k=0}^{0}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}=a_1\left(\theta-\frac{\Delta\Psi}{2}\right)$$

$$g\left(\theta-\frac{\Delta\theta}{2}\right)=\sum_{k=0}^{0}b_{2k+1}\left(\theta-\frac{\Delta\theta}{2}\right)^{2k+1}=b_1\left(\theta-\frac{\Delta\theta}{2}\right)$$

$$(A.2)$$

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$$\int_0^{\Delta\Psi} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \int_0^{\Delta\Psi} a_1\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \left[\frac{a_1}{2}\theta^2 - a_1\frac{\Delta\Psi}{2}\theta\right]\Bigg|_0^{\Delta\Psi} = 0 \quad \forall a_1$$

$$\int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \int_0^{\Delta\theta} b_1\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \left[\frac{b_1}{2}\theta^2 - b_1\frac{\Delta\theta}{2}\theta\right]\Bigg|_0^{\Delta\theta} = 0 \quad \forall b_1$$
$$(A.3)$$

$$COD_{avg} + COD_{max}a_1\left(\Delta\Psi - \frac{\Delta\Psi}{2}\right) = 0 \rightarrow a_1 = -\frac{2}{\Delta\Psi}\frac{COD_{avg}}{COD_{max}}$$

$$CSD_{avg} + CSD_{max}b_1\left(\Delta\theta - \frac{\Delta\theta}{2}\right) = 0 \rightarrow b_1 = -\frac{2}{\Delta\theta}\frac{CSD_{avg}}{CSD_{max}}$$
$$(A.4)$$

$$\sum_{k=0}^0 \frac{a_{2k+1}}{2(k+1)}\frac{\min(\Delta\theta,\Delta\Phi)^{2(k+1)}}{\Delta\theta} = \frac{a_1}{2}\frac{\min(\Delta\theta,\Delta\Phi)^2}{\Delta\theta}$$

$$\sum_{k=0}^0 \frac{b_{2k+1}}{2(k+1)}\frac{\Delta\theta^{2(k+1)}}{\Delta\theta} = \frac{b_1}{2}\Delta\theta$$
$$(A.5)$$

n = 2

$$f(\theta) = \sum_{k=0}^1 a_{2k+1}\theta^{2k+1} = a_1\theta + a_3\theta^3$$

$$g(\theta) = \sum_{k=0}^1 b_{2k+1}\theta^{2k+1} = b_1\theta + b_3\theta^3$$
$$(A.6)$$

n = 3

$$f(\theta) = \sum_{k=0}^2 a_{2k+1}\theta^{2k+1} = a_1\theta + a_3\theta^3 + a_5\theta^5$$

$$g(\theta) = \sum_{k=0}^1 b_{2k+1}\theta^{2k+1} = b_1\theta + b_3\theta^3 + b_5\theta^5$$
$$(A.7)$$