Constitutive modeling for laminates with fiber/matrix interface cracks under transverse loading

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Abstract

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1. Introduction

Review damage characterization [1, 2, 3]

Shear-lag [4]

Stress-transfer [5, 6]

Stress-based variational [7, 8, 9]+angle ply [9]

Displacement-based variational [10]

Self-consistent approach [11]

Finite strip approach [12]

Continuum damage mechanics [13, 14, 15]

Discrete damage mechanics [16]

Cylindrical model [17]

Reissner mixed variational principle [18]

Development of glob-loc model [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]

2. Derivation of constitutive relations

2.1. Reference frames

Local reference frame of k-th layer: index 1 is the in-plane longitudinal or fiber or 0°-direction; index 2 is the in-plane transverse or 90°-direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0°direction; index y is the in-plane transverse direction; index z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Scalar density Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \tag{1}$$

V_a is equal to the product of total crack surface S_C and average crack opening u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \tag{2}$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\begin{aligned}\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} &= \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \cancel{w}}{L_{lam} \cancel{w} t_{90^\circ}} R_f \Delta\theta = \frac{1}{n2L}\frac{1}{k2L} R_f \Delta\theta = \\ &= \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nkR_f}\frac{\Delta\theta}{\pi}\end{aligned} \tag{3}$$

Second-order tensor density

Fourth-order tensor density

2.3. Homogenization

$$\sigma_{ij}^{avg} = \frac{1}{V}\int_V \sigma_{ij} dV \quad \varepsilon_{ij}^{avg} = \frac{1}{V}\int_V \varepsilon_{ij} dV \tag{4}$$

$$\underset{\sim}{\tilde{\sigma}}_k^{avg} = \underset{\approx}{\tilde{Q}}_k \left(\underset{\sim}{\tilde{\varepsilon}}_k^{avg} - \underset{\sim}{\tilde{\alpha}}_k \Delta T\right) \tag{5}$$

$$\underset{\sim}{\tilde{\sigma}}_{LAM} = \underset{\sim}{\tilde{\sigma}}^{avg} = \frac{1}{V}\int_V \underset{\sim}{\tilde{\sigma}} dV = \frac{1}{V}\sum_{k=1}^{N}\int_{V_k} \underset{\sim}{\tilde{\sigma}} dV_k = \sum_{k=1}^{N} \underset{\sim}{\tilde{\sigma}}_k^{avg} \frac{t_k}{h} \tag{6}$$

$$\varepsilon_{ij}^{avg} = \frac{1}{V}\int_V \varepsilon_{ij} dV = \frac{1}{V}\int_V \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) dV = \frac{1}{V}\int_S \frac{1}{2}\left(u_i n_j + u_j n_i\right) dS \tag{7}$$

$$\begin{aligned}
\varepsilon_{ij}^{avg} =& \frac{1}{V}\int_S \frac{1}{2}\left(u_i n_j + u_j n_i\right) dS = \\
=& \frac{1}{V}\int_{S_B} \frac{1}{2}\left(u_i n_j + u_j n_i\right) dS + \frac{1}{V}\int_{S_C} \frac{1}{2}\left(u_i n_j + u_j n_i\right) dS = \\
=& \varepsilon_{ij}^{applied} + \beta_{ij}
\end{aligned} \tag{8}$$

$$\underset{\sim}{\tilde{\varepsilon}}_k^{avg} = \underset{\sim}{\tilde{\varepsilon}}_k^{applied} + \underset{\sim}{\tilde{\beta}}_k = \underset{\sim}{\tilde{\varepsilon}}_k^{LAM} + \underset{\sim}{\tilde{\beta}}_k \tag{9}$$

$$\underset{\sim}{\tilde{\sigma}}_{LAM} = \frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k \left(\underset{\sim}{\tilde{\varepsilon}}_k^{LAM} - \underset{\sim}{\tilde{\alpha}}_k \Delta T + \underset{\sim}{\tilde{\beta}}_k\right) \tag{10}$$

$$\underset{\sim}{\tilde{\sigma}}_{LAM} = \underset{\approx}{Q}_{eff}^{LAM} \underset{\sim}{\tilde{\varepsilon}}^{LAM} \tag{11}$$

$$\underset{\approx}{Q}_{eff}^{LAM} \underset{\sim}{\tilde{\varepsilon}}^{LAM} = \left(\frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k\right) \underset{\sim}{\tilde{\varepsilon}}^{LAM} + \frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k \underset{\sim}{\tilde{\beta}}_k \tag{12}$$

$$\underset{\approx}{Q}_{eff}^{LAM} \underset{\sim}{\tilde{\varepsilon}}^{LAM} \cdot \frac{\underset{\sim}{\tilde{\varepsilon}}^{LAM}}{\underset{\sim}{\tilde{\varepsilon}}^{LAM} \cdot \underset{\sim}{\tilde{\varepsilon}}^{LAM}} = \left(\frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k\right) \underset{\sim}{\tilde{\varepsilon}}^{LAM} \cdot \frac{\underset{\sim}{\tilde{\varepsilon}}^{LAM}}{\underset{\sim}{\tilde{\varepsilon}}^{LAM} \cdot \underset{\sim}{\tilde{\varepsilon}}^{LAM}} + \left(\frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k \underset{\sim}{\tilde{\beta}}_k\right) \cdot \frac{\underset{\sim}{\tilde{\varepsilon}}^{LAM}}{\underset{\sim}{\tilde{\varepsilon}}^{LAM} \cdot \underset{\sim}{\tilde{\varepsilon}}^{LAM}} \tag{13}$$

$$\underset{\approx}{Q}_{eff}^{LAM} = \left(\frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k\right) + \frac{1}{h}\sum_{k=1}^{N} t_k \underset{\approx}{\tilde{Q}}_k \frac{\underset{\sim}{\tilde{\beta}}_k \cdot \underset{\sim}{\tilde{\varepsilon}}^{LAM}}{\underset{\sim}{\tilde{\varepsilon}}^{LAM} \cdot \underset{\sim}{\tilde{\varepsilon}}^{LAM}} \tag{14}$$

$$\underset{\approx}{T}_k = \begin{bmatrix} \cos^2(\theta_k) & \sin^2(\theta_k) & 2\cos(\theta_k)\sin(\theta_k) \\ \sin^2(\theta_k) & \cos^2(\theta_k) & -2\cos(\theta_k)\sin(\theta_k) \\ -\cos(\theta_k)\sin(\theta_k) & \cos(\theta_k)\sin(\theta_k) & \cos^2(\theta_k) - \sin^2(\theta_k) \end{bmatrix} \tag{15}$$

$$\underline{\underline{\tilde{Q}}}_k = \underline{\underline{T}}_k^{-1} \underline{\underline{Q}}_k \left(\underline{\underline{T}}_k^{-1}\right)^T \tag{16}$$

$$\begin{aligned}\underline{\underline{Q}}^{LAM}_{eff} &= \left(\frac{1}{h}\sum_{k=1}^{N} t_k \underline{\underline{T}}_k^{-1} \underline{\underline{Q}}_k \left(\underline{\underline{T}}_k^{-1}\right)^T\right) + \frac{1}{h}\sum_{k=1}^{N} t_k \underline{\underline{T}}_k^{-1} \underline{\underline{Q}}_k \left(\underline{\underline{T}}_k^{-1}\right)^T \frac{\underline{\underline{T}}_k^{-1} \underline{\tilde{\beta}}_k \left(\underline{\underline{T}}_k^{-1}\right)^T \cdot \underline{\tilde{\varepsilon}}^{LAM}}{\underline{\tilde{\varepsilon}}^{LAM} \cdot \underline{\tilde{\varepsilon}}^{LAM}} = \\ &= \frac{1}{h}\sum_{k=1}^{N} t_k \underline{\underline{T}}_k^{-1} \underline{\underline{Q}}_k \left(\underline{\underline{T}}_k^{-1}\right)^T \left(1 + \frac{\underline{\underline{T}}_k^{-1} \underline{\tilde{\beta}}_k \left(\underline{\underline{T}}_k^{-1}\right)^T \cdot \underline{\tilde{\varepsilon}}^{LAM}}{\underline{\tilde{\varepsilon}}^{LAM} \cdot \underline{\tilde{\varepsilon}}^{LAM}}\right)\end{aligned} \tag{17}$$

$$Q^{LAM}_{eff,lmij} \tilde{\varepsilon}^{LAM}_{ij} = \left(\frac{1}{h}\sum_{k=1}^{N} t_k \tilde{Q}_{k,lmij}\right) \tilde{\varepsilon}^{LAM}_{ij} + \frac{1}{h}\sum_{k=1}^{N} t_k \tilde{Q}_{k,lmij} \tilde{\beta}_{k,ij} \tag{18}$$

$$Q^{LAM}_{eff,lmij} \tilde{\varepsilon}^{LAM}_{ij} \left(\tilde{\varepsilon}^{LAM}_{ij}\right)^{-1} = \left(\frac{1}{h}\sum_{k=1}^{N} t_k \tilde{Q}_{k,lmij}\right) \tilde{\varepsilon}^{LAM}_{ij} \left(\tilde{\varepsilon}^{LAM}_{ij}\right)^{-1} + \frac{1}{h}\left(\sum_{k=1}^{N} t_k \tilde{Q}_{k,lmij} \tilde{\beta}_{k,ij}\right) \left(\tilde{\varepsilon}^{LAM}_{ij}\right)^{-1} \tag{19}$$

$$Q^{LAM}_{eff,lmij} = \left(\frac{1}{h}\sum_{k=1}^{N} t_k \tilde{Q}_{k,lmij}\right) + \frac{1}{h}\sum_{k=1}^{N} t_k \left(\tilde{Q}_{k,lmij} \tilde{\beta}_{k,ij} \left(\tilde{\varepsilon}^{LAM}_{ij}\right)^{-1}\right) \tag{20}$$

$$Q^{LAM}_{eff,lmij} = \frac{1}{h}\sum_{k=1}^{N} t_k \tilde{Q}_{k,lmij} \left(\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \tilde{\beta}_{k,ij} \left(\tilde{\varepsilon}^{LAM}_{ij}\right)^{-1}\right) \tag{21}$$

$$\begin{aligned}&Q^{LAM}_{eff,lmij} = \\ &\frac{1}{h}\sum_{k=1}^{N} t_k T^{-1}_{k,lmrs} Q_{k,rspq} \left(\underline{\underline{T}}_k^{-1}\right)^T_{pqij} \left(\frac{1}{2}\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + T^{-1}_{k,ip} \tilde{\beta}_{k,pq} \left(\underline{\underline{T}}_k^{-1}\right)^T_{qj} T^{-1}_{k,lmrs} \left(\tilde{\varepsilon}^{LAM}_{pq}\right)^{-1} \left(\underline{\underline{T}}_k^{-1}\right)^T_{qj}\right)\end{aligned} \tag{22}$$

2.4. Exact expression of the Vakulenko-Kachanov tensor

In the local reference frame of k-th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \tag{23}$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \tag{24}$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \tag{25}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$\begin{aligned}
\beta_{11} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_1 + u_1 n_1 \right) dS = \frac{1}{V_k} \int_{S_C} \cancelto{0}{u_1} n_1 dS = 0 \\
\beta_{22} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_2 + u_2 n_2 \right) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\
\beta_{33} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_3 + u_3 n_3 \right) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\
\beta_{12} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancelto{0}{u_1} n_2 + u_2 \cancelto{0}{n_1} \right) dS = 0 \\
\beta_{13} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancelto{0}{u_1} n_3 + u_3 \cancelto{0}{n_1} \right) dS = 0 \\
\beta_{23} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_3 + u_3 n_2 \right) dS \\
\beta_{21} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 \cancelto{0}{n_1} + \cancelto{0}{u_1} n_2 \right) dS = 0 \\
\beta_{31} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 \cancelto{0}{n_1} + \cancelto{0}{u_1} n_3 \right) dS = 0 \\
\beta_{32} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_2 + u_2 n_3 \right) dS = \beta_{23}
\end{aligned} \tag{26}$$

Only 3 independent components of the tensor β_{ij} remain: β_{22}, β_{33} and β_{23}. Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f \left(-n_2^m\right) dS \right] \\
\beta_{33} =& \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f \left(-n_3^m\right) dS \right] \\
\beta_{23} =& \frac{1}{V_k} \int_{S_C} \left(u_2 n_3 + u_3 n_2\right) dS = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} \left(u_2^m n_3^m + u_3^m n_2^m\right) dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f\right) dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f \left(-n_3^m\right) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f \left(-n_2^m\right) dS \right]
\end{aligned}
\tag{27}
$$

The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$
S_C^m = S_C^f = n_D R_f \Delta\theta \tag{28}
$$

With Eq. 28, we can recast Eq. 27 as

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \frac{1}{L_{lam} w t_{90^\circ}}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \frac{1}{L_{lam}}\frac{n_D R_f}{t_{90^\circ}}\left[\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] \\
\beta_{33} =& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right) n_3^m d\theta\right] \\
\beta_{23} =& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_3^m d\theta + \frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right) n_2^m d\theta\right]
\end{aligned}
\tag{29}
$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$
\begin{aligned}
u_2^m - u_2^f =& \left(u_r^m - u_r^f\right)\cos(\theta) - \left(u_\theta^m - u_\theta^f\right)\sin(\theta) = \\
=& COD(\theta)\cos(\theta) - CSD(\theta)\sin(\theta) \\
u_3^m - u_3^f =& \left(u_r^m - u_r^f\right)\sin(\theta) + \left(u_\theta^m - u_\theta^f\right)\cos(\theta) = \\
=& COD(\theta)\sin(\theta) + CSD(\theta)\cos(\theta)
\end{aligned}
\tag{30}
$$

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ:

$$
\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned}
\tag{31}
$$

Thus, Eq. 29 becomes

$$
\begin{aligned}
&\beta_{22} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(\cos^2\left(\theta\right) - \cos\left(\theta\right)\sin\left(\theta\right)\right) - CSD\left(\theta\right)\left(\sin\left(\theta\right)\cos\left(\theta\right) - \sin^2\left(\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(1 + \cos\left(2\theta\right) - \sin\left(2\theta\right)\right) + CSD\left(\theta\right)\left(1 - \cos\left(2\theta\right) - \sin\left(2\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\cos\left(2\theta\right)\right] d\theta \\
&\beta_{33} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(\sin\left(\theta\right)\cos\left(\theta\right) + \sin^2\left(\theta\right)\right) + CSD\left(\theta\right)\left(\cos^2\left(\theta\right) + \cos\left(\theta\right)\sin\left(\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(1 + \sin\left(2\theta\right) - \cos\left(2\theta\right)\right) + CSD\left(\theta\right)\left(1 + \sin\left(2\theta\right) + \cos\left(2\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\cos\left(2\theta\right)\right] d\theta \\
&\beta_{23} = \\
&\quad = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right)\left(2\sin\left(\theta\right)\cos\left(\theta\right) + \cos^2\left(\theta\right) - \sin^2\left(\theta\right)\right) + \\
&\qquad - \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD\left(\theta\right)\left(\sin^2\left(\theta\right) - \cos^2\left(\theta\right) + 2\cos\left(\theta\right)\sin\left(\theta\right)\right) d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2\left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\sin\left(2\theta\right)\right] d\theta
\end{aligned}
\tag{32}
$$

2.5. Analytical modeling of $COD\left(\theta\right)$ and $CSD\left(\theta\right)$

The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of θ, the angular coordinate along the crack which varies between 0 and $\Delta\theta$. Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be expressed as the sum of their average value and a term, respectively $\delta COD\left(\theta\right)$

and $\delta CSD\left(\theta\right)$, that represents the variation of the function from its average:

$$\begin{aligned} COD\left(\theta\right) &= COD_{avg} + \delta COD\left(\theta\right) \\ CSD\left(\theta\right) &= CSD_{avg} + \delta CSD\left(\theta\right). \end{aligned} \tag{33}$$

By defining $\Delta\Psi$

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right), \tag{34}$$

we introduce at this point an approximation and assume that the functions $\delta COD\left(\theta\right)$ and $\delta CSD\left(\theta\right)$ can be expressed as the product of the maximum value of the displacement and a function, respectively $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$:

$$\begin{aligned} COD\left(\theta\right) &= COD_{avg} + \delta COD\left(\theta\right) = COD_{avg} + COD_{max} f\left(\theta - \frac{\Delta\Psi}{2}\right) \\ CSD\left(\theta\right) &= CSD_{avg} + \delta CSD\left(\theta\right) = CSD_{avg} + CSD_{max} g\left(\theta - \frac{\Delta\theta}{2}\right), \end{aligned} \tag{35}$$

where $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ are assumed to be odd functions over their respective integration domain $[0, \Delta\Psi]$ and $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = 0 \qquad \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = 0. \tag{36}$$

We assume the two functions $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ to be two odd polynomials of degree $2n-1$:

$$\begin{aligned} f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \begin{cases} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} & 0 \leq \theta \leq \Delta\Psi \\ 0 \quad otherwise & \end{cases} \\ g\left(\theta - \frac{\Delta\theta}{2}\right) &= \begin{cases} \sum_{k=0}^{n-1} b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} & 0 \leq \theta \leq \Delta\theta \\ 0 \quad otherwise & \end{cases} \end{aligned} \tag{37}$$

which satisfy by construction the conditions expressed in Equation 36. The coefficients a_{2k+1} and b_{2k+1} are determined by imposing that

$$\begin{aligned} COD\left(\Delta\Psi\right) &= 0 \\ CSD\left(\Delta\theta\right) &= 0. \end{aligned} \tag{38}$$

The explicit construction of the polynomials $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ for $n = 1, 2, 3$ (or degree $2n - 1 = 1, 3, 5$) is reported in Appendix A.

We recall the expressions of the non-zero components of the Vakulenko-Kachanov tensor

$$
\begin{aligned}
&\beta_{22} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta \\
&\beta_{33} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta \\
&\beta_{23} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \sin\left(2\theta\right) \right] d\theta
\end{aligned}
\tag{39}
$$

and proceed to the integration of the different summands:

1.

$$
\begin{aligned}
&\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD\left(\theta\right)d\theta = \\
&=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}\right)d\theta = \\
&=\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD_{avg}d\theta + \\
&+\frac{1}{\Delta\theta}\int_0^{\Delta\Psi} COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1} d\theta + \\
&+\cancelto{0}{\frac{1}{\Delta\theta}\int_{\Delta\Psi}^{\Delta\theta} COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1} d\theta} \quad = \\
&=\frac{1}{\Delta\theta}\left[COD_{avg}\theta\right]\Bigg|_0^{\Delta\theta}+\frac{1}{\Delta\theta}\left[COD_{max}\sum_{k=0}^{n-1}\frac{a_{2k+1}}{2\left(k+1\right)}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2(k+1)}\right]\Bigg|_0^{\Delta\Psi} = \\
&=COD_{avg}+COD_{max}\sum_{k=0}^{n-1}\frac{a_{2k+1}}{2\left(k+1\right)\Delta\theta}\left(\left(\frac{\Delta\Psi}{2}\right)^{2(k+1)}-\left(-\frac{\Delta\Psi}{2}\right)^{2(k+1)}\right) = \\
&=COD_{avg}
\end{aligned}
\tag{40}
$$

2.

$$
\frac{1}{\Delta\theta}\int_0^{\Delta\theta} CSD\left(\theta\right)d\theta = CSD_{avg} \tag{41}
$$

3.

$$
\begin{aligned}
&\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD\left(\theta\right)\sin\left(2\theta\right)d\theta = \\
&=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\theta^{2k+1}\right)\sin\left(2\theta\right)d\theta = \\
&=-\frac{1}{2\Delta\theta}COD_{avg}\left[\cos\left(2\theta\right)\right]\Bigg|_0^{\Delta\theta} + \\
&+\frac{1}{\Delta\theta}\left[COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin\left(\frac{1+mod\left(i,2\right)}{2}\pi-2\theta\right)\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{(k-i)!}\theta^{k-i}\right)\right]\Bigg|_0^{\Delta\Psi} = \\
&=\frac{1-\cos\left(2\Delta\theta\right)}{2\Delta\theta}COD_{avg} + \\
&+\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin\left(\frac{1+mod\left(i,2\right)}{2}\pi-2\Delta\Psi\right)\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right)
\end{aligned}
\tag{42}
$$

4.

$$
\begin{aligned}
&\frac{1}{\Delta\theta}\int_0^{\Delta\theta} CSD\,(\theta)\sin{(2\theta)}\,d\theta = \\
=&\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(CSD_{avg}+CSD_{max}\sum_{k=0}^{n-1} b_{2k+1}\theta^{2k+1}\right)\sin{(2\theta)}\,d\theta = \\
=&-\frac{1}{2\Delta\theta}CSD_{avg}\left[\cos{(2\theta)}\right]\Bigg|_0^{\Delta\theta} + \\
+&\frac{1}{\Delta\theta}\left[CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1+mod\,(i,2)}{2}\pi-2\theta)\left(\sum_{k=i}^{2n-1} b_k\frac{k!}{(k-i)!}\theta^{k-i}\right)\right]\Bigg|_0^{\Delta\Psi} = \\
=&\frac{1-\cos{(2\Delta\theta)}}{2\Delta\theta}CSD_{avg}+ \\
+&\frac{1}{\Delta\theta}CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1+mod\,(i,2)}{2}\pi-2\Delta\Psi)\left(\sum_{k=i}^{2n-1} b_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right)
\end{aligned}
\tag{43}
$$

5.

$$
\begin{aligned}
&\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD\,(\theta)\cos{(2\theta)}\,d\theta = \\
=&\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1}\right)\cos{(2\theta)}\,d\theta = \\
=&\frac{1}{2\Delta\theta}COD_{avg}\left[\sin{(2\theta)}\right]\Bigg|_0^{\Delta\theta} + \\
+&\frac{1}{\Delta\theta}\left[COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i}\sin(\frac{1+mod\,(i+1,2)}{2}\pi-2\theta)\left(\sum_{k=i}^{2n-1} a_k\frac{k!}{(k-i)!}\theta^{k-i}\right)\right]\Bigg|_0^{\Delta\Psi} = \\
=&\frac{\sin{(2\Delta\theta)}}{2\Delta\theta}COD_{avg}+ \\
+&\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i}\sin(\frac{1+mod\,(i+1,2)}{2}\pi-2\Delta\Psi)\left(\sum_{k=i}^{2n-1} a_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right)
\end{aligned}
\tag{44}
$$

6.

$$
\begin{aligned}
&\frac{1}{\Delta\theta}\int_0^{\Delta\theta} CSD\left(\theta\right)\cos\left(2\theta\right)d\theta = \\
=&\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(CSD_{avg}+CSD_{max}\sum_{k=0}^{n-1} b_{2k+1}\theta^{2k+1}\right)\cos\left(2\theta\right)d\theta = \\
=&-\frac{1}{2\Delta\theta}CSD_{avg}\left[\cos\left(2\theta\right)\right]\Bigg|_0^{\Delta\theta}+ \\
+&\frac{1}{\Delta\theta}\left[CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^i\sin(\frac{1+mod\left(i+1,2\right)}{2}\pi-2\theta)\left(\sum_{k=i}^{2n-1} b_k\frac{k!}{(k-i)!}\theta^{k-i}\right)\right]\Bigg|_0^{\Delta\Psi} = \\
=&\frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}CSD_{avg}+ \\
+&\frac{1}{\Delta\theta}CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^i\sin(\frac{1+mod\left(i+1,2\right)}{2}\pi-2\Delta\Psi)\left(\sum_{k=i}^{2n-1} b_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right)
\end{aligned}
\tag{45}
$$

Finally we can compute the expressions of the components of the Vakulenko-Kachanov tensor:

1. β_{22}

$$
\begin{aligned}
&\beta_{22} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\,(\theta) + CSD\,(\theta)}{2} \left(1 - \sin\left(2\theta\right)\right) + \frac{COD\,(\theta) - CSD\,(\theta)}{2} \cos\left(2\theta\right) \right] d\theta \\
&= \frac{\rho_D}{2} \left(COD_{avg} + CSD_{avg}\right) + \\
&- \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} - \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&+ \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&- \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) = \\
&= \frac{\rho_D}{2} COD_{avg} \left(1 - \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} + \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} CSD_{avg} \left(1 - \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} - \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) \right) \cdot \\
&\qquad\qquad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&- \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) \right) \cdot \\
&\qquad\qquad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)
\end{aligned}
\tag{46}
$$

2. β_{33}

$$
\begin{aligned}
&\beta_{33} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\,(\theta) + CSD\,(\theta)}{2} \left(1 + \sin\left(2\theta\right)\right) - \frac{COD\,(\theta) - CSD\,(\theta)}{2} \cos\left(2\theta\right) \right] d\theta = \\
&= \frac{\rho_D}{2} \left(COD_{avg} + CSD_{avg}\right) + \\
&+ \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} + \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&- \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&+ \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) = \\
&= \frac{\rho_D}{2} COD_{avg} \left(1 + \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} - \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} CSD_{avg} \left(1 - \frac{1 + \cos\left(2\Delta\theta\right)}{2\Delta\theta} + \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) \right) \cdot \\
&\qquad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&+ \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) \right) \cdot \\
&\qquad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)
\end{aligned} \tag{47}
$$

3. β_{23}

$$
\begin{aligned}
&\beta_{23} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD\,(\theta) + CSD\,(\theta)}{2} \cos\left(2\theta\right) + \frac{COD\,(\theta) - CSD\,(\theta)}{2} \sin\left(2\theta\right) \right] d\theta = \\
&= \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} - \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) = \\
&= \frac{\rho_D}{2} COD_{avg} \left(\frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} + \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}\right) + \\
&- \frac{\rho_D}{2} CSD_{avg} \left(\frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} - \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right)\right) \cdot \\
&\qquad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right)\right) \cdot \\
&\qquad \cdot \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right)
\end{aligned}
\tag{48}
$$

2.6. Application to UD composite

2.7. Application to $[0^\circ_{mk}, 90^\circ_k, 0^\circ_{mk}]$ laminate

3. Computations with the Finite Element Method (FEM)

3.1. Models of Representative Volume Element (RVE)

3.2. Details of FEM implementation

4. Results and discussion

5. Conclusions

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Appendix A. Explicit expressions for $f(\theta)$ and $g(\theta)$

In the following, recall that

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right). \tag{A.1}$$

n = 1

$$\begin{aligned} f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^{0} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1\left(\theta - \frac{\Delta\Psi}{2}\right) \\ g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^{0} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1\left(\theta - \frac{\Delta\theta}{2}\right) \end{aligned} \tag{A.2}$$

$$\begin{aligned} \int_0^{\Delta\Psi} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta &= \int_0^{\Delta\Psi} a_1\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \left[\frac{a_1}{2}\theta^2 - a_1\frac{\Delta\Psi}{2}\theta\right]\Bigg|_0^{\Delta\Psi} = 0 \quad \forall a_1 \\ \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta &= \int_0^{\Delta\theta} b_1\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \left[\frac{b_1}{2}\theta^2 - b_1\frac{\Delta\theta}{2}\theta\right]\Bigg|_0^{\Delta\theta} = 0 \quad \forall b_1 \end{aligned} \tag{A.3}$$

$$COD_{avg} + COD_{max} a_1 \left(\Delta\Psi - \frac{\Delta\Psi}{2} \right) = 0 \rightarrow a_1 = -\frac{2}{\Delta\Psi} \frac{COD_{avg}}{COD_{max}}$$
$$CSD_{avg} + CSD_{max} b_1 \left(\Delta\theta - \frac{\Delta\theta}{2} \right) = 0 \rightarrow b_1 = -\frac{2}{\Delta\theta} \frac{CSD_{avg}}{CSD_{max}} \tag{A.4}$$

$$\begin{aligned}
&\sum_{i=0}^{1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=0}^{1-i} a_k \left(1-(k+1)\right)!\Delta\Psi^k\right) = \\
&= \left(-\frac{1}{2}\right)^{1} \sin\left(\frac{1+mod\,(0,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=0}^{1} a_k \left(1-(k+1)\right)!\Delta\Psi^k\right) + \\
&+ \left(-\frac{1}{2}\right)^{2} \sin\left(\frac{1+mod\,(1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=0}^{0} a_k \left(1-(k+1)\right)!\Delta\Psi^k\right) = \\
&= -\frac{1}{2}\cos\left(2\Delta\Psi\right) \left(\sum_{k=0}^{1} a_k \left(1-(k+1)\right)!\Delta\Psi^k\right) + \\
&\quad + \frac{1}{4}\sin\left(2\Delta\Psi\right) \left(\sum_{k=0}^{0} a_k \left(1-(k+1)\right)!\Delta\Psi^k\right) = \\
&= -\frac{1}{2}\cos\left(2\Delta\Psi\right) \left(a_0 + a_1 \left(n-(k+1)\right)!\Delta\Psi^k\right) + \\
&\quad + \frac{1}{4}\sin\left(2\Delta\Psi\right) \left(\sum_{k=0}^{0} a_k \left(n-(k+1)\right)!\Delta\Psi^k\right) =
\end{aligned} \tag{A.5}$$

n = 2

$$f\left(\theta - \frac{\Delta\Psi}{2}\right) = \sum_{k=0}^{1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) + a_3 \left(\theta - \frac{\Delta\Psi}{2}\right)^3$$
$$g\left(\theta - \frac{\Delta\theta}{2}\right) = \sum_{k=0}^{1} b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1 \left(\theta - \frac{\Delta\theta}{2}\right) + b_3 \left(\theta - \frac{\Delta\theta}{2}\right)^3 \tag{A.6}$$

$\mathbf{n = 3}$

$$\begin{aligned} f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^{2} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = \\ &= a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) + a_3 \left(\theta - \frac{\Delta\Psi}{2}\right)^3 + a_5 \left(\theta - \frac{\Delta\Psi}{2}\right)^5 \\ g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^{1} b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = \\ &= b_1 \left(\theta - \frac{\Delta\theta}{2}\right) + b_3 \left(\theta - \frac{\Delta\theta}{2}\right)^3 + b_5 \left(\theta - \frac{\Delta\theta}{2}\right)^5 \end{aligned} \tag{A.7}$$