[T1]

"$\langle r_{k-1}, p_k\rangle = \langle b, p_k\rangle$":
Since $\forall j < k$, $\langle p_k, p_j\rangle_A = 0$, p_k is A-orthogonal to the entire subspace $\text{span}\{p_0, p_1, \cdots, p_{k-1}\}$. Now from the fact that $r_{k-1} = b - Ax_{k-1}$ where $x_{k-1} \in \text{span}\{p_0, p_1, \cdots, p_{k-1}\}$,

$$\langle r_{k-1}, p_k\rangle = \langle b, p_k\rangle - \langle x_k, p_k\rangle_A = \langle b, p_k\rangle$$

This shall prove the first equality about α_k. The second follows from the fact $\langle r_k, p_k\rangle = \|r\|^2$ which follows from the fact that $\langle r_k, p_k\rangle = 0$.

"$-\frac{<\vec{r}_{k-1}, A\vec{p}_{k-1}>}{<\vec{p}_{k-1}, A\vec{p}_{k-1}>} = \frac{<\vec{r}_{k-1}, \vec{r}_{k-1}>}{<\vec{r}_{k-2}, \vec{r}_{k-2}>}$":
Since $r_{k-1} = r_{k-2} - \alpha_{k-1}Ap_{k-1}$, the numerator

$$\langle r_{k-1}, Ap_{k-1}\rangle = \left\langle r_{k-1}, \frac{r_{k-2} - r_{k-1}}{\alpha_{k-1}}\right\rangle = \frac{1}{\alpha_{k-1}}(\langle r_{k-1}, r_{k-2}\rangle - \|r_{k-1}\|^2)$$

The last term equals to $\frac{-1}{\alpha_{k-1}}\|r_{k-1}\|^2$ since $\{r_k\}$ had been proven to be an orthogonal set. Combined with the formula from part (a), $\alpha_{k-1} = \frac{\|r_{k-2}\|^2}{\langle p_{k-1}, Ap_{k-1}\rangle}$, the whole expression becomes

$$\begin{aligned} -\frac{\langle r_{k-1}, Ap_{k-1}\rangle}{\langle p_{k-1}, Ap_{k-1}\rangle} &= \frac{\|r_{k-1}\|^2}{\alpha_{k-1}\langle p_{k-1}, Ap_{k-1}\rangle} \\ &= \frac{\|r_{k-1}\|^2}{\|r_{k-2}\|^2} \end{aligned}$$

[C1/2]

Taking (temporarily) the conclusion from Assignment 4, $\omega = 0.8$ and 2 relaxation operations were applied at each level, we produce the following table that shows the time (in second) required to preform a Pre-Conditioned Conjugate Gradient (PCCG) with maximal Multi-Grid level (also the index of the finest grid) ranging from 5 to 7 and corresponding minimal Multi-Grid level ranging from 0 to 4, 5 or 6, respectively. The time required by un-pre-conditioned Conjugate Gradient was also attached for comparison. Note that since the tolerance was set to be the same as `1e-06`, the precision of the following test results was the same.

	CG	0	1	2	3	4	5	6
5	0.0335198	0.0174213	0.0188152	0.0222806	0.0323056	0.0495342		
6	0.277031	0.0719253	0.0808236	0.0972585	0.136014	0.227178	0.371307	
7	2.26964	0.301327	0.343037	0.409965	0.563305	0.92027	1.69528	2.93866

Observe that despite that we have to perform more computation within one V-cycle with minimal level 0 at each three maximal level, the time required to perform PCCG is still smaller than those with a larger minimal level. Also notice that all these numbers are significantly smaller than that for un-pre-conditioned CG. This has explained that the method gains an efficiency boost when the pre-conditioner is applied at the finest level. Here's a similarly labeled table with the content replaced by iteration count:

	CG	0	1	2	3	4	5	6
5	102	5	6	8	13	24		
6	202	5	6	8	12	23	46	
7	396	5	6	8	12	22	44	90

However, it's worth noting that the computational resources acquired for one iteration at max-min multi-grid level (0, 5) is different from that of one iteration without pre-conditioner or other min multi-grid level.

[C3/4]

Here's some precious test results from running the test. (takes 8 minutes with $N = 1$ and when $N = 5$......) The table shows the optimal parameter sets and the time using them.

The following is test result with $N = 1$, slight error could be expected.

max level	min level	ω	relaxation	time
5	0	0.7	2	0.01781
6	0	1	2	0.075065
7	0	1	2	0.315877

The following is the test results (from running overnight) with $N = 5$

max level	min level	ω	relaxation	time
5	1	1	2	0.0181482
6	0	0.8	2	0.0817066
7	0	0.9	2	0.336129

I also collected the data when running with $N = 5$. It'll be attached as `MultiParameterTest_data.csv` since the LaTeXpackage `tabular` doesn't support cross pages table. From the above result we can conservatively conclude that agreeing with the conclusion from Assignment 4, $\omega = 0.8$, 2 relaxation operations and min Multi-Grid level 0 will be the optimal parameter on average.