Additional Exercise 3.21 (a),(b) [Boyd & Vandenberghe, 2017]

Ans: The observation from Problem 4.26 [Boyd & Vandenberghe, 2004] is for $x \in \mathbb{R}^n, y, z \in \mathbb{R}$,

$$x^T x \leq yz, y \geq 0, z \geq 0 \Leftrightarrow \left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2 \leq y + z, y \geq 0, z \geq 0$$

This is true since

$$\begin{aligned} \left\| \begin{bmatrix} 2x \\ y - z \end{bmatrix} \right\|_2^2 - (y+z)^2 &= 4\|x\|_2^2 + y^2 - 2yz + z^2 - (y^2 + 2yz + z^2) \\ &= 4(x^T x - yz) \end{aligned}$$

Back to Additional Exercise 3.21, we know now the following two constraints are equivalent.

$$y \leq \sqrt{z_1 z_2}, y, z_1, z_2 \geq 0 \Leftrightarrow \left\| \begin{bmatrix} 2y \\ z_1 - z_2 \end{bmatrix} \right\|_2 \leq z_1 + z_2, y, z_1, z_2 \geq 0$$

Claim: "For $n = 2^k, k \in \mathbb{N}$, the constraint $y \leq (z_1 z_2 \cdots z_n)^{1/n}, z_1, z_2, \cdots, z_n \geq 0$ can expressed as SOC constraint. " The base case $k = 1$ is done as above. Now assuming the statement is true for $n = 2^k$, in order to express the following in SOC constraint,

$$y \leq (z_1 z_2 \cdots z_n s_1 s_2 \cdots s_n)^{1/2n}, z_1, z_2, \cdots, z_n, s_1, s_2, \cdots, s_n \geq 0$$

consider $t_i = \sqrt{z_i s_i}, i = 1, 2, \cdots, n$; this can be done by setting constraints

$$t_i \geq 0, t_i^2 = \frac{1}{2}(z_i + s_i)^2 - \frac{1}{2}(z_i^2 + s_i^2)$$

Additional Problem 1

Ans:

Additional Problem 2

Ans:

Additional Problem 3

Ans:

Additional Exercise 3.11 [Boyd & Vandenberghe, 2017]

Ans: