In-class Exercise 2.4
Eigenvalues, eigenvectors and solvability
AMATH 301
University of Washington
Jakob Kotas

1. In the 2.4 video it was found that the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$$

has eigenvalues:

$$\lambda_1 = 2, \;\; \lambda_2 = 4$$

with associated eigenvectors:

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \;\; v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(a) Use this visualization to understand the meaning of the eigenvalues and eigenvectors for this matrix.

(b) Construct S, the matrix whose columns are the eigenvectors of A, and Λ, the diagonal matrix whose elements are the corresponding eigenvalues of A, and verify that $AS = S\Lambda$ for this system.

$$S = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \;\; \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$AS = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix}$$

$$S\Lambda = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix}$$

(c) Compute A^5 using $A = S\Lambda S^{-1}$. You can use the following general formula for inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$A^5 = (S\Lambda S^{-1})^5 = S\Lambda^5 S^{-1}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}^5\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{(3)(1) - (1)(1)}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^5 & 0 \\ 0 & 4^5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & 1024 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 32 & -32 \\ -1024 & 3072 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -928 & 2976 \\ -992 & 3040 \end{bmatrix}$$

$$= \begin{bmatrix} -464 & 1488 \\ -496 & 1520 \end{bmatrix}$$

2. Let A be the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

(a) Find the two eigenvalues λ of A by hand using $\det(A - \lambda I) = 0$, along with the formula for determinant of a 2×2 matrix:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$$

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)(-4-\lambda) - (2)(3) = 0$$

$$\lambda^2 + 3\lambda - 10 = 0$$

$$(\lambda + 5)(\lambda - 2) = 0$$

$$\boxed{\lambda_1 = -5, \quad \lambda_2 = 2}$$

(b) Find the two associated eigenvectors v of A by hand using $(A - \lambda I)v = 0$. Since eigenvectors are only defined up to a multiplicative constant, let's scale ours so that the first component is 1. (If the first component is 0, scale so that the second component is 1 instead.)

$\underline{\lambda_1 = -5\text{:}}$

$$\begin{bmatrix} 1-(-5) & 2 \\ 3 & -4-(-5) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Both rows are a scalar multiple of the equation

$$3x + y = 0$$

$$y = -3x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -3x \end{bmatrix} = x \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

and ignoring the constant x gives an eigenvector of

$$\boxed{v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}}.$$

$\underline{\lambda_2 = 2\text{:}}$

$$\begin{bmatrix} 1-2 & 2 \\ 3 & -4-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Both rows are a scalar multiple of the equation

$$-x + 2y = 0$$

$$x = 2y$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix} = y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2y \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

and ignoring the constant $2y$ gives an eigenvector of

$$\boxed{v_2 = \begin{bmatrix} 1 \\ \frac{1}{2}. \end{bmatrix}}$$

(c) Check your answers to (a) and (b) using `np.linalg.eig`.

3. Consider the matrix

$$A = \begin{bmatrix} -1 & 1.5 & 2.5 \\ 0 & 0.5 & 2.5 \\ 0 & 2.5 & 0.5 \end{bmatrix}.$$

You can take as a given that its eigenvectors are:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Use this visualization to find the associated eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with those eigenvectors.

(b) Also find the eigenvalues using the formula $Ax = \lambda x$.

$$Av_i = \lambda_i v_i$$

$$\underbrace{\begin{bmatrix} -1 & 1.5 & 2.5 \\ 0 & 0.5 & 2.5 \\ 0 & 2.5 & 0.5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{v_1} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3v_1, \; \boxed{\lambda_1 = 3}$$

$$\underbrace{\begin{bmatrix} -1 & 1.5 & 2.5 \\ 0 & 0.5 & 2.5 \\ 0 & 2.5 & 0.5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}_{v_2} = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = -2v_2, \; \boxed{\lambda_2 = -2}$$

$$\underbrace{\begin{bmatrix} -1 & 1.5 & 2.5 \\ 0 & 0.5 & 2.5 \\ 0 & 2.5 & 0.5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_3} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -1v_3, \; \boxed{\lambda_3 = -1}$$

(c) Power iteration is an iterative method for approximating the eigenvector associated with the largest eigenvalue (in absolute value) of a square matrix A. As with many of the iterative algorithms we have seen, it requires certain properties on A for convergence to be guaranteed. It can be shown that this A matrix satisfies these properties.

Starting with an initial guess vector $x_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, use Python to perform the iteration:

$$x_{n+1} = \frac{Ax_n}{||Ax_n||_2}$$

20 times. Then print the resulting approximate eigenvector x_{20}. Also find its corresponding eigenvalue λ, which can be found using the formula:

$$\lambda = \frac{x^T Ax}{x^T x}.$$

This provides some numerical evidence for one of the three eigenvectors we were given.