Computing energy release rates using the Virtual Crack Closure Technique with the Finite Element Method: analytical discussion

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Abstract.   The effect of crack tip orientation and elements' size in its vicinity on mode splitting in the Virtual Crack Closure Technique is analyzed by means of analytical derivations.

(i) The total energy release rate is shown to have no direct dependence on the debond angular size, but only an indirect one through the FEM solution of the crack displacement field in the crack tip neighborhood. It thus can be computed using unrotated forces and displacements, i.e. aligned with the global and not with the local refere

(ii) Mode I and mode II energy release rate are expressed as a function of the crack displacement and it is shown as in [5] that crack tip forces depend linearly on both components of the crack displacements. However, what in [5] is an assumption used to reverse-engineer the relationship, here the dependency is fully derived in terms of the underlying FEM discretization. It is furthermore shown that crack tip forces do not only depend on crack displacements, but also on the resultant of forces of the 4 elements connected to the crack tip, which represent the influence of the rest of the domain.

(iii) A new vectorial formulation of the VCCT is proposed, which can be applied to elements of different order and in the presence of quarter-tip singularity (based on the work in [6]). It provides a general framework for the analysis of curved cracks analyzed with FEM. It furthermore expresses directly the dependence on FEM discretization and solution. By implementing the equations provided alongside the classic FEM, it can provide a native formulation which does not to the extraction of internal or reaction forces in the post-processing phase.

(iv) Drawing upon the asymptotic expression of the displacement derived in [7] and presented in [8], it is shown that mode I and mode II for a curved crack behaves as $A \log{(\delta)} + B$, where δ is the angular discretization at the crack tip. The result is confirmed by the numerical results.

List of acronyms

VCCT	Virtual Crack Closure Technique
BEM	Boundary Element Method
FEM	Finite Element Method

List of symbols

G_I	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate
G_{II}	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate
G_{TOT}	$\left[\frac{J}{m^2}\right]$	Total energy release rate
$G_{I,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate in $r-\theta$ reference frame
$G_{II,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate in $r-\theta$ reference frame
$G_{TOT,r\theta}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate in $r-\theta$ reference frame
$\widetilde{G}_{I,xy}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate of equivalent crack in $x-y$ reference frame
$\widetilde{G}_{II,xy}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate of equivalent crack in $x-y$ reference frame
$\widetilde{G}_{TOT,xy}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate of equivalent crack in $x-y$ reference frame
R_f	$[\mu m]$	Fiber radius
a	$[\mu m]$	Debond size
Δa	$[\mu m]$	Debond increment
$\Delta\theta$	$[rad]$	Half debond angular size
δ	$[rad]$	Angular size of element at the interface close to the crack tip
$u_{x,[A-Z]}$	$[\mu m]$	Displacement along x of a point labeled with a letter in [A-Z]
$u_{y,[A-Z]}$	$[\mu m]$	Displacement along y of a point labeled with a letter in [A-Z]
u_x	$[\mu m]$	Displacement along x-direction
u_y	$[\mu m]$	Displacement along y-direction
u_r	$[\mu m]$	Displacement along r-direction
u_θ	$[\mu m]$	Displacement along θ-direction
$F_{x,[A-Z]}$	$[\mu m]$	Force along x at a point labeled with a letter in [A-Z]
$F_{y,[A-Z]}$	$[\mu m]$	Force along y at a point labeled with a letter in [A-Z]
F_x	$[\mu m]$	Force along x-direction
F_y	$[\mu m]$	Force along y-direction
F_r	$[\mu m]$	Force along r-direction
F_θ	$[\mu m]$	Force along θ-direction
$\underline{\underline{R}}$	$[-]$	Rotation matrix

1. FEM formulation with quadrilateral elements

$$\Pi\left(u_i\right) = \int_V \sigma_{ij}\varepsilon_{ij}dV + \int_V \rho\ddot{u}_i u_i dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \tag{1}$$

$$\Pi\left(u_i\right) = \int_V \sigma_{ij}\varepsilon_{ij}dV - \int_V F_i^T u_i dV - \int_S f_i^T u_i dS \tag{2}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

$$\underline{\varepsilon}\left(x,y\right) = \underline{\underline{\widetilde{B}}} \cdot \underline{u}\left(x,y\right) \tag{4}$$

$$\underline{\underline{\widetilde{B}}} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \tag{5}$$

$$\underline{\sigma} = \underline{\underline{D}} \cdot \underline{\varepsilon} \tag{6}$$

$$\underline{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} \qquad \underline{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} \tag{7}$$

$$\underline{\underline{D}} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{bmatrix} \qquad \text{with } G = \frac{E}{2\left(1+\nu\right)} \text{ for an isotropic material} \tag{8}$$

$$\begin{aligned} E_1 &= \frac{E}{1-\nu^2} \quad E_2 = \nu E_1 \quad \text{for plane stress} \\ E_1 &= \frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)} \quad E_2 = \frac{\nu E_1}{1-\nu} \quad \text{for plane strain} \end{aligned} \tag{9}$$

$$\begin{aligned} \Pi\left(\underline{u}\right) &= \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{\underline{D}} \cdot \underline{\varepsilon}dV - \int_V \underline{F}^T \underline{u}dV - \int_S \underline{f}^T \underline{u}dS = \\ &= \frac{1}{2}\int_V \underline{u}^T \underline{\underline{\widetilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\widetilde{B}}} \cdot \underline{u}dV - \int_V \underline{F}^T \underline{u}dV - \int_S \underline{f}^T \underline{u}dS \end{aligned} \tag{10}$$

$$\delta\Pi\left(\delta\underline{u}\right) = 0 \tag{11}$$

$$\delta\Pi\left(\delta\underline{u}\right)=\Pi\left(\underline{u}+\delta\underline{u}\right)-\Pi\left(\underline{u}\right)=$$

$$=\frac{1}{2}\int_{V}\left(\underline{u}+\delta\underline{u}\right)^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\left(\underline{u}+\delta\underline{u}\right)dV-\int_{V}\underline{F}^{T}\left(\underline{u}+\delta\underline{u}\right)dV-\int_{S}\underline{f}^{T}\left(\underline{u}+\delta\underline{u}\right)dS+$$

$$-\frac{1}{2}\int_{V}\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\underline{u}dV+\int_{V}\underline{F}^{T}\underline{u}dV+\int_{S}\underline{f}^{T}\underline{u}dS=$$

$$=\frac{1}{2}\int_{V}\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\underline{u}dV+\frac{1}{2}\int_{V}\delta\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\delta\underline{u}dV\quad\nearrow^{\approx\,0}\quad+$$

$$+\frac{1}{2}\int_{V}\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\delta\underline{u}dV+\frac{1}{2}\int_{V}\delta\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\underline{u}dV+$$

$$-\int_{V}\underline{F}^{T}\underline{u}dV-\int_{V}\underline{F}^{T}\delta\underline{u}dV+$$

$$-\int_{S}\underline{f}^{T}\underline{u}dS-\int_{S}\underline{f}^{T}\delta\underline{u}dS+$$

$$-\frac{1}{2}\int_{V}\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\underline{u}dV+\int_{V}\underline{F}^{T}\underline{u}dV+\int_{S}\underline{f}^{T}\underline{u}dS=$$

$$=\int_{V}\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\delta\underline{u}dV-\int_{V}\underline{F}^{T}\delta\underline{u}dV-\int_{S}\underline{f}^{T}\delta\underline{u}dS$$

$$(12)$$

$$\int_{V}\delta\underline{u}^{T}\underline{\widetilde{\underline{B}}}^{T}\underline{\underline{D}}\cdot\underline{\widetilde{\underline{B}}}\cdot\underline{u}dV-\int_{V}\delta\underline{u}^{T}\underline{F}dV-\int_{S}\delta\underline{u}^{T}\underline{f}dS=0 \qquad (13)$$

$$\underline{u}=\begin{bmatrix}u_{x}\left(x,y\right)\\u_{y}\left(x,y\right)\end{bmatrix}\qquad\underline{u}_{N}=\begin{bmatrix}u_{1,x}\\u_{1,y}\\u_{2,x}\\u_{2,y}\\u_{3,x}\\u_{3,y}\\u_{4,x}\\u_{4,y}\end{bmatrix}\text{ or }\underline{u}_{N}=\begin{bmatrix}u_{1,x}\\u_{1,y}\\u_{2,x}\\u_{2,y}\\u_{3,x}\\u_{3,y}\\u_{4,x}\\u_{4,y}\\u_{5,x}\\u_{5,y}\\u_{6,x}\\u_{6,y}\\u_{7,x}\\u_{7,y}\\u_{8,x}\\u_{8,y}\end{bmatrix} \qquad (14)$$

$$\underline{u}=\underline{\underline{N}}\cdot\underline{u}_{N} \qquad (15)$$

$$\underline{\underline{N}}=\begin{bmatrix}N_{1}&0&N_{2}&0&N_{3}&0&N_{4}&0\\0&N_{1}&0&N_{2}&0&N_{3}&0&N_{4}\end{bmatrix} \qquad (16)$$

$$\underline{\underline{N}}=\begin{bmatrix}N_{1}&0&N_{2}&0&N_{3}&0&N_{4}&0&N_{5}&0&N_{6}&0&N_{7}&0&N_{8}&0\\0&N_{1}&0&N_{2}&0&N_{3}&0&N_{4}&0&N_{5}&0&N_{6}&0&N_{7}&0&N_{8}\end{bmatrix} \qquad (17)$$

$$\begin{cases} N_1 = N_1\,(\xi,\eta) \\ N_2 = N_2\,(\xi,\eta) \\ N_3 = N_3\,(\xi,\eta) \\ N_4 = N_4\,(\xi,\eta) \end{cases} \quad \text{with} \quad \begin{cases} \xi = \xi\,(x,y) \\ \eta = \eta\,(x,y) \end{cases} \quad \text{for isoparametric elements} \qquad (18)$$

$$\underline{\underline{B}} = \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \qquad (19)$$

$$\underline{\underline{B}} = \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} =$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_5}{\partial x} & 0 & \frac{\partial N_6}{\partial x} & 0 & \frac{\partial N_7}{\partial x} & 0 & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} & 0 & \frac{\partial N_5}{\partial y} & 0 & \frac{\partial N_6}{\partial y} & 0 & \frac{\partial N_7}{\partial y} & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial y} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial y} & \frac{\partial N_6}{\partial x} & \frac{\partial N_7}{\partial y} & \frac{\partial N_7}{\partial x} & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{bmatrix}$$

$$(20)$$

$$\delta\underline{u} = \delta\left(\underline{\underline{N}} \cdot \underline{u}_N\right) = \underline{\underline{N}}\delta\underline{u}_N \qquad (21)$$

$$\int_V \delta u_N{}^T \underline{N}^T \underline{\underline{\widetilde{B}}}^T \underline{D} \cdot \underline{\underline{\widetilde{B}}} \cdot \underline{N} \cdot u_N dV - \int_V \delta u_N{}^T \underline{N}^T \underline{F}dV - \int_S \delta u_N{}^T \underline{N}^T \underline{f}dS = 0 \qquad (22)$$

$$\underline{\delta u_N}^T \left(\int_V \underline{\underline{B}}^T \underline{D} \cdot \underline{\underline{B}}dV \cdot \underline{u}_N - \int_V \underline{\underline{N}}^T \underline{F}dV - \int_S \underline{\underline{N}}^T \underline{f}dS \right) = 0 \qquad (23)$$

$$\underline{k} \cdot \underline{u}_N = \underline{F}_N \quad \underline{k} = \int_V \left(\underline{\underline{B}}^T \underline{D} \cdot \underline{\underline{B}} \right) dV \quad \underline{F}_N = \int_V \underline{\underline{N}}^T \underline{F}dV + \int_S \underline{\underline{N}}^T \underline{f}dS \qquad (24)$$

$$\begin{cases} N_1\,(\xi,\eta) = \frac{1}{4}\,(1-\xi)\,(1-\eta) \\ N_2\,(\xi,\eta) = \frac{1}{4}\,(1+\xi)\,(1-\eta) \\ N_3\,(\xi,\eta) = \frac{1}{4}\,(1+\xi)\,(1+\eta) \\ N_4\,(\xi,\eta) = \frac{1}{4}\,(1-\xi)\,(1+\eta) \end{cases}$$

$$\begin{cases} N_1\,(\xi,\eta) = \frac{1}{4}\,(1-\xi)\,(1-\eta)\,(-\xi-\eta-1) \\ N_2\,(\xi,\eta) = \frac{1}{4}\,(1+\xi)\,(1-\eta)\,(\xi-\eta-1) \\ N_3\,(\xi,\eta) = \frac{1}{4}\,(1+\xi)\,(1+\eta)\,(\xi+\eta-1) \\ N_4\,(\xi,\eta) = \frac{1}{4}\,(1-\xi)\,(1+\eta)\,(-\xi+\eta-1) \\ N_5\,(\xi,\eta) = \frac{1}{2}\,(1-\xi^2)\,(1-\eta) \\ N_6\,(\xi,\eta) = \frac{1}{2}\,(1+\xi)\,(1-\eta^2) \\ N_7\,(\xi,\eta) = \frac{1}{2}\,(1-\xi^2)\,(1+\eta) \\ N_8\,(\xi,\eta) = \frac{1}{2}\,(1-\xi)\,(1-\eta^2) \end{cases} \qquad (25)$$

$$\begin{cases} \frac{\partial N_1(\xi,\eta)}{\partial \xi} = -\frac{1}{4}\,(1-\eta) \\ \frac{\partial N_2(\xi,\eta)}{\partial \xi} = \frac{1}{4}\,(1-\eta) \\ \frac{\partial N_3(\xi,\eta)}{\partial \xi} = \frac{1}{4}\,(1+\eta) \\ \frac{\partial N_4(\xi,\eta)}{\partial \xi} = -\frac{1}{4}\,(1+\eta) \\ \frac{\partial N_1(\xi,\eta)}{\partial \eta} = -\frac{1}{4}\,(1-\xi) \\ \frac{\partial N_2(\xi,\eta)}{\partial \eta} = -\frac{1}{4}\,(1+\xi) \\ \frac{\partial N_3(\xi,\eta)}{\partial \eta} = \frac{1}{4}\,(1+\xi) \\ \frac{\partial N_4(\xi,\eta)}{\partial \eta} = \frac{1}{4}\,(1-\xi) \end{cases} \qquad (26)$$

$$
\begin{cases}
\frac{\partial N_1(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1-\eta)(2\xi+\eta) \\
\frac{\partial N_2(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1-\eta)(2\xi-\eta) \\
\frac{\partial N_3(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1+\eta)(2\xi+\eta) \\
\frac{\partial N_4(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1+\eta)(2\xi-\eta) \\
\frac{\partial N_5(\xi,\eta)}{\partial \xi} = -\xi(1-\eta) \\
\frac{\partial N_6(\xi,\eta)}{\partial \xi} = \frac{1}{2}\left(1-\eta^2\right) \\
\frac{\partial N_7(\xi,\eta)}{\partial \xi} = -\xi(1+\eta) \\
\frac{\partial N_8(\xi,\eta)}{\partial \xi} = -\frac{1}{2}\left(1-\eta^2\right) \\
\frac{\partial N_1(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1-\xi)(2\eta+\xi) \\
\frac{\partial N_2(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1+\xi)(2\eta-\xi) \\
\frac{\partial N_3(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1+\xi)(2\eta+\xi) \\
\frac{\partial N_4(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1-\xi)(2\eta-\xi) \\
\frac{\partial N_5(\xi,\eta)}{\partial \eta} = -\frac{1}{2}\left(1-\xi^2\right) \\
\frac{\partial N_6(\xi,\eta)}{\partial \eta} = -\eta(1+\xi) \\
\frac{\partial N_7(\xi,\eta)}{\partial \eta} = \frac{1}{2}\left(1-\xi^2\right) \\
\frac{\partial N_8(\xi,\eta)}{\partial \eta} = -\eta(1-\xi)
\end{cases}
\tag{27}
$$

$$
\underline{p} = \underline{\underline{N}} \cdot \underline{p}_N
\tag{28}
$$

$$
\underline{p} = \begin{bmatrix} x \\ y \end{bmatrix}
\qquad
\underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix}
\quad \text{or} \quad
\underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \\ x_5 \\ y_5 \\ x_6 \\ y_6 \\ x_7 \\ y_7 \\ x_8 \\ y_8 \end{bmatrix}
\tag{29}
$$

$$
\begin{aligned}
x = x(\xi,\eta) &= N_1(\xi,\eta)\,x_1 + N_2(\xi,\eta)\,x_2 + N_3(\xi,\eta)\,x_3 + N_4(\xi,\eta)\,x_4 \\
y = y(\xi,\eta) &= N_1(\xi,\eta)\,y_1 + N_2(\xi,\eta)\,y_2 + N_3(\xi,\eta)\,y_3 + N_4(\xi,\eta)\,y_4
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
x = x(\xi,\eta) =\,& N_1(\xi,\eta)\,x_1 + N_2(\xi,\eta)\,x_2 + N_3(\xi,\eta)\,x_3 + N_4(\xi,\eta)\,x_4 + \\
& + N_5(\xi,\eta)\,x_5 + N_6(\xi,\eta)\,x_6 + N_7(\xi,\eta)\,x_7 + N_8(\xi,\eta)\,x_8 \\
y = y(\xi,\eta) =\,& N_1(\xi,\eta)\,y_1 + N_2(\xi,\eta)\,y_2 + N_3(\xi,\eta)\,y_3 + N_4(\xi,\eta)\,y_4 + \\
& + N_5(\xi,\eta)\,y_5 + N_6(\xi,\eta)\,y_6 + N_7(\xi,\eta)\,y_7 + N_8(\xi,\eta)\,y_8
\end{aligned}
\tag{31}
$$

$$\begin{cases} \frac{\partial x}{\partial \xi} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi}x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi}x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi}x_4 \\ \frac{\partial x}{\partial \eta} = \frac{\partial N_1(\xi,\eta)}{\partial \eta}x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta}x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta}x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta}x_4 \\ \frac{\partial y}{\partial \xi} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi}y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi}y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi}y_4 \\ \frac{\partial y}{\partial \eta} = \frac{\partial N_1(\xi,\eta)}{\partial \eta}y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta}y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta}y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta}y_4 \end{cases} \tag{32}$$

$$\begin{cases} \frac{\partial x}{\partial \xi} = & \frac{\partial N_1(\xi,\eta)}{\partial \xi}x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi}x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi}x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi}x_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \xi}x_5 + \frac{\partial N_6(\xi,\eta)}{\partial \xi}x_6 + \frac{\partial N_7(\xi,\eta)}{\partial \xi}x_7 + \frac{\partial N_8(\xi,\eta)}{\partial \xi}x_8 \\ \frac{\partial x}{\partial \eta} = & \frac{\partial N_1(\xi,\eta)}{\partial \eta}x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta}x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta}x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta}x_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \eta}x_5 + \frac{\partial N_6(\xi,\eta)}{\partial \eta}x_6 + \frac{\partial N_7(\xi,\eta)}{\partial \eta}x_7 + \frac{\partial N_8(\xi,\eta)}{\partial \eta}x_8 \\ \frac{\partial y}{\partial \xi} = & \frac{\partial N_1(\xi,\eta)}{\partial \xi}y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi}y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi}y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi}y_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \xi}y_5 + \frac{\partial N_6(\xi,\eta)}{\partial \xi}y_6 + \frac{\partial N_7(\xi,\eta)}{\partial \xi}y_7 + \frac{\partial N_8(\xi,\eta)}{\partial \xi}y_8 \\ \frac{\partial y}{\partial \eta} = & \frac{\partial N_1(\xi,\eta)}{\partial \eta}y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta}y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta}y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta}y_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \eta}y_5 + \frac{\partial N_6(\xi,\eta)}{\partial \eta}y_6 + \frac{\partial N_7(\xi,\eta)}{\partial \eta}y_7 + \frac{\partial N_8(\xi,\eta)}{\partial \eta}y_8 \end{cases} \tag{33}$$

$$\begin{cases} \frac{\partial N_1(\xi,\eta)}{\partial x} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_1(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_2(\xi,\eta)}{\partial x} = \frac{\partial N_2(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_2(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_3(\xi,\eta)}{\partial x} = \frac{\partial N_3(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_3(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_4(\xi,\eta)}{\partial x} = \frac{\partial N_4(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_4(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_1(\xi,\eta)}{\partial y} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_1(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_2(\xi,\eta)}{\partial y} = \frac{\partial N_2(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_2(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_3(\xi,\eta)}{\partial y} = \frac{\partial N_3(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_3(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_4(\xi,\eta)}{\partial y} = \frac{\partial N_4(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_4(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \end{cases} \tag{34}$$

$$\begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial x} & \frac{\partial N_1(\xi,\eta)}{\partial y} \\ \frac{\partial N_2(\xi,\eta)}{\partial x} & \frac{\partial N_2(\xi,\eta)}{\partial y} \\ \frac{\partial N_3(\xi,\eta)}{\partial x} & \frac{\partial N_3(\xi,\eta)}{\partial y} \\ \frac{\partial N_4(\xi,\eta)}{\partial x} & \frac{\partial N_4(\xi,\eta)}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial \xi} & \frac{\partial N_1(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_2(\xi,\eta)}{\partial \xi} & \frac{\partial N_2(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_3(\xi,\eta)}{\partial \xi} & \frac{\partial N_3(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_4(\xi,\eta)}{\partial \xi} & \frac{\partial N_4(\xi,\eta)}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \tag{35}$$

$$\underline{e}_\xi = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} \qquad \underline{e}_\eta = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{36}$$

$$\underline{e}_x = \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial x} \end{bmatrix} \qquad \underline{e}_y = \begin{bmatrix} \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial y} \end{bmatrix} \tag{37}$$

$$\underline{\underline{J}} = \left[\underline{e}_\xi | \underline{e}_\eta\right] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad \underline{\underline{J}}^{-1} = \left[\underline{e}_x | \underline{e}_y\right] = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \tag{38}$$

$$\underline{\underline{J}}^{-1} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix} \tag{39}$$

$$\underline{\underline{g}} = \underline{\underline{J}}^T \underline{\underline{J}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta} \end{bmatrix} \tag{40}$$

$$g = det\left(\underline{\underline{g}}\right) =$$

$$= \left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right) \tag{41}$$

$$dV\left(x, y\right) = \sqrt{g}dV\left(\xi, \eta\right) \tag{42}$$

$$\underline{\underline{k_e}} = \int_{V_e(x,y)} \left(\underline{B}^T \underline{\underline{D}} \cdot \underline{B}\right) dV_e\left(x, y\right) = \int_{V_e(\xi,\eta)} \left(\underline{B}^T \underline{\underline{D}} \cdot \underline{B}\right) \sqrt{g}dV_e\left(\xi, \eta\right)$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(\underline{B}^T \underline{\underline{D}} \cdot \underline{B}\right)\sqrt{g}d\xi d\eta \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \left(\underline{B}^T\left(\xi_i, \eta_j\right) \cdot \underline{\underline{D}} \cdot \underline{B}\left(\xi_i, \eta_j\right)\sqrt{g}\right) \tag{43}$$

$$k_e = \begin{bmatrix} k_{e|11} & k_{e|12} & k_{e|13} & k_{e|14} & k_{e|15} & k_{e|16} & k_{e|17} & k_{e|18} \\ k_{e|12} & k_{e|22} & k_{e|23} & k_{e|24} & k_{e|25} & k_{e|26} & k_{e|27} & k_{e|28} \\ k_{e|13} & k_{e|23} & k_{e|33} & k_{e|34} & k_{e|35} & k_{e|36} & k_{e|37} & k_{e|38} \\ k_{e|14} & k_{e|24} & k_{e|34} & k_{e|44} & k_{e|45} & k_{e|46} & k_{e|47} & k_{e|48} \\ k_{e|15} & k_{e|25} & k_{e|35} & k_{e|45} & k_{e|55} & k_{e|56} & k_{e|57} & k_{e|58} \\ k_{e|16} & k_{e|26} & k_{e|36} & k_{e|46} & k_{e|56} & k_{e|66} & k_{e|67} & k_{e|68} \\ k_{e|17} & k_{e|27} & k_{e|37} & k_{e|47} & k_{e|57} & k_{e|67} & k_{e|77} & k_{e|78} \\ k_{e|18} & k_{e|28} & k_{e|38} & k_{e|48} & k_{e|58} & k_{e|68} & k_{e|78} & k_{e|88} \end{bmatrix} \tag{44}$$

2. VCCT for first order quadrilateral elements

2.1. Definition of crack tip reference frame

Figure 1. Schematic representation of the discretized crack tip geometry for 1st order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \tag{46}$$

2.2. Calculation of displacements and reaction forces

$$u_x = u_{x,M} - u_{x,F} \qquad u_y = u_{y,M} - u_{y,F} \tag{47}$$

$$u_r = \cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y \qquad u_\theta = -\sin(\Delta\theta)\, u_x + \cos(\Delta\theta)\, u_y \tag{48}$$

$$F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT} \tag{49}$$

$$
\left\{
\begin{aligned}
&\left(k_{e,M|11} + k_{e,M|33}\right) u_{x,MCT} + \left(k_{e,M|12} + k_{e,M|34}\right) u_{y,MCT}+ \\
&+k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + \left(k_{M|17} + k_{M|35}\right) u_{N,MC|7} + \left(k_{M|18} + k_{M|36}\right) u_{N,MC|8}+ \\
&+ \sum_{i=5}^{6} k_{M|1i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|3i} u_{N,MB|i} + k_{M|31} u_{x,NCOI} + k_{M|32} u_{y,NCOI} = 0 \\[6pt]
&\left(k_{e,M|21} + k_{e,M|43}\right) u_{x,MCT} + \left(k_{e,M|22} + k_{e,M|44}\right) u_{y,MCT}+ \\
&+k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + \left(k_{M|27} + k_{M|45}\right) u_{N,MC|7} + \left(k_{M|28} + k_{M|46}\right) u_{N,MC|8}+ \\
&+ \sum_{i=5}^{6} k_{M|2i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|4i} u_{N,MB|i} + k_{M|41} u_{x,NCOI} + k_{M|42} u_{y,NCOI} = 0 \\[6pt]
&\left(k_{e,F|77} + k_{e,F|55}\right) u_{x,FCT} + \left(k_{e,F|78} + k_{e,F|56}\right) u_{y,FCT}+ \\
&+k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + \left(k_{F|71} + k_{F|53}\right) u_{N,FC|1} + \left(k_{F|72} + k_{F|54}\right) u_{N,FC|2}+ \\
&+ \sum_{i=2}^{3} k_{F|7i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|5i} u_{N,FB|i} + k_{F|57} u_{x,NCOI} + k_{F|58} u_{y,NCOI} = 0 \\[6pt]
&\left(k_{e,F|87} + k_{e,F|65}\right) u_{x,FCT} + \left(k_{e,F|88} + k_{e,F|66}\right) u_{y,FCT}+ \\
&+k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + \left(k_{F|81} + k_{F|63}\right) u_{N,FC|1} + \left(k_{F|82} + k_{F|64}\right) u_{N,FC|2}+ \\
&+ \sum_{i=2}^{3} k_{F|8i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|6i} u_{N,FB|i} + k_{F|67} u_{x,NCOI} + k_{F|68} u_{y,NCOI} = 0 \\[6pt]
&u_{x,FCT} - u_{x,MCT} - u_{x,DUMMY} = 0 \\
&u_{y,FCT} - u_{y,MCT} - u_{y,DUMMY} = 0 \\[6pt]
&u_{x,DUMMY} = 0 \\
&u_{y,DUMMY} = 0
\end{aligned}
\right.
\tag{50}
$$

$$
\left\{
\begin{aligned}
&\left(k_{e,M|11} + k_{e,M|33}\right) u_{x,MCT} + \left(k_{e,M|12} + k_{e,M|34}\right) u_{y,MCT}+ \\
&+k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + \left(k_{M|17} + k_{M|35}\right) u_{N,MC|7} + \left(k_{M|18} + k_{M|36}\right) u_{N,MC|8}+ \\
&+ \sum_{i=5}^{6} k_{M|1i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|3i} u_{N,MB|i} + k_{M|31} u_{x,NCOI} + k_{M|32} u_{y,NCOI} = 0 \\[6pt]
&\left(k_{e,M|21} + k_{e,M|43}\right) u_{x,MCT} + \left(k_{e,M|22} + k_{e,M|44}\right) u_{y,MCT}+ \\
&+k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + \left(k_{M|27} + k_{M|45}\right) u_{N,MC|7} + \left(k_{M|28} + k_{M|46}\right) u_{N,MC|8}+ \\
&+ \sum_{i=5}^{6} k_{M|2i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|4i} u_{N,MB|i} + k_{M|41} u_{x,NCOI} + k_{M|42} u_{y,NCOI} = 0 \\[6pt]
&\left(k_{e,F|77} + k_{e,F|55}\right) u_{x,MCT} + \left(k_{e,F|78} + k_{e,F|56}\right) u_{y,MCT}+ \\
&+k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + \left(k_{F|71} + k_{F|53}\right) u_{N,FC|1} + \left(k_{F|72} + k_{F|54}\right) u_{N,FC|2}+ \\
&+ \sum_{i=2}^{3} k_{F|7i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|5i} u_{N,FB|i} + k_{F|57} u_{x,NCOI} + k_{F|58} u_{y,NCOI} = 0 \\[6pt]
&\left(k_{e,F|87} + k_{e,F|65}\right) u_{x,MCT} + \left(k_{e,F|88} + k_{e,F|66}\right) u_{y,MCT}+ \\
&+k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + \left(k_{F|81} + k_{F|63}\right) u_{N,FC|1} + \left(k_{F|82} + k_{F|64}\right) u_{N,FC|2}+ \\
&+ \sum_{i=2}^{3} k_{F|8i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|6i} u_{N,FB|i} + k_{F|67} u_{x,NCOI} + k_{F|68} u_{y,NCOI} = 0 \\[6pt]
&u_{x,FCT} = u_{x,MCT} \\
&u_{y,FCT} = u_{y,MCT} \\[6pt]
&R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\
&R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{aligned}
\right.
\tag{51}
$$

$$\begin{cases}
\left(k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}\right) u_{x,MCT} + \left(k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}\right) u_{y,MCT} + \\[4pt]
+ k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F} + \\[4pt]
+ \left(k_{M|31} + k_{F|57}\right) u_{x,NCOI} + \left(k_{M|32} + k_{F|58}\right) u_{y,NCOI} + \\[4pt]
+ \left(k_{M|17} + k_{M|35}\right) u_{N,MC|7} + \left(k_{M|18} + k_{M|36}\right) u_{N,MC|8} + \left(k_{F|71} + k_{F|53}\right) u_{N,FC|1} + \left(k_{F|72} + k_{F|54}\right) u_{N,FC|2} + \\[4pt]
+ \sum_{i=5}^{6} k_{M|1i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|3i} u_{N,MB|i} + \sum_{i=2}^{3} k_{F|7i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|5i} u_{N,FB|i} = 0 \\[10pt]
\left(k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}\right) u_{x,MCT} + \left(k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}\right) u_{y,MCT} + \\[4pt]
+ k_{e,M|23} u_{x,M} + k_{e,M|24} u_{y,M} + k_{e,F|85} u_{x,F} + k_{e,F|86} u_{y,F} + \\[4pt]
+ \left(k_{M|41} + k_{F|67}\right) u_{x,NCOI} + \left(k_{M|42} + k_{F|68}\right) u_{y,NCOI} + \\[4pt]
+ \left(k_{M|27} + k_{M|45}\right) u_{N,MC|7} + \left(k_{M|28} + k_{M|46}\right) u_{N,MC|8} + \left(k_{F|81} + k_{F|63}\right) u_{N,FC|1} + \left(k_{F|82} + k_{F|64}\right) u_{N,FC|2} + \\[4pt]
+ \sum_{i=2}^{3} k_{F|8i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|6i} u_{N,FB|i} + \sum_{i=5}^{6} k_{M|2i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|4i} u_{N,MB|i} = 0 \\[10pt]
u_{x,FCT} = u_{x,MCT} \\[4pt]
u_{y,FCT} = u_{y,MCT} \\[8pt]
R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\[4pt]
R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{cases} \tag{52}$$

$$\begin{cases}
u_{y,MCT} = -\dfrac{k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} u_{x,MCT} + \\[6pt]
-\dfrac{k_{e,M|13} u_{x,M} + k_{e,M|14} u_{y,M} + k_{e,F|75} u_{x,F} + k_{e,F|76} u_{y,F}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\[6pt]
-\dfrac{\left(k_{M|31} + k_{F|57}\right) u_{x,NCOI} + \left(k_{M|32} + k_{F|58}\right) u_{y,NCOI}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\[6pt]
-\dfrac{\left(k_{M|17} + k_{M|35}\right) u_{N,MC|7} + \left(k_{M|18} + k_{M|36}\right) u_{N,MC|8} + \left(k_{F|71} + k_{F|53}\right) u_{N,FC|1} + \left(k_{F|72} + k_{F|54}\right) u_{N,FC|2}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} + \\[6pt]
-\dfrac{\sum_{i=5}^{6} k_{M|1i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|3i} u_{N,MB|i} + \sum_{i=2}^{3} k_{F|7i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|5i} u_{N,FB|i}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \\[10pt]
\left[\left(k_{e,M|21} + k_{e,M|43} + k_{e,F|87} + k_{e,F|65}\right) + \dfrac{k_{e,M|11} + k_{e,M|33} + k_{e,F|77} + k_{e,F|55}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \left(k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}\right)\right] u_{x,MCT} + \\[6pt]
+ \left(k_{e,M|23} - \dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|13}\right) u_{x,M} + \\[6pt]
+ \left(k_{e,M|24} - \dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|14}\right) u_{y,M} + \\[6pt]
+ \left(k_{e,F|85} - \dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|75}\right) u_{x,F} + \\[6pt]
+ \left(k_{e,F|86} - \dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} k_{e,M|76}\right) u_{y,F} + \\[6pt]
+ \left[\left(k_{M|41} + k_{F|67}\right) - \dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \left(k_{M|31} + k_{F|57}\right)\right] u_{x,NCOI} + \\[6pt]
+ \left[\left(k_{M|42} + k_{F|68}\right) - \dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \left(k_{M|32} + k_{F|58}\right)\right] u_{y,NCOI} + \\[6pt]
+ \left(k_{M|27} + k_{M|45}\right) u_{N,MC|7} + \left(k_{M|28} + k_{M|46}\right) u_{N,MC|8} + \left(k_{F|81} + k_{F|63}\right) u_{N,FC|1} + \left(k_{F|82} + k_{F|64}\right) u_{N,FC|2} + \\[6pt]
-\dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \left[\left(k_{M|17} + k_{M|35}\right) u_{N,MC|7} + \left(k_{M|18} + k_{M|36}\right) u_{N,MC|8}\right] + \\[6pt]
-\dfrac{k_{e,M|22} + k_{e,M|44} + k_{e,F|88} + k_{e,F|66}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} \left[\left(k_{F|71} + k_{F|53}\right) u_{N,FC|1} + \left(k_{F|72} + k_{F|54}\right) u_{N,FC|2}\right] \\[6pt]
+ \sum_{i=2}^{3} k_{F|8i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|6i} u_{N,FB|i} + \sum_{i=5}^{6} k_{M|2i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|4i} u_{N,MB|i} + \\[6pt]
-\dfrac{\sum_{i=5}^{6} k_{M|1i} u_{N,MC|i} + \sum_{i=7}^{8} k_{M|3i} u_{N,MB|i} + \sum_{i=2}^{3} k_{F|7i} u_{N,FC|i} + \sum_{i=1}^{2} k_{F|5i} u_{N,FB|i}}{k_{e,M|12} + k_{e,M|34} + k_{e,F|78} + k_{e,F|56}} = 0 \\[10pt]
u_{x,FCT} = u_{x,MCT} \\[4pt]
u_{y,FCT} = u_{y,MCT} \\[8pt]
R_{x,DUMMY} = R_{x,FCT} = -R_{x,MCT} = F_{x,CT} \\[4pt]
R_{y,DUMMY} = R_{y,FCT} = -R_{y,MCT} = F_{y,CT}
\end{cases} \tag{53}$$

$$
\begin{cases}
u_{y,MCT} = -\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}u_{x,MCT}+ \\[4pt]
\quad -\frac{k_{e,M|13}u_{x,M}+k_{e,M|14}u_{y,M}+k_{e,F|75}u_{x,F}+k_{e,F|76}u_{y,F}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}+ \\[4pt]
\quad -\frac{(k_{M|31}+k_{F|57})u_{x,NCOI}+(k_{M|32}+k_{F|58})u_{y,NCOI}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}+ \\[4pt]
\quad -\frac{(k_{M|17}+k_{M|35})u_{N,MC|7}+(k_{M|18}+k_{M|36})u_{N,MC|8}+(k_{F|71}+k_{F|53})u_{N,FC|1}+(k_{F|72}+k_{F|54})u_{N,FC|2}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}+ \\[4pt]
\quad -\frac{\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} \\[12pt]
\left[(k_{e,M|21}+k_{e,M|43}+k_{e,F|87}+k_{e,F|65})+\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}(k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66})\right]u_{x,MCT}+ \\[4pt]
\quad +\left(k_{e,M|23}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|13}\right)u_x+ \\[4pt]
\quad +\left(k_{e,M|24}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|14}\right)u_y+ \\[4pt]
\quad +\left(k_{e,M|23}+k_{e,F|85}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}(k_{e,M|13}+k_{e,M|75})\right)\underbrace{u_{x,F}}_{\approx 0}+ \\[4pt]
\quad +\left(k_{e,M|24}+k_{e,F|86}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}(k_{e,M|14}+k_{e,M|76})\right)\underbrace{u_{y,F}}_{\approx 0}+ \\[4pt]
\quad +\left[(k_{M|41}+k_{F|67})-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}(k_{M|31}+k_{F|57})\right]u_{x,NCOI}+ \\[4pt]
\quad +\left[(k_{M|42}+k_{F|68})-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}(k_{M|32}+k_{F|58})\right]u_{y,NCOI}+ \\[4pt]
\quad +(k_{M|27}+k_{M|45})u_{N,MC|7}+(k_{M|28}+k_{M|46})u_{N,MC|8}+(k_{F|81}+k_{F|63})u_{N,FC|1}+(k_{F|82}+k_{F|64})u_{N,FC|2}+ \\[4pt]
\quad -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left[(k_{M|17}+k_{M|35})u_{N,MC|7}+(k_{M|18}+k_{M|36})u_{N,MC|8}\right]+ \\[4pt]
\quad -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left[(k_{F|71}+k_{F|53})u_{N,FC|1}+(k_{F|72}+k_{F|54})u_{N,FC|2}\right] \\[4pt]
\quad +\sum_{i=2}^{3}k_{F|8i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|6i}u_{N,FB|i}+\sum_{i=5}^{6}k_{M|2i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|4i}u_{N,MB|i}+ \\[4pt]
\quad -\frac{\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}=0 \\[12pt]
u_{x,FCT}=u_{x,MCT} \\[2pt]
u_{y,FCT}=u_{y,MCT} \\[10pt]
F_{x,CT}\quad=R_{x,FCT}= \\[2pt]
\quad=(k_{e,F|77}+k_{e,F|55})u_{x,FCT}+(k_{e,F|78}+k_{e,F|56})u_{y,FCT}+ \\[2pt]
\quad+k_{e,F|75}\underbrace{u_{x,F}}_{\approx 0}+k_{e,F|76}\underbrace{u_{y,F}}_{\approx 0}+ \\[2pt]
\quad+\sum_{i=1}^{4}k_{e,F|7i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8}k_{e,F|5i}u_{N,FB|i} \\[4pt]
F_{y,CT}\quad=R_{y,FCT}= \\[2pt]
\quad=(k_{e,F|87}+k_{e,F|65})u_{x,FCT}+(k_{e,F|88}+k_{e,F|66})u_{y,FCT}+ \\[2pt]
\quad+k_{e,F|85}\underbrace{u_{x,F}}_{\approx 0}+k_{e,F|86}\underbrace{u_{y,F}}_{\approx 0}+ \\[2pt]
\quad+\sum_{i=1}^{4}k_{e,F|8i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8}k_{e,F|6i}u_{N,FB|i}
\end{cases}
\tag{54}
$$

$$
\begin{cases}
F_{x,CT}\quad=(k_{e,F|77}+k_{e,F|55})u_{x,MCT}+(k_{e,F|78}+k_{e,F|56})u_{y,MCT}+ \\
\qquad+\sum_{i=1}^{4}k_{e,F|7i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8}k_{e,F|5i}u_{N,FB|i} \\
F_{y,CT}\quad=(k_{e,F|87}+k_{e,F|65})u_{x,MCT}+(k_{e,F|88}+k_{e,F|66})u_{y,MCT}+ \\
\qquad+\sum_{i=1}^{4}k_{e,F|8i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8}k_{e,F|6i}u_{N,FB|i}
\end{cases}
\tag{55}
$$

$$
\begin{cases}
F_{x,CT}\quad=K_{xx}u_x+K_{xy}u_y+ \\
\qquad+\sum_{i=1}^{4}K_{FC,x|i}u_{N,FC|i}+\sum_{i=1,i\neq(3,4,5,6)}^{8}K_{FB,x|i}u_{N,FB|i}+ \\
\qquad+\sum_{i=5}^{8}K_{FC,x|i}u_{N,MC|i}+\sum_{i=7}^{8}K_{MB,x|i}u_{N,FB|i} \\
F_{y,CT}\quad=K_{yx}u_x+K_{yy}u_y+ \\
\qquad+\sum_{i=1}^{4}K_{FC,y|i}u_{N,FC|i}+\sum_{i=1,i\neq(3,4,5,6)}^{8}K_{FB,y|i}u_{N,FB|i}+ \\
\qquad+\sum_{i=5}^{8}K_{FC,y|i}u_{N,MC|i}+\sum_{i=7}^{8}K_{MB,y|i}u_{N,FB|i}
\end{cases}
\tag{56}
$$

$$\begin{cases} F_{x,CT} &= K_{xx}u_x + K_{xy}u_y + \widetilde{F}_x \\ F_{y,CT} &= K_{yx}u_x + K_{yy}u_y + \widetilde{F}_y \end{cases} \tag{57}$$

2.3. Calculation of energy release rates

$$\begin{aligned} G_{I,r\theta} &= \frac{1}{2}\frac{F_r u_r}{R_f \delta} = \\ &= \frac{1}{2R_f \delta}\left(\cos\left(\Delta\theta\right)F_x + \sin\left(\Delta\theta\right)F_y\right)\left(\cos\left(\Delta\theta\right)u_x + \sin\left(\Delta\theta\right)u_y\right) = \\ &= \frac{1}{2R_f \delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) \end{aligned} \tag{58}$$

$$\begin{aligned} G_{II,r\theta} &= \frac{1}{2}\frac{F_\theta u_\theta}{R_f \delta} = \\ &= \frac{1}{2R_f \delta}\left(-\sin\left(\Delta\theta\right)F_x + \cos\left(\Delta\theta\right)F_y\right)\left(-\sin\left(\Delta\theta\right)u_x + \cos\left(\Delta\theta\right)u_y\right) = \\ &= \frac{1}{2R_f \delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right) \end{aligned} \tag{59}$$

$$\begin{aligned} G_{TOT,r\theta} &= G_{I,r\theta} + G_{II,r\theta} = \\ &= \frac{1}{2R_f \delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) + \\ &\quad + \frac{1}{2R_f \delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right) = \\ &= \frac{1}{2R_f \delta}\left(\underbrace{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)}_{1}F_x u_x\right) + \\ &\quad + \frac{1}{2R_f \delta}\left(\underbrace{\left(\left(F_x u_y + F_y u_x\right) - \left(F_x u_y + F_y u_x\right)\right)}_{0}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\right) + \\ &\quad + \frac{1}{2R_f \delta}\left(\underbrace{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)}_{1}F_y u_y\right) = \\ &= \frac{1}{2}\frac{F_x u_x}{R_f \delta} + \frac{1}{2}\frac{F_y u_y}{R_f \delta} = \\ &= \widetilde{G}_{I,xy} + \widetilde{G}_{II,xy} = \widetilde{G}_{TOT,xy} \end{aligned} \tag{60}$$

$$\begin{aligned} G_{I,r\theta} &= \frac{1}{2R_f \delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) = \\ &= \cos^2\left(\Delta\theta\right)\frac{F_x u_x}{2R_f \delta} + \left(\frac{F_x u_y}{2R_f \delta} + \frac{F_y u_x}{2R_f \delta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\frac{F_y u_y}{2R_f \delta} = \\ &= \cos^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} + \left(\widetilde{G}_{I,xy}\frac{u_y}{u_x} + \widetilde{G}_{II,xy}\frac{u_x}{u_y}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\widetilde{G}_{II,xy} \end{aligned} \tag{61}$$

$$G_{II,r\theta} = \frac{1}{2R_f\delta} \left(\sin^2\left(\Delta\theta\right) F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right) F_y u_y\right) =$$

$$= \sin^2\left(\Delta\theta\right)\frac{F_x u_x}{2R_f\delta} - \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)\frac{F_y u_y}{2R_f\delta} = \qquad (62)$$

$$= \sin^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} - \left(\widetilde{G}_{I,xy}\frac{u_y}{u_x} + \widetilde{G}_{II,xy}\frac{u_x}{u_y}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)\widetilde{G}_{II,xy}$$

2.4. Sensitivity analysis of the FEM solution

$$\begin{cases} F_{x,CT} &= K_{xx}u_x + K_{xy}u_y + \widetilde{F}_x \\ F_{y,CT} &= K_{yx}u_x + K_{yy}u_y + \widetilde{F}_y \end{cases} \qquad (63)$$

$$G_{I,r\theta} \sim \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)\left(K_{xx}u_x^2 + K_{xy}u_y u_x + \widetilde{F}_x u_x\right) +$$

$$+ \frac{1}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\left(K_{xx}u_x u_y + K_{xy}u_y^2 + \widetilde{F}_x u_y\right) +$$

$$+ \frac{1}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\left(K_{yx}u_x^2 + K_{yy}u_y u_x + \widetilde{F}_y u_x\right) + \qquad (64)$$

$$+ \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)\left(K_{yx}u_y u_x + K_{yy}u_y^2 + \widetilde{F}_y\right)$$

$$G_{II,r\theta} \sim \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right) k_x u_x^2\left(\Delta\theta\right) +$$

$$- \frac{1}{2R_f\delta}\left(k_x + k_y\right) u_x\left(\Delta\theta\right) u_y\left(\Delta\theta\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \qquad (65)$$

$$+ \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right) k_y u_y^2\left(\Delta\theta\right)$$

$$G_{TOT,r\theta} \sim \frac{1}{2R_f\delta}\left(k_x u_x^2\left(\Delta\theta\right) + k_y u_y^2\left(\Delta\theta\right)\right) \qquad (66)$$

$$\frac{\partial G_{I,r\theta}}{\partial\Delta\theta} \sim \frac{1}{R_f\delta}\cos^2\left(\Delta\theta\right) k_x u_x\left(\Delta\theta\right)\frac{\partial u_x\left(\Delta\theta\right)}{\partial\Delta\theta} +$$

$$+ \frac{1}{2R_f\delta}\left(k_x + k_y\right)\left(\frac{\partial u_x\left(\Delta\theta\right)}{\partial\Delta\theta}u_y\left(\Delta\theta\right) + u_x\left(\Delta\theta\right)\frac{\partial u_y\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) +$$

$$+ \frac{1}{R_f\delta}\sin^2\left(\Delta\theta\right) k_y u_y\left(\Delta\theta\right)\frac{\partial u_y\left(\Delta\theta\right)}{\partial\Delta\theta} + \qquad (67)$$

$$+ \frac{1}{2R_f\delta}\left(k_y u_y^2\left(\Delta\theta\right) - k_x u_x^2\left(\Delta\theta\right)\right)\sin\left(2\Delta\theta\right) +$$

$$+ \frac{1}{2R_f\delta}\left(k_x u_x\left(\Delta\theta\right) u_y\left(\Delta\theta\right) + k_y u_y\left(\Delta\theta\right) u_x\left(\Delta\theta\right)\right)\cos\left(2\Delta\theta\right)$$

$$\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f\delta} \sin^2(\Delta\theta)\, k_x u_x(\Delta\theta)\, \frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} +$$

$$-\frac{1}{2R_f\delta}(k_x + k_y)\left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} u_y(\Delta\theta) + u_x(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) +$$

$$+\frac{1}{R_f\delta}\cos^2(\Delta\theta)\, k_y u_y(\Delta\theta)\,\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \tag{68}$$

$$+\frac{1}{2R_f\delta}\left(k_x u_x^2(\Delta\theta) - k_y u_y^2(\Delta\theta)\right)\sin(2\Delta\theta) +$$

$$-\frac{1}{2R_f\delta}\left(k_x u_x(\Delta\theta) u_y(\Delta\theta) + k_y u_y(\Delta\theta) u_x(\Delta\theta)\right)\cos(2\Delta\theta)$$

$$\frac{\partial G_{TOT,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f\delta}\left(k_x u_x(\Delta\theta)\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + k_y u_y(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta}\right) \tag{69}$$

2.5. Discretization error

$$u_r = \cos(\Delta\theta - \delta)\, u_x + \sin(\Delta\theta - \delta)\, u_y \qquad u_\theta = -\sin(\Delta\theta - \delta)\, u_x + \cos(\Delta\theta - \delta)\, u_y \tag{70}$$

$$F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT} \tag{71}$$

$$\widetilde{G}_{I,r\theta} = \frac{1}{2}\frac{F_r u_r}{R_f\delta} =$$

$$= \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\cos(\Delta\theta - \delta)\, u_x + \sin(\Delta\theta - \delta)\, u_y\right) =$$

$$= \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y\right)\cos(\delta) +$$

$$+ \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\sin(\Delta\theta)\, u_x - \cos(\Delta\theta)\, u_y\right)\sin(\delta) =$$

$$= \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta)\, F_x u_x + (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_y\right)\cos(\delta) +$$

$$+ \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)\, F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_x\right)\sin(\delta) =$$

$$= G_{I,r\theta}\cos(\delta) +$$

$$+ \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)\, F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_x\right)\sin(\delta) =$$

$$= G_{I,r\theta}\cos(\delta) - \frac{1}{2R_f\delta}F_r u_\theta \sin\delta$$

$$\tag{72}$$

$$\lim_{\delta \to 0} \widetilde{G}_{I,r\theta} = \lim_{\delta \to 0} G_{I,r\theta} \cos(\delta) +$$

$$+ \lim_{\delta \to 0} \frac{1}{2R_f \delta} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \sin(\delta) =$$

$$= G_{I,r\theta} +$$

$$+ \frac{1}{2R_f} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right) \lim_{\delta \to 0} \frac{\sin(\delta)}{\delta} =$$

$$= G_{I,r\theta} + \frac{1}{2R_f} \left(-\cos^2(\Delta\theta) F_x u_y + (F_x u_x - F_y u_y) \cos(\Delta\theta) \sin(\Delta\theta) + \sin^2(\Delta\theta) F_y u_x \right)$$

$$= G_{I,r\theta} - \frac{1}{2R_f} F_r u_\theta$$

$$(73)$$

$$\widetilde{G}_{II,r\theta} = \frac{1}{2} \frac{F_\theta u_\theta}{R_f \delta} =$$

$$= \frac{1}{2R_f \delta} \left(-\sin(\Delta\theta) F_x + \cos(\Delta\theta) F_y \right) \left(-\sin(\Delta\theta - \delta) u_x + \cos(\Delta\theta - \delta) u_y \right) =$$

$$= \frac{1}{2R_f \delta} \left(\sin^2(\Delta\theta) \cos(\delta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) \cos(\delta) + \cos^2(\Delta\theta) \cos(\delta) F_y u_y \right) +$$

$$+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) \sin(\delta) + \cos^2(\Delta\theta) \sin(\delta) F_y u_x - \sin^2(\Delta\theta) \sin(\delta) F_x u_y \right) =$$

$$= \frac{1}{2R_f \delta} \left(\sin^2(\Delta\theta) F_x u_x - (F_x u_y + F_y u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_y \right) \cos(\delta) +$$

$$+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) =$$

$$= G_{II,r\theta} \cos(\delta) +$$

$$+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) =$$

$$= G_{II,r\theta} \cos(\delta) + \frac{1}{2R_f \delta} F_\theta u_r \sin\delta$$

$$(74)$$

$$\lim_{\delta \to 0} \widetilde{G}_{II,r\theta} = \lim_{\delta \to 0} G_{II,r\theta} \cos(\delta) +$$

$$+ \lim_{\delta \to 0} \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \sin(\delta) =$$

$$= G_{II,r\theta} +$$

$$+ \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) \lim_{\delta \to 0} \frac{\sin(\delta)}{\delta}$$

$$= G_{II,r\theta} + \frac{1}{2R_f \delta} \left((F_y u_y - F_x u_x) \cos(\Delta\theta) \sin(\Delta\theta) + \cos^2(\Delta\theta) F_y u_x - \sin^2(\Delta\theta) F_x u_y \right) =$$

$$= G_{II,r\theta} + \frac{1}{2R_f} F_\theta u_r$$

$$(75)$$

$$\widetilde{G}_{TOT,r\theta} = \widetilde{G}_{I,r\theta} + \widetilde{G}_{II,r\theta} =$$
$$= (G_{I,r\theta} + G_{II,r\theta})\cos(\delta) +$$
$$+ \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)F_xu_y + (F_xu_x - F_yu_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)F_yu_x\right)\sin(\delta) +$$
$$+ \frac{1}{2R_f\delta}\left((F_yu_y - F_xu_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)F_yu_x - \sin^2(\Delta\theta)F_xu_y\right)\sin(\delta) =$$
$$= (G_{I,r\theta} + G_{II,r\theta})\cos(\delta) + \frac{1}{2R_f\delta}(F_yu_x - F_xu_y)\sin(\delta) =$$
$$= G_{TOT,r\theta}\cos(\delta) + \frac{1}{2R_f\delta}(F_yu_x - F_xu_y)\sin(\delta) =$$
$$= G_{TOT,r\theta}\cos(\delta) + \frac{1}{2R_f\delta}(F_\theta u_r - F_r u_\theta)\sin(\delta)$$

$$(76)$$

$$\lim_{\delta\to 0}\widetilde{G}_{TOT,r\theta} = \lim_{\delta\to 0}G_{TOT,r\theta}\cos(\delta) + \lim_{\delta\to 0}\frac{1}{2R_f\delta}(F_yu_x - F_xu_y)\sin(\delta) =$$
$$= G_{TOT,r\theta} + \frac{1}{2R_f}(F_yu_x - F_xu_y)\lim_{\delta\to 0}\frac{\sin(\delta)}{\delta} - 0$$
$$= G_{TOT,r\theta} + \frac{1}{2R_f}(F_yu_x - F_xu_y) =$$
$$= G_{TOT,r\theta} + \frac{1}{2R_f}(F_\theta u_r - F_r u_\theta)$$

$$(77)$$

2.6. Contact region

$$u_r = 0 \tag{78}$$

$$\cos(\Delta\theta)\,u_x + \sin(\Delta\theta)\,u_y = 0 \tag{79}$$

$$u_y = -\frac{u_x}{\tan(\Delta\theta)} \tag{80}$$

$$u_\theta = -\sin(\Delta\theta)\,u_x - \frac{\cos^2(\Delta\theta)}{\sin(\Delta\theta)}u_x =$$
$$= -\frac{u_x}{\sin(\Delta\theta)} \tag{81}$$

$$F_r = \cos(\Delta\theta)\,F_{x,CT} + \sin(\Delta\theta)\,F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\,F_{x,CT} + \cos(\Delta\theta)\,F_{y,CT} \tag{82}$$

$$G_{I,r\theta} = \frac{1}{2}\frac{F_r u_r}{R_f\delta} = 0 \tag{83}$$

$$
\begin{aligned}
G_{II,r\theta} &= \frac{1}{2}\frac{F_\theta u_\theta}{R_f \delta} = \\
&= \frac{1}{2R_f \delta}\left(-\sin\left(\Delta\theta\right)F_x + \cos\left(\Delta\theta\right)F_y\right)\left(-\frac{u_x}{\sin\left(\Delta\theta\right)}\right) = \\
&= \frac{1}{2R_f \delta}\left(F_x u_x - \frac{F_y u_x}{\tan\left(\Delta\theta\right)}\right) \\
&= \frac{1}{2R_f \delta}\left(F_x - \frac{F_y}{\tan\left(\Delta\theta\right)}\right)u_x
\end{aligned}
\tag{84}
$$

3. VCCT for second order quadrilateral elements

3.1. Definition of crack tip reference frame

Figure 2. Schematic representation of the discretized crack tip geometry for 2nd order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos\left(\Delta\theta\right) & \sin\left(\Delta\theta\right) \\ -\sin\left(\Delta\theta\right) & \cos\left(\Delta\theta\right) \end{bmatrix} \qquad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^{T} = \begin{bmatrix} \cos\left(\Delta\theta\right) & -\sin\left(\Delta\theta\right) \\ \sin\left(\Delta\theta\right) & \cos\left(\Delta\theta\right) \end{bmatrix} \qquad (85)$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \qquad (86)$$

3.2. Calculation of displacements and reaction forces

$$\begin{aligned} u_{x,1} = u_{x,M1} - u_{x,F1} \qquad u_{y,1} = u_{y,M1} - u_{y,F1} \\ u_{x,2} = u_{x,M2} - u_{x,F2} \qquad u_{y,2} = u_{y,M2} - u_{y,F2} \end{aligned} \qquad (87)$$

$$\begin{aligned} u_{r,1} = \cos\left(\Delta\theta\right) u_{x,1} + \sin\left(\Delta\theta\right) u_{y,1} \qquad u_{\theta,1} = -\sin\left(\Delta\theta\right) u_{x,1} + \cos\left(\Delta\theta\right) u_{y,1} \\ u_{r,2} = \cos\left(\Delta\theta\right) u_{x,2} + \sin\left(\Delta\theta\right) u_{y,2} \qquad u_{\theta,2} = -\sin\left(\Delta\theta\right) u_{x,2} + \cos\left(\Delta\theta\right) u_{y,2} \end{aligned} \qquad (88)$$

$$F_{r,1} = \cos\left(\Delta\theta\right) F_{x,1} + \sin\left(\Delta\theta\right) F_{y,1} \qquad F_{\theta,1} = -\sin\left(\Delta\theta\right) F_{x,1} + \cos\left(\Delta\theta\right) F_{y,1}$$
$$F_{r,2} = \cos\left(\Delta\theta\right) F_{x,2} + \sin\left(\Delta\theta\right) F_{y,2} \qquad F_{\theta,2} = -\sin\left(\Delta\theta\right) F_{x,2} + \cos\left(\Delta\theta\right) F_{y,2} \tag{89}$$

3.3. Calculation of energy release rates

$$
\begin{aligned}
G_{I,r\theta} = & \frac{1}{2R_f\delta}\left(F_{r,1}u_{r,1} + F_{r,2}u_{r,2}\right) = \\
= & \frac{1}{2R_f\delta}\left(\cos\left(\Delta\theta\right) F_{x,1} + \sin\left(\Delta\theta\right) F_{y,1}\right)\left(\cos\left(\Delta\theta\right) u_{x,1} + \sin\left(\Delta\theta\right) u_{y,1}\right) + \\
& + \frac{1}{2R_f\delta}\left(\cos\left(\Delta\theta\right) F_{x,2} + \sin\left(\Delta\theta\right) F_{y,2}\right)\left(\cos\left(\Delta\theta\right) u_{x,2} + \sin\left(\Delta\theta\right) u_{y,2}\right) + \\
= & \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right) F_{x,1}u_{x,1} + \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right) F_{y,1}u_{y,1}\right) + \\
& + \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right) F_{x,2}u_{x,2} + \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right) F_{y,2}u_{y,2}\right)
\end{aligned}
\tag{90}
$$

$$
\begin{aligned}
G_{II,r\theta} = & \frac{1}{2R_f\delta}\left(F_{\theta,1}u_{\theta,1} + F_{\theta,2}u_{\theta,2}\right) = \\
= & \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right) F_{x,1} + \cos\left(\Delta\theta\right) F_{y,1}\right)\left(-\sin\left(\Delta\theta\right) u_{x,1} + \cos\left(\Delta\theta\right) u_{y,1}\right) + \\
& + \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right) F_{x,2} + \cos\left(\Delta\theta\right) F_{y,2}\right)\left(-\sin\left(\Delta\theta\right) u_{x,2} + \cos\left(\Delta\theta\right) u_{y,2}\right) = \\
= & \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right) F_{x,1}u_{x,1} - \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right) F_{y,1}u_{y,1}\right) + \\
& + \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right) F_{x,2}u_{x,2} - \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right) F_{y,2}u_{y,2}\right)
\end{aligned}
\tag{91}
$$

$$
\begin{aligned}
G_{TOT,r\theta} =\,& G_{I,r\theta} + G_{II,r\theta} = \\
=\,& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_{x,1}u_{x,1} + \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_{y,1}u_{y,1}\right) + \\
& + \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_{x,2}u_{x,2} + \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_{y,2}u_{y,2}\right) + \\
& + \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_{x,1}u_{x,1} - \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_{y,1}u_{y,1}\right) + \\
& + \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_{x,2}u_{x,2} - \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_{y,2}u_{y,2}\right) = \\
=\,& \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right) + \\
& + \frac{1}{2R_f\delta}\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) + \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
& + \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right) + \\
& - \frac{1}{2R_f\delta}\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) + \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) = \\
=\,& \frac{1}{2R_f\delta}\left(\overbrace{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)}^{1}\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2}\right)\right) + \\
& + \frac{1}{2R_f\delta}\left(\overbrace{\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) - \left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right)}^{0}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
& + \frac{1}{2R_f\delta}\left(\overbrace{\left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right) - \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)}^{0}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
& + \frac{1}{2R_f\delta}\left(\overbrace{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)}^{1}\left(F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right)\right) = \\
=\,& \frac{1}{2}\frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{R_f\delta} + \frac{1}{2}\frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{R_f\delta} = \\
=\,& \widetilde{G}_{I,xy} + \widetilde{G}_{II,xy} = \widetilde{G}_{TOT,xy}
\end{aligned}
$$

$$(92)$$

$$G_{I,r\theta} = \cos^2\left(\Delta\theta\right)\frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{2R_f\delta} +$$

$$+ \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) +$$

$$+ \sin^2\left(\Delta\theta\right)\frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{2R_f\delta} =$$

$$= \cos^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} + \sin^2\left(\Delta\theta\right)\widetilde{G}_{II,xy} +$$

$$+ \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) \tag{93}$$

$$G_{II,r\theta} = \sin^2\left(\Delta\theta\right)\frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{2R_f\delta} +$$

$$- \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) +$$

$$+ \cos^2\left(\Delta\theta\right)\frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{2R_f\delta} =$$

$$= \sin^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} + \cos^2\left(\Delta\theta\right)\widetilde{G}_{II,xy} +$$

$$- \frac{F_{x,1}u_{y,1} + F_{y,1}u_{x,1} + F_{x,2}u_{y,2} + F_{y,2}u_{x,2}}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) \tag{94}$$

3.4. Sensitivity analysis of the FEM solution

$$\begin{array}{cc} F_{x,1} \sim k_{x,1}u_{x,1} & F_{y,1} \sim k_{y,1}u_{y,1} \\ F_{x,2} \sim k_{x,2}u_{x,2} & F_{y,2} \sim k_{y,2}u_{y,2} \end{array} \tag{95}$$

$$G_{I,r\theta} \sim \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2\right) +$$

$$+ \frac{1}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}u_{y,1} + k_{y,1}u_{y,1}u_{x,1} + k_{x,2}u_{x,2}u_{y,2} + k_{y,2}u_{y,2}u_{x,2}\right) +$$

$$+ \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)\left(k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2\right) \tag{96}$$

$$G_{II,r\theta} \sim \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2\right) +$$

$$- \frac{1}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}u_{y,1} + k_{y,1}u_{y,1}u_{x,1} + k_{x,2}u_{x,2}u_{y,2} + k_{y,2}u_{y,2}u_{x,2}\right) +$$

$$+ \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)\left(k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2\right) \tag{97}$$

$$G_{TOT,r\theta} \sim \frac{1}{2R_f\delta}\left(\left(k_{x,1}u_{x,1}^2\left(\Delta\theta\right) + k_{x,2}u_{x,2}^2\left(\Delta\theta\right)\right) + \left(k_{y,1}u_{y,1}^2\left(\Delta\theta\right) + k_{y,2}u_{y,2}^2\left(\Delta\theta\right)\right)\right) \tag{98}$$

$$\frac{\partial G_{I,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f\delta}\cos^2(\Delta\theta)\left(k_{x,1}u_{x,1}(\Delta\theta)\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} + k_{x,2}u_{x,2}(\Delta\theta)\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta}\right) +$$

$$+ \frac{1}{2R_f\delta}(k_{x,1}+k_{y,1})\left(\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta}u_{y,1}(\Delta\theta) + u_{x,1}(\Delta\theta)\frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) +$$

$$+ \frac{1}{2R_f\delta}(k_{x,2}+k_{y,2})\left(\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta}u_{y,2}(\Delta\theta) + u_{x,2}(\Delta\theta)\frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) +$$

$$+ \frac{1}{R_f\delta}\sin^2(\Delta\theta)\left(k_{y,1}u_{y,1}(\Delta\theta)\frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} + k_{y,2}u_{y,2}(\Delta\theta)\frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta}\right) +$$

$$+ \frac{1}{2R_f\delta}\left(\left(k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2\right) - \left(k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2\right)\right)\sin(2\Delta\theta) +$$

$$+ \frac{1}{2R_f\delta}\left((k_{x,1}+k_{y,1})u_{x,1}u_{y,1} + (k_{x,2}+k_{y,2})u_{x,2}u_{y,2}\right)\cos(2\Delta\theta)$$

$$(99)$$

$$\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f\delta}\sin^2(\Delta\theta)\left(k_{x,1}u_{x,1}(\Delta\theta)\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta} + k_{x,2}u_{x,2}(\Delta\theta)\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta}\right) +$$

$$- \frac{1}{2R_f\delta}(k_{x,1}+k_{y,1})\left(\frac{\partial u_{x,1}(\Delta\theta)}{\partial \Delta\theta}u_{y,1}(\Delta\theta) + u_{x,1}(\Delta\theta)\frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) +$$

$$- \frac{1}{2R_f\delta}(k_{x,2}+k_{y,2})\left(\frac{\partial u_{x,2}(\Delta\theta)}{\partial \Delta\theta}u_{y,2}(\Delta\theta) + u_{x,2}(\Delta\theta)\frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) +$$

$$+ \frac{1}{R_f\delta}\cos^2(\Delta\theta)\left(k_{y,1}u_{y,1}(\Delta\theta)\frac{\partial u_{y,1}(\Delta\theta)}{\partial \Delta\theta} + k_{y,2}u_{y,2}(\Delta\theta)\frac{\partial u_{y,2}(\Delta\theta)}{\partial \Delta\theta}\right) +$$

$$+ \frac{1}{2R_f\delta}\left(\left(k_{x,1}u_{x,1}^2 + k_{x,2}u_{x,2}^2\right) - \left(k_{y,1}u_{y,1}^2 + k_{y,2}u_{y,2}^2\right)\right)\sin(2\Delta\theta) +$$

$$- \frac{1}{2R_f\delta}\left((k_{x,1}+k_{y,1})u_{x,1}u_{y,1} + (k_{x,2}+k_{y,2})u_{x,2}u_{y,2}\right)\cos(2\Delta\theta)$$

$$(100)$$

$$G_{TOT,r\theta} \sim \frac{1}{R_f\delta}\left(\left(k_{x,1}u_{x,1}\frac{\partial u_{x,1}}{\partial \Delta\theta} + k_{x,2}u_{x,2}\frac{\partial u_{x,2}}{\partial \Delta\theta}\right) + \left(k_{y,1}u_{y,1}\frac{\partial u_{y,1}}{\partial \Delta\theta} + k_{y,2}u_{y,2}\frac{\partial u_{y,2}}{\partial \Delta\theta}\right)\right) \quad (101)$$

4. A vectorial formulation of the VCCT

4.1. Vectorial formulation

m is the order of the element's shape functions, and $p, q = 1$ represents the crack tip

$$\underline{\underline{R}}_{\Delta\theta} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}_{\Delta\theta}^{-1} = \underline{\underline{R}}^T = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{102}$$

$$\underline{\underline{P}}_{\delta}(p) = \begin{bmatrix} \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \\ -\sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \end{bmatrix} \tag{103}$$

$$\underline{\underline{P}}_{\delta}^{-1} = \underline{\underline{P}}_{\delta}^T = \begin{bmatrix} \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & -\sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \\ \sin\left(\left(1 + \frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1 + \frac{1-p}{m}\right)\delta\right) \end{bmatrix} \tag{104}$$

$$\underline{\underline{Q}}_{\delta}(q) = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & \sin\left(\frac{q-1}{m}\delta\right) \\ -\sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \tag{105}$$

$$\underline{\underline{Q}}_{\delta}^{-1} = \underline{\underline{Q}}_{\delta}^T = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & -\sin\left(\frac{q-1}{m}\delta\right) \\ \sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \tag{106}$$

$$\frac{\partial \underline{\underline{R}}_{\Delta\theta}}{\partial \Delta\theta} = \underline{\underline{D}} \cdot \underline{\underline{R}}_{\Delta\theta} \tag{107}$$

$$\frac{\partial \underline{\underline{P}}_{\delta}}{\partial \delta} = \left(1 + \frac{1-p}{m}\right) \underline{\underline{D}} \cdot \underline{\underline{P}}_{\delta} \tag{108}$$

$$\frac{\partial \underline{\underline{Q}}_{\delta}}{\partial \delta} = \frac{q-1}{m} \underline{\underline{D}} \cdot \underline{\underline{Q}}_{\delta} \tag{109}$$

$$\underline{\underline{D}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{110}$$

$$\underline{F}_{xy} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \qquad \underline{F}_{r\theta} = \begin{bmatrix} F_r \\ F_\theta \end{bmatrix} \tag{111}$$

$$\underline{u}_{xy} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad \underline{u}_{r\theta} = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \tag{112}$$

$$\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} \tag{113}$$

$$\underline{F}_{r\theta} = \underline{\underline{Q}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{F}_{xy} \qquad \underline{u}_{r\theta} = \underline{\underline{P}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{u}_{xy} \tag{114}$$

$$\underline{F}_{xy} = \underline{\underline{K}}_{xy} \underline{u}_{xy} + \widetilde{\underline{F}}_{xy} = \underline{\underline{K}}_{xy} \underline{u}_{xy} + \widetilde{\underline{\underline{K}}}_N \underline{u}_N \tag{115}$$

$$G_{TOT} = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} \underline{u}_{r\theta,p}^T \underline{\underline{T}}_{pq}^T \underline{F}_{r\theta,q} =$$

$$= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{F}_{r\theta,q}\underline{u}_{r\theta,p}^T \underline{\underline{T}}_{pq}\right) = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\begin{bmatrix} t_{pq|11}F_{r,q}u_{r,p} & t_{pq|12}F_{r,q}u_{\theta,p} \\ t_{pq|21}F_{\theta,q}u_{r,p} & t_{pq|22}F_{\theta,q}u_{\theta,p} \end{bmatrix}\right) =$$

$$= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{F}_{xy,q}\underline{u}_{xy,p}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) \tag{116}$$

$$\underline{\underline{T}}_{pq} = \begin{cases} \underline{\underline{I}} \ for \ p = q < 2 \\ \underline{\underline{0}} \ otherwise \end{cases} \quad \text{for } 1^{st} \text{ order quadrilateral elements}$$

$$= \begin{cases} \underline{\underline{I}} \ for \ p = q < 3 \\ \underline{\underline{0}} \ otherwise \end{cases} \quad \text{for } 2^{nd} \text{ order quadrilateral elements}$$

$$= \begin{cases} \underline{\underline{I}} \ for \ p = q < 4 \\ \underline{\underline{0}} \ otherwise \end{cases} \quad \text{for } 3^{rd} \text{ order quadrilateral elements}$$

$$= \begin{cases} \left(14 - \frac{33\pi}{8}\right)\underline{\underline{I}} \ for \ p = 1, q = 1 \\ \left(-52 + \frac{33\pi}{2}\right)\underline{\underline{I}} \ for \ p = 1, q = 2 \\ \left(17 - \frac{21\pi}{4}\right)\underline{\underline{I}} \ for \ p = 2, q = 1 \\ \left(-\frac{7}{2} + \frac{21\pi}{16}\right)\underline{\underline{I}} \ for \ p = 2, q = 2 \\ \left(8 - \frac{21\pi}{8}\right)\underline{\underline{I}} \ for \ p = 1, q = 3 \\ \left(-32 + \frac{21\pi}{2}\right)\underline{\underline{I}} \ for \ p = 2, q = 3 \\ \underline{\underline{0}} \ otherwise \end{cases} \quad \text{for } 2^{nd} \text{ order quarter-point quadrilateral elements} \tag{117}$$

$$= \begin{cases} \left(-11187 + \frac{7155\pi}{2}\right)\underline{\underline{I}} \ for \ p = 1, q = 1 \\ \left(38556 - \frac{24543\pi}{2}\right)\underline{\underline{I}} \ for \ p = 1, q = 2 \\ \left(-53055 + \frac{33777\pi}{2}\right)\underline{\underline{I}} \ for \ p = 1, q = 3 \\ \left(\frac{11396}{3} - \frac{9575\pi}{8}\right)\underline{\underline{I}} \ for \ p = 2, q = 1 \\ \left(-12936 + \frac{33003\pi}{8}\right)\underline{\underline{I}} \ for \ p = 2, q = 2 \\ \left(17988 - \frac{45837\pi}{8}\right)\underline{\underline{I}} \ for \ p = 2, q = 3 \\ \left(-\frac{8453}{3} + \frac{3595\pi}{4}\right)\underline{\underline{I}} \ for \ p = 3, q = 1 \\ \left(9804 - \frac{12411\pi}{4}\right)\underline{\underline{I}} \ for \ p = 3, q = 2 \\ \left(-13587 + \frac{17289\pi}{4}\right)\underline{\underline{I}} \ for \ p = 3, q = 3 \\ \left(6948 - \frac{17685\pi}{8}\right)\underline{\underline{I}} \ for \ p = 1, q = 4 \\ \left(-23976 + \frac{60993\pi}{8}\right)\underline{\underline{I}} \ for \ p = 2, q = 4 \\ \left(33372 - \frac{84807\pi}{8}\right)\underline{\underline{I}} \ for \ p = 3, q = 4 \\ \underline{\underline{0}} \ otherwise \end{cases} \quad \text{for } 3^{rd} \text{ order quarter-point quadrilateral elements}$$

$$G_{TOT} = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \left(\underline{\underline{K}}_{xy}\underline{u}_{xy} + \underline{\widetilde{F}}_{xy}\right)\underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) =$$

$$= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{F}}_{xy}\underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) \tag{118}$$

$$\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{F}_{xy,q}\underline{u}_{xy,p}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) \tag{119}$$

$$\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} = \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{F}}_{xy} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) =$$

$$= \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \underline{u}_{N} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) \tag{120}$$

4.2. Sensitivity analysis

$$\frac{\partial \underline{G}}{\partial \delta} = -\frac{1}{2R_f\delta^2} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) - \frac{1}{2R_f\delta^2} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \underline{u}_{N} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{D}}^T \underline{\underline{T}}_{pq}^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \underline{u}_{N} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{D}}^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{DQ}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{u}_{xy} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{DQ}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \underline{u}_{N} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \frac{\partial \underline{u}_{N}}{\partial \delta} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \underline{u}_{xy} \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \underline{u}_{N} \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) =$$

$$= \frac{1}{\delta} \underline{G} + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \left(\underline{\underline{K}}_{xy} \underline{u}_{xy} + \underline{\widetilde{K}}_{N} \underline{u}_{N}\right) \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{D}}^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{DQ}}_\delta \underline{\underline{R}}_{\Delta\theta} \left(\underline{\underline{K}}_{xy} \underline{u}_{xy} + \underline{\widetilde{K}}_{N} \underline{u}_{N}\right) \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\underline{K}}_{xy} \frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) + \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \frac{\partial \underline{u}_{N}}{\partial \delta} \underline{u}_{xy}^T \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) +$$

$$+ \frac{1}{2R_f\delta} \sum_{p=1}^{m+1} \sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_\delta \underline{\underline{R}}_{\Delta\theta} \underline{\widetilde{K}}_{N} \underline{u}_{N} \frac{\partial \underline{u}_{xy}^T}{\partial \delta} \underline{\underline{R}}_{\Delta\theta}^T \underline{\underline{P}}_\delta^T \underline{\underline{T}}_{pq}^T\right) \tag{121}$$

Following Comninou

$$u\left(\delta\right) \sim \sqrt{\delta}\left(\sin, \cos\right)\left(\epsilon \log \delta\right) \qquad \epsilon = \frac{1}{2\pi} \log\left(\frac{1-\beta}{1+\beta}\right) \tag{122}$$

$$\underline{u}_{xy}, \underline{u}_{N} \sim u\left(\delta\right) \sim \sqrt{\delta}\left(\sin, \cos\right)\left(\epsilon \log \delta\right) \xrightarrow{\delta \to 0} 0 \tag{123}$$

$$\underline{u}_{xy}\underline{u}_{xy}^T, \underline{u}_{N}\underline{u}_{xy}^T \sim u^2\left(\delta\right) \sim \delta\left(\sin^2, \cos^2, \sin\cdot\cos\right)\left(\epsilon \log \delta\right) \xrightarrow{\delta \to 0} 0 \tag{124}$$

$$\frac{\partial \underline{u}_{xy}}{\partial \delta} \underline{u}_{xy}^T, \frac{\partial \underline{u}_{N}}{\partial \delta} \underline{u}_{xy}^T \sim -\frac{1}{2}\left(\sin^2, \cos^2, \sin\cdot\cos\right)\left(\epsilon \log \delta\right) + \left(-\sin^2, \cos^2, \pm\sin\cdot\cos\right)\left(\epsilon \log \delta\right) \xrightarrow{\delta \to 0} finite \tag{125}$$

$$\underline{G} \sim \frac{1}{\delta} \underline{u}_{xy} \underline{u}_{xy}^T \sim \frac{1}{\delta} u^2\left(\delta\right) \sim \left(\sin^2, \cos^2, \sin\cdot\cos\right)\left(\epsilon \log \delta\right) \xrightarrow{\delta \to 0} finite \tag{126}$$

$$\underline{\underline{P}}_\delta, \underline{\underline{Q}}_\delta \xrightarrow{\delta \to 0} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{127}$$

$$\lim_{\delta \to 0} \frac{\partial \underline{G}}{\partial \delta} \sim \frac{1}{\delta}\left(\underline{F}(\delta)^{\,0} + \underline{C}\right) \tag{128}$$

$$\lim_{\delta \to 0} \frac{\partial \underline{G}}{\partial \delta} \sim \frac{1}{\delta} \xrightarrow{\int d\delta} \lim_{\delta \to 0} G \sim A \log(\delta) + B \tag{129}$$

$$\frac{\partial \underline{G}}{\partial \Delta \theta} = \frac{1}{2R_f \delta} Diag\left(\underline{DR}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_\delta^T\right) + \frac{1}{2R_f \delta} Diag\left(\underline{DR}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_N\underline{u}_N\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_\delta^T\right) +$$

$$+ \frac{1}{2R_f \delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{D}}^T\underline{\underline{R}}_\delta^T\right) + \frac{1}{2R_f \delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_N\underline{u}_N\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{D}}^T\underline{\underline{R}}_\delta^T\right) +$$

$$+ \frac{1}{2R_f \delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\frac{\partial \underline{u}_{xy}}{\partial \Delta\theta}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_\delta^T\right) + \frac{1}{2R_f \delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_N\frac{\partial \underline{u}_N}{\partial \Delta\theta}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_\delta^T\right) +$$

$$+ \frac{1}{2R_f \delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\frac{\partial \underline{u}_{xy}^T}{\partial \Delta\theta}\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_\delta^T\right) + \frac{1}{2R_f \delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_N\underline{u}_N\frac{\partial \underline{u}_{xy}^T}{\partial \Delta\theta}\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_\delta^T\right) \tag{130}$$

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