Exercise 2.7 [Boyd & Vandenberghe, 2004]

Voronoi description of halfspace. Let a and b be distinct points in $\mathbb{R}^n$. Shpow that the set of all points that are closer (in Euclidean norm) to a than b, *i.e.*$\{x \mid \|x-a\|_2 \le \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^t x \le d$. Draw a picture.

Ans: Denote $x = [x_1, \cdots, x_n], a = [a_1, \cdots, a_n], b = [b_1, \cdots, b_n]$, the quiterium $\|x-a\|_2 \le \|x-b\|_2$ is equivalent to

$$(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2 \le (x_1 - b_1)^2 + \cdots + (x_n - b_n)^2$$

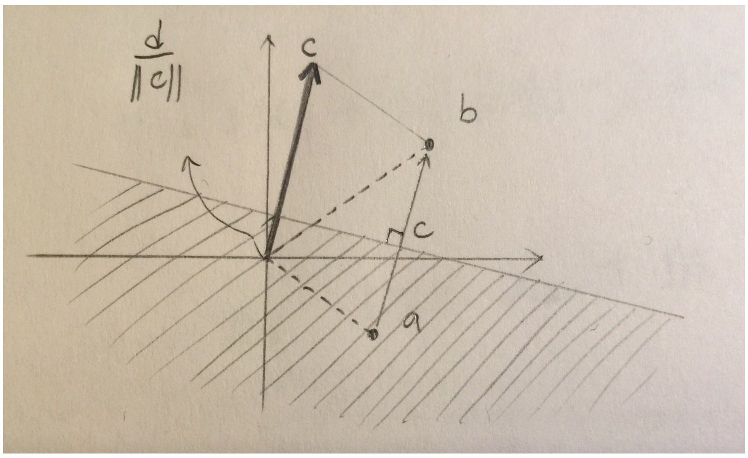
Organize the inequality for a bit (subtract $x_1^2 + \cdots + x_n^2$ for both sides and move terms around) and we get

$$(-2a_1 + 2b_1)x_1 + \cdots + (-2a_n + 2b_n)x_n \le (b_1^2 - a_1^2) + \cdots + (b_n^2 - a_n^2)$$

Let $c = -a + b \in \mathbb{R}^n$, $d = \frac{1}{2}(\|b\|_2^2 - \|a\|_2^2) \in \mathbb{R}$ and divide the inequality by 2, we get

$$c^T x \le d$$

as in the desired form. $\square$



Exercise 2.12 [Boyd & Vandenberghe, 2004]

Which of the following sets are convex?
(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \forall y \in S\}$$

where $S \subseteq \mathbb{R}^n$.

Ans: Observe this set (hereby denoted as D) is the intersection of halfspaces (as proved in **2.7**)

$$D = \bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$

Since intersection of convex sets is also convex, YES, the set D is convex. □

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathrm{dist}(x, S) \leq \mathrm{dist}(x, T)\}$$

where $S, T \subseteq \mathbb{R}^n$, and

$$\mathrm{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$$

Ans: NO. Consider counterexample with $S = \{z \mid \|z\|_2 \geq 2\}$ and $T = \{0\}$; the set described in the problem (hereby denoted as E) is the complement set of unit disk

$$E = \{x \mid \|x\|_2 \geq 1\}$$

This is true since for $x \in \mathbb{R}^n$, $\mathrm{dist}(x, \{0\}) = \|x\|_2$ and

$$\mathrm{dist}(x, S) = \begin{cases} 2 - \|x\|_2, \|x\|_2 \leq 2 \\ 0, \|x\|_2 > 2 \end{cases}$$

And certainly the complement set of unit disk is not convex. □

(f) [HUL93, volume 1, page 93] The set $\{x \mid x + S_2 \subseteq S_1\}$ where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

Ans: YES. Take $x_1, x_2 \in F := \{x \mid x + S_2 \subseteq S_1\}$ and arbitrary $y \in S_2$; we have $x_1 + y, x_2 + y \in S_1$ since $x_1, x_2 \in F$. Now for $\theta \in [0, 1]$, since S_1 is convex,

$$\theta(x_1 + y) + (1 - \theta)(x_2 + y) = \theta x_1 + (1 - \theta)x_2 + y \in S_1$$

Since y was chosen arbitrarily, $\theta x_1 + (1 - \theta)x_2 + S_2 \subseteq S_1$; that is, $\theta x_1 + (1 - \theta)x_2 \in F$. □

Exercise 2.16 [Boyd & Vandenberghe, 2004]

Show that if S_1 and S_2 are convex sets in $\mathbb{R}^m \times \mathbb{R}^n$[1], then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n, (x, y_1) \in S_1, (x, y_2) \in S_2\}.$$

Ans: Take $(x, y_1 + y_2), (z, y_3 + y_4) \in S$ (where $(x, y_1), (z, y_3) \in S_1, (x, y_2); (z, y_4) \in S_2$) and $\theta \in [0, 1]$; our concern is whether

$$v := \theta(x, y_1 + y_2) + (1 - \theta)(z, y_3 + y_4) = (\theta x + (1 - \theta)z, \theta y_1 + (1 - \theta)y_3 + \theta y_2 + (1 - \theta)y_4)$$

is in S. Now that both S_1 and S_2 are convex, we are safe to claim

$$\theta(x, y_1) + (1 - \theta)(z, y_3) = (\theta x + (1 - \theta)z, \theta y_1 + (1 - \theta)y_3) \in S_1$$

$$\theta(x, y_2) + (1 - \theta)(z, y_4) = (\theta x + (1 - \theta)z, \theta y_2 + (1 - \theta)y_4) \in S_2$$

Note the vector v is exactly the direct sum of these 2 vectors, henceforth it's in S. □

[1] I believe this was a typo in the textbook.

Additional Problem 1

(a) The objective can be written as

$$\sum_{i=1}^{m}(u_c^2+v_c^2-R^2-2u_iu_c-2v_iv_c+u_i^2+v_i^2)^2=\sum_{i=1}^{m}(-2u_ix_1-2v_ix_2+x_3+u_i^2+v_i^2)^2$$

Let $b=[-u_1^2-v_1^2,\cdots,-u_m^2-v_m^2]^T$, $a_i=[-2u_i,-2v_i,1]$ and $A=\begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ (or $A=[-2u,-2v,\mathbf{1}]$ where $\mathbf{1}$ denotes the constant column vector with entries all 1), then we get the objective is the same as

$$\|Ax-b\|_2^2$$

(b) Aiming to solve $A^TAx=A^Tb$, first observe that

$$A^TA=\begin{bmatrix} 4\|u\|^2 & 4u^Tv & -2\sum u_i \\ 4u^Tv & 4\|v\|^2 & -2\sum v_i \\ -2\sum u_i & -2\sum v_i & m \end{bmatrix}, A^Tb=\begin{bmatrix} 2\sum u_i(u_i^2+v_i^2) \\ 2\sum v_i(u_i^2+v_i^2) \\ -\|u\|^2-\|v\|^2 \end{bmatrix}$$

Now take the first 2 equations from $A^TAx=A^Tb$ and sum them up, we get

$$4\sum(u_i+v_i)u_i\hat{x}_1+4\sum(u_i+v_i)v_i\hat{x}_2-2\sum(u_i+v_i)\hat{x}_3=2\sum(u_i+v_i)(u_i^2+v_i^2)$$

Lump the terms to one side and complete the squares when needed,

$$\begin{aligned}0&=\sum(u_i+v_i)(-2u_i\hat{x}_1-2v_i\hat{x}_2+\hat{x}_3+u_i^2+v_i^2)\\&=\sum(u_i+v_i)((\hat{x}_1-u_i)^2+(\hat{x}_2-v_i)^2-(\hat{x}_1^2+\hat{x}_2^2-\hat{x}_3))\end{aligned}$$

Since $(\hat{x}_1-u_i)^2+(\hat{x}_2-v_i)^2\geq 0$ is always true, in order to make the sum zero we need

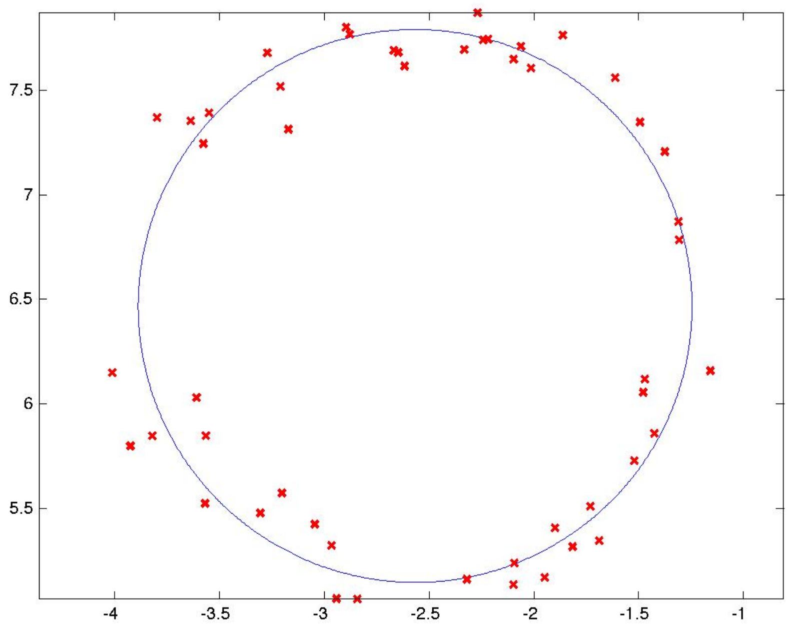
$$\hat{x}_1^2+\hat{x}_2^2-\hat{x}_3\geq 0$$

.

(c) Attached are the MATLAB codes and the plot.

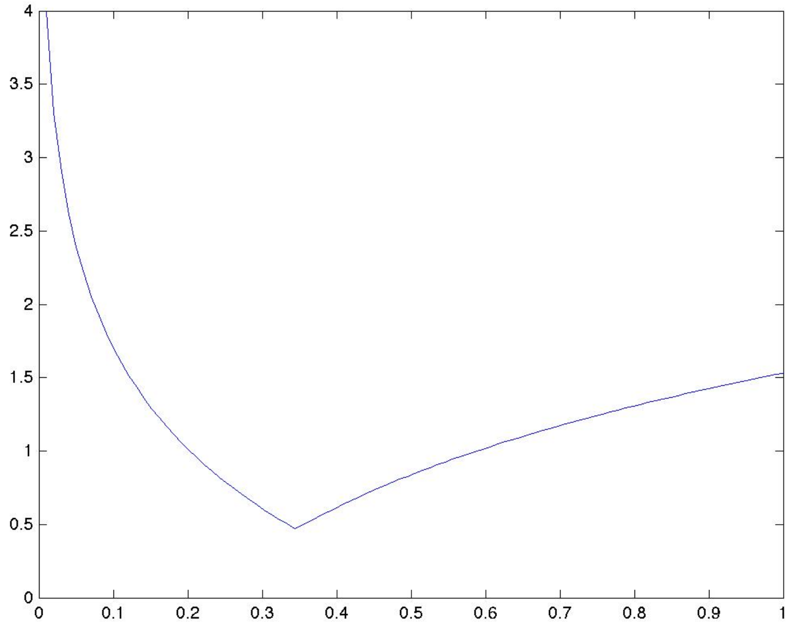
```
circlefit;
A = [-2*u, -2*v, ones(50,1)];
b = -u.^2-v.^2;
x = A\b;
R = sqrt(x(1)^2+x(2)^2-x(3));
theta = linspace(0,2*pi, 1000);
circle_x = x(1) + R*cos(theta);
circle_y = x(2) + R*sin(theta);
```

```
handle = plot(circle_x, circle_y, 'b-');
hold on
handle = plot(u,v,'rx', 'LineWidth', 2);
axis equal
saveas(handle,'EE236P1','jpg');
```

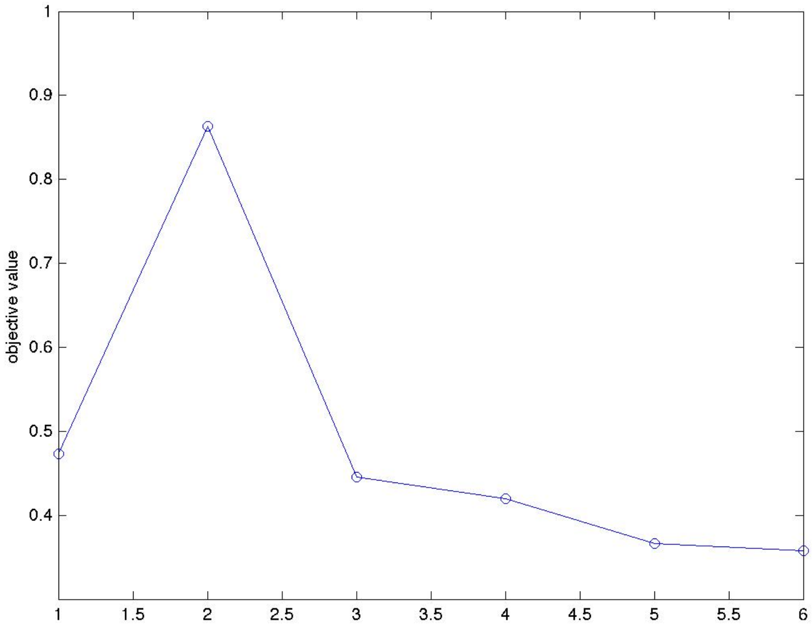


Additional Problem 1

(a) Minimizer seems to fall at 0.34 with minumum 0.5. (However, looking closer to the data, it seems $\gamma_{min} \approx 0.3434$, $f_0(0.3434) = 0.4732$)



The followings are the plot of all the results and the MATLAB codes. We can see least-square with saturation appears to be the worse method.




```
hold off
clear all;
clc;
illumdata;
[n,m] = size(A);
sample_size = 100;

%(a) Equal lamp powers
f0 = @(p) max(abs(log(A*p)));
fa = @(gamma) f0(gamma*ones(m,1));
gamma = linspace(0,1,sample_size);
fa_gamma = zeros(1,sample_size);
for i=1:sample_size
fa_gamma(i) = fa(gamma(i));
end
handle_a = plot(gamma, fa_gamma,'b');
saveas(handle_a, 'hw1_p2_a','jpg');

[obja,index] = min(fa_gamma);
pa = gamma(index)*ones(m,1);

%(b) Least-squares with saturation
pb = A\ones(n,1);
for i=1:m
```

```
    if pb(i) > 1
        pb(i) = 1;
    end
    if pb(i) < 0
        pb(i) = 0;
    end
end
objb = f0(pb);

%(c) Regularized least-squares
for rho=linspace(0,1,21)
    pc = (A'*A + rho*eye(m))\(A'*ones(n,1)+0.5*rho*ones(m,1));
    if pc > zeros(m,1) & pc < ones(m,1)
        break
    end
end
objc = f0(pc);

%(d) Chebyshev approximation
cvx_begin
    variable pd(m)
    minimize( norm(A*pd-1, inf) )
    subject to
        pd <= ones(m,1)
        pd >= zeros(m,1)
cvx_end
objd = f0(pd);

%(e) Piecewise-linear approximation
h_pwl = @(u) max([u;2/0.5-1/0.5/0.5*u;2/0.8-1/0.8/0.8*u;2-u]);
cvx_begin
    variable pe(m)
    minimize( max(h_pwl((A*pe)')) )
    subject to
        pe <= ones(m,1)
        pe >= zeros(m,1)
cvx_end
obje = f0(pe);

%(f) Exact solution
cvx_begin
    variable pf(m)
    minimize( max(max(A*pf, inv_pos(A*pf))) )
```

```
    subject to
        pf <= ones(m,1)
        pf >= zeros(m,1)
cvx_end
objf = f0(pf);

% plot
obj = [obja;objb;objc;objd;obje;objf];
handle = plot(1:6,obj, 'o-');
saveas(handle, 'hw1_p2', 'jpg');
```