A APPENDIX: PROOF OF THEOREM 2

Proof. For each basic model of a pair of items S_n^k, S_m^k, let $z_{nm}^* = S_n^* - S_m^*$, and $l_b(z_{nm}^*) = l_b(S_n^*, S_m^*)$ in Eq. 5, we expand the BPR loss $l_b(z_{nm}^k)$ around point z_{nm}^{ens} by Taylor expansion with Lagrange type reminder,

$$\begin{aligned} l_b(z_{nm}^k) &= l_b(z_{nm}^{ens}) + \frac{\partial l_b(z_{nm}^{ens})}{\partial z}(z_{nm}^k - z_{nm}^{ens}) + \frac{1}{2!}\frac{\partial^2 l_b(\tilde{z}_{nm}^{ens})}{(\partial z)^2} \\ &:= l_b(z_{nm}^{ens}) - B_{nm}^k + A_{nm}^k \end{aligned} \tag{20}$$

Where $\tilde{z}_{nm}^{ens}$ is an interpolation point between z_{nm}^{ens} and z_{nm}^k. According to Eq. 5, we have

$$B_{nm}^k = [1 - \sigma(z_{nm}^{ens})](z_{nm}^k - z_{nm}^{ens})$$

With the limitation that $\sum_{k=1}^K w_n^k = 1$ and $|w_n^k - w_m^k| \le \delta$, the weighted sum of B_{nm}^k is

$$\begin{aligned} \sum_{k=1}^K w_n^k B_{nm}^k &= [1 - \sigma(z_{nm}^{ens})]\sum_{k=1}^K w_n^k(z_{nm}^k - z_{nm}^{ens}) \\ &= [1 - \sigma(z_{nm}^{ens})]\left(\sum_{k=1}^K w_n^k z_{nm}^k - z_{nm}^{ens}\right) \\ &= [1 - \sigma(z_{nm}^{ens})]\left(-\sum_{k=1}^K w_n^k S_m^k + S_m^{ens}\right) \\ &\le \left|1 - \sigma(z_{nm}^{ens})\right|\left|-\sum_{k=1}^K w_n^k S_m^k + S_m^{ens}\right| \\ &< \left|S_m^{ens} - \sum_{k=1}^K w_n^k S_m^k\right| \\ &= \left|\sum_{k=1}^K (w_n^k - w_m^k) S_m^k\right| \\ &\le \sum_{k=1}^K |w_n^k - w_m^k||S_m^k| \le \sigma \sum_{k=1}^K S_m^k \end{aligned}$$

Sum both sides of Eq. 20 with weights, we can get

$$\begin{aligned} \sum_{k=1}^K w_n^k l_b(z_{nm}^k) &= \sum_{k=1}^K w_n^k l_b(z_{nm}^{ens}) - \sum_{k=1}^K w_n^k B_{nm}^k + \sum_{k=1}^K w_n^k A_{nm}^k \\ &> l_b(z_{nm}^{ens}) - \sigma\sum_{k=1}^K S_m^k + \sum_{k=1}^K w_n^k A_{nm}^k \end{aligned}$$

Therefore,

$$l_b(S_n^{ens}, S_m^{ens}) = l_b(z_{nm}^{ens}) < \sum_{k=1}^K w_n^k l_b(z_{nm}^k) + \sigma\sum_{k=1}^K S_m^k - \sum_{k=1}^K w_n^k A_{nm}^k$$

Proof done. □

B APPENDIX: PROOF OF THEOREM 3

Proof. To simplify the proof, we define the score difference $z_{n:N} := [z_{n+1}, ..., z_N] = [S_n - S_{n+1}, ..., S_n - S_N]$ and the logarithm pseudo-sigmoid function,

$$g_n(z_{n:N}) := \log\left(1 + \sum_{m=n+1}^N \exp(-z_{nm})\right)$$

For each base model of a list of items $\mathbf{S}^k = \{S_n^k | n \in \{1, 2, ..., N\}\}$, the PL loss is $l_{p-l}(\mathbf{S}^k) = \sum_{n=1}^N g_n(z_{n:N}^k)$. Firstly, we expand $g_n(z_{n:N}^k)$ around point $z_{n:N}^{ens}$ by Taylor expansion with Lagrange type reminder,

$$\begin{aligned} g_n(z_{n:N}^k) =& g_n(z_{n:N}^{ens}) + [\nabla g_n(z_{n:N}^{ens})]^T [z_{n:N}^k - z_{n:N}^{ens}] \\ &+ \frac{1}{2!}[z_{n:N}^k - z_{n:N}^{ens}]^T H_n(\tilde{z}_{n:N}^{ens})[z_{n:N}^k - z_{n:N}^{ens}] \\ :=& g_n(z_{n:N}^{ens}) - B_n^k + A_n^k \end{aligned} \tag{21}$$

Calculate the first-order and second-order conduction of $g_n(z_{n:N})$, we have

$$B_n^k = \frac{\sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})[z_{nm}^k - z_{nm}^{ens}]}{1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})}$$

$$A_n^k = \frac{\left(\sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})(z_{nm}^k - z_{nm}^{ens})\right)^2}{2(1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens}))^2}$$

With the limitation that $\sum_{k=1}^K w_n^k = 1$ and $w_n^k \ge 0$, the weighted sum of the n-th item, B_n^k, of K base models will be

$$\begin{aligned} \sum_{k=1}^K w_n^k B_n^k &= \frac{\sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})\sum_{k=1}^K w_n^k[z_{nm}^k - z_{nm}^{ens}]}{1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})} \\ &= \frac{\sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})\left(\sum_{k=1}^K w_n^k S_m^k - S_m^{ens}\right)}{1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})} \\ &= \frac{\sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})\left(\sum_{k=1}^K w_n^k S_m^k - \sum_{k=1}^K w_m^k S_m^k\right)}{1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})} \\ &\le \frac{\sum_{m=n+1}^N \delta \cdot \exp(-\tilde{z}_{nm}^{ens})\left(\sum_{k=1}^K S_m^k\right)}{1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})} \\ &\le \frac{\sum_{m=n+1}^N \delta \cdot \exp(-\tilde{z}_{nm}^{ens}) S_{\text{sum}}^{\max}}{1 + \sum_{m=n+1}^N \exp(-\tilde{z}_{nm}^{ens})} \\ &< \delta \cdot S_{\text{sum}}^{\max} \end{aligned}$$

Where $S_{\text{sum}}^{\max}$ is defined as

$$S_{\text{sum}}^{\max} = \max_{m=1}^N \sum_{k=1}^K S_m^k$$

Therefore,

$$\sum_{k=1}^K w_n^k g_n(z_{n:N}^k) > g_n(z_{n:N}^{ens}) - \delta S_{\text{sum}}^{\max} + \sum_{k=1}^K w_n^k A_n^k$$

Sum from $n = 1$ to $n = N$,

$$\sum_{k=1}^{K}\sum_{n=1}^{N} w_n^k g_n(z_{n:N}^k) > \sum_{n=1}^{N} g_n(z_{n:N}^{ens}) - \delta N S_{\text{sum}}^{\max} + \sum_{n=1}^{N}\sum_{k=1}^{K} A_n^k$$

$$\sum_{k=1}^{K} w_{max}^k l_{p-l}(\mathrm{S}^k) > l_{p-l}(\mathrm{S}^{ens}) - \delta N S_{\text{sum}}^{\max} + \sum_{n=1}^{N}\sum_{k=1}^{K} w_n^k A_n^k$$

I.e.,

$$l_{p-l}(\mathrm{S}^{ens}) < \sum_{k=1}^{K} w_{max}^k l_{p-l}(\mathrm{S}^k) + \delta N S_{\text{sum}}^{\max} - \sum_{n=1}^{N}\sum_{k=1}^{K} w_n^k A_n^k$$

Proof done. ◻