Additional Exercise 12.6 [Boyd & Vandenberghe, 2017]

Ans: (a) Assuming in the discretization θ^{tar} remains to be a node. Since $M \geq \max\{|G(\theta)| \mid |\theta_k - \theta^{tar}| \geq \Delta\}$ is equivalent to $M^2 \geq \max\{|G(\theta)|^2 \mid |\theta_k - \theta^{tar}| \geq \Delta\}$, we can write the SOCP

$$\begin{array}{rl} \text{minimize} & t \\ \text{subjected to} & G(\theta^{tar}, \omega) = 1 \\ & |G(\theta_k, \omega)|^2 \leq t \text{ for } k \text{ such that } |\theta_k - \theta^{tar}| \geq \Delta \end{array}$$

(b) The code associated with this problem is attached here.

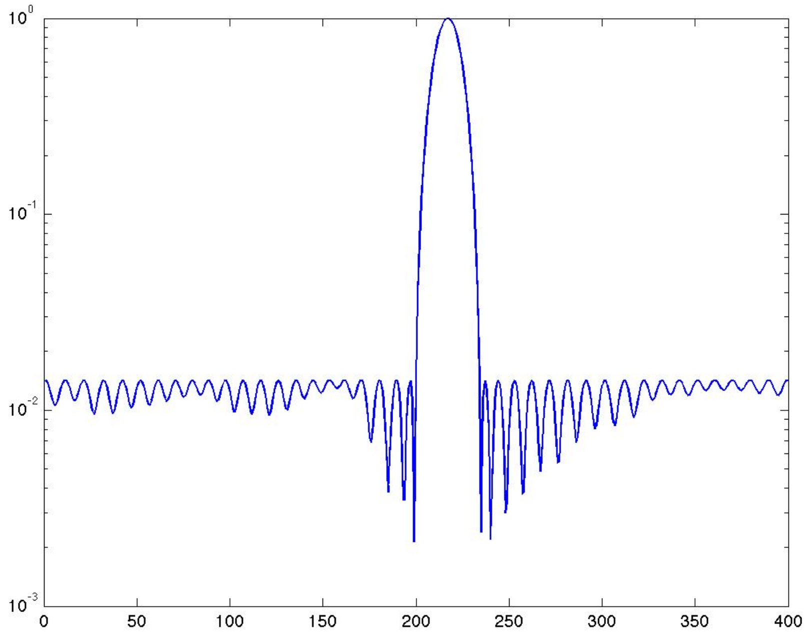
```
rand('state',0);
n = 40;
x = 30 * rand(1,n);
y = 30 * rand(1,n);

N = 400;
theta = linspace(-pi,pi,N)';
theta_tar = pi/12;
[~, tar_k] = min(abs(theta-theta_tar));
theta_tar_approx = theta(tar_k);
Delta = pi/12;
outside_index = (abs(theta-theta_tar)>= Delta);
G = @(omega) exp(1i*(cos(theta)*x+sin(theta)*y))*omega;
G_tar = @(omega) exp(1i*(cos(theta_tar_approx)*x+sin(theta_tar_approx)*y))*omega;

cvx_begin
    variable omega(n) complex
    variable t
    minimize t;
    subject to
        diag(outside_index)*abs(G(omega)) <= t*ones(N,1)
        G_tar(omega) == 1
cvx_end

h=semilogy(abs(G(omega)),'b-','LineWidth',1.2);
saveas(h, 'hw6P126','jpg');
```

The graph of $|G(\theta)|$ with logarithm scaling is attached here.


10^0
10^{-1}
10^{-2}
10^{-3}
0
50
100
150
200
250
300
350
400

Additional Exercise 4.4 [Boyd & Vandenberghe, 2017]

Ans: The KKT condition to the problem is ① $x^T x - t = 0$, ② (No inequality constraint. Skipped.), ③ (No inequality constraint. Skipped.), ④ the vanishing gradient (w.r.t. x and t) of Lagrangian,

$$\sum_{k=1}^{m} -4\left(t - 2y_k^T x + \|y_k\|_2^2 - d_k^2\right) y_k + 2\nu x = 0$$

$$\sum_{k=1}^{m} 2\left(t - 2y_k^T x + \|y_k\|_2^2 - d_k^2\right) - \nu = 0$$

Rearrange the above two equations we can solve for x and t once ν is given. The Lagrangian dual function is

$$\begin{aligned} g(\nu) &= \inf_{x,t}\left(\sum_{k=1}^{m}\left(t - 2y_k^T x + \|y_k\|_2^2 - d_k^2\right)^2 + \nu(x^T x - t)\right) \\ &= \inf_{x,t}\left(\sum_{k=1}^{m}\left(\|x - y_k\|_2^2 - d_k^2 - x^T x + t\right)^2 + \nu(x^T x - t)\right) \end{aligned}$$

Exercise 4.43 (b, c) [Boyd & Vandenberghe, 2004]

Ans: (b) Note that the smallest eigenvalue $\lambda_m(x) = \lambda_{min}(A(x))$ also has a similar relation

$$\lambda_{min}(A) \geq s \Longleftrightarrow A \succeq sI$$

Therefore the problem can be formulated as

$$\begin{array}{rl} \text{minimize} & t - s \\ \text{subjected to} & A(x) \preceq tI \\ & sI \preceq A(x) \end{array}$$

(c) We need to do the problem

$$\begin{array}{rl} \text{minimize} & \lambda/\gamma \\ \text{subjected to} & A(x) \preceq \lambda I \\ & \gamma I \preceq A(x) \\ & 0 \prec \gamma I \end{array}$$

Following the hint to change the variable, $y = \frac{x}{\gamma}, t = \frac{\lambda}{\gamma}, s = \frac{1}{\gamma}$, then $A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n = \gamma(sA_0 + y_1 A_1 + \cdots + y_n A_n)$, and the problem becomes

$$\begin{array}{rl} \text{minimize} & t \\ \text{subjected to} & sA_0 + y_1 A_1 + \cdots + y_n A_n \preceq tI \\ & I \preceq sA_0 + y_1 A_1 + \cdots + y_n A_n \end{array}$$

$\square$

Exercise 5.19 [Boyd & Vandenberghe, 2004]

Ans: (a) The optimal value is

$$\max\left\{\sum_{i=1}^{n} y_i x_i \mid y_i \in [0,1], \sum_{i=1}^{n} y_i = r\right\}$$

Since this is linear in x, the optimal value can only be lumping the weights (y_i's) on the largest r components of x.
(b) The Lagrangian dual function is (note the original problem is maximizing $x^T y$, so I translated it to minimizing $-x^T y$.)

$$\begin{aligned} g(\lambda_1, \lambda_2, \nu; x) &= \inf_y -x^T y - \lambda_1^T y + \lambda_2^T (y - \mathbf{1}) + \nu(\mathbf{1}^T y - r) \\ &= \inf_y y^T(-x - \lambda_1 + \lambda_2 + \nu\mathbf{1}) - \mathbf{1}^T \lambda_2 - \nu r \\ &= \begin{cases} -\mathbf{1}^T \lambda_2 - \nu r, & -x - \lambda_1 + \lambda_2 + \nu\mathbf{1} = 0 \\ \infty, & \text{else} \end{cases} \end{aligned}$$

The dual problem is then

$$\begin{aligned} \text{maximize} \quad & -\mathbf{1}^T \lambda_2 - \nu r \\ \text{subjected to} \quad & -x - \lambda_1 + \lambda_2 + \nu\mathbf{1} = 0 \\ & \lambda_1 \succeq 0 \\ & \lambda_2 \succeq 0 \end{aligned}$$

Or if we let $\lambda_2 = u, \nu = t$ and combine the two constraints of λ_1,

$$\begin{aligned} \text{maximize} \quad & -\mathbf{1}^T u - tr \\ \text{subjected to} \quad & -x + u + t\mathbf{1} \succeq 0 \\ & u \succeq 0 \end{aligned}$$

This is essentially the same as given in the problem. The three conditions $rt + \mathbf{1}^T u \leq \alpha, t\mathbf{1} + u \succeq x, u \succeq 0$ just mean that the optimal value of the dual LP (which is the same as the primal LP, which is $f(x)$) is less than or equal to α.

(c) Here the constraint $\sum_{i=1}^{\lfloor 0.1n \rfloor} x_{[i]} \leq 0.8$ yields $r = \lfloor 0.1n \rfloor, \alpha = 0.8$; that is, the problem can be formulated using the technique discussed above as

$$
\begin{array}{rl}
\text{minimize} & x^T \Sigma x \\
\text{subjected to} & \bar{p}^T x \geq r_{min} \\
& \mathbf{1}^T x = 1 \\
& x \succeq 0 \\
& \lfloor 0.1n \rfloor t + \mathbf{1}^T u \leq 0.8 \\
& t\mathbf{1} + u \succeq x \\
& u \succeq 0
\end{array}
$$

with variables $x \in \mathbb{R}^n, t \in \mathbb{R}, u \in \mathbb{R}^n$. $\square$

Exercise 5.21 (a, b, c) [Boyd & Vandenberghe, 2004]

Ans: (a) For sure $\frac{d^2}{dx^2}e^{-x} = (-1)^2 e^{-x} = e^{-x} > 0$, the cost function is convex. Let $f_1(x, y) = x^2/y$.

$$\nabla f_1(x, y) = \begin{bmatrix} 2x/y & -x^2/y^2 \end{bmatrix}$$
$$H_{f_1}(x, y) = \begin{bmatrix} \frac{2}{y} & \frac{-2x}{y^2} \\ \frac{-2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}$$

and $H_{f_1}(x, y) \succeq 0$ since $\frac{2}{y} > 0$ and $\det(H_{f_1}(x, y)) = \frac{4x^2}{y^4} - \frac{4x^2}{y^4} = 0 \geq 0$. The constraint function is essentially ruling $x = 0$ since the domain $\mathcal{D}$ restricts $y > 0$; with this observation, we conclude the optimal value to the problem is $e^{-0} = 1$.

(b) First observe the Lagrangian is always positive for nonnegative λ,

$$L(x, y, \lambda) = e^{-x} + \lambda \frac{x^2}{y} \geq 0$$

since $e^{-x} > 0$ and $x^2/y \geq 0$. Now consider $y = \lambda x^4$ as $x \to \infty$, then

$$L(x, y, \lambda, \nu) = e^{-x} + \lambda \frac{x^2}{y} = e^{-x} + \frac{1}{x^2} \to 0$$

That is, the Lagrangian dual function

$$g(\lambda) = \inf_{x,y} L(x, y, \lambda) = 0$$

is the zero constant function regardless of the input λ. Hence any $\lambda \geq 0$ can be optimal solution to the dual problem

$$\begin{aligned} \text{minimize} \quad & g(\lambda) \\ \text{subjected to} \quad & \lambda \geq 0 \end{aligned}$$

and the optimal value $d^* = 0$. Notice a optimal duality gap $p^* - d^* = 1 - 0 = 1$.

(c) Note that the Slater's condition does not hold for this problem since there exists no x and y such that $x^2/y < 0$ and $y > 0$. $\square$

Additional Exercise 4.30 [Boyd & Vandenberghe, 2017]

Ans: (a) The Lagrangian dual function is

$$\begin{aligned} g(\lambda) &= \inf_{x,y}\left(c^T x + \frac{1}{\mu}\sum_{i=1}^m \log(1+e^{\mu y_i}) + \lambda^T(Ax-b-y)\right) \\ &= -\lambda^T b + \inf_{x,y}\left(x^T(c+A^T\lambda) + \frac{1}{\mu}\sum_{i=1}^m \log(1+e^{\mu y_i}) - \lambda^T y\right) \end{aligned}$$

The y part is minimized when

$$\frac{1}{\mu}\cdot\frac{\mu e^{\mu y_i}}{1+e^{\mu y_i}} - \lambda_i = 0 \text{ for } i = 1,\cdots,m$$

or

$$y_i = \frac{1}{\mu}\log\left(\frac{\lambda_i}{1-\lambda_i}\right) \text{ for } i = 1,\cdots m$$

therefore,

$$\begin{aligned} g(\lambda) &= -\lambda^T b + \inf_{x,y}\left(x^T(c+A^T\lambda) + \frac{1}{\mu}\sum_{i=1}^m \log(1+e^{\mu y_i}) - \lambda^T y\right) \\ &= -\lambda^T b + \frac{1}{\mu}\sum_{i=1}^m \log\left(\frac{1}{1-\lambda}\right) - \sum_{i=1}^m \frac{\lambda_i}{\mu}\log\left(\frac{\lambda_i}{1-\lambda_i}\right) + \inf_x x^T(c+A^T\lambda) \\ &= -\lambda^T b - \frac{1}{\mu}\sum_{i=1}^m ((1-\lambda_i)\log(1-\lambda_i) + \lambda_i\log(\lambda)) + \inf_x x^T(c+A^T\lambda) \\ &= \begin{cases} -b^T\lambda + \frac{1}{\mu}\sum\limits_{i=1}^m \log\left(\lambda_i^{\lambda_i}(1-\lambda_i)^{1-\lambda_i}\right), & c + A^T\lambda = 0 \\ -\infty, & \text{else} \end{cases} \end{aligned}$$

with the domain restriction that $\lambda_i \in (0,1)$ for $i = 1,\cdots,m$.

(b) (Note I used λ as my dual variable where it's z in the problem.) The Lagrangian dual of the dual linear program is

$$\tilde{g}(z) = \begin{cases} -b^T z, & A^T z + c = 0 \\ -\infty, & \text{else} \end{cases}$$

Assuming strong duality (assume so because the linear program and its dual has it), the optimal value of problem (25) is $q^* = g(\lambda^*)$ for some λ^*. Now since z^* is an optimal solution to the dual linear program, we see it provides an upper bound

$$q^* \le g(z^*) = -b^T z^* + \frac{1}{\mu}\sum_{i=1}^m \log\left(z_i^{*z_i^*}(1-z_i^*)^{1-z_i^*}\right) \le p^* + \frac{1}{\mu}\sum_{i=1}^m \log 2$$

the last inequality follows from that $0 \preceq z^* \preceq \mathbf{1}$. The other side of the inequality required is not much but from the fact that $p^* \le g(\lambda)$ whenever $g(\lambda)$ is defined. □