Computing energy release rates using the Virtual Crack Closure Technique with the Finite Element Method: analytical discussion

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Abstract. The effect of debond size and crack tip orientation of mode splitting in the Virtual Crack Closure Technique is analyzed by means of analytical derivations. The total energy release rate is shown to have no direct dependence on the debond angular size, but only an indirect one through the FEM solution of the crack displacement field in the crack tip neighbourhood.

List of acronyms

VCCT	Virtual Crack Closure Technique
BEM	Boundary Element Method
FEM	Finite Element Method

List of symbols

G_I	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate
G_{II}	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate
G_{TOT}	$\left[\frac{J}{m^2}\right]$	Total energy release rate
$G_{I,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate in $r-\theta$ reference frame
$G_{II,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate in $r-\theta$ reference frame
$G_{TOT,r\theta}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate in $r-\theta$ reference frame
$\widetilde{G}_{I,xy}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate of equivalent crack in $x-y$ reference frame
$\widetilde{G}_{II,xy}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate of equivalent crack in $x-y$ reference frame
$\widetilde{G}_{TOT,xy}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate of equivalent crack in $x-y$ reference frame
R_f	$[\mu m]$	Fiber radius
a	$[\mu m]$	Debond size
Δa	$[\mu m]$	Debond increment
$\Delta\theta$	$[rad]$	Half debond angular size
δ	$[rad]$	Angular size of element at the interface close to the crack tip
$u_{x,[A-Z]}$	$[\mu m]$	Displacement along x of a point labeled with a letter in [A-Z]
$u_{y,[A-Z]}$	$[\mu m]$	Displacement along y of a point labeled with a letter in [A-Z]
u_x	$[\mu m]$	Displacement along x-direction
u_y	$[\mu m]$	Displacement along y-direction
u_r	$[\mu m]$	Displacement along r-direction
u_θ	$[\mu m]$	Displacement along θ-direction
$F_{x,[A-Z]}$	$[\mu m]$	Force along x at a point labeled with a letter in [A-Z]
$F_{y,[A-Z]}$	$[\mu m]$	Force along y at a point labeled with a letter in [A-Z]
F_x	$[\mu m]$	Force along x-direction
F_y	$[\mu m]$	Force along y-direction
F_r	$[\mu m]$	Force along r-direction
F_θ	$[\mu m]$	Force along θ-direction
$\underline{\underline{R}}$	$[-]$	Rotation matrix

1. FEM formulation with quadrilateral elements

$$\Pi\left(u_i\right) = \int_V \sigma_{ij}\varepsilon_{ij} dV + \int_V \rho \ddot{u}_i u_i dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \tag{1}$$

$$\Pi\left(u_i\right) = \int_V \sigma_{ij}\varepsilon_{ij} dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

$$\underline{\varepsilon}\left(x, y\right) = \underline{\underline{\widetilde{B}}} \cdot \underline{u}\left(x, y\right) \tag{4}$$

$$\underline{\underline{\widetilde{B}}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{5}$$

$$\underline{\sigma} = \underline{\underline{D}} \cdot \underline{\varepsilon} \tag{6}$$

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{7}$$

$$\underline{\underline{D}} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{bmatrix} \qquad \text{with } G = \frac{E}{2\left(1+\nu\right)} \text{ for an isotropic material} \tag{8}$$

$$\begin{aligned} E_1 &= \frac{E}{1-\nu^2} \quad E_2 = \nu E_1 \quad \text{for plane stress} \\ E_1 &= \frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)} \quad E_2 = \frac{\nu E_1}{1-\nu} \quad \text{for plane strain} \end{aligned} \tag{9}$$

$$\begin{aligned} \Pi\left(\underline{u}\right) =& \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{\underline{D}} \cdot \underline{\varepsilon} dV - \int_V \underline{F}^T \underline{u} dV - \int_S \underline{f}^T \underline{u} dS = \\ =& \frac{1}{2}\int_V \underline{u}^T \underline{\underline{\widetilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\widetilde{B}}} \cdot \underline{u} dV - \int_V \underline{F}^T \underline{u} dV - \int_S \underline{f}^T \underline{u} dS \end{aligned} \tag{10}$$

$$\delta\Pi\left(\delta\underline{u}\right) = 0 \tag{11}$$

$$
\begin{aligned}
\delta\Pi\left(\delta\underline{u}\right) =& \Pi\left(\underline{u}+\delta\underline{u}\right)-\Pi\left(\underline{u}\right)= \\
=&\frac{1}{2}\int_V\left(\underline{u}+\underline{\delta u}\right)^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\left(\underline{u}+\underline{\delta u}\right)dV-\int_V\underline{F}^T\left(\underline{u}+\underline{\delta u}\right)dV-\int_S\underline{f}^T\left(\underline{u}+\underline{\delta u}\right)dS+ \\
&-\frac{1}{2}\int_V\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV+\int_V\underline{F}^T\underline{u}dV+\int_S\underline{f}^T\underline{u}dS= \\
=&\cancel{\frac{1}{2}\int_V\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV}+\cancelto{\approx 0}{\frac{1}{2}\int_V\underline{\delta u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{\delta u}dV}+ \\
&+\frac{1}{2}\int_V\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{\delta u}dV+\frac{1}{2}\int_V\underline{\delta u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV+ \\
&-\cancel{\int_V\underline{F}^T\underline{u}dV}-\int_V\underline{F}^T\underline{\delta u}dV+ \\
&-\cancel{\int_S\underline{f}^T\underline{u}dS}-\int_S\underline{f}^T\underline{\delta u}dS+ \\
&-\cancel{\frac{1}{2}\int_V\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV}+\cancel{\int_V\underline{F}^T\underline{u}dV}+\cancel{\int_S\underline{f}^T\underline{u}dS}= \\
=&\int_V\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{\delta u}dV-\int_V\underline{F}^T\underline{\delta u}dV-\int_S\underline{f}^T\underline{\delta u}dS
\end{aligned}
\tag{12}
$$

$$
\int_V\underline{\delta u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV-\int_V\underline{\delta u}^T\underline{F}dV-\int_S\underline{\delta u}^T\underline{f}dS=0 \tag{13}
$$

$$
\underline{u}=\begin{bmatrix}u_x\left(x,y\right)\\u_y\left(x,y\right)\end{bmatrix}\qquad\underline{u}_N=\begin{bmatrix}u_{1,x}\\u_{1,y}\\u_{2,x}\\u_{2,y}\\u_{3,x}\\u_{3,y}\\u_{4,x}\\u_{4,y}\end{bmatrix}\ \text{or}\ \underline{u}_N=\begin{bmatrix}u_{1,x}\\u_{1,y}\\u_{2,x}\\u_{2,y}\\u_{3,x}\\u_{3,y}\\u_{4,x}\\u_{4,y}\\u_{5,x}\\u_{5,y}\\u_{6,x}\\u_{6,y}\\u_{7,x}\\u_{7,y}\\u_{8,x}\\u_{8,y}\end{bmatrix} \tag{14}
$$

$$
\underline{u}=\underline{\underline{N}}\cdot\underline{u}_N \tag{15}
$$

$$
\underline{\underline{N}}=\begin{bmatrix}N_1&0&N_2&0&N_3&0&N_4&0\\0&N_1&0&N_2&0&N_3&0&N_4\end{bmatrix} \tag{16}
$$

$$
\underline{\underline{N}}=\begin{bmatrix}N_1&0&N_2&0&N_3&0&N_4&0&N_5&0&N_6&0&N_7&0&N_8&0\\0&N_1&0&N_2&0&N_3&0&N_4&0&N_5&0&N_6&0&N_7&0&N_8\end{bmatrix} \tag{17}
$$

$$\begin{cases} N_1 = N_1(\xi,\eta) \\ N_2 = N_2(\xi,\eta) \\ N_3 = N_3(\xi,\eta) \\ N_4 = N_4(\xi,\eta) \end{cases} \quad \text{with} \quad \begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases} \quad \text{for isoparametric elements} \tag{18}$$

$$\underline{\underline{B}} = \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \tag{19}$$

$$\begin{aligned} \underline{\underline{B}} =& \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} = \\ =& \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_5}{\partial x} & 0 & \frac{\partial N_6}{\partial x} & 0 & \frac{\partial N_7}{\partial x} & 0 & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} & 0 & \frac{\partial N_5}{\partial y} & 0 & \frac{\partial N_6}{\partial y} & 0 & \frac{\partial N_7}{\partial y} & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial y} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial y} & \frac{\partial N_6}{\partial x} & \frac{\partial N_7}{\partial y} & \frac{\partial N_7}{\partial x} & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{bmatrix} \end{aligned} \tag{20}$$

$$\delta \underline{u} = \delta\left(\underline{\underline{N}} \cdot \underline{u_N}\right) = \underline{\underline{N}} \delta \underline{u}_N \tag{21}$$

$$\int_V \underline{\delta u_N}^T \underline{\underline{N}}^T \underline{\underline{\widetilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} \cdot \underline{u_N} dV - \int_V \underline{\delta u_N}^T \underline{\underline{N}}^T \underline{F} dV - \int_S \underline{\delta u_N}^T \underline{\underline{N}}^T \underline{f} dS = 0 \tag{22}$$

$$\underline{\delta u_N}^T \left(\int_V \underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}} dV \cdot \underline{u_N} - \int_V \underline{\underline{N}}^T \underline{F} dV - \int_S \underline{\underline{N}}^T \underline{f} dS \right) = 0 \tag{23}$$

$$\underline{\underline{k}} \cdot \underline{u}_N = \underline{F}_N \quad \underline{\underline{k}} = \int_V \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}} \right) dV \quad \underline{F}_N = \int_V \underline{\underline{N}}^T \underline{F} dV + \int_S \underline{\underline{N}}^T \underline{f} dS \tag{24}$$

$$\begin{cases} N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{cases} \qquad \begin{cases} N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \\ N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\ N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\ N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \\ N_5(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1-\eta) \\ N_6(\xi,\eta) = \frac{1}{2}(1+\xi)\left(1-\eta^2\right) \\ N_7(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1+\eta) \\ N_8(\xi,\eta) = \frac{1}{2}(1-\xi)\left(1-\eta^2\right) \end{cases} \tag{25}$$

$$\begin{cases} \frac{\partial N_1(\xi,\eta)}{\partial \xi} = -\frac{1}{4}(1-\eta) \\ \frac{\partial N_2(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1-\eta) \\ \frac{\partial N_3(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1+\eta) \\ \frac{\partial N_4(\xi,\eta)}{\partial \xi} = -\frac{1}{4}(1+\eta) \\ \frac{\partial N_1(\xi,\eta)}{\partial \eta} = -\frac{1}{4}(1-\xi) \\ \frac{\partial N_2(\xi,\eta)}{\partial \eta} = -\frac{1}{4}(1+\xi) \\ \frac{\partial N_3(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1+\xi) \\ \frac{\partial N_4(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1-\xi) \end{cases} \tag{26}$$

$$
\begin{cases}
\frac{\partial N_1(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)\left(-\xi-\eta-1\right) \\
\frac{\partial N_2(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)\left(\xi-\eta-1\right) \\
\frac{\partial N_3(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\left(\xi+\eta-1\right) \\
\frac{\partial N_4(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\left(-\xi+\eta-1\right) \\
\frac{\partial N_5(\xi,\eta)}{\partial \xi} = \frac{1}{2}\left(1-\xi^2\right)\left(1-\eta\right) \\
\frac{\partial N_6(\xi,\eta)}{\partial \xi} = \frac{1}{2}\left(1+\xi\right)\left(1-\eta^2\right) \\
\frac{\partial N_7(\xi,\eta)}{\partial \xi} = \frac{1}{2}\left(1-\xi^2\right)\left(1+\eta\right) \\
\frac{\partial N_8(\xi,\eta)}{\partial \xi} = \frac{1}{2}\left(1-\xi\right)\left(1-\eta^2\right) \\
\frac{\partial N_1(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right)\left(-\xi-\eta-1\right) \\
\frac{\partial N_2(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right)\left(\xi-\eta-1\right) \\
\frac{\partial N_3(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right)\left(\xi+\eta-1\right) \\
\frac{\partial N_4(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right)\left(-\xi+\eta-1\right) \\
\frac{\partial N_5(\xi,\eta)}{\partial \eta} = \frac{1}{2}\left(1-\xi^2\right)\left(1-\eta\right) \\
\frac{\partial N_6(\xi,\eta)}{\partial \eta} = \frac{1}{2}\left(1+\xi\right)\left(1-\eta^2\right) \\
\frac{\partial N_7(\xi,\eta)}{\partial \eta} = \frac{1}{2}\left(1-\xi^2\right)\left(1+\eta\right) \\
\frac{\partial N_8(\xi,\eta)}{\partial \eta} = \frac{1}{2}\left(1-\xi\right)\left(1-\eta^2\right)
\end{cases}
\tag{27}
$$

$$
\underline{p} = \underline{\underline{N}} \cdot \underline{p}_N \tag{28}
$$

$$
\underline{p} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \text{ or } \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \\ x_5 \\ y_5 \\ x_6 \\ y_6 \\ x_7 \\ y_7 \\ x_8 \\ y_8 \end{bmatrix} \tag{29}
$$

$$
\begin{aligned}
x = x\left(\xi,\eta\right) = N_1\left(\xi,\eta\right)x_1 + N_2\left(\xi,\eta\right)x_2 + N_3\left(\xi,\eta\right)x_3 + N_4\left(\xi,\eta\right)x_4 \\
y = y\left(\xi,\eta\right) = N_1\left(\xi,\eta\right)y_1 + N_2\left(\xi,\eta\right)y_2 + N_3\left(\xi,\eta\right)y_3 + N_4\left(\xi,\eta\right)y_4
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
x = x\left(\xi,\eta\right) =& N_1\left(\xi,\eta\right)x_1 + N_2\left(\xi,\eta\right)x_2 + N_3\left(\xi,\eta\right)x_3 + N_4\left(\xi,\eta\right)x_4 + \\
&+ N_5\left(\xi,\eta\right)x_5 + N_6\left(\xi,\eta\right)x_6 + N_7\left(\xi,\eta\right)x_7 + N_8\left(\xi,\eta\right)x_8 \\
y = y\left(\xi,\eta\right) =& N_1\left(\xi,\eta\right)y_1 + N_2\left(\xi,\eta\right)y_2 + N_3\left(\xi,\eta\right)y_3 + N_4\left(\xi,\eta\right)y_4 + \\
&+ N_5\left(\xi,\eta\right)y_5 + N_6\left(\xi,\eta\right)y_6 + N_7\left(\xi,\eta\right)y_7 + N_8\left(\xi,\eta\right)y_8
\end{aligned}
\tag{31}
$$

$$\begin{cases} \frac{\partial N_1(\xi,\eta)}{\partial x} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_1(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_2(\xi,\eta)}{\partial x} = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right) \\ \frac{\partial N_3(\xi,\eta)}{\partial x} = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right) \\ \frac{\partial N_4(\xi,\eta)}{\partial x} = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right) \\ \frac{\partial N_1(\xi,\eta)}{\partial y} = \frac{1}{4}\left(1-\xi\right)\left(1-\eta\right) \\ \frac{\partial N_2(\xi,\eta)}{\partial y} = \frac{1}{4}\left(1+\xi\right)\left(1-\eta\right) \\ \frac{\partial N_3(\xi,\eta)}{\partial y} = \frac{1}{4}\left(1+\xi\right)\left(1+\eta\right) \\ \frac{\partial N_4(\xi,\eta)}{\partial y} = \frac{1}{4}\left(1-\xi\right)\left(1+\eta\right) \end{cases} \tag{32}$$

$$\underline{e}_\xi = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} \quad \underline{e}_\eta = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{33}$$

$$\underline{e}_x = \begin{bmatrix} \frac{\partial xi}{\partial x} \\ \frac{\partial eta}{\partial x} \end{bmatrix} \quad \underline{e}_y = \begin{bmatrix} \frac{\partial xi}{\partial y} \\ \frac{\partial eta}{\partial y} \end{bmatrix} \tag{34}$$

$$\underline{\underline{J}} = \left[\underline{e}_\xi | \underline{e}_\eta\right] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{35}$$

$$\underline{\underline{g}} = \underline{\underline{J}}^T \underline{\underline{J}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta} \end{bmatrix} \tag{36}$$

$$\begin{aligned} g =& det\left(\underline{\underline{g}}\right) = \\ =& \left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right) \end{aligned} \tag{37}$$

$$dV\left(x,y\right) = \sqrt{g} dV\left(\xi,\eta\right) \tag{38}$$

$$\underline{\underline{k_e}} = \int_{V_e(x,y)} \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}\right) dV_e\left(x,y\right) = \int_{V_e(\xi,\eta)} \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}\right) \sqrt{g} dV_e\left(\xi,\eta\right) \tag{39}$$

$$\begin{cases} k_{e|11} = \\ k_{e|12} = \\ k_{e|22} = \\ k_{e|21} = k_{e|12} \end{cases} \tag{40}$$

2. VCCT for first order quadrilateral elements

2.1. Definition of crack tip reference frame

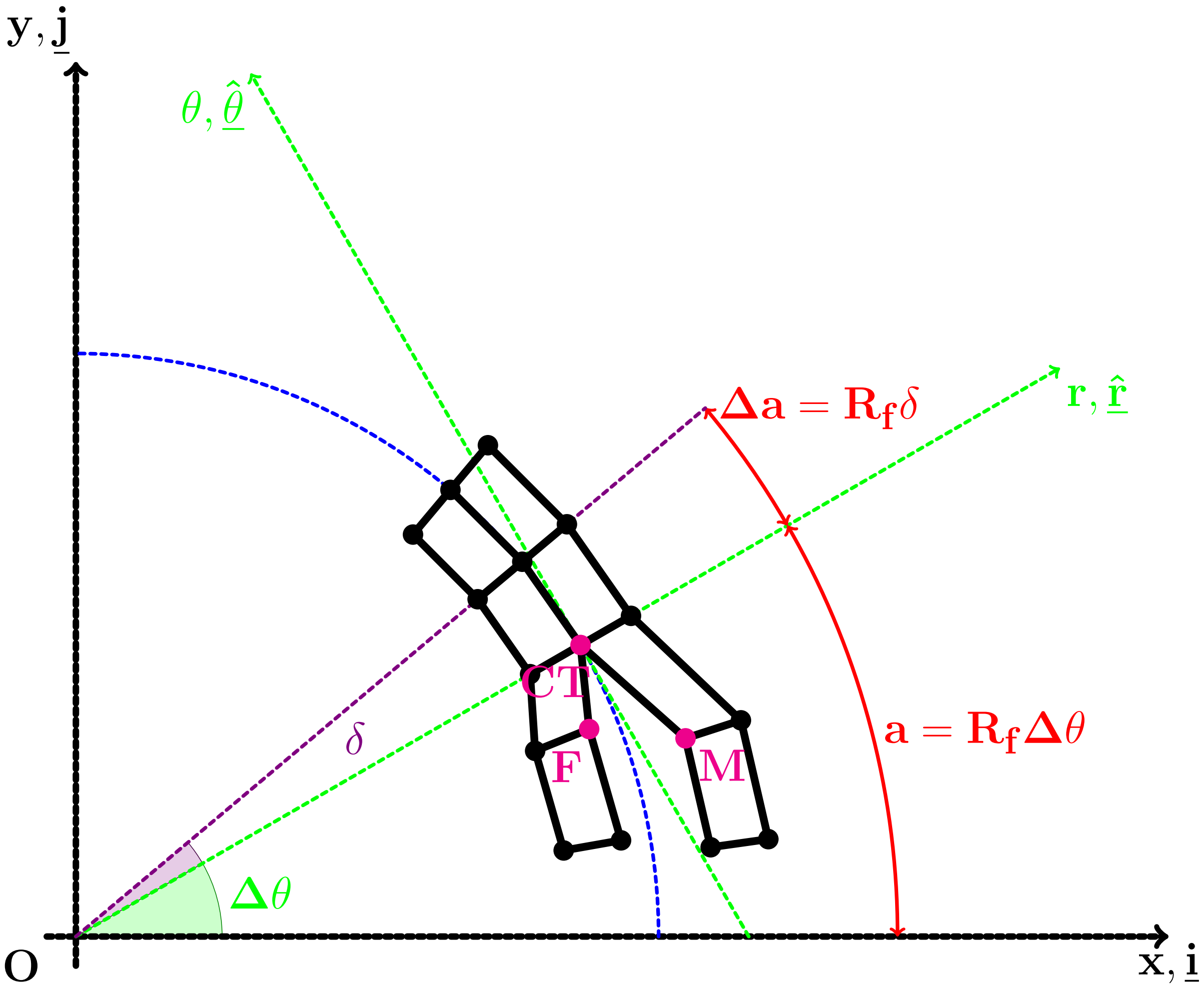

Figure 1. Schematic representation of the discretized crack tip geometry for 1^{st} order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^{T} = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{41}$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \tag{42}$$

2.2. Calculation of displacements and reaction forces

$$u_x = u_{x,M} - u_{x,F} \qquad u_y = u_{y,M} - u_{y,F} \tag{43}$$

$$u_r = \cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y \qquad u_\theta = -\sin(\Delta\theta)\, u_x + \cos(\Delta\theta)\, u_y \tag{44}$$

$$F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT} \tag{45}$$

2.3. Calculation of energy release rates

$$\begin{aligned}
G_{I,r\theta} =& \frac{1}{2}\frac{F_r u_r}{R_f \delta} = \\
=& \frac{1}{2R_f\delta}\left(\cos\left(\Delta\theta\right)F_x + \sin\left(\Delta\theta\right)F_y\right)\left(\cos\left(\Delta\theta\right)u_x + \sin\left(\Delta\theta\right)u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right)
\end{aligned} \tag{46}$$

$$\begin{aligned}
G_{II,r\theta} =& \frac{1}{2}\frac{F_\theta u_\theta}{R_f \delta} = \\
=& \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right)F_x + \cos\left(\Delta\theta\right)F_y\right)\left(-\sin\left(\Delta\theta\right)u_x + \cos\left(\Delta\theta\right)u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right)
\end{aligned} \tag{47}$$

$$\begin{aligned}
G_{TOT,r\theta} =& G_{I,r\theta} + G_{II,r\theta} = \\
=& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) + \\
&+ \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)} F_x u_x\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cancelto{0}{\left(\left(F_x u_y + F_y u_x\right) - \left(F_x u_y + F_y u_x\right)\right)} \cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)} F_y u_y\right) = \\
=& \frac{1}{2}\frac{F_x u_x}{R_f\delta} + \frac{1}{2}\frac{F_y u_y}{R_f\delta} = \\
=& \widetilde{G}_{I,xy} + \widetilde{G}_{II,xy} = \widetilde{G}_{TOT,xy}
\end{aligned} \tag{48}$$

$$\begin{aligned}
G_{I,r\theta} =& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) = \\
&= \cos^2\left(\Delta\theta\right)\frac{F_x u_x}{2R_f\delta} + \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\frac{F_y u_y}{2R_f\delta} = \\
&= \cos^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} + \left(\widetilde{G}_{I,xy}\frac{u_y}{u_x} + \widetilde{G}_{II,xy}\frac{u_x}{u_y}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\widetilde{G}_{II,xy}
\end{aligned} \tag{49}$$

$$
\begin{aligned}
G_{II,r\theta} =& \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta)\,F_x u_x - (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_y\right) = \\
=& \sin^2(\Delta\theta)\,\frac{F_x u_x}{2R_f\delta} - \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,\frac{F_y u_y}{2R_f\delta} = \\
=& \sin^2(\Delta\theta)\,\widetilde{G}_{I,xy} - \left(\widetilde{G}_{I,xy}\frac{u_y}{u_x} + \widetilde{G}_{II,xy}\frac{u_x}{u_y}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,\widetilde{G}_{II,xy}
\end{aligned}
\tag{50}
$$

2.4. Sensitivity analysis of the FEM solution

$$
F_x \sim k_x u_x \qquad F_y \sim k_y u_y \tag{51}
$$

$$
\begin{aligned}
G_{I,r\theta} \sim& \frac{1}{2R_f\delta}\cos^2(\Delta\theta)\,k_x u_x^2(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}(k_x + k_y)\,u_x(\Delta\theta)\,u_y(\Delta\theta)\cos(\Delta\theta)\sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}\sin^2(\Delta\theta)\,k_y u_y^2(\Delta\theta)
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
G_{II,r\theta} \sim& \frac{1}{2R_f\delta}\sin^2(\Delta\theta)\,k_x u_x^2(\Delta\theta) + \\
&- \frac{1}{2R_f\delta}(k_x + k_y)\,u_x(\Delta\theta)\,u_y(\Delta\theta)\cos(\Delta\theta)\sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}\cos^2(\Delta\theta)\,k_y u_y^2(\Delta\theta)
\end{aligned}
\tag{53}
$$

$$
G_{TOT,r\theta} \sim \frac{1}{2R_f\delta}\left(k_x u_x^2(\Delta\theta) + k_y u_y^2(\Delta\theta)\right) \tag{54}
$$

$$
\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial\Delta\theta} \sim& \frac{1}{R_f\delta}\cos^2(\Delta\theta)\,k_x u_x(\Delta\theta)\,\frac{\partial u_x(\Delta\theta)}{\partial\Delta\theta} + \\
&+ \frac{1}{2R_f\delta}(k_x + k_y)\left(\frac{\partial u_x(\Delta\theta)}{\partial\Delta\theta}u_y(\Delta\theta) + u_x(\Delta\theta)\,\frac{\partial u_y(\Delta\theta)}{\partial\Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \\
&+ \frac{1}{R_f\delta}\sin^2(\Delta\theta)\,k_y u_y(\Delta\theta)\,\frac{\partial u_y(\Delta\theta)}{\partial\Delta\theta} + \\
&+ \frac{1}{2R_f\delta}\left(k_y u_y^2(\Delta\theta) - k_x u_x^2(\Delta\theta)\right)\sin(2\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}\left(k_x u_x(\Delta\theta)\,u_y(\Delta\theta) + k_y u_y(\Delta\theta)\,u_x(\Delta\theta)\right)\cos(2\Delta\theta)
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta}\sin^2(\Delta\theta)\, k_x u_x(\Delta\theta)\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + \\
& - \frac{1}{2R_f\delta}(k_x+k_y)\left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} u_y(\Delta\theta) + u_x(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \\
& + \frac{1}{R_f\delta}\cos^2(\Delta\theta)\, k_y u_y(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \\
& + \frac{1}{2R_f\delta}\left(k_x u_x^2(\Delta\theta) - k_y u_y^2(\Delta\theta)\right)\sin(2\Delta\theta) + \\
& - \frac{1}{2R_f\delta}\left(k_x u_x(\Delta\theta)\, u_y(\Delta\theta) + k_y u_y(\Delta\theta)\, u_x(\Delta\theta)\right)\cos(2\Delta\theta)
\end{aligned}
\tag{56}
$$

$$
\frac{\partial G_{TOT,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f\delta}\left(k_x u_x(\Delta\theta)\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + k_y u_y(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta}\right)
\tag{57}
$$

2.5. Discretization error

$$
u_r = \cos(\Delta\theta-\delta)\, u_x + \sin(\Delta\theta-\delta)\, u_y \qquad u_\theta = -\sin(\Delta\theta-\delta)\, u_x + \cos(\Delta\theta-\delta)\, u_y
\tag{58}
$$

$$
F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT}
\tag{59}
$$

$$
\begin{aligned}
\widetilde{G}_{I,r\theta} =& \frac{1}{2}\frac{F_r u_r}{R_f\delta} = \\
=& \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\cos(\Delta\theta-\delta)\, u_x + \sin(\Delta\theta-\delta)\, u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y\right)\cos(\delta) + \\
& + \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\sin(\Delta\theta)\, u_x - \cos(\Delta\theta)\, u_y\right)\sin(\delta) = \\
=& \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta)\, F_x u_x + (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_y\right)\cos(\delta) + \\
& + \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)\, F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_x\right)\sin(\delta) = \\
=& G_{I,r\theta}\cos(\delta) + \\
& + \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)\, F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_x\right)\sin(\delta)
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
G_{II,r\theta} =& \frac{1}{2}\frac{F_\theta u_\theta}{R_f\delta} = \\
=& \frac{1}{2R_f\delta}\left(-\sin(\Delta\theta)\, F_x + \cos(\Delta\theta)\, F_y\right)\left(-\sin(\Delta\theta-\delta)\, u_x + \cos(\Delta\theta-\delta)\, u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta)\, F_x u_x - (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\, F_y u_y\right)
\end{aligned}
\tag{61}
$$

2.6. Contact region

$$u_r = 0 \tag{62}$$

$$\cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y = 0 \tag{63}$$

$$u_y = -\frac{u_x}{\tan(\Delta\theta)} \tag{64}$$

$$\begin{aligned} u_\theta =& -\sin(\Delta\theta)\, u_x - \frac{\cos^2(\Delta\theta)}{\sin(\Delta\theta)} u_x = \\ =& -\frac{u_x}{\sin(\Delta\theta)} \end{aligned} \tag{65}$$

$$F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT} \tag{66}$$

$$G_{I,r\theta} = \frac{1}{2}\frac{F_r u_r}{R_f \delta} = 0 \tag{67}$$

$$\begin{aligned} G_{II,r\theta} =& \frac{1}{2}\frac{F_\theta u_\theta}{R_f \delta} = \\ =& \frac{1}{2R_f\delta}\left(-\sin(\Delta\theta)\, F_x + \cos(\Delta\theta)\, F_y\right)\left(-\frac{u_x}{\sin(\Delta\theta)}\right) = \\ =& \frac{1}{2R_f\delta}\left(F_x u_x - \frac{F_y u_x}{\tan(\Delta\theta)}\right) \\ =& \frac{1}{2R_f\delta}\left(F_x - \frac{F_y}{\tan(\Delta\theta)}\right) u_x \end{aligned} \tag{68}$$

3. VCCT for second order quadrilateral elements

3.1. Definition of crack tip reference frame

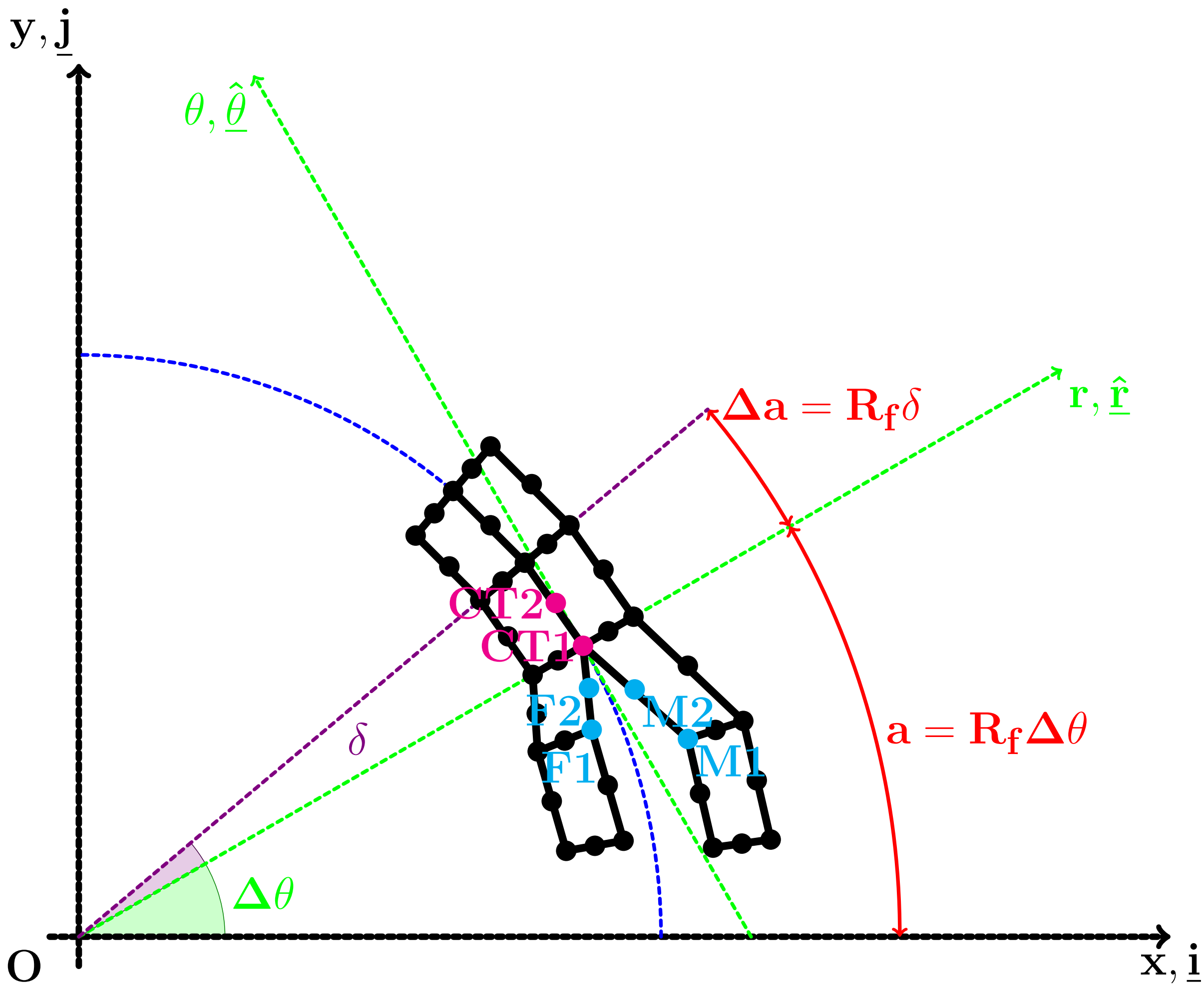

Figure 2. Schematic representation of the discretized crack tip geometry for 2^{nd} order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^{T} = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{69}$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \tag{70}$$

3.2. Calculation of displacements and reaction forces

$$\begin{aligned} u_{x,1} &= u_{x,M1} - u_{x,F1} \qquad & u_{y,1} &= u_{y,M1} - u_{y,F1} \\ u_{x,2} &= u_{x,M2} - u_{x,F2} \qquad & u_{y,2} &= u_{y,M2} - u_{y,F2} \end{aligned} \tag{71}$$

$$\begin{aligned} u_{r,1} &= \cos(\Delta\theta)\, u_{x,1} + \sin(\Delta\theta)\, u_{y,1} \qquad & u_{\theta,1} &= -\sin(\Delta\theta)\, u_{x,1} + \cos(\Delta\theta)\, u_{y,1} \\ u_{r,2} &= \cos(\Delta\theta)\, u_{x,2} + \sin(\Delta\theta)\, u_{y,2} \qquad & u_{\theta,2} &= -\sin(\Delta\theta)\, u_{x,2} + \cos(\Delta\theta)\, u_{y,2} \end{aligned} \tag{72}$$

$$
\begin{aligned}
F_{r,1} &= \cos\left(\Delta\theta\right) F_{x,1} + \sin\left(\Delta\theta\right) F_{y,1} \qquad & F_{\theta,1} &= -\sin\left(\Delta\theta\right) F_{x,1} + \cos\left(\Delta\theta\right) F_{y,1} \\
F_{r,2} &= \cos\left(\Delta\theta\right) F_{x,2} + \sin\left(\Delta\theta\right) F_{y,2} \qquad & F_{\theta,2} &= -\sin\left(\Delta\theta\right) F_{x,2} + \cos\left(\Delta\theta\right) F_{y,2}
\end{aligned}
\tag{73}
$$

3.3. Calculation of energy release rates

$$
\begin{aligned}
G_{I,r\theta} =& \frac{1}{2R_f\delta}\left(F_{r,1}u_{r,1} + F_{r,2}u_{r,2}\right) = \\
=& \frac{1}{2R_f\delta}\left(\cos\left(\Delta\theta\right) F_{x,1} + \sin\left(\Delta\theta\right) F_{y,1}\right)\left(\cos\left(\Delta\theta\right) u_{x,1} + \sin\left(\Delta\theta\right) u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cos\left(\Delta\theta\right) F_{x,2} + \sin\left(\Delta\theta\right) F_{y,2}\right)\left(\cos\left(\Delta\theta\right) u_{x,2} + \sin\left(\Delta\theta\right) u_{y,2}\right) + \\
=& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right) F_{x,1}u_{x,1} + \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right) F_{y,1}u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right) F_{x,2}u_{x,2} + \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right) F_{y,2}u_{y,2}\right)
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
G_{II,r\theta} =& \frac{1}{2R_f\delta}\left(F_{\theta,1}u_{\theta,1} + F_{\theta,2}u_{\theta,2}\right) = \\
=& \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right) F_{x,1} + \cos\left(\Delta\theta\right) F_{y,1}\right)\left(-\sin\left(\Delta\theta\right) u_{x,1} + \cos\left(\Delta\theta\right) u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right) F_{x,2} + \cos\left(\Delta\theta\right) F_{y,2}\right)\left(-\sin\left(\Delta\theta\right) u_{x,2} + \cos\left(\Delta\theta\right) u_{y,2}\right) = \\
=& \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right) F_{x,1}u_{x,1} - \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right) F_{y,1}u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right) F_{x,2}u_{x,2} - \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right) F_{y,2}u_{y,2}\right)
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
G_{TOT,r\theta} =& G_{I,r\theta} + G_{II,r\theta} = \\
=& \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta)\,F_{x,1}u_{x,1} + \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\,F_{y,1}u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta)\,F_{x,2}u_{x,2} + \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\,F_{y,2}u_{y,2}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta)\,F_{x,1}u_{x,1} - \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_{y,1}u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta)\,F_{x,2}u_{x,2} - \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_{y,2}u_{y,2}\right) = \\
=& \frac{1}{2R_f\delta}\cos^2(\Delta\theta)\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) + \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)\cos(\Delta\theta)\sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}\sin^2(\Delta\theta)\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right) + \\
&- \frac{1}{2R_f\delta}\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) + \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)\cos(\Delta\theta)\sin(\Delta\theta) = \\
=& \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2(\Delta\theta) + \sin^2(\Delta\theta)\right)}\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2}\right)\right) + \\
&+ \frac{1}{2R_f\delta}\cancelto{0}{\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) - \left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right)\right)}\cos(\Delta\theta)\sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}\cancelto{0}{\left(\left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right) - \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)}\cos(\Delta\theta)\sin(\Delta\theta) + \\
&+ \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2(\Delta\theta) + \sin^2(\Delta\theta)\right)}\left(F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right)\right) = \\
=& \frac{1}{2}\frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{R_f\delta} + \frac{1}{2}\frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{R_f\delta} = \\
=& \widetilde{G}_{I,xy} + \widetilde{G}_{II,xy} = \widetilde{G}_{TOT,xy}
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
G_{I,r\theta} = & \cos^2(\Delta\theta)\,\frac{F_{x,1}u_{x,1}+F_{x,2}u_{x,2}}{2R_f\delta}+\\
& +\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)+\\
& +\sin^2(\Delta\theta)\,\frac{F_{y,1}u_{y,1}+F_{y,2}u_{y,2}}{2R_f\delta}=\\
= & \cos^2(\Delta\theta)\,\widetilde{G}_{I,xy}+\sin^2(\Delta\theta)\,\widetilde{G}_{II,xy}+\\
& +\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)
\end{aligned}
\tag{77}
$$

$$
\begin{aligned}
G_{II,r\theta} = & \sin^2(\Delta\theta)\,\frac{F_{x,1}u_{x,1}+F_{x,2}u_{x,2}}{2R_f\delta}+\\
& -\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)+\\
& +\cos^2(\Delta\theta)\,\frac{F_{y,1}u_{y,1}+F_{y,2}u_{y,2}}{2R_f\delta}=\\
= & \sin^2(\Delta\theta)\,\widetilde{G}_{I,xy}+\cos^2(\Delta\theta)\,\widetilde{G}_{II,xy}+\\
& -\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)
\end{aligned}
\tag{78}
$$

3.4. Sensitivity analysis of the FEM solution

$$
\begin{aligned}
F_{x,1} &\sim k_{x,1}u_{x,1} & \qquad F_{y,1} &\sim k_{y,1}u_{y,1}\\
F_{x,2} &\sim k_{x,2}u_{x,2} & \qquad F_{y,2} &\sim k_{y,2}u_{y,2}
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
G_{I,r\theta} \sim & \frac{1}{2R_f\delta}\cos^2(\Delta\theta)\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)+\\
& +\frac{1}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)\left(k_{x,1}u_{x,1}u_{y,1}+k_{y,1}u_{y,1}u_{x,1}+k_{x,2}u_{x,2}u_{y,2}+k_{y,2}u_{y,2}u_{x,2}\right)+\\
& +\frac{1}{2R_f\delta}\sin^2(\Delta\theta)\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
G_{II,r\theta} \sim & \frac{1}{2R_f\delta}\sin^2(\Delta\theta)\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)+\\
& -\frac{1}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)\left(k_{x,1}u_{x,1}u_{y,1}+k_{y,1}u_{y,1}u_{x,1}+k_{x,2}u_{x,2}u_{y,2}+k_{y,2}u_{y,2}u_{x,2}\right)+\\
& +\frac{1}{2R_f\delta}\cos^2(\Delta\theta)\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)
\end{aligned}
\tag{81}
$$

$$
G_{TOT,r\theta} \sim \frac{1}{2R_f\delta}\left(\left(k_{x,1}u_{x,1}^2(\Delta\theta)+k_{x,2}u_{x,2}^2(\Delta\theta)\right)+\left(k_{y,1}u_{y,1}^2(\Delta\theta)+k_{y,2}u_{y,2}^2(\Delta\theta)\right)\right)
\tag{82}
$$

$$
\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta}\cos^2\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{x,2}u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&+\frac{1}{2R_f\delta}\left(k_{x,1}+k_{y,1}\right)\left(\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,1}\left(\Delta\theta\right)+u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&+\frac{1}{2R_f\delta}\left(k_{x,2}+k_{y,2}\right)\left(\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,2}\left(\Delta\theta\right)+u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&+\frac{1}{R_f\delta}\sin^2\left(\Delta\theta\right)\left(k_{y,1}u_{y,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{y,2}u_{y,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&+\frac{1}{2R_f\delta}\left(\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)-\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)\right)\sin\left(2\Delta\theta\right)+\\
&+\frac{1}{2R_f\delta}\left(\left(k_{x,1}+k_{y,1}\right)u_{x,1}u_{y,1}+\left(k_{x,2}+k_{y,2}\right)u_{x,2}u_{y,2}\right)\cos\left(2\Delta\theta\right)
\end{aligned}
\tag{83}
$$

$$
\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta}\sin^2\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{x,2}u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&-\frac{1}{2R_f\delta}\left(k_{x,1}+k_{y,1}\right)\left(\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,1}\left(\Delta\theta\right)+u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&-\frac{1}{2R_f\delta}\left(k_{x,2}+k_{y,2}\right)\left(\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,2}\left(\Delta\theta\right)+u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&+\frac{1}{R_f\delta}\cos^2\left(\Delta\theta\right)\left(k_{y,1}u_{y,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{y,2}u_{y,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&+\frac{1}{2R_f\delta}\left(\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)-\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)\right)\sin\left(2\Delta\theta\right)+\\
&-\frac{1}{2R_f\delta}\left(\left(k_{x,1}+k_{y,1}\right)u_{x,1}u_{y,1}+\left(k_{x,2}+k_{y,2}\right)u_{x,2}u_{y,2}\right)\cos\left(2\Delta\theta\right)
\end{aligned}
\tag{84}
$$

$$
G_{TOT,r\theta} \sim \frac{1}{R_f\delta}\left(\left(k_{x,1}u_{x,1}\frac{\partial u_{x,1}}{\partial\Delta\theta}+k_{x,2}u_{x,2}\frac{\partial u_{x,2}}{\partial\Delta\theta}\right)+\left(k_{y,1}u_{y,1}\frac{\partial u_{y,1}}{\partial\Delta\theta}+k_{y,2}u_{y,2}\frac{\partial u_{y,2}}{\partial\Delta\theta}\right)\right) \tag{85}
$$

4. Trigonometric interpolation

By inspecting the results of the previous sections, an interpolation of the BEM results is performed manually by trial and error. The following procedure is followed:

(i) a basis function is selected;

(ii) the frequency of the trigonometric basis function is calculated based on the position of the maximum value;

(iii) the amplitude is calculated based on the maximum value.

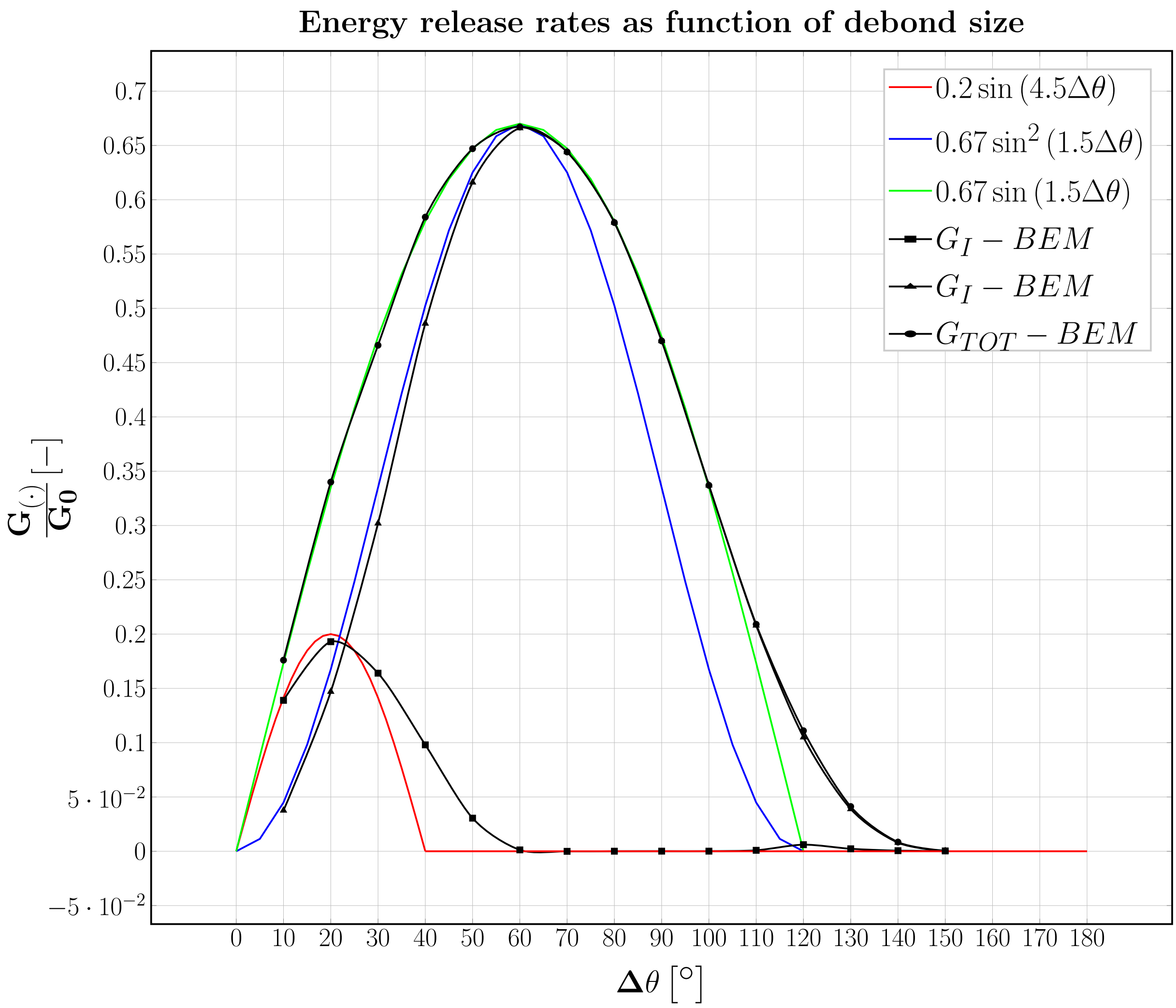

Figure 3. Interpolation of BEM results by manual trial and error.

The good agreement suggests that trigonometric functions are very good candidates for energy release rate data interpolation. The effect of different parameters can thus modeled as the functional variation of frequency (period) and amplitude of the trigonometric basis functions.

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