[T1]

"if": Assume $\rho(A) = |\lambda_{max}| < 1$. Since $\|N\|_F$ is a fixed number, we could find $\mu \geq 0$ large enough such that

$$\rho(A) + \frac{\|N\|_F}{1+\mu} < 1$$

with this we shall get

$$\|A^k\|_2 \leq (1+\mu)^{n-1} \left(\rho(A) + \frac{\|N\|_F}{1+\mu} \right)^k \to 0 \text{ as } k \to \infty$$

"and only if": Suppose $\rho(A) \geq 1$ for contradiction; take v be the corresponding eigenvector to the eigenvalue λ_{max}, then

$$\|A^k v\|_2 = |\lambda_{max}|^k \|v\|_2 \geq \|v\|_2 > 0$$

this shall contradict the condition $\|A^k\|_2 \to 0$ as $k \to \infty$.

[T2]

(a) Suppose $A = M - N$ is singular; take $x \in N(A)$, then

$$Ax = Mx - Nx = 0$$

and $Mx = Nx$. Assuming M is non-singular, apply the inverse on both side we see

$$x = M^{-1}Mx = M^{-1}Nx$$

namely x is an eigenvector of $M^{-1}N$ with eigenvalue 1, contradicts the condition $\rho(M^{-1}N) < 1$.

(b) No. Consider the absurd example: $A = I - I = O$ is the zero matrix, written as the difference of two identity matrices, $b = 0$ is the zero vector. In this case with any given initial guess x^0, the iteration gives

$$x^{k+1} = I^{-1}\left(Ix^k + 0\right) = x^k = \cdots = x^0$$

Nothing diverges.

[T3]

Suppose for contradiction that $\rho(M^{-1}N) \geq 1$; take take v be the corresponding eigenvector to the eigenvalue λ_{max} (which is the largest eigenvalue in magnitude). Set $b = 0$ and $x^0 = v$, then the iteration gives

$$\begin{aligned} x^1 &= M^{-1}Nx^0 = \lambda_{max} v \\ &\vdots \\ x^{k+1} &= M^{-1}Nx^k = \lambda_{max}^{k+1} v \end{aligned}$$

And unless $\lambda_{max} = 1$, this will be a never converging iteration given $|\lambda_{max}| \geq 1$. In the case of $\lambda_{max} = 1$, just consider $b = Mv$, then with the same initial guess $x^0 = v$, we have

$$\begin{aligned} x^1 &= M^{-1}Nx^0 + M^{-1}Mv = 2v \\ &\vdots \\ x^{k+1} &= M^{-1}Nx^k + v = (k+2)v \end{aligned}$$

and this shall diverge and contradicts the assumption.

[T4]

The iteration can be formulated as

$$x^{k+1} = I^{-1}(I - \alpha A)x^k + \alpha b$$

and the iterative matrix is $I - \alpha A$. According to [T3] (with b replaced by αb which is still a constant vector), if $\rho(M^{-1}N) \geq 1$ then there exist scenarios such that the iteration diverges. Now suppose A has both positive and negative eigenvalues, denoted by λ^+, λ^-, then

$$1 - \alpha\lambda^+, 1 - \alpha\lambda^- \in \sigma(I - \alpha A)$$

for any given $\alpha \neq 0$, one of which is greater than 1; therefore, $\rho(M^{-1}N) > 1$.

[C1/2]

The results from running my program (also attached) are

```
XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
Gauss-Jacobi Iterations  = 743
Residual norm (L2)       = 0.000994929
Residual norm (Inf)      = 0.00198206

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
SOR Iterations           = 380
Residual norm (L2)       = 0.000995524
Residual norm (Inf)      = 0.00256255

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 24
Y-Wavenumber  : 24
Omega         : 1
Gauss-Jacobi Iterations  = 6
Residual norm (L2)       = 0.000803458
Residual norm (Inf)      = 0.00952229

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 24
Y-Wavenumber  : 24
Omega         : 1
SOR Iterations           = 8
Residual norm (L2)       = 0.000666814
Residual norm (Inf)      = 0.00286167
```

[C3]

Some other results from running my program are

► increasing mesh size ◄

```
XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 100
Y-Panel Count : 100
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
Gauss-Jacobi Iterations  = 2980
Residual norm (L2)       = 0.000998181
Residual norm (Inf)      = 0.00199636

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 100
Y-Panel Count : 100
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
SOR Iterations           = 1499
Residual norm (L2)       = 0.00099629
Residual norm (Inf)      = 0.00234244

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 200
Y-Panel Count : 200
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
Gauss-Jacobi Iterations  = 11926
Residual norm (L2)       = 0.000999976
Residual norm (Inf)      = 0.00199995

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 200
Y-Panel Count : 200
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
SOR Iterations           = 5972
Residual norm (L2)       = 0.000999479
Residual norm (Inf)      = 0.00218814
```

▶ varying relaxation parameter ◀

```
XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 0.25
Gauss-Jacobi Iterations  = 2982
Residual norm (L2)       = 0.000998567
Residual norm (Inf)      = 0.00198929

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 0.25
SOR Iterations           = 2613
Residual norm (L2)       = 0.000999976
Residual norm (Inf)      = 0.00210963

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 0.5
Gauss-Jacobi Iterations  = 1489
Residual norm (L2)       = 0.000998673
Residual norm (Inf)      = 0.0019895

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 0.5
SOR Iterations           = 1122
Residual norm (L2)       = 0.000998006
Residual norm (Inf)      = 0.00224027

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
```

```
Omega          : 0.75
Gauss-Jacobi Iterations  = 992
Residual norm (L2)       = 0.000994834
Residual norm (Inf)      = 0.00198185

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 0.75
SOR Iterations           = 626
Residual norm (L2)       = 0.000998346
Residual norm (Inf)      = 0.00239897

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
Gauss-Jacobi Iterations  = 743
Residual norm (L2)       = 0.000994929
Residual norm (Inf)      = 0.00198206

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1
SOR Iterations           = 380
Residual norm (L2)       = 0.000995524
Residual norm (Inf)      = 0.00256255

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1.25
Gauss-Jacobi Iterations  = 1816
Residual norm (L2)       = nan
Residual norm (Inf)      = inf

XXXX  SOR  2D Operator Test Output XXXX
```

```
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1.25
SOR Iterations           = 235
Residual norm (L2)       = 0.00099727
Residual norm (Inf)      = 0.00267221

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1.5
Gauss-Jacobi Iterations  = 1055
Residual norm (L2)       = nan
Residual norm (Inf)      = inf

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1.5
SOR Iterations           = 144
Residual norm (L2)       = 0.000973898
Residual norm (Inf)      = 0.00237577

XXXX  Gauss-Jacobi 2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1.75
Gauss-Jacobi Iterations  = 796
Residual norm (L2)       = nan
Residual norm (Inf)      = inf

XXXX  SOR  2D Operator Test Output XXXX
X-Panel Count : 50
Y-Panel Count : 50
X-Wavenumber  : 1
Y-Wavenumber  : 1
Omega         : 1.75
SOR Iterations           = 80
```

```
Residual norm (L2)        = 0.000958905
Residual norm (Inf)       = 0.002221
```

We can conclude that increasing mesh size slows down the convergence drastically. On the other hand, Gauss-Jacobi seems to work the best at $\omega = 1$ while SOR can work better with over-relaxation with $\omega = 1.75$.