

Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

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Abstract

Keywords: Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

1. Introduction

Main ref [1]

2. Derivation of constitutive relations

2.1. Reference frames

5 **Local reference frame of k -th layer:** index 1 is the in-plane longitudinal or fiber or 0° -direction; index 2 is the in-plane transverse or 90° -direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0° direction; index y is the in-plane transverse direction; index
10 z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \quad (1)$$

V_a is equal to the product of total crack surface S_C and average crack opening

15 u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \quad (2)$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C . It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\begin{aligned} \rho_D &= \frac{\text{total area of debonds}}{\text{total layer volume}} = \frac{n_D w R_f \Delta \theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \mathcal{W}}{L_{lam} \mathcal{W} t_{90^\circ}} R_f \Delta \theta = \frac{1}{n 2L} \frac{1}{k 2L} R_f \Delta \theta = \\ &= \frac{1}{nk 4L^2} R_f \Delta \theta = \frac{V_f}{nk \pi R_f^2} R_f \Delta \theta = \frac{V_f}{nk R_f} \frac{\Delta \theta}{\pi} \end{aligned} \quad (3)$$

20 2.3. Vakulenko-Kachanov tensor

In the local reference frame of k -th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \quad (4)$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \quad (5)$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_i n_j + u_j n_i) dS \quad (6)$$

25 Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$\begin{aligned}
\beta_{11} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_1 n_1 + u_1 n_1) dS = \frac{1}{V_k} \int_{S_C} \cancel{u_1} \vec{n}_1^0 dS = 0 \\
\beta_{22} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_2 + u_2 n_2) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_3 + u_3 n_3) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\
\beta_{12} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\cancel{u_1} \vec{n}_2^0 + u_2 \cancel{n_1})^0 dS = 0 \\
\beta_{13} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (\cancel{u_1} \vec{n}_3^0 + u_3 \cancel{n_1})^0 dS = 0 \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 n_3 + u_3 n_2) dS \\
\beta_{21} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_2 \cancel{n_1}^0 + \cancel{u_1} \vec{n}_2^0) dS = 0 \\
\beta_{31} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 \cancel{n_1}^0 + \cancel{u_1} \vec{n}_3^0) dS = 0 \\
\beta_{32} &= \frac{1}{V_k} \int_{S_C} \frac{1}{2} (u_3 n_2 + u_2 n_3) dS = \beta_{23}
\end{aligned} \tag{7}$$

Only 3 independent components of the tensor β_{ij} remain: β_{22} , β_{33} and β_{23} . Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f (-n_2^m) dS \right] \\
\beta_{33} &= \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f (-n_3^m) dS \right] \\
\beta_{23} &= \frac{1}{V_k} \int_{S_C} (u_2 n_3 + u_3 n_2) dS = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} (u_2^m n_3^m + u_3^m n_2^m) dS + \int_{S_C^f} (u_2^f n_3^f + u_3^f n_2^f) dS \right] = \\
&= \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f (-n_3^m) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f (-n_2^m) dS \right]
\end{aligned} \tag{8}$$

30 The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$S_C^m = S_C^f = n_D R_f \Delta \theta \tag{9}$$

With Eq. 9, we can recast Eq. 8 as

$$\begin{aligned}
\beta_{22} &= \frac{1}{V_k} \left[n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam} w t_{90^\circ}} \left[n_D R_f w \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \frac{1}{L_{lam}} \frac{n_D R_f}{t_{90^\circ}} \left[\int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] = \\
&= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_2^m d\theta \right] \\
\beta_{33} &= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_3^m d\theta \right] \\
\beta_{23} &= \rho_D \left[\frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_2^m - u_2^f) n_3^m d\theta + \frac{1}{\Delta\theta} \int_0^{\Delta\theta} (u_3^m - u_3^f) n_2^m d\theta \right]
\end{aligned} \tag{10}$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$\begin{aligned}
u_2^m - u_2^f &= (u_r^m - u_r^f) \cos(\theta) - (u_\theta^m - u_\theta^f) \sin(\theta) = \\
&= COD(\theta) \cos(\theta) - CSD(\theta) \sin(\theta) \\
u_3^m - u_3^f &= (u_r^m - u_r^f) \sin(\theta) + (u_\theta^m - u_\theta^f) \cos(\theta) = \\
&= COD(\theta) \sin(\theta) + CSD(\theta) \cos(\theta)
\end{aligned} \tag{11}$$

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ :

$$\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned} \tag{12}$$

Thus, Eq. 10 becomes

$$\begin{aligned}
\beta_{22} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (\cos^2(\theta) - \cos(\theta) \sin(\theta)) - CSD(\theta) (\sin(\theta) \cos(\theta) - \sin^2(\theta))] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (1 + \cos(2\theta) - \sin(2\theta)) + CSD(\theta) (1 - \cos(2\theta) - \sin(2\theta))] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{33} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (\sin(\theta) \cos(\theta) + \sin^2(\theta)) + CSD(\theta) (\cos^2(\theta) + \cos(\theta) \sin(\theta))] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} [COD(\theta) (1 + \sin(2\theta) - \cos(2\theta)) + CSD(\theta) (1 + \sin(2\theta) + \cos(2\theta))] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{23} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD(\theta) (2 \sin(\theta) \cos(\theta) + \cos^2(\theta) - \sin^2(\theta)) + \\
&\quad - \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) (\sin^2(\theta) - \cos^2(\theta) + 2 \cos(\theta) \sin(\theta)) d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta \\
&\hspace{15em} (13)
\end{aligned}$$

2.4. Analytical modeling of $COD(\theta)$ and $CSD(\theta)$

40 The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of θ , the angular coordinate along the crack which varies between 0 and $\Delta\theta$. Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be expressed as the sum of their average value and a term, respectively $\delta COD(\theta)$ and $\delta CSD(\theta)$, that represents the variation of the function from its average:

45

$$\begin{aligned}
COD(\theta) &= COD_{avg} + \delta COD(\theta) \\
CSD(\theta) &= CSD_{avg} + \delta CSD(\theta).
\end{aligned} \tag{14}$$

We introduce at this point an approximation and assume that the functions $\delta COD(\theta)$ and $\delta CSD(\theta)$ can be expressed as the product of the maximum value and a function, respectively $f(\theta)$ and $g(\theta)$:

$$\begin{aligned}
COD(\theta) &= COD_{avg} + \delta COD(\theta) = COD_{avg} + COD_{max} f(\theta) \\
CSD(\theta) &= CSD_{avg} + \delta CSD(\theta) = CSD_{avg} + CSD_{max} g(\theta),
\end{aligned} \tag{15}$$

where $f(\theta)$ and $g(\theta)$ are assumed to be odd functions over the integration
50 domain $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f(\theta) d\theta = 0 \quad \int_0^{\Delta\theta} g(\theta) d\theta = 0. \tag{16}$$

$$f(\theta) = \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \quad g(\theta) = \sum_{k=0}^{n-1} b_{2k+1} \theta^{2k+1} \tag{17}$$

$$\begin{aligned}
\beta_{22} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 - \sin(2\theta)) + \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{33} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD(\theta) + CSD(\theta)}{2} (1 + \sin(2\theta)) - \frac{COD(\theta) - CSD(\theta)}{2} \cos(2\theta) \right] d\theta \\
\beta_{23} &= \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD(\theta) + CSD(\theta)}{2} \cos(2\theta) + \frac{COD(\theta) - CSD(\theta)}{2} \sin(2\theta) \right] d\theta
\end{aligned} \tag{18}$$

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\min(\Delta\theta, \Delta\Phi)} COD(\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \right) d\theta = \\
& = \frac{1}{\Delta\theta} \left[COD_{avg} \theta + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \theta^{2(k+1)} \right] \Bigg|_0^{\min(\Delta\theta, \Delta\Phi)} = \\
& = COD_{avg} + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \frac{\min(\Delta\theta, \Delta\Phi)^{2(k+1)}}{\Delta\theta}
\end{aligned} \tag{19}$$

$$\frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD(\theta) d\theta = CSD_{avg} + CSD_{max} \sum_{k=0}^{n-1} \frac{b_{2k+1}}{2(k+1)} \Delta\theta^{2k+1} \tag{20}$$

$$\begin{aligned}
& \frac{1}{\Delta\theta} \int_0^{\min(\Delta\theta, \Delta\Phi)} COD(\theta) \sin(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1} \right) \sin(2\theta) d\theta = \\
& = \frac{1}{\Delta\theta} \left[-\frac{1}{2} COD_{avg} \cos(2\theta) + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2(k+1)} \theta^{2(k+1)} \right] \Bigg|_0^{\min(\Delta\theta, \Delta\Phi)} = \\
& \tag{21}
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\Delta\theta} (1 + \sin(2\theta) - \cos(2\theta)) d\theta = \left[\theta - \frac{1}{2} \cos(2\theta) - \frac{1}{2} \sin(2\theta) \right]_0^{\Delta\theta} = \\
& = \Delta\theta + \frac{1}{2} (1 - \cos(2\Delta\theta)) - \frac{1}{2} \sin(2\Delta\theta) = \\
& = \Delta\theta - \frac{1}{2} \left(\sqrt{2} \sin\left(2\Delta\theta + \frac{\pi}{4}\right) - 1 \right) \\
& \tag{22}
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\Delta\theta} f(\theta) (1 + \sin(2\theta) - \cos(2\theta)) d\theta = \\
& = \int_0^{\Delta\theta} f(\theta) d\theta - \sqrt{2} \int_0^{\Delta\theta} f(\theta) \sin\left(\frac{\pi}{4} - 2\theta\right) d\theta \\
& \int_0^{\Delta\theta} g(\theta) (1 + \sin(2\theta) - \cos(2\theta)) d\theta = \\
& = \int_0^{\Delta\theta} g(\theta) d\theta - \sqrt{2} \int_0^{\Delta\theta} g(\theta) \sin\left(\frac{\pi}{4} - 2\theta\right) d\theta
\end{aligned} \tag{23}$$

$$\begin{aligned}
& \int_0^{\Delta\theta} f(\theta) \sin\left(\frac{\pi}{4} - 2\theta\right) d\theta = \\
& = \left[\sum_{i=0}^n \left[\left(a_0 (1 - H(i-1)) + \sum_{k=0}^{n-1} a_{2k+1} \theta^{2k+1-i} \right) \frac{1}{2} (-1)^i \sin\left(\left(1 - \frac{1}{2} \bmod \left(\frac{i+1}{2} \right) \right) \pi - \left(\frac{\pi}{4} - 2\theta \right) \right) \right] \right]_0^{\Delta\theta} = \\
& = \sum_{i=0}^n \left[\left(a_0 (1 - H(i-1)) + \sum_{k=0}^{n-1} a_{2k+1} \Delta\theta^{2k+1-i} \right) \frac{1}{2} (-1)^i \sin\left(\left(1 - \frac{1}{2} \bmod \left(\frac{i+1}{2} \right) \right) \pi - \left(\frac{\pi}{4} - 2\Delta\theta \right) \right) \right] + \\
& + \frac{\sqrt{2}}{4} a_0 \\
& \int_0^{\Delta\theta} g(\theta) \sin\left(\frac{\pi}{4} - 2\theta\right) d\theta = \\
& = \left[\sum_{i=0}^n \left[\left(b_0 (1 - H(i-1)) + \sum_{k=0}^{n-1} b_{2k+1} \theta^{2k+1-i} \right) \frac{1}{2} (-1)^i \sin\left(\left(1 - \frac{1}{2} \bmod \left(\frac{i+1}{2} \right) \right) \pi - \left(\frac{\pi}{4} - 2\theta \right) \right) \right] \right]_0^{\Delta\theta} = \\
& = \sum_{i=0}^n \left[\left(b_0 (1 - H(i-1)) + \sum_{k=0}^{n-1} b_{2k+1} \Delta\theta^{2k+1-i} \right) \frac{1}{2} (-1)^i \sin\left(\left(1 - \frac{1}{2} \bmod \left(\frac{i+1}{2} \right) \right) \pi - \left(\frac{\pi}{4} - 2\Delta\theta \right) \right) \right] + \\
& + \frac{\sqrt{2}}{4} b_0
\end{aligned} \tag{24}$$

$$\begin{aligned}
\beta_{22} = & \rho_D \left[1 - \frac{1}{2\Delta\theta} \left(\sqrt{2} \sin \left(2\Delta\theta + \frac{\pi}{4} \right) - 1 \right) \right] (COD_{avg} - CSD_{avg}) + \\
& - \frac{1}{2} \rho_D (a_0 COD_{max} - b_0 CSD_{max}) + \\
& - \sqrt{2} \sum_{i=0}^n \left((a_0 COD_{max} - b_0 CSD_{max}) (1 - H(i-1)) + \sum_{k=0}^{n-1} (a_{2k+1} COD_{max} - b_{2k+1} CSD_{max}) \Delta\theta^{2k+1-i} \right) \times \\
& \times \frac{1}{2} (-1)^i \sin \left(\left(1 - \frac{1}{2} \bmod \left(\frac{i+1}{2} \right) \right) \pi - \left(\frac{\pi}{4} - 2\Delta\theta \right) \right)
\end{aligned} \tag{25}$$

References

- [1] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), *Comprehensive Composite Materials II*, Elsevier, Oxford, 2018, pp. 192 – 220. doi:<https://doi.org/10.1016/B978-0-12-803581-8.09910-0>.