

Exercise 3.18 [Boyd & Vandenberghe, 2004]

Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

- (a) $f(X) = \text{tr}(X^{-1})$ is convex on $\text{dom } f = \mathbf{S}_{++}^n$.
 (b) $f(X) = (\det X)^{1/n}$ is concave on $\text{dom } f = \mathbf{S}_{++}^n$.

Ans: Here only does part (a).

(a) It suffices to show $g(t) = \text{tr}((X + tZ)^{-1})$ is convex w.r.t. $t \in [t_0, t_1]$ where $X \in \mathbf{S}_{++}^n$, $Z \in \mathbf{S}^n$ and $[t_0, t_1]$ is an interval such that $X + tZ$ remains positive definite. Use this [property of positive definite matrix](#), write

$$X = S^{-1}S^{-T}, Z = S^{-1}LS^{-T}$$

then

$$\begin{aligned} g(t) &= \text{tr}((X + tZ)^{-1}) \\ &= \text{tr}((S^{-1}(I + tL)S^{-T})^{-1}) \\ &= \text{tr}(S(I + tL)^{-1}S^T) \\ &= \text{tr}(S^T S(I + tL)^{-1}) \\ g'(t) &= \text{tr}\left(S^T S \frac{d}{dt}(I + tL)^{-1}\right) \\ &= -\text{tr}(S^T S(I + tL)^{-1}L(I + tL)^{-1}) \end{aligned}$$

Here I made use of the identity

$$\begin{aligned} I &= A^{-1}(t)A(t) \\ O &= \left(\frac{d}{dt}A^{-1}(t)\right)A(t) + A^{-1}(t)A'(t) \\ \frac{d}{dt}A^{-1}(t) &= -A^{-1}(t)A'(t)A^{-1}(t) \end{aligned}$$

Now that $I + tL$ and L are both diagonal, they commute.

$$\begin{aligned} g'(t) &= -\text{tr}(S^T S(I + tL)^{-1}L(I + tL)^{-1}) \\ &= -\text{tr}(S^T SL(I + tL)^{-2}) \\ g''(t) &= \text{tr}(S^T SL((I + tL)^{-2}L(I + tL)^{-1} + (I + tL)^{-1}L(I + tL)^{-2})) \\ &= 2\text{tr}(SL^2(I + tL)^{-3}S^T) \end{aligned}$$

Since t is such that $X + tZ = S^{-1}(I + tL)S^{-T} \succ 0$, $I + tL$ must be positive diagonal matrix, and $(I + tL)^{-3} \succ 0$; alongside with $L^2 \succeq 0$, we see $L^2(I + tL)^{-3} \succeq 0$, therefore

$$SL^2(I + tL)^{-3}S^T \succeq 0$$

and $g''(t) \geq 0$ (i.e. g is convex). □

Exercise 3.19 [Boyd & Vandenberghe, 2004]

Nonnegative weighted sums and integrals.

(a) Show that $f(x) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of x , where $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the i th largest component of x . (You can use the fact that $f(x) = \sum_{i=1}^k x_{[i]}$ is convex on \mathbb{R}^n .)

(b) Let $T(x, \omega)$ denote the trigonometric polynomial

$$T(x, \omega) = x_1 + x_2 \cos \omega + x_3 \cos 2\omega + \dots + x_n \cos(n-1)\omega.$$

Show that the function

$$f(x) = - \int_0^{2x} \log T(x, \omega) d\omega$$

is convex on $\{x \in \mathbb{R}^n \mid T(x, \omega) > 0, 0 \leq \omega \leq 2\pi\}$.

Ans: Here only does part (a).

(a) Let $\beta_i = \alpha_i - \alpha_{i+1}$; then observe

$$\begin{aligned} f(x) &= \sum_{i=1}^r \alpha_i x_{[i]} \\ &= \alpha_r \sum_{i=1}^r x_{[i]} + (\alpha_{r-1} - \alpha_r) \sum_{i=1}^{r-1} x_{[i]} + \dots + (\alpha_1 - \alpha_2) \sum_{i=1}^1 x_{[i]} \\ &= \alpha_r \sum_{i=1}^r x_{[i]} + \beta_{r-1} \sum_{i=1}^{r-1} x_{[i]} + \dots + \beta_1 \sum_{i=1}^1 x_{[i]} \end{aligned}$$

is a linear combination of convex functions with nonnegative coefficients and therefore convex. \square

Exercise 3.22 [Boyd & Vandenberghe, 2004]

Composition rules. Show that the following functions are convex.

- (a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^T x + b_i}))$ on $\text{dom } f = \{x \mid \sum_{i=1}^m e^{a_i^T x + b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^n e^{y_i})$ is convex.
- (b) $f(x, u, v) = -\sqrt{uv - x^T x}$ on $\text{dom } f = \{(x, u, v) \mid uv > x^T x, u, v > 0\}$. Use the fact that $x^T x/u$ is convex in (x, u) for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex on \mathbb{R}_{++}^2 .
- (c) $f(x, u, v) = -\log(uv - x^T x)$ on $\text{dom } f = \{(x, u, v) \mid uv > x^T x, u, v > 0\}$.
- (d) $f(x, t) = -(t^p - \|x\|_p^p)^{1/p}$ where $p > 1$ and $\text{dom } f = \{(x, t) \mid t \geq \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x, u) for $u > 0$ (see exercise 3.23), and that $-x^{1/p}y^{1-1/p}$ is convex on \mathbb{R}_+^2 (see exercise 3.16).
- (e) $f(x, t) = -\log(t^p - \|x\|_p^p)$ where $p > 1$ and $\text{dom } f = \{(x, t) \mid t > \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x, u) for $u > 0$ (see exercise 3.23).

Ans: Here only does part (c).

(c) Let $g(x, u, v) = uv - x^T x$, then $\nabla g(x, u, v) = [-2x^T, v, u]$ and

$$H_g(x, u, v) = \begin{bmatrix} -2I & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

and for any $(x, u, v) \in \text{dom } f$,

$$[x^T, u, v] \begin{bmatrix} -2I & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} x \\ u \\ v \end{bmatrix} = -2x^T x + 2uv > 0$$

That is, g is convex on $\text{dom } f$. Now that $h = -\log(\cdot)$ is convex and nonincreasing, we conclude $f = -\log(g(\cdot))$ is convex on $\text{dom } f$ as well. \square

Additional Exercise 2.5 [Boyd & Vandenberghe, 2017]

A *perspective composition rule* [Marèchal]. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function with $f(0) \leq 0$.

(a) Show that the perspective $tf(x/t)$, with domain $\{(x, t) \mid t > 0, x/t \in \mathbf{dom} f\}$, is nonincreasing as a function of t .

(b) Let g be concave and positive on its domain. Show that the function

$$h(x) = g(x)f(x/g(x)), \quad \mathbf{dom} h = \{x \in \mathbf{dom} g \mid x/g(x) \in \mathbf{dom} f\}$$

is convex.

(c) As an example, show that

$$h(x) = \frac{x^T x}{\left(\prod_{k=1}^n x_k\right)^{1/n}}, \quad \mathbf{dom} h = \mathbb{R}_{++}^n$$

is convex. *Ans:* Here only does part (a) and (b).

Additional Exercise 2.30 [Boyd & Vandenberghe, 2017]

Huber penalty. The infimal convolution of two functions f and g on \mathbb{R}^n is defined as

$$h(x) = \inf_y (f(y) + g(x - y))$$

(see exercise 2.17). Show that the infimal convolution of $f(x) = \|x\|_1$ and $g(x) = (1/2)\|x\|_2^2$, *i.e.* , the function

$$h(x) = \inf_y (f(y) + g(x - y)) = \inf_y \left(\|y\|_1 + \frac{1}{2}\|x - y\|_2^2 \right)$$

is the *Huber penalty*

$$h(x) = \sum_{i=1}^n \phi(x_i), \quad \phi(u) = \begin{cases} u^2/2, & |u| \leq 1 \\ |u| - 1/2, & |u| > 1 \end{cases}$$

Ans:

Additional Exercise 2.31 [Boyd & Vandenberghe, 2017]

Suppose the function $h : \mathbb{R} \rightarrow \mathbb{R}$ is convex, nondecreasing, with $\text{dom } h = \mathbb{R}$, and $h(t) = h(0)$ for $t \leq 0$.

(a) Show that the function $f(x) = h(\|x\|_2)$ is convex on \mathbb{R}^n .

(b) Show that the conjugate of f is $f^*(y) = h^*(\|y\|_2)$.

(c) As an example, derive the conjugate of $f(x) = (1/p)\|x\|_2^p$ for $p > 1$, by applying the result of part (b) with the function

$$h(t) = \frac{1}{p} \max\{0, t\}^p = \begin{cases} \frac{1}{p} t^p, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Ans:

(a) Let $g(x) = \|x\|_2$, then

$$\begin{aligned} \nabla g(x) &= \frac{x^T}{\|x\|_2} \\ H_g(x) &= \frac{1}{\|x\|_2^2} \left(I\|x\|_2 - \frac{x}{\|x\|_2} x^T \right) \\ &= \frac{1}{\|x\|_2} \left(I - \frac{1}{\|x\|_2^2} x x^T \right) \\ v^T H_g(x) v &= \frac{1}{\|x\|_2} \left(\|v\|_2 - \frac{(x^T v)^2}{\|x\|_2^2} \right) \geq 0 \quad (\text{by Cauchy inequality}) \end{aligned}$$

Since h is convex and nondecreasing and $g = \|\cdot\|_2$ is convex, the composition $f = h \circ g$ must be convex, too.

(b) By definition,

$$\begin{aligned} f^*(y) &= \sup_{x \in \mathbb{R}^n} x^T y - f(x) \\ &= \sup_{x \in \mathbb{R}^n} x^T y - h(\|x\|_2) \\ &= \max \left\{ \sup_{x \neq 0} \|x\|_2 \left(\frac{x}{\|x\|_2} \right)^T y - h(\|x\|_2), -h(0) \right\} \\ &= \sup_{x \neq 0} \|x\|_2 \left(\frac{x}{\|x\|_2} \right)^T y - h(\|x\|_2) \quad (\text{because } -h \text{ is nonincreasing}) \\ &= \sup_{x \neq 0} \|x\|_2 \|y\|_2 - h(\|x\|_2) \\ &= \sup_{t > 0} t \|y\|_2 - h(t) \end{aligned}$$

The last equality follows from that the first term is maximized when x is parallel to y . Now observe the definition of h^*

$$h^*(\|y\|_2) = \sup_{t \in \mathbb{R}} t \|y\|_2 - h(t)$$

This differs from $f^*(y)$ only by the range of the minimizing variable

Additional Exercise 3.17 [Boyd & Vandenberghe, 2017]

Minimum fuel optimal control. Solve the minimum fuel optimal control problem described in exercise 4.16 of *Convex Optimization*, for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, x_{des} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, N = 30.$$

You can do this by forming the LP you found in your solution of exercise 4.16, or more directly using CVX. Plot the actuator signal $u(t)$ as a function of time t .

Ans: