In-class Exercise 2.1
Direct Solution Methods for Ax=b
AMATH 301
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1. Solve the system of equations:

$$\begin{cases} 5x_1 + x_2 + x_3 &= -14 \\ 3x_2 + 4x_3 &= 25 \\ 4x_1 + 3x_3 &= 5 \end{cases}$$

by performing Naive Gaussian elimination (no partial pivoting) on the matrix-vector system:

$$\underbrace{\begin{bmatrix} 5 & 1 & 1 \\ 0 & 3 & 4 \\ 4 & 0 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -14 \\ 25 \\ 5 \end{bmatrix}}_{b}$$

Permitted row operations are:

- Replace a row with itself plus a nonzero multiple of the pivot row
- Multiply a row by a nonzero constant
- Swap two rows (not allowed for Naive Gaussian elimination, but allowed for Gaussian elimination with partial pivoting)

2. Perform Gaussian elimination on the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$$

while keeping track of the row operations in the premultiplying matrices L_1, L_2, $\cdots$.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}}_{A} \quad \begin{array}{l} R_2 \leftarrow R_2 - \frac{4}{1}R_1 \\ R_3 \leftarrow R_3 - \frac{7}{1}R_1 \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -21 \end{bmatrix} \quad R_3 \leftarrow R_3 - \frac{-6}{-3}R_2$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{L_2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix}}_{U}$$

(a) Use Python to find L_2^{-1}. Does this make sense? Premultiply both sides by L_2^{-1}.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -7 & 0 & 1 \end{bmatrix}}_{L_1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{L_2^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix}}_{U}$$

(b) Use Python to find L_1^{-1}. Does this make sense? Premultiply both sides by L_1^{-1}.

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}}_{L_1^{-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{L_2^{-1}} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix}}_{U}$$

(c) Use Python to find $L = L_1^{-1}L_2^{-1}$ so that $A = LU$. (It can be shown in general that the product of lower-triangular matrices is lower-triangular, and the inverse of a lower-triangular matrix is also lower triangular).

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -9 \end{bmatrix}}_{U}$$

3. (a) Show that Naive Gaussian elimination fails on the matrix-vector system:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -8 \\ -19 \\ 0 \end{bmatrix}}_{b}$$

even though the system has a unique solution:

$$x = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}.$$

(b) Amend the Naive Gaussian elimination code from #1 to perform partial pivoting: interchange rows so that every time a pivot is chosen, the largest number in that column (in absolute value) is the pivot.

(c) Use the improved Gauss elimination code to solve $Ax = b$. Confirm using the built-in function `np.linalg.solve`.

Visualization of #3

4. Show that the built-in function `np.linalg.solve` fails to solve the system:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 13 \\ 19 \end{bmatrix}}_{b}$$

even though

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

solves $Ax = b$. Why is this happening? Is this failure due to the same reason as problem #3?

Visualization of #4