Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

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Abstract

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1. Introduction

Main ref [1]

2. Derivation of constitutive relations

2.1. Reference frames

5   **Local reference frame of k-th layer:**   index 1 is the in-plane longitudinal or fiber or 0°-direction; index 2 is the in-plane transverse or 90°-direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0°direction; index y is the in-plane transverse direction; index
10   z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \qquad (1)$$

V_a is equal to the product of total crack surface S_C and average crack opening u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \tag{2}$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \not{w}}{L_{lam} \not{w} t_{90^\circ}} R_f \Delta\theta = \frac{1}{n2L} \frac{1}{k2L} R_f \Delta\theta =$$
$$= \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nkR_f} \frac{\Delta\theta}{\pi} \tag{3}$$

2.3. Vakulenko-Kachanov tensor

In the local reference frame of k-th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \tag{4}$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \tag{5}$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \tag{6}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

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$$\beta_{11} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_1 + u_1 n_1 \right) dS = \frac{1}{V_k} \int_{S_C} u_1 n_1^0 dS = 0$$

$$\beta_{22} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_2 + u_2 n_2 \right) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS$$

$$\beta_{33} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_3 + u_3 n_3 \right) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS$$

$$\beta_{12} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_2^0 + u_2 n_1^0 \right) dS = 0$$

$$\beta_{13} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_3^0 + u_3 n_1^0 \right) dS = 0 \tag{7}$$

$$\beta_{23} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_3 + u_3 n_2 \right) dS$$

$$\beta_{21} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_1^0 + u_1 n_2^0 \right) dS = 0$$

$$\beta_{31} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_1^0 + u_1 n_3^0 \right) dS = 0$$

$$\beta_{32} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_2 + u_2 n_3 \right) dS = \beta_{23}$$

Only 3 independent components of the tensor β_{ij} remain: β_{22}, β_{33} and β_{23}.
Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$\beta_{22} = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f \left(-n_2^m \right) dS \right]$$

$$\beta_{33} = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f \left(-n_3^m \right) dS \right]$$

$$\beta_{23} = \frac{1}{V_k} \int_{S_C} \left(u_2 n_3 + u_3 n_2 \right) dS =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} \left(u_2^m n_3^m + u_3^m n_2^m \right) dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f \right) dS \right] =$$

$$= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f \left(-n_3^m \right) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f \left(-n_2^m \right) dS \right]$$

$$(8)$$

The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta\theta \tag{9}$$

With Eq. 9, we can recast Eq. 8 as

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$$\beta_{22} = \frac{1}{V_k} \left[n_D R_f w \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] =$$

$$= \frac{1}{L_{lam} w t_{90°}} \left[n_D R_f w \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] =$$

$$= \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90°}} \left[\int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] =$$

$$= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right]$$

$$\beta_{33} = \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_3^m - u_3^f \right) n_3^m d\theta \right]$$

$$\beta_{23} = \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_3^m - u_3^f \right) n_2^m d\theta \right]$$

(10)

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$u_2^m - u_2^f = \left(u_r^m - u_r^f \right) \cos(\theta) - \left(u_\theta^m - u_\theta^f \right) \sin(\theta) =$$

$$= COD(\theta)\cos(\theta) - CSD(\theta)\sin(\theta)$$

$$u_3^m - u_3^f = \left(u_r^m - u_r^f \right) \sin(\theta) + \left(u_\theta^m - u_\theta^f \right) \cos(\theta) =$$

$$= COD(\theta)\sin(\theta) + CSD(\theta)\cos(\theta)$$

(11)

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ:

$$n_2^m = \cos(\theta) - \sin(\theta)$$

$$n_3^m = \sin(\theta) + \cos(\theta)$$

(12)

Thus, Eq. 10 becomes

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(\sin\left(\theta\right)\cos\left(\theta\right) - \sin^2\left(\theta\right) \right) + CSD\left(\theta\right) \left(\cos^2\left(\theta\right) - \cos\left(\theta\right)\sin\left(\theta\right) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left(COD\left(\theta\right) - CSD\left(\theta\right) \right) \left(-1 + \sin\left(2\theta\right) - \cos\left(2\theta\right) \right) d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left(COD\left(\theta\right) - CSD\left(\theta\right) \right) \left(\sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right) - 1 \right) d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(\sin\left(\theta\right)\cos\left(\theta\right) + \sin^2\left(\theta\right) \right) + CSD\left(\theta\right) \left(\cos^2\left(\theta\right) + \cos\left(\theta\right)\sin\left(\theta\right) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(1 + \sin\left(2\theta\right) - \cos\left(2\theta\right) \right) + CSD\left(\theta\right) \left(1 + \sin\left(2\theta\right) + \cos\left(2\theta\right) \right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[\left(COD\left(\theta\right) + CSD\left(\theta\right) \right) \left(1 + \sin\left(2\theta\right) \right) - \left(COD\left(\theta\right) - CSD\left(\theta\right) \right) \cos\left(2\theta\right) \right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right) \left(2\sin\left(\theta\right)\cos\left(\theta\right) + \cos^2\left(\theta\right) - \sin^2\left(\theta\right) \right) +$$

$$- \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD\left(\theta\right) \left(\sin^2\left(\theta\right) - \cos^2\left(\theta\right) + 2\cos\left(\theta\right)\sin\left(\theta\right) \right) d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\left(COD\left(\theta\right) - CSD\left(\theta\right) \right) \sin\left(2\theta\right) + \left(COD\left(\theta\right) + CSD\left(\theta\right) \right) \cos\left(2\theta\right) \right] d\theta$$

$$(13)$$

2.4. Analytical modeling of $COD\left(\theta\right)$ and $CSD\left(\theta\right)$

$$COD\left(\theta\right) = COD_{avg} + \delta COD\left(\theta\right) = COD_{avg} + COD_{max} f\left(\theta\right)$$

$$CSD\left(\theta\right) = CSD_{avg} + \delta CSD\left(\theta\right) = CSD_{avg} + CSD_{max} g\left(\theta\right)$$

$$(14)$$

$$\int_0^{\Delta\theta} f\left(\theta\right) d\theta = 0 \qquad \int_0^{\Delta\theta} g\left(\theta\right) d\theta = 0 \qquad (15)$$

$$f(\theta) = a_0 + \sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1} \quad g(\theta) = b_0 + \sum_{k=0}^{n-1} b_{2k+1}\theta^{2k+1} \tag{16}$$

$$COD(\Delta\Phi) = COD_{avg} + COD_{max}\left(a_0 + \sum_{k=0}^{n-1} a_{2k+1}\Delta\Phi^{2k+1}\right) = 0$$

$$CSD(\Delta\Theta) = CSD_{avg} + CSD_{max}\left(b_0 + \sum_{k=0}^{n-1} b_{2k+1}\Delta\Theta^{2k+1}\right) = 0 \tag{17}$$

$$a_0 = -\left(\frac{COD_{avg}}{COD_{max}} + \sum_{k=0}^{n-1} a_{2k+1}\Delta\Phi^{2k+1}\right)$$

$$b_0 = -\left(\frac{CSD_{avg}}{CSD_{max}} + \sum_{k=0}^{n-1} b_{2k+1}\Delta\Theta^{2k+1}\right) \tag{18}$$

$$\beta_{22} = \rho_D COD_{avg}\frac{1}{\Delta\theta}\int_0^{\Delta\theta}(1 + \sin(2\theta) - \cos(2\theta))\,d\theta +$$

$$+ \rho_D COD_{max}\frac{1}{\Delta\theta}\int_0^{\Delta\theta} f(\theta)(1 + \sin(2\theta) - \cos(2\theta))\,d\theta +$$

$$- \rho_D CSD_{avg}\frac{1}{\Delta\theta}\int_0^{\Delta\theta}(1 + \sin(2\theta) - \cos(2\theta))\,d\theta + \tag{19}$$

$$- \rho_D CSD_{max}\frac{1}{\Delta\theta}\int_0^{\Delta\theta} g(\theta)(1 + \sin(2\theta) - \cos(2\theta))\,d\theta$$

$$\int_0^{\Delta\theta}(1 + \sin(2\theta) - \cos(2\theta))\,d\theta = \left[\theta - \frac{1}{2}\cos(2\theta) - \frac{1}{2}\sin(2\theta)\right]_0^{\Delta\theta} =$$

$$= \Delta\theta + \frac{1}{2}(1 - \cos(2\Delta\theta)) - \frac{1}{2}\sin(2\Delta\theta) =$$

$$= \Delta\theta - \frac{1}{2}\left(\sqrt{2}\sin\left(2\Delta\theta + \frac{\pi}{4}\right) - 1\right) \tag{20}$$

$$\int_0^{\Delta\theta} f(\theta)\left(1+\sin(2\theta)-\cos(2\theta)\right)d\theta =$$

$$= \underbrace{\int_0^{\Delta\theta} f(\theta)\,d\theta}_{0} - \sqrt{2}\int_0^{\Delta\theta} f(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta$$

$$\int_0^{\Delta\theta} g(\theta)\left(1+\sin(2\theta)-\cos(2\theta)\right)d\theta =$$

$$= \underbrace{\int_0^{\Delta\theta} g(\theta)\,d\theta}_{0} - \sqrt{2}\int_0^{\Delta\theta} g(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta$$

$$(21)$$

$$\int_0^{\Delta\theta} f(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta =$$

$$= \left[\sum_{i=0}^{n}\left[\left(a_0(1-H(i-1))+\sum_{k=0}^{n-1}a_{2k+1}\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\theta\right)\right)\right]\right]_0^{\Delta\theta} =$$

$$= \sum_{i=0}^{n}\left[\left(a_0(1-H(i-1))+\sum_{k=0}^{n-1}a_{2k+1}\Delta\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right)\right] +$$

$$+\frac{\sqrt{2}}{4}a_0$$

$$\int_0^{\Delta\theta} g(\theta)\sin\left(\frac{\pi}{4}-2\theta\right)d\theta =$$

$$= \left[\sum_{i=0}^{n}\left[\left(b_0(1-H(i-1))+\sum_{k=0}^{n-1}b_{2k+1}\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\theta\right)\right)\right]\right]_0^{\Delta\theta} =$$

$$= \sum_{i=0}^{n}\left[\left(b_0(1-H(i-1))+\sum_{k=0}^{n-1}b_{2k+1}\Delta\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i\sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right)\right] +$$

$$+\frac{\sqrt{2}}{4}b_0$$

$$(22)$$

$$\beta_{22} = \rho_D \left[1 - \frac{1}{2\Delta\theta} \left(\sqrt{2}\sin\left(2\Delta\theta + \frac{\pi}{4}\right) - 1 \right) \right] (COD_{avg} - CSD_{avg}) +$$
$$- \frac{1}{2} \rho_D \left(a_0 COD_{max} - b_0 CSD_{max} \right) +$$
$$- \sqrt{2} \sum_{i=0}^{n} \left(\left(a_0 COD_{max} - b_0 CSD_{max} \right) \left(1 - H\left(i-1\right) \right) + \sum_{k=0}^{n-1} \left(a_{2k+1} COD_{max} - b_{2k+1} CSD_{max} \right) \Delta\theta^{2k+1-i} \right) \times$$
$$\times \frac{1}{2} (-1)^i \sin\left(\left(1 - \frac{1}{2} \bmod\left(\frac{i+1}{2}\right) \right) \pi - \left(\frac{\pi}{4} - 2\Delta\theta\right) \right)$$
$$\tag{23}$$

References

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