Exercise 3.18 [Boyd & Vandenberghe, 2004]

Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.
(a) $f(X) = \mathbf{tr}(X^{-1})$ is convex on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.
(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

Ans: Here only does part (a).
(a) Let $Z \in \mathbf{S}^n_{++}$, $V \in \mathbf{S}^n$, and $(a, b) \subseteq \mathbb{R}$, an interval such that $\forall t \in (a, b), Z + tV \in \mathbf{S}^n_{++}$.
Define $g(t) = f(Z + tV) = \mathbf{tr}((Z + tV)^{-1})$.

Exercise 3.19 [Boyd & Vandenberghe, 2004]

Nonnegative weighted sums and integrals.
(a) Show that $f(x) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ is a convex function of x, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of x. (You can use the fact that $f(x) = \sum_{i=1}^{k} x_{[i]}$ os convex on $\mathbb{R}^n$.)
(b) Let $T(x, \omega)$ denote the trigonometric polynomial

$$T(x, \omega) = x_1 + x_2 \cos\omega + x_3 \cos 2\omega + \cdots + x_n \cos(n-1)\omega.$$

Show that the function

$$f(x) = -\int_0^{2x} \log T(x, \omega) d\omega$$

is convex on $\{x \in \mathbb{R}^n \mid T(x, \omega) > 0, 0 \leq \omega \leq 2\pi\}$.

Ans: Here only does part (a).

Exercise 3.22 [Boyd & Vandenberghe, 2004]

Composition rules. Show that the following functions are convex.
(a) $f(x) = -\log(-\log(\sum_{i=1}^{m} e^{a_i^T x + b_i}))$ on $\mathbf{dom}\, f = \{x \mid \sum_{i=1}^{m} e^{a_i^T x + b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^{n} e^{y_i})$ is convex.
(b) $f(x, u, v) = -\sqrt{uv - x^T x}$ on $\mathbf{dom}\, f = \{(x, u, v) \mid uv > x^T x, u, v > 0\}$. Use the fact that $x^T x/u$ is convex in (x, u) for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex on $\mathbb{R}^2_{++}$.
(c) $f(x, u, v) = -\log(uv - x^T x)$ on $\mathbf{dom}\, f = \{(x, u, v) \mid uv > x^T x, u, v > 0\}$.
(d) $f(x, t) = -(t^p - \|x\|_p^p)^{1/p}$ where $p > 1$ and $\mathbf{dom}\, f = \{(x, t) \mid t \geq \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x, u) for $u > 0$ (see exercise 3.23), and that $-x^{1/p} y^{1-1/p}$ is convex on $\mathbb{R}^2_+$ (see exercise 3.16).
(e) $f(x, t) = -\log(t^p - \|x\|_p^p)$ where $p > 1$ and $\mathbf{dom}\, f = \{(x, t) \mid t > \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x, u) for $u > 0$ (see exercise 3.23).

Ans: Here only does part (c).

Additional Exercise 2.5 [Boyd & Vandenberghe, 2017]

A perspective composition rule [Marèchal]. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function with $f(0) \leq 0$.
(a) Show that the perspective $tf(x/t)$, with domain $\{(x,t) \mid t > 0, x/t \in \mathbf{dom}\, f\}$, is nonincreasing as a function of t.
(b) Let g be concave and positive on its domain. Show that the function

$$h(x) = g(x)f(x/g(x)), \quad \mathbf{dom}\, h = \{x \in \mathbf{dom}\, g \mid x/g(x) \in \mathbf{dom}\, f\}$$

is convex.
(c) As an example, show that

$$h(x) = \frac{x^T x}{(\prod_{k=1}^{n} x_k)^{1/n}}, \quad \mathbf{dom}\, h = \mathbb{R}^n_{++}$$

is convex. *Ans:* Here only does part (a) and (b).

Additional Exercise 2.30 [Boyd & Vandenberghe, 2017]

Huber penalty. The infimal convolution of two functions f and g on $\mathbb{R}^n$ is defined as

$$h(x) = \inf_y (f(y) + g(x - y))$$

(see exercise 2.17). Show that the infimal convolution of $f(x) = \|x\|_1$ and $g(x) = (1/2)\|x\|_2^2$, *i.e.*, the function

$$h(x) = \inf_y (f(y) + g(x - y)) = \inf_y \left(\|y\|_1 + \frac{1}{2}\|x - y\|_2^2 \right)$$

is the *Huber penalty*

$$h(x) = \sum_{i=1}^{n} \phi(x_i), \quad \phi(u) = \begin{cases} u^2/2, |u| \le 1 \\ |u| - 1/2, |u| > 1 \end{cases}$$

Ans:

Additional Exercise 2.31 [Boyd & Vandenberghe, 2017]

Suppose the function $h : \mathbb{R} \to \mathbb{R}$ is convex, nondecreasing, with $\mathbf{dom}\, h = \mathbb{R}$, and $h(t) = h(0)$ for $t \leq 0$.
(a) Show that the function $f(x) = h(\|x\|_2)$ is convex on $\mathbb{R}^n$.
(b) Show that the conjugate of f is $f^*(y) = h^*(\|y\|_2)$.
(c) As an example, derive the conjugate of $f(x) = (1/p)\|x\|_2^p$ for $p > 1$, by applying the result of part (b) with the function

$$h(t) = \frac{1}{p}\max\{0, t\}^p = \begin{cases} \frac{1}{p}t^p, t \geq 0 \\ 0, t < 0 \end{cases}$$

. *Ans:*

Additional Exercise 3.17 [Boyd & Vandenberghe, 2017]

Minimum fuel optimal control. Solve the minimum fuel optimal control problem described in exercise 4.16 of *Convex Optimization*, for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, x_{des} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, N = 30.$$

You can do this by forming the LP you found in your solution of exercise 4.16, or more directly using CVX. Plot the actuator signal $u(t)$ as a function of time t.

Ans: