# Problem 1

(a) Since $\mathcal{A}$ is infinite, we can take an countably infinite family of distinct elements $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$. ("Distinct" meaning

$$m \neq n \Rightarrow A_m \neq A_n$$

Construct a monotinic family of sets $\{B_n\}_{n\in\mathbb{N}}$ by following

$$B_n = \bigcap_{k=n}^{\infty} A_k$$

## Problem 3

(a)

"$A = \bigcap_{m\in\mathbb{N}} \bigcup_{n=m}^{\infty} A_n$":

"$\subseteq$":

Fix an arbitrary $x \in A$ and an $m \in \mathbb{N}$. Since $x \in A_n$ for infinitely many $n \in \mathbb{N}$ and $m$ is just a finite number, we can always find $n_m$ such that $n_m \geq m$ and $x \in A_{n_m}$. That is,

$$x \in A_{n_m} \subseteq \bigcup_{n=m}^{\infty} A_n$$

Now since $x$ is in every set that's being taken intersection of,

$$x \in \bigcap_{m\in\mathbb{N}} \bigcup_{n=m}^{\infty} A_n$$

"$\supseteq$":

Fix an arbitrary $x \in \bigcap_{m\in\mathbb{N}} \bigcup_{n=m}^{\infty} A_n$. Suppose (for contradiction) that $x \in A_n$ for only fintely many (possibly zero) $n \in \mathbb{N}$. Since there're only finite number of these $n$'s, we can take their upper bound $N$, then

$$k > N \Rightarrow x \notin A_k$$

This leads to $x \notin \bigcup_{n=N}^{\infty} A_n$ and hence

$$x \notin \bigcap_{m\in\mathbb{N}} \bigcup_{n=m}^{\infty} A_n$$

which contradicts our assumption.

"$A \in \mathcal{A}$":

Following the above claim; first, since $\sigma$-algebra is closed under countable unions,

$$\bigcup_{n=m}^{\infty} A_n \in \mathcal{A} \text{ for } m \in \mathbb{N}$$

then since $\sigma$-algebra is closed under countable intersections,

$$\bigcap_{m\in\mathbb{N}} \bigcup_{n=m}^{\infty} A_n \in \mathcal{A}$$

$\square$

(b)

Denote $B_m = \bigcup_{n=m}^{\infty} A_n$. Since $A = \bigcap_{m\in\mathbb{N}} B_m$,

$$\mu(A) \leq \mu(B_m) \text{ for } m \in \mathbb{N}$$

Now

$$\begin{aligned}
\mu(A) &\leq \inf_{m\in\mathbb{N}} \mu(B_m) \\
&= \inf_{m\in\mathbb{N}} \mu\left(\bigcup_{n=m}^{\infty} A_n\right) \\
&\leq \inf_{m\in\mathbb{N}} \sum_{n=m}^{\infty} \mu(A_n) \\
&= \lim_{m\to\infty} \sum_{n=m}^{\infty} \mu(A_n) \\
&= 0
\end{aligned}$$