Problem 1

V is a vector space and $S \subseteq V$.
(a) W is a subspace of V is W is a vector space with the same operations (addition and scalar multiplication) defined on V.
(b) $span(S)$ is the set of all linear combinations of vectors in S.
(c) S is linearly dependent if there exists $v_1, \cdots, v_n \in S$ and scalars $a_1, \cdots, a_n \in F$ such that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

and not all a_i's are zero.
(d) S is a basis of V if S is linearly independent and $span(S) = V$.
(e) $\dim(V)$ is the unique number of vectors in each basis of V.

Problem 2

V is a vector vector space, $S \subseteq V$ is linearly independent and $v \in V \setminus S$. Prove that $S \cup \{v\}$ is linearly dependent if and only if $v \in span(S)$.
<u>"$\Rightarrow$"</u> Suppose $S \cup \{v\}$ is linearly dependent, take the vectors $v_1, \cdots, v_n \in S \cup \{v\}$ and scalar $a_1, \cdots, a_n$ such that

$$a_1 v_1 + \cdots + a_n v_n = 0$$

Without loss of generality we can assume v is among these vectors, say, $v_1 = v$. If $a_1 = 0$, then we get

$$a_2 v_2 + \cdots + a_n v_n = 0$$

with $v_2, \cdots, v_n \in S$ and not all $a_2, \cdots, a_n$ are zero. This contradicts the assumption that S is linearly independent. If $a_1 \neq 0$, then we can divide the equation by a_1 and get

$$v + \frac{a_2}{a_1} v_2 + \cdots + \frac{a_n}{a_1} v_n = 0$$

Therefore $v = -\frac{a_2}{a_1} v_2 + \cdots + \frac{a_n}{a_1} v_n$ is a linear combination of vectors $v_2, \cdots, v_n \in S$ and $v \in span(S)$.

<u>"$\Leftarrow$"</u> Suppose $v \in span(S)$; write $v = c_1 u_1 + \cdots + c_n u_n$ where $u_1, \cdots, u_n \in S$ and $c_1, \cdots, c_n$ are scalars. We see

$$\begin{aligned} 0 &= -v + c_1 u_1 + \cdots + c_n u_n \\ &= (-1)v + c_1 u_1 + \cdots + c_n u_u \end{aligned}$$

is a linear combination of vectors $v, u_1, \cdots, u_n \in S \cup \{v\}$ with not all zero coefficients. In particular, the coefficient of v is $(-1) \neq 0$. $\square$

Problem 3

(a) ① The zero matrix $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ certainly is a diagonal matrix with zeros as diagonal entries. For diagonal matrices $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$ and scalar $c \in \mathbb{R}$, ② the sum $A + B = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}$ and ③ the scalar multiple $cA = \begin{bmatrix} ca_1 & 0 \\ 0 & ca_2 \end{bmatrix}$ are both diagonal. Therefore W is closed under addition and scalar multiplication. By Theorem in textbook, W is a subspace of $M_{2\times 2}(\mathbb{R})$.
(b, c) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq W$ is a basis since

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow a = b = 0 \text{ (linear independence)}$$

and

$$\forall A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in W, A = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in span(S) \text{ (} S \text{ spans } W \text{)}$$

Also, $dim(W) = \#S = 2$.
(d) Consider $T = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$; $S \cup T$ is a basis of $M_{2\times 2}(\mathbb{R})$ because

$$a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c \\ d & b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Leftrightarrow a = b = c = d = 0$$

and

$$\forall A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{R}), A = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in span(S \cup T)$$

Problem 4

(a) V is a vector space and W_1, W_2 are subspaces of V. Prove that if W_1 and W_2 are finite dimensional then so is $W_1 + W_2$.
Take basis $\beta_1 = \{v_1, \cdots, v_n\}, \beta_2 = \{u_1, \cdots, u_m\}$ of W_1, W_2, respectively, then

$$W_1 = \{a_1 v_1 + \cdots a_n v_n \mid a_1, \cdots, a_n \in F\}, W_2 = \{c_1 u_1 + \cdots + c_m u_m \mid c_1, \cdots, c_m \in F\}$$

Now by definition of summed spaces,

$$W_1 + W_2 = \{(a_1 v_1 + \cdots + a_n v_n) + (c_1 u_1 + \cdots + c_m u_m) \mid a_1, \cdots, a_n, c_1, \cdots, c_m \in F\} = span(\beta_1 \cup \beta_2)$$

is generated by a finite set $\beta_1 \cup \beta_2$. By theorem in textbook, we can reduce the generating set $\{v_1, \cdots, v_n, u_1, \cdots, u_m\}$ to a basis, which can contain at most $n + m$ vectors and is

henceforth finite.
(b) Take basis $\beta_0 = \{t_1, \cdots, t_r\}$ of $W_1 \cap W_2$; extend β_0 to a basis of W_1 as $\{t_1, \cdots, t_r, v_{r+1}, \cdots, v_n\}$ (note the dimension of W_1 is still n as in the previous part), extend it similarly with W_2 and get $\{t_1, \cdots, t_r, u_{r+1}, \cdots, u_m\}$. With the same notion in part (a), we get

$$W_1 + W_2 = span(\{t_1, \cdots, t_r, v_{r+1}, \cdots, v_n, u_{r+1}, \cdots, u_m\})$$

Suppose $a_1t_1 + \cdots + a_rt_r + b_{r+1}v_{r+1} + \cdots + b_nv_n + c_{r+1}u_{r+1} + \cdots c_mu_m = 0$, then

$$u = c_{r+1}u_{r+1} + \cdots c_mu_m = -(a_1t_1 + \cdots + a_rt_r) - (b_{r+1}v_{r+1} + \cdots + b_nv_n) \in W_1 \cap W_2$$

Since $t_1, \cdots, t_r$ is a basis of $W_1 \cap W_2$, we can represent $u = \alpha_1t_1 + \cdots + \alpha_rt_r$, and

$$u = \alpha_1t_1 + \cdots + \alpha_rt_r = -(a_1t_1 + \cdots + a_rt_r) - (b_{r+1}v_{r+1} + \cdots + b_nv_n)$$

Or $(\alpha_1+a_1)t_1+\cdots+(\alpha_r+a_r)t_r+b_{r+1}v_{r+1}+\cdots+b_nv_n = 0$. Since $t_1, \cdots, t_r, v_{r+1}, \cdots, v_n$ is a basis of W_1, we must get $b_{r+1} = \cdots = b_n = 0$. Similarly, we shall have $c_{r+1} = \cdots = c_m = 0$. That is, we have

$$a_1t_1 + \cdots + a_rt_r = 0$$

and again since $t_1, \cdots, t_r$ is linearly independent, we have $a_1 = \cdots = a_r = 0$. We just proved $\{t_1, \cdots, t_r, v_{r+1}, \cdots, v_n, u_{r+1}, \cdots, u_m\}$ is a linearly independent set. My proof to this part is a little tedious; perhaps the instructor will have a shorter proof.

(c) Consider $V = \mathbb{R}^3$, $W_1 = \{x\text{-}y \text{ plane}\} = span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$, $W_2 = \{y\text{-}z \text{ plane}\} = span\left(\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$. In this case, $W_1 \cap W_2 = \{y\text{-axis}\} = span\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \neq \{0\}$ and indeed $W_1 + W_2 = span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \mathbb{R}^3$.

Problem 5

(a) False. $\begin{bmatrix}-1\\1\\2\\1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\\-1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ is equivalent to solving $\left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & 1\\0 & 1 & 3\\0 & 1 & 0\end{array}\right]$ (by elementary matrix operations), which is not solvable.
(b) True.
(c) False. $dim(\mathbb{R}^2) = 2$; there can't be a linearly independent subset of $\mathbb{R}^2$ of more than 2 vectors.
(d) False. Consider 1-dimensional vector space $V = span(v)$; every vector is a multiple of the other.
(e) False. It is an "affine space" of $\mathbb{R}^4$, but definitely not a subspace of $\mathbb{R}^4$. For instance,

it doesn't contain zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(f) True.

(g) False. The inequality should be reversed. Linear independent subset should always contain less (or equal number of) vectors than spanning set.

(h) True.

(i) False. Consider $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, since

$$1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

it's linearly dependent. However, the subset $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \subseteq S$ is easily the basis of $\mathbb{R}^2$ and linearly independent.