

On the meaning of the normalisation parameter G_0 and the normalised Energy Release Rates in the fiber/matrix interface problem

Luca Di Stasio^{1,2,a}, Janis Varna^{2,b} and Zoubir Ayadi^{1,c}

¹SI2M, IJL, EEIGM, Université de Lorraine, 6 Rue Bastien Lepage, F-54010 Nancy, France

²Division of Polymer Engineering, Luleå University of Technology, SE-97187 Luleå, Sweden

E-mail: ^aluca.di-stasio@univ-lorraine.fr, ^bjanis.varna@ltu.se, ^czoubir.ayadi@univ-lorraine.fr

Abstract.

List of acronyms

BEM	Boundary Element Method
ERR	Energy Release Rate (here used as synonym of SERR)
FEM	Finite Element Method
SERR	Strain Energy Release Rate
VCCT	Virtual Crack Closure Technique

List of symbols

1. Introduction

The numerical analysis of the fiber/matrix interface has focused on the determination of mode mixity through the calculation of mode I and mode II energy release rates. In early papers, where the problem was solved in the complex domain by means of the Airy function and conformal transformations, it was shown that the stress field was due to oscillate in a very small region close to the crack tip. Thus, the evaluation of stresses and stress intensity factors represents an arduous obstacle when to the discrete counterpart of the problem.

Due to their definition as rates of energy change, the calculation of ERRs fits better into the frameworks of discrete procedures, whether the Finite Element Method (FEM) or the Boundary Element Method (BEM). Different variations of the same principle has been derived over the years, namely the Crack Closure Technique, the Crack Closure Integral, the Virtual Crack Closure Technique and the Virtual Crack Closure Integral. The idea at the core of these methods is that, given that the crack is propagating in a linear elastic medium, the energy released by the creation of a unit area of crack's surfaces is equal to the work needed to close the new created surfaces back together.

Energy Release Rates have been so far reported in a normalized form, where a reference energy release rate G_0 is used as normalization parameter. In [?], the authors claim that the use of

$$G_0 = \frac{1 + k_m}{8\mu_m} \sigma_0^2 R_f \pi \quad (1)$$

would make the results comparable between different material systems. In equation 1, k_m is the Kolosov constant for the matrix, which is equal to $3 - 4\nu$ for plane strain and $\frac{3-\nu}{1+\nu}$ for plane stress conditions, μ_m is the shear modulus of the matrix, σ_0 is the applied stress at the boundary and R_f is the radius of the inclusion.

In [?], the same normalization parameter is used to analyze the effect of a neighbouring fiber on the Energy Release Rates. A similar use can be found in [?] It seems that the first appearance of G_0 can be retrieved in [?], soon followed by [?] where a parametric study of Toya's analytical results is performed and tabulated. However, in Toya the normalization is performed for the maximum? stress at the crack tip, and its actual formulation is *formula from toya*.

The question thus arises: what is the meaning of G_0 ? And consequently, what is the meaning of the normalized Energy Release Rates? How does the selection of this peculiar normalization value make results comparable across different material systems and ply geometries? In this brief note we will try to answer these questions.

2. Analysis of G_0

Consider an infinite plate made of an isotropic homogeneous material with a central crack of length $2a$ in its center. We assume that the material is the same as matrix constituent of a composite, and thus we will use the pedix m to identify the properties. Under an applied remote traction σ_0 perpendicular to the crack faces, the mode I Stress Intensity Factor (SIF) reads [?]

$$K_I^{CC} = \sigma_0 \sqrt{\pi a}. \quad (2)$$

Across the literature on Linear Elastic Fracture Mechanics, in textbooks and handbooks, the expression in eq. 2 has assumed the role of reference for the expression of Stress Intensity Factors for all possible modes of fracture and geometries. Thus, the SIF for any other arbitrary geometrical configuration of crack and surrounding structure can be expressed as

$$K_{mode} = \beta K_{ref} \quad \text{where} \quad K_{ref} = K_I^{CC} = \sigma_0 \sqrt{\pi a} \quad (3)$$

or, equivalently,

$$K_{mode} = \beta \sigma_0 \sqrt{\pi a} \quad (4)$$

where β is the shape factor. Thus, evaluation of the SIF for a new configuration reduces to the determination of its shape factor.

The corresponding energy release rate can be computed to be

$$G_{ref} = G_I^{CC} = \frac{(K_I^{CC})^2}{E_m^*}. \quad (5)$$

For plane strain,

$$E_m^* = \frac{E_m}{1 - \nu^2} \quad \text{and thus} \quad G_{ref} = \frac{1 - \nu^2}{E_m} \sigma_0^2 \pi a, \quad (6)$$

while for plane stress,

$$E_m^* = E_m \quad \text{and thus} \quad G_{ref} = \frac{\sigma_0^2 \pi a}{E_m}. \quad (7)$$

The energy release rate for the generic configuration thus reads

$$G_{mode} = \beta G_{ref}, \quad (8)$$

which particularizes to $G_{mode} = \beta \frac{1 - \nu^2}{E_m} \sigma_0^2 \pi a$ for plane strain, and $G_{mode} = \beta \frac{\sigma_0^2 \pi a}{E_m}$ for plane stress.

A homogeneous isotropic material has only 2 independent parameters that determine its elastic behavior; thus it holds

$$G_m = \frac{E_m}{2(1 + \nu_m)}. \quad (9)$$

Let us consider once again the reference energy release rate for plane strain states and rework its expression

$$\begin{aligned} G_{ref} &= \frac{1 - \nu_m^2}{E_m} \sigma_0^2 \pi a = \\ &= \frac{(1 - \nu_m)(1 + \nu_m)}{E_m} \sigma_0^2 \pi a = \\ &= \frac{(1 - \nu_m)}{2G_m} \sigma_0^2 \pi a = \\ &= \frac{4(1 - \nu_m)}{4 - 2G_m} \sigma_0^2 \pi a = \\ &= \frac{1 + 3 - 4\nu_m}{8G_m} \sigma_0^2 \pi a = \\ &= \frac{1 + k_m}{8G_m} \sigma_0^2 \pi a \end{aligned} \quad (10)$$

where $k_m = 3 - 4\nu_m$ is the Kolosov's constant for plane strain. Similarly, for plane stress

$$\begin{aligned}
G_{ref} &= \frac{\sigma_0^2 \pi a}{E_m} = \\
&= \frac{\sigma_0^2 \pi a}{2G_m(1+\nu_m)} = \\
&= \frac{4}{4} \frac{1}{2G_m(1+\nu_m)} \sigma_0^2 \pi a \\
&= \frac{1}{8G_m} \frac{4+\nu_m-\nu_m}{1+\nu_m} \sigma_0^2 \pi a \\
&= \frac{1}{8G_m} \left(\frac{1+\nu_m}{1+\nu_m} + \frac{3-\nu_m}{1+\nu_m} \right) \sigma_0^2 \pi a = \\
&= \frac{1 + \frac{3-\nu_m}{1+\nu_m}}{8G_m} \sigma_0^2 \pi a = \\
&= \frac{1+k_m}{8G_m} \sigma_0^2 \pi a
\end{aligned} \tag{11}$$

where $k_m = \frac{3-\nu_m}{1+\nu_m}$ is the Kolosov's constant for plane stress. Recalling now eq. 1 and comparing with eqs. 10 and 11, we can now identify a clear physical meaning: G_0 is the mode I energy release rate for a central crack of total length equal to the inclusion diameter $2R_f$ in an infinite plate made of the same material as the matrix. Furthermore, a historical purpose can be identified, as this formulation of G_0 allow for the expression of the energy release rates for the fiber/matrix interface problem as

$$G_I = \beta_I G_{ref} \quad G_{II} = \beta_{II} G_{ref}, \tag{12}$$

where β_I and β_{II} are respectively the mode I and mode II shape parameters and can now be identified as the functions $\frac{G_I}{G_0}$ and $\frac{G_{II}}{G_0}$ that represent a main part of the results of the fiber/matrix interface problem.

3. A new formulation of G_0 for a multi-scale analysis of transverse cracking

Given the elastic properties of the transversely isotropic UD ply $E_1, E_2, \nu_{12}, \nu_{23}$, for a 90° ply under transverse tension the cross section along the direction of the load coincides with the plane of transversal isotropy. It is thus possible, for a system in plane strain, to define equivalent isotropic Young's modulus and Poisson's ratio as follows. The effective Young's modulus and Poisson's ratio in plane strain in the plane of isotropy are defined as

$$E^* = \frac{E_2}{1 - \nu_{21}\nu_{12}} \quad \nu^* = \frac{\nu_{23} + \nu_{21}\nu_{12}}{1 + \nu_{23}} \tag{13}$$

recalling the definition of the effective plane strain properties

$$E^* = \frac{E}{1 - \nu^2} \quad \nu^* = \frac{\nu}{1 - \nu} \tag{14}$$

by equating the corresponding expression in Eq. 13 and 14, we obtain the equivalent isotropic values as

$$E = \frac{E_2(1 + 2\nu_{23} + \nu_{21}\nu_{12})}{(1 + \nu_{23})^2} \quad \nu = \frac{\nu_{23} + \nu_{21}\nu_{12}}{1 + \nu_{23}} \tag{15}$$

In reference to the expressions in 5 and 5, we can thus introduce a reference G_0 as

$$\begin{aligned}
G_0 &= \frac{\sigma_0^2}{E^*} \pi R_f \quad \text{for a stress or force controlled test} \\
G_0 &= E^* \varepsilon_0^2 \pi R_f \quad \text{for a strain or displacement controlled test}
\end{aligned} \tag{16}$$

which represents the Energy Release Rate of an elastic isotropic infinite plate, with properties equivalent to those of the transversely isotropic ply in its plane of symmetry, with a central crack of length $2R_f$. What's more, G_0 now synthesizes the effect of material properties and magnitude of the applied load, in fact

$$G_0 = G_0(\varepsilon_0, V_f, E_{1f}, E_{2f}, E_m, nu_{12f}, nu_{23f}, nu_m, G_{12f}, G_{23f}) \tag{17}$$

With such definition, G_0 can be used as a normalization factor for both fiber/matrix interface debonds as well as transverse cracks, modelled in 2D as well as in 3D. It furthermore implies the definition of regimes of behavior, as shown in the following assuming the same V_f , same materials and the same applied strain.

$$\left\{ \begin{array}{l}
\frac{G_I}{G_{TOT}} \approx 1, \frac{G_I}{G_0} \approx 1 \\
\rightarrow \text{the crack (debond or transverse)} \sim \text{a transverse central crack in an infinite medium} \\
\frac{G_I}{G_{TOT}} \approx 1, \frac{G_I}{G_0} \sim \frac{B_f \Delta \theta}{B_f} \text{ or } \sim \frac{a}{R_f} \\
\rightarrow \text{the crack (debond or transverse)} \sim \text{a transverse central crack of a different size in an infinite medium} \\
\frac{G_I}{G_{TOT}} \approx 1, \frac{G_I}{G_0} \sim \sec\left(\frac{\pi}{2} \frac{R_f \Delta \theta}{t_k}\right) \left(1 - 0.025 \left(\frac{R_f \Delta \theta}{t_k}\right)^2 + 0.06 \left(\frac{R_f \Delta \theta}{t_k}\right)^4\right)^2 \\
\text{or } \sim \sec\left(\frac{\pi}{2} \frac{a}{t_k}\right) \left(1 - 0.025 \left(\frac{a}{t_k}\right)^2 + 0.06 \left(\frac{a}{t_k}\right)^4\right)^2 \quad 2t_k = \text{ply thickness} \\
\rightarrow \text{the crack (debond or transverse)} \sim \text{a transverse central crack in a finite size plate} \\
\frac{G_I}{G_{TOT}} < 1, \frac{G_{TOT}}{G_0} \approx 1 \\
\rightarrow \text{global ERR} \sim \text{a transverse central crack in an infinite medium, but mode ratio is different} \\
\frac{G_I}{G_{TOT}} < 1, \frac{G_{TOT}}{G_0} \sim \frac{B_f \Delta \theta}{B_f} \text{ or } \sim \frac{a}{R_f} \\
\rightarrow \text{global ERR} \sim \text{a transverse central crack of a different size in an infinite medium, but mode ratio is different} \\
\frac{G_I}{G_{TOT}} < 1, \frac{G_{TOT}}{G_0} \sim \sec\left(\frac{\pi}{2} \frac{R_f \Delta \theta}{t_k}\right) \left(1 - 0.025 \left(\frac{R_f \Delta \theta}{t_k}\right)^2 + 0.06 \left(\frac{R_f \Delta \theta}{t_k}\right)^4\right)^2 \\
\text{or } \sim \sec\left(\frac{\pi}{2} \frac{a}{t_k}\right) \left(1 - 0.025 \left(\frac{a}{t_k}\right)^2 + 0.06 \left(\frac{a}{t_k}\right)^4\right)^2 \quad 2t_k = \text{ply thickness} \\
\rightarrow \text{global ERR} \sim \text{a transverse central crack in a finite size plate, but mode ratio is different}
\end{array} \right. \tag{18}$$

References

- [1] Griffiths R. 1994 Stiffness matrix of the four-node quadrilateral element in closed form *Int. J. Numer. Meth. Eng.* **57** (2) 109–143
- [2] Krueger R. 2004 Virtual crack closure technique: History, approach, and applications *Appl. Mech. Rev.* **57** (2) 109–143
- [3] ABAQUS 2016 ABAQUS 2016 Analysis User's Manual *Online Documentation Help: Dassault Systèmes*
- [4] Rice J. R. 1968 A Path Independent Integral and the Approximate Analysis of Strain Concentration by Notches and Cracks *J. Appl. Mech.* **35** 379–386