[T1]

(a) Inspect the recursive definition. For $r = 0$,

$$\forall b \in \mathbb{R}^n, \tilde{H}(0)b = x^0 = 0$$

therefore $\tilde{H}(0) = O$ is the zero operator; for $r = 1, 2, \cdots$

$$\forall b \in \mathbb{R}^n, \tilde{H}(r)b = P^{-1}b + P^{-1}Q\tilde{H}(r-1)b = (P^{-1} + P^{-1}Q\tilde{H}(r-1))b$$

$$\tilde{H}(r) = P^{-1} + P^{-1}Q\tilde{H}(r-1)$$

therefore, with the inductive hypothesis that $\tilde{H}(r-1)$ is a symmetric operator, so is $\tilde{H}(r)$.

(b) For weighted Gauss-Jacobi, the iterative step is defined by

$$x^k = \omega D^{-1}(b - Rx^{k-1}) + (1-\omega)x^{k-1}$$

where D consists of the diagonal entries of A and R the negative of all off diagonal entries and ω the relaxation parameter; therefore, the recursive definition for $\tilde{H}$ is $\tilde{H}(0) = O$,

$$\tilde{H}(r) = \omega D^{-1} + ((1-\omega)I - \omega D^{-1}R)\tilde{H}(r-1), r = 1, 2, \cdots$$

Expand the recursive definition we get

$$\begin{aligned}
\tilde{H}(r) &= \sum_{k=0}^{r-1}((1-\omega)I - \omega D^{-1}R)^k \omega D^{-1} \\
&\to \omega D^{-1}(I - ((1-\omega)I - \omega D^{-1}R))^{-1} \qquad (\text{as } r \to \infty) \\
&= \omega D^{-1}(\omega I - \omega D^{-1}R)^{-1} \\
&= (D - R)^{-1} = A^{-1}
\end{aligned}$$

Here we relied on one assumption that $\rho((1-\omega)I - \omega D^{-1}R) < 1$ for the convergence of the power series. This conclusion confirms that weighted Gauss-Jacobi solves the linear problem $Ax = b$.

[T2]

(a) In this case, we will need to store interior grids of size $(2^1-1) \times (2^1-1), (2^2-1) \times (2^2-1), \cdots, (2^{K+1}-1) \times (2^{K+1}-1)$; summing these numbers up we get

$$\begin{aligned}
\sum_{k=1}^{K+1}(2^k-1)^2 &= \sum_{k=1}^{K+1} 2^{2k} - 2 \cdot 2^k + 1 \\
&= \frac{4(4^{K+1}-1)}{4-1} - 2\frac{2(2^{K+1}-1)}{2-1} + K + 1 \\
&= \frac{4}{3}(4^{K+1}-1) - 4(2^{K+1}-1) + K + 1 \\
&= \left(\frac{1}{3} \cdot 2^{K+1} - 1\right) 2^{K+3} + K + \frac{11}{3}
\end{aligned}$$

(some number that doesn't look so good...)

(b) Assume 1 relaxation is applied at each level; (simply multiply the final result with the number of relaxation will give the desired answer in other cases) since in one single V cycle we go from level K down to level 0, (note no relaxation operation is applied at level 0) and back to level K, in total we will need to apply $2K$ relaxation operations. (note: the size of the relaxation operations are different among different level!)

(c) Let γ be the number of relaxation operation applied at each level. Since the relaxation operation is a pentadiagonal operator, for a fixed size grid, the arithmetic operation per each grid node is $\mathcal{O}(1)$. (ignore the edge cases for all good!); therefore, in order to compare the numerical resources used, we need only compare the number of total grid nodes involved. The number of grid nodes involved for a V cycle is (number taken from part (a)), assuming γ relaxation operations are applied at each level:

$$\gamma \sum_{k=2}^{K+1} (2^k - 1)^2 = \gamma \left(\frac{1}{3} \cdot 2^{K+1} - 1 \right) 2^{K+3} + K\gamma + \frac{8}{3}\gamma$$

(Note that no operation is done at the lowest level, so the summation starts from $k = 2$) On the other hand, the number of grid nodes involved for one relaxation operation applied on the finest grid is $(2^{K+1} - 1)^2 = 4^{K+1} - 2^{K+2} + 1$, and the ratio between these two is

$$\frac{\frac{1}{3}\gamma \left(2^{K+1} - 3\right) 2^{K+3} + K\gamma + \frac{8}{3}\gamma}{(2^{K+1} - 1)^2}$$

Something you wouldn't really want to evaluate... but APPROXIMATELY, assuming K is large, is $\frac{4}{3}\gamma$.

[C2]

The following is the results of running my program.

```
XXXX MultiGrid2d XXXX

Vcycle : 1Residual Norm : 0.78113
Vcycle : 2Residual Norm : 0.154505
Vcycle : 3Residual Norm : 0.0305834
Vcycle : 4Residual Norm : 0.0060601
Vcycle : 5Residual Norm : 0.0012019
Vcycle : 6Residual Norm : 0.000238576

XXXX MultiGrid2d Test Output XXXX
X-Panel Count    : 64
Y-Panel Count    : 64
X-Wavenumber     : 1
Y-Wavenumber     : 1
Omega            : 0.75
```

```
RelaxationCount  : 2
MultiGrid V-cycles      = 6
Residual norm (L2)      = 0.000238576
Residual norm (Inf)     = 6.90996e-05
```

[C3]

Here's the table of the V-cycle count times relaxation count (this number multiplied by $\frac{4}{3}$ will be equivalent to the number of relaxation applied at finest level, see [T2] part (c)) with $\omega = 0.1, 0.2, \cdots, 0.9$ (shown along the vertical axis) and relaxation count varying from 2 to 6 (shown along the horizontal axis). See the minimum appears at $\omega = 0.7, 0.8$ with relaxation count equals 2.

	2	3	4	5	6
0.1	146	147	148	150	150
0.2	72	72	76	75	78
0.3	48	48	52	55	60
0.4	36	36	40	45	48
0.5	28	30	36	40	42
0.6	24	27	32	35	42
0.7	20	27	32	35	42
0.8	20	24	28	35	36
0.9	22	24	28	30	36

Here's one more set of data generated with max multi-grid level 7; see the minimum also appears around $\omega = 0.7, 0.8$ with 2 relaxations.

	2	3	4	5	6
0.1	160	159	160	160	162
0.2	78	78	80	80	84
0.3	52	54	56	55	60
0.4	38	39	44	45	54
0.5	30	33	36	40	48
0.6	26	30	32	40	42
0.7	22	27	32	35	42
0.8	20	24	32	35	42
0.9	20	24	28	35	36