Exercise 3.55 [Boyd & Vandenberghe, 2004]

Ans: (a) Differentiate $f(x) = \int_{-\infty}^{x} e^{-h(t)} dt$ and get

$$f'(x) = e^{-h(x)}, f''(x) = -h'(x)e^{-h(x)}$$

Therefore for $x \in \mathbf{dom}\, h$, $h'(x) \geq 0$,

$$f''(x)f(x) = -h'(x)e^{-h(x)} \int_{-\infty}^{x} e^{-h(t)} dt \leq 0 \leq (f'(x))^2$$

since exponentials and their integrals are always positive.

(b) Apply the inequality $h(t) \geq h(x) + h'(x)(t-x)$ to the integral,

$$\begin{aligned}
\int_{-\infty}^{x} e^{-h(t)} dx &\leq \int_{-\infty}^{x} e^{-h(x)-h'(x)(t-x)} dt \\
&= e^{-h(x)} \int_{-\infty}^{x} e^{-h'(x)(t-x)} dt \\
&= e^{-h(x)} \left(\frac{1}{-h'(x)} \right) \int_{-\infty}^{0} e^{s} ds \text{ (change of variable with } s = -h'(x)(t-x)) \\
&= \frac{e^{-h(x)}}{-h'(x)} \cdot 1
\end{aligned}$$

Now that $h'(x) < 0$, we get

$$f''(x)f(x) \leq -h'(x)e^{-h(x)} \left(\frac{e^{-h(x)}}{-h'(x)} \right) = e^{-2h(x)} = (f'(x))^2$$

□

Additional Exercise 3.5 [Boyd & Vandenberghe, 2017]

Ans: First observe the objective can be written as an over all maximization among $i \times j$ quotients,

$$f_0(x) = \frac{\max_{i=1,\cdots,m} a_i^T x + b_i}{\min_{j=1,\cdots,p} c_j^T x + d_j} = \max_{\substack{i=1,\cdots,m \\ j=1,\cdots,p}} \frac{a_i^T x + b_i}{c_j^T x + d_j}$$

since for fixed i_0 and j_0,

$$\frac{\max_{i=1,\cdots,m} a_i^T x + b_i}{\min_{j=1,\cdots,p} c_j^T x + d_j} \geq \frac{a_{i_0}^T x + b_{i_0}}{\min_{j=1,\cdots,p} c_j^T x + d_j} \geq \frac{a_{i_0}^T x + b_{i_0}}{c_{j_0}^T x + d_{j_0}}$$

(the reverse inequality follows from the combination of maximizing index of numerator and minimizing index of denominator is within the range of the $i \times j$ index pairs.) Let

$$y_j = \frac{x}{c_j^T x + d_j}, z_j = \frac{1}{c_j^T x + d_j}$$

then the following 2 problems are equivalent (identical trick as in textbook §4.3.2),

$$\begin{cases} \text{minimize} & \dfrac{a_i^T x + b_i}{c_j^T x + d_j} \\ \text{subject to} & Fx \preceq g \end{cases} \qquad \begin{cases} \text{minimize} & a_i^T y_j + b_i z_j \\ \text{subject to} & Fy_j - gz_j \preceq 0 \end{cases}$$

Therefore, combine them all together, we can get the equivalent LP for the original problem,

$$\begin{cases} \text{minimize} & t \\ \text{subject to} & a_i^T y_j - b_i z_j \leq t, \text{ for } i = 1, \cdots, m, j = 1, \cdots, p \\ & Fy_j - gz_j \preceq 0, \text{ for } j = 1, \cdots, p \\ \text{with variables} & y_j, z_j, t \end{cases}$$

$\square$

Exercise 4.21 [Boyd & Vandenberghe, 2004]

Ans: Here only does part (b).
Consider Lagrange multiplier,

$$\begin{aligned} c &= \lambda 2A(x^* - x_c) \\ x^* &= \frac{1}{2\lambda}A^{-1}c + x_c \end{aligned}$$

(Note that $A \in \mathbf{S}^n_{++}$ must be invertible.) Since x^* must be on the boundary of the ellipsoid $\{x \mid (x - x_c)^T A(x - x_c) \leq 1\}$,

$$\begin{aligned} 1 &= (x^* - x_c)^T A(x^* - x_c) \\ &= \frac{1}{(2\lambda)^2}c^T A^{-1}c \end{aligned}$$

Solve for λ,

$$\lambda = \frac{1}{2}\sqrt{c^T A^{-1}c}$$

and

$$x^* = \frac{1}{\sqrt{c^T A^{-1}c}}A^{-1}c + x_c$$

(Note how the final formula assumes $c \neq 0$.) $\square$

Exercise 4.25 [Boyd & Vandenberghe, 2004]

Ans: Without loss of generality, first look into the constraint induced by the first ellipsoid; we are looking for $a \in \mathbb{R}^n, b \in R$ such that

$$a^T x + b > 0 \text{ for } x = P_1 u + q_1 \text{ with } \|u\|_2 \leq 1$$

or,

$$0 < a^T(P_1 u + q_1) + b = u^T P_1 a + q_1^T a + b$$

Rewrite the inequality we get

$$-b - q_1^T a < u^T P_1 a \leq \|P_1 a\|_2$$

where the last inequality is from Cauchy's. Combine such inequalities for all ellipsoids, we get the SOCP feasibility problem,

$$\begin{cases} \text{find} & a, b \\ \text{subject to} & -b - q_i^T a \leq \|P_i a\|_2, \text{ for } i = 1, \cdots, K \\ & b + q_i^T a \leq \|P_i a\|_2, \text{ for } i = K+1, \cdots, L \end{cases}$$

□

Additional Exercise 7.9 [Boyd & Vandenberghe, 2017]

Ans: (a) It is to show the objective function

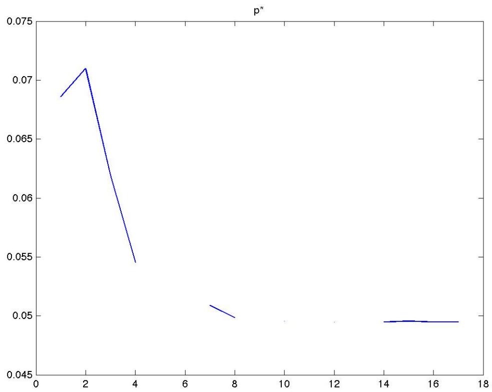
$$g(x) = \max_{k=1,\cdots,N} \|f_k(x) - y_k\|_2$$

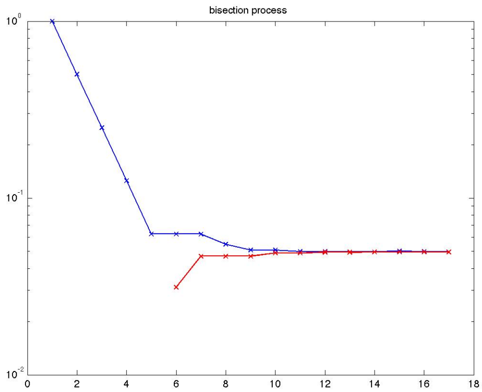
is a quasiconvex function. Take $t \in \mathbb{R}$, the t-sublevelset

$$\begin{aligned}
\{x \mid g(x) \leq t\} &= \{x \mid \|f_k(x) - y_k\|_2 \leq t, k = 1, \cdots N\} \\
&= \bigcap_{i=1}^{N} \{x \mid \|f_k(x) - y_k\|_2 \leq t\} \\
&= \bigcap_{i=1}^{N} \left\{x \mid \left\| \frac{1}{c_k^T x + d_k}(A_k x + b_k) - y_k \right\|_2 \leq t \right\} \\
&= \bigcap_{i=1}^{N} \left\{x \mid \left\|(A_k x + b_k) - y_k(c_k^T x + d_k)\right\|_2 \leq t(c_k^T x + d_k)\right\}
\end{aligned}$$

is the intersection of N convex cones and therefore convex. (Note the assumption $c_k^T x + d > 0$ is used in the last equality.)

(b) Here's the graph of p^* and the upper/lower bound during the bisection process. Note the p^* curve is missing some parts because some subproblems were infeasible; there was no solution and CVX outputted NaN.






```
%% Problem A7.9
P = zeros(3,4,4);
P(:,:,1) = [1 0 0 0; 0 1 0 0; 0 0 1 0];
P(:,:,2) = [1 0 0 0; 0 0 1 0; 0 -1 0 10];
P(:,:,3) = [1 1 1 -10; -1 1 1 0; -1 -1 1 10];
P(:,:,4) = [0 1 1 0; 0 -1 1 0; -1 0 0 10];
```

```
y = zeros(2,4);
y(:,1) = [.98; .93];
y(:,2) = [1.01; 1.01];
y(:,3) = [.95; 1.05];
y(:,4) = [2.04; 0];

f = @(x,k) (P(1:2,:,k)*[x;1])/(P(3,:,k)*[x;1]);
g = @(x) max([norm(f(x,1)-y(:,1)), norm(f(x,2)-y(:,2)), norm(f(x,3)-y(:,3)), norm(f(x


phi_k = @(t,x,k) norm(P(1:2,:,k)*[x;1]-y(:,k)*P(3,:,k)*[x;1]) - t*P(3,:,k)*[x;1];
phi = @(t,x) max([phi_k(t,x,1), phi_k(t,x,2), phi_k(t,x,3), phi_k(t,x,4)]);
TOL = 1e-4;
l = 0;  % objective function is always nonnegative
u = 1;  % random upper bound
num_iter = 0;
while u-l > TOL
    num_iter = num_iter + 1;
    t = 0.5 * (u + l);
    cvx_begin
        variable x(3);
        minimize 1;
        subject to
            phi(t,x) <= 0;
    cvx_end
    p_star(num_iter) = g(x);
    u_data(num_iter) = u;
    l_data(num_iter) = l;
    if cvx_optval == 1
        u = t;
    else
        l = t;
    end
end
h1 = plot(p_star,'LineWidth', 1.1);
title('p*');
saveas(h1,'A79_pstar','jpg');

figure;
semilogy(u_data,'b-x','LineWidth',1.05);
hold on;
h2 = semilogy(l_data,'r-x','LineWidth',1.05);
title('bisection process');
```

```
saveas(h2, 'A79_bisection', 'jpg');
```

Additional Exercise 14.8 [Boyd & Vandenberghe, 2017]

Ans:
(a) First write all the constraints in discretized form. The glide slope constraint can be viewed as

$$[0,0,1]p_k \geq \alpha \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} p_k \right\|_2 \text{ for } k = 0, 1, \cdots, K$$

The position and velocity requires more work.

$$\begin{aligned}
v_k &= v_{k-1} + \left(\frac{h}{m}\right) f_{k-1} - hge_3 \\
&= v_{k-2} + \left(\frac{h}{m}\right) f_{k-2} - hge_3 + \left(\frac{h}{m}\right) f_{k-1} - hge_3 \\
&= \cdots \\
&= v_1 + \frac{h}{m}\sum_{i=1}^{k-1} f_i - (k-1)hge_3 \\
p_{k+1} &= p_k + \frac{h}{2}(v_k + v_{k+1}) \\
&= p_{k-1} + \frac{h}{2}(v_{k-1} + v_k) + \frac{h}{2}(v_k + v_{k+1}) \\
&= \cdots \\
&= p_1 + \frac{h}{2}v_1 + h(v_2 + \cdots + v_k) + \frac{h}{2}v_{k+1} \\
&= p_1 + \frac{h}{2}v_1 + (k-1)hv_1 + h\sum_{j=2}^{k}\left(\frac{h}{m}\sum_{i=1}^{j-1} f_i - (j-1)hge_3\right) + \frac{h}{2}v_1 + \frac{h}{2}\left(\frac{h}{m}\sum_{i=1}^{k} f_i - khge_3\right) \\
&= p_1 + khv_1 - \frac{k(k+1)}{2}h^2 ge_3 + \frac{h^2}{m}\sum_{j=2}^{k}\sum_{i=1}^{j-1} f_i + \frac{h^2}{2m}\sum_{i=1}^{k} f_i
\end{aligned}$$

Therefore the constraints for touch down is

$$\begin{aligned}
0 &= v_{K+1} \\
&= v_1 + \frac{h}{m}\sum_{i=1}^{K} f_i - Khge_3 \\
&= v_0 + \frac{h}{m} f\mathbf{1} - Khge_3 \\
0 &= p_{K+1} \\
&= p_1 + Khv_1 - \frac{K(K+1)}{2}h^2 ge_3 + \frac{h^2}{m}\sum_{j=2}^{K}\sum_{i=1}^{j-1} f_i + \frac{h^2}{2m}\sum_{i=1}^{K} f_i \\
&= p_1 + Khv_1 - \frac{K(K+1)}{2}h^2 ge_3 + \frac{h^2}{m} f \begin{bmatrix} K-1 \\ \vdots \\ 2 \\ 1 \\ 0 \end{bmatrix} + \frac{h^2}{2m} f\mathbf{1} \\
&= p_1 + Khv_1 - \frac{K(K+1)}{2}h^2 ge_3 + \frac{h^2}{m} f \begin{bmatrix} K-0.5 \\ \vdots \\ 2.5 \\ 1.5 \\ 0.5 \end{bmatrix}
\end{aligned}$$

where $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and $f = [f_1, f_2, \cdots, f_K] \in \mathbb{R}^{3\times K}$. Aside of the constraints, the total fuel use can also be expressed in discretized form since

$$\int_0^{T_t d} \gamma \|f(t)\|_2 dt \approx \sum_{i=1}^{K} \gamma \|f_k\|_2 h$$

The fuel descent problem can be now formalized

$$\begin{cases}
\text{minimize} & \sum_{i=1}^{K} \gamma h \|f_k\|_2 \\
\text{subject to} & v_1 + \frac{h}{m}\sum_{i=1}^{K} f_i - Khge_3 = 0 \\
& p_1 + Khv_1 - \frac{K(K+1)}{2}h^2 ge_3 + \frac{h^2}{m}\sum_{j=2}^{K}\sum_{i=1}^{j-1} f_i + \frac{h^2}{2m}\sum_{i=1}^{K} f_i = 0 \\
& [0,0,1]p_k \geq \alpha \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} p_k \right\|_2 \text{ for } k = 0, 1, \cdots, K \\
& p_{k+1} = p_1 + khv_1 - \frac{k(k+1)}{2}h^2 ge_3 + \frac{h^2}{m}\sum_{j=2}^{k}\sum_{i=1}^{j-1} f_i + \frac{h^2}{2m}\sum_{i=1}^{k} f_i \text{ for } k = 0, 1, \cdots, K
\end{cases}$$

(b) The problem for solving minimum time descent is

$$\begin{cases} \text{minimize} & K \\ \text{subject to} & v_1 + \frac{h}{m}\sum\limits_{i=1}^{K} f_i - Khge_3 = 0 \\ & p_1 + Khv_1 - \frac{K(K+1)}{2}h^2ge_3 + \frac{h^2}{m}\sum\limits_{j=2}^{K}\sum\limits_{i=1}^{j-1} f_i + \frac{h^2}{2m}\sum\limits_{i=1}^{K} f_i = 0 \\ & [0,0,1]p_k \geq \alpha \left\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} p_k \right\|_2 \text{ for } k = 0, 1, \cdots, K \\ & p_{k+1} = p_1 + khv_1 - \frac{k(k+1)}{2}h^2ge_3 + \frac{h^2}{m}\sum\limits_{j=2}^{k}\sum\limits_{i=1}^{j-1} f_i + \frac{h^2}{2m}\sum\limits_{i=1}^{k} f_i \text{ for } k = 0, 1, \cdots, K \end{cases}$$

Without any thoughts, this problem can be solved by solving at most $\aleph_0$ (jk, but also not) feasibility problems with $K = 1, 2, \cdots$.

(c)