

Computing energy release rates using the Virtual Crack Closure Technique with the Finite Element Method: analytical discussion

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Abstract. The effect of crack tip orientation and elements' size in its vicinity on mode splitting in the Virtual Crack Closure Technique is analyzed by means of analytical derivations.

- (i) The total energy release rate is shown to have no direct dependence on the debond angular size, but only an indirect one through the FEM solution of the crack displacement field in the crack tip neighborhood. It thus can be computed using unrotated forces and displacements, i.e. aligned with the global and not with the local reference.
- (ii) Mode I and mode II energy release rate are expressed as a function of the crack displacement and it is shown as in [5] that crack tip forces depend linearly on both components of the crack displacements. However, what in [5] is an assumption used to reverse-engineer the relationship, here the dependency is fully derived in terms of the underlying FEM discretization. It is furthermore shown that crack tip forces do not only depend on crack displacements, but also on the resultant of forces of the 4 elements connected to the crack tip, which represent the influence of the rest of the domain.
- (iii) A new vectorial formulation of the VCCT is proposed, which can be applied to elements of different order and in the presence of quarter-tip singularity (based on the work in [6]). It provides a general framework for the analysis of curved cracks analyzed with FEM. It furthermore expresses directly the dependence on FEM discretization and solution. By implementing the equations provided alongside the classic FEM, it can provide a native formulation which does not require the extraction of internal or reaction forces in the post-processing phase.
- (iv) Drawing upon the asymptotic expression of the displacement derived in [7] and presented in [8], it is shown that mode I and mode II for a curved crack behaves as $A \log(\delta) + B$, where δ is the angular discretization at the crack tip. The result is confirmed by the numerical results.

List of acronyms

VCCT	Virtual Crack Closure Technique
BEM	Boundary Element Method
FEM	Finite Element Method

List of symbols

G_I	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate
G_{II}	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate
G_{TOT}	$\left[\frac{J}{m^2}\right]$	Total energy release rate
$G_{I,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate in $r - \theta$ reference frame
$G_{II,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate in $r - \theta$ reference frame
$G_{TOT,r\theta}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate in $r - \theta$ reference frame
$\tilde{G}_{I,xy}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate of equivalent crack in $x - y$ reference frame
$\tilde{G}_{II,xy}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate of equivalent crack in $x - y$ reference frame
$\tilde{G}_{TOT,xy}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate of equivalent crack in $x - y$ reference frame
R_f	$[\mu m]$	Fiber radius
a	$[\mu m]$	Debond size
Δa	$[\mu m]$	Debond increment
$\Delta \theta$	$[rad]$	Half debond angular size
δ	$[rad]$	Angular size of element at the interface close to the crack tip
$u_{x,[A-Z]}$	$[\mu m]$	Displacement along x of a point labeled with a letter in [A-Z]
$u_{y,[A-Z]}$	$[\mu m]$	Displacement along y of a point labeled with a letter in [A-Z]
u_x	$[\mu m]$	Displacement along x -direction
u_y	$[\mu m]$	Displacement along y -direction
u_r	$[\mu m]$	Displacement along r -direction
u_θ	$[\mu m]$	Displacement along θ -direction
$F_{x,[A-Z]}$	$[\mu m]$	Force along x at a point labeled with a letter in [A-Z]
$F_{y,[A-Z]}$	$[\mu m]$	Force along y at a point labeled with a letter in [A-Z]
F_x	$[\mu m]$	Force along x -direction
F_y	$[\mu m]$	Force along y -direction
F_r	$[\mu m]$	Force along r -direction
F_θ	$[\mu m]$	Force along θ -direction
$\underline{\underline{R}}$	$[-]$	Rotation matrix

1. FEM formulation with quadrilateral elements

$$\Pi(u_i) = \int_V \sigma_{ij} \varepsilon_{ij} dV + \int_V \rho \ddot{u}_i u_i dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \quad (1)$$

$$\Pi(u_i) = \int_V \sigma_{ij} \varepsilon_{ij} dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \quad (2)$$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (3)$$

$$\underline{\underline{\varepsilon}}(x, y) = \underline{\underline{\tilde{B}}} \cdot \underline{u}(x, y) \quad (4)$$

$$\underline{\underline{\tilde{B}}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \quad (5)$$

$$\underline{\underline{\sigma}} = \underline{\underline{D}} \cdot \underline{\underline{\varepsilon}} \quad (6)$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \quad \underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \quad (7)$$

$$\underline{\underline{D}} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{bmatrix} \quad \text{with } G = \frac{E}{2(1+\nu)} \text{ for an isotropic material} \quad (8)$$

$$\begin{aligned} E_1 &= \frac{E}{1-\nu^2} & E_2 &= \nu E_1 & \text{for plane stress} \\ E_1 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} & E_2 &= \frac{\nu E_1}{1-\nu} & \text{for plane strain} \end{aligned} \quad (9)$$

$$\begin{aligned} \Pi(\underline{u}) &= \frac{1}{2} \int_V \underline{\underline{\varepsilon}}^T \underline{\underline{D}} \cdot \underline{\underline{\varepsilon}} dV - \int_V \underline{F}^T \underline{u} dV - \int_S \underline{f}^T \underline{u} dS = \\ &= \frac{1}{2} \int_V \underline{u}^T \underline{\underline{\tilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\tilde{B}}} \cdot \underline{u} dV - \int_V \underline{F}^T \underline{u} dV - \int_S \underline{f}^T \underline{u} dS \end{aligned} \quad (10)$$

$$\delta \Pi(\delta \underline{u}) = 0 \quad (11)$$

$$\begin{aligned}
\delta \Pi(\delta \underline{u}) &= \Pi(\underline{u} + \delta \underline{u}) - \Pi(\underline{u}) = \\
&= \frac{1}{2} \int_V (\underline{u} + \delta \underline{u})^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot (\underline{u} + \delta \underline{u}) dV - \int_V \underline{F}^T (\underline{u} + \delta \underline{u}) dV - \int_S \underline{f}^T (\underline{u} + \delta \underline{u}) dS + \\
&- \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \int_V \underline{F}^T \underline{u} dV + \int_S \underline{f}^T \underline{u} dS = \\
&= \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \frac{1}{2} \int_V \delta \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \delta \underline{u} dV \xrightarrow{\approx 0} + \\
&+ \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \delta \underline{u} dV + \frac{1}{2} \int_V \delta \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \\
&- \int_V \underline{F}^T \underline{u} dV - \int_V \underline{F}^T \delta \underline{u} dV + \\
&- \int_S \underline{f}^T \underline{u} dS - \int_S \underline{f}^T \delta \underline{u} dS + \\
&- \frac{1}{2} \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV + \int_V \underline{F}^T \underline{u} dV + \int_S \underline{f}^T \underline{u} dS = \\
&= \int_V \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \delta \underline{u} dV - \int_V \underline{F}^T \delta \underline{u} dV - \int_S \underline{f}^T \delta \underline{u} dS
\end{aligned} \tag{12}$$

$$\int_V \delta \underline{u}^T \underline{\tilde{B}}^T \underline{D} \cdot \underline{\tilde{B}} \cdot \underline{u} dV - \int_V \delta \underline{u}^T \underline{F} dV - \int_S \delta \underline{u}^T \underline{f} dS = 0 \tag{13}$$

$$\underline{u} = \begin{bmatrix} u_x(x, y) \\ u_y(x, y) \end{bmatrix} \quad \underline{u}_N = \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{2,x} \\ u_{2,y} \\ u_{3,x} \\ u_{3,y} \\ u_{4,x} \\ u_{4,y} \end{bmatrix} \quad \text{or} \quad \underline{u}_N = \begin{bmatrix} u_{1,x} \\ u_{1,y} \\ u_{2,x} \\ u_{2,y} \\ u_{3,x} \\ u_{3,y} \\ u_{4,x} \\ u_{4,y} \\ u_{5,x} \\ u_{5,y} \\ u_{6,x} \\ u_{6,y} \\ u_{7,x} \\ u_{7,y} \\ u_{8,x} \\ u_{8,y} \end{bmatrix} \tag{14}$$

$$\underline{u} = \underline{\underline{N}} \cdot \underline{u}_N \tag{15}$$

$$\underline{\underline{N}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{16}$$

$$\underline{\underline{N}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \tag{17}$$

$$\begin{cases} N_1 = N_1(\xi, \eta) \\ N_2 = N_2(\xi, \eta) \\ N_3 = N_3(\xi, \eta) \\ N_4 = N_4(\xi, \eta) \end{cases} \quad \text{with} \quad \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases} \quad \text{for isoparametric elements} \quad (18)$$

$$\underline{\underline{B}} = \underline{\underline{\tilde{B}}} \cdot \underline{\underline{N}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \quad (19)$$

$$\begin{aligned} \underline{\underline{B}} &= \underline{\underline{\tilde{B}}} \cdot \underline{\underline{N}} = \\ &= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_5}{\partial x} & 0 & \frac{\partial N_6}{\partial x} & 0 & \frac{\partial N_7}{\partial x} & 0 & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} & 0 & \frac{\partial N_5}{\partial y} & 0 & \frac{\partial N_6}{\partial y} & 0 & \frac{\partial N_7}{\partial y} & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial y} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial y} & \frac{\partial N_6}{\partial x} & \frac{\partial N_7}{\partial y} & \frac{\partial N_7}{\partial x} & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{bmatrix} \end{aligned} \quad (20)$$

$$\delta \underline{u} = \delta (\underline{\underline{N}} \cdot \underline{u}_N) = \underline{\underline{N}} \delta \underline{u}_N \quad (21)$$

$$\int_V \delta \underline{u}_N^T \underline{\underline{N}}^T \underline{\underline{\tilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\tilde{B}}} \cdot \underline{\underline{N}} \cdot \underline{u}_N dV - \int_V \delta \underline{u}_N^T \underline{\underline{N}}^T \underline{F} dV - \int_S \delta \underline{u}_N^T \underline{\underline{N}}^T \underline{f} dS = 0 \quad (22)$$

$$\delta \underline{u}_N^T \left(\int_V \underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}} dV \cdot \underline{u}_N - \int_V \underline{\underline{N}}^T \underline{F} dV - \int_S \underline{\underline{N}}^T \underline{f} dS \right) = 0 \quad (23)$$

$$\underline{k} \cdot \underline{u}_N = \underline{F}_N \quad \underline{k} = \int_V (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) dV \quad \underline{F}_N = \int_V \underline{\underline{N}}^T \underline{F} dV + \int_S \underline{\underline{N}}^T \underline{f} dS \quad (24)$$

$$\begin{cases} N_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{cases} \quad \begin{cases} N_1(\xi, \eta) = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \\ N_2(\xi, \eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\ N_3(\xi, \eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\ N_4(\xi, \eta) = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \\ N_5(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1-\eta) \\ N_6(\xi, \eta) = \frac{1}{2}(1+\xi)(1-\eta^2) \\ N_7(\xi, \eta) = \frac{1}{2}(1-\xi^2)(1+\eta) \\ N_8(\xi, \eta) = \frac{1}{2}(1-\xi)(1-\eta^2) \end{cases} \quad (25)$$

$$\begin{cases} \frac{\partial N_1(\xi, \eta)}{\partial \xi} = -\frac{1}{4}(1-\eta) \\ \frac{\partial N_2(\xi, \eta)}{\partial \xi} = \frac{1}{4}(1-\eta) \\ \frac{\partial N_3(\xi, \eta)}{\partial \xi} = \frac{1}{4}(1+\eta) \\ \frac{\partial N_4(\xi, \eta)}{\partial \xi} = -\frac{1}{4}(1+\eta) \\ \frac{\partial N_1(\xi, \eta)}{\partial \eta} = -\frac{1}{4}(1-\xi) \\ \frac{\partial N_2(\xi, \eta)}{\partial \eta} = -\frac{1}{4}(1+\xi) \\ \frac{\partial N_3(\xi, \eta)}{\partial \eta} = \frac{1}{4}(1+\xi) \\ \frac{\partial N_4(\xi, \eta)}{\partial \eta} = \frac{1}{4}(1-\xi) \end{cases} \quad (26)$$

$$\left\{ \begin{array}{l} \frac{\partial N_1(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 - \eta) (2\xi + \eta) \\ \frac{\partial N_2(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 - \eta) (2\xi - \eta) \\ \frac{\partial N_3(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 + \eta) (2\xi + \eta) \\ \frac{\partial N_4(\xi, \eta)}{\partial \xi} = \frac{1}{4} (1 + \eta) (2\xi - \eta) \\ \frac{\partial N_5(\xi, \eta)}{\partial \xi} = -\xi (1 - \eta) \\ \frac{\partial N_6(\xi, \eta)}{\partial \xi} = \frac{1}{2} (1 - \eta^2) \\ \frac{\partial N_7(\xi, \eta)}{\partial \xi} = -\xi (1 + \eta) \\ \frac{\partial N_8(\xi, \eta)}{\partial \xi} = -\frac{1}{2} (1 - \eta^2) \\ \frac{\partial N_1(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 - \xi) (2\eta + \xi) \\ \frac{\partial N_2(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 + \xi) (2\eta - \xi) \\ \frac{\partial N_3(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 + \xi) (2\eta + \xi) \\ \frac{\partial N_4(\xi, \eta)}{\partial \eta} = \frac{1}{4} (1 - \xi) (2\eta - \xi) \\ \frac{\partial N_5(\xi, \eta)}{\partial \eta} = -\frac{1}{2} (1 - \xi^2) \\ \frac{\partial N_6(\xi, \eta)}{\partial \eta} = -\eta (1 + \xi) \\ \frac{\partial N_7(\xi, \eta)}{\partial \eta} = \frac{1}{2} (1 - \xi^2) \\ \frac{\partial N_8(\xi, \eta)}{\partial \eta} = -\eta (1 - \xi) \end{array} \right. \quad (27)$$

$$\underline{p} = \underline{\underline{N}} \cdot \underline{p}_N \quad (28)$$

$$\underline{p} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \quad \text{or} \quad \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \\ x_5 \\ y_5 \\ x_6 \\ y_6 \\ x_7 \\ y_7 \\ x_8 \\ y_8 \end{bmatrix} \quad (29)$$

$$\begin{aligned} x &= x(\xi, \eta) = N_1(\xi, \eta) x_1 + N_2(\xi, \eta) x_2 + N_3(\xi, \eta) x_3 + N_4(\xi, \eta) x_4 \\ y &= y(\xi, \eta) = N_1(\xi, \eta) y_1 + N_2(\xi, \eta) y_2 + N_3(\xi, \eta) y_3 + N_4(\xi, \eta) y_4 \end{aligned} \quad (30)$$

$$\begin{aligned} x &= x(\xi, \eta) = N_1(\xi, \eta) x_1 + N_2(\xi, \eta) x_2 + N_3(\xi, \eta) x_3 + N_4(\xi, \eta) x_4 + \\ &\quad + N_5(\xi, \eta) x_5 + N_6(\xi, \eta) x_6 + N_7(\xi, \eta) x_7 + N_8(\xi, \eta) x_8 \\ y &= y(\xi, \eta) = N_1(\xi, \eta) y_1 + N_2(\xi, \eta) y_2 + N_3(\xi, \eta) y_3 + N_4(\xi, \eta) y_4 + \\ &\quad + N_5(\xi, \eta) y_5 + N_6(\xi, \eta) y_6 + N_7(\xi, \eta) y_7 + N_8(\xi, \eta) y_8 \end{aligned} \quad (31)$$

$$\begin{aligned}
g &= \det \left(\underline{\underline{g}} \right) = \\
&= \left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \eta} \right) - \left(\frac{\partial x}{\partial \eta} \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta} \frac{\partial y}{\partial \xi} \right) \left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} \right)
\end{aligned} \tag{41}$$

$$dV(x, y) = \sqrt{g} dV(\xi, \eta) \tag{42}$$

$$\begin{aligned}
\underline{\underline{k}}_e &= \int_{V_e(x, y)} (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) dV_e(x, y) = \int_{V_e(\xi, \eta)} (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) \sqrt{g} dV_e(\xi, \eta) \\
&= \int_{-1}^1 \int_{-1}^1 (\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}) \sqrt{g} d\xi d\eta \approx \sum_{i=1}^N \sum_{j=1}^N w_i w_j (\underline{\underline{B}}^T(\xi_i, \eta_j) \cdot \underline{\underline{D}} \cdot \underline{\underline{B}}(\xi_i, \eta_j) \sqrt{g})
\end{aligned} \tag{43}$$

$$k_e = \begin{bmatrix} k_{e|11} & k_{e|12} & k_{e|13} & k_{e|14} & k_{e|15} & k_{e|16} & k_{e|17} & k_{e|18} \\ k_{e|12} & k_{e|22} & k_{e|23} & k_{e|24} & k_{e|25} & k_{e|26} & k_{e|27} & k_{e|28} \\ k_{e|13} & k_{e|23} & k_{e|33} & k_{e|34} & k_{e|35} & k_{e|36} & k_{e|37} & k_{e|38} \\ k_{e|14} & k_{e|24} & k_{e|34} & k_{e|44} & k_{e|45} & k_{e|46} & k_{e|47} & k_{e|48} \\ k_{e|15} & k_{e|25} & k_{e|35} & k_{e|45} & k_{e|55} & k_{e|56} & k_{e|57} & k_{e|58} \\ k_{e|16} & k_{e|26} & k_{e|36} & k_{e|46} & k_{e|56} & k_{e|66} & k_{e|67} & k_{e|68} \\ k_{e|17} & k_{e|27} & k_{e|37} & k_{e|47} & k_{e|57} & k_{e|67} & k_{e|77} & k_{e|78} \\ k_{e|18} & k_{e|28} & k_{e|38} & k_{e|48} & k_{e|58} & k_{e|68} & k_{e|78} & k_{e|88} \end{bmatrix} \tag{44}$$

2.1. Definition of crack tip reference frame



Figure 1. Schematic representation of the discretized crack tip geometry for 1st order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad (45)$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \quad (46)$$

2.2. Calculation of displacements and reaction forces

$$u_x = u_{x,M} - u_{x,F} \quad u_y = u_{y,M} - u_{y,F} \quad (47)$$

$$u_r = \cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y \quad u_\theta = -\sin(\Delta\theta) u_x + \cos(\Delta\theta) u_y \quad (48)$$

$$F_r = \cos(\Delta\theta) F_{x,CT} + \sin(\Delta\theta) F_{y,CT} \quad F_\theta = -\sin(\Delta\theta) F_{x,CT} + \cos(\Delta\theta) F_{y,CT} \quad (49)$$

$$\left\{ \begin{array}{l}
(k_{e,M|11} + k_{e,M|33}) u_{x,MCT} + (k_{e,M|12} + k_{e,M|34}) u_{y,MCT} + \\
+ k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + (k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + k_{M|31} u_{x,NCOI} + k_{M|32} u_{y,NCOI} = 0 \\
\\
(k_{e,M|21} + k_{e,M|43}) u_{x,MCT} + (k_{e,M|22} + k_{e,M|44}) u_{y,MCT} + \\
+ k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + k_{M|41} u_{x,NCOI} + k_{M|42} u_{y,NCOI} = 0 \\
\\
(k_{e,F|77} + k_{e,F|55}) u_{x,FCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,FCT} + \\
+ k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i} + k_{F|57} u_{x,NCOI} + k_{F|58} u_{y,NCOI} = 0 \\
\\
(k_{e,F|87} + k_{e,F|65}) u_{x,FCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,FCT} + \\
+ k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + k_{F|67} u_{x,NCOI} + k_{F|68} u_{y,NCOI} = 0 \\
\\
u_{x,FCT} - u_{x,MCT} - u_{x,DUMMY} = 0 \\
u_{y,FCT} - u_{y,MCT} - u_{y,DUMMY} = 0 \\
\\
u_{x,DUMMY} = 0 \\
u_{y,DUMMY} = 0
\end{array} \right. \quad (50)$$

$$\left\{ \begin{array}{l}
(k_{e,M|11} + k_{e,M|33}) u_{x,MCT} + (k_{e,M|12} + k_{e,M|34}) u_{y,MCT} + \\
+ k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + (k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + k_{M|31} u_{x,NCOI} + k_{M|32} u_{y,NCOI} = 0 \\
\\
(k_{e,M|21} + k_{e,M|43}) u_{x,MCT} + (k_{e,M|22} + k_{e,M|44}) u_{y,MCT} + \\
+ k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + \\
+ \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + k_{M|41} u_{x,NCOI} + k_{M|42} u_{y,NCOI} = 0 \\
\\
(k_{e,F|77} + k_{e,F|55}) u_{x,MCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,MCT} + \\
+ k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i} + k_{F|57} u_{x,NCOI} + k_{F|58} u_{y,NCOI} = 0 \\
\\
(k_{e,F|87} + k_{e,F|65}) u_{x,MCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,MCT} + \\
+ k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
+ \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + k_{F|67} u_{x,NCOI} + k_{F|68} u_{y,NCOI} = 0 \\
\\
u_{x,FCT} = u_{x,MCT} \\
u_{y,FCT} = u_{y,MCT} \\
\\
R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{array} \right. \quad (51)$$

$$\left\{ \begin{aligned}
& (k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}) u_{x,MCT} + (k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}) u_{y,MCT} + \\
& + k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + \\
& + (k_{M|31} + k_{F|57}) u_{x,NCOI} + (k_{M|32} + k_{F|58}) u_{y,NCOI} + \\
& + (k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2} + \\
& + \sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i} = 0 \\
\\
& (k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}) u_{x,MCT} + (k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}) u_{y,MCT} + \\
& + k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + \\
& + (k_{M|41} + k_{F|67}) u_{x,NCOI} + (k_{M|42} + k_{F|68}) u_{y,NCOI} + \\
& + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
& + \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} = 0 \\
\\
& u_{x,FCT} = u_{x,MCT} \\
& u_{y,FCT} = u_{y,MCT} \\
\\
& R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
& R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{aligned} \right. \tag{52}$$

$$\left\{ \begin{aligned}
& u_{y,MCT} = -\frac{k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} u_{x,MCT} + \\
& -\frac{k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\
& -\frac{(k_{M|31} + k_{F|57}) u_{x,NCOI} + (k_{M|32} + k_{F|58}) u_{y,NCOI}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\
& -\frac{(k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8} + (k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \\
\\
& \left[(k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}) + \frac{k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} (k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}) \right] u_{x,MCT} + \\
& + \left(k_{e,M|23} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|13} \right) u_{x,M} + \\
& + \left(k_{e,M|24} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|14} \right) u_{y,M} + \\
& + \left(k_{e,F|85} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|75} \right) u_{x,F} + \\
& + \left(k_{e,F|86} - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|76} \right) u_{y,F} + \\
& + \left[(k_{M|41} + k_{F|67}) - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} (k_{M|31} + k_{F|57}) \right] u_{x,NCOI} + \\
& + \left[(k_{M|42} + k_{F|68}) - \frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} (k_{M|32} + k_{F|58}) \right] u_{y,NCOI} + \\
& + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
& -\frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} [(k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8}] + \\
& -\frac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} [(k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2}] \\
& + \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} = 0 \\
\\
& u_{x,FCT} = u_{x,MCT} \\
& u_{y,FCT} = u_{y,MCT} \\
\\
& R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
& R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{aligned} \right. \tag{53}$$

$$\left\{ \begin{aligned}
& u_{y,MCT} = -\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} u_{x,MCT} + \\
& -\frac{k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
& -\frac{(k_{M|31}+k_{F|57}) u_{x,NCOI} + (k_{M|32}+k_{F|58}) u_{y,NCOI}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
& -\frac{(k_{M|17}+k_{M|35}) u_{N,MC|7} + (k_{M|18}+k_{M|36}) u_{N,MC|8} + (k_{F|71}+k_{F|53}) u_{N,FC|1} + (k_{F|72}+k_{F|54}) u_{N,FC|2}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} \\
& \left[(k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}) + \frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}) \right] u_{x,MCT} + \\
& + \left(k_{e,M|23} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} k_{e,M|13} \right) u_x + \\
& + \left(k_{e,M|24} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} k_{e,M|14} \right) u_y + \\
& + \left(k_{e,M|23} + k_{e,F|85} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{e,M|13} + k_{e,M|75}) \right) \overset{\text{approx}}{\approx} 0 + \\
& + \left(k_{e,M|24} + k_{e,F|86} - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{e,M|14} + k_{e,M|76}) \right) \overset{\text{approx}}{\approx} 0 + \\
& + \left[(k_{M|41} + k_{F|67}) - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{M|31} + k_{F|57}) \right] u_{x,NCOI} + \\
& + \left[(k_{M|42} + k_{F|68}) - \frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} (k_{M|32} + k_{F|58}) \right] u_{y,NCOI} + \\
& + (k_{M|27} + k_{M|45}) u_{N,MC|7} + (k_{M|28} + k_{M|46}) u_{N,MC|8} + (k_{F|81} + k_{F|63}) u_{N,FC|1} + (k_{F|82} + k_{F|64}) u_{N,FC|2} + \\
& -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} [(k_{M|17} + k_{M|35}) u_{N,MC|7} + (k_{M|18} + k_{M|36}) u_{N,MC|8}] + \\
& -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} [(k_{F|71} + k_{F|53}) u_{N,FC|1} + (k_{F|72} + k_{F|54}) u_{N,FC|2}] \\
& + \sum_{i=2}^3 k_{F|8i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|6i} u_{N,FB|i} + \sum_{i=5}^6 k_{M|2i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|4i} u_{N,MB|i} + \\
& -\frac{\sum_{i=5}^6 k_{M|1i} u_{N,MC|i} + \sum_{i=7}^8 k_{M|3i} u_{N,MB|i} + \sum_{i=2}^3 k_{F|7i} u_{N,FC|i} + \sum_{i=1}^2 k_{F|5i} u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} = 0 \\
& u_{x,FCT} = u_{x,MCT} \\
& u_{y,FCT} = u_{y,MCT} \\
& F_{x,CT} = R_{x,FCT} = \\
& = (k_{e,F|77} + k_{e,F|55}) u_{x,FCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,FCT} + \\
& + k_{e,F|75} \overset{\text{approx}}{\approx} 0 + k_{e,F|76} \overset{\text{approx}}{\approx} 0 + \\
& + \sum_{i=1}^4 k_{e,F|7i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|5i} u_{N,FB|i} \\
& F_{y,CT} = R_{y,FCT} = \\
& = (k_{e,F|87} + k_{e,F|65}) u_{x,FCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,FCT} + \\
& + k_{e,F|85} \overset{\text{approx}}{\approx} 0 + k_{e,F|86} \overset{\text{approx}}{\approx} 0 + \\
& + \sum_{i=1}^4 k_{e,F|8i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|6i} u_{N,FB|i}
\end{aligned} \right. \tag{54}$$

$$\left\{ \begin{aligned}
F_{x,CT} &= (k_{e,F|77} + k_{e,F|55}) u_{x,MCT} + (k_{e,F|78} + k_{e,F|56}) u_{y,MCT} + \\
& + \sum_{i=1}^4 k_{e,F|7i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|5i} u_{N,FB|i} \\
F_{y,CT} &= (k_{e,F|87} + k_{e,F|65}) u_{x,MCT} + (k_{e,F|88} + k_{e,F|66}) u_{y,MCT} + \\
& + \sum_{i=1}^4 k_{e,F|8i} u_{N,FC|i} + \sum_{i=1, i \neq (5,6)}^8 k_{e,F|6i} u_{N,FB|i}
\end{aligned} \right. \tag{55}$$

$$\left\{ \begin{aligned}
F_{x,CT} &= K_{xx} u_x + K_{xy} u_y + \\
& + \sum_{i=1}^4 K_{FC,x|i} u_{N,FC|i} + \sum_{i=1, i \neq (3,4,5,6)}^8 K_{FB,x|i} u_{N,FB|i} + \\
& + \sum_{i=5}^8 K_{FC,x|i} u_{N,MC|i} + \sum_{i=7}^8 K_{MB,x|i} u_{N,FB|i} \\
F_{y,CT} &= K_{yx} u_x + K_{yy} u_y + \\
& + \sum_{i=1}^4 K_{FC,y|i} u_{N,FC|i} + \sum_{i=1, i \neq (3,4,5,6)}^8 K_{FB,y|i} u_{N,FB|i} + \\
& + \sum_{i=5}^8 K_{FC,y|i} u_{N,MC|i} + \sum_{i=7}^8 K_{MB,y|i} u_{N,FB|i}
\end{aligned} \right. \tag{56}$$

$$\begin{cases} F_{x,CT} &= K_{xx}u_x + K_{xy}u_y + \tilde{F}_x \\ F_{y,CT} &= K_{yx}u_x + K_{yy}u_y + \tilde{F}_y \end{cases} \quad (57)$$

2.3. Calculation of energy release rates

$$\begin{aligned} G_{I,r\theta} &= \frac{1}{2} \frac{F_r u_r}{R_f \delta} = \\ &= \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y) = \\ &= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) \end{aligned} \quad (58)$$

$$\begin{aligned} G_{II,r\theta} &= \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} = \\ &= \frac{1}{2R_f \delta} (-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y) (-\sin(\Delta\theta) u_x + \cos(\Delta\theta) u_y) = \\ &= \frac{1}{2R_f \delta} (\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y) \end{aligned} \quad (59)$$

$$\begin{aligned} G_{TOT,r\theta} &= G_{I,r\theta} + G_{II,r\theta} = \\ &= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) + \\ &\quad + \frac{1}{2R_f \delta} (\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y) = \\ &= \frac{1}{2R_f \delta} \left(\begin{array}{c} \xrightarrow{1} \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) F_x u_x \end{array} \right) + \\ &\quad + \frac{1}{2R_f \delta} \left(\begin{array}{c} \xrightarrow{0} \\ ((F_x u_y + F_y u_x) - (F_x u_y + F_y u_x)) \cos(\Delta\theta) \sin(\Delta\theta) \end{array} \right) + \\ &\quad + \frac{1}{2R_f \delta} \left(\begin{array}{c} \xrightarrow{1} \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) F_y u_y \end{array} \right) = \\ &= \frac{1}{2} \frac{F_x u_x}{R_f \delta} + \frac{1}{2} \frac{F_y u_y}{R_f \delta} = \\ &= \tilde{G}_{I,xy} + \tilde{G}_{II,xy} = \tilde{G}_{TOT,xy} \end{aligned} \quad (60)$$

$$\begin{aligned} G_{I,r\theta} &= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) = \\ &= \cos^2(\Delta\theta) \frac{F_x u_x}{2R_f \delta} + \left(\frac{F_x u_y}{2R_f \delta} + \frac{F_y u_x}{2R_f \delta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) \frac{F_y u_y}{2R_f \delta} = \\ &= \cos^2(\Delta\theta) \tilde{G}_{I,xy} + \left(\tilde{G}_{I,xy} \frac{u_y}{u_x} + \tilde{G}_{II,xy} \frac{u_x}{u_y} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) \tilde{G}_{II,xy} \end{aligned} \quad (61)$$

$$\begin{aligned}
G_{II,r\theta} &= \frac{1}{2R_f\delta} \left(\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y \right) = \\
&= \sin^2(\Delta\theta) \frac{F_x u_x}{2R_f\delta} - \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) \frac{F_y u_y}{2R_f\delta} = \\
&= \sin^2(\Delta\theta) \tilde{G}_{I,xy} - \left(\tilde{G}_{I,xy} \frac{u_y}{u_x} + \tilde{G}_{II,xy} \frac{u_x}{u_y} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) \tilde{G}_{II,xy}
\end{aligned} \tag{62}$$

2.4. Sensitivity analysis of the FEM solution

$$\begin{cases} F_{x,CT} &= K_{xx} u_x + K_{xy} u_y + \tilde{F}_x \\ F_{y,CT} &= K_{yx} u_x + K_{yy} u_y + \tilde{F}_y \end{cases} \tag{63}$$

$$\begin{aligned}
G_{I,r\theta} &\sim \frac{1}{2R_f\delta} \cos^2(\Delta\theta) \left(K_{xx} u_x^2 + K_{xy} u_y u_x + \tilde{F}_x u_x \right) + \\
&+ \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) \left(K_{xx} u_x u_y + K_{xy} u_y^2 + \tilde{F}_x u_y \right) + \\
&+ \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) \left(K_{yx} u_x^2 + K_{yy} u_y u_x + \tilde{F}_y u_x \right) + \\
&+ \frac{1}{2R_f\delta} \sin^2(\Delta\theta) \left(K_{yx} u_y u_x + K_{yy} u_y^2 + \tilde{F}_y \right)
\end{aligned} \tag{64}$$

$$\begin{aligned}
G_{II,r\theta} &\sim \frac{1}{2R_f\delta} \sin^2(\Delta\theta) k_x u_x^2(\Delta\theta) + \\
&- \frac{1}{2R_f\delta} (k_x + k_y) u_x(\Delta\theta) u_y(\Delta\theta) \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta} \cos^2(\Delta\theta) k_y u_y^2(\Delta\theta)
\end{aligned} \tag{65}$$

$$G_{TOT,r\theta} \sim \frac{1}{2R_f\delta} (k_x u_x^2(\Delta\theta) + k_y u_y^2(\Delta\theta)) \tag{66}$$

$$\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial \Delta\theta} &\sim \frac{1}{R_f\delta} \cos^2(\Delta\theta) k_x u_x(\Delta\theta) \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + \\
&+ \frac{1}{2R_f\delta} (k_x + k_y) \left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} u_y(\Delta\theta) + u_x(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \frac{1}{R_f\delta} \sin^2(\Delta\theta) k_y u_y(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \\
&+ \frac{1}{2R_f\delta} (k_y u_y^2(\Delta\theta) - k_x u_x^2(\Delta\theta)) \sin(2\Delta\theta) + \\
&+ \frac{1}{2R_f\delta} (k_x u_x(\Delta\theta) u_y(\Delta\theta) + k_y u_y(\Delta\theta) u_x(\Delta\theta)) \cos(2\Delta\theta)
\end{aligned} \tag{67}$$

$$\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f \delta} \sin^2(\Delta\theta) k_x u_x(\Delta\theta) \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + \\
& - \frac{1}{2R_f \delta} (k_x + k_y) \left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} u_y(\Delta\theta) + u_x(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{R_f \delta} \cos^2(\Delta\theta) k_y u_y(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \\
& + \frac{1}{2R_f \delta} (k_x u_x^2(\Delta\theta) - k_y u_y^2(\Delta\theta)) \sin(2\Delta\theta) + \\
& - \frac{1}{2R_f \delta} (k_x u_x(\Delta\theta) u_y(\Delta\theta) + k_y u_y(\Delta\theta) u_x(\Delta\theta)) \cos(2\Delta\theta)
\end{aligned} \tag{68}$$

$$\frac{\partial G_{TOT,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f \delta} \left(k_x u_x(\Delta\theta) \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + k_y u_y(\Delta\theta) \frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} \right) \tag{69}$$

2.5. Discretization error

$$u_r = \cos(\Delta\theta - \delta) u_x + \sin(\Delta\theta - \delta) u_y \quad u_\theta = -\sin(\Delta\theta - \delta) u_x + \cos(\Delta\theta - \delta) u_y \tag{70}$$

$$F_r = \cos(\Delta\theta) F_{x,CT} + \sin(\Delta\theta) F_{y,CT} \quad F_\theta = -\sin(\Delta\theta) F_{x,CT} + \cos(\Delta\theta) F_{y,CT} \tag{71}$$

$$\begin{aligned}
\tilde{G}_{I,r\theta} &= \frac{1}{2} \frac{F_r u_r}{R_f \delta} = \\
&= \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\cos(\Delta\theta - \delta) u_x + \sin(\Delta\theta - \delta) u_y) = \\
&= \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y) \cos(\delta) + \\
&\quad + \frac{1}{2R_f \delta} (\cos(\Delta\theta) F_x + \sin(\Delta\theta) F_y) (\sin(\Delta\theta) u_x - \cos(\Delta\theta) u_y) \sin(\delta) = \\
&= \frac{1}{2R_f \delta} (\cos^2(\Delta\theta) F_x u_x + (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_y) \cos(\delta) + \\
&\quad + \frac{1}{2R_f \delta} (-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x) \sin(\delta) = \\
&= G_{I,r\theta} \cos(\delta) + \\
&\quad + \frac{1}{2R_f \delta} (-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x) \sin(\delta) = \\
&= G_{I,r\theta} \cos(\delta) - \frac{1}{2R_f \delta} F_r u_\theta \sin \delta
\end{aligned} \tag{72}$$

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_{I,r\theta} &= \lim_{\delta \rightarrow 0} G_{I,r\theta} \cos(\delta) + \\
&+ \lim_{\delta \rightarrow 0} \frac{1}{2R_f \delta} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \sin(\delta) = \\
&= G_{I,r\theta} + \\
&+ \frac{1}{2R_f} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \lim_{\delta \rightarrow 0} \frac{\sin(\delta)}{\delta} = \\
&= G_{I,r\theta} + \frac{1}{2R_f} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \\
&= G_{I,r\theta} - \frac{1}{2R_f} F_r u_\theta
\end{aligned} \tag{73}$$

$$\begin{aligned}
\tilde{G}_{II,r\theta} &= \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} = \\
&= \frac{1}{2R_f \delta} \left(-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y \right) \left(-\sin(\Delta\theta - \delta) u_x + \cos(\Delta\theta - \delta) u_y \right) = \\
&= \frac{1}{2R_f \delta} \left(\sin^2(\Delta\theta) \cos(\delta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) \cos(\delta) + \cos^2(\Delta\theta) \cos(\delta) F_y u_y \right) + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) \sin(\delta) + \cos^2(\Delta\theta) \sin(\delta) F_y u_x - \sin^2(\Delta\theta) \sin(\delta) F_x u_y \right) = \\
&= \frac{1}{2R_f \delta} \left(\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y \right) \cos(\delta) + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= G_{II,r\theta} \cos(\delta) + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= G_{II,r\theta} \cos(\delta) + \frac{1}{2R_f \delta} F_\theta u_r \sin \delta
\end{aligned} \tag{74}$$

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_{II,r\theta} &= \lim_{\delta \rightarrow 0} G_{II,r\theta} \cos(\delta) + \\
&+ \lim_{\delta \rightarrow 0} \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= G_{II,r\theta} + \\
&+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \lim_{\delta \rightarrow 0} \frac{\sin(\delta)}{\delta} = \\
&= G_{II,r\theta} + \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) = \\
&= G_{II,r\theta} + \frac{1}{2R_f} F_\theta u_r
\end{aligned} \tag{75}$$

$$\begin{aligned}
\tilde{G}_{TOT,r\theta} &= \tilde{G}_{I,r\theta} + \tilde{G}_{II,r\theta} = \\
&= (G_{I,r\theta} + G_{II,r\theta}) \cos(\delta) + \\
&+ \frac{1}{2R_f\delta} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \sin(\delta) + \\
&+ \frac{1}{2R_f\delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) = \\
&= (G_{I,r\theta} + G_{II,r\theta}) \cos(\delta) + \frac{1}{2R_f\delta} (F_y u_x - F_x u_y) \sin(\delta) = \\
&= G_{TOT,r\theta} \cos(\delta) + \frac{1}{2R_f\delta} (F_y u_x - F_x u_y) \sin(\delta) = \\
&= G_{TOT,r\theta} \cos(\delta) + \frac{1}{2R_f\delta} (F_\theta u_r - F_r u_\theta) \sin(\delta)
\end{aligned} \tag{76}$$

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \tilde{G}_{TOT,r\theta} &= \lim_{\delta \rightarrow 0} G_{TOT,r\theta} \cos(\delta) + \lim_{\delta \rightarrow 0} \frac{1}{2R_f\delta} (F_y u_x - F_x u_y) \sin(\delta) = \\
&= G_{TOT,r\theta} + \frac{1}{2R_f} (F_y u_x - F_x u_y) \lim_{\delta \rightarrow 0} \frac{\sin(\delta)}{\delta} - 0 \\
&= G_{TOT,r\theta} + \frac{1}{2R_f} (F_y u_x - F_x u_y) = \\
&= G_{TOT,r\theta} + \frac{1}{2R_f} (F_\theta u_r - F_r u_\theta)
\end{aligned} \tag{77}$$

2.6. Contact region

$$u_r = 0 \tag{78}$$

$$\cos(\Delta\theta) u_x + \sin(\Delta\theta) u_y = 0 \tag{79}$$

$$u_y = -\frac{u_x}{\tan(\Delta\theta)} \tag{80}$$

$$\begin{aligned}
u_\theta &= -\sin(\Delta\theta) u_x - \frac{\cos^2(\Delta\theta)}{\sin(\Delta\theta)} u_x = \\
&= -\frac{u_x}{\sin(\Delta\theta)}
\end{aligned} \tag{81}$$

$$F_r = \cos(\Delta\theta) F_{x,CT} + \sin(\Delta\theta) F_{y,CT} \quad F_\theta = -\sin(\Delta\theta) F_{x,CT} + \cos(\Delta\theta) F_{y,CT} \tag{82}$$

$$G_{I,r\theta} = \frac{1}{2} \frac{F_r u_r}{R_f \delta} = 0 \tag{83}$$

$$\begin{aligned}
G_{II,r\theta} &= \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} = \\
&= \frac{1}{2R_f \delta} (-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y) \left(-\frac{u_x}{\sin(\Delta\theta)} \right) = \\
&= \frac{1}{2R_f \delta} \left(F_x u_x - \frac{F_y u_x}{\tan(\Delta\theta)} \right) \\
&= \frac{1}{2R_f \delta} \left(F_x - \frac{F_y}{\tan(\Delta\theta)} \right) u_x
\end{aligned} \tag{84}$$

3. VCCT for second order quadrilateral elements

3.1. Definition of crack tip reference frame



Figure 2. Schematic representation of the discretized crack tip geometry for 2^{nd} order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad (85)$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \quad (86)$$

3.2. Calculation of displacements and reaction forces

$$\begin{aligned} u_{x,1} &= u_{x,M1} - u_{x,F1} & u_{y,1} &= u_{y,M1} - u_{y,F1} \\ u_{x,2} &= u_{x,M2} - u_{x,F2} & u_{y,2} &= u_{y,M2} - u_{y,F2} \end{aligned} \quad (87)$$

$$\begin{aligned} u_{r,1} &= \cos(\Delta\theta) u_{x,1} + \sin(\Delta\theta) u_{y,1} & u_{\theta,1} &= -\sin(\Delta\theta) u_{x,1} + \cos(\Delta\theta) u_{y,1} \\ u_{r,2} &= \cos(\Delta\theta) u_{x,2} + \sin(\Delta\theta) u_{y,2} & u_{\theta,2} &= -\sin(\Delta\theta) u_{x,2} + \cos(\Delta\theta) u_{y,2} \end{aligned} \quad (88)$$

$$\begin{aligned}
F_{r,1} &= \cos(\Delta\theta) F_{x,1} + \sin(\Delta\theta) F_{y,1} & F_{\theta,1} &= -\sin(\Delta\theta) F_{x,1} + \cos(\Delta\theta) F_{y,1} \\
F_{r,2} &= \cos(\Delta\theta) F_{x,2} + \sin(\Delta\theta) F_{y,2} & F_{\theta,2} &= -\sin(\Delta\theta) F_{x,2} + \cos(\Delta\theta) F_{y,2}
\end{aligned} \tag{89}$$

3.3. Calculation of energy release rates

$$\begin{aligned}
G_{I,r\theta} &= \frac{1}{2R_f\delta} (F_{r,1}u_{r,1} + F_{r,2}u_{r,2}) = \\
&= \frac{1}{2R_f\delta} (\cos(\Delta\theta) F_{x,1} + \sin(\Delta\theta) F_{y,1}) (\cos(\Delta\theta) u_{x,1} + \sin(\Delta\theta) u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (\cos(\Delta\theta) F_{x,2} + \sin(\Delta\theta) F_{y,2}) (\cos(\Delta\theta) u_{x,2} + \sin(\Delta\theta) u_{y,2}) + \\
&= \frac{1}{2R_f\delta} (\cos^2(\Delta\theta) F_{x,1}u_{x,1} + (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,1}u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (\cos^2(\Delta\theta) F_{x,2}u_{x,2} + (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,2}u_{y,2})
\end{aligned} \tag{90}$$

$$\begin{aligned}
G_{II,r\theta} &= \frac{1}{2R_f\delta} (F_{\theta,1}u_{\theta,1} + F_{\theta,2}u_{\theta,2}) = \\
&= \frac{1}{2R_f\delta} (-\sin(\Delta\theta) F_{x,1} + \cos(\Delta\theta) F_{y,1}) (-\sin(\Delta\theta) u_{x,1} + \cos(\Delta\theta) u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (-\sin(\Delta\theta) F_{x,2} + \cos(\Delta\theta) F_{y,2}) (-\sin(\Delta\theta) u_{x,2} + \cos(\Delta\theta) u_{y,2}) = \\
&= \frac{1}{2R_f\delta} (\sin^2(\Delta\theta) F_{x,1}u_{x,1} - (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,1}u_{y,1}) + \\
&\quad + \frac{1}{2R_f\delta} (\sin^2(\Delta\theta) F_{x,2}u_{x,2} - (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,2}u_{y,2})
\end{aligned} \tag{91}$$

$$\begin{aligned}
G_{TOT,r\theta} &= G_{I,r\theta} + G_{II,r\theta} = \\
&= \frac{1}{2R_f\delta} \left(\cos^2(\Delta\theta) F_{x,1}u_{x,1} + (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,1}u_{y,1} \right) + \\
&\quad + \frac{1}{2R_f\delta} \left(\cos^2(\Delta\theta) F_{x,2}u_{x,2} + (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,2}u_{y,2} \right) + \\
&\quad + \frac{1}{2R_f\delta} \left(\sin^2(\Delta\theta) F_{x,1}u_{x,1} - (F_{x,1}u_{y,1} + F_{y,1}u_{x,1}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,1}u_{y,1} \right) + \\
&\quad + \frac{1}{2R_f\delta} \left(\sin^2(\Delta\theta) F_{x,2}u_{x,2} - (F_{x,2}u_{y,2} + F_{y,2}u_{x,2}) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,2}u_{y,2} \right) = \\
&= \frac{1}{2R_f\delta} \cos^2(\Delta\theta) (F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}) + \\
&\quad + \frac{1}{2R_f\delta} ((F_{x,1}u_{y,1} + F_{x,1}u_{y,1}) + (F_{y,2}u_{x,2} + F_{y,2}u_{x,2})) \cos(\Delta\theta) \sin(\Delta\theta) + \\
&\quad + \frac{1}{2R_f\delta} \sin^2(\Delta\theta) (F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}) + \\
&\quad - \frac{1}{2R_f\delta} ((F_{x,1}u_{y,1} + F_{x,1}u_{y,1}) + (F_{y,2}u_{x,2} + F_{y,2}u_{x,2})) \cos(\Delta\theta) \sin(\Delta\theta) = \\
&= \frac{1}{2R_f\delta} \left(\begin{array}{c} \nearrow 1 \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) \quad (F_{x,1}u_{x,1} + F_{x,2}u_{x,2}) \end{array} \right) + \\
&\quad + \frac{1}{2R_f\delta} \begin{array}{c} \nearrow 0 \\ ((F_{x,1}u_{y,1} + F_{x,1}u_{y,1}) - (F_{x,1}u_{y,1} + F_{x,1}u_{y,1})) \quad \cos(\Delta\theta) \sin(\Delta\theta) + \\ \nearrow 0 \end{array} \\
&\quad + \frac{1}{2R_f\delta} \begin{array}{c} \nearrow 0 \\ ((F_{y,2}u_{x,2} + F_{y,2}u_{x,2}) - (F_{y,2}u_{x,2} + F_{y,2}u_{x,2})) \quad \cos(\Delta\theta) \sin(\Delta\theta) + \\ \nearrow 1 \end{array} \\
&\quad + \frac{1}{2R_f\delta} \left(\begin{array}{c} \nearrow 1 \\ (\cos^2(\Delta\theta) + \sin^2(\Delta\theta)) \quad (F_{y,1}u_{y,1} + F_{y,2}u_{y,2}) \end{array} \right) = \\
&= \frac{1}{2} \frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{R_f\delta} + \frac{1}{2} \frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{R_f\delta} = \\
&= \tilde{G}_{I,xy} + \tilde{G}_{II,xy} = \tilde{G}_{TOT,xy}
\end{aligned}$$

$$\begin{aligned}
G_{I,r\theta} &= \cos^2(\Delta\theta) \frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{2R_f\delta} + \\
&+ \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \sin^2(\Delta\theta) \frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{2R_f\delta} = \\
&= \cos^2(\Delta\theta) \tilde{G}_{I,xy} + \sin^2(\Delta\theta) \tilde{G}_{II,xy} + \\
&+ \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta)
\end{aligned} \tag{93}$$

$$\begin{aligned}
G_{II,r\theta} &= \sin^2(\Delta\theta) \frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{2R_f\delta} + \\
&- \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) + \\
&+ \cos^2(\Delta\theta) \frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{2R_f\delta} = \\
&= \sin^2(\Delta\theta) \tilde{G}_{I,xy} + \cos^2(\Delta\theta) \tilde{G}_{II,xy} + \\
&- \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta)
\end{aligned} \tag{94}$$

3.4. Sensitivity analysis of the FEM solution

$$\begin{aligned}
F_{x,1} &\sim k_{x,1}u_{x,1} & F_{y,1} &\sim k_{y,1}u_{y,1} \\
F_{x,2} &\sim k_{x,2}u_{x,2} & F_{y,2} &\sim k_{y,2}u_{y,2}
\end{aligned} \tag{95}$$

$$\begin{aligned}
G_{I,r\theta} &\sim \frac{1}{2R_f\delta} \cos^2(\Delta\theta) (k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2) + \\
&+ \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) (k_{x,1}u_{x,1}u_{y,1} + k_{y,1}u_{y,1}u_{x,1} + k_{x,2}u_{x,2}u_{y,2} + k_{y,2}u_{y,2}u_{x,2}) + \\
&+ \frac{1}{2R_f\delta} \sin^2(\Delta\theta) (k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2)
\end{aligned} \tag{96}$$

$$\begin{aligned}
G_{II,r\theta} &\sim \frac{1}{2R_f\delta} \sin^2(\Delta\theta) (k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2) + \\
&- \frac{1}{2R_f\delta} \cos(\Delta\theta) \sin(\Delta\theta) (k_{x,1}u_{x,1}u_{y,1} + k_{y,1}u_{y,1}u_{x,1} + k_{x,2}u_{x,2}u_{y,2} + k_{y,2}u_{y,2}u_{x,2}) + \\
&+ \frac{1}{2R_f\delta} \cos^2(\Delta\theta) (k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2)
\end{aligned} \tag{97}$$

$$G_{TOT,r\theta} \sim \frac{1}{2R_f\delta} ((k_{x,1}u_{x,1}^2(\Delta\theta) + k_{x,2}u_{x,2}^2(\Delta\theta)) + (k_{y,1}u_{y,1}^2(\Delta\theta) + k_{y,2}u_{y,2}^2(\Delta\theta))) \tag{98}$$

$$\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta} \cos^2(\Delta\theta) \left(k_{x,1} u_{x,1}(\Delta\theta) \frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} + k_{x,2} u_{x,2}(\Delta\theta) \frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& + \frac{1}{2R_f\delta} (k_{x,1} + k_{y,1}) \left(\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} u_{y,1}(\Delta\theta) + u_{x,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{2R_f\delta} (k_{x,2} + k_{y,2}) \left(\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} u_{y,2}(\Delta\theta) + u_{x,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{R_f\delta} \sin^2(\Delta\theta) \left(k_{y,1} u_{y,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} + k_{y,2} u_{y,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& + \frac{1}{2R_f\delta} ((k_{y,1} u_{y,1}^2 + k_{y,2} u_{y,2}^2) - (k_{x,1} u_{x,1}^2 + k_{x,2} u_{x,2}^2)) \sin(2\Delta\theta) + \\
& + \frac{1}{2R_f\delta} ((k_{x,1} + k_{y,1}) u_{x,1} u_{y,1} + (k_{x,2} + k_{y,2}) u_{x,2} u_{y,2}) \cos(2\Delta\theta)
\end{aligned} \tag{99}$$

$$\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta} \sin^2(\Delta\theta) \left(k_{x,1} u_{x,1}(\Delta\theta) \frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} + k_{x,2} u_{x,2}(\Delta\theta) \frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& - \frac{1}{2R_f\delta} (k_{x,1} + k_{y,1}) \left(\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} u_{y,1}(\Delta\theta) + u_{x,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& - \frac{1}{2R_f\delta} (k_{x,2} + k_{y,2}) \left(\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta} u_{y,2}(\Delta\theta) + u_{x,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) \cos(\Delta\theta) \sin(\Delta\theta) + \\
& + \frac{1}{R_f\delta} \cos^2(\Delta\theta) \left(k_{y,1} u_{y,1}(\Delta\theta) \frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} + k_{y,2} u_{y,2}(\Delta\theta) \frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta} \right) + \\
& + \frac{1}{2R_f\delta} ((k_{x,1} u_{x,1}^2 + k_{x,2} u_{x,2}^2) - (k_{y,1} u_{y,1}^2 + k_{y,2} u_{y,2}^2)) \sin(2\Delta\theta) + \\
& - \frac{1}{2R_f\delta} ((k_{x,1} + k_{y,1}) u_{x,1} u_{y,1} + (k_{x,2} + k_{y,2}) u_{x,2} u_{y,2}) \cos(2\Delta\theta)
\end{aligned} \tag{100}$$

$$G_{TOT,r\theta} \sim \frac{1}{R_f\delta} \left(\left(k_{x,1} u_{x,1} \frac{\partial u_{x,1}}{\partial \Delta\theta} + k_{x,2} u_{x,2} \frac{\partial u_{x,2}}{\partial \Delta\theta} \right) + \left(k_{y,1} u_{y,1} \frac{\partial u_{y,1}}{\partial \Delta\theta} + k_{y,2} u_{y,2} \frac{\partial u_{y,2}}{\partial \Delta\theta} \right) \right) \tag{101}$$

4. A vectorial formulation of the VCCT

4.1. Vectorial formulation

$$\underline{\underline{R}}_{\Delta\theta} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad \underline{\underline{R}}_{\Delta\theta}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \quad (102)$$

$$\underline{\underline{Q}}_{\delta}(p) = \begin{bmatrix} \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \\ -\sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \end{bmatrix} \quad (103)$$

$$\underline{\underline{Q}}_{\delta}^{-1} = \underline{\underline{Q}}_{\delta}^T = \begin{bmatrix} \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & -\sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \\ \sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \end{bmatrix} \quad (104)$$

$$\underline{\underline{P}}_{\delta}(p) = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & \sin\left(\frac{q-1}{m}\delta\right) \\ -\sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \quad (105)$$

$$\underline{\underline{P}}_{\delta}^{-1} = \underline{\underline{P}}_{\delta}^T = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & -\sin\left(\frac{q-1}{m}\delta\right) \\ \sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \quad (106)$$

$$\frac{\partial \underline{\underline{R}}_{\Delta\theta}}{\partial \Delta\theta} = \underline{\underline{D}} \cdot \underline{\underline{R}}_{\Delta\theta} \quad (107)$$

$$\frac{\partial \underline{\underline{Q}}_{\delta}}{\partial \delta} = \left(1 + \frac{1-p}{m}\right) \underline{\underline{D}} \cdot \underline{\underline{Q}}_{\delta} \quad (108)$$

$$\underline{\underline{D}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (109)$$

$$\underline{\underline{F}}_{xy} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \quad \underline{\underline{F}}_{r\theta} = \begin{bmatrix} F_r \\ F_{\theta} \end{bmatrix} \quad (110)$$

$$\underline{\underline{u}}_{xy} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad \underline{\underline{u}}_{r\theta} = \begin{bmatrix} u_r \\ u_{\theta} \end{bmatrix} \quad (111)$$

$$\underline{\underline{G}} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} \quad (112)$$

$$\underline{\underline{F}}_{r\theta} = \underline{\underline{P}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{\underline{F}}_{xy} \quad \underline{\underline{u}}_{r\theta} = \underline{\underline{Q}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{\underline{u}}_{xy} \quad (113)$$

$$\underline{\underline{F}}_{xy} = \underline{\underline{K}}_{xy} \underline{\underline{u}}_{xy} + \tilde{\underline{\underline{F}}}_{xy} = \underline{\underline{K}}_{xy} \underline{\underline{u}}_{xy} + \tilde{\underline{\underline{K}}}_N \underline{\underline{u}}_N \quad (114)$$

$$\begin{aligned} G_{TOT} &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \underline{\underline{u}}_{r\theta,p}^T \underline{\underline{T}}_{pq}^T \underline{\underline{F}}_{r\theta,q} = \\ &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{\underline{F}}_{r\theta,q} \underline{\underline{u}}_{r\theta,p}^T \underline{\underline{T}}_{pq}^T \right) = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\begin{bmatrix} t_{pq|11} F_{r,q} u_{r,p} & t_{pq|12} F_{r,q} u_{\theta,p} \\ t_{pq|21} F_{\theta,q} u_{r,p} & t_{pq|22} F_{\theta,q} u_{\theta,p} \end{bmatrix} \right) = \\ &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr \left(\underline{\underline{R}}_{\Delta\theta} \underline{\underline{F}}_{xy,q} \underline{\underline{u}}_{xy,p}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{R}}_{\delta}^T \underline{\underline{T}}_{pq}^T \right) \end{aligned} \quad (115)$$

where m is the order of the element's shape functions, and $p, q = 1$ represents the crack tip.

$$\begin{aligned}
T_{pq} &= \begin{cases} \underline{\underline{I}} & \text{for } p = q < 2 \\ \underline{\underline{0}} & \text{otherwise} \end{cases} && \text{for } 1^{st} \text{ order quadrilateral elements} \\
&= \begin{cases} \underline{\underline{I}} & \text{for } p = q < 3 \\ \underline{\underline{0}} & \text{otherwise} \end{cases} && \text{for } 2^{nd} \text{ order quadrilateral elements} \\
&= \begin{cases} \underline{\underline{I}} & \text{for } p = q < 4 \\ \underline{\underline{0}} & \text{otherwise} \end{cases} && \text{for } 3^{rd} \text{ order quadrilateral elements} \\
&= \begin{cases} \begin{pmatrix} (14 - \frac{33\pi}{8}) \underline{\underline{I}} & \text{for } p = 1, q = 1 \\ (-52 + \frac{33\pi}{2}) \underline{\underline{I}} & \text{for } p = 1, q = 2 \\ (17 - \frac{21\pi}{4}) \underline{\underline{I}} & \text{for } p = 2, q = 1 \\ (-\frac{7}{2} + \frac{21\pi}{16}) \underline{\underline{I}} & \text{for } p = 2, q = 2 \\ (8 - \frac{21\pi}{8}) \underline{\underline{I}} & \text{for } p = 1, q = 3 \\ (-32 + \frac{21\pi}{2}) \underline{\underline{I}} & \text{for } p = 2, q = 3 \\ \underline{\underline{0}} & \text{otherwise} \end{pmatrix} && \text{for } 2^{nd} \text{ order quarter-point quadrilateral elements} \\ \begin{pmatrix} (-11187 + \frac{7155\pi}{2}) \underline{\underline{I}} & \text{for } p = 1, q = 1 \\ (38556 - \frac{24543\pi}{2}) \underline{\underline{I}} & \text{for } p = 1, q = 2 \\ (-53055 + \frac{33777\pi}{2}) \underline{\underline{I}} & \text{for } p = 1, q = 3 \\ (\frac{11396}{3} - \frac{9575\pi}{8}) \underline{\underline{I}} & \text{for } p = 2, q = 1 \\ (-12936 + \frac{33003\pi}{8}) \underline{\underline{I}} & \text{for } p = 2, q = 2 \\ (17988 - \frac{45837\pi}{8}) \underline{\underline{I}} & \text{for } p = 2, q = 3 \\ (-\frac{8453}{3} + \frac{3595\pi}{4}) \underline{\underline{I}} & \text{for } p = 3, q = 1 \\ (9804 - \frac{12411\pi}{4}) \underline{\underline{I}} & \text{for } p = 3, q = 2 \\ (-13587 + \frac{17289\pi}{4}) \underline{\underline{I}} & \text{for } p = 3, q = 3 \\ (6948 - \frac{17685\pi}{8}) \underline{\underline{I}} & \text{for } p = 1, q = 4 \\ (-23976 + \frac{60993\pi}{8}) \underline{\underline{I}} & \text{for } p = 2, q = 4 \\ (33372 - \frac{84807\pi}{8}) \underline{\underline{I}} & \text{for } p = 3, q = 4 \\ \underline{\underline{0}} & \text{otherwise} \end{pmatrix} && \text{for } 3^{rd} \text{ order quarter-point quadrilateral elements} \end{cases} \quad (116)
\end{aligned}$$

$$\begin{aligned}
G_{TOT} &= \frac{1}{2R_f\delta} \underline{\underline{u}}_{r\theta}^T \underline{\underline{F}}_{r\theta} = \\
&= \frac{1}{2R_f\delta} Tr(\underline{\underline{F}}_{r\theta} \underline{\underline{u}}_{r\theta}^T) = \frac{1}{2R_f\delta} Tr\left(\begin{bmatrix} \underline{\underline{F}}_r \underline{\underline{u}}_r & \underline{\underline{F}}_r \underline{\underline{u}}_\theta \\ \underline{\underline{F}}_\theta \underline{\underline{u}}_r & \underline{\underline{F}}_\theta \underline{\underline{u}}_\theta \end{bmatrix}\right) = \\
&= \frac{1}{2R_f\delta} Tr(\underline{\underline{R}}_{\Delta\theta} \underline{\underline{F}}_{xy} \underline{\underline{u}}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{R}}_\delta^T) \quad (117)
\end{aligned}$$

$$\begin{aligned}
G_{TOT} &= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{R}}_{\Delta\theta} \left(\underline{\underline{K}}_{xy} \underline{\underline{u}}_{xy} + \tilde{\underline{\underline{F}}}_{xy}\right) \underline{\underline{u}}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{R}}_\delta^T\right) = \\
&= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{\underline{u}}_{xy} \underline{\underline{u}}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{R}}_\delta^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{R}}_{\Delta\theta} \tilde{\underline{\underline{F}}}_{xy} \underline{\underline{u}}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{R}}_\delta^T\right) \quad (118)
\end{aligned}$$

$$\underline{\underline{G}} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{R}}_{\Delta\theta} \underline{\underline{F}}_{xy} \underline{\underline{u}}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{R}}_\delta^T\right) \quad (119)$$

$$\begin{aligned}
\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} &= \frac{1}{2R_f\delta \sum_{p=1}^{m+1} \sum_{q=1}^{m+1}} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{F}}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) = \\
&= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right)
\end{aligned} \tag{120}$$

4.2. Sensitivity analysis

$$\begin{aligned}
\frac{\partial \underline{G}}{\partial \delta} &= -\frac{1}{2R_f\delta^2} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) - \frac{1}{2R_f\delta^2} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \\
&+ \frac{1}{2R_f \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \delta} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \underline{D}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \underline{D}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \frac{\partial \underline{u}_N}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \underline{u}_{xy} \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) = \\
&= \frac{1}{\delta} \underline{G} + \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\frac{1}{2R_f\delta} \underline{R}_{\Delta\theta} \left(\underline{K}_{xy} \underline{u}_{xy} + \tilde{\underline{K}}_N \underline{u}_N \right) \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \underline{D}^T \right) + \\
&+ \frac{1}{R_f\delta} \text{Diag} \left(\underline{R}_{\Delta\theta} \underline{K}_{xy} \frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \frac{\partial \underline{u}_N}{\partial \delta} \underline{u}_{xy}^T \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right) + \\
&+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \text{Diag} \left(\underline{R}_{\Delta\theta} \tilde{\underline{K}}_N \underline{u}_N \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{R}_{\Delta\theta}^T \underline{R}_{\delta}^T \right)
\end{aligned} \tag{121}$$

Following Comninou

$$u(\delta) \sim \sqrt{\delta} (\sin, \cos) (\epsilon \log \delta) \quad \epsilon = \frac{1}{2\pi} \log \left(\frac{1-\beta}{1+\beta} \right) \tag{122}$$

$$\underline{u}_{xy}, \underline{u}_N \sim u(\delta) \sim \sqrt{\delta} (\sin, \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} 0 \tag{123}$$

$$\underline{u}_{xy} \underline{u}_{xy}^T, \underline{u}_N \underline{u}_{xy}^T \sim u^2(\delta) \sim \delta (\sin^2, \cos^2, \sin \cdot \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} 0 \tag{124}$$

$$\frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T, \frac{\partial \underline{u}_N}{\partial \delta} \underline{u}_{xy}^T \sim -\frac{1}{2} (\sin^2, \cos^2, \sin \cdot \cos) (\epsilon \log \delta) + (-\sin^2, \cos^2, \pm \sin \cdot \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} \text{finite} \tag{125}$$

$$\underline{G} \sim \frac{1}{\delta} \underline{u}_{xy} \underline{u}_{xy}^T \sim \frac{1}{\delta} u^2(\delta) \sim (\sin^2, \cos^2, \sin \cdot \cos) (\epsilon \log \delta) \xrightarrow{\delta \rightarrow 0} \text{finite} \tag{126}$$

$$\lim_{\delta \rightarrow 0} \frac{\partial \underline{G}}{\partial \delta} \sim \frac{1}{\delta} \left(\underline{F}(\delta) \xrightarrow{0} + \underline{C} \right) \tag{127}$$

$$\lim_{\delta \rightarrow 0} \frac{\partial G}{\partial \delta} \sim \frac{1}{\delta} \xrightarrow{f d\delta} \lim_{\delta \rightarrow 0} G \sim A \log(\delta) + B \quad (128)$$

$$\begin{aligned} \frac{\partial G}{\partial \Delta \theta} = & \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{D}} \underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} u_{xy} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{D}} \underline{\underline{R}}_{\Delta \theta} \tilde{\underline{\underline{K}}}_N u_N u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \\ & + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} u_{xy} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{D}}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \tilde{\underline{\underline{K}}}_N u_N u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{D}}^T \underline{\underline{R}}_{\delta}^T \right) + \\ & + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} \frac{\partial u_{xy}}{\partial \Delta \theta} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \tilde{\underline{\underline{K}}}_N \frac{\partial u_N}{\partial \Delta \theta} u_{xy}^T \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \\ & + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \underline{\underline{K}}_{xy} u_{xy} \frac{\partial u_{xy}^T}{\partial \Delta \theta} \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) + \frac{1}{2R_f \delta} \text{Diag} \left(\underline{\underline{R}}_{\Delta \theta} \tilde{\underline{\underline{K}}}_N u_N \frac{\partial u_{xy}^T}{\partial \Delta \theta} \underline{\underline{R}}_{\Delta \theta}^T \underline{\underline{R}}_{\delta}^T \right) \end{aligned} \quad (129)$$

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