Computing energy release rates using the Virtual Crack Closure Technique with the Finite Element Method: analytical discussion

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Abstract. The effect of crack tip orientation and elements' size in its vicinity on mode splitting in the Virtual Crack Closure Technique is analyzed by means of analytical derivations.

(i) The total energy release rate is shown to have no direct dependence on the debond angular size, but only an indirect one through the FEM solution of the crack displacement field in the crack tip neighborhood. It thus can be computed using unrotated forces and displacements, i.e. aligned with the global and not with the local refere

(ii) Mode I and mode II energy release rate are expressed as a function of the crack displacement and it is shown as in [5] that crack tip forces depend linearly on both components of the crack displacements. However, what in [5] is an assumption used to reverse-engineer the relationship, here the dependency is fully derived in terms of the underlying FEM discretization. It is furthermore shown that crack tip forces do not only depend on crack displacements, but also on the resultant of forces of the 4 elements connected to the crack tip, which represent the influence of the rest of the domain.

(iii) A new vectorial formulation of the VCCT is proposed, which can be applied to elements of different order and in the presence of quarter-tip singularity (based on the work in [6]). It provides a general framework for the analysis of curved cracks analyzed with FEM. It furthermore expresses directly the dependence on FEM discretization and solution. By implementing the equations provided alongside the classic FEM, it can provide a native formulation which does not to the extraction of internal or reaction forces in the post-processing phase.

(iv) Drawing upon the asymptotic expression of the displacement derived in [7] and presented in [8], it is shown that mode I and mode II for a curved crack behaves as $A\log(\delta) + B$, where δ is the angular discretization at the crack tip. The result is confirmed by the numerical results.

List of acronyms

VCCT	Virtual Crack Closure Technique
BEM	Boundary Element Method
FEM	Finite Element Method

List of symbols

G_I	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate
G_{II}	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate
G_{TOT}	$\left[\frac{J}{m^2}\right]$	Total energy release rate
$G_{I,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate in $r-\theta$ reference frame
$G_{II,r\theta}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate in $r-\theta$ reference frame
$G_{TOT,r\theta}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate in $r-\theta$ reference frame
$\widetilde{G}_{I,xy}$	$\left[\frac{J}{m^2}\right]$	Mode I energy release rate of equivalent crack in $x-y$ reference frame
$\widetilde{G}_{II,xy}$	$\left[\frac{J}{m^2}\right]$	Mode II energy release rate of equivalent crack in $x-y$ reference frame
$\widetilde{G}_{TOT,xy}$	$\left[\frac{J}{m^2}\right]$	Total energy release rate of equivalent crack in $x-y$ reference frame
R_f	$[\mu m]$	Fiber radius
a	$[\mu m]$	Debond size
Δa	$[\mu m]$	Debond increment
$\Delta\theta$	$[rad]$	Half debond angular size
δ	$[rad]$	Angular size of element at the interface close to the crack tip
$u_{x,[A-Z]}$	$[\mu m]$	Displacement along x of a point labeled with a letter in [A-Z]
$u_{y,[A-Z]}$	$[\mu m]$	Displacement along y of a point labeled with a letter in [A-Z]
u_x	$[\mu m]$	Displacement along x-direction
u_y	$[\mu m]$	Displacement along y-direction
u_r	$[\mu m]$	Displacement along r-direction
u_θ	$[\mu m]$	Displacement along θ-direction
$F_{x,[A-Z]}$	$[\mu m]$	Force along x at a point labeled with a letter in [A-Z]
$F_{y,[A-Z]}$	$[\mu m]$	Force along y at a point labeled with a letter in [A-Z]
F_x	$[\mu m]$	Force along x-direction
F_y	$[\mu m]$	Force along y-direction
F_r	$[\mu m]$	Force along r-direction
F_θ	$[\mu m]$	Force along θ-direction
$\underline{\underline{R}}$	$[-]$	Rotation matrix

1. FEM formulation with quadrilateral elements

$$\Pi\left(u_i\right)=\int_V \sigma_{ij}\varepsilon_{ij}dV+\int_V \rho\ddot{u}_i u_i dV-\int_V F_i^T u_i dV-\int_S f_i^T u_i dS \tag{1}$$

$$\Pi\left(u_i\right)=\int_V \sigma_{ij}\varepsilon_{ij}dV-\int_V F_i^T u_i dV-\int_S f_i^T u_i dS \tag{2}$$

$$\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \tag{3}$$

$$\underline{\varepsilon}\left(x,y\right)=\underline{\underline{\widetilde{B}}}\cdot\underline{u}\left(x,y\right) \tag{4}$$

$$\underline{\underline{\widetilde{B}}}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix} \tag{5}$$

$$\underline{\sigma}=\underline{\underline{D}}\cdot\underline{\varepsilon} \tag{6}$$

$$\underline{\sigma}=\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}\qquad \underline{\varepsilon}=\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} \tag{7}$$

$$\underline{\underline{D}}=\begin{bmatrix}E_1 & E_2 & 0\\ E_2 & E_1 & 0\\ 0 & 0 & G\end{bmatrix}\qquad \text{with } G=\frac{E}{2\left(1+\nu\right)} \text{ for an isotropic material} \tag{8}$$

$$\begin{aligned}E_1&=\frac{E}{1-\nu^2}\quad E_2=\nu E_1\quad \text{for plane stress}\\ E_1&=\frac{E\left(1-\nu\right)}{\left(1+\nu\right)\left(1-2\nu\right)}\quad E_2=\frac{\nu E_1}{1-\nu}\quad \text{for plane strain}\end{aligned} \tag{9}$$

$$\begin{aligned}\Pi\left(\underline{u}\right)=&\frac{1}{2}\int_V \underline{\varepsilon}^T\underline{\underline{D}}\cdot\underline{\varepsilon}dV-\int_V \underline{F}^T\underline{u}dV-\int_S \underline{f}^T\underline{u}dS=\\ &=\frac{1}{2}\int_V \underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV-\int_V \underline{F}^T\underline{u}dV-\int_S \underline{f}^T\underline{u}dS\end{aligned} \tag{10}$$

$$\delta\Pi\left(\delta\underline{u}\right)=0 \tag{11}$$

$$
\begin{aligned}
\delta\Pi\left(\delta\underline{u}\right) =& \Pi\left(\underline{u}+\delta\underline{u}\right)-\Pi\left(\underline{u}\right)= \\
=& \frac{1}{2}\int_V \left(\underline{u}+\delta\underline{u}\right)^T \underline{\underline{\widetilde{B}}}^T \underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\left(\underline{u}+\delta\underline{u}\right)dV-\int_V \underline{F}^T\left(\underline{u}+\delta\underline{u}\right)dV-\int_S \underline{f}^T\left(\underline{u}+\delta\underline{u}\right)dS+ \\
&-\frac{1}{2}\int_V \underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV+\int_V \underline{F}^T\underline{u}dV+\int_S \underline{f}^T\underline{u}dS= \\
=& \cancel{\frac{1}{2}\int_V \underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV}+\cancelto{\approx 0}{\frac{1}{2}\int_V \delta\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\delta\underline{u}dV}+ \\
&+\frac{1}{2}\int_V \underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\delta\underline{u}dV+\frac{1}{2}\int_V \delta\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV+ \\
&-\cancel{\int_V \underline{F}^T\underline{u}dV}-\int_V \underline{F}^T\delta\underline{u}dV+ \\
&-\cancel{\int_S \underline{f}^T\underline{u}dS}-\int_S \underline{f}^T\delta\underline{u}dS+ \\
&-\cancel{\frac{1}{2}\int_V \underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV}+\cancel{\int_V \underline{F}^T\underline{u}dV}+\cancel{\int_S \underline{f}^T\underline{u}dS}= \\
=& \int_V \underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\delta\underline{u}dV-\int_V \underline{F}^T\delta\underline{u}dV-\int_S \underline{f}^T\delta\underline{u}dS
\end{aligned}
\tag{12}
$$

$$
\int_V \delta\underline{u}^T\underline{\underline{\widetilde{B}}}^T\underline{\underline{D}}\cdot\underline{\underline{\widetilde{B}}}\cdot\underline{u}dV-\int_V \delta\underline{u}^T\underline{F}dV-\int_S \delta\underline{u}^T\underline{f}dS=0 \tag{13}
$$

$$
\underline{u}=\begin{bmatrix}u_x\left(x,y\right)\\u_y\left(x,y\right)\end{bmatrix}\qquad \underline{u}_N=\begin{bmatrix}u_{1,x}\\u_{1,y}\\u_{2,x}\\u_{2,y}\\u_{3,x}\\u_{3,y}\\u_{4,x}\\u_{4,y}\end{bmatrix}\ \text{or}\ \underline{u}_N=\begin{bmatrix}u_{1,x}\\u_{1,y}\\u_{2,x}\\u_{2,y}\\u_{3,x}\\u_{3,y}\\u_{4,x}\\u_{4,y}\\u_{5,x}\\u_{5,y}\\u_{6,x}\\u_{6,y}\\u_{7,x}\\u_{7,y}\\u_{8,x}\\u_{8,y}\end{bmatrix} \tag{14}
$$

$$
\underline{u}=\underline{\underline{N}}\cdot\underline{u}_N \tag{15}
$$

$$
\underline{\underline{N}}=\begin{bmatrix}N_1&0&N_2&0&N_3&0&N_4&0\\0&N_1&0&N_2&0&N_3&0&N_4\end{bmatrix} \tag{16}
$$

$$
\underline{\underline{N}}=\begin{bmatrix}N_1&0&N_2&0&N_3&0&N_4&0&N_5&0&N_6&0&N_7&0&N_8&0\\0&N_1&0&N_2&0&N_3&0&N_4&0&N_5&0&N_6&0&N_7&0&N_8\end{bmatrix} \tag{17}
$$

$$\begin{cases} N_1 = N_1(\xi,\eta) \\ N_2 = N_2(\xi,\eta) \\ N_3 = N_3(\xi,\eta) \\ N_4 = N_4(\xi,\eta) \end{cases} \quad \text{with} \quad \begin{cases} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{cases} \quad \text{for isoparametric elements} \tag{18}$$

$$\underline{\underline{B}} = \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \tag{19}$$

$$\begin{aligned} \underline{\underline{B}} =& \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} = \\ =& \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 & \frac{\partial N_5}{\partial x} & 0 & \frac{\partial N_6}{\partial x} & 0 & \frac{\partial N_7}{\partial x} & 0 & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} & 0 & \frac{\partial N_5}{\partial y} & 0 & \frac{\partial N_6}{\partial y} & 0 & \frac{\partial N_7}{\partial y} & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} & \frac{\partial N_5}{\partial y} & \frac{\partial N_5}{\partial x} & \frac{\partial N_6}{\partial y} & \frac{\partial N_6}{\partial x} & \frac{\partial N_7}{\partial y} & \frac{\partial N_7}{\partial x} & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{bmatrix} \end{aligned} \tag{20}$$

$$\delta \underline{u} = \delta\left(\underline{\underline{N}} \cdot \underline{u_N}\right) = \underline{\underline{N}} \delta \underline{u}_N \tag{21}$$

$$\int_V \underline{\delta u_N}^T \underline{\underline{N}}^T \underline{\underline{\widetilde{B}}}^T \underline{\underline{D}} \cdot \underline{\underline{\widetilde{B}}} \cdot \underline{\underline{N}} \cdot \underline{u_N} dV - \int_V \underline{\delta u_N}^T \underline{\underline{N}}^T \underline{F} dV - \int_S \underline{\delta u_N}^T \underline{\underline{N}}^T \underline{f} dS = 0 \tag{22}$$

$$\underline{\delta u_N}^T \left(\int_V \underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}} dV \cdot \underline{u_N} - \int_V \underline{\underline{N}}^T \underline{F} dV - \int_S \underline{\underline{N}}^T \underline{f} dS \right) = 0 \tag{23}$$

$$\underline{\underline{k}} \cdot \underline{u}_N = \underline{F}_N \quad \underline{\underline{k}} = \int_V \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}\right) dV \quad \underline{F}_N = \int_V \underline{\underline{N}}^T \underline{F} dV + \int_S \underline{\underline{N}}^T \underline{f} dS \tag{24}$$

$$\begin{cases} N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta) \end{cases} \qquad \begin{cases} N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1) \\ N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1) \\ N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \\ N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1) \\ N_5(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1-\eta) \\ N_6(\xi,\eta) = \frac{1}{2}(1+\xi)\left(1-\eta^2\right) \\ N_7(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1+\eta) \\ N_8(\xi,\eta) = \frac{1}{2}(1-\xi)\left(1-\eta^2\right) \end{cases} \tag{25}$$

$$\begin{cases} \frac{\partial N_1(\xi,\eta)}{\partial \xi} = -\frac{1}{4}(1-\eta) \\ \frac{\partial N_2(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1-\eta) \\ \frac{\partial N_3(\xi,\eta)}{\partial \xi} = \frac{1}{4}(1+\eta) \\ \frac{\partial N_4(\xi,\eta)}{\partial \xi} = -\frac{1}{4}(1+\eta) \\ \frac{\partial N_1(\xi,\eta)}{\partial \eta} = -\frac{1}{4}(1-\xi) \\ \frac{\partial N_2(\xi,\eta)}{\partial \eta} = -\frac{1}{4}(1+\xi) \\ \frac{\partial N_3(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1+\xi) \\ \frac{\partial N_4(\xi,\eta)}{\partial \eta} = \frac{1}{4}(1-\xi) \end{cases} \tag{26}$$

$$
\begin{cases}
\frac{\partial N_1(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1-\eta\right)\left(2\xi+\eta\right) \\
\frac{\partial N_2(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1-\eta\right)\left(2\xi-\eta\right) \\
\frac{\partial N_3(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1+\eta\right)\left(2\xi+\eta\right) \\
\frac{\partial N_4(\xi,\eta)}{\partial \xi} = \frac{1}{4}\left(1+\eta\right)\left(2\xi-\eta\right) \\
\frac{\partial N_5(\xi,\eta)}{\partial \xi} = -\xi\left(1-\eta\right) \\
\frac{\partial N_6(\xi,\eta)}{\partial \xi} = \frac{1}{2}\left(1-\eta^2\right) \\
\frac{\partial N_7(\xi,\eta)}{\partial \xi} = -\xi\left(1+\eta\right) \\
\frac{\partial N_8(\xi,\eta)}{\partial \xi} = -\frac{1}{2}\left(1-\eta^2\right) \\
\frac{\partial N_1(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1-\xi\right)\left(2\eta+\xi\right) \\
\frac{\partial N_2(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1+\xi\right)\left(2\eta-\xi\right) \\
\frac{\partial N_3(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1+\xi\right)\left(2\eta+\xi\right) \\
\frac{\partial N_4(\xi,\eta)}{\partial \eta} = \frac{1}{4}\left(1-\xi\right)\left(2\eta-\xi\right) \\
\frac{\partial N_5(\xi,\eta)}{\partial \eta} = -\frac{1}{2}\left(1-\xi^2\right) \\
\frac{\partial N_6(\xi,\eta)}{\partial \eta} = -\eta\left(1+\xi\right) \\
\frac{\partial N_7(\xi,\eta)}{\partial \eta} = \frac{1}{2}\left(1-\xi^2\right) \\
\frac{\partial N_8(\xi,\eta)}{\partial \eta} = -\eta\left(1-\xi\right)
\end{cases}
\tag{27}
$$

$$
\underline{p} = \underline{\underline{N}} \cdot \underline{p}_N \tag{28}
$$

$$
\underline{p} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{bmatrix} \text{ or } \underline{p}_N = \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \\ x_5 \\ y_5 \\ x_6 \\ y_6 \\ x_7 \\ y_7 \\ x_8 \\ y_8 \end{bmatrix} \tag{29}
$$

$$
\begin{aligned}
x = x\left(\xi,\eta\right) = N_1\left(\xi,\eta\right)x_1 + N_2\left(\xi,\eta\right)x_2 + N_3\left(\xi,\eta\right)x_3 + N_4\left(\xi,\eta\right)x_4 \\
y = y\left(\xi,\eta\right) = N_1\left(\xi,\eta\right)y_1 + N_2\left(\xi,\eta\right)y_2 + N_3\left(\xi,\eta\right)y_3 + N_4\left(\xi,\eta\right)y_4
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
x = x\left(\xi,\eta\right) =& N_1\left(\xi,\eta\right)x_1 + N_2\left(\xi,\eta\right)x_2 + N_3\left(\xi,\eta\right)x_3 + N_4\left(\xi,\eta\right)x_4 + \\
&+ N_5\left(\xi,\eta\right)x_5 + N_6\left(\xi,\eta\right)x_6 + N_7\left(\xi,\eta\right)x_7 + N_8\left(\xi,\eta\right)x_8 \\
y = y\left(\xi,\eta\right) =& N_1\left(\xi,\eta\right)y_1 + N_2\left(\xi,\eta\right)y_2 + N_3\left(\xi,\eta\right)y_3 + N_4\left(\xi,\eta\right)y_4 + \\
&+ N_5\left(\xi,\eta\right)y_5 + N_6\left(\xi,\eta\right)y_6 + N_7\left(\xi,\eta\right)y_7 + N_8\left(\xi,\eta\right)y_8
\end{aligned}
\tag{31}
$$

$$\begin{cases} \frac{\partial x}{\partial \xi} = \frac{\partial N_1(\xi,\eta)}{\partial \xi} x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi} x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi} x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi} x_4 \\ \frac{\partial x}{\partial \eta} = \frac{\partial N_1(\xi,\eta)}{\partial \eta} x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta} x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta} x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta} x_4 \\ \frac{\partial y}{\partial \xi} = \frac{\partial N_1(\xi,\eta)}{\partial \xi} y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi} y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi} y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi} y_4 \\ \frac{\partial y}{\partial \eta} = \frac{\partial N_1(\xi,\eta)}{\partial \eta} y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta} y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta} y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta} y_4 \end{cases} \tag{32}$$

$$\begin{cases} \frac{\partial x}{\partial \xi} = & \frac{\partial N_1(\xi,\eta)}{\partial \xi} x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi} x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi} x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi} x_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \xi} x_5 + \frac{\partial N_6(\xi,\eta)}{\partial \xi} x_6 + \frac{\partial N_7(\xi,\eta)}{\partial \xi} x_7 + \frac{\partial N_8(\xi,\eta)}{\partial \xi} x_8 \\ \frac{\partial x}{\partial \eta} = & \frac{\partial N_1(\xi,\eta)}{\partial \eta} x_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta} x_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta} x_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta} x_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \eta} x_5 + \frac{\partial N_6(\xi,\eta)}{\partial \eta} x_6 + \frac{\partial N_7(\xi,\eta)}{\partial \eta} x_7 + \frac{\partial N_8(\xi,\eta)}{\partial \eta} x_8 \\ \frac{\partial y}{\partial \xi} = & \frac{\partial N_1(\xi,\eta)}{\partial \xi} y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \xi} y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \xi} y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \xi} y_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \xi} y_5 + \frac{\partial N_6(\xi,\eta)}{\partial \xi} y_6 + \frac{\partial N_7(\xi,\eta)}{\partial \xi} y_7 + \frac{\partial N_8(\xi,\eta)}{\partial \xi} y_8 \\ \frac{\partial y}{\partial \eta} = & \frac{\partial N_1(\xi,\eta)}{\partial \eta} y_1 + \frac{\partial N_2(\xi,\eta)}{\partial \eta} y_2 + \frac{\partial N_3(\xi,\eta)}{\partial \eta} y_3 + \frac{\partial N_4(\xi,\eta)}{\partial \eta} y_4 + \\ & + \frac{\partial N_5(\xi,\eta)}{\partial \eta} y_5 + \frac{\partial N_6(\xi,\eta)}{\partial \eta} y_6 + \frac{\partial N_7(\xi,\eta)}{\partial \eta} y_7 + \frac{\partial N_8(\xi,\eta)}{\partial \eta} y_8 \end{cases} \tag{33}$$

$$\begin{cases} \frac{\partial N_1(\xi,\eta)}{\partial x} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_1(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_2(\xi,\eta)}{\partial x} = \frac{\partial N_2(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_2(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_3(\xi,\eta)}{\partial x} = \frac{\partial N_3(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_3(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_4(\xi,\eta)}{\partial x} = \frac{\partial N_4(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_4(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_1(\xi,\eta)}{\partial y} = \frac{\partial N_1(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_1(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_2(\xi,\eta)}{\partial y} = \frac{\partial N_2(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_2(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_3(\xi,\eta)}{\partial y} = \frac{\partial N_3(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_3(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \\ \frac{\partial N_4(\xi,\eta)}{\partial y} = \frac{\partial N_4(\xi,\eta)}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_4(\xi,\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} \end{cases} \tag{34}$$

$$\begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial x} & \frac{\partial N_1(\xi,\eta)}{\partial y} \\ \frac{\partial N_2(\xi,\eta)}{\partial x} & \frac{\partial N_2(\xi,\eta)}{\partial y} \\ \frac{\partial N_3(\xi,\eta)}{\partial x} & \frac{\partial N_3(\xi,\eta)}{\partial y} \\ \frac{\partial N_4(\xi,\eta)}{\partial x} & \frac{\partial N_4(\xi,\eta)}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1(\xi,\eta)}{\partial \xi} & \frac{\partial N_1(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_2(\xi,\eta)}{\partial \xi} & \frac{\partial N_2(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_3(\xi,\eta)}{\partial \xi} & \frac{\partial N_3(\xi,\eta)}{\partial \eta} \\ \frac{\partial N_4(\xi,\eta)}{\partial \xi} & \frac{\partial N_4(\xi,\eta)}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \tag{35}$$

$$\underline{e}_\xi = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} \quad \underline{e}_\eta = \begin{bmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \end{bmatrix} \tag{36}$$

$$\underline{e}_x = \begin{bmatrix} \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial x} \end{bmatrix} \quad \underline{e}_y = \begin{bmatrix} \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial y} \end{bmatrix} \tag{37}$$

$$\underline{\underline{J}} = \left[\underline{e}_\xi | \underline{e}_\eta\right] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad \underline{\underline{J}}^{-1} = \left[\underline{e}_x | \underline{e}_y\right] = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} \tag{38}$$

$$\underline{\underline{J}}^{-1} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \frac{1}{\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix} \tag{39}$$

$$\underline{\underline{g}} = \underline{\underline{J}}^T \underline{\underline{J}} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta} \\ \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta} \end{bmatrix} \tag{40}$$

$$
\begin{aligned}
g =& det\left(\underline{\underline{g}}\right) = \\
=& \left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \eta}\right) - \left(\frac{\partial x}{\partial \eta}\frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta}\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial x}{\partial \xi}\frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi}\frac{\partial y}{\partial \eta}\right)
\end{aligned}
\tag{41}
$$

$$
dV\left(x,y\right) = \sqrt{g}dV\left(\xi,\eta\right) \tag{42}
$$

$$
\begin{aligned}
\underline{\underline{k_e}} &= \int_{V_e(x,y)} \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}\right) dV_e\left(x,y\right) = \int_{V_e(\xi,\eta)} \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}\right) \sqrt{g} dV_e\left(\xi,\eta\right) \\
&= \int_{-1}^{1}\int_{-1}^{1} \left(\underline{\underline{B}}^T \underline{\underline{D}} \cdot \underline{\underline{B}}\right) \sqrt{g} d\xi d\eta \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \left(\underline{\underline{B}}^T\left(\xi_i,\eta_j\right) \cdot \underline{\underline{D}} \cdot \underline{\underline{B}}\left(\xi_i,\eta_j\right)\sqrt{g}\right)
\end{aligned}
\tag{43}
$$

$$
k_e = \begin{bmatrix}
k_{e|11} & k_{e|12} & k_{e|13} & k_{e|14} & k_{e|15} & k_{e|16} & k_{e|17} & k_{e|18} \\
k_{e|12} & k_{e|22} & k_{e|23} & k_{e|24} & k_{e|25} & k_{e|26} & k_{e|27} & k_{e|28} \\
k_{e|13} & k_{e|23} & k_{e|33} & k_{e|34} & k_{e|35} & k_{e|36} & k_{e|37} & k_{e|38} \\
k_{e|14} & k_{e|24} & k_{e|34} & k_{e|44} & k_{e|45} & k_{e|46} & k_{e|47} & k_{e|48} \\
k_{e|15} & k_{e|25} & k_{e|35} & k_{e|45} & k_{e|55} & k_{e|56} & k_{e|57} & k_{e|58} \\
k_{e|16} & k_{e|26} & k_{e|36} & k_{e|46} & k_{e|56} & k_{e|66} & k_{e|67} & k_{e|68} \\
k_{e|17} & k_{e|27} & k_{e|37} & k_{e|47} & k_{e|57} & k_{e|67} & k_{e|77} & k_{e|78} \\
k_{e|18} & k_{e|28} & k_{e|38} & k_{e|48} & k_{e|58} & k_{e|68} & k_{e|78} & k_{e|88}
\end{bmatrix} \tag{44}
$$

2. VCCT for first order quadrilateral elements

2.1. Definition of crack tip reference frame

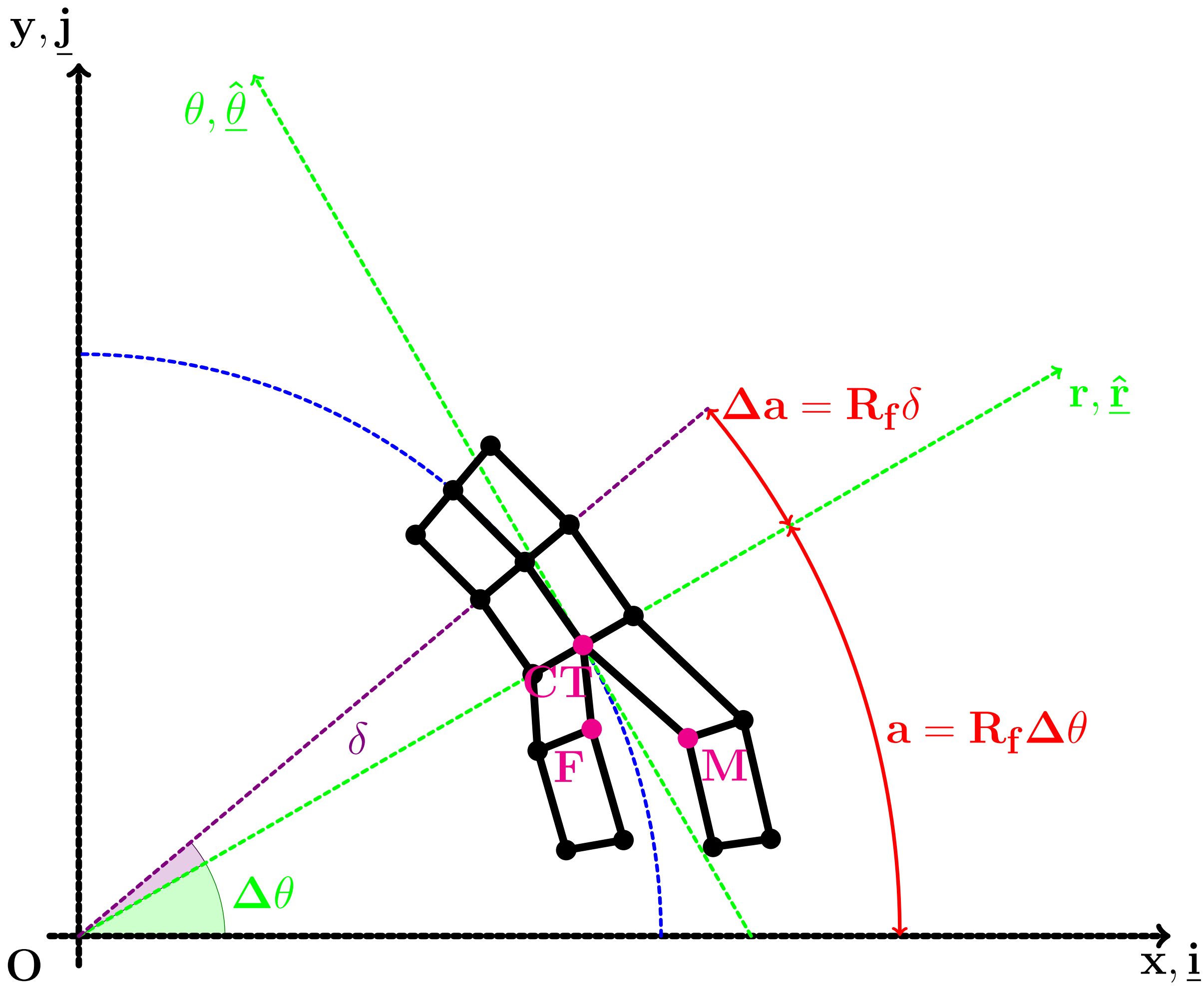

Figure 1. Schematic representation of the discretized crack tip geometry for 1^{st} order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^{T} = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \tag{46}$$

2.2. Calculation of displacements and reaction forces

$$u_x = u_{x,M} - u_{x,F} \qquad u_y = u_{y,M} - u_{y,F} \tag{47}$$

$$u_r = \cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y \qquad u_\theta = -\sin(\Delta\theta)\, u_x + \cos(\Delta\theta)\, u_y \tag{48}$$

$$F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT} \tag{49}$$

$$
\begin{cases}
\left(k_{e,M|11}+k_{e,M|33}\right)u_{x,MCT}+\left(k_{e,M|12}+k_{e,M|34}\right)u_{y,MCT}+ \\
+k_{e,M|13}u_{x,M}+k_{e,M|14}u_{y,M}+\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}+ \\
+\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+k_{M|31}u_{x,NCOI}+k_{M|32}u_{y,NCOI}=0 \\[1ex]
\left(k_{e,M|21}+k_{e,M|43}\right)u_{x,MCT}+\left(k_{e,M|22}+k_{e,M|44}\right)u_{y,MCT}+ \\
+k_{e,M|23}u_{x,M}+k_{e,M|24}u_{y,M}+\left(k_{M|27}+k_{M|45}\right)u_{N,MC|7}+\left(k_{M|28}+k_{M|46}\right)u_{N,MC|8}+ \\
+\sum_{i=5}^{6}k_{M|2i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|4i}u_{N,MB|i}+k_{M|41}u_{x,NCOI}+k_{M|42}u_{y,NCOI}=0 \\[1ex]
\left(k_{e,F|77}+k_{e,F|55}\right)u_{x,FCT}+\left(k_{e,F|78}+k_{e,F|56}\right)u_{y,FCT}+ \\
+k_{e,F|75}u_{x,F}+k_{e,F|76}u_{y,F}+\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}+ \\
+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}+k_{F|57}u_{x,NCOI}+k_{F|58}u_{y,NCOI}=0 \\[1ex]
\left(k_{e,F|87}+k_{e,F|65}\right)u_{x,FCT}+\left(k_{e,F|88}+k_{e,F|66}\right)u_{y,FCT}+ \\
+k_{e,F|85}u_{x,F}+k_{e,F|86}u_{y,F}+\left(k_{F|81}+k_{F|63}\right)u_{N,FC|1}+\left(k_{F|82}+k_{F|64}\right)u_{N,FC|2}+ \\
+\sum_{i=2}^{3}k_{F|8i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|6i}u_{N,FB|i}+k_{F|67}u_{x,NCOI}+k_{F|68}u_{y,NCOI}=0 \\[1ex]
u_{x,FCT}-u_{x,MCT}-u_{x,DUMMY}=0 \\
u_{y,FCT}-u_{y,MCT}-u_{y,DUMMY}=0 \\[1ex]
u_{x,DUMMY}=0 \\
u_{y,DUMMY}=0
\end{cases}
\tag{50}
$$

$$
\begin{cases}
\left(k_{e,M|11}+k_{e,M|33}\right)u_{x,MCT}+\left(k_{e,M|12}+k_{e,M|34}\right)u_{y,MCT}+ \\
+k_{e,M|13}u_{x,M}+k_{e,M|14}u_{y,M}+\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}+ \\
+\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+k_{M|31}u_{x,NCOI}+k_{M|32}u_{y,NCOI}=0 \\[1ex]
\left(k_{e,M|21}+k_{e,M|43}\right)u_{x,MCT}+\left(k_{e,M|22}+k_{e,M|44}\right)u_{y,MCT}+ \\
+k_{e,M|23}u_{x,M}+k_{e,M|24}u_{y,M}+\left(k_{M|27}+k_{M|45}\right)u_{N,MC|7}+\left(k_{M|28}+k_{M|46}\right)u_{N,MC|8}+ \\
+\sum_{i=5}^{6}k_{M|2i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|4i}u_{N,MB|i}+k_{M|41}u_{x,NCOI}+k_{M|42}u_{y,NCOI}=0 \\[1ex]
\left(k_{e,F|77}+k_{e,F|55}\right)u_{x,MCT}+\left(k_{e,F|78}+k_{e,F|56}\right)u_{y,MCT}+ \\
+k_{e,F|75}u_{x,F}+k_{e,F|76}u_{y,F}+\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}+ \\
+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}+k_{F|57}u_{x,NCOI}+k_{F|58}u_{y,NCOI}=0 \\[1ex]
\left(k_{e,F|87}+k_{e,F|65}\right)u_{x,MCT}+\left(k_{e,F|88}+k_{e,F|66}\right)u_{y,MCT}+ \\
+k_{e,F|85}u_{x,F}+k_{e,F|86}u_{y,F}+\left(k_{F|81}+k_{F|63}\right)u_{N,FC|1}+\left(k_{F|82}+k_{F|64}\right)u_{N,FC|2}+ \\
+\sum_{i=2}^{3}k_{F|8i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|6i}u_{N,FB|i}+k_{F|67}u_{x,NCOI}+k_{F|68}u_{y,NCOI}=0 \\[1ex]
u_{x,FCT}=u_{x,MCT} \\
u_{y,FCT}=u_{y,MCT} \\[1ex]
R_{x,DUMMY}=R_{x,FCT}=-R_{x,MCT}=F_{x,CT} \\
R_{y,DUMMY}=R_{y,FCT}=-R_{y,MCT}=F_{y,CT}
\end{cases}
\tag{51}
$$

$$
\left\{
\begin{array}{l}
\left(k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}\right)u_{x,MCT}+\left(k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}\right)u_{y,MCT}+\\
+k_{e,M|13}u_{x,M}+k_{e,M|14}u_{y,M}+k_{e,F|75}u_{x,F}+k_{e,F|76}u_{y,F}+\\
+\left(k_{M|31}+k_{F|57}\right)u_{x,NCOI}+\left(k_{M|32}+k_{F|58}\right)u_{y,NCOI}+\\
+\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}+\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}+\\
+\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}=0\\
\\
\left(k_{e,M|21}+k_{e,M|43}+k_{e,F|87}+k_{e,F|65}\right)u_{x,MCT}+\left(k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}\right)u_{y,MCT}+\\
+k_{e,M|23}u_{x,M}+k_{e,M|24}u_{y,M}+k_{e,F|85}u_{x,F}+k_{e,F|86}u_{y,F}+\\
+\left(k_{M|41}+k_{F|67}\right)u_{x,NCOI}+\left(k_{M|42}+k_{F|68}\right)u_{y,NCOI}+\\
+\left(k_{M|27}+k_{M|45}\right)u_{N,MC|7}+\left(k_{M|28}+k_{M|46}\right)u_{N,MC|8}+\left(k_{F|81}+k_{F|63}\right)u_{N,FC|1}+\left(k_{F|82}+k_{F|64}\right)u_{N,FC|2}+\\
+\sum_{i=2}^{3}k_{F|8i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|6i}u_{N,FB|i}+\sum_{i=5}^{6}k_{M|2i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|4i}u_{N,MB|i}=0\\
\\
u_{x,FCT}=u_{x,MCT}\\
u_{y,FCT}=u_{y,MCT}\\
\\
R_{x,DUMMY}=R_{x,FCT}=-R_{x,MCT}=F_{x,CT}\\
R_{y,DUMMY}=R_{y,FCT}=-R_{y,MCT}=F_{y,CT}
\end{array}
\right.
\tag{52}
$$

$$
\left\{
\begin{array}{l}
u_{y,MCT}=-\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}u_{x,MCT}+\\
-\frac{k_{e,M|13}u_{x,M}+k_{e,M|14}u_{y,M}+k_{e,F|75}u_{x,F}+k_{e,F|76}u_{y,F}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}+\\
-\frac{\left(k_{M|31}+k_{F|57}\right)u_{x,NCOI}+\left(k_{M|32}+k_{F|58}\right)u_{y,NCOI}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}+\\
-\frac{\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}+\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}+\\
-\frac{\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\\
\\
\left[\left(k_{e,M|21}+k_{e,M|43}+k_{e,F|87}+k_{e,F|65}\right)+\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}\right)\right]u_{x,MCT}+\\
+\left(k_{e,M|23}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|13}\right)u_{x,M}+\\
+\left(k_{e,M|24}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|14}\right)u_{y,M}+\\
+\left(k_{e,F|85}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|75}\right)u_{x,F}+\\
+\left(k_{e,F|86}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|76}\right)u_{y,F}+\\
+\left[\left(k_{M|41}+k_{F|67}\right)-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{M|31}+k_{F|57}\right)\right]u_{x,NCOI}+\\
+\left[\left(k_{M|42}+k_{F|68}\right)-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{M|32}+k_{F|58}\right)\right]u_{y,NCOI}+\\
+\left(k_{M|27}+k_{M|45}\right)u_{N,MC|7}+\left(k_{M|28}+k_{M|46}\right)u_{N,MC|8}+\left(k_{F|81}+k_{F|63}\right)u_{N,FC|1}+\left(k_{F|82}+k_{F|64}\right)u_{N,FC|2}+\\
-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left[\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}\right]+\\
-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left[\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}\right]\\
+\sum_{i=2}^{3}k_{F|8i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|6i}u_{N,FB|i}+\sum_{i=5}^{6}k_{M|2i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|4i}u_{N,MB|i}+\\
-\frac{\sum_{i=5}^{6}k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8}k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3}k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2}k_{F|5i}u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}=0\\
\\
u_{x,FCT}=u_{x,MCT}\\
u_{y,FCT}=u_{y,MCT}\\
\\
R_{x,DUMMY}=R_{x,FCT}=-R_{x,MCT}=F_{x,CT}\\
R_{y,DUMMY}=R_{y,FCT}=-R_{y,MCT}=F_{y,CT}
\end{array}
\right.
\tag{53}
$$

$$
\left\{
\begin{array}{ll}
 & u_{y,MCT} = -\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} u_{x,MCT} + \\
 & -\frac{k_{e,M|13}u_{x,M}+k_{e,M|14}u_{y,M}+k_{e,F|75}u_{x,F}+k_{e,F|76}u_{y,F}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
 & -\frac{\left(k_{M|31}+k_{F|57}\right)u_{x,NCOI}+\left(k_{M|32}+k_{F|58}\right)u_{y,NCOI}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
 & -\frac{\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}+\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} + \\
 & -\frac{\sum_{i=5}^{6} k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8} k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3} k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2} k_{F|5i}u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} \\
 & \\
 & \left[\left(k_{e,M|21}+k_{e,M|43}+k_{e,F|87}+k_{e,F|65}\right)+\frac{k_{e,M|11}+k_{e,M|33}+k_{e,F|77}+k_{e,F|55}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}\right)\right] u_{x,MCT} + \\
 & +\left(k_{e,M|23}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|13}\right) u_x + \\
 & +\left(k_{e,M|24}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}k_{e,M|14}\right) u_y + \\
 & +\left(k_{e,M|23}+k_{e,F|85}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{e,M|13}+k_{e,M|75}\right)\right) \cancelto{\approx 0}{u_{x,F}} + \\
 & +\left(k_{e,M|24}+k_{e,F|86}-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{e,M|14}+k_{e,M|76}\right)\right) \cancelto{\approx 0}{u_{y,F}} + \\
 & +\left[\left(k_{M|41}+k_{F|67}\right)-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{M|31}+k_{F|57}\right)\right] u_{x,NCOI} + \\
 & +\left[\left(k_{M|42}+k_{F|68}\right)-\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left(k_{M|32}+k_{F|58}\right)\right] u_{y,NCOI} + \\
 & +\left(k_{M|27}+k_{M|45}\right)u_{N,MC|7}+\left(k_{M|28}+k_{M|46}\right)u_{N,MC|8}+\left(k_{F|81}+k_{F|63}\right)u_{N,FC|1}+\left(k_{F|82}+k_{F|64}\right)u_{N,FC|2} + \\
 & -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left[\left(k_{M|17}+k_{M|35}\right)u_{N,MC|7}+\left(k_{M|18}+k_{M|36}\right)u_{N,MC|8}\right] + \\
 & -\frac{k_{e,M|22}+k_{e,M|44}+k_{e,F|88}+k_{e,F|66}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}}\left[\left(k_{F|71}+k_{F|53}\right)u_{N,FC|1}+\left(k_{F|72}+k_{F|54}\right)u_{N,FC|2}\right] \\
 & +\sum_{i=2}^{3} k_{F|8i}u_{N,FC|i}+\sum_{i=1}^{2} k_{F|6i}u_{N,FB|i}+\sum_{i=5}^{6} k_{M|2i}u_{N,MC|i}+\sum_{i=7}^{8} k_{M|4i}u_{N,MB|i} + \\
 & -\frac{\sum_{i=5}^{6} k_{M|1i}u_{N,MC|i}+\sum_{i=7}^{8} k_{M|3i}u_{N,MB|i}+\sum_{i=2}^{3} k_{F|7i}u_{N,FC|i}+\sum_{i=1}^{2} k_{F|5i}u_{N,FB|i}}{k_{e,M|12}+k_{e,M|34}+k_{e,F|78}+k_{e,F|56}} = 0 \\
 & \\
 & u_{x,FCT} = u_{x,MCT} \\
 & u_{y,FCT} = u_{y,MCT} \\
 & \\
F_{x,CT} & = R_{x,FCT} = \\
 & = \left(k_{e,F|77}+k_{e,F|55}\right)u_{x,FCT}+\left(k_{e,F|78}+k_{e,F|56}\right)u_{y,FCT} + \\
 & +k_{e,F|75}\cancelto{\approx 0}{u_{x,F}}+k_{e,F|76}\cancelto{\approx 0}{u_{y,F}} + \\
 & +\sum_{i=1}^{4} k_{e,F|7i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8} k_{e,F|5i}u_{N,FB|i} \\
F_{y,CT} & = R_{y,FCT} = \\
 & = \left(k_{e,F|87}+k_{e,F|65}\right)u_{x,FCT}+\left(k_{e,F|88}+k_{e,F|66}\right)u_{y,FCT} + \\
 & +k_{e,F|85}\cancelto{\approx 0}{u_{x,F}}+k_{e,F|86}\cancelto{\approx 0}{u_{y,F}} + \\
 & +\sum_{i=1}^{4} k_{e,F|8i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8} k_{e,F|6i}u_{N,FB|i}
\end{array}
\right.
\tag{54}
$$

$$
\begin{cases}
F_{x,CT} & = \left(k_{e,F|77}+k_{e,F|55}\right)u_{x,MCT}+\left(k_{e,F|78}+k_{e,F|56}\right)u_{y,MCT} + \\
 & +\sum_{i=1}^{4} k_{e,F|7i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8} k_{e,F|5i}u_{N,FB|i} \\
F_{y,CT} & = \left(k_{e,F|87}+k_{e,F|65}\right)u_{x,MCT}+\left(k_{e,F|88}+k_{e,F|66}\right)u_{y,MCT} + \\
 & +\sum_{i=1}^{4} k_{e,F|8i}u_{N,FC|i}+\sum_{i=1,i\neq(5,6)}^{8} k_{e,F|6i}u_{N,FB|i}
\end{cases}
\tag{55}
$$

$$
\begin{cases}
F_{x,CT} & = K_{xx}u_x+K_{xy}u_y + \\
 & +\sum_{i=1}^{4} K_{FC,x|i}u_{N,FC|i}+\sum_{i=1,i\neq(3,4,5,6)}^{8} K_{FB,x|i}u_{N,FB|i} + \\
 & +\sum_{i=5}^{8} K_{FC,x|i}u_{N,MC|i}+\sum_{i=7}^{8} K_{MB,x|i}u_{N,FB|i} \\
F_{y,CT} & = K_{yx}u_x+K_{yy}u_y + \\
 & +\sum_{i=1}^{4} K_{FC,y|i}u_{N,FC|i}+\sum_{i=1,i\neq(3,4,5,6)}^{8} K_{FB,y|i}u_{N,FB|i} + \\
 & +\sum_{i=5}^{8} K_{FC,y|i}u_{N,MC|i}+\sum_{i=7}^{8} K_{MB,y|i}u_{N,FB|i}
\end{cases}
\tag{56}
$$

$$\begin{cases} F_{x,CT} & = K_{xx}u_x + K_{xy}u_y + \widetilde{F}_x \\ F_{y,CT} & = K_{yx}u_x + K_{yy}u_y + \widetilde{F}_y \end{cases} \tag{57}$$

2.3. Calculation of energy release rates

$$\begin{aligned} G_{I,r\theta} =& \frac{1}{2}\frac{F_r u_r}{R_f \delta} = \\ =& \frac{1}{2R_f\delta}\left(\cos\left(\Delta\theta\right)F_x + \sin\left(\Delta\theta\right)F_y\right)\left(\cos\left(\Delta\theta\right)u_x + \sin\left(\Delta\theta\right)u_y\right) = \\ =& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) \end{aligned} \tag{58}$$

$$\begin{aligned} G_{II,r\theta} =& \frac{1}{2}\frac{F_\theta u_\theta}{R_f \delta} = \\ =& \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right)F_x + \cos\left(\Delta\theta\right)F_y\right)\left(-\sin\left(\Delta\theta\right)u_x + \cos\left(\Delta\theta\right)u_y\right) = \\ =& \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right) \end{aligned} \tag{59}$$

$$\begin{aligned} G_{TOT,r\theta} =& G_{I,r\theta} + G_{II,r\theta} = \\ =& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) + \\ &+ \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right) = \\ =& \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)} F_x u_x\right) + \\ &+ \frac{1}{2R_f\delta}\left(\cancelto{0}{\left(\left(F_x u_y + F_y u_x\right) - \left(F_x u_y + F_y u_x\right)\right)} \cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\right) + \\ &+ \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)} F_y u_y\right) = \\ =& \frac{1}{2}\frac{F_x u_x}{R_f\delta} + \frac{1}{2}\frac{F_y u_y}{R_f\delta} = \\ =& \widetilde{G}_{I,xy} + \widetilde{G}_{II,xy} = \widetilde{G}_{TOT,xy} \end{aligned} \tag{60}$$

$$\begin{aligned} G_{I,r\theta} =& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_x u_x + \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_y\right) = \\ =& \cos^2\left(\Delta\theta\right)\frac{F_x u_x}{2R_f\delta} + \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\frac{F_y u_y}{2R_f\delta} = \\ =& \cos^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} + \left(\widetilde{G}_{I,xy}\frac{u_y}{u_x} + \widetilde{G}_{II,xy}\frac{u_x}{u_y}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\widetilde{G}_{II,xy} \end{aligned} \tag{61}$$

$$
\begin{aligned}
G_{II,r\theta} =& \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_x u_x - \left(F_x u_y + F_y u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_y\right) = \\
=& \sin^2\left(\Delta\theta\right)\frac{F_x u_x}{2R_f\delta} - \left(\frac{F_x u_y}{2R_f\delta} + \frac{F_y u_x}{2R_f\delta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)\frac{F_y u_y}{2R_f\delta} = \\
=& \sin^2\left(\Delta\theta\right)\widetilde{G}_{I,xy} - \left(\widetilde{G}_{I,xy}\frac{u_y}{u_x} + \widetilde{G}_{II,xy}\frac{u_x}{u_y}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)\widetilde{G}_{II,xy}
\end{aligned} \tag{62}
$$

2.4. Sensitivity analysis of the FEM solution

$$
\begin{cases}
F_{x,CT} &= K_{xx}u_x + K_{xy}u_y + \widetilde{F}_x \\
F_{y,CT} &= K_{yx}u_x + K_{yy}u_y + \widetilde{F}_y
\end{cases} \tag{63}
$$

$$
\begin{aligned}
G_{I,r\theta} \sim& \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)\left(K_{xx}u_x^2 + K_{xy}u_y u_x + \widetilde{F}_x u_x\right) + \\
&+ \frac{1}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\left(K_{xx}u_x u_y + K_{xy}u_y^2 + \widetilde{F}_x u_y\right) + \\
&+ \frac{1}{2R_f\delta}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)\left(K_{yx}u_x^2 + K_{yy}u_y u_x + \widetilde{F}_y u_x\right) + \\
&+ \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)\left(K_{yx}u_y u_x + K_{yy}u_y^2 + \widetilde{F}_y\right)
\end{aligned} \tag{64}
$$

$$
\begin{aligned}
G_{II,r\theta} \sim& \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)k_x u_x^2\left(\Delta\theta\right) + \\
&- \frac{1}{2R_f\delta}\left(k_x + k_y\right)u_x\left(\Delta\theta\right)u_y\left(\Delta\theta\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
&+ \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)k_y u_y^2\left(\Delta\theta\right)
\end{aligned} \tag{65}
$$

$$
G_{TOT,r\theta} \sim \frac{1}{2R_f\delta}\left(k_x u_x^2\left(\Delta\theta\right) + k_y u_y^2\left(\Delta\theta\right)\right) \tag{66}
$$

$$
\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial\Delta\theta} \sim& \frac{1}{R_f\delta}\cos^2\left(\Delta\theta\right)k_x u_x\left(\Delta\theta\right)\frac{\partial u_x\left(\Delta\theta\right)}{\partial\Delta\theta} + \\
&+ \frac{1}{2R_f\delta}\left(k_x + k_y\right)\left(\frac{\partial u_x\left(\Delta\theta\right)}{\partial\Delta\theta}u_y\left(\Delta\theta\right) + u_x\left(\Delta\theta\right)\frac{\partial u_y\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
&+ \frac{1}{R_f\delta}\sin^2\left(\Delta\theta\right)k_y u_y\left(\Delta\theta\right)\frac{\partial u_y\left(\Delta\theta\right)}{\partial\Delta\theta} + \\
&+ \frac{1}{2R_f\delta}\left(k_y u_y^2\left(\Delta\theta\right) - k_x u_x^2\left(\Delta\theta\right)\right)\sin\left(2\Delta\theta\right) + \\
&+ \frac{1}{2R_f\delta}\left(k_x u_x\left(\Delta\theta\right)u_y\left(\Delta\theta\right) + k_y u_y\left(\Delta\theta\right)u_x\left(\Delta\theta\right)\right)\cos\left(2\Delta\theta\right)
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial \Delta\theta} \sim & \frac{1}{R_f\delta}\sin^2(\Delta\theta)\, k_x u_x(\Delta\theta)\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + \\
& - \frac{1}{2R_f\delta}(k_x+k_y)\left(\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta}u_y(\Delta\theta) + u_x(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \\
& + \frac{1}{R_f\delta}\cos^2(\Delta\theta)\, k_y u_y(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta} + \\
& + \frac{1}{2R_f\delta}\left(k_x u_x^2(\Delta\theta) - k_y u_y^2(\Delta\theta)\right)\sin(2\Delta\theta) + \\
& - \frac{1}{2R_f\delta}\left(k_x u_x(\Delta\theta)\, u_y(\Delta\theta) + k_y u_y(\Delta\theta)\, u_x(\Delta\theta)\right)\cos(2\Delta\theta)
\end{aligned} \tag{68}
$$

$$
\frac{\partial G_{TOT,r\theta}}{\partial \Delta\theta} \sim \frac{1}{R_f\delta}\left(k_x u_x(\Delta\theta)\frac{\partial u_x(\Delta\theta)}{\partial \Delta\theta} + k_y u_y(\Delta\theta)\frac{\partial u_y(\Delta\theta)}{\partial \Delta\theta}\right) \tag{69}
$$

2.5. Discretization error

$$
u_r = \cos(\Delta\theta-\delta)\, u_x + \sin(\Delta\theta-\delta)\, u_y \qquad u_\theta = -\sin(\Delta\theta-\delta)\, u_x + \cos(\Delta\theta-\delta)\, u_y \tag{70}
$$

$$
F_r = \cos(\Delta\theta)\, F_{x,CT} + \sin(\Delta\theta)\, F_{y,CT} \qquad F_\theta = -\sin(\Delta\theta)\, F_{x,CT} + \cos(\Delta\theta)\, F_{y,CT} \tag{71}
$$

$$
\begin{aligned}
\widetilde{G}_{I,r\theta} =& \frac{1}{2}\frac{F_r u_r}{R_f\delta} = \\
=& \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\cos(\Delta\theta-\delta)\, u_x + \sin(\Delta\theta-\delta)\, u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\cos(\Delta\theta)\, u_x + \sin(\Delta\theta)\, u_y\right)\cos(\delta) + \\
& + \frac{1}{2R_f\delta}\left(\cos(\Delta\theta)\, F_x + \sin(\Delta\theta)\, F_y\right)\left(\sin(\Delta\theta)\, u_x - \cos(\Delta\theta)\, u_y\right)\sin(\delta) = \\
=& \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta)\, F_x u_x + (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_y\right)\cos(\delta) + \\
& + \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)\, F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_x\right)\sin(\delta) = \\
=& G_{I,r\theta}\cos(\delta) + \\
& + \frac{1}{2R_f\delta}\left(-\cos^2(\Delta\theta)\, F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\, F_y u_x\right)\sin(\delta) = \\
=& G_{I,r\theta}\cos(\delta) - \frac{1}{2R_f\delta}F_r u_\theta \sin\delta
\end{aligned} \tag{72}
$$

$$
\begin{aligned}
\lim_{\delta\to 0}\widetilde{G}_{I,r\theta} =& \lim_{\delta\to 0} G_{I,r\theta}\cos(\delta) + \\
+\lim_{\delta\to 0}\frac{1}{2R_f\delta}&\left(-\cos^2(\Delta\theta)\,F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\,F_y u_x\right)\sin(\delta) = \\
=& G_{I,r\theta} + \\
+\frac{1}{2R_f}&\left(-\cos^2(\Delta\theta)\,F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\,F_y u_x\right)\lim_{\delta\to 0}\frac{\sin(\delta)}{\delta} = \\
=& G_{I,r\theta} + \frac{1}{2R_f}\left(-\cos^2(\Delta\theta)\,F_x u_y + (F_x u_x - F_y u_y)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta)\,F_y u_x\right) \\
=& G_{I,r\theta} - \frac{1}{2R_f}F_r u_\theta
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
\widetilde{G}_{II,r\theta} =& \frac{1}{2}\frac{F_\theta u_\theta}{R_f\delta} = \\
=& \frac{1}{2R_f\delta}\left(-\sin(\Delta\theta)\,F_x + \cos(\Delta\theta)\,F_y\right)\left(-\sin(\Delta\theta-\delta)\,u_x + \cos(\Delta\theta-\delta)\,u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta)\cos(\delta)\,F_x u_x - (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta)\cos(\delta) + \cos^2(\Delta\theta)\cos(\delta)\,F_y u_y\right) + \\
+& \frac{1}{2R_f\delta}\left((F_y u_y - F_x u_x)\cos(\Delta\theta)\sin(\Delta\theta)\sin(\delta) + \cos^2(\Delta\theta)\sin(\delta)\,F_y u_x - \sin^2(\Delta\theta)\sin(\delta)\,F_x u_y\right) = \\
=& \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta)\,F_x u_x - (F_x u_y + F_y u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_y\right)\cos(\delta) + \\
+& \frac{1}{2R_f\delta}\left((F_y u_y - F_x u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_x - \sin^2(\Delta\theta)\,F_x u_y\right)\sin(\delta) = \\
=& G_{II,r\theta}\cos(\delta) + \\
+& \frac{1}{2R_f\delta}\left((F_y u_y - F_x u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_x - \sin^2(\Delta\theta)\,F_x u_y\right)\sin(\delta) = \\
=& G_{II,r\theta}\cos(\delta) + \frac{1}{2R_f\delta}F_\theta u_r\sin\delta
\end{aligned}
\tag{74}
$$

$$
\begin{aligned}
\lim_{\delta\to 0}\widetilde{G}_{II,r\theta} =& \lim_{\delta\to 0} G_{II,r\theta}\cos(\delta) + \\
+\lim_{\delta\to 0}\frac{1}{2R_f\delta}&\left((F_y u_y - F_x u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_x - \sin^2(\Delta\theta)\,F_x u_y\right)\sin(\delta) = \\
=& G_{II,r\theta} + \\
+\frac{1}{2R_f\delta}&\left((F_y u_y - F_x u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_x - \sin^2(\Delta\theta)\,F_x u_y\right)\lim_{\delta\to 0}\frac{\sin(\delta)}{\delta} \\
=& G_{II,r\theta} + \frac{1}{2R_f\delta}\left((F_y u_y - F_x u_x)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta)\,F_y u_x - \sin^2(\Delta\theta)\,F_x u_y\right) = \\
=& G_{II,r\theta} + \frac{1}{2R_f}F_\theta u_r
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
\widetilde{G}_{TOT,r\theta} = \widetilde{G}_{I,r\theta} + \widetilde{G}_{II,r\theta} = \\
&= \left(G_{I,r\theta} + G_{II,r\theta}\right)\cos\left(\delta\right) + \\
&+ \frac{1}{2R_f\delta}\left(-\cos^2\left(\Delta\theta\right)F_x u_y + \left(F_x u_x - F_y u_y\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_y u_x\right)\sin\left(\delta\right) + \\
&+ \frac{1}{2R_f\delta}\left(\left(F_y u_y - F_x u_x\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_y u_x - \sin^2\left(\Delta\theta\right)F_x u_y\right)\sin\left(\delta\right) = \\
&= \left(G_{I,r\theta} + G_{II,r\theta}\right)\cos\left(\delta\right) + \frac{1}{2R_f\delta}\left(F_y u_x - F_x u_y\right)\sin\left(\delta\right) = \\
&= G_{TOT,r\theta}\cos\left(\delta\right) + \frac{1}{2R_f\delta}\left(F_y u_x - F_x u_y\right)\sin\left(\delta\right) = \\
&= G_{TOT,r\theta}\cos\left(\delta\right) + \frac{1}{2R_f\delta}\left(F_\theta u_r - F_r u_\theta\right)\sin\left(\delta\right)
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
\lim_{\delta\to 0}\widetilde{G}_{TOT,r\theta} = \lim_{\delta\to 0} G_{TOT,r\theta}\cos\left(\delta\right) + \lim_{\delta\to 0}\frac{1}{2R_f\delta}\left(F_y u_x - F_x u_y\right)\sin\left(\delta\right) = \\
&= G_{TOT,r\theta} + \frac{1}{2R_f}\left(F_y u_x - F_x u_y\right)\lim_{\delta\to 0}\frac{\sin\left(\delta\right)}{\delta} - 0 \\
&= G_{TOT,r\theta} + \frac{1}{2R_f}\left(F_y u_x - F_x u_y\right) = \\
&= G_{TOT,r\theta} + \frac{1}{2R_f}\left(F_\theta u_r - F_r u_\theta\right)
\end{aligned}
\tag{77}
$$

2.6. Contact region

$$
u_r = 0 \tag{78}
$$

$$
\cos\left(\Delta\theta\right)u_x + \sin\left(\Delta\theta\right)u_y = 0 \tag{79}
$$

$$
u_y = -\frac{u_x}{\tan\left(\Delta\theta\right)} \tag{80}
$$

$$
\begin{aligned}
u_\theta =& -\sin\left(\Delta\theta\right)u_x - \frac{\cos^2\left(\Delta\theta\right)}{\sin\left(\Delta\theta\right)}u_x = \\
=& -\frac{u_x}{\sin\left(\Delta\theta\right)}
\end{aligned}
\tag{81}
$$

$$
F_r = \cos\left(\Delta\theta\right)F_{x,CT} + \sin\left(\Delta\theta\right)F_{y,CT} \qquad F_\theta = -\sin\left(\Delta\theta\right)F_{x,CT} + \cos\left(\Delta\theta\right)F_{y,CT} \tag{82}
$$

$$
G_{I,r\theta} = \frac{1}{2}\frac{F_r u_r}{R_f\delta} = 0 \tag{83}
$$

$$
\begin{aligned}
G_{II,r\theta} =& \frac{1}{2}\frac{F_\theta u_\theta}{R_f \delta} = \\
=& \frac{1}{2R_f\delta}\left(-\sin\left(\Delta\theta\right)F_x + \cos\left(\Delta\theta\right)F_y\right)\left(-\frac{u_x}{\sin\left(\Delta\theta\right)}\right) = \\
=& \frac{1}{2R_f\delta}\left(F_x u_x - \frac{F_y u_x}{\tan\left(\Delta\theta\right)}\right) \\
=& \frac{1}{2R_f\delta}\left(F_x - \frac{F_y}{\tan\left(\Delta\theta\right)}\right)u_x
\end{aligned}
\tag{84}
$$

3. VCCT for second order quadrilateral elements

3.1. Definition of crack tip reference frame

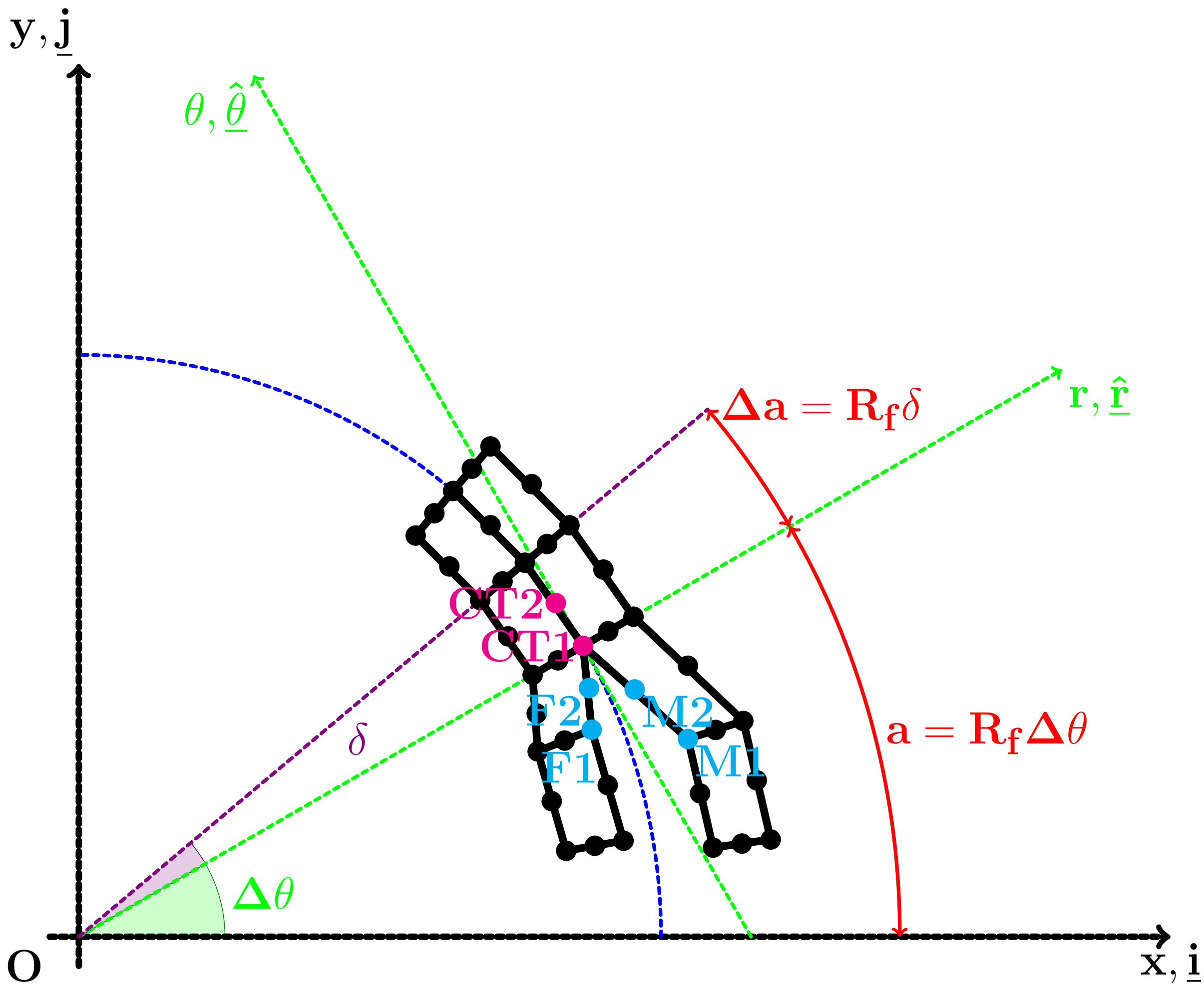

Figure 2. Schematic representation of the discretized crack tip geometry for 2^{nd} order quadrilateral elements.

$$\underline{\underline{R}} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}^{-1} = \underline{\underline{R}}^{T} = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{85}$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = \underline{\underline{R}} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{R}}^{-1} \begin{bmatrix} r \\ \theta \end{bmatrix} \tag{86}$$

3.2. Calculation of displacements and reaction forces

$$\begin{aligned} u_{x,1} &= u_{x,M1} - u_{x,F1} \qquad & u_{y,1} &= u_{y,M1} - u_{y,F1} \\ u_{x,2} &= u_{x,M2} - u_{x,F2} \qquad & u_{y,2} &= u_{y,M2} - u_{y,F2} \end{aligned} \tag{87}$$

$$\begin{aligned} u_{r,1} &= \cos(\Delta\theta)\, u_{x,1} + \sin(\Delta\theta)\, u_{y,1} \qquad & u_{\theta,1} &= -\sin(\Delta\theta)\, u_{x,1} + \cos(\Delta\theta)\, u_{y,1} \\ u_{r,2} &= \cos(\Delta\theta)\, u_{x,2} + \sin(\Delta\theta)\, u_{y,2} \qquad & u_{\theta,2} &= -\sin(\Delta\theta)\, u_{x,2} + \cos(\Delta\theta)\, u_{y,2} \end{aligned} \tag{88}$$

$$\begin{aligned} F_{r,1} &= \cos(\Delta\theta) F_{x,1} + \sin(\Delta\theta) F_{y,1} \qquad & F_{\theta,1} &= -\sin(\Delta\theta) F_{x,1} + \cos(\Delta\theta) F_{y,1} \\ F_{r,2} &= \cos(\Delta\theta) F_{x,2} + \sin(\Delta\theta) F_{y,2} \qquad & F_{\theta,2} &= -\sin(\Delta\theta) F_{x,2} + \cos(\Delta\theta) F_{y,2} \end{aligned} \tag{89}$$

3.3. Calculation of energy release rates

$$\begin{aligned} G_{I,r\theta} =& \frac{1}{2R_f\delta}\left(F_{r,1}u_{r,1} + F_{r,2}u_{r,2}\right) = \\ =& \frac{1}{2R_f\delta}\left(\cos(\Delta\theta) F_{x,1} + \sin(\Delta\theta) F_{y,1}\right)\left(\cos(\Delta\theta) u_{x,1} + \sin(\Delta\theta) u_{y,1}\right) + \\ &+ \frac{1}{2R_f\delta}\left(\cos(\Delta\theta) F_{x,2} + \sin(\Delta\theta) F_{y,2}\right)\left(\cos(\Delta\theta) u_{x,2} + \sin(\Delta\theta) u_{y,2}\right) + \\ =& \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta) F_{x,1}u_{x,1} + \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,1}u_{y,1}\right) + \\ &+ \frac{1}{2R_f\delta}\left(\cos^2(\Delta\theta) F_{x,2}u_{x,2} + \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \sin^2(\Delta\theta) F_{y,2}u_{y,2}\right) \end{aligned} \tag{90}$$

$$\begin{aligned} G_{II,r\theta} =& \frac{1}{2R_f\delta}\left(F_{\theta,1}u_{\theta,1} + F_{\theta,2}u_{\theta,2}\right) = \\ =& \frac{1}{2R_f\delta}\left(-\sin(\Delta\theta) F_{x,1} + \cos(\Delta\theta) F_{y,1}\right)\left(-\sin(\Delta\theta) u_{x,1} + \cos(\Delta\theta) u_{y,1}\right) + \\ &+ \frac{1}{2R_f\delta}\left(-\sin(\Delta\theta) F_{x,2} + \cos(\Delta\theta) F_{y,2}\right)\left(-\sin(\Delta\theta) u_{x,2} + \cos(\Delta\theta) u_{y,2}\right) = \\ =& \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta) F_{x,1}u_{x,1} - \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,1}u_{y,1}\right) + \\ &+ \frac{1}{2R_f\delta}\left(\sin^2(\Delta\theta) F_{x,2}u_{x,2} - \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos(\Delta\theta)\sin(\Delta\theta) + \cos^2(\Delta\theta) F_{y,2}u_{y,2}\right) \end{aligned} \tag{91}$$

$$
\begin{aligned}
G_{TOT,r\theta} =& G_{I,r\theta} + G_{II,r\theta} = \\
=& \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_{x,1}u_{x,1} + \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_{y,1}u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cos^2\left(\Delta\theta\right)F_{x,2}u_{x,2} + \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)F_{y,2}u_{y,2}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_{x,1}u_{x,1} - \left(F_{x,1}u_{y,1} + F_{y,1}u_{x,1}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_{y,1}u_{y,1}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\sin^2\left(\Delta\theta\right)F_{x,2}u_{x,2} - \left(F_{x,2}u_{y,2} + F_{y,2}u_{x,2}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \cos^2\left(\Delta\theta\right)F_{y,2}u_{y,2}\right) = \\
=& \frac{1}{2R_f\delta}\cos^2\left(\Delta\theta\right)\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right) + \\
&+ \frac{1}{2R_f\delta}\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) + \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
&+ \frac{1}{2R_f\delta}\sin^2\left(\Delta\theta\right)\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2} + F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right) + \\
&- \frac{1}{2R_f\delta}\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) + \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) = \\
=& \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)}\left(F_{x,1}u_{x,1} + F_{x,2}u_{x,2}\right)\right) + \\
&+ \frac{1}{2R_f\delta}\cancelto{0}{\left(\left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right) - \left(F_{x,1}u_{y,1} + F_{x,1}u_{y,1}\right)\right)}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
&+ \frac{1}{2R_f\delta}\cancelto{0}{\left(\left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right) - \left(F_{y,2}u_{x,2} + F_{y,2}u_{x,2}\right)\right)}\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right) + \\
&+ \frac{1}{2R_f\delta}\left(\cancelto{1}{\left(\cos^2\left(\Delta\theta\right) + \sin^2\left(\Delta\theta\right)\right)}\left(F_{y,1}u_{y,1} + F_{y,2}u_{y,2}\right)\right) = \\
=& \frac{1}{2}\frac{F_{x,1}u_{x,1} + F_{x,2}u_{x,2}}{R_f\delta} + \frac{1}{2}\frac{F_{y,1}u_{y,1} + F_{y,2}u_{y,2}}{R_f\delta} = \\
=& \widetilde{G}_{I,xy} + \widetilde{G}_{II,xy} = \widetilde{G}_{TOT,xy}
\end{aligned}
\qquad (92)
$$

$$
\begin{aligned}
G_{I,r\theta} =& \cos^2(\Delta\theta)\,\frac{F_{x,1}u_{x,1}+F_{x,2}u_{x,2}}{2R_f\delta}+\\
&+\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)+\\
&+\sin^2(\Delta\theta)\,\frac{F_{y,1}u_{y,1}+F_{y,2}u_{y,2}}{2R_f\delta}=\\
=& \cos^2(\Delta\theta)\,\widetilde{G}_{I,xy}+\sin^2(\Delta\theta)\,\widetilde{G}_{II,xy}+\\
&+\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
G_{II,r\theta} =& \sin^2(\Delta\theta)\,\frac{F_{x,1}u_{x,1}+F_{x,2}u_{x,2}}{2R_f\delta}+\\
&-\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)+\\
&+\cos^2(\Delta\theta)\,\frac{F_{y,1}u_{y,1}+F_{y,2}u_{y,2}}{2R_f\delta}=\\
=& \sin^2(\Delta\theta)\,\widetilde{G}_{I,xy}+\cos^2(\Delta\theta)\,\widetilde{G}_{II,xy}+\\
&-\frac{F_{x,1}u_{y,1}+F_{y,1}u_{x,1}+F_{x,2}u_{y,2}+F_{y,2}u_{x,2}}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)
\end{aligned}
\tag{94}
$$

3.4. Sensitivity analysis of the FEM solution

$$
\begin{array}{ll}
F_{x,1}\sim k_{x,1}u_{x,1} & F_{y,1}\sim k_{y,1}u_{y,1}\\
F_{x,2}\sim k_{x,2}u_{x,2} & F_{y,2}\sim k_{y,2}u_{y,2}
\end{array}
\tag{95}
$$

$$
\begin{aligned}
G_{I,r\theta} \sim& \frac{1}{2R_f\delta}\cos^2(\Delta\theta)\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)+\\
&+\frac{1}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)\left(k_{x,1}u_{x,1}u_{y,1}+k_{y,1}u_{y,1}u_{x,1}+k_{x,2}u_{x,2}u_{y,2}+k_{y,2}u_{y,2}u_{x,2}\right)+\\
&+\frac{1}{2R_f\delta}\sin^2(\Delta\theta)\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)
\end{aligned}
\tag{96}
$$

$$
\begin{aligned}
G_{II,r\theta} \sim& \frac{1}{2R_f\delta}\sin^2(\Delta\theta)\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)+\\
&-\frac{1}{2R_f\delta}\cos(\Delta\theta)\sin(\Delta\theta)\left(k_{x,1}u_{x,1}u_{y,1}+k_{y,1}u_{y,1}u_{x,1}+k_{x,2}u_{x,2}u_{y,2}+k_{y,2}u_{y,2}u_{x,2}\right)+\\
&+\frac{1}{2R_f\delta}\cos^2(\Delta\theta)\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)
\end{aligned}
\tag{97}
$$

$$
G_{TOT,r\theta}\sim\frac{1}{2R_f\delta}\left(\left(k_{x,1}u_{x,1}^2(\Delta\theta)+k_{x,2}u_{x,2}^2(\Delta\theta)\right)+\left(k_{y,1}u_{y,1}^2(\Delta\theta)+k_{y,2}u_{y,2}^2(\Delta\theta)\right)\right)
\tag{98}
$$

$$
\begin{aligned}
\frac{\partial G_{I,r\theta}}{\partial\Delta\theta} \sim & \frac{1}{R_f\delta}\cos^2\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{x,2}u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&+\frac{1}{2R_f\delta}\left(k_{x,1}+k_{y,1}\right)\left(\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,1}\left(\Delta\theta\right)+u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&+\frac{1}{2R_f\delta}\left(k_{x,2}+k_{y,2}\right)\left(\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,2}\left(\Delta\theta\right)+u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&+\frac{1}{R_f\delta}\sin^2\left(\Delta\theta\right)\left(k_{y,1}u_{y,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{y,2}u_{y,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&+\frac{1}{2R_f\delta}\left(\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)-\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)\right)\sin\left(2\Delta\theta\right)+\\
&+\frac{1}{2R_f\delta}\left(\left(k_{x,1}+k_{y,1}\right)u_{x,1}u_{y,1}+\left(k_{x,2}+k_{y,2}\right)u_{x,2}u_{y,2}\right)\cos\left(2\Delta\theta\right)
\end{aligned}
\tag{99}
$$

$$
\begin{aligned}
\frac{\partial G_{II,r\theta}}{\partial\Delta\theta} \sim & \frac{1}{R_f\delta}\sin^2\left(\Delta\theta\right)\left(k_{x,1}u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{x,2}u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&-\frac{1}{2R_f\delta}\left(k_{x,1}+k_{y,1}\right)\left(\frac{\partial u_{x,1}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,1}\left(\Delta\theta\right)+u_{x,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&-\frac{1}{2R_f\delta}\left(k_{x,2}+k_{y,2}\right)\left(\frac{\partial u_{x,2}\left(\Delta\theta\right)}{\partial\Delta\theta}u_{y,2}\left(\Delta\theta\right)+u_{x,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)\cos\left(\Delta\theta\right)\sin\left(\Delta\theta\right)+\\
&+\frac{1}{R_f\delta}\cos^2\left(\Delta\theta\right)\left(k_{y,1}u_{y,1}\left(\Delta\theta\right)\frac{\partial u_{y,1}\left(\Delta\theta\right)}{\partial\Delta\theta}+k_{y,2}u_{y,2}\left(\Delta\theta\right)\frac{\partial u_{y,2}\left(\Delta\theta\right)}{\partial\Delta\theta}\right)+\\
&+\frac{1}{2R_f\delta}\left(\left(k_{x,1}u_{x,1}^2+k_{x,2}u_{x,2}^2\right)-\left(k_{y,1}u_{y,1}^2+k_{y,2}u_{y,2}^2\right)\right)\sin\left(2\Delta\theta\right)+\\
&-\frac{1}{2R_f\delta}\left(\left(k_{x,1}+k_{y,1}\right)u_{x,1}u_{y,1}+\left(k_{x,2}+k_{y,2}\right)u_{x,2}u_{y,2}\right)\cos\left(2\Delta\theta\right)
\end{aligned}
\tag{100}
$$

$$
G_{TOT,r\theta} \sim \frac{1}{R_f\delta}\left(\left(k_{x,1}u_{x,1}\frac{\partial u_{x,1}}{\partial\Delta\theta}+k_{x,2}u_{x,2}\frac{\partial u_{x,2}}{\partial\Delta\theta}\right)+\left(k_{y,1}u_{y,1}\frac{\partial u_{y,1}}{\partial\Delta\theta}+k_{y,2}u_{y,2}\frac{\partial u_{y,2}}{\partial\Delta\theta}\right)\right) \tag{101}
$$

4. A vectorial formulation of the VCCT

4.1. Vectorial formulation

m is the order of the element's shape functions, and $p, q = 1$ represents the crack tip

$$\underline{\underline{R}}_{\Delta\theta} = \begin{bmatrix} \cos(\Delta\theta) & \sin(\Delta\theta) \\ -\sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \qquad \underline{\underline{R}}_{\Delta\theta}^{-1} = \underline{\underline{R}}^{T} = \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{bmatrix} \tag{102}$$

$$\underline{\underline{P}}_{\delta}(p) = \begin{bmatrix} \cos\left(\left(1+\frac{1-p}{m}\right)\delta\right) & \sin\left(\left(1+\frac{1-p}{m}\right)\delta\right) \\ -\sin\left(\left(1+\frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1+\frac{1-p}{m}\right)\delta\right) \end{bmatrix} \tag{103}$$

$$\underline{\underline{P}}_{\delta}^{-1} = \underline{\underline{P}}_{\delta}^{T} = \begin{bmatrix} \cos\left(\left(1+\frac{1-p}{m}\right)\delta\right) & -\sin\left(\left(1+\frac{1-p}{m}\right)\delta\right) \\ \sin\left(\left(1+\frac{1-p}{m}\right)\delta\right) & \cos\left(\left(1+\frac{1-p}{m}\right)\delta\right) \end{bmatrix} \tag{104}$$

$$\underline{\underline{Q}}_{\delta}(q) = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & \sin\left(\frac{q-1}{m}\delta\right) \\ -\sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \tag{105}$$

$$\underline{\underline{Q}}_{\delta}^{-1} = \underline{\underline{Q}}_{\delta}^{T} = \begin{bmatrix} \cos\left(\frac{q-1}{m}\delta\right) & -\sin\left(\frac{q-1}{m}\delta\right) \\ \sin\left(\frac{q-1}{m}\delta\right) & \cos\left(\frac{q-1}{m}\delta\right) \end{bmatrix} \tag{106}$$

$$\frac{\partial \underline{\underline{R}}_{\Delta\theta}}{\partial \Delta\theta} = \underline{\underline{D}} \cdot \underline{\underline{R}}_{\Delta\theta} \tag{107}$$

$$\frac{\partial \underline{\underline{P}}_{\delta}}{\partial \delta} = \left(1 + \frac{1-p}{m}\right) \underline{\underline{D}} \cdot \underline{\underline{P}}_{\delta} \tag{108}$$

$$\frac{\partial \underline{\underline{Q}}_{\delta}}{\partial \delta} = \frac{q-1}{m} \underline{\underline{D}} \cdot \underline{\underline{Q}}_{\delta} \tag{109}$$

$$\underline{\underline{D}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{110}$$

$$\underline{F}_{xy} = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \qquad \underline{F}_{r\theta} = \begin{bmatrix} F_r \\ F_\theta \end{bmatrix} \tag{111}$$

$$\underline{u}_{xy} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad \underline{u}_{r\theta} = \begin{bmatrix} u_r \\ u_\theta \end{bmatrix} \tag{112}$$

$$\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} \tag{113}$$

$$\underline{F}_{r\theta} = \underline{\underline{Q}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{F}_{xy} \qquad \underline{u}_{r\theta} = \underline{\underline{P}}_{\delta} \underline{\underline{R}}_{\Delta\theta} \underline{u}_{xy} \tag{114}$$

$$\underline{F}_{xy} = \underline{\underline{K}}_{xy} \underline{u}_{xy} + \underline{\widetilde{F}}_{xy} = \underline{\underline{K}}_{xy} \underline{u}_{xy} + \underline{\underline{\widetilde{K}}}_{N} \underline{u}_{N} \tag{115}$$

$$
\begin{aligned}
G_{TOT} &= \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} \underline{u}^T_{r\theta,p}\underline{\underline{T}}^T_{pq}\underline{F}_{r\theta,q} = \\
&= \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Tr\left(\underline{F}_{r\theta,q}\underline{u}^T_{r\theta,p}\underline{\underline{T}}^T_{pq}\right) = \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Tr\left(\begin{bmatrix} t_{pq|11}F_{r,q}u_{r,p} & t_{pq|12}F_{r,q}u_{\theta,p} \\ t_{pq|21}F_{\theta,q}u_{r,p} & t_{pq|22}F_{\theta,q}u_{\theta,p}\end{bmatrix}\right) = \\
&= \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{F}_{xy,q}\underline{u}^T_{xy,p}\underline{\underline{R}}^T_{\Delta\theta}\underline{\underline{P}}^T_{\delta}\underline{\underline{T}}^T_{pq}\right)
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
\underline{\underline{T}}_{pq} &= \begin{cases} \underline{\underline{I}}\ for\ p = q < 2 \\ \underline{\underline{0}}\ otherwise \end{cases} && \text{for } 1^{st} \text{ order quadrilateral elements} \\
&= \begin{cases} \underline{\underline{I}}\ for\ p = q < 3 \\ \underline{\underline{0}}\ otherwise \end{cases} && \text{for } 2^{nd} \text{ order quadrilateral elements} \\
&= \begin{cases} \underline{\underline{I}}\ for\ p = q < 4 \\ \underline{\underline{0}}\ otherwise \end{cases} && \text{for } 3^{rd} \text{ order quadrilateral elements} \\
&= \begin{cases} \left(14 - \frac{33\pi}{8}\right)\underline{\underline{I}}\ for\ p = 1, q = 1 \\ \left(-52 + \frac{33\pi}{2}\right)\underline{\underline{I}}\ for\ p = 1, q = 2 \\ \left(17 - \frac{21\pi}{4}\right)\underline{\underline{I}}\ for\ p = 2, q = 1 \\ \left(-\frac{7}{2} + \frac{21\pi}{16}\right)\underline{\underline{I}}\ for\ p = 2, q = 2 \\ \left(8 - \frac{21\pi}{8}\right)\underline{\underline{I}}\ for\ p = 1, q = 3 \\ \left(-32 + \frac{21\pi}{2}\right)\underline{\underline{I}}\ for\ p = 2, q = 3 \\ \underline{\underline{0}}\ otherwise \end{cases} && \text{for } 2^{nd} \text{ order quarter-point quadrilateral elements} \\
&= \begin{cases} \left(-11187 + \frac{7155\pi}{2}\right)\underline{\underline{I}}\ for\ p = 1, q = 1 \\ \left(38556 - \frac{24543\pi}{2}\right)\underline{\underline{I}}\ for\ p = 1, q = 2 \\ \left(-53055 + \frac{33777\pi}{2}\right)\underline{\underline{I}}\ for\ p = 1, q = 3 \\ \left(\frac{11396}{3} - \frac{9575\pi}{8}\right)\underline{\underline{I}}\ for\ p = 2, q = 1 \\ \left(-12936 + \frac{33003\pi}{8}\right)\underline{\underline{I}}\ for\ p = 2, q = 2 \\ \left(17988 - \frac{45837\pi}{8}\right)\underline{\underline{I}}\ for\ p = 2, q = 3 \\ \left(-\frac{8453}{3} + \frac{3595\pi}{4}\right)\underline{\underline{I}}\ for\ p = 3, q = 1 \\ \left(9804 - \frac{12411\pi}{4}\right)\underline{\underline{I}}\ for\ p = 3, q = 2 \\ \left(-13587 + \frac{17289\pi}{4}\right)\underline{\underline{I}}\ for\ p = 3, q = 3 \\ \left(6948 - \frac{17685\pi}{8}\right)\underline{\underline{I}}\ for\ p = 1, q = 4 \\ \left(-23976 + \frac{60993\pi}{8}\right)\underline{\underline{I}}\ for\ p = 2, q = 4 \\ \left(33372 - \frac{84807\pi}{8}\right)\underline{\underline{I}}\ for\ p = 3, q = 4 \\ \underline{\underline{0}}\ otherwise \end{cases} && \text{for } 3^{rd} \text{ order quarter-point quadrilateral elements}
\end{aligned}
\tag{117}
$$

$$
\begin{aligned}
G_{TOT} &= \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\left(\underline{\underline{K}}_{xy}\underline{u}_{xy} + \underline{\widetilde{F}}_{xy}\right)\underline{u}^T_{xy}\underline{\underline{R}}^T_{\Delta\theta}\underline{\underline{P}}^T_{\delta}\underline{\underline{T}}^T_{pq}\right) = \\
&= \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}^T_{xy}\underline{\underline{R}}^T_{\Delta\theta}\underline{\underline{P}}^T_{\delta}\underline{\underline{T}}^T_{pq}\right) + \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Tr\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{\widetilde{F}}_{xy}\underline{u}^T_{xy}\underline{\underline{R}}^T_{\Delta\theta}\underline{\underline{P}}^T_{\delta}\underline{\underline{T}}^T_{pq}\right)
\end{aligned}
\tag{118}
$$

$$
\underline{G} = \begin{bmatrix} G_I \\ G_{II} \end{bmatrix} = \frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1} Diag\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{F}_{xy,q}\underline{u}^T_{xy,p}\underline{\underline{R}}^T_{\Delta\theta}\underline{\underline{P}}^T_{\delta}\underline{\underline{T}}^T_{pq}\right)
\tag{119}
$$

$$\underline{G}=\begin{bmatrix}G_I\\G_{II}\end{bmatrix}=\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{P}}_{\delta}^{T}\underline{\underline{T}}_{pq}^{T}\right)+\frac{1}{2R_f\delta}Diag\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{F}}}_{xy}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{P}}_{\delta}^{T}\underline{\underline{T}}_{pq}^{T}\right)=$$
$$=\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{Q}}_{\delta}\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{P}}_{\delta}^{T}\underline{\underline{T}}_{pq}^{T}\right)+\frac{1}{2R_f\delta}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right) \tag{120}$$

4.2. Sensitivity analysis

$$\begin{aligned}\frac{\partial\underline{G}}{\partial\delta}=&-\frac{1}{2R_f\delta^2}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)-\frac{1}{2R_f\delta^2}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)+\\&+\frac{1}{2R_f\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}\delta}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\underline{\underline{D}}^{T}\right)+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\underline{\underline{D}}^{T}\right)\\&+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\frac{\partial\underline{u}_{xy}}{\partial\delta}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\frac{\partial\underline{u}_{N}}{\partial\delta}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)+\\&+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\frac{\partial\underline{u}_{xy}^{T}}{\partial\delta}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\frac{\partial\underline{u}_{xy}^{T}}{\partial\delta}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)=\\&=\frac{1}{\delta}\underline{G}+\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\frac{1}{2R_f\delta}\underline{\underline{R}}_{\Delta\theta}\left(\underline{\underline{K}}_{xy}\underline{u}_{xy}+\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\right)\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\underline{\underline{D}}^{T}\right)+\\&+\frac{1}{R_f\delta}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\frac{\partial\underline{u}_{xy}}{\partial\delta}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\frac{\partial\underline{u}_{N}}{\partial\delta}\underline{u}_{xy}^{T}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)+\\&+\frac{1}{2R_f\delta}\sum_{p=1}^{m+1}\sum_{q=1}^{m+1}Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\frac{\partial\underline{u}_{xy}^{T}}{\partial\delta}\underline{\underline{R}}_{\Delta\theta}^{T}\underline{\underline{R}}_{\delta}^{T}\right)\end{aligned} \tag{121}$$

Following Comninou

$$u(\delta)\sim\sqrt{\delta}\,(\sin,\cos)\,(\epsilon\log\delta)\qquad\epsilon=\frac{1}{2\pi}\log\left(\frac{1-\beta}{1+\beta}\right) \tag{122}$$

$$\underline{u}_{xy},\underline{u}_{N}\sim u(\delta)\sim\sqrt{\delta}\,(\sin,\cos)\,(\epsilon\log\delta)\xrightarrow{\delta\to0}0 \tag{123}$$

$$\underline{u}_{xy}\underline{u}_{xy}^{T},\underline{u}_{N}\underline{u}_{xy}^{T}\sim u^2(\delta)\sim\delta\left(\sin^2,\cos^2,\sin\cdot\cos\right)(\epsilon\log\delta)\xrightarrow{\delta\to0}0 \tag{124}$$

$$\frac{\partial\underline{u}_{xy}}{\partial\delta}\underline{u}_{xy}^{T},\frac{\partial\underline{u}_{N}}{\partial\delta}\underline{u}_{xy}^{T}\sim-\frac{1}{2}\left(\sin^2,\cos^2,\sin\cdot\cos\right)(\epsilon\log\delta)+\left(-\sin^2,\cos^2,\pm\sin\cdot\cos\right)(\epsilon\log\delta)\xrightarrow{\delta\to0}finite \tag{125}$$

$$\underline{G}\sim\frac{1}{\delta}\underline{u}_{xy}\underline{u}_{xy}^{T}\sim\frac{1}{\delta}u^2(\delta)\sim\left(\sin^2,\cos^2,\sin\cdot\cos\right)(\epsilon\log\delta)\xrightarrow{\delta\to0}finite \tag{126}$$

$$\lim_{\delta \to 0} \frac{\partial \underline{G}}{\partial \delta} \sim \frac{1}{\delta}\left(\cancelto{0}{\underline{F}(\delta)} + \underline{C}\right) \tag{127}$$

$$\lim_{\delta \to 0} \frac{\partial \underline{G}}{\partial \delta} \sim \frac{1}{\delta} \quad \xrightarrow{\int d\delta} \quad \lim_{\delta \to 0} G \sim A\log(\delta) + B \tag{128}$$

$$\begin{aligned}
\frac{\partial \underline{G}}{\partial \Delta\theta} =& \frac{1}{2R_f\delta} Diag\left(\underline{\underline{DR}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_{\delta}^T\right) + \frac{1}{2R_f\delta} Diag\left(\underline{\underline{DR}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_{\delta}^T\right) + \\
&+ \frac{1}{2R_f\delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{D}}^T\underline{\underline{R}}_{\delta}^T\right) + \frac{1}{2R_f\delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{D}}^T\underline{\underline{R}}_{\delta}^T\right) + \\
&+ \frac{1}{2R_f\delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\frac{\partial \underline{u}_{xy}}{\partial \Delta\theta}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_{\delta}^T\right) + \frac{1}{2R_f\delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\frac{\partial \underline{u}_{N}}{\partial \Delta\theta}\underline{u}_{xy}^T\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_{\delta}^T\right) + \\
&+ \frac{1}{2R_f\delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{K}}_{xy}\underline{u}_{xy}\frac{\partial \underline{u}_{xy}^T}{\partial \Delta\theta}\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_{\delta}^T\right) + \frac{1}{2R_f\delta} Diag\left(\underline{\underline{R}}_{\Delta\theta}\underline{\underline{\widetilde{K}}}_{N}\underline{u}_{N}\frac{\partial \underline{u}_{xy}^T}{\partial \Delta\theta}\underline{\underline{R}}_{\Delta\theta}^T\underline{\underline{R}}_{\delta}^T\right)
\end{aligned} \tag{129}$$

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