[T1]

(a) Use the knowledge that $\langle u_0, u\rangle = \cos\theta$,

$$\begin{aligned}|\epsilon|^2 &= \|u_0\|^2 + \langle u_0, u\rangle^2 \|u\|^2 - 2\langle u_0, u\rangle^2 \\ &= 1 - \cos^2\theta = \sin^2\theta\end{aligned}$$

(b) First note that $\langle \epsilon v, u\rangle = \langle u_0 - \langle u_0, u\rangle u, u\rangle = \langle u_0, u\rangle - \langle u_0, u\rangle \|u\|^2 = 0$. Now since both u_0 and u are normalized, we have $\lambda = \langle u, Au\rangle$ and

$$\begin{aligned}\lambda_0 &= \langle u_0, Au_0\rangle \\ &= \langle \epsilon v + \langle u_0, u\rangle u, \epsilon Av + \langle u_0, u\rangle Au\rangle \\ &= \epsilon^2 \langle v, Av\rangle + \langle u_0, u\rangle \langle u, \epsilon Av\rangle + \langle u_0, u\rangle \langle \epsilon v, Au\rangle + \langle u_0, u\rangle^2 \langle u, Au\rangle \\ &= \mathcal{O}(\epsilon^2) + 2\langle u_0, u\rangle \langle \epsilon v, \lambda u\rangle + \cos^2\theta\lambda \\ &= \mathcal{O}(\epsilon^2) + 0 + (1-\epsilon^2)\lambda = \lambda + \mathcal{O}(\epsilon^2)\end{aligned}$$

(c) Even if A is not symmetric, $\langle u_0, u\rangle^2 \lambda = \lambda + \mathcal{O}(\epsilon^2)$ and $\langle \epsilon v, u\rangle = 0$ still holds. The only problem caused by this asymmetry is $\langle u, \epsilon Av\rangle \neq \langle Au, \epsilon v\rangle$ and this will cause a $\mathcal{O}(\epsilon)$ error since we lose control to $\langle u, Av\rangle$ in this case.

[T2]

(a) Observe for each Eigenpair (λ_i, v_i), $(Av_i = \lambda_i v_i)$

$$v_i = \lambda_i A^{-1} v_i$$

That is, $A^{-1}v_i = \frac{1}{\lambda_i} v_i$ and $\left(\frac{1}{\lambda_i}, v_i\right)$ is an Eigenpair of A^{-1}.

(b) Similarly, consider each Eigenpair (λ_i, v_i), write $p(x) = \sum_{j=0}^{r} \alpha_j x^j$,

$$p(A)v_i = \sum_{j=0}^{r} \alpha_j A^j v_i = \sum_{j=0}^{r} \alpha_j \lambda_i^j v_i = p(\lambda_i)v_i$$

That is, $(p(\lambda_i), v_i)$ is an Eigenpair of $p(A)$.

(c) Similarly, since $p(A)v_i = p(\lambda_i)v_i$, if $q(\lambda_i) \neq 0$,

$$\frac{p(\lambda_i)}{q(\lambda_i)} q(A)v_i = \frac{p(\lambda_i)}{q(\lambda_i)} q(\lambda_i)v_i = p(\lambda_i)v_i$$

Therefore $q(A)^{-1}p(A)v_i = q(A)^{-1}p(\lambda_i)v_i = \frac{p(\lambda_i)}{q(\lambda_i)} v_i$ and $(\frac{p(\lambda_i)}{q(\lambda_i)}, v_i)$ is an Eigenpair of $q(A)^{-1}p(A)$. In the other case that $q(\lambda_i) = 0$, we find that

$$q(A)v_i = q(\lambda_i)v_i = 0$$

[C1]