In-class Exercise 3.1
Numerical differentiation
AMATH 301
University of Washington
Jakob Kotas

Centered finite-difference formulas:

$$\frac{df}{dt} = \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t} + O((\Delta t)^2)$$

$$\frac{df}{dt} = \frac{-f(t+2\Delta t) + 8f(t+\Delta t) - 8f(t-\Delta t) + f(t-2\Delta t)}{12\Delta t} + O((\Delta t)^4)$$

Forward finite-difference formulas:

$$\frac{df}{dt} = \frac{f(t+\Delta t) - f(t)}{\Delta t} + O(\Delta t)$$

$$\frac{df}{dt} = \frac{-3f(t) + 4f(t+\Delta t) - f(t+2\Delta t)}{2\Delta t} + O((\Delta t)^2)$$

Backward finite-difference formulas:

$$\frac{df}{dt} = \frac{f(t) - f(t-\Delta t)}{\Delta t} + O(\Delta t)$$

$$\frac{df}{dt} = \frac{3f(t) - 4f(t-\Delta t) + f(t-2\Delta t)}{2\Delta t} + O((\Delta t)^2)$$

1. Let $f(t) = \sqrt{t}$. Approximate $f'(a)$ using the centered finite-difference formulas with a and Δt given below and try both the $O((\Delta t)^2$ and $O((\Delta t)^4)$ formula. In each case find the relative error, that is:

$$\text{relative error } = \left| \frac{f'_{\text{approx}}(a) - f'_{\text{true}}(a)}{f'_{\text{true}}(t)} \right|.$$

   (a) $a = 4$, $\Delta t = 0.01$
   (b) $a = 4$, $\Delta t = 10^{-4}$
   (c) $a = 4$, $\Delta t = 10^{-20}$
   (d) $a = 0.03$, $\Delta t = 0.01$
   (e) In the first formula above (low-order centered first derivative), the $O((\Delta t)^2)$ term contains $f'''(c)$ for some c on $[a-\Delta t, a+\Delta t]$. Find $f'''(t)$ and consider the value of $|f'''(t)|$ in $[a-\Delta t, a+\Delta t]$. Does this explain part (d)?
   Visualization for #1

2. In a real problem, we would not be able to do the error calculation above because we don't know f'_{true}. In that case, one thing we can do instead is find $f'(a)$ with a larger Δt and $f'(a)$ again with a smaller Δt. Then the relative error is:

$$\text{relative error } \approx \left| \frac{f'_L(a) - f'_S(a)}{f'_S(a)} \right|$$

   where $f'_L(a)$ is the approximation using the larger Δt and $f'_S(a)$ is the approximation using the smaller Δt.

   Let $f(t) = \dfrac{\sin(t^2)}{\ln(t) + e^{\sqrt{t}}}$.

(a) Approximate $f'(2)$ using the higher-order $O((\Delta t)^4)$ centered finite-difference formula with $\Delta t = 0.02$ and $\Delta t = 0.01$. Approximate the relative error without actually finding $f'(t)$ by hand.

(b) Use a for-loop to iteratively decrease Δt by half, starting at $\Delta t = 0.02$, until you reach an approximate relative error (relative error between successive steps) of $< 10^{-8}$.

3. In some cases, rather than being given a function that we can sample anywhere, we only have a discrete data set. Consider the data:

t	-0.1	0	0.1	0.2	0.3	0.4	0.5
$f(t)$	3.25	0.5	-2	-4	-5	-4	1

Approximate $f'(a)$ at $a = -0.1, 0, 0.1, \cdots, 0.5$ using either the centered finite-difference formula of order $O((\Delta t)^2)$. At the endpoints, use the forward or backward finite-difference formula of order $O((\Delta t)^2)$.