In-class Exercise 3.2
Numerical integration
AMATH 301
University of Washington
Jakob Kotas

- (Centered) Riemann Sum (aka Composite Midpoint Rule):

$$\int_a^b f(x)dx = \left[\sum_{k=0}^{N-1} f\left(\frac{x_k + x_{k+1}}{2}\right) h\right] + O(h^2)$$

$$= \left(f\left(x_{1/2}\right) + f\left(x_{3/2}\right) + \cdots + f\left(x_{N-1/2}\right)\right) h + O(h^2)$$

where $a = x_0$, $b = x_N$, $x_k = x_0 + hk$, $h = \dfrac{b-a}{N}$ small.

- (Composite) Trapezoid Rule:

$$\int_a^b f(x)dx = \left[\sum_{k=0}^{N-1} \frac{f(x_k) + f(x_{k+1})}{2} h\right] + O(h^2)$$

$$= \Big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)\Big)\frac{h}{2} + O(h^2)$$

- (Composite) Simpson's Rule:

$$\int_a^b f(x)dx = \left[\sum_{k=0}^{N-1} \left(f(x_k) + 4f\left(\frac{x_k + x_{k+1}}{2}\right) + f(x_{k+1})\right)\frac{h}{3}\right] + O(h^4)$$

$$= \Big(f(x_0) + 4f(x_{1/2}) + 2f(x_1) + 4f(x_{3/2}) + 2f(x_2) + \cdots + 2f(x_{N-1}) + 4f(x_{N-1/2}) + f(x_N)\Big)\frac{h}{3} + O(h^4)$$

1. Approximate $\int_0^4 \sqrt{t}dt$ using each of the three rules above with $N = 8$. Report the relative error of each rule. Then repeat with $N = 80$.

$$\text{relative error} = \left|\frac{\int_{\text{approx}} - \int_{\text{true}}}{\int_{\text{true}}}\right|$$

2. The Standard Normal Distribution (Normal with mean $\mu = 0$, standard deviation $\sigma = 1$) is an important quantity in probability and statistics:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

   It is known that the antiderivative of $f(x)$ is not expressible in terms of elementary functions.

   Approximate $\int_{-1}^1 f(x)dx$, which is the probability that a Standard Normally-Distributed random variable is between -1 and 1. Do this with Simpson's rule by initially using $N = 2$ subintervals, and successively halving the interval until the relative error between steps

$$\text{relative error} \approx \left|\frac{\int_S - \int_L}{\int_S}\right|$$

is less than 10^{-8}. Here, $\int_S$ represents the integral with the smaller step size (more subintervals) and $\int_L$ is the integral with larger step size (fewer subintervals).

Compare the result to the true value using $\Phi(x)$, the cumulative probability distribution for the Standard Normal Distribution:

$$\Phi(b) = \int_{-\infty}^{b} f(x)dx = \frac{1}{2}\left(1 + \text{erf}\left(\frac{b}{\sqrt{2}}\right)\right)$$

where $\text{erf}(x)$ is the "error function." Use `math.erf`.

3. If, as in the Kutz video, we have the Trapezoid rule formula:

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12}f''(c)$$

Why is it stated that the Trapezoid Rule is $O(h^2)$?

> The Trapezoid rule formula for a single trapezoid does have error $O(h^3)$. Because we are considering *composite* rules, in general error compounds. With $N = \frac{b-a}{h}$ subdivisions, the overall error is $O(h^2)$.

4. How should we proceed if only a discrete data set is given rather than $f(x)$ that we can sample anywhere?

> Typically the easiest is to use the Trapezoid rule as it does not involve x_k where k is a non-integer power. Or, you could widen the subintervals so that alternating data points are the edges or midpoints of subintervals to use Simpson's rule, but that only works with an even number of data points.