Problem 1 (Exercise 1.11.1)

"$f+g$ is log-convex":
Refer to `http://math.stackexchange.com/questions/665768/how-to-prove-that-the-sum-of-two-lo`
Fix $x, y \in [0,1]$, then from log-convexity of f and g, there exist constants a, b, c, d such that $\forall z \in [x, y]$,

$$f(z) \leq e^{az+b} \text{ and } g(z) \leq e^{cz+d}$$

with equality holds at both endpoints. Summing both inequality we get

$$(f+g)(z) \leq e^{az+b} + e^{cz+d}$$

with equality holds at endpoints. WLOG assume $a \geq c$, we can show now that RHS is log-convex by differentiating

$$\begin{aligned}\frac{d}{dx} \log\left(e^{az+b} + e^{cdz+d}\right) &= \frac{ae^{az+b} + ce^{cz+d}}{e^{az+b} + e^{cz+d}} \\ &= c + \frac{a-c}{1 + e^{(c-a)z+d-b}}\end{aligned}$$

which is an increasing function of z.

"fg is log-convex": Observe that

$$\log(fg) = \log(f) + \log(g)$$

is a convex function since its sum of two convex function.

"$\max(f, g)$ is log-convex": Use notation $f_1 = f$, $f_2 = g$. Fix $x, y \in [0,1]$, $\theta \in (0,1)$; use monotonicity of log and the log-convexity of both functions,

$$\begin{aligned}\log\max(f_1, f_2)(\theta x + (1-\theta)y) &\leq \log f_i(\theta x + (1-\theta)y) && \text{for } i = 1, 2 \\ &\leq \theta \log f_i(x) + (1-\theta)\log f_i(y) && \text{for } i = 1, 2\end{aligned}$$

$\square$

Problem 2 (Exercise 1.11.4)

Refer to `https://en.wikipedia.org/wiki/Hadamard_three-lines_theorem`
Define

$$M(\sigma) := \sup_{t\in\mathbb{R}} |f(\sigma + it)|$$

Problem 3 (Exercise 1.11.5)

Observe that the inequality is a special case of Hölder inequality, of which the equality is equivalent to the condition

$$|f|^{p_0} = C|f|^{p_1} \text{ for some constant } C$$

Now that we have $p_0 < p_1$, this is to say that $|f|^{p_1 - p_0} = C$ is a constant; however, if we applied instead Hölder inequality on a smaller domain and just sum up over the partition. $\square$

Problem 4 (Exercise 1.11.8)

The first inequality follows from Chebyshev's inequality. Now denote $X = \{x_1, \cdots, x_n\}$; to prove the second inequality, we first WLOG assume $f = \sum_{i=1}^{n} f_i 1_{\{x_i\}}$ where $|f_n| \geq |f_{n-1}| \geq \cdots \geq |f_1| \geq 0$. Observe that

$$\|f\|_{p,\infty} = \sup_{t>0} t\lambda_f(t)^{1/p}$$

should attain maximum at $|f_n|, \cdots, |f_2|$ or $|f_1|$, so if we normalize $\|f\|_{p,\infty} = 1$, then we know

$$\max_{1\leq i\leq n} |f_i|(n-i+1)^{\frac{1}{p}} \leq 1$$

therefore for $i = 1, \cdots, n$, $f_i \leq (n-k+1)^{1/p}$. Now back to the inequality,

$$\begin{aligned}
\|f\|_p &= \left(\sum_{i=1}^{n} |f_i|^p\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{n} (n-i+1)^{-\frac{1}{p}}\right)^{\frac{1}{p}} \\
&= \left(\sum_{k=1}^{n} k^{-\frac{1}{p}}\right)^{\frac{1}{p}} \\
&\leq (\log())^{\frac{1}{p}} \\
&\leq
\end{aligned}$$

Problem 5 (Exercise 1.11.11)

"(ii) ⇒ (i)": WLOG assume $f \geq 0$. Take $E_t = \{x \in X : f(x) \geq t\}$, then from assumption (ii),

$$t\lambda_t f(t) \leq \int f 1_{E_t} d\mu \leq C' \lambda_f(t)^{\frac{1}{p'}}$$

Divide both side with $\lambda_f(t)^{\frac{1}{p}}$, then

$$\forall t > 0, t\lambda_f(t)^{\frac{1}{p}} \leq C'$$

and taking supremum shows that $\|f\|_{L^{p,\infty}(X)} \leq C'$.

"(i) ⇒ (ii)":

Problem 6 (Exercise 1.11.13)

"(i) $\Rightarrow$ (ii)": By Exercise 1.11.11, assuming (i) (which is the same as (i) in previous problem), there exists $C' > 0$ such that for all $E \subseteq X$ with finite measure,

$$\left| \int_X f 1_E d\mu \right| \le C' \mu(E)^{\frac{1}{p'}}$$

Take $E' = E$, then of course the conditions holds for (ii).

"(ii) $\Rightarrow$ (i)": I will prove (ii) of Exercise 1.11.11 by assuming (ii) of this problem; then by equivalence given in previous problem, that will prove (i) for this problem.

Fix $E \subseteq X$ with finite measure, we aim to find a constant C' independent of E such that

$$\left| \int f 1_E d\mu \right| \le C' \mu(E)^{\frac{1}{p'}}$$

Denote $E_0 = E$. Take from the assumption that there exists $E_1 \subseteq E_0$ with $\mu(E_1) \ge \frac{1}{2}\mu(E)$ and

$$\left| \int f 1_{E_1} d\mu \right| \le C \mu(E)^{\frac{1}{p'}}$$

and successively take $E_2, E_3, \cdots$ and denote $F_k = \bigcup_{i=1}^{k} E_i$ such that $E_{k+1} \subseteq E \setminus F_k$ (therefore E_i's are disjoint) with

$$\mu(E_{k+1}) \ge \frac{1}{2}\mu(E \setminus F_k)$$

and

$$\left| \int f 1_{E_{k+1}} d\mu \right| \le C \mu(E \setminus F_k)^{\frac{1}{p'}}$$

With these conditions we see that $\mu(E \setminus F_k) \le \frac{1}{2^k}\mu(E)$ and therefore

$$\begin{aligned} \left| \int f 1_{F_k} d\mu \right| &\le \sum_{i=1}^{k} \left| \int f 1_{E_i} d\mu \right| \\ &\le \sum_{i=1}^{k} \frac{C}{2^k} \mu(E)^{\frac{1}{p'}} \le 2C\mu(E) \end{aligned}$$

Exhaust $k \to \infty$ then we know $F_k \nearrow E$ and therefore by DCT $\int f 1_{F_k} d\mu \to \int f 1_E d\mu$ and the bound from RHS remains solid

$$\left| \int f 1_E d\mu \right| \le 2C \mu(E)^{\frac{1}{p'}}$$