In-class Exercise 8.4
Boundary value problems
AMATH 301
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1. We wish to solve the two-point boundary value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^t, \;\; y(0) = 2, \;\; y(1) = 3.$$

   (a) Solve instead the initial value problem:

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = e^t, \;\; y(0) = 2, \;\; y'(0) = \alpha$$

   where $\alpha = 0$. Use `scipy.integrate.solve_ivp`. Print the value of $y(1)$ that results. Is it too low or too high?

   (b) Try different values of α until $y(1)$ gets close to the desired value of 3. This is known as the shooting method.

2. We wish to solve the two-point boundary value problem:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + t^2y = 0, \;\; y(0) = 3, \;\; y(2) = 5.$$

   (a) Show that the initial value problem:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + t^2y = 0, \;\; y(0) = 3, \;\; y'(0) = \alpha$$

   has $y(2) < 5$ for $\alpha = -5$ and $y(2) > 5$ for $\alpha = 5$. Use `scipy.integrate.solve_ivp`.

   (b) Conceptual question: What iterative algorithm could you use to approximate the correct value of α? (Hint: let $g(\alpha)$ be the function that returns $y(2)$ for a given value of α.)

   (c) Conceptual question: In order for the shooting method to work as described, we had to assume that the solution curves never cross. Is this a fair assumption in all cases?

   (d) Plot the solutions for all values of α in $-5, -4, -3, \cdots 5$. Then extend the domain to $0 \leq t \leq 5$. What are the implications of this graph on $g(\alpha)$?