Disclaimer: This homework was finished through an intense discussion and long hours of working together with Howard Heaton and Maria Ntekoume.

Exercise 1

(1) Suppose for contradiction,

$$\exists K \subset\subset \Omega, \forall N \in \mathbb{N} \cup \{0\}, \exists \varphi \in \mathcal{D}_K, |T\varphi| > C\|\varphi\|_N$$

Consider specifically when $C = N$, take $\{\varphi_N\}_{N=0}^{\infty} \subseteq \mathcal{D}_K$ with

$$|T\varphi_N| > N\|\varphi_N\|_N$$

By scaling (using linearity of T), we can WLOG assume that $|T\varphi_N| = 1$, and

$$\|\varphi_N\|_N < \frac{1}{N}$$

Observe that for any multi-index α, $\{\varphi_N\}_{N>|\alpha|}$ converges to $0_{\mathcal{D}_K}$ w.r.t. $\|\cdot\|_{|\alpha|}$ norm; therefore, adding the fact that supp$\varphi_N \subseteq K$, we conclude $\{\varphi_N\}$ converges to $0_{\mathcal{D}_K}$ in $\mathcal{D}(\Omega)$ w.r.t. the topology inscribed in the problem set. Now by continuity of T, we should have

$$T(\varphi_N) \to T(0_{\mathcal{D}_K}) = 0$$

(last equality comes from linearity of T.) This contradicts our assumption that $|T\varphi_N| = 1$ for all N. $\square$

(2) $T\varphi = \int \partial^2_{x_1x_1}\varphi dx$ should be a distribution of order 2. Fix any $K \subset\subset \Omega$,

$$|T\varphi| \leq \int_\Omega |\partial^2_{x_1x_1}\varphi| dx \leq |\Omega| \|\varphi\|_2$$

To show that "<u>T has order exactly 2</u>", suppose there exists $N < 2$ with $C > 0$ such that $\forall \varphi \in \mathcal{D}, |T\varphi| \leq C\|\varphi\|_N$. Take a function φ such that $T\varphi = 1$, pick $x_0 \in int(\text{supp}(\varphi))$, let $\varphi_\epsilon(x) = \epsilon^2\varphi(x_0 + \frac{x-x_0}{\epsilon})$ for $\epsilon \in (0,1)$. This function φ_ϵ has a smaller support than φ. Note that

$$\int_\Omega \partial^2_{x_1x_1}\varphi_\epsilon dx = \int_\Omega \frac{\epsilon^2}{\epsilon^2}\partial^2_{x_1x_1}\varphi dx = 1$$

However,

$$\|\varphi_\epsilon\|_{L^\infty} = \epsilon^2\|\varphi\|_{L^\infty} \qquad \|\partial_{x_i}\varphi_\epsilon\|_{L^\infty} = \epsilon\|\partial_{x_i}\varphi\|_{L^\infty} \text{ for any } i = 1, \cdots, d$$

By letting $\epsilon \to 0$ we get a sequence that will eventually contradict the following condition for T to be of order $N < 2$

$$1 = |T\varphi_\epsilon| \leq C\|\varphi_\epsilon\|_N \leq \epsilon\|\varphi\|_N$$

$S\varphi = \sum_{k=1}^{\infty} D^k\varphi(\frac{1}{k})$ for $\varphi \in ((0,2))$ should be a distribution of order ∞. Consider $K_m = [\frac{1}{m}, 1] \subset\subset (0,2)$, for $\varphi \in \mathcal{D}_{K_m}$,

$$S\varphi = \sum_{k=1}^{m-1} D^k\varphi(\frac{1}{k})$$

Roughly speaking since this involves the $(m-1)$-th derivative of φ, the distribution S needs to have order at least $m-1$. With construction like previously, once we fixed the domain we can always find φ_ϵ such that its $(m-1)$-th derivative remains of order 1, its $\|\cdot\|_{m-2}$ norm tends to zero, then we can argue that S on the compact set K needs to have order at least $m-1$.

For distribution of order 0, since it is a bounded linear functional on C_c^∞ which is a linear subspace of C_c, we apply Hahn-Banach Theorem to extend T to all continuous function with compact support. Applying Riesz representation theorem for bounded linear functional gives us a signed Radon measure. Note that Radon measure gives finite measure for compact sets. $\square$

(3) First, let us show that "<u>T is of order 0</u>". Fix $K \subset\subset \Omega$, take $\eta \in \mathcal{D}$ a positive test function that is constantly 1 on K (Urysohn's Lemma). For any $\varphi \in \mathcal{K}$, $\text{supp}\varphi \subseteq K$,

$$\forall x \in \Omega, \varphi(x) \leq \|\varphi\|_0 \eta(x)$$

Since T is a positive distribution, $T(\|\varphi\|_0\eta - \varphi) \geq 0$; this leads to

$$\|\varphi\|_0 T(\eta) \geq T(\varphi)$$

Similarly since $-\varphi \leq \|\varphi\|_0\eta$, $T(\|\varphi\|_0\eta + \varphi) \geq 0$, and

$$\|\varphi\|_0 T(\eta) \geq -T(\varphi)$$

We conclude that $|T(\varphi)| \leq T(\eta)\|\varphi\|_0$ and this $T(\eta) \geq 0$ is the constant associated with this compact set K. Use problem (2), T extended to the whole $C_c(\Omega)$ is equivalent to a signed Radon measure μ_T. We assumed T is a positive distribution, that is, $\forall \varphi \in \mathcal{D}$

$$\varphi \geq 0 \Rightarrow \int_\Omega \varphi d\mu_T = T\varphi \geq 0$$

Now approximate general $C_c(\Omega)$ functions with $C_c^\infty(\Omega)$ functions and utilize continuity of the extended linear functional, we have $\forall f \in C_c(\Omega)$,

$$\varphi \geq 0 \Rightarrow \int_\Omega \varphi d\mu_T = T\varphi \geq 0$$

This shows that μ_T satisfies the definition of being a positive measure. $\square$

(4)(5) I'd like to invoke a theorem I found in [3]. It states every distribution of compact support can be written as

$$T = \sum_{|\alpha|\leq N} D^\alpha g_\alpha$$

that is

$$T\varphi = \sum_{|\alpha|\leq N} (-1)^{|\alpha|} \int_\Omega D^\alpha \varphi g_\alpha dx$$

where N is a finite number (which becomes T's order) and $g_\alpha dx$ are measures on $\mathbb{R}^d$. With this result, we can easily see that for a fixed compact set $K \subset\subset \Omega$,

$$|T\varphi| \leq \sum_{|\alpha|<N} \mu_{g_\alpha}(K)\|D^\alpha \varphi\|_{L^\infty} \leq (\text{some constant depending on } K \text{ and } g_\alpha)\|\varphi\|_N$$

(Here μ_{g_α} denotes the measure $\mu_{g_\alpha}(E) = \int_E g_\alpha dx$.) This sums up problem (4).

For problem (5), we see that these μ_{g_α}'s must be measures on the finite support of T. In other words, we have

$$T\varphi = \sum_{x_i\in \text{supp}(T)} \sum_{|\alpha|\leq N} g_\alpha(x_i) D^\alpha \varphi$$

In other words, a linear combination of some derivatives of φ at these finite points. □

(6) Consider a sequence of radially symmetric mollifiers $\{\eta_k\}_{k\in\mathbb{N}}$. Let f_k be the function defined by

$$f_k(x) = T(\sigma_x(\eta_k)) \text{ where } \sigma_x(\eta_k)(y) = \eta_k(x-y)$$

Some would denote $f = \eta_k * T$. To show that f_k is a C^∞ function, utilize the smoothness of η_k and pass the derivative through linearity of T:

$$\partial_{x_j} D^\alpha f_k(x) = \lim_{h\to 0} \frac{T\sigma_{x+h} D^\alpha \eta_k - T\sigma_x D^\alpha \eta_k}{h} \text{ for } j = 1, \cdots, d$$

The limit exist because η_k is smooth and the convergence of shifting by h is uniform. Take induction on the multi-index α; we conclude that f_k is smooth.

...This is Lemma 9 on p.551 on [3] I just don't have enough time to type them...

Exercise 2

(1) This solution was done with help from [1]. By *flattening*, we consider the harmonic function on an *annulus* $A_{r_1,r_2} := B_{r_2}(0) \setminus B_{r_1}(0)$:

$$v(x) = \frac{\log(r_2/|x|)}{\log(r_2/r_1)} \min_{\partial B_{r_1}(0)} u + \frac{\log(|x|/r_1)}{\log(r_2/r_1)} \min_{\partial B_{r_2}(0)} u \tag{1}$$

Since this expression v is a linear combination of the fundamental solutions (logarithm of 2-norm) on the domain excluding their singularities, it is a harmonic function on A_{r_1,r_2}. Now that

$$v(\partial B_{r_1}(0)) = \left\{\min_{\partial B_{r_1}(0)} u\right\} \qquad v(\partial B_{r_2}(0) = \left\{\min_{\partial B_{r_2}(0)} u\right\}$$

we have

$$v\mid_{\partial A_{r_1,r_2}} \leq u\mid_{\partial A_{r_1,r_2}}$$

by minimum principle of superharmonic functions, $v(x) \leq u(x)$ over all $x \in A_{r_1,r_2}$. Now since u is bounded from below by $m = 0$, we see that in the above expression (1),

$$\min_{\partial B_{r_2}(0)} u \to m \text{ as } r_2 \to \infty$$

and expression (1) becomes a constant $\min_{B_{r_1}(0)} u$. Hence for all $x \in_{r_2} (0)^c$,

$$u(x) \geq \min_{\partial B_{r_1}} u$$

Taking $r_1 \to 0$, we get

$$u(x) \geq \liminf_{r_1 \to 0} \min_{\partial B_{r_1}(0)} u = u(0)$$

Not that the last equality only relies on the continuity of u. Again, this is true for all $x \neq 0$, so we found that u attains global minimum at the origin. By strong minimum principle for superharmonic functions, (the harmonic function with the same boundary data on, say, a ball, will be necessarily smaller than u, but u attains minimum in the interior, which makes this harmonic function squeezed to attain minimum too and hence a constant; now the boundary data must also be constant. This is true for balls of all balls of different radii. Idea taken from [2].) we conclude that u must be a constant. $\square$

In the cases when $d \geq 3$, we can easily take a nonzero nonnegative function $f \geq 0$ and consider the following Poisson equation

$$-\Delta u = f$$

Take the fundamental solution $\Phi(x) = \frac{1}{d(d-2)\omega_d |x|^{d-2}}$, notice how it is positive (on where it's defined) when $d \geq 3$. The convolution $u = \Phi * f$ solves the above Poisson equation and is therefore superharmonic ($-\Delta u = f \geq 0$). Now that u is a result from convoluting a positive function with a nonzero nonnegative function, it must be nonzero nonnegative.

(2) Apply Green's formula once the derivative is taken inside the integral:

$$
\begin{aligned}
\frac{d}{dr}\int_{\partial B_1(0)} u(rx)u(\frac{r}{x})dx &= \int_{\partial B_1(0)} u(rx)\nabla u(\tfrac{x}{r})\cdot\left(-\frac{1}{r^2}x\right) + u(\tfrac{x}{r})\nabla u(rx)\cdot x d\mathcal{H}^{d-1} \\
&= \int_{\partial B_1(0)} -\frac{u(rx)}{r^2}\nabla u(\tfrac{x}{r})\cdot n + u(\tfrac{x}{r})\nabla u(rx)\cdot n d\mathcal{H}^{d-1} \\
&= \iint_{\partial B_1(0)} -\frac{u(rx)}{r^2}\frac{\Delta u(\frac{x}{r})}{r} - \frac{1}{r}\nabla u(rx)\cdot\nabla u(\tfrac{x}{r})dx \\
&+ \iint_{\partial B_1(0)} u(\tfrac{x}{r})r\Delta u(rx) + \frac{1}{r}\nabla u(\tfrac{x}{r})\cdot\nabla u(rx)dx \\
&= 0
\end{aligned}
$$

(Note that the first and third term vanishes because u is harmonic, and the second and fourth term cancel each other.) Now suppose u vanishes on $B_\rho(0)$ for some $\rho > 0$. Pick $a \in (0,\rho)$, we can show "$\underline{\text{for any } c > a,\ \|u\|_{L^2(B_c(0))} = 0}$":
Consider the shifted harmonic function $v(x) = u(cx)$,

$$
\begin{aligned}
0 &= \int_{\partial B_1(0)} u(ax)u(\tfrac{c^2}{a}x)d\mathcal{H}^{d-1} && (u \text{ vanishes on } B_a(0)) \\
&= \int_{\partial B_1(0)} v(\tfrac{a}{c}x)v(\tfrac{c}{a}x)d\mathcal{H}^{d-1} \\
&= \int_{\partial B_1(0)} v(rx)v(\tfrac{x}{r})d\mathcal{H}^{d-1} && (\text{denote } r = \tfrac{a}{c}) \\
&= \int_{\partial B_1(0)} v(x)^2 d\mathcal{H}^{d-1} && (\text{this expression is constant in } r) \\
&= \int_{\partial B_1(0)} u(cx)^2 d\mathcal{H}^{d-1}
\end{aligned}
$$

We now conclude that u is zero for the entire domain since its zero on spheres of all radii. $\square$

Exercise 3

(1) Consider the following identity: for any $x, y \in \mathbb{R}^d$ satisfying $|x| = r < \rho = |y|$,

$$\frac{\rho - r}{(\rho + r)^{d-1}} \leq \frac{\rho^2 - r^2}{|x-y|^d} \leq \frac{\rho + r}{(\rho - r)^{d-1}} \tag{2}$$

Since $u \geq 0$ is a harmonic function, we can invoke Poisson formula

$$\begin{aligned}
u(x) &= \int_{\partial B_\rho(0)} \frac{\rho^2 - |x|^2}{\omega_d \rho |x - \xi|^d} u(\xi) d\mathcal{H}^{d-1} && \text{(Poisson formula)} \\
&= \int_{\partial B_1(0)} \frac{1 - |x|^2}{\omega_d |x - \xi|^d} u(\xi) d\mathcal{H}^{d-1} && (\rho = 1) \\
&\leq \int_{\partial B_1(0)} \frac{1 + 1/2}{\omega_d (1/2)^{d-1}} u(\xi) d\mathcal{H}^{d-1} && \text{(from (2), } |x| \text{ ranges from 0 to } \tfrac{1}{2}) \\
&= \frac{\frac{3}{2}\omega_d \cdot 1^{d-1}}{\omega_d (1/2)^{d-1}} u(0) && \text{(Mean Value Property)} \\
&= 3 \cdot 2^{d-2} u(0)
\end{aligned}$$

Let $C(d) = 3 \cdot 2^{d-2} \leq 2^d$; since x was taken arbitrarily in $B_{1/2}(0)$, we have $\sup_{B_{1/2}(0)} u \leq C(d) u(0)$ as desired. □

On a side note, use the other part of inequality (2), with similar approach we get

$$\frac{2^{d-2}}{3^{d-1}} u(0) \leq \inf_{B_{1/2}(0)} u$$

(2) Take $v = u - m \geq 0$ and $w = M - u \geq 0$ be the shifted nonnegative harmonic functions where

$$m = \inf_{B_1(0)} u \qquad M = \sup_{B_1(0)} u$$

(Assuming the above values are both finite!) Invoke the inequality in class, we have

$$\sup_{B_{\frac{1}{2}}(0)} u - m \leq C(d) \left(\inf_{B_{\frac{1}{2}}(0)} u - m \right) \tag{3}$$

$$M - \inf_{B_{\frac{1}{2}}(0)} u \leq C(d) \left(M - \sup_{B_{\frac{1}{2}}(0)} u \right) \tag{4}$$

Let us also denote

$$K = \sup_{B_{\frac{1}{2}}(0)} u \qquad k = \inf_{B_{\frac{1}{2}}(0)} u$$

then summing inequalities (3) and (4) shows

$$\begin{aligned}
K-m &\leq C(d)(k-m) \\
M-k &\leq C(d)(M-K) \\
K-k+M-k &\leq C(d)(M-m)-C(d)(K-k) \\
K-k &\leq \frac{C(d)-1}{C(d)+1}(M-m)
\end{aligned}$$

Here this constant $\mathcal{C}=\frac{C(d)-1}{C(d)+1} \leq \frac{C(d)}{C(d)+1}$ as described in the problem set.

(3) To prove local $C^{0,\alpha}$-boundedness, first fix a $K \subset\subset \Omega$; let $r = dist(K, \partial\Omega)$, take the finite (sub)covering $\{B_r(x_i)\}_{i=1}^N$ for K. We split the proof into 2 cases:
Case ①: For $x, y \in B_r(x_i)$ for some $i = 1, \cdots, N$. Find a $k \in \mathbb{N}$ such that

$$\frac{r}{2^{k+1}} \leq |x-y| \leq \frac{r}{2^k}$$

Use the oscillation decay estimate from part (2), we have

$$\frac{|u(x)-u(y)|}{|x-y|^\alpha} \leq \frac{\operatorname{osc}_{B_{\frac{r}{2^k}}(x_i)} u}{(r/2^{k+1})^\alpha} \leq \frac{\mathcal{C}^k \operatorname{osc}_{B_r(x_i)} u}{r^\alpha \mathcal{C}^{k+1}} \leq \frac{2}{r^\alpha \mathcal{C}} \sup_{B_1(0)} |u|$$

Case ②: For any $x, y \in B_1(0)$, connect them with $x = x_0, x_1, \cdots, x_K = y$ such that $\frac{r}{2} < |x_i - x_{i-1}| < r$ and each pair x_i, x_{i-1} are both in $B_r(x_j)$ for some j. Note that we need convexity of the domain to draw a straight line between x, y so we can make the following estimate on the size K of this sequence

$$\frac{Kr}{2} < |x-y| < Kr$$

Now the Hölder condition can be provided:

$$\begin{aligned}
|u(x)-u(y)| &\leq \sum_{i=2}^K |u(x_i)-u(x_{i-1})| \\
&\leq \frac{2}{r^\alpha \mathcal{C}} \sup_{B_1(0)} |u| \left(\sum_{i=2}^K |x_i - x_{i-1}|^\alpha \right) && \text{(from case ①)} \\
&< \frac{2}{r^\alpha \mathcal{C}} \sup_{B_1(0)} |u| \cdot K r^\alpha && (|x_i - x_{i-1}| < r) \\
&< \frac{4|x-y|}{\mathcal{C} r} \sup_{B_1(0)} |u| && \left(K < \tfrac{2|x-y|}{r}\right) \\
&< \frac{4 \cdot 2^{1-\alpha} |x-y|^\alpha}{\mathcal{C} r} \sup_{B_1(0)} |u| && (|x-y| < 2)
\end{aligned}$$

We conclude that
$$M(d) = \frac{4 \cdot 2^{1-\alpha}|x-y|^{\alpha}}{\mathcal{C}r}$$
which only depends on α, K, and d. $\square$

(4) Integral formula of harmonic functions $D^{\alpha}u$ gives
$$\partial_{x_i} D^{\alpha}u(x) = \frac{1}{\omega_d}\int_{\partial B_1(x)} (D^{\alpha}u)\nu_i d\mathcal{H}^{d-1}$$

Denote

$$M(r) = \max_{\partial B_r(0)} u = \max_{B_r(0)} u \qquad m(r) = \min_{\partial B_r(0)} u = \min_{B_r(0)} u$$

Apply Harnack's inequality on $u - m(4r)$, we have the following inequality

$$\begin{aligned} m(r) - m(4r) &\geq \frac{1}{2^d}(M(r) - m(4r)) \\ m(r) &\geq \frac{1}{2^d}M(r) + \left(1 - \frac{1}{2^d}\right) m(4r) \\ M(r) &\leq 2^d m(r) + (2^d - 1)m(4r) \end{aligned}$$

Even if we assume u has polynomial growth, find a polynomial p so $m(4r) \leq p(4)m(r)$ for large r, I still don't see how the approximation should work. The only inequality we have is $u(x) \geq f(|x|)$, so $m(r) \geq f(r)$. The following is Harnack's inequality applied to $M(4r) - u$. As far as I can see, nothing gives any bound like $M(r) \leq C_1 f(r) + C_2$.

$$\begin{aligned} M(4r) - M(r) &\geq \frac{1}{2^d}(M(4r) - m(r)) \\ M(r) &\leq \frac{1}{2^d}m(r) + \left(1 - \frac{1}{2^d}\right) M(4r) \\ M(r) &\leq \frac{1}{2^d}f(r) + \left(1 - \frac{1}{2^d}\right) M(4r) \end{aligned}$$

References

[1] How to prove liouvilletheorem for subharmonic functions. https://math.stackexchange.com/questions/1743841/how-to-prove-liouvilles-theorem-for-subharmonic-functions.

[2] Maximum principle for subharmonic functions. https://math.stackexchange.com/questions/1489107/r
principle-for-subharmonic-functions.

[3] Peter D. Lax. *Functional Analysis.* 2002.