Constitutive modeling for laminates with fiber/matrix interface cracks under transverse loading

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Abstract

Keywords: Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

1. Introduction

Main ref [1, 2, 3, 4]

2. Derivation of constitutive relations

2.1. Reference frames

5 **Local reference frame of k-th layer:** index 1 is the in-plane longitudinal or fiber or 0°-direction; index 2 is the in-plane transverse or 90°-direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0°direction; index y is the in-plane transverse direction; index
10 z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \tag{1}$$

V_a is equal to the product of total crack surface S_C and average crack opening

u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \qquad (2)$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \cancel{w}}{L_{lam} \cancel{w} t_{90^\circ}} R_f \Delta\theta = \frac{1}{n2L} \frac{1}{k2L} R_f \Delta\theta =$$
$$= \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nkR_f} \frac{\Delta\theta}{\pi}$$
$$(3)$$

2.3. Homogenization

$$\sigma_{ij}^{avg} = \frac{1}{V} \int_V \sigma_{ij} dV \quad \varepsilon_{ij}^{avg} = \frac{1}{V} \int_V \varepsilon_{ij} dV \qquad (4)$$

$$\underline{\tilde{\sigma}}_k^{avg} = \underline{\underline{\tilde{Q}}}_k \left(\underline{\tilde{\varepsilon}}_k^{avg} - \underline{\tilde{\alpha}}_k \Delta T \right) \qquad (5)$$

$$\underline{\tilde{\sigma}}_{LAM} = \underline{\tilde{\sigma}}^{avg} = \frac{1}{V} \int_V \underline{\tilde{\sigma}} dV = \frac{1}{V} \sum_{k=1}^{N} \int_{V_k} \underline{\tilde{\sigma}} dV_k = \sum_{k=1}^{N} \underline{\tilde{\sigma}}_k^{avg} \frac{t_k}{h} \qquad (6)$$

$$\varepsilon_{ij}^{avg} = \frac{1}{V} \int_V \varepsilon_{ij} dV = \frac{1}{V} \int_V \frac{1}{2} \left(u_{i,j} + u_{j,i} \right) dV = \frac{1}{V} \int_S \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \quad (7)$$

$$\varepsilon_{ij}^{avg} = \frac{1}{V} \int_S \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS =$$
$$= \frac{1}{V} \int_{S_B} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS + \frac{1}{V} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS = \qquad (8)$$
$$= \varepsilon_{ij}^{applied} + \beta_{ij}$$

$$\underline{\tilde{\varepsilon}}_k^{avg} = \underline{\tilde{\varepsilon}}_k^{applied} + \underline{\tilde{\beta}}_k \qquad (9)$$

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2.4. Vakulenko-Kachanov tensor

In the local reference frame of k-th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \tag{10}$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \tag{11}$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \tag{12}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$
\begin{aligned}
\beta_{11} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_1 + u_1 n_1 \right) dS = \frac{1}{V_k} \int_{S_C} u_1^{\ 0} n_1 dS = 0 \\
\beta_{22} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_2 + u_2 n_2 \right) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_3 + u_3 n_3 \right) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\
\beta_{12} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1^{\ 0} n_2 + u_2 n_1^{\ 0} \right) dS = 0 \\
\beta_{13} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1^{\ 0} n_3 + u_3 n_1^{\ 0} \right) dS = 0 \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_3 + u_3 n_2 \right) dS \\
\beta_{21} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_1^{\ 0} + u_1^{\ 0} n_2 \right) dS = 0 \\
\beta_{31} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_1^{\ 0} + u_1^{\ 0} n_3 \right) dS = 0 \\
\beta_{32} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_2 + u_2 n_3 \right) dS = \beta_{23}
\end{aligned} \tag{13}
$$

Only 3 independent components of the tensor β_{ij} remain: β_{22}, β_{33} and β_{23}. Split total crack surface S_C into total matrix crack surface S_C^m and total fiber

crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f (-n_2^m) dS \right] \\
\beta_{33} =& \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f (-n_3^m) dS \right] \\
\beta_{23} =& \frac{1}{V_k} \int_{S_C} (u_2 n_3 + u_3 n_2) \, dS = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} (u_2^m n_3^m + u_3^m n_2^m) \, dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f \right) dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f (-n_3^m) \, dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f (-n_2^m) \, dS \right]
\end{aligned}
\tag{14}
$$

The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$
S_C^m = S_C^f = n_D R_f \Delta\theta
\tag{15}
$$

With Eq. 14, we can recast Eq. 13 as

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k} \left[n_D R_f w \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] = \\
=& \frac{1}{L_{lam} w t_{90°}} \left[n_D R_f w \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] = \\
=& \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90°}} \left[\int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] = \\
=& \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_2^m d\theta \right] \\
\beta_{33} =& \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_3^m - u_3^f \right) n_3^m d\theta \right] \\
\beta_{23} =& \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_2^m - u_2^f \right) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(u_3^m - u_3^f \right) n_2^m d\theta \right]
\end{aligned}
\tag{16}
$$

We can express the displacement jumps at the interface as a function of
the local Crack Opening Displacement (COD) and Crack Sliding Displacement
(CSD) as

$$
\begin{aligned}
u_2^m - u_2^f =& \left(u_r^m - u_r^f \right) \cos\left(\theta\right) - \left(u_\theta^m - u_\theta^f \right) \sin\left(\theta\right) = \\
=& COD\left(\theta\right) \cos\left(\theta\right) - CSD\left(\theta\right) \sin\left(\theta\right) \\
u_3^m - u_3^f =& \left(u_r^m - u_r^f \right) \sin\left(\theta\right) + \left(u_\theta^m - u_\theta^f \right) \cos\left(\theta\right) = \\
=& COD\left(\theta\right) \sin\left(\theta\right) + CSD\left(\theta\right) \cos\left(\theta\right)
\end{aligned}
\tag{17}
$$

where θ is the local angular coordinate at the interface. We can similarly
express n_2^m and n_3^m as a function of θ:

$$
\begin{aligned}
n_2^m =& \cos\left(\theta\right) - \sin\left(\theta\right) \\
n_3^m =& \sin\left(\theta\right) + \cos\left(\theta\right)
\end{aligned}
\tag{18}
$$

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Thus, Eq. 15 becomes

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(\cos^2\left(\theta\right) - \cos\left(\theta\right)\sin\left(\theta\right)\right) - CSD\left(\theta\right) \left(\sin\left(\theta\right)\cos\left(\theta\right) - \sin^2\left(\theta\right)\right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(1 + \cos\left(2\theta\right) - \sin\left(2\theta\right)\right) + CSD\left(\theta\right) \left(1 - \cos\left(2\theta\right) - \sin\left(2\theta\right)\right) \right] d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(\sin\left(\theta\right)\cos\left(\theta\right) + \sin^2\left(\theta\right)\right) + CSD\left(\theta\right) \left(\cos^2\left(\theta\right) + \cos\left(\theta\right)\sin\left(\theta\right)\right) \right] d\theta =$$

$$= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right) \left(1 + \sin\left(2\theta\right) - \cos\left(2\theta\right)\right) + CSD\left(\theta\right) \left(1 + \sin\left(2\theta\right) + \cos\left(2\theta\right)\right) \right] d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right) \left(2\sin\left(\theta\right)\cos\left(\theta\right) + \cos^2\left(\theta\right) - \sin^2\left(\theta\right)\right) +$$

$$- \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD\left(\theta\right) \left(\sin^2\left(\theta\right) - \cos^2\left(\theta\right) + 2\cos\left(\theta\right)\sin\left(\theta\right)\right) d\theta =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \sin\left(2\theta\right) \right] d\theta$$

$$(19)$$

40   *2.5. Analytical modeling of $COD\left(\theta\right)$ and $CSD\left(\theta\right)$*

The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of θ, the angular coordinate along the crack which varies between 0 and $\Delta\theta$. Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be
45   expressed as the sum of their average value and a term, respectively $\delta COD\left(\theta\right)$

and $\delta CSD(\theta)$, that represents the variation of the function from its average:

$$COD(\theta) = COD_{avg} + \delta COD(\theta)$$
$$CSD(\theta) = CSD_{avg} + \delta CSD(\theta). \tag{20}$$

By defining $\Delta\Psi$

$$\Delta\Psi = \min(\Delta\theta, \Delta\Phi), \tag{21}$$

we introduce at this point an approximation and assume that the functions $\delta COD(\theta)$ and $\delta CSD(\theta)$ can be expressed as the product of the maximum value of the displacement and a function, respectively $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$:

$$COD(\theta) = COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max}f\left(\theta - \frac{\Delta\Psi}{2}\right)$$
$$CSD(\theta) = CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max}g\left(\theta - \frac{\Delta\theta}{2}\right), \tag{22}$$

where $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ are assumed to be odd functions over their respective integration domain $[0, \Delta\Psi]$ and $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = 0 \quad \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = 0. \tag{23}$$

We assume the two functions $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ to be two odd polynomials of degree $2n - 1$:

$$f\left(\theta - \frac{\Delta\Psi}{2}\right) = \begin{cases} \sum_{k=0}^{n-1} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} & 0 \le \theta \le \Delta\Psi \\ 0 & otherwise \end{cases}$$
$$g\left(\theta - \frac{\Delta\theta}{2}\right) = \begin{cases} \sum_{k=0}^{n-1} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} & 0 \le \theta \le \Delta\theta \\ 0 & otherwise \end{cases} \tag{24}$$

which satisfy by construction the conditions expressed in Equation 22. The coefficients a_{2k+1} and b_{2k+1} are determined by imposing that

$$COD(\Delta\Psi) = 0$$
$$CSD(\Delta\theta) = 0. \tag{25}$$

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The explicit construction of the polynomials $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ for $n = 1, 2, 3$ (or degree $2n - 1 = 1, 3, 5$) is reported in Appendix A.

We recall the expressions of the non-zero components of the Vakulenko-Kachanov

60   tensor

$$\beta_{22} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta$$

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta$$

$$\beta_{23} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \sin\left(2\theta\right) \right] d\theta$$

$$(26)$$

and proceed to the integration of the different summands:

1.

$$\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD\left(\theta\right)d\theta =$$

$$=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}\right)d\theta =$$

$$=\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD_{avg}d\theta+$$

$$+\frac{1}{\Delta\theta}\int_0^{\Delta\Psi} COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}d\theta+$$

$$+\frac{1}{\Delta\theta}\int_{\Delta\Psi}^{\Delta\theta} COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}d\theta \quad = $$

$$=\frac{1}{\Delta\theta}\left[COD_{avg}\theta\right]\bigg|_0^{\Delta\theta}+\frac{1}{\Delta\theta}\left[COD_{max}\sum_{k=0}^{n-1}\frac{a_{2k+1}}{2\left(k+1\right)}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2(k+1)}\right]\bigg|_0^{\Delta\Psi}=$$

$$=COD_{avg}+COD_{max}\sum_{k=0}^{n-1}\frac{a_{2k+1}}{2\left(k+1\right)\Delta\theta}\left(\left(\frac{\Delta\Psi}{2}\right)^{2(k+1)}-\left(-\frac{\Delta\Psi}{2}\right)^{2(k+1)}\right)=$$

$$=COD_{avg}$$

$$(27)$$

2.

$$\frac{1}{\Delta\theta}\int_0^{\Delta\theta} CSD\left(\theta\right)d\theta = CSD_{avg} \qquad (28)$$

3.

$$\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD\left(\theta\right)\sin\left(2\theta\right)d\theta =$$

$$=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg}+COD_{max}\sum_{k=0}^{n-1}a_{2k+1}\theta^{2k+1}\right)\sin\left(2\theta\right)d\theta =$$

$$=-\frac{1}{2\Delta\theta}COD_{avg}\left[\cos\left(2\theta\right)\right]\bigg|_0^{\Delta\theta}+$$

$$+\frac{1}{\Delta\theta}\left[COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1+mod\left(i,2\right)}{2}\pi-2\theta)\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{\left(k-i\right)!}\theta^{k-i}\right)\right]\bigg|_0^{\Delta\Psi}=$$

$$=\frac{1-\cos\left(2\Delta\theta\right)}{2\Delta\theta}COD_{avg}+$$

$$+\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1+mod\left(i,2\right)}{2}\pi-2\Delta\Psi)\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{\left(k-i\right)!}\Delta\Psi^{k-i}\right)$$

$$(29)$$

4.

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD\left(\theta\right) \sin\left(2\theta\right) d\theta =$$

$$= \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(CSD_{avg} + CSD_{max} \sum_{k=0}^{n-1} b_{2k+1} \theta^{2k+1} \right) \sin\left(2\theta\right) d\theta =$$

$$= -\frac{1}{2\Delta\theta} CSD_{avg} \left[\cos\left(2\theta\right)\right] \Big|_0^{\Delta\theta} +$$

$$+ \frac{1}{\Delta\theta} \left[CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+mod\left(i,2\right)}{2}\pi - 2\theta\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \theta^{k-i} \right) \right] \Big|_0^{\Delta\Psi} =$$

$$= \frac{1-\cos\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} +$$

$$+ \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+mod\left(i,2\right)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)$$

$$\tag{30}$$

5.

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right) \cos\left(2\theta\right) d\theta =$$

$$= \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \right) \cos\left(2\theta\right) d\theta =$$

$$= \frac{1}{2\Delta\theta} COD_{avg} \left[\sin\left(2\theta\right)\right] \Big|_0^{\Delta\theta} +$$

$$+ \frac{1}{\Delta\theta} \left[COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1+mod\left(i+1,2\right)}{2}\pi - 2\theta\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \theta^{k-i} \right) \right] \Big|_0^{\Delta\Psi} =$$

$$= \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} +$$

$$+ \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1+mod\left(i+1,2\right)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)$$

$$\tag{31}$$

6.

$$\frac{1}{\Delta\theta}\int_0^{\Delta\theta} CSD\left(\theta\right)\cos\left(2\theta\right)d\theta =$$

$$=\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(CSD_{avg}+CSD_{max}\sum_{k=0}^{n-1}b_{2k+1}\theta^{2k+1}\right)\cos\left(2\theta\right)d\theta =$$

$$=-\frac{1}{2\Delta\theta}CSD_{avg}\left[\cos\left(2\theta\right)\right]\Big|_0^{\Delta\theta}+$$

$$+\frac{1}{\Delta\theta}\left[CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^i\sin(\frac{1+mod\left(i+1,2\right)}{2}\pi-2\theta)\left(\sum_{k=i}^{2n-1}b_k\frac{k!}{(k-i)!}\theta^{k-i}\right)\right]\Big|_0^{\Delta\Psi}=$$

$$=\frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta}CSD_{avg}+$$

$$+\frac{1}{\Delta\theta}CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^i\sin(\frac{1+mod\left(i+1,2\right)}{2}\pi-2\Delta\Psi)\left(\sum_{k=i}^{2n-1}b_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right)$$

$$(32)$$

Finally we can compute the expressions of the components of the Vakulenko-Kachanov tensor:

1. β_{22}

$\beta_{22} =$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta$$

$$= \frac{\rho_D}{2} \left(COD_{avg} + CSD_{avg}\right) +$$

$$- \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} - \frac{\rho_D}{2} \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} +$$

$$- \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin(\frac{1 + mod\left(i,2\right)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) +$$

$$- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin(\frac{1 + mod\left(i,2\right)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) +$$

$$+ \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} COD_{avg} +$$

$$+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin(\frac{1 + mod\left(i+1,2\right)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) +$$

$$- \frac{\rho_D}{2} \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} CSD_{avg} +$$

$$- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i} \sin(\frac{1 + mod\left(i+1,2\right)}{2}\pi - 2\Delta\Psi) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) =$$

$$= \frac{\rho_D}{2} COD_{avg} \left(1 - \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} + \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} \right) +$$

$$+ \frac{\rho_D}{2} CSD_{avg} \left(1 - \frac{1 - \cos\left(2\Delta\theta\right)}{2\Delta\theta} - \frac{\sin\left(2\Delta\theta\right)}{2\Delta\theta} \right) +$$

$$+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1 + mod\left(i+1,2\right)}{2}\pi - 2\Delta\Psi \right) - \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\left(i,2\right)}{2}\pi - 2\Delta\Psi \right) \right) \cdot$$

$$\cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right) +$$

$$- \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^{i} \sin\left(\frac{1 + mod\left(i+1,2\right)}{2}\pi - 2\Delta\Psi \right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod\left(i,2\right)}{2}\pi - 2\Delta\Psi \right) \right) \cdot$$

$$\cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i} \right)$$

$$(33)$$

2. β_{33}

$$\beta_{33} =$$

$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\,(\theta) + CSD\,(\theta)}{2}\,(1 + \sin(2\theta)) - \frac{COD\,(\theta) - CSD\,(\theta)}{2}\cos(2\theta) \right] d\theta =$$

$$= \frac{\rho_D}{2}\,(COD_{avg} + CSD_{avg}) +$$

$$+ \frac{\rho_D}{2}\frac{1 - \cos(2\Delta\theta)}{2\Delta\theta}COD_{avg} + \frac{\rho_D}{2}\frac{1 - \cos(2\Delta\theta)}{2\Delta\theta}CSD_{avg} +$$

$$+ \frac{\rho_D}{2}\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi)\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right) +$$

$$+ \frac{\rho_D}{2}\frac{1}{\Delta\theta}CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi)\left(\sum_{k=i}^{2n-1}b_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right) +$$

$$- \frac{\rho_D}{2}\frac{\sin(2\Delta\theta)}{2\Delta\theta}COD_{avg} +$$

$$- \frac{\rho_D}{2}\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i}\sin(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi)\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right) +$$

$$+ \frac{\rho_D}{2}\frac{\sin(2\Delta\theta)}{2\Delta\theta}CSD_{avg} +$$

$$+ \frac{\rho_D}{2}\frac{1}{\Delta\theta}CSD_{max}\sum_{i=0}^{2n-1}\left(-\frac{1}{2}\right)^{i}\sin(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi)\left(\sum_{k=i}^{2n-1}b_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right) =$$

$$= \frac{\rho_D}{2}COD_{avg}\left(1 + \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} - \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) +$$

$$+ \frac{\rho_D}{2}CSD_{avg}\left(1 - \frac{1 + \cos(2\Delta\theta)}{2\Delta\theta} + \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) +$$

$$+ \frac{\rho_D}{2}COD_{max}\frac{1}{\Delta\theta}\sum_{i=0}^{2n-1}\left(\left(-\frac{1}{2}\right)^{i+1}\sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i}\sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right)\right)\cdot$$

$$\cdot\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right) +$$

$$+ \frac{\rho_D}{2}CSD_{max}\frac{1}{\Delta\theta}\sum_{i=0}^{2n-1}\left(\left(-\frac{1}{2}\right)^{i}\sin\left(\frac{1 + mod\,(i+1,2)}{2}\pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1}\sin\left(\frac{1 + mod\,(i,2)}{2}\pi - 2\Delta\Psi\right)\right)\cdot$$

$$\cdot\left(\sum_{k=i}^{2n-1}a_k\frac{k!}{(k-i)!}\Delta\Psi^{k-i}\right) \tag{34}$$

3. β_{23}

$$\beta_{23} =$$
$$= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta =$$
$$= \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} COD_{avg} +$$
$$+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) +$$
$$+ \frac{\rho_D}{2} \frac{\sin(2\Delta\theta)}{2\Delta\theta} CSD_{avg} +$$
$$+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod(i+1,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) +$$
$$+ \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} COD_{avg} - \frac{\rho_D}{2} \frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} CSD_{avg} +$$
$$+ \frac{\rho_D}{2} \frac{1}{\Delta\theta} COD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) +$$
$$- \frac{\rho_D}{2} \frac{1}{\Delta\theta} CSD_{max} \sum_{i=0}^{2n-1} \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod(i,2)}{2}\pi - 2\Delta\Psi\right) \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) =$$
$$= \frac{\rho_D}{2} COD_{avg} \left(\frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} + \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) +$$
$$- \frac{\rho_D}{2} CSD_{avg} \left(\frac{1 - \cos(2\Delta\theta)}{2\Delta\theta} - \frac{\sin(2\Delta\theta)}{2\Delta\theta}\right) +$$
$$+ \frac{\rho_D}{2} COD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod(i+1,2)}{2}\pi - 2\Delta\Psi\right) + \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod(i,2)}{2}\pi - 2\Delta\Psi\right)\right) \cdot$$
$$\cdot \left(\sum_{k=i}^{2n-1} a_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right) +$$
$$+ \frac{\rho_D}{2} CSD_{max} \frac{1}{\Delta\theta} \sum_{i=0}^{2n-1} \left(\left(-\frac{1}{2}\right)^i \sin\left(\frac{1 + mod(i+1,2)}{2}\pi - 2\Delta\Psi\right) - \left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1 + mod(i,2)}{2}\pi - 2\Delta\Psi\right)\right) \cdot$$
$$\cdot \left(\sum_{k=i}^{2n-1} b_k \frac{k!}{(k-i)!} \Delta\Psi^{k-i}\right)$$

$$(35)$$

2.6. Non-dimensional crack displacements and crack compliance tensor

References

[1] P. Lundmark, J. Varna, Modeling thermo-mechanical properties of damaged laminates, Key Engineering Materials 251-252 (2003) 381–388. doi:10.4028/www.scientific.net/kem.251-252.381.

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[2] P. Lundmark, J. Varna, Crack face sliding effect on stiffness of laminates with ply cracks, Composites Science and Technology 66 (10) (2006) 1444–1454. `doi:10.1016/j.compscitech.2005.08.016`.

[3] J. Varna, Modelling mechanical performance of damaged laminates, Journal of Composite Materials 47 (20-21) (2012) 2443–2474. `doi:10.1177/0021998312469241`.

[4] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), Comprehensive Composite Materials II, Elsevier, Oxford, 2018, pp. 192 – 220. `doi:10.1016/B978-0-12-803581-8.09910-0`.

Appendix A. Explicit expressions for $f(\theta)$ and $g(\theta)$

In the following, recall that

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right). \tag{A.1}$$

n = 1

$$
f\left(\theta - \frac{\Delta\Psi}{2}\right) = \sum_{k=0}^{0} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1\left(\theta - \frac{\Delta\Psi}{2}\right)
$$
$$
g\left(\theta - \frac{\Delta\theta}{2}\right) = \sum_{k=0}^{0} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1\left(\theta - \frac{\Delta\theta}{2}\right) \tag{A.2}
$$

$$
\int_0^{\Delta\Psi} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \int_0^{\Delta\Psi} a_1\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \left[\frac{a_1}{2}\theta^2 - a_1\frac{\Delta\Psi}{2}\theta\right]\Bigg|_0^{\Delta\Psi} = 0 \quad \forall a_1
$$
$$
\int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \int_0^{\Delta\theta} b_1\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \left[\frac{b_1}{2}\theta^2 - b_1\frac{\Delta\theta}{2}\theta\right]\Bigg|_0^{\Delta\theta} = 0 \quad \forall b_1
$$
$$\tag{A.3}$$

$$COD_{avg} + COD_{max}a_1\left(\Delta\Psi - \frac{\Delta\Psi}{2}\right) = 0 \to a_1 = -\frac{2}{\Delta\Psi}\frac{COD_{avg}}{COD_{max}}$$

$$CSD_{avg} + CSD_{max}b_1\left(\Delta\theta - \frac{\Delta\theta}{2}\right) = 0 \to b_1 = -\frac{2}{\Delta\theta}\frac{CSD_{avg}}{CSD_{max}}$$

$$(A.4)$$

$$\sum_{i=0}^{1}\left(-\frac{1}{2}\right)^{i+1}\sin\left(\frac{1+mod\,(i,2)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{1-i}a_k\left(1-(k+1)\right)!\Delta\Psi^k\right) =$$

$$= \left(-\frac{1}{2}\right)^{1}\sin\left(\frac{1+mod\,(0,2)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{1}a_k\left(1-(k+1)\right)!\Delta\Psi^k\right) +$$

$$+ \left(-\frac{1}{2}\right)^{2}\sin\left(\frac{1+mod\,(1,2)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{0}a_k\left(1-(k+1)\right)!\Delta\Psi^k\right) =$$

$$= -\frac{1}{2}\cos\left(2\Delta\Psi\right)\left(\sum_{k=0}^{1}a_k\left(1-(k+1)\right)!\Delta\Psi^k\right) +$$

$$+ \frac{1}{4}\sin\left(2\Delta\Psi\right)\left(\sum_{k=0}^{0}a_k\left(1-(k+1)\right)!\Delta\Psi^k\right) =$$

$$= -\frac{1}{2}\cos\left(2\Delta\Psi\right)\left(a_0 + a_1\left(n-(k+1)\right)!\Delta\Psi^k\right) +$$

$$+ \frac{1}{4}\sin\left(2\Delta\Psi\right)\left(\sum_{k=0}^{0}a_k\left(n-(k+1)\right)!\Delta\Psi^k\right) =$$

$$(A.5)$$

n = 2

$$f\left(\theta - \frac{\Delta\Psi}{2}\right) = \sum_{k=0}^{1}a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1\left(\theta - \frac{\Delta\Psi}{2}\right) + a_3\left(\theta - \frac{\Delta\Psi}{2}\right)^{3}$$

$$g\left(\theta - \frac{\Delta\theta}{2}\right) = \sum_{k=0}^{1}b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1\left(\theta - \frac{\Delta\theta}{2}\right) + b_3\left(\theta - \frac{\Delta\theta}{2}\right)^{3}$$

$$(A.6)$$

n = 3

$$f\left(\theta - \frac{\Delta\Psi}{2}\right) = \sum_{k=0}^{2} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} =$$

$$= a_1\left(\theta - \frac{\Delta\Psi}{2}\right) + a_3\left(\theta - \frac{\Delta\Psi}{2}\right)^3 + a_5\left(\theta - \frac{\Delta\Psi}{2}\right)^5$$

$$g\left(\theta - \frac{\Delta\theta}{2}\right) = \sum_{k=0}^{1} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} =$$

$$= b_1\left(\theta - \frac{\Delta\theta}{2}\right) + b_3\left(\theta - \frac{\Delta\theta}{2}\right)^3 + b_5\left(\theta - \frac{\Delta\theta}{2}\right)^5$$

$$(\text{A.7})$$