

# Constitutive modeling for laminates with fiber/matrix interface cracks under transverse loading

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## Abstract

*Keywords:* Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

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## 1. Introduction

Main ref [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

## 2. Derivation of constitutive relations

### 2.1. Reference frames

- 5 **Local reference frame of  $k$ -th layer:** index 1 is the in-plane longitudinal or fiber or  $0^\circ$ -direction; index 2 is the in-plane transverse or  $90^\circ$ -direction; index 3 is the out-of-plane or through-the-thickness direction.

**Global reference frame of laminate:** index  $x$  is the in-plane longitudinal or laminate  $0^\circ$  direction; index  $y$  is the in-plane transverse direction; index  
10  $z$  is the out-of-plane or through-the-thickness direction.

### 2.2. Crack density

**Principle 1.** Normalized volume of cracks  $V_{an}$  is the ratio of cracked volume  $V_a$  to material volume  $V$

$$V_{an} = \frac{V_a}{V} \quad (1)$$

$V_a$  is equal to the product of total crack surface  $S_C$  and average crack opening

15  $u_a$

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \quad (2)$$

The ratio  $\frac{S_C}{V}$  has a size of  $\frac{1}{length}$  and correspond to the crack density  $\rho_C$ . It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\begin{aligned} \rho_D &= \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta \theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \mathcal{W}}{L_{lam} \mathcal{W} t_{90^\circ}} R_f \Delta \theta = \frac{1}{n_2 L} \frac{1}{k_2 L} R_f \Delta \theta = \\ &= \frac{1}{nk 4 L^2} R_f \Delta \theta = \frac{V_f}{nk \pi R_f^2} R_f \Delta \theta = \frac{V_f}{nk R_f} \frac{\Delta \theta}{\pi} \end{aligned} \quad (3)$$

20 2.3. Homogenization

$$\sigma_{ij}^{avg} = \frac{1}{V} \int_V \sigma_{ij} dV \quad \varepsilon_{ij}^{avg} = \frac{1}{V} \int_V \varepsilon_{ij} dV \quad (4)$$

$$\tilde{\sigma}_k^{avg} = \tilde{Q}_k (\tilde{\varepsilon}_k^{avg} - \tilde{\alpha}_k \Delta T) \quad (5)$$

$$\tilde{\sigma}_{LAM} = \tilde{\sigma}^{avg} = \frac{1}{V} \int_V \tilde{\sigma} dV = \frac{1}{V} \sum_{k=1}^N \int_{V_k} \tilde{\sigma} dV_k = \sum_{k=1}^N \tilde{\sigma}_k^{avg} \frac{t_k}{h} \quad (6)$$

$$\varepsilon_{ij}^{avg} = \frac{1}{V} \int_V \varepsilon_{ij} dV = \frac{1}{V} \int_V \frac{1}{2} (u_{i,j} + u_{j,i}) dV = \frac{1}{V} \int_S \frac{1}{2} (u_i n_j + u_j n_i) dS \quad (7)$$

$$\begin{aligned} \varepsilon_{ij}^{avg} &= \frac{1}{V} \int_S \frac{1}{2} (u_i n_j + u_j n_i) dS = \\ &= \frac{1}{V} \int_{S_B} \frac{1}{2} (u_i n_j + u_j n_i) dS + \frac{1}{V} \int_{S_C} \frac{1}{2} (u_i n_j + u_j n_i) dS = \\ &= \varepsilon_{ij}^{applied} + \beta_{ij} \end{aligned} \quad (8)$$

$$\tilde{\underline{\varepsilon}}_k^{avg} = \tilde{\underline{\varepsilon}}_k^{applied} + \tilde{\underline{\beta}}_k = \tilde{\underline{\varepsilon}}_k^{LAM} + \tilde{\underline{\beta}}_k \quad (9)$$

$$\tilde{\underline{\sigma}}_{LAM} = \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \left( \tilde{\underline{\varepsilon}}_k^{LAM} - \tilde{\underline{\alpha}}_k \Delta T + \tilde{\underline{\beta}}_k \right) \quad (10)$$

$$\tilde{\underline{\sigma}}_{LAM} = Q_{\underline{\varepsilon}_{eff}}^{LAM} \tilde{\underline{\varepsilon}}^{LAM} \quad (11)$$

$$Q_{\underline{\varepsilon}_{eff}}^{LAM} \tilde{\underline{\varepsilon}}^{LAM} = \left( \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \right) \tilde{\underline{\varepsilon}}^{LAM} + \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \tilde{\underline{\beta}}_k \quad (12)$$

$$Q_{\underline{\varepsilon}_{eff}}^{LAM} \tilde{\underline{\varepsilon}}^{LAM} \cdot \frac{\tilde{\underline{\varepsilon}}^{LAM}}{\tilde{\underline{\varepsilon}}^{LAM} \cdot \tilde{\underline{\varepsilon}}^{LAM}} = \left( \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \right) \tilde{\underline{\varepsilon}}^{LAM} \cdot \frac{\tilde{\underline{\varepsilon}}^{LAM}}{\tilde{\underline{\varepsilon}}^{LAM} \cdot \tilde{\underline{\varepsilon}}^{LAM}} + \left( \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \tilde{\underline{\beta}}_k \right) \cdot \frac{\tilde{\underline{\varepsilon}}^{LAM}}{\tilde{\underline{\varepsilon}}^{LAM} \cdot \tilde{\underline{\varepsilon}}^{LAM}} \quad (13)$$

$$Q_{\underline{\varepsilon}_{eff}}^{LAM} = \left( \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \right) + \frac{1}{h} \sum_{k=1}^N t_k \tilde{\underline{Q}}_{\underline{\varepsilon}_k} \frac{\tilde{\underline{\beta}}_k \cdot \tilde{\underline{\varepsilon}}^{LAM}}{\tilde{\underline{\varepsilon}}^{LAM} \cdot \tilde{\underline{\varepsilon}}^{LAM}} \quad (14)$$

$$\underline{T}_k = \begin{bmatrix} \cos^2(\theta_k) & \sin^2(\theta_k) & 2 \cos(\theta_k) \sin(\theta_k) \\ \sin^2(\theta_k) & \cos^2(\theta_k) & -2 \cos(\theta_k) \sin(\theta_k) \\ -\cos(\theta_k) \sin(\theta_k) & \cos(\theta_k) \sin(\theta_k) & \cos^2(\theta_k) - \sin^2(\theta_k) \end{bmatrix} \quad (15)$$

$$\tilde{\underline{Q}}_{\underline{\varepsilon}_k} = T_k^{-1} Q_{\underline{\varepsilon}_k} \left( T_k^{-1} \right)^T \quad (16)$$

$$\begin{aligned} Q_{\underline{\varepsilon}_{eff}}^{LAM} &= \left( \frac{1}{h} \sum_{k=1}^N t_k T_k^{-1} Q_{\underline{\varepsilon}_k} \left( T_k^{-1} \right)^T \right) + \frac{1}{h} \sum_{k=1}^N t_k T_k^{-1} Q_{\underline{\varepsilon}_k} \left( T_k^{-1} \right)^T \frac{\tilde{\underline{\beta}}_k \cdot \tilde{\underline{\varepsilon}}^{LAM}}{\tilde{\underline{\varepsilon}}^{LAM} \cdot \tilde{\underline{\varepsilon}}^{LAM}} = \\ &= \frac{1}{h} \sum_{k=1}^N t_k T_k^{-1} Q_{\underline{\varepsilon}_k} \left( T_k^{-1} \right)^T \left( 1 + \frac{\tilde{\underline{\beta}}_k \cdot \tilde{\underline{\varepsilon}}^{LAM}}{\tilde{\underline{\varepsilon}}^{LAM} \cdot \tilde{\underline{\varepsilon}}^{LAM}} \right) \end{aligned} \quad (17)$$

#### 2.4. Vakulenko-Kachanov tensor

In the local reference frame of  $k$ -th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \quad (18)$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \quad (19)$$

25 Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_i n_j + u_j n_i) dS \quad (20)$$

Expand the expression for each component and simplify based on the fact that  $u_1 = 0$ :

$$\begin{aligned} \beta_{11} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_1 n_1 + u_1 n_1) dS = \frac{1}{V_k} \int_{S_C} \mathbf{u}_1^0 n_1 dS = 0 \\ \beta_{22} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_2 + u_2 n_2) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\ \beta_{33} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_3 + u_3 n_3) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\ \beta_{12} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\mathbf{u}_1^0 n_2 + u_2 \mathbf{p}_1^0) dS = 0 \\ \beta_{13} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\mathbf{u}_1^0 n_3 + u_3 \mathbf{p}_1^0) dS = 0 \\ \beta_{23} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_3 + u_3 n_2) dS \\ \beta_{21} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 \mathbf{p}_1^0 + \mathbf{u}_1^0 n_2) dS = 0 \\ \beta_{31} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 \mathbf{p}_1^0 + \mathbf{u}_1^0 n_3) dS = 0 \\ \beta_{32} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_2 + u_2 n_3) dS = \beta_{23} \end{aligned} \quad (21)$$

Only 3 independent components of the tensor  $\beta_{ij}$  remain:  $\beta_{22}$ ,  $\beta_{33}$  and  $\beta_{23}$ . Split total crack surface  $S_C$  into total matrix crack surface  $S_C^m$  and total fiber

30 crack surface  $S_C^f$  and remember that  $n_i^f = -n_i^m$  for  $i = 2, 3$

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[ \int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f (-n_2^m) dS \right] \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[ \int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f (-n_3^m) dS \right] \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} (u_2 n_3 + u_3 n_2) dS = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} (u_2^m n_3^m + u_3^m n_2^m) dS + \int_{S_C^f} (u_2^f n_3^f + u_3^f n_2^f) dS \right] = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f (-n_3^m) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f (-n_2^m) dS \right] \\
&\hspace{15em} (22)
\end{aligned}$$

The total matrix debonded surface  $S_C^m$  is equal to the total fiber debonded surface  $S_C^f$  and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta \theta \quad (23)$$

With Eq. 23, we can recast Eq. 22 as

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \left[ n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam} w t_{90^\circ}} \left[ n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90^\circ}} \left[ \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] \\
\beta_{33} &= \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_3^m d\theta \right] \\
\beta_{23} &= \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_2^m d\theta \right]
\end{aligned} \tag{24}$$

We can express the displacement jumps at the interface as a function of  
35 the local Crack Opening Displacement (COD) and Crack Sliding Displacement  
(CSD) as

$$\begin{aligned}
u_2^m - u_2^f &= (u_r^m - u_r^f) \cos(\theta) - (u_\theta^m - u_\theta^f) \sin(\theta) = \\
&= COD(\theta) \cos(\theta) - CSD(\theta) \sin(\theta) \\
u_3^m - u_3^f &= (u_r^m - u_r^f) \sin(\theta) + (u_\theta^m - u_\theta^f) \cos(\theta) = \\
&= COD(\theta) \sin(\theta) + CSD(\theta) \cos(\theta)
\end{aligned} \tag{25}$$

where  $\theta$  is the local angular coordinate at the interface. We can similarly  
express  $n_2^m$  and  $n_3^m$  as a function of  $\theta$ :

$$\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned} \tag{26}$$

Thus, Eq. 24 becomes

$$\begin{aligned}
& \beta_{22} = \\
& = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (\cos^2(\theta) - \cos(\theta) \sin(\theta)) - CSD(\theta) (\sin(\theta) \cos(\theta) - \sin^2(\theta))] d\theta = \\
& = \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (1 + \cos(2\theta) - \sin(2\theta)) + CSD(\theta) (1 - \cos(2\theta) - \sin(2\theta))] d\theta = \\
& = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[ \frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
& \beta_{33} = \\
& = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (\sin(\theta) \cos(\theta) + \sin^2(\theta)) + CSD(\theta) (\cos^2(\theta) + \cos(\theta) \sin(\theta))] d\theta = \\
& = \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (1 + \sin(2\theta) - \cos(2\theta)) + CSD(\theta) (1 + \sin(2\theta) + \cos(2\theta))] d\theta = \\
& = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[ \frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
& \beta_{23} = \\
& = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) (2 \sin(\theta) \cos(\theta) + \cos^2(\theta) - \sin^2(\theta)) + \\
& \quad - \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) (\sin^2(\theta) - \cos^2(\theta) + 2 \cos(\theta) \sin(\theta)) d\theta = \\
& = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[ \frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta \\
& \hspace{15em} (27)
\end{aligned}$$

#### 40 2.5. Analytical modeling of $COD(\theta)$ and $CSD(\theta)$

The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of  $\theta$ , the angular coordinate along the crack which varies between 0 and  $\Delta\theta$ . Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be  
45 expressed as the sum of their average value and a term, respectively  $\delta COD(\theta)$

and  $\delta CSD(\theta)$ , that represents the variation of the function from its average:

$$\begin{aligned} COD(\theta) &= COD_{avg} + \delta COD(\theta) \\ CSD(\theta) &= CSD_{avg} + \delta CSD(\theta). \end{aligned} \quad (28)$$

By defining  $\Delta\Psi$

$$\Delta\Psi = \min(\Delta\theta, \Delta\Phi), \quad (29)$$

we introduce at this point an approximation and assume that the functions  $\delta COD(\theta)$  and  $\delta CSD(\theta)$  can be expressed as the product of the maximum value of the displacement and a function, respectively  $f\left(\theta - \frac{\Delta\Psi}{2}\right)$  and  $g\left(\theta - \frac{\Delta\theta}{2}\right)$ :

$$\begin{aligned} COD(\theta) &= COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max} f\left(\theta - \frac{\Delta\Psi}{2}\right) \\ CSD(\theta) &= CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max} g\left(\theta - \frac{\Delta\theta}{2}\right), \end{aligned} \quad (30)$$

where  $f\left(\theta - \frac{\Delta\Psi}{2}\right)$  and  $g\left(\theta - \frac{\Delta\theta}{2}\right)$  are assumed to be odd functions over their respective integration domain  $[0, \Delta\Psi]$  and  $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = 0 \quad \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = 0. \quad (31)$$

We assume the two functions  $f\left(\theta - \frac{\Delta\Psi}{2}\right)$  and  $g\left(\theta - \frac{\Delta\theta}{2}\right)$  to be two odd polynomials of degree  $2n - 1$ :

$$\begin{aligned} f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \begin{cases} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} & 0 \leq \theta \leq \Delta\Psi \\ 0 & otherwise \end{cases} \\ g\left(\theta - \frac{\Delta\theta}{2}\right) &= \begin{cases} \sum_{k=0}^{n-1} b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} & 0 \leq \theta \leq \Delta\theta \\ 0 & otherwise \end{cases} \end{aligned} \quad (32)$$

which satisfy by construction the conditions expressed in Equation 31. The coefficients  $a_{2k+1}$  and  $b_{2k+1}$  are determined by imposing that

$$\begin{aligned} COD(\Delta\Psi) &= 0 \\ CSD(\Delta\theta) &= 0. \end{aligned} \quad (33)$$



The explicit construction of the polynomials  $f\left(\theta - \frac{\Delta\Psi}{2}\right)$  and  $g\left(\theta - \frac{\Delta\theta}{2}\right)$  for  $n = 1, 2, 3$  (or degree  $2n - 1 = 1, 3, 5$ ) is reported in Appendix A.

We recall the expressions of the non-zero components of the Vakulenko-Kachanov

<sup>60</sup> tensor

$$\begin{aligned}
\beta_{22} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[ \frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{33} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[ \frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{23} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[ \frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta
\end{aligned} \tag{34}$$

and proceed to the integration of the different summands:

1.

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left( COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left( \theta - \frac{\Delta\Psi}{2} \right)^{2k+1} \right) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD_{avg} d\theta + \\
& + \frac{1}{\Delta\theta} \int_0^{\Delta\Psi} COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left( \theta - \frac{\Delta\Psi}{2} \right)^{2k+1} d\theta + \\
& + \frac{1}{\Delta\theta} \int_{\Delta\Psi}^{\Delta\theta} COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left( \theta - \frac{\Delta\Psi}{2} \right)^{2k+1} d\theta = \\
& = \frac{1}{\Delta\theta} [COD_{avg}\theta] \Big|_0^{\Delta\theta} + \frac{1}{\Delta\theta} \left[ COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \left( \theta - \frac{\Delta\Psi}{2} \right)^{2(k+1)} \right] \Big|_0^{\Delta\Psi} = \\
& = COD_{avg} + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)\Delta\theta} \left( \left( \frac{\Delta\Psi}{2} \right)^{2(k+1)} - \left( -\frac{\Delta\Psi}{2} \right)^{2(k+1)} \right) = \\
& = COD_{avg}
\end{aligned} \tag{35}$$

2.

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) d\theta = CSD_{avg} \tag{36}$$

3.

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) \sin(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left( COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \right) \sin(2\theta) d\theta = \\
& = -\frac{1}{2\Delta\theta} COD_{avg} [\cos(2\theta)] \Big|_0^{\Delta\theta} + \\
& + \frac{1}{\Delta\theta} \left[ COD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^{i+1} \sin\left( \frac{1+\text{mod}(i,2)}{2} \pi - 2\theta \right) \left( \sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \theta^{k-i} \right) \right] \Big|_0^{\Delta\Psi} = \\
& = \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} COD_{avg} + \\
& + \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^{i+1} \sin\left( \frac{1+\text{mod}(i,2)}{2} \pi - 2\Delta\Psi \right) \left( \sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)
\end{aligned} \tag{37}$$

4.

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) \sin(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left( CSD_{avg} + CSD_{max} \sum_{k=0}^{n-1} b_{2k+1} \theta^{2k+1} \right) \sin(2\theta) d\theta = \\
& = -\frac{1}{2\Delta\theta} CSD_{avg} [\cos(2\theta)] \Big|_0^{\Delta\theta} + \\
& + \frac{1}{\Delta\theta} \left[ CSD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\theta\right) \left( \sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \theta^{k-i} \right) \right] \Big|_0^{\Delta\theta} = \\
& = \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
& + \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\Delta\Psi\right) \left( \sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)
\end{aligned} \tag{38}$$

5.

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) \cos(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left( COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \right) \cos(2\theta) d\theta = \\
& = \frac{1}{2\Delta\theta} COD_{avg} [\sin(2\theta)] \Big|_0^{\Delta\theta} + \\
& + \frac{1}{\Delta\theta} \left[ COD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^i \sin\left(\frac{1+\text{mod}(i+1,2)}{2}\pi - 2\theta\right) \left( \sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \theta^{k-i} \right) \right] \Big|_0^{\Delta\theta} = \\
& = \frac{\sin(2\Delta\theta)}{2\Delta\theta} COD_{avg} + \\
& + \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^i \sin\left(\frac{1+\text{mod}(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left( \sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)
\end{aligned} \tag{39}$$

6.

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) \cos(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left( CSD_{avg} + CSD_{max} \sum_{k=0}^{n-1} b_{2k+1} \theta^{2k+1} \right) \cos(2\theta) d\theta = \\
& = -\frac{1}{2\Delta\theta} CSD_{avg} [\cos(2\theta)] \Big|_0^{\Delta\theta} + \\
& + \frac{1}{\Delta\theta} \left[ CSD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^i \sin\left( \frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\theta \right) \left( \sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \theta^{k-i} \right) \right] \Big|_0^{\Delta\theta} = \\
& = \frac{\sin(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
& + \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left( -\frac{1}{2} \right)^i \sin\left( \frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\theta \right) \left( \sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\theta^{k-i} \right) \\
& \hspace{15em} (40)
\end{aligned}$$

Finally we can compute the expressions of the components of the Vakulenko-Kachanov tensor:

1.  $\beta_{22}$

$$\begin{aligned}
\beta_{22} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[ \frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
&= \frac{\rho_D}{2} (COD_{avg} + CSD_{avg}) + \\
&\quad - \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} COD_{avg} - \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
&\quad - \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&\quad - \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&\quad + \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} COD_{avg} + \\
&\quad + \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&\quad - \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
&\quad - \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) = \\
&= \frac{\rho_D}{2} COD_{avg} \left(1 - \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} + \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) + \\
&\quad + \frac{\rho_D}{2} CSD_{avg} \left(1 - \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} - \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) + \\
&\quad + \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left( \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \right) \\
&\quad \cdot \left( \sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) + \\
&\quad - \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left( \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \right) \\
&\quad \cdot \left( \sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)
\end{aligned} \tag{41}$$

$$\begin{aligned}
\beta_{33} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[ \frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta = \\
&= \frac{\rho_D}{2} (COD_{avg} + CSD_{avg}) + \\
&+ \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} COD_{avg} + \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&- \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} COD_{avg} + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) = \\
&= \frac{\rho_D}{2} COD_{avg} \left(1 + \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} - \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} CSD_{avg} \left(1 - \frac{1 + \cos(2\Delta\theta)}{2\Delta\theta} + \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left( \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) \right) \cdot \\
&\quad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left( \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + \text{mod}(i+1, 2)}{2} \pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + \text{mod}(i, 2)}{2} \pi - 2\Delta\Psi\right) \right) \cdot \\
&\quad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right)
\end{aligned} \tag{42}$$

### 3. $\beta_{23}$

$$\begin{aligned}
\beta_{23} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[ \frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta = \\
&= \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} COD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1+\text{mod}(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1+\text{mod}(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} COD_{avg} - \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} CSD_{avg} + \\
&+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) = \\
&= \frac{\rho_D}{2} COD_{avg} \left(\frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} + \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) + \\
&- \frac{\rho_D}{2} CSD_{avg} \left(\frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} - \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) + \\
&+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^i \sin\left(\frac{1+\text{mod}(i+1,2)}{2}\pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\Delta\Psi\right)\right) \cdot \\
&\quad \cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) + \\
&+ \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^i \sin\left(\frac{1+\text{mod}(i+1,2)}{2}\pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\Delta\Psi\right)\right) \cdot \\
&\quad \cdot \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right)
\end{aligned} \tag{43}$$

2.6. *Application to UD composite*

2.7. *Application to  $[0_{mk}^\circ, 90_k^\circ, 0_{mk}^\circ]$  laminate*

### 3. Computations with the Finite Element Method (FEM)

70 3.1. *Models of Representative Volume Element (RVE)*

3.2. *Details of FEM implementation*

### 4. Results and discussion

### 5. Conclusions

### References

- 75 [1] P. Lundmark, J. Varna, Modeling thermo-mechanical properties of damaged laminates, *Key Engineering Materials* 251-252 (2003) 381–388. doi:10.4028/www.scientific.net/kem.251-252.381.
- [2] P. Lundmark, J. Varna, Crack face sliding effect on stiffness of laminates with ply cracks, *Composites Science and Technology* 66 (10) (2006) 1444–  
80 1454. doi:10.1016/j.compscitech.2005.08.016.
- [3] J. Varna, Modelling mechanical performance of damaged laminates, *Journal of Composite Materials* 47 (20-21) (2012) 2443–2474. doi:10.1177/0021998312469241.
- [4] M. S. Loukil, J. Varna, Z. Ayadi, Applicability of solutions for periodic intralaminar crack distributions to non-uniformly damaged laminates,  
85 *Journal of Composite Materials* 47 (3) (2012) 287–301. doi:10.1177/0021998312440126.
- [5] M. S. Loukil, W. Hussain, A. Kirti, A. Pupurs, J. Varna, Thermoelastic constants of symmetric laminates with cracks in 90-layer: application of  
90 simple models, *Plastics, Rubber and Composites* 42 (4) (2013) 157–166. doi:10.1179/1743289811y.00000000064.



- [6] M. S. Loukil, J. Varna, Z. Ayadi, Engineering expressions for thermo-elastic constants of laminates with high density of transverse cracks, *Composites Part A: Applied Science and Manufacturing* 48 (2013) 37–46. doi:10.1016/j.compositesa.2012.12.012.
- [7] J. Varna, M. S. Loukil, Effective transverse modulus of a damaged layer: Potential for predicting symmetric laminate stiffness degradation, *Journal of Composite Materials* 51 (14) (2016) 1945–1959. doi:10.1177/0021998316658965.
- [8] A. Pupurs, J. Varna, M. Loukil, H. B. Kahla, D. Mattsson, Effective stiffness concept in bending modeling of laminates with damage in surface layers, *Composites Part A: Applied Science and Manufacturing* 82 (2016) 244–252. doi:10.1016/j.compositesa.2015.11.012.
- [9] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), *Comprehensive Composite Materials II*, Elsevier, Oxford, 2018, pp. 192 – 220. doi:10.1016/B978-0-12-803581-8.09910-0.
- [10] M. S. Loukil, J. Varna, Effective shear modulus of a damaged ply in laminate stiffness analysis: Determination and validation, *Journal of Composite Materials* (2019) 002199831987436doi:10.1177/0021998319874369.
- [11] M. S. Loukil, J. Varna, Crack face sliding displacement (CSD) as an input in exact GLOB-LOC expressions for in-plane elastic constants of symmetric damaged laminates, *International Journal of Damage Mechanics* (2019) 105678951986600doi:10.1177/1056789519866000.

## Appendix A. Explicit expressions for $f(\theta)$ and $g(\theta)$

In the following, recall that

$$\Delta\Psi = \min(\Delta\theta, \Delta\Phi). \quad (\text{A.1})$$

$$\mathbf{n} = \mathbf{1}$$

$$\begin{aligned} f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^0 a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) \\ g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^0 b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1 \left(\theta - \frac{\Delta\theta}{2}\right) \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \int_0^{\Delta\Psi} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta &= \int_0^{\Delta\Psi} a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \left[ \frac{a_1}{2} \theta^2 - a_1 \frac{\Delta\Psi}{2} \theta \right] \Big|_0^{\Delta\Psi} = 0 \quad \forall a_1 \\ \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta &= \int_0^{\Delta\theta} b_1 \left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \left[ \frac{b_1}{2} \theta^2 - b_1 \frac{\Delta\theta}{2} \theta \right] \Big|_0^{\Delta\theta} = 0 \quad \forall b_1 \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} COD_{avg} + COD_{max} a_1 \left(\Delta\Psi - \frac{\Delta\Psi}{2}\right) &= 0 \rightarrow a_1 = -\frac{2}{\Delta\Psi} \frac{COD_{avg}}{COD_{max}} \\ CSD_{avg} + CSD_{max} b_1 \left(\Delta\theta - \frac{\Delta\theta}{2}\right) &= 0 \rightarrow b_1 = -\frac{2}{\Delta\theta} \frac{CSD_{avg}}{CSD_{max}} \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \sum_{i=0}^1 \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+\text{mod}(i,2)}{2}\pi - 2\Delta\Psi\right) &\left(\sum_{k=0}^{1-i} a_k (1-(k+1))! \Delta\Psi^k\right) = \\ &= \left(-\frac{1}{2}\right)^1 \sin\left(\frac{1+\text{mod}(0,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=0}^1 a_k (1-(k+1))! \Delta\Psi^k\right) + \\ &+ \left(-\frac{1}{2}\right)^2 \sin\left(\frac{1+\text{mod}(1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=0}^0 a_k (1-(k+1))! \Delta\Psi^k\right) = \\ &= -\frac{1}{2} \cos(2\Delta\Psi) \left(\sum_{k=0}^1 a_k (1-(k+1))! \Delta\Psi^k\right) + \\ &+ \frac{1}{4} \sin(2\Delta\Psi) \left(\sum_{k=0}^0 a_k (1-(k+1))! \Delta\Psi^k\right) = \\ &= -\frac{1}{2} \cos(2\Delta\Psi) (a_0 + a_1 (n-(k+1))! \Delta\Psi^k) + \\ &+ \frac{1}{4} \sin(2\Delta\Psi) \left(\sum_{k=0}^0 a_k (n-(k+1))! \Delta\Psi^k\right) = \end{aligned} \quad (\text{A.5})$$

**n = 2**

$$\begin{aligned}
f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^1 a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) + a_3 \left(\theta - \frac{\Delta\Psi}{2}\right)^3 \\
g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^1 b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1 \left(\theta - \frac{\Delta\theta}{2}\right) + b_3 \left(\theta - \frac{\Delta\theta}{2}\right)^3
\end{aligned}
\tag{A.6}$$

**n = 3**

$$\begin{aligned}
f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^2 a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = \\
&= a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) + a_3 \left(\theta - \frac{\Delta\Psi}{2}\right)^3 + a_5 \left(\theta - \frac{\Delta\Psi}{2}\right)^5 \\
g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^1 b_{2k+1} \left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = \\
&= b_1 \left(\theta - \frac{\Delta\theta}{2}\right) + b_3 \left(\theta - \frac{\Delta\theta}{2}\right)^3 + b_5 \left(\theta - \frac{\Delta\theta}{2}\right)^5
\end{aligned}
\tag{A.7}$$