Stiffness reduction in UD and cross-ply laminates due to fiber/matrix interface cracks

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Abstract

Keywords: Fiber Reinforced Polymer Composite (FRPC), Debonding, Finite element analysis (FEA)

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1. Introduction

Main ref [1]

2. Derivation of constitutive relations

2.1. Reference frames

Local reference frame of k-th layer: index 1 is the in-plane longitudinal or fiber or 0°-direction; index 2 is the in-plane transverse or 90°-direction; index 3 is the out-of-plane or through-the-thickness direction.

Global reference frame of laminate: index x is the in-plane longitudinal or laminate 0°direction; index y is the in-plane transverse direction; index z is the out-of-plane or through-the-thickness direction.

2.2. Crack density

Principle 1. Normalized volume of cracks V_{an} is the ratio of cracked volume V_a to material volume V

$$V_{an} = \frac{V_a}{V} \tag{1}$$

V_a is equal to the product of total crack surface S_C and average crack opening u_a

$$V_{an} = \frac{S_C u_a}{V} = \frac{S_C}{V} u_a = \rho_C u_a \tag{2}$$

The ratio $\frac{S_C}{V}$ has a size of $\frac{1}{length}$ and correspond to the crack density ρ_C. It means: product of crack density and average crack opening is equal to normalized volume of cracks.

Applying the previous Principle to debonds, we have:

$$\begin{aligned}\rho_D = \frac{\text{total area of debonds}}{\text{total layer volume}} =& \frac{n_D w R_f \Delta\theta}{L_{lam} w t_{90^\circ}} = \frac{n_D \cancel{w}}{L_{lam} \cancel{w} t_{90^\circ}} R_f \Delta\theta = \frac{1}{n2L}\frac{1}{k2L} R_f \Delta\theta = \\ =& \frac{1}{nk4L^2} R_f \Delta\theta = \frac{V_f}{nk\pi R_f^2} R_f \Delta\theta = \frac{V_f}{nkR_f}\frac{\Delta\theta}{\pi}\end{aligned} \tag{3}$$

2.3. Vakulenko-Kachanov tensor

In the local reference frame of k-th layer, the outer normal at crack faces has components:

$$n_1 = 0 \quad n_2 \neq 0 \quad n_3 \neq 0 \tag{4}$$

while crack face displacement has components:

$$u_1 = 0 \quad u_2 \neq 0 \quad u_3 \neq 0 \tag{5}$$

Definition of Vakulenko-Kachanov tensor:

$$\beta_{ij} = \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_i n_j + u_j n_i \right) dS \tag{6}$$

Expand the expression for each component and simplify based on the fact that $u_1 = 0$:

$$
\begin{aligned}
\beta_{11} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_1 n_1 + u_1 n_1 \right) dS = \frac{1}{V_k} \int_{S_C} \cancelto{0}{u_1} n_1 dS = 0 \\
\beta_{22} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_2 + u_2 n_2 \right) dS = \frac{1}{V_k} \int_{S_C} u_2 n_2 dS \\
\beta_{33} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_3 + u_3 n_3 \right) dS = \frac{1}{V_k} \int_{S_C} u_3 n_3 dS \\
\beta_{12} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancelto{0}{u_1} n_2 + u_2 \cancelto{0}{n_1} \right) dS = 0 \\
\beta_{13} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(\cancelto{0}{u_1} n_3 + u_3 \cancelto{0}{n_1} \right) dS = 0 \\
\beta_{23} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 n_3 + u_3 n_2 \right) dS \\
\beta_{21} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_2 \cancelto{0}{n_1} + \cancelto{0}{u_1} n_2 \right) dS = 0 \\
\beta_{31} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 \cancelto{0}{n_1} + \cancelto{0}{u_1} n_3 \right) dS = 0 \\
\beta_{32} =& \frac{1}{V_k} \int_{S_C} \frac{1}{2} \left(u_3 n_2 + u_2 n_3 \right) dS = \beta_{23}
\end{aligned}
\tag{7}
$$

Only 3 independent components of the tensor β_{ij} remain: β_{22}, β_{33} and β_{23}. Split total crack surface S_C into total matrix crack surface S_C^m and total fiber crack surface S_C^f and remember that $n_i^f = -n_i^m$ for $i = 2, 3$

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k} \int_{S_C} u_2 n_2 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f n_2^f dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_2^m dS + \int_{S_C^f} u_2^f \left(-n_2^m\right) dS \right] \\
\beta_{33} =& \frac{1}{V_k} \int_{S_C} u_3 n_3 dS = \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f n_3^f dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_3^m n_3^m dS + \int_{S_C^f} u_3^f \left(-n_3^m\right) dS \right] \\
\beta_{23} =& \frac{1}{V_k} \int_{S_C} \left(u_2 n_3 + u_3 n_2\right) dS = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} \left(u_2^m n_3^m + u_3^m n_2^m\right) dS + \int_{S_C^f} \left(u_2^f n_3^f + u_3^f n_2^f\right) dS \right] = \\
=& \frac{1}{V_k} \left[\int_{S_C^m} u_2^m n_3^m dS + \int_{S_C^f} u_2^f \left(-n_3^m\right) dS + \int_{S_C^m} u_3^m n_2^m dS + \int_{S_C^f} u_3^f \left(-n_2^m\right) dS \right]
\end{aligned}
\tag{8}
$$

The total matrix debonded surface S_C^m is equal to the total fiber debonded surface S_C^f and equal to:

$$
S_C^m = S_C^f = n_D R_f \Delta\theta \tag{9}
$$

With Eq. 9, we can recast Eq. 8 as

$$
\begin{aligned}
\beta_{22} =& \frac{1}{V_k}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \frac{1}{L_{lam} w t_{90^\circ}}\left[n_D R_f w \int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \frac{1}{L_{lam}}\frac{n_D R_f}{t_{90^\circ}}\left[\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] = \\
=& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_2^m d\theta\right] \\
\beta_{33} =& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right) n_3^m d\theta\right] \\
\beta_{23} =& \rho_D\left[\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_2^m - u_2^f\right) n_3^m d\theta + \frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(u_3^m - u_3^f\right) n_2^m d\theta\right]
\end{aligned}
\tag{10}
$$

We can express the displacement jumps at the interface as a function of the local Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) as

$$
\begin{aligned}
u_2^m - u_2^f =& \left(u_r^m - u_r^f\right)\cos(\theta) - \left(u_\theta^m - u_\theta^f\right)\sin(\theta) = \\
=& COD(\theta)\cos(\theta) - CSD(\theta)\sin(\theta) \\
u_3^m - u_3^f =& \left(u_r^m - u_r^f\right)\sin(\theta) + \left(u_\theta^m - u_\theta^f\right)\cos(\theta) = \\
=& COD(\theta)\sin(\theta) + CSD(\theta)\cos(\theta)
\end{aligned}
\tag{11}
$$

where θ is the local angular coordinate at the interface. We can similarly express n_2^m and n_3^m as a function of θ:

$$
\begin{aligned}
n_2^m &= \cos(\theta) - \sin(\theta) \\
n_3^m &= \sin(\theta) + \cos(\theta)
\end{aligned}
\tag{12}
$$

Thus, Eq. 10 becomes

$$
\begin{aligned}
&\beta_{22} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(\cos^2\left(\theta\right) - \cos\left(\theta\right)\sin\left(\theta\right)\right) - CSD\left(\theta\right)\left(\sin\left(\theta\right)\cos\left(\theta\right) - \sin^2\left(\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(1 + \cos\left(2\theta\right) - \sin\left(2\theta\right)\right) + CSD\left(\theta\right)\left(1 - \cos\left(2\theta\right) - \sin\left(2\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\cos\left(2\theta\right)\right] d\theta \\
&\beta_{33} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(\sin\left(\theta\right)\cos\left(\theta\right) + \sin^2\left(\theta\right)\right) + CSD\left(\theta\right)\left(\cos^2\left(\theta\right) + \cos\left(\theta\right)\sin\left(\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{2\Delta\theta} \int_0^{\Delta\theta} \left[COD\left(\theta\right)\left(1 + \sin\left(2\theta\right) - \cos\left(2\theta\right)\right) + CSD\left(\theta\right)\left(1 + \sin\left(2\theta\right) + \cos\left(2\theta\right)\right)\right] d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\cos\left(2\theta\right)\right] d\theta \\
&\beta_{23} = \\
&\quad = \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right)\left(2\sin\left(\theta\right)\cos\left(\theta\right) + \cos^2\left(\theta\right) - \sin^2\left(\theta\right)\right) + \\
&\qquad - \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} CSD\left(\theta\right)\left(\sin^2\left(\theta\right) - \cos^2\left(\theta\right) + 2\cos\left(\theta\right)\sin\left(\theta\right)\right) d\theta = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2\left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2}\cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2}\sin\left(2\theta\right)\right] d\theta
\end{aligned}
\tag{13}
$$

2.4. Analytical modeling of $COD\left(\theta\right)$ and $CSD\left(\theta\right)$

The Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) are in general a function of θ, the angular coordinate along the crack which varies between 0 and $\Delta\theta$. Without making any approximation, the Crack Opening Displacement (COD) and Crack Sliding Displacement (CSD) can be expressed as the sum of their average value and a term, respectively $\delta COD\left(\theta\right)$

and $\delta CSD\left(\theta\right)$, that represents the variation of the function from its average:

$$\begin{aligned} COD\left(\theta\right) &= COD_{avg} + \delta COD\left(\theta\right) \\ CSD\left(\theta\right) &= CSD_{avg} + \delta CSD\left(\theta\right). \end{aligned} \tag{14}$$

By defining $\Delta\Psi$

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right), \tag{15}$$

we introduce at this point an approximation and assume that the functions $\delta COD\left(\theta\right)$ and $\delta CSD\left(\theta\right)$ can be expressed as the product of the maximum value of the displacement and a function, respectively $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$:

$$\begin{aligned} COD\left(\theta\right) &= COD_{avg} + \delta COD\left(\theta\right) = COD_{avg} + COD_{max} f\left(\theta - \frac{\Delta\Psi}{2}\right) \\ CSD\left(\theta\right) &= CSD_{avg} + \delta CSD\left(\theta\right) = CSD_{avg} + CSD_{max} g\left(\theta - \frac{\Delta\theta}{2}\right), \end{aligned} \tag{16}$$

where $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ are assumed to be odd functions over their respective integration domain $[0, \Delta\Psi]$ and $[0, \Delta\theta]$

$$\int_0^{\Delta\theta} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = 0 \qquad \int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = 0. \tag{17}$$

We assume the two functions $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ to be two odd polynomials of degree $2n-1$:

$$\begin{aligned} f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \begin{cases} \sum_{k=0}^{n-1} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} & 0 \le \theta \le \Delta\Psi \\ 0 \quad otherwise \end{cases} \\ g\left(\theta - \frac{\Delta\theta}{2}\right) &= \begin{cases} \sum_{k=0}^{n-1} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} & 0 \le \theta \le \Delta\theta \\ 0 \quad otherwise \end{cases} \end{aligned} \tag{18}$$

which satisfy by construction the conditions expressed in Equation 17. The coefficients a_{2k+1} and b_{2k+1} are determined by imposing that

$$\begin{aligned} COD\left(\Delta\Psi\right) &= 0 \\ CSD\left(\Delta\theta\right) &= 0. \end{aligned} \tag{19}$$

The explicit construction of the polynomials $f\left(\theta - \frac{\Delta\Psi}{2}\right)$ and $g\left(\theta - \frac{\Delta\theta}{2}\right)$ for $n = 1, 2, 3$ (or degree $2n - 1 = 1, 3, 5$) is reported in Appendix A.

$$
\begin{aligned}
&\beta_{22} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 - \sin\left(2\theta\right)\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta \\
&\beta_{33} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \left(1 + \sin\left(2\theta\right)\right) - \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \cos\left(2\theta\right) \right] d\theta \\
&\beta_{23} = \\
&= \rho_D \frac{1}{\Delta\theta} \int_0^{\Delta\theta} 2 \left[\frac{COD\left(\theta\right) + CSD\left(\theta\right)}{2} \cos\left(2\theta\right) + \frac{COD\left(\theta\right) - CSD\left(\theta\right)}{2} \sin\left(2\theta\right) \right] d\theta
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
&\frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD\left(\theta\right) d\theta = \\
&\quad = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} \left(COD_{avg} + COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} \right) d\theta = \\
&\quad = \frac{1}{\Delta\theta} \int_0^{\Delta\theta} COD_{avg} d\theta + \\
&\quad + \frac{1}{\Delta\theta} \int_0^{\Delta\Psi} COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} d\theta + \\
&\quad + \overbrace{\frac{1}{\Delta\theta} \int_{\Delta\Psi}^{\Delta\theta} COD_{max} \sum_{k=0}^{n-1} a_{2k+1} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} d\theta}^{0} \quad = \\
&\quad = \frac{1}{\Delta\theta} \left[COD_{avg} \theta \right] \Bigg|_0^{\Delta\theta} + \frac{1}{\Delta\theta} \left[COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2\left(k+1\right)} \left(\theta - \frac{\Delta\Psi}{2}\right)^{2(k+1)} \right] \Bigg|_0^{\Delta\Psi} = \\
&\quad = COD_{avg} + COD_{max} \sum_{k=0}^{n-1} \frac{a_{2k+1}}{2\left(k+1\right)\Delta\theta} \left(\left(\frac{\Delta\Psi}{2}\right)^{2(k+1)} - \left(-\frac{\Delta\Psi}{2}\right)^{2(k+1)} \right) = \\
&\quad = COD_{avg}
\end{aligned}
\tag{21}
$$

$$\frac{1}{\Delta\theta}\int_0^{\Delta\theta} CSD\left(\theta\right)d\theta = CSD_{avg} \tag{22}$$

$$\begin{aligned}
&\frac{1}{\Delta\theta}\int_0^{\Delta\theta} COD\left(\theta\right)\sin\left(2\theta\right)d\theta = \\
=&\frac{1}{\Delta\theta}\int_0^{\Delta\theta}\left(COD_{avg} + COD_{max}\sum_{k=0}^{n-1} a_{2k+1}\theta^{2k+1}\right)\sin\left(2\theta\right)d\theta = \\
=&-\frac{1}{2\Delta\theta}COD_{avg}\left[\cos\left(2\theta\right)\right]\Bigg|_0^{\Delta\theta} + \\
+&\frac{1}{\Delta\theta}\left[COD_{max}\sum_{i=0}^{n}\left(-\frac{1}{2}\right)^{i+1}\sin(\frac{1+mod\left(i,2\right)}{2}\pi - 2\theta)\left(\sum_{k=0}^{n-i} a_k\left(n-(k+1)\right)!\theta^k\right)\right]\Bigg|_0^{\Delta\Psi} = \\
=&\frac{1-\cos\left(2\Delta\theta\right)}{2\Delta\theta}COD_{avg} + \\
+&\frac{1}{\Delta\theta}COD_{max}\sum_{i=0}^{n}\left(-\frac{1}{2}\right)^{i+1}\sin\left(\frac{1+mod\left(i,2\right)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{n-i} a_k\left(n-(k+1)\right)!\Delta\Psi^k\right)
\end{aligned} \tag{23}$$

References

[1] J. Varna, 2.10 crack separation based models for microcracking, in: P. W. Beaumont, C. H. Zweben (Eds.), Comprehensive Composite Materials II, Elsevier, Oxford, 2018, pp. 192 – 220. doi:https://doi.org/10.1016/B978-0-12-803581-8.09910-0.

Appendix A. Explicit expressions for $f\left(\theta\right)$ and $g\left(\theta\right)$

In the following, recall that

$$\Delta\Psi = \min\left(\Delta\theta, \Delta\Phi\right). \tag{A.1}$$

n = 1

$$\begin{aligned}
f\left(\theta - \frac{\Delta\Psi}{2}\right) &= \sum_{k=0}^{0} a_{2k+1}\left(\theta - \frac{\Delta\Psi}{2}\right)^{2k+1} = a_1\left(\theta - \frac{\Delta\Psi}{2}\right) \\
g\left(\theta - \frac{\Delta\theta}{2}\right) &= \sum_{k=0}^{0} b_{2k+1}\left(\theta - \frac{\Delta\theta}{2}\right)^{2k+1} = b_1\left(\theta - \frac{\Delta\theta}{2}\right)
\end{aligned} \tag{A.2}$$

$$\int_0^{\Delta\Psi} f\left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \int_0^{\Delta\Psi} a_1 \left(\theta - \frac{\Delta\Psi}{2}\right) d\theta = \left[\frac{a_1}{2}\theta^2 - a_1 \frac{\Delta\Psi}{2}\theta\right]\Bigg|_0^{\Delta\Psi} = 0 \quad \forall a_1$$

$$\int_0^{\Delta\theta} g\left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \int_0^{\Delta\theta} b_1 \left(\theta - \frac{\Delta\theta}{2}\right) d\theta = \left[\frac{b_1}{2}\theta^2 - b_1 \frac{\Delta\theta}{2}\theta\right]\Bigg|_0^{\Delta\theta} = 0 \quad \forall b_1 \tag{A.3}$$

$$COD_{avg} + COD_{max} a_1 \left(\Delta\Psi - \frac{\Delta\Psi}{2}\right) = 0 \rightarrow a_1 = -\frac{2}{\Delta\Psi}\frac{COD_{avg}}{COD_{max}}$$

$$CSD_{avg} + CSD_{max} b_1 \left(\Delta\theta - \frac{\Delta\theta}{2}\right) = 0 \rightarrow b_1 = -\frac{2}{\Delta\theta}\frac{CSD_{avg}}{CSD_{max}} \tag{A.4}$$

$$\begin{aligned}
&\sum_{i=0}^{1}\left(-\frac{1}{2}\right)^{i+1} \sin\left(\frac{1+mod\,(i,2)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{1-i} a_k\,(1-(k+1))!\Delta\Psi^k\right) = \\
&= \left(-\frac{1}{2}\right)^{1} \sin\left(\frac{1+mod\,(0,2)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{1} a_k\,(1-(k+1))!\Delta\Psi^k\right) + \\
&+ \left(-\frac{1}{2}\right)^{2} \sin\left(\frac{1+mod\,(1,2)}{2}\pi - 2\Delta\Psi\right)\left(\sum_{k=0}^{0} a_k\,(1-(k+1))!\Delta\Psi^k\right) = \\
&= -\frac{1}{2}\cos\left(2\Delta\Psi\right)\left(\sum_{k=0}^{1} a_k\,(1-(k+1))!\Delta\Psi^k\right) + \\
&\quad + \frac{1}{4}\sin\left(2\Delta\Psi\right)\left(\sum_{k=0}^{0} a_k\,(1-(k+1))!\Delta\Psi^k\right) = \\
&= -\frac{1}{2}\cos\left(2\Delta\Psi\right)\left(a_0 + a_1\,(n-(k+1))!\Delta\Psi^k\right) + \\
&\quad + \frac{1}{4}\sin\left(2\Delta\Psi\right)\left(\sum_{k=0}^{0} a_k\,(n-(k+1))!\Delta\Psi^k\right) =
\end{aligned} \tag{A.5}$$

n = 2

$$f\left(\theta-\frac{\Delta\Psi}{2}\right)=\sum_{k=0}^{1}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}=a_1\left(\theta-\frac{\Delta\Psi}{2}\right)+a_3\left(\theta-\frac{\Delta\Psi}{2}\right)^3$$

$$g\left(\theta-\frac{\Delta\theta}{2}\right)=\sum_{k=0}^{1}b_{2k+1}\left(\theta-\frac{\Delta\theta}{2}\right)^{2k+1}=b_1\left(\theta-\frac{\Delta\theta}{2}\right)+b_3\left(\theta-\frac{\Delta\theta}{2}\right)^3 \tag{A.6}$$

n = 3

$$\begin{aligned}f\left(\theta-\frac{\Delta\Psi}{2}\right)&=\sum_{k=0}^{2}a_{2k+1}\left(\theta-\frac{\Delta\Psi}{2}\right)^{2k+1}=\\&=a_1\left(\theta-\frac{\Delta\Psi}{2}\right)+a_3\left(\theta-\frac{\Delta\Psi}{2}\right)^3+a_5\left(\theta-\frac{\Delta\Psi}{2}\right)^5\\g\left(\theta-\frac{\Delta\theta}{2}\right)&=\sum_{k=0}^{1}b_{2k+1}\left(\theta-\frac{\Delta\theta}{2}\right)^{2k+1}=\\&=b_1\left(\theta-\frac{\Delta\theta}{2}\right)+b_3\left(\theta-\frac{\Delta\theta}{2}\right)^3+b_5\left(\theta-\frac{\Delta\theta}{2}\right)^5\end{aligned} \tag{A.7}$$