In-class Exercise 8.1
Initial value problems
AMATH 301
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1. Consider the initial value problem (ordinary differential equation with initial condition):

$$\frac{dy}{dt} = \underbrace{y - 5t}_{f(t,y)}, \qquad y(0) = 3.$$

   (a) Plot a slope (direction) field– that is, slopes at many values of (t, y).

   (b) Use Euler's method:

$$y_{n+1} = y_n + \Delta t f(t_n, y_n)$$

   with a time-step of $\Delta t = 0.25$, iteratively until reaching $t = 2$.

   (c) Show by hand that $y(t) = -2e^t + 5t + 5$ solves the IVP. Then plot it in Python. (Don't worry about how we found this if you haven't taken an ODEs class).

$$y = -2e^t + 5t + 5$$

   Take a derivative:

$$y' = -2e^t + 5$$

   Plug into ODE:

$$y - 5t = (-2e^t + 5t + 5) - 5t = -2e^t + 5 = y'$$

   Plug in initial condition:

$$y(0) = -2e^0 + 5(0) + 5 = 3$$

   (d) Reduce Δt to 0.05 and see if the approximation from Euler's method improves (global error is reduced).

2. Consider the initial value problem:

$$\frac{dy}{dt} = \underbrace{-y}_{f(t,y)}, \qquad y(0) = 1.$$

   (a) Plot the slope field, analytical solution $y(t) = e^{-t}$, and approximate solution using Euler's method with a time-step of $\Delta t = 2$, iteratively until reaching $t = 20$. Does Euler's method appear stable? What if $\Delta t = 4$? $\Delta t = 0.5$?

   (b) Try using "Backwards Euler's Method:"

$$y_{n+1} = y_n + \Delta t f(t_{n+1}, y_{n+1})$$

   with $\Delta = 2$ instead. Is the method stable?

$$y_{n+1} = y_n + \Delta t(-y_{n+1})$$
$$(1 + \Delta t)y_{n+1} = y_n$$
$$y_{n+1} = \frac{y_n}{1 + \Delta t}$$

3. The equation of motion for a mass on a spring (simple harmonic motion) from physics is:

$$\frac{d^2y}{dt^2} = -y$$

Since this involves a second derivative, we call it a "second-order" ODE. Consider the initial condition $y(0) = 1$, $y'(0) = 0$.

(a) Write the second-order ODE as a system of two first-order ODEs.

Let $z = dy/dt$. Then:

$$\begin{aligned} \frac{dy}{dt} &= z \\ \frac{dz}{dt} &= -y \end{aligned}$$

Or, to write in vector notation:

$$\frac{d}{dt}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -y \end{bmatrix}$$

The initial condition is:

$$\begin{bmatrix} y(0) \\ z(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) Solve using Euler's method with a time-step of $\Delta t = 0.1$ to a final time of $t = 20$. Compare to the analytical solution of $y(t) = \cos(t)$.

(c) Solve using the built-in solver `solve_ivp` in the `scipy.integrate` library. Designate `RK45` (4th-order Runge-Kutta) as the method.

4. We can see evidence of resonance (feedback) in the driven harmonic oscillator:

$$\frac{d^2y}{dt^2} = -y + \cos(\alpha t)$$

where α is some constant. Using `solve_ivp`, investigate what happens for $\alpha = 1.3, 1.1, 1.01$.