Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

Luca Di Stasio[a], Janis Varna[a]

[a] *Luleå University of Technology, University Campus, SE-97187 Luleå, Sweden*

Abstract

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1. Introduction

2. Derivation of constitutive relations

2.1. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \tag{1}$$

V_a is equal to the product of total crack surface S_C and average crack opening u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \tag{2}$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \cancel{w}}{L_{lam} \cancel{w} t_{90^\circ}} R_f \Delta\theta = \frac{1}{n2L}\frac{1}{k2L} R_f \Delta\theta = \\ = \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nkR_f}\frac{\Delta\theta}{\pi} \tag{3}$$

2.2. Vakulenko-Kachanov tensor

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_i n_j + u_j n_i\right) dS \tag{4}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$\begin{aligned}
\beta_{11} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_1 n_1 + u_1 n_1\right) dS = \frac{1}{V_k}\int_{S_C} \cancelto{0}{u_1 n_1} dS = 0 \\
\beta_{22} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_2 n_2 + u_2 n_2\right) dS = \frac{1}{V_k}\int_{S_C} u_2 n_2 dS \\
\beta_{33} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_3 n_3 + u_3 n_3\right) dS = \frac{1}{V_k}\int_{S_C} u_3 n_3 dS \\
\beta_{12} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(\cancelto{0}{u_1 n_2} + u_2 n_1\right) dS = \frac{1}{2}\frac{1}{V_k}\int_{S_C} u_2 n_1 dS \\
\beta_{13} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(\cancelto{0}{u_1 n_3} + u_3 n_1\right) dS = \frac{1}{2}\frac{1}{V_k}\int_{S_C} u_3 n_1 dS \\
\beta_{23} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_2 n_3 + u_3 n_2\right) dS \\
\beta_{21} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_2 n_1 + \cancelto{0}{u_1 n_2}\right) dS = \beta_{12} \\
\beta_{31} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_3 n_1 + \cancelto{0}{u_1 n_3}\right) dS = \beta_{13} \\
\beta_{32} =& \frac{1}{V_k}\int_{S_C} \frac{1}{2}\left(u_3 n_2 + u_2 n_3\right) dS = \beta_{23}
\end{aligned} \tag{5}$$

Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k}\int_{S_C} u_2 n_2 dS = \frac{1}{V_k}\left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS\right] = \\
=& \frac{1}{V_k}\left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f \left(-n_2^m\right) dS\right] \\
\beta_{33} =& \frac{1}{V_k}\int_{S_C} u_3 n_3 dS = \frac{1}{V_k}\left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS\right] = \\
=& \frac{1}{V_k}\left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f \left(-n_3^m\right) dS\right] \\
\beta_{23} =& \frac{1}{V_k}\int_{S_C} \left(u_2 n_3 + u_3 n_2\right) dS = \\
=& \frac{1}{V_k}\left[\int_{S_C^m} \left(u_2^m n_3^m + u_3^m n_2^m\right) dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f\right) dS\right] = \\
=& \frac{1}{V_k}\left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f \left(-n_3^m\right) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f \left(-n_2^m\right) dS\right]
\end{aligned}
\tag{6}
$$

The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$
S_C^m = S_C^f = n_D R_f \Delta\theta \tag{7}
$$

With Eq. 7, we can recast Eq. 6 as

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \frac{1}{L_{lam} w t_{90^\circ}}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90^\circ}}\left[\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] \\
\beta_{33} =& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right) n_3^m d\theta\right] \\
\beta_{23} =& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_3^m d\theta + \frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right) n_2^m d\theta\right]
\end{aligned}
\tag{8}
$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$
\begin{aligned}
u_2^m - u_2^f =& \left(u_r^m - u_r^f\right)\cos(\theta) - \left(u_\theta^m - u_\theta^f\right)\sin(\theta) = \\
=& COD(\theta)\cos(\theta) - CSD(\theta)\sin(\theta) \\
u_3^m - u_3^f =& \left(u_r^m - u_r^f\right)\sin(\theta) + \left(u_\theta^m - u_\theta^f\right)\cos(\theta) = \\
=& COD(\theta)\sin(\theta) + CSD(\theta)\cos(\theta)
\end{aligned}
\tag{9}
$$

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ:

$$
\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned}
\tag{10}
$$

Thus, Eq. 8 becomes

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(\sin(\theta)\cos(\theta) - \sin^2(\theta)\right) + CSD(\theta) \left(\cos^2(\theta) - \cos(\theta)\sin(\theta)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left(COD(\theta) - CSD(\theta)\right) \left(-1 + \sin(2\theta) - \cos(2\theta)\right) d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left(COD(\theta) - CSD(\theta)\right) \left(\sqrt{2}\sin\left(2\theta - \frac{\pi}{4}\right) - 1\right) d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(\sin(\theta)\cos(\theta) + \sin^2(\theta)\right) + CSD(\theta) \left(\cos^2(\theta) + \cos(\theta)\sin(\theta)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD(\theta) \left(1 + \sin(2\theta) - \cos(2\theta)\right) + CSD(\theta) \left(1 + \sin(2\theta) + \cos(2\theta)\right)\right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[\left(COD(\theta) + CSD(\theta)\right)\left(1 + \sin(2\theta)\right) - \left(COD(\theta) - CSD(\theta)\right)\cos(2\theta)\right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) \left(2\sin(\theta)\cos(\theta) + \cos^2(\theta) - \sin^2(\theta)\right) +$$

$$- \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) \left(\sin^2(\theta) - \cos^2(\theta) + 2\cos(\theta)\sin(\theta)\right) d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\left(COD(\theta) - CSD(\theta)\right)\sin(2\theta) + \left(COD(\theta) + CSD(\theta)\right)\cos(2\theta)\right] d\theta \tag{11}$$

2.3. Analytical modeling of $COD(\theta)$ and $CSD(\theta)$

$$\begin{aligned} COD(\theta) &= COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max} f(\theta) \\ CSD(\theta) &= CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max} g(\theta) \end{aligned} \tag{12}$$

$$\int_0^{\Delta\theta} f(\theta)\, d\theta = 0 \qquad \int_0^{\Delta\theta} g(\theta)\, d\theta = 0 \tag{13}$$

$$f(\theta) = a_0 + \sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1} \quad g(\theta) = b_0 + \sum_{k=0}^{n-1} b_{2k+1}\theta^{2k+1} \tag{14}$$

$$\begin{aligned} COD(\Delta\Phi) &= COD_{avg} + COD_{max}\left(a_0 + \sum_{k=0}^{n-1} a_{2k+1}\Delta\Phi^{2k+1}\right) = 0 \\ CSD(\Delta\Theta) &= CSD_{avg} + CSD_{max}\left(b_0 + \sum_{k=0}^{n-1} b_{2k+1}\Delta\Theta^{2k+1}\right) = 0 \end{aligned} \tag{15}$$

$$\begin{aligned} a_0 &= -\left(\frac{COD_{avg}}{COD_{max}} + \sum_{k=0}^{n-1} a_{2k+1}\Delta\Phi^{2k+1}\right) \\ b_0 &= -\left(\frac{CSD_{avg}}{CSD_{max}} + \sum_{k=0}^{n-1} b_{2k+1}\Delta\Theta^{2k+1}\right) \end{aligned} \tag{16}$$

$$\begin{aligned} \beta_{22} =& \rho_D COD_{avg}\frac{1}{\Delta\theta}\int_0^{\Delta\theta}(1+\sin(2\theta)-\cos(2\theta))\,d\theta + \\ &+ \rho_D COD_{max}\frac{1}{\Delta\theta}\int_0^{\Delta\theta} f(\theta)(1+\sin(2\theta)-\cos(2\theta))\,d\theta + \\ &- \rho_D CSD_{avg}\frac{1}{\Delta\theta}\int_0^{\Delta\theta}(1+\sin(2\theta)-\cos(2\theta))\,d\theta + \\ &- \rho_D CSD_{max}\frac{1}{\Delta\theta}\int_0^{\Delta\theta} g(\theta)(1+\sin(2\theta)-\cos(2\theta))\,d\theta \end{aligned} \tag{17}$$

$$\begin{aligned} \int_0^{\Delta\theta}(1+\sin(2\theta)-\cos(2\theta))\,d\theta &= \left[\theta - \frac{1}{2}\cos(2\theta) - \frac{1}{2}\sin(2\theta)\right]_0^{\Delta\theta} = \\ &= \Delta\theta + \frac{1}{2}(1-\cos(2\Delta\theta)) - \frac{1}{2}\sin(2\Delta\theta) = \\ &= \Delta\theta - \frac{1}{2}\left(\sqrt{2}\sin\left(2\Delta\theta + \frac{\pi}{4}\right) - 1\right) \end{aligned} \tag{18}$$

$$
\begin{aligned}
&\int_0^{\Delta\theta} f(\theta)\left(1+\sin(2\theta)-\cos(2\theta)\right) d\theta = \\
&\qquad = \cancelto{0}{\int_0^{\Delta\theta} f(\theta)\, d\theta} - \sqrt{2}\int_0^{\Delta\theta} f(\theta)\sin\left(\frac{\pi}{4}-2\theta\right) d\theta \\
&\int_0^{\Delta\theta} g(\theta)\left(1+\sin(2\theta)-\cos(2\theta)\right) d\theta = \\
&\qquad = \cancelto{0}{\int_0^{\Delta\theta} g(\theta)\, d\theta} - \sqrt{2}\int_0^{\Delta\theta} g(\theta)\sin\left(\frac{\pi}{4}-2\theta\right) d\theta
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\int_0^{\Delta\theta} f(\theta)\sin\left(\frac{\pi}{4}-2\theta\right) d\theta = \\
&= \left[\sum_{i=0}^{n}\left[\left(a_0\left(1-H(i-1)\right)+\sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i \sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\theta\right)\right)\right]\right]_0^{\Delta\theta} = \\
&= \sum_{i=0}^{n}\left[\left(a_0\left(1-H(i-1)\right)+\sum_{k=0}^{n-1} a_{2k+1}\Delta\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i \sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right)\right] + \\
&+\frac{\sqrt{2}}{4}a_0 \\
&\int_0^{\Delta\theta} g(\theta)\sin\left(\frac{\pi}{4}-2\theta\right) d\theta = \\
&= \left[\sum_{i=0}^{n}\left[\left(b_0\left(1-H(i-1)\right)+\sum_{k=0}^{n-1} b_{2k+1}\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i \sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\theta\right)\right)\right]\right]_0^{\Delta\theta} = \\
&= \sum_{i=0}^{n}\left[\left(b_0\left(1-H(i-1)\right)+\sum_{k=0}^{n-1} b_{2k+1}\Delta\theta^{2k+1-i}\right)\frac{1}{2}(-1)^i \sin\left(\left(1-\frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi-\left(\frac{\pi}{4}-2\Delta\theta\right)\right)\right] + \\
&+\frac{\sqrt{2}}{4}b_0
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\beta_{22} =& \rho_D \left[1 - \frac{1}{2\Delta\theta}\left(\sqrt{2}\sin\left(2\Delta\theta + \frac{\pi}{4}\right) - 1\right)\right]\left(COD_{avg} - CSD_{avg}\right) + \\
& - \frac{1}{2}\rho_D\left(a_0 COD_{max} - b_0 CSD_{max}\right) + \\
-\sqrt{2}\sum_{i=0}^{n}&\left(\left(a_0 COD_{max} - b_0 CSD_{max}\right)\left(1 - H\left(i-1\right)\right) + \sum_{k=0}^{n-1}\left(a_{2k+1}COD_{max} - b_{2k+1}CSD_{max}\right)\Delta\theta^{2k+1-i}\right) \times \\
& \times \frac{1}{2}\left(-1\right)^{i}\sin\left(\left(1 - \frac{1}{2}\bmod\left(\frac{i+1}{2}\right)\right)\pi - \left(\frac{\pi}{4} - 2\Delta\theta\right)\right)
\end{aligned}
\tag{21}
$$