1 Derivation

HJB is

$$0 = \rho\theta(\frac{C_t^{\frac{1-\gamma}{\theta}}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1)dt + E[\frac{dV_t}{V_t}]$$

Define G_t such that

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma}\rho^\theta G_t^\theta$$

Denote μ_C and σ_C the geometric drift of C_t, and μ_G, σ_G the geometric drift and volatility of G_t and apply Ito.

$$0 = \frac{\theta}{G} - \theta\rho + (1-\gamma)\mu_C + \theta\mu_G - \frac{1}{2}(1-\gamma)\gamma\sigma_C'\sigma_C + \theta(1-\gamma)\sigma_G'\sigma_C + \frac{1}{2}\theta(\theta-1)\sigma_G^2$$

By taking FOC for consumption, one can check that G corresponds to the wealth ratio

2 Long run risk model

2.1 Derivation

We now assume that the evolution of consumption is driven by two state variables μ_t and σ_t:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1-\sigma_t)dt + \nu_\sigma dZ_t^\sigma
\end{aligned}$$

We write $G_t = G(\mu, \sigma)$ and we get the PDE

$$\begin{aligned}
0 = &\frac{\theta}{G} - \rho\theta + (1-\gamma)(\mu - \frac{1}{2}\gamma\nu_D^2\sigma) \\
&+ \kappa_\mu\theta(\bar{\mu} - \mu)\frac{\partial_\mu G}{G} + \kappa_\sigma\theta(1-\sigma)\frac{\partial_\sigma G}{G} \\
&+ \frac{1}{2}\nu_\mu^2\sigma(\theta\frac{\partial_\mu^2 G}{G} + \theta(\theta-1)(\frac{\partial_\mu G}{G})^2) + \frac{1}{2}\nu_\sigma^2(\theta\frac{\partial_\sigma^2 G}{G} + \theta(\theta-1)(\frac{\partial_\sigma G}{G})^2)
\end{aligned}$$

2.2 Numerical Methods

We can use two methods

2.2.1 Finite Difference Method

We can discretize this PDE on a grid using a Finite Difference Scheme.

- We upwind the first derivative, i.e. we approximate $\partial_x G$ by a forward difference when the drift of the x variable is positive and a backward difference when the drift is negative.
- At the border of the grid, the PDE involves the value of G outside the grid through the second derivative. To get the value of G at these nodes, we apply a boundary counditions: state variables have reflecting boundaries at the borders. This gives a supplementary condition of the form $\partial G = 0$ at the borders[1]

To sum up the scheme is

$$\begin{aligned}
\Delta_\mu G &= (\mu_i \leq \bar{\mu})\frac{G_{i+1,j} - G_{i,j}}{\Delta\mu} + \mu_i > \bar{\mu})\frac{G_{i,j} - G_{i-1,j}}{\Delta\mu} \\
\Delta_\mu^2 G &= \frac{G_{i+1,j} + G_{i-1,j} - 2G_{i,j}}{(\Delta\mu)^2} \\
\Delta_\sigma G &= (\sigma_j \leq 1)\frac{G_{i+1,j} - G_{i,j}}{G_{i,j}\Delta\mu} + (\sigma_j > 1)\frac{G_{i,j} - G_{i-1,j}}{G_{i,j}\Delta\mu} \\
\Delta_\sigma^2 G &= \frac{G_{i,j+1} + G_{i,j-1} - 2G_{i,j}}{(\Delta\sigma)^2} \\
0 &= \frac{\theta}{G_{i,j}} - \rho\theta + (1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j \\
&+ \theta\kappa_{\mu_i}(\bar{\mu} - \mu_i)\frac{\Delta_\mu G}{G_{ij}} + \theta\kappa_{\sigma_i}(1-\sigma_j)\frac{\Delta_\sigma G}{G_{ij}} \\
&+ \frac{\theta}{2}\nu_{\mu_i}^2\sigma_j(\frac{\Delta_\mu^2 G}{G_{ij}} + (\theta-1)(\frac{\Delta_\mu G}{G_{ij}})^2) + \frac{\theta}{2}\nu_{\sigma_j}^2(\frac{\Delta_\sigma^2 G}{G_{ij}} + (\theta-1)(\frac{\Delta_\sigma G}{G_{ij}})^2)
\end{aligned}$$

Denote Y the vector of $(G_{ij})_{1\leq i,j\leq n}$. The scheme defines a function F such that $F(Y) = 0$. We can solve for Y using one of these methods:

1. Use a non linear solver for the system $F(Y) = 0$. These algorithms start with an initial guess, and update based on the Jacobian of F. In some PDE (not this one), you need a good initial guess. A technique is to solve the PDE for $\theta = 1$, and use the solution at an initial guess for other values of θ.
2. Use an ODE solver for the system $F(Y) = \dot{Y}$. The solution when $T \to +\infty$ is the solution of the PDE. This method is called “the method of lines”.
3.

[1] At the frontier we have

$$dG_t = G'(x)\sigma(x)\sqrt{dt} + G'(x)\mu(x)dt + \frac{1}{2}G''(x)\sigma^2(x)dt + o(dt)$$

Since dG_t is $O(dt)$, we must have $G'(x) = 0$, at least when $\sigma(x) \neq 0$.At the points $j = 0$, the volatility is zero. At the frontier $\sigma = 0$, the term involving the second derivative disappears naturally so we don't need to add a reflecting boundary conditon

Both methods require the jacobian of F, which can generally be automatically computed using automatic or numerical differenciation. A less robust method solves G by iterating over time

$$\frac{G_{n+1} - G_n}{\Delta t} = F(G_n)$$

This method can be seen as a special case of a non linear solver (fixed point method) or as a special case of an ODE solver (i.e. Euler method). The criterion for the convergence of this method is that F_{ij} is decreasing in $G_{i,j}$ (Barles Souganadis theorem). This is not the case if $\theta < 0$ (which the relevant case in BKY 2007 with $\gamma = 7.5$ and $\psi = 1.5$)

2.2.2 Spectral Method (= collocation method)

We can also solve the PDE by looking at solutions of the form

$$V(\mu, \sigma) = \sum_{kl} a_{kl} \phi_k(\mu) \psi_l(\sigma)$$

where ϕ, ψ denote any basis of functions (splines or chebyshev polynomials). The value function is characterized by its coordinates a_{kl} on this basis. Writing the PDE on a grid gives a non linear system in term of the coordinates, which we can solve. We use the same border condition and initial guess as the Finite Difference method.

2.3 Comparison

Correspondences	
$\Delta c_{t+1} = \mu + x_t + \sigma_t \eta_{t+1}$	$\frac{dC_t}{C_t} = \mu_t dt + \nu_D \sqrt{\sigma_t} dZ_t$
$x_{t+1} = \rho x_t + \phi_e \sigma_t e_{t+1}$	$d\mu_t = \kappa_\mu (\bar{\mu} - \mu_t) dt + \nu_\mu \sqrt{\sigma_t} dZ_t^\mu$
$\sigma_{t+1}^2 = \sigma^2 + \nu_1(\sigma_t^2 - \sigma^2) + \sigma_w w_{t+1}$	$d\sigma_t = \kappa_\sigma (1 - \sigma_t) dt + \nu_\sigma dZ_t^\sigma$

Correspondences		
growth rate	μ	$\bar{\mu} = \mu \times 12$
volatility	σ^2	$\nu_D = \sigma \times \sqrt{12}$
growth persistence	ρ	$\kappa_\mu = (1 - \rho) \times 12$
volatility persistence	ν_1	$\kappa_\sigma = (1 - \nu_1) \times 12$
growth rate volatility	φ_e	$\nu_\mu = (\varphi_e \sigma 12) \times \sqrt{12}$
volatility volatility	σ_w	$\nu_\sigma = (\sigma_w / \overline{\sigma}^2) \times \sqrt{12}$
time discount	δ	$\rho = (1 - \delta) \times 12$

BY 2004		
Name	Discrete Time	Continuous Time
growth rate	$\mu = 0.0015$	$\overline{\mu} = 0.018$
volatility	$\sigma = 0.0078$	$\nu_D = 0.027$
growth persistence	$\rho = 0.979$	$\kappa_\mu = 0.252$
volatility persistence	$\nu_1 = 0.987$	$\kappa_\sigma = 0.156$
growth rate volatility	$\phi_e = 0.044$	$\nu_\mu = 0.0143$
volatility volatility	$\sigma_w = 0.0000023$	$\nu_\sigma = 0.131$
time discount	$\delta = 0.998$	$\rho = 0.024$

BKY 2007		
Name	Discrete Time	Continuous Time
mean growth rate	$\mu = 0.0015$	$\overline{\mu} = 0.018$
mean volatility	$\sigma = 0.0072$	$\nu_D = 0.025$
growth persistence	$\rho = 0.975$	$\kappa_\mu = 0.3$
volatility persistence	$\nu_1 = 0.999$	$\kappa_\sigma = 0.012$
growth rate volatility	$\phi_e = 0.038$	$\nu_\mu = 0.0114$
volatility volatility	$\sigma_w = 0.00000283$	$\nu_\sigma = 0.189$
time discount	$\delta = 0.9989$	$\rho = 0.0132$

3 Loglinearization

- $$\begin{aligned} 0 &= \theta e^{-\overline{g}}(1 - A_1(\mu - \overline{\mu}) - A_2(\sigma - 1)) - \rho\theta + (1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma \\ &+ \theta\kappa_\mu(\bar{\mu} - \mu)A_1 + \theta\kappa_\sigma(1-\sigma)A_2 \\ &+ \frac{\theta^2}{2}\nu_\mu^2\sigma A_1^2 + \frac{\theta}{2}\nu_\sigma^2 A_2^2 \end{aligned}$$

  We obtain

  $$\begin{aligned} A_1 &= \frac{1 - \frac{1}{\psi}}{e^{-\overline{g}} + \kappa_\mu} \\ A_2 &= \frac{1}{2}\frac{(\theta - \frac{\theta}{\psi})(\theta - \frac{\theta}{\psi} - 1)\nu_D^2 + A_1^2\theta^2\nu_\mu^2}{\theta(e^{-\overline{g}} + \kappa_\sigma)} \end{aligned}$$

  BY obtains

  $$\begin{aligned} A_1 &= \frac{1 - \frac{1}{\psi}}{1 - \frac{1-\kappa_\mu}{1+e^{-\overline{g}}}} \\ A_2 &= \frac{1}{2}\frac{(\theta - \frac{\theta}{\psi})^2\nu_D^2 + A_1^2\theta^2(\frac{e^{-\overline{g}}}{1+e^{-\overline{g}}})^2\nu_\mu^2}{\theta(1 - \frac{e^{-\overline{g}}}{1+e^{-\overline{g}}}(1-\kappa_\sigma))} \end{aligned}$$

- The HJB equation can be obtained by applying Euler equation to the return of the wealth portfolio

$$
\begin{aligned}
\frac{EdR}{dt} &= \frac{1}{G} + \mu_G + \mu_C + \sigma_G' \sigma_C \\
\sigma_{dR} &= \sigma_G + \sigma_C \\
r &= \theta\rho + \frac{\theta}{\psi}\mu_C + (1-\theta)EdR_C + \frac{1}{2}(\frac{\theta}{\psi} - 1)\sigma_C^2 - \frac{1}{2}\theta\sigma_{R_c}^2 + \frac{\theta}{\psi}(1-\theta)\sigma_C\sigma_{dR_C} \\
\kappa &= \frac{\theta}{\psi}\sigma_C + (1-\theta)\sigma_{dR}
\end{aligned}
$$

We would obtain the same than Bansal Yaron by transforming the return:

$$
\begin{aligned}
R_{t+1} &= \frac{C_{t+1} + W_{t+1}}{W_t} \\
&= \frac{C_t}{W_t}(\frac{C_{t+1}}{C_t} + \frac{W_{t+1}}{C_t}) \\
&= \frac{1}{G_t}\frac{C_{t+1}}{C_t}(1 + G_{t+1})
\end{aligned}
$$

$$
d\ln R = \kappa_0 dt + \kappa_1(d\ln G + \ln G dt) - \ln G dt + d\ln C
$$

- In continuous time the dividend is known at t

$$
\begin{aligned}
R_{t+1} &= \frac{C_t + W_{t+1}}{W_t} \\
&= \frac{C_t}{W_t}(1 + \frac{C_{t+1}}{C_t}\frac{W_{t+1}}{C_{t+1}}) \\
&= \frac{1}{G_t}(1 + \frac{C_{t+1}}{C_t}G_{t+1})
\end{aligned}
$$

$$
d\ln R = \kappa_0 dt + \kappa_1(d\ln G + \ln G dt + d\ln C) - \ln G dt
$$