Additional Exercise 3.21 [Boyd & Vandenberghe, 2017]

Norm approximation via SOCP, for ℓ_p-norms with rational p.

(a) Use the observation at the beginning of exercise 4.26 in *Convex Optimization* to express the constraint

$$y \leq \sqrt{z_1 z_2}, y, z_1, z_2 \geq 0,$$

with variables y, z_1, z_2 as a second-order cone constraint. Then extend your result to the constraint

$$y \leq (z_1 z_2 \cdots z_n)^{1/n}, y \geq 0, z \succeq 0,$$

where n is a positive integer, and the variables are $y \in \mathbb{R}$ and $z \in \mathbb{R}^n$. First assume that n is a power of two, and then generalize your formulation to arbitrary positive integers.

(b) Express the constraint

$$f(x) \leq t$$

as a second-order cone constraint, for the following two convex functions f:

$$f(x) = \begin{cases} x^\alpha, x \geq 0 \\ 0, x < 0 \end{cases},$$

where α is a rational and greater than or equal to one, and

$$f(x) = x^\alpha, \mathbf{dom}\, f = \mathbb{R}_{++},$$

where α is rational and negative.

(c) Formulate the norm approximation problem

$$\text{minimize} \|Ax - b\|_p$$

as a second-order cone program, where p is a rational number greater than or equal to one. The variable in the optimization problem is $x \in \mathbb{R}^n$. The matrix $A \in \mathbb{R}^{m \times n}$ and the vector $b \in \mathbb{R}^m$ are given. For an m-vector y, the norm $\|y\|_p$ is defined as

$$\|y\|_p = \left(\sum_{k=1}^{m} |y_k|^p \right)^{1/p}$$

where $p \geq 1$.

Ans: Here only presents solutions to part (a) and (b).
(a) The observation from Problem 4.26 [Boyd & Vandenberghe, 2004] is for $x \in \mathbb{R}^n, y, z \in \mathbb{R}$,

$$x^T x \leq yz, y \geq 0, z \geq 0 \Leftrightarrow \left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, y \geq 0, z \geq 0$$

This is true since

$$\left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2^2 - (y+z)^2 = 4\|x\|_2^2 + y^2 - 2yz + z^2 - (y^2 + 2yz + z^2)$$
$$= 4x^T x - 4yz$$

Back to Additional Exercise 3.21, first we claim: "For $n = 2^k, k \in \mathbb{N}$, the constraint

$$y \leq (z_1 z_2 \cdots z_n)^{1/n}, z_1, z_2, \cdots, z_n \geq 0$$

can expressed as SOC constraints. " The base case $k = 1$ is done as described above. Now assuming the statement is true for $n = 2^k$, in order to express the following in SOC constraint,

$$y \leq (z_1 z_2 \cdots z_n s_1 s_2 \cdots s_n)^{1/2n}, z_1, z_2, \cdots, z_n, s_1, s_2, \cdots, s_n \geq 0$$

introduce n new variables: $t_i = \sqrt{z_i s_i}, i = 1, 2, \cdots, n$; this can be done by setting constraints

$$t_i \geq 0, t_i^2 = \frac{1}{2}(z_i + s_i)^2 - \frac{1}{2}(z_i^2 + s_i^2)$$

Then the geometric mean condition is now

$$y \leq (t_1^2 t_2^2 \cdots t_n^2)^{1/2n} = (t_1 t_2 \cdots t_n)^{1/n}$$

From inductive hypothesis, this can be expressed as SOC constraints. (To be more specific, in order to find the equivalent SOC constraints for the original inequality of the geometric mean of 2^k variables, we can write a recursive program to introduce new variables and conditions. We omit the tedious notations here.) We've proved our claim. For cases when n is not a power of two,

Additional Problem 1

Ans:

Additional Problem 2

Ans:

Additional Problem 3

Ans:

Additional Exercise 3.11 [Boyd & Vandenberghe, 2017]

Formulate the following optimization problems as semidefinite programs. The variable is $x \in \mathbb{R}^n$; $F(x)$ is defined as

$$F(x) = F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n$$

with $F_i \in \mathbf{S}^m$. The domain of f in each subproblem is $\mathbf{dom}\, f = \{x \in \mathbb{R}^n \mid F(x) \succ 0\}$.

(a) Minimize $f(x) = c^T F(x)^{-1} c$ where $c \in \mathbb{R}^m$.

(b) Minimize $f(x) = \max_{i=1,\cdots,K} c_i^T F(x)^{-1} c_i$ where $c_i \in \mathbb{R}^m, i = 1, \cdots, K$.

(c) Minimize $f(x) = \sup\limits_{\|c\|_2 \leq 1} c^T F(x)^{-1} c$.

(d) Minimize $f(x) = \mathbf{E}(c^T F(x)^{-1} c)$ where c is a random vector with mean $\mathbf{E}c = \bar{c}$ and covariance $\mathbf{E}(c - \bar{c})(c - \bar{c})^T = S$.

Ans: