

# Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

Luca Di Stasio<sup>a</sup>, Janis Varna<sup>a</sup>

<sup>a</sup>*Luleå University of Technology, University Campus, SE-97187 Luleå, Sweden*

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## Abstract

*Keywords:* Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

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## 1. Introduction

## 2. Derivation of constitutive relations

### 2.1. Crack density

**Principle 1.** Normalized volume of cracks  $V_{an}$  is the ratio of cracked volume  
5  $V_a$  to material volume  $V$

$$V_{an} = \frac{V_a}{V} \quad (1)$$

$V_a$  is equal to the product of total crack surface  $S_C$  and average crack opening  
 $u_a$

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \quad (2)$$

The ratio  $\frac{S_C}{V}$  has a size of  $\frac{1}{length}$  and correspond to the crack density  $\rho_C$ . It  
means: product of crack density and average crack opening is equal to normal-  
10 ized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\begin{aligned}
\rho_D &= \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta \theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \mathcal{W}}{L_{lam} \mathcal{W} t_{90^\circ}} R_f \Delta \theta = \frac{1}{n 2L} \frac{1}{k 2L} R_f \Delta \theta = \\
&= \frac{1}{nk 4L^2} R_f \Delta \theta = \frac{V_f}{nk \pi R_f^2} R_f \Delta \theta = \frac{V_f}{nk R_f} \frac{\Delta \theta}{\pi}
\end{aligned} \tag{3}$$

## 2.2. Vakulenko-Kachanov tensor

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_i n_j + u_j n_i) dS \tag{4}$$

Expand the expression for each component and simplify based on the fact

15 that  $u_1 = 0$ :

$$\begin{aligned}
\beta_{11} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_1 n_1 + u_1 n_1) dS = \frac{1}{V_k} \int_{S_C} \cancel{u_1} \mathbf{n}_1^0 dS = 0 \\
\beta_{22} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_2 + u_2 n_2) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_3 + u_3 n_3) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\
\beta_{12} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\cancel{u_1} \mathbf{n}_2^0 + u_2 n_1) dS = \frac{1}{2} \frac{1}{V_k} \int_{S_C} u_2 n_1 dS \\
\beta_{13} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\cancel{u_1} \mathbf{n}_3^0 + u_3 n_1) dS = \frac{1}{2} \frac{1}{V_k} \int_{S_C} u_3 n_1 dS \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_3 + u_3 n_2) dS \\
\beta_{21} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_1 + \cancel{u_1} \mathbf{n}_2^0) dS = \beta_{12} \\
\beta_{31} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_1 + \cancel{u_1} \mathbf{n}_3^0) dS = \beta_{13} \\
\beta_{32} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_2 + u_2 n_3) dS = \beta_{23}
\end{aligned} \tag{5}$$

Split total crack surface  $S_C$  into total matrix crack surface  $S_C^m$  and total fiber crack surface  $S_C^f$  and remember that  $n_i^f = -n_i^m$  for  $i = 2, 3$

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[ \int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f (-n_2^m) dS \right] \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[ \int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f (-n_3^m) dS \right] \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} (u_2 n_3 + u_3 n_2) dS = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} (u_2^m n_3^m + u_3^m n_2^m) dS + \int_{S_C^f} (u_2^f n_3^f + u_3^f n_2^f) dS \right] = \\
&= \frac{1}{V_k} \left[ \int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f (-n_3^m) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f (-n_2^m) dS \right]
\end{aligned} \tag{6}$$

The total matrix debonded surface  $S_C^m$  is equal to the total fiber debonded surface  $S_C^f$  and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta \theta \tag{7}$$

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With Eq. 7, we can recast Eq. 6 as

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \left[ n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam} w t_{90^\circ}} \left[ n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90^\circ}} \left[ \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] \\
\beta_{33} &= \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_3^m d\theta \right] \\
\beta_{23} &= \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_2^m d\theta \right]
\end{aligned} \tag{8}$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$\begin{aligned}
u_2^m - u_2^f &= (u_r^m - u_r^f) \cos(\theta) - (u_\theta^m - u_\theta^f) \sin(\theta) = \\
&= COD(\theta) \cos(\theta) - CSD(\theta) \sin(\theta) \\
u_3^m - u_3^f &= (u_r^m - u_r^f) \sin(\theta) + (u_\theta^m - u_\theta^f) \cos(\theta) = \\
&= COD(\theta) \sin(\theta) + CSD(\theta) \cos(\theta)
\end{aligned} \tag{9}$$

where  $\theta$  is the local angular coordinate at the interface. We can similarly  
25 express  $n_2^m$  and  $n_3^m$  as a function of  $\theta$ :

$$\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned} \tag{10}$$

Thus, Eq. 9 becomes

$$\beta_{22} = \rho_D \left[ \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (COD(\theta) (\sin(\theta) \cos(\theta) - \sin^2(\theta)) + CSD(\theta) (\cos^2(\theta) - \cos(\theta) \sin(\theta))) d\theta \right] \tag{11}$$

### 2.3. Modeling $COD(\theta)$ and $CSD(\theta)$

$$COD(\theta) = COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max} f(\theta) \quad (12)$$

$$CSD(\theta) = CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max} g(\theta)$$

$$\int_0^{\Delta\theta} f(\theta) d\theta = 0 \quad \int_0^{\Delta\theta} g(\theta) d\theta = 0 \quad (13)$$

$$f(\theta) = a_0 + \sum_{k=0}^n a_{2k+1} \theta^{2k+1} \quad g(\theta) = b_0 + \sum_{k=0}^n b_{2k+1} \theta^{2k+1} \quad (14)$$

$$COD(\Delta\Phi) = COD_{avg} + COD_{max} \left( a_0 + \sum_{k=0}^n a_{2k+1} \Delta\Phi^{2k+1} \right) = 0$$

$$CSD(\Delta\Theta) = CSD_{avg} + CSD_{max} \left( b_0 + \sum_{k=0}^n b_{2k+1} \Delta\Theta^{2k+1} \right) = 0 \quad (15)$$

$$a_0 = - \left( \frac{COD_{avg}}{COD_{max}} + \sum_{k=0}^n a_{2k+1} \Delta\Phi^{2k+1} \right)$$

$$b_0 = - \left( \frac{CSD_{avg}}{CSD_{max}} + \sum_{k=0}^n b_{2k+1} \Delta\Theta^{2k+1} \right) \quad (16)$$

$$\begin{aligned} \beta_{22} = & \frac{1}{2} \frac{1}{L} \frac{R_f}{t_{90^\circ}} COD_{avg} \int_0^{\Delta\theta} (1 + \sin(2\theta) - \cos(2\theta)) d\theta + \\ & + \frac{1}{2} \frac{1}{L} \frac{R_f}{t_{90^\circ}} COD_{max} \int_0^{\Delta\theta} f(\theta) (1 + \sin(2\theta) - \cos(2\theta)) d\theta + \\ & - \frac{1}{2} \frac{1}{L} \frac{R_f}{t_{90^\circ}} CSD_{avg} \int_0^{\Delta\theta} (1 + \sin(2\theta) - \cos(2\theta)) d\theta + \\ & - \frac{1}{2} \frac{1}{L} \frac{R_f}{t_{90^\circ}} CSD_{max} \int_0^{\Delta\theta} g(\theta) (1 + \sin(2\theta) - \cos(2\theta)) d\theta \end{aligned} \quad (17)$$

$$\begin{aligned} \int_0^{\Delta\theta} (1 + \sin(2\theta) - \cos(2\theta)) d\theta &= \left[ \theta - \frac{1}{2} \cos(2\theta) - \frac{1}{2} \sin(2\theta) \right]_0^{\Delta\theta} = \\ &= \Delta\theta + \frac{1}{2} (1 - \cos(2\Delta\theta)) - \frac{1}{2} \sin(2\Delta\theta) = \\ &= \Delta\theta - \frac{1}{2} \left( \sqrt{2} \sin\left(2\Delta\theta + \frac{\pi}{4}\right) - 1 \right) \end{aligned} \quad (18)$$

$$\begin{aligned}
& \int_0^{\Delta\theta} f(\theta) (1 + \sin(2\theta) - \cos(2\theta)) d\theta = \\
& = \int_{\theta}^{\Delta\theta} f(\theta) d\theta - \sqrt{2} \int_0^{\Delta\theta} f(\theta) \sin\left(\frac{\pi}{4} - 2\Delta\theta\right) d\theta
\end{aligned} \tag{19}$$

$$\int_0^{\Delta\theta} f(\theta) \sin\left(\frac{\pi}{4} - 2\Delta\theta\right) d\theta = \left[ \sum_{k=0}^n (-1)^{2k+1} \sin\left(\pi - \left(\frac{\pi}{4} - 2\Delta\theta\right)\right) \right]_0^{\Delta\theta} \tag{20}$$