Exercise 2.7 [Boyd & Vandenberghe, 2004]

Voronoi description of halfspace. Let a and b be distinct points in $\mathbb{R}^n$. Shpow that the set of all points that are closer (in Euclidean norm) to a than b, *i.e.*$\{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^t x \leq d$. Draw a picture.

Ans: Denote $x = [x_1, \cdots, x_n], a = [a_1, \cdots, a_n], b = [b_1, \cdots, b_n]$, the quiterium $\|x-a\|_2 \leq \|x-b\|_2$ is equivalent to

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 \leq (x_1 - b_1)^2 + \cdots + (x_n - b_n)^2$$

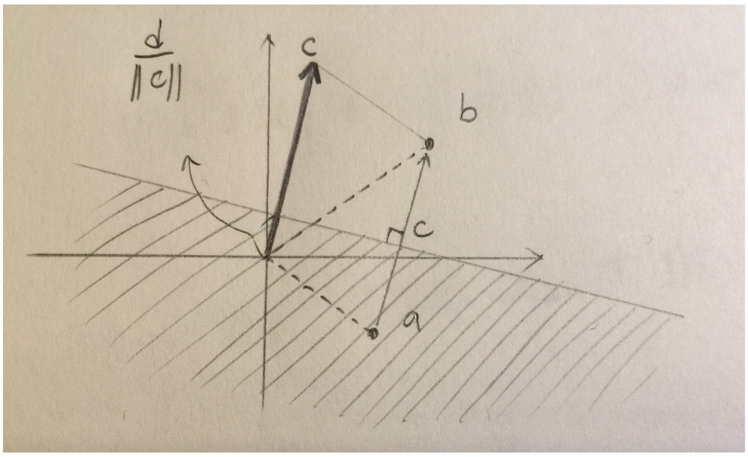
Organize the inequality for a bit (subtract $x_1^2 + \cdots + x_n^2$ for both sides and move terms around) and we get

$$(-2a_1 + 2b_1)x_1 + \cdots + (-2a_n + 2b_n)x_n \leq (b_1^2 - a_1^2) + \cdots + (b_n^2 - a_n^2)$$

Let $c = -a + b \in \mathbb{R}^n$, $d = \frac{1}{2}(\|b\|_2^2 - \|a\|_2^2) \in \mathbb{R}$ and divide the inequality by 2, we get

$$c^T x \leq d$$

as in the desired form. $\square$



Exercise 2.12 [Boyd & Vandenberghe, 2004]

Which of the following sets are convex?
(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2 \forall y \in S\}$$

where $S \subseteq \mathbb{R}^n$.

Ans: Observe this set (hereby denoted as D) is the intersection of halfspaces (as proved in **2.7**)

$$D = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$$

Since intersection of convex sets is also convex, YES, the set D is convex. □

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathrm{dist}(x, S) \le \mathrm{dist}(x, T)\}$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathrm{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$$

Ans: NO. Consider counterexample with $S = \{z \mid \|z\|_2 \ge 2\}$ and $T = \{0\}$; the set described in the problem (hereby denoted as E) is the complement set of unit disk

$$E = \{x \mid \|x\|_2 \ge 1\}$$

This is true since for $x \in \mathbb{R}^n$, $\mathrm{dist}(x, \{0\}) = \|x\|_2$ and

$$\mathrm{dist}(x, S) = \begin{cases} 2 - \|x\|_2, \|x\|_2 \le 2 \\ 0, \|x\|_2 > 2 \end{cases}$$

And certainly the complement set of unit disk is not convex. □

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

Ans: YES. Take $x_1, x_2 \in F := \{x \mid x + S_2 \subseteq S_1\}$ and arbitrary $y \in S_2$; we have $x_1 + y, x_2 + y \in S_1$ since $x_1, x_2 \in F$. Now for $\theta \in [0, 1]$, since S_1 is convex,

$$\theta(x_1 + y) + (1 - \theta)(x_2 + y) = \theta x_1 + (1 - \theta)x_2 + y \in S_1$$

Since y was chosen arbitrarily, $\theta x_1 + (1 - \theta)x_2 + S_2 \subseteq S_1$; that is, $\theta x_1 + (1 - \theta)x_2 \in F$. □

Exercise 2.16 [Boyd & Vandenberghe, 2004]

Show that if S_1 and S_2 are convex sets in $\mathbb{R}^m \times \mathbb{R}^n$[1], then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}.$$

Ans: Take $(x, y_1 + y_2), (z, y_3 + y_4) \in S$ (where $(x, y_1), (z, y_3) \in S_1, (x, y_2); (z, y_4) \in S_2$) and $\theta \in [0, 1]$; our concern is whether

$$v := \theta(x, y_1 + y_2) + (1 - \theta)(z, y_3 + y_4) = (\theta x + (1 - \theta)z, \theta y_1 + (1 - \theta)y_3 + \theta y_2 + (1 - \theta)y_4)$$

is in S. Now that both S_1 and S_2 are convex, we are safe to claim

$$\theta(x, y_1) + (1 - \theta)(z, y_3) = (\theta x + (1 - \theta)z, \theta y_1 + (1 - \theta)y_3) \in S_1$$

$$\theta(x, y_2) + (1 - \theta)(z, y_4) = (\theta x + (1 - \theta)z, \theta y_2 + (1 - \theta)y_4) \in S_2$$

Note the vector v is exactly the direct sum of these 2 vectors, henceforth it's in S. $\square$

[1] I believe this was a typo in the textbook.