Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

Luca Di Stasio[a], Janis Varna[a]

[a]Luleå University of Technology, University Campus, SE-97187 Luleå, Sweden

Abstract

Keywords: Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

1. Introduction

Main ref [1]

2. Derivation of constitutive relations

2.1. Reference frames

5 **Local reference frame of k-th layer:** index 1 is the in-plane longitudinal or fiber or 0°-direction; index 2 is the in-plane transverse or 90°-direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0°direction; index y is the in-plane transverse direction; index 10 z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \tag{1}$$

V_a is equal to the product of total crack surface S_C and average crack opening u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \tag{2}$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90°}} = \frac{n_D \cancel{w}}{L_{lam} \cancel{w} t_{90°}} R_f \Delta\theta = \frac{1}{n2L} \frac{1}{k2L} R_f \Delta\theta =$$
$$= \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nkR_f} \frac{\Delta\theta}{\pi} \tag{3}$$

2.3. Vakulenko-Kachanov tensor

In the local reference frame of k-th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \tag{4}$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \tag{5}$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \tag{6}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$\beta_{11} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_1 + u_1 n_1\right) dS = \frac{1}{V_k} \int_{S_C} u_1 n_1^0 dS = 0$$

$$\beta_{22} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_2 + u_2 n_2\right) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS$$

$$\beta_{33} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_3 + u_3 n_3\right) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS$$

$$\beta_{12} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_2^0 + u_2 n_1^0\right) dS = 0$$

$$\beta_{13} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_3^0 + u_3 n_1^0\right) dS = 0 \qquad (7)$$

$$\beta_{23} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_3 + u_3 n_2\right) dS$$

$$\beta_{21} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_1^0 + u_1 n_2^0\right) dS = 0$$

$$\beta_{31} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_1^0 + u_1 n_3^0\right) dS = 0$$

$$\beta_{32} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_2 + u_2 n_3\right) dS = \beta_{23}$$

Only 3 independent components of the tensor β_{ij} remain: β_{22}, β_{33} and β_{23}. Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$\beta_{22} = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f \left(-n_2^m \right) dS \right]$$

$$\beta_{33} = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f \left(-n_3^m \right) dS \right]$$

$$\beta_{23} = \frac{1}{V_k} \int_{S_C} \left(u_2 n_3 + u_3 n_2 \right) dS =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} \left(u_2^m n_3^m + u_3^m n_2^m \right) dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f \right) dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f \left(-n_3^m \right) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f \left(-n_2^m \right) dS \right]$$

$$(8)$$

The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta\theta \qquad (9)$$

With Eq. 9, we can recast Eq. 8 as

$$\beta_{22} = \frac{1}{V_k}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right)n_2^m d\theta\right] =$$

$$=\frac{1}{L_{lam}wt_{90°}}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right)n_2^m d\theta\right] =$$

$$=\frac{1}{L_{lam}}\frac{n_D R_f}{t_{90°}}\left[\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right)n_2^m d\theta\right] =$$

$$=\rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right)n_2^m d\theta\right]$$

$$\beta_{33} = \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right)n_3^m d\theta\right]$$

$$\beta_{23} = \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right)n_3^m d\theta + \frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right)n_2^m d\theta\right]$$

(10)

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$u_2^m - u_2^f = \left(u_r^m - u_r^f\right)\cos\left(\theta\right) - \left(u_\theta^m - u_\theta^f\right)\sin\left(\theta\right) =$$

$$=COD\left(\theta\right)\cos\left(\theta\right) - CSD\left(\theta\right)\sin\left(\theta\right)$$

$$u_3^m - u_3^f = \left(u_r^m - u_r^f\right)\sin\left(\theta\right) + \left(u_\theta^m - u_\theta^f\right)\cos\left(\theta\right) =$$

$$=COD\left(\theta\right)\sin\left(\theta\right) + CSD\left(\theta\right)\cos\left(\theta\right)$$

(11)

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ:

$$n_2^m = \cos\left(\theta\right) - \sin\left(\theta\right)$$

$$n_3^m = \sin\left(\theta\right) + \cos\left(\theta\right)$$

(12)

Thus, Eq. 10 becomes

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(\cos^2(\theta) - \cos(\theta)\sin(\theta) \right) - CSD(\theta) \left(\sin(\theta)\cos(\theta) - \sin^2(\theta) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(1 + \cos(2\theta) - \sin(2\theta) \right) + CSD(\theta) \left(1 - \cos(2\theta) - \sin(2\theta) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} \left(1 - \sin(2\theta) \right) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(\sin(\theta)\cos(\theta) + \sin^2(\theta) \right) + CSD(\theta) \left(\cos^2(\theta) + \cos(\theta)\sin(\theta) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(1 + \sin(2\theta) - \cos(2\theta) \right) + CSD(\theta) \left(1 + \sin(2\theta) + \cos(2\theta) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} \left(1 + \sin(2\theta) \right) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) \left(2\sin(\theta)\cos(\theta) + \cos^2(\theta) - \sin^2(\theta) \right) +$$

$$- \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) \left(\sin^2(\theta) - \cos^2(\theta) + 2\cos(\theta)\sin(\theta) \right) d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta$$

$$(13)$$

2.4. Analytical modeling of $COD(\theta)$ and $CSD(\theta)$

$$COD(\theta) = COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max} f(\theta)$$
$$(14)$$
$$CSD(\theta) = CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max} g(\theta)$$

$$\int_0^{\Delta\theta} f(\theta) \, d\theta = 0 \qquad \int_0^{\Delta\theta} g(\theta) \, d\theta = 0 \qquad (15)$$

$$f(\theta) = \sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1} \quad g(\theta) = \sum_{k=0}^{n-1} b_{2k+1}\theta^{2k+1} \tag{16}$$

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2}(1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2}\cos(2\theta) \right] d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2}(1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2}\cos(2\theta) \right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2\left[\frac{COD(\theta) + CSD(\theta)}{2}\cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2}\sin(2\theta) \right] d\theta \tag{17}$$

$$\frac{1}{\Delta\theta} \int_0^{\min(\Delta\theta,\Delta\Phi)} COD(\theta)\, d\theta =$$

$$= \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1} \right) d\theta =$$

$$= \frac{1}{\Delta\theta} \left[COD_{avg}\theta + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)}\theta^{2(k+1)} \right] \Bigg|_0^{\min(\Delta\theta,\Delta\Phi)} = \tag{18}$$

$$= COD_{avg} + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \frac{\min(\Delta\theta,\Delta\Phi)^{2(k+1)}}{\Delta\theta}$$

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta)\, d\theta = CSD_{avg} + CSD_{max} \sum_{k=0}^{n-1} \frac{b_{2k+1}}{2(k+1)}\Delta\theta^{2k+1} \tag{19}$$

$$\frac{1}{\Delta\theta}\int_0^{\min(\Delta\theta,\Delta\Phi)} COD(\theta)\sin(2\theta)\,d\theta =$$

$$=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\theta^{2k+1}\right)\sin(2\theta)\,d\theta =$$

$$=\frac{1}{\Delta\theta}\left[-\frac{1}{2}COD_{avg}\cos(2\theta)+COD_{max}\sum_{k=0}^{n-1}\frac{a_{2k+1}}{2(k+1)}\theta^{2(k+1)}\right]\Bigg|_0^{\min(\Delta\theta,\Delta\Phi)} =$$

$$(20)$$

$$\int_0^{\Delta\theta}(1+\sin(2\theta)-\cos(2\theta))\,d\theta =\left[\theta-\frac{1}{2}\cos(2\theta)-\frac{1}{2}\sin(2\theta)\right]_0^{\Delta\theta} =$$

$$=\Delta\theta+\frac{1}{2}(1-\cos(2\Delta\theta))-\frac{1}{2}\sin(2\Delta\theta) =$$

$$=\Delta\theta-\frac{1}{2}\left(\sqrt{2}\sin\left(2\Delta\theta+\frac{\pi}{4}\right)-1\right)$$

$$(21)$$

$$\int_0^{\Delta\theta}f(\theta)(1+\sin(2\theta)-\cos(2\theta))\,d\theta =$$

$$=\cancelto{0}{\int_0^{\Delta\theta}f(\theta)\,d\theta}-\sqrt{2}\int_0^{\Delta\theta}f(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta$$

$$\int_0^{\Delta\theta}g(\theta)(1+\sin(2\theta)-\cos(2\theta))\,d\theta =$$

$$=\cancelto{0}{\int_0^{\Delta\theta}g(\theta)\,d\theta}-\sqrt{2}\int_0^{\Delta\theta}g(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta$$

$$(22)$$

$$\int_0^{\Delta\theta} f(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta =$$

$$= \left[\sum_{i=0}^{n}\left[\left(a_0\left(1-H\left(i-1\right)\right)+\sum_{k=0}^{n-1}a_{2k+1}\theta^{2k+1-i}\right)\frac{1}{2}\left(-1\right)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\theta\right)\right)\right]\right]_0^{\Delta\theta} =$$

$$= \sum_{i=0}^{n}\left[\left(a_0\left(1-H\left(i-1\right)\right)+\sum_{k=0}^{n-1}a_{2k+1}\Delta\theta^{2k+1-i}\right)\frac{1}{2}\left(-1\right)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right)\right] +$$

$$+ \frac{\sqrt{2}}{4}a_0$$

$$\int_0^{\Delta\theta} g(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta =$$

$$= \left[\sum_{i=0}^{n}\left[\left(b_0\left(1-H\left(i-1\right)\right)+\sum_{k=0}^{n-1}b_{2k+1}\theta^{2k+1-i}\right)\frac{1}{2}\left(-1\right)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\theta\right)\right)\right]\right]_0^{\Delta\theta} =$$

$$= \sum_{i=0}^{n}\left[\left(b_0\left(1-H\left(i-1\right)\right)+\sum_{k=0}^{n-1}b_{2k+1}\Delta\theta^{2k+1-i}\right)\frac{1}{2}\left(-1\right)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right)\right] +$$

$$+ \frac{\sqrt{2}}{4}b_0 \tag{23}$$

$$\beta_{22} = \rho_D\left[1-\frac{1}{2\Delta\theta}\left(\sqrt{2}\sin\left(2\Delta\theta+\frac{\pi}{4}\right)-1\right)\right]\left(COD_{avg}-CSD_{avg}\right) +$$

$$- \frac{1}{2}\rho_D\left(a_0 COD_{max}-b_0 CSD_{max}\right) +$$

$$- \sqrt{2}\sum_{i=0}^{n}\left(\left(a_0 COD_{max}-b_0 CSD_{max}\right)\left(1-H\left(i-1\right)\right)+\sum_{k=0}^{n-1}\left(a_{2k+1}COD_{max}-b_{2k+1}CSD_{max}\right)\Delta\theta^{2k+1-i}\right)\times$$

$$\times\frac{1}{2}\left(-1\right)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right) \tag{24}$$

40 **References**

[1] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), Comprehensive Composite Materials II, Elsevier, Oxford, 2018, pp. 192 – 220. doi:https://doi.org/10.1016/B978-0-12-803581-8.09910-0.