1 Derivation

HJB is

$$0 = \rho\theta V_t\left(\frac{C_t^{\frac{1-\gamma}{\theta}}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1\right) + \frac{E[dV_t]}{dt}$$

Let's define G_t such that

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} G_t$$

G is directly related to the wealth/consumption ratio:

$$WC_t = \frac{G_t^{1/\theta}}{\rho}$$

By Ito, denoting μ_C and σ_C the geometric drift of C_t, and μ_G, σ_G the arithmetic drift and volatility of G_t

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C'\sigma_C) + \mu_G + \sigma_G'(1-\gamma)\sigma_C$$

2 Long run risk model

2.1 Derivation

We now assume that the evolution of consumption is driven by two state variables μ_t and σ_t:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t} dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t} dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1 - \sigma_t)dt + \nu_\sigma\sqrt{\sigma_t} dZ_t^\sigma
\end{aligned}$$

We write $G_t = G(\mu, \sigma)$ and we get the PDE

$$\begin{aligned}
0 = {} & \rho\theta[G^{1-\frac{1}{\theta}} - G] + G((1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma) \\
& + \kappa_\mu(\bar{\mu} - \mu)\frac{\partial G}{\partial \mu} + \kappa_\sigma(1-\sigma)\frac{\partial G}{\partial \sigma} \\
& + \frac{1}{2}\nu_\mu^2\sigma\frac{\partial^2 G}{\partial \mu^2} + \frac{1}{2}\nu_\sigma^2\sigma\frac{\partial^2 G}{\partial \sigma^2}
\end{aligned}$$

2.1.1 Log linearization method

We guess $G = e^{A_0 + A_1\mu_t + A_2\sigma_t}$ and linearize the non linear term in G

$$G^{-\frac{1}{\theta}} \approx \overline{G}(1 - \frac{1}{\theta}(A_1(\mu_t - \overline{\mu}) + A_2(\sigma_t - 1))$$

We can then solve for A_0, A_1, A_2 by setting the constant term, the term in μ_t and the term in σ_t to zero in the linearized PDE. We obtain

$$
\begin{aligned}
0 &= -\rho\overline{G}A_1 + 1 - \gamma - \kappa_\mu A_1 \\
0 &= -\rho\overline{G}A_2 - \frac{1}{2}(1-\gamma)\gamma\nu_D^2 + \kappa_\sigma(1-\sigma)A_2 + \frac{1}{2}\nu_\mu^2 A_1^2 + \frac{1}{2}\nu_\sigma^2 A_2^2
\end{aligned}
$$

Bansal Yaron (2004) find

$$
\begin{aligned}
A1 &= \theta\frac{1-\frac{1}{\psi}}{1-0.997e^{-\kappa_\mu}} \\
A2 &= 0.5\theta^2\frac{(1-\frac{1}{\psi})^2 + (A_1 0.997\frac{\nu_\mu}{\nu_D})^2}{1-0.997e^{-\kappa_\sigma}}
\end{aligned}
$$

2.1.2 Finite Difference Method

We can discretize this PDE on a grid using a Finite Difference Scheme.

- We upwind the first derivative, i.e. we approximate ∂G by a forward difference when the drift is positive and a backward difference when the drift is negative.
- At the border of the grid, the PDE involves the value of G outside the grid (through the second derivative). To get the value of G at these nodes, we apply a boundary counditions: state variables have reflecting boundaries at the borders. This gives a supplementary condition of the form $\partial G = 0$ at the borders[1]

To handle boundary conditions, To sum up the scheme is

$$
\begin{aligned}
0 = {} & \rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}] + G_{ij}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
& + (\kappa_{\mu_i}(\bar{\mu}-\mu_i))^+\frac{G_{i+1,j}-G_{i,j}}{\Delta\mu} + (\kappa_{\mu_i}(\bar{\mu}-\mu_i))^-\frac{G_{i,j}-G_{i-1,j}}{\Delta\mu} \\
& + (\kappa_{\sigma_j}(1-\sigma_j))^+\frac{G_{i,j+1}-G_{i,j}}{\Delta\sigma} + (\kappa_{\sigma_j}(1-\sigma_j))^-\frac{G_{i,j}-G_{i,j-1}}{\Delta\sigma} \\
& + \frac{1}{2}\nu_{\mu_i}^2\sigma_j\frac{G_{i+1,j}-2G_{i,j}+G_{i-1,j}}{(\Delta\mu)^2} + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j\frac{G_{i,j+1}-2G_{i,j}+G_{i,j-1}}{(\Delta\sigma)^2}
\end{aligned}
$$

Denote Y the vector of $(G_{ij})_{1\le i,j\le n}$. The scheme defines a function F such that $F(Y)=0$. We can solve for Y using one of these methods:

[1] At the frontier we have

$$dG_t = G'(x)\sigma(x)\sqrt{dt} + G'(x)\mu(x)dt + \frac{1}{2}G''(x)\sigma^2(x)dt + o(dt)$$

Since dG_t is $O(dt)$, we must have $G'(x)=0$, at least when $\sigma(x)\neq 0$.At the points $j=0$, the volatility is zero. At the frontier $\sigma=0$, the term involving the second derivative disappears naturally so we don't need to add a reflecting boundary conditon

1. Use a non linear solver for the system $F(Y) = 0$. These algorithms start with an initial guess, and update based on the Jacobian of F. In some PDE (not this one), you need a good initial guess. A technique is to solve the PDE for $\theta = 1$, and use the solution at an initial guess for other values of θ.

2. Use an ODE solver for the system $F(Y) = \dot{Y}$. The solution when $T \to +\infty$ is the solution of the PDE. This method is called "the method of lines".

Both methods require the jacobian of F, which can generally be automatically computed using automatic or numerical differenciation. A solution that would *not* work is to solve for G by iterating over time

$$\frac{G_{n+1} - G_n}{\Delta t} = F(G_n)$$

This method can be seen as a special case of a non linear solver (fixed point method) or as a special case of an ODE solver (in this context, it is called the Euler method). The criterion for the convergence of this method is that F is monotonous in G (Barles Souganadis theorem). This is not the case here, due to the non linear term $\rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}]$.

2.1.3 Spectral Method (= collocation method)

We can also solve the PDE by looking at solutions of the form

$$V(\mu, \sigma) = \sum_{kl} a_{kl} \phi_k(\mu) \psi_l(\sigma)$$

where ϕ, ψ denote any basis of functions (splines or chebyshev polynomials). The value function is characterized by its coordinates a_{kl} on this basis. Writing the PDE on a grid gives a non linear system in term of the coordinates, which we can solve. We use the same border condition and initial guess as the Finite Difference method.

2.2 Comparison

Correspondances			
mean growth rate	μ	$\bar{\mu}$	$\mu = \bar{\mu}$
mean volatility	σ^2	ν_D	$\sqrt{\sigma^2} = \nu_D$
growth persistence	ρ	κ_μ	$-\log(\rho) = \kappa_\mu$
volatility persistence	ν_1	κ_σ	$-\log(\nu_1) = \kappa_\sigma$
growth rate volatility	φ_e	ν_μ	$\varphi_e \times \sqrt{\sigma^2} = \nu_\mu$
volatility volatility	σ_w	ν_σ	$\sigma_w / \sigma^2 = \nu_\sigma$
time discount	δ	ρ	$-\log(\delta) = \rho$

BY 2004		
Name	Discrete Time	Continuous Time
mean growth rate	$\mu = 0.0015$	$\overline{\mu} = 0.0015$
mean volatility	$\sigma = 0.0078$	$\nu_D = 0.0078$
growth persistence	$\rho = 0.979$	$\kappa_\mu = 0.0212$
volatility persistence	$\nu_1 = 0.987$	$\kappa_\sigma = 0.0131$
growth rate volatility	$\phi_e = 0.044$	$\nu_\mu = 0.0003432$
volatility volatility	$\sigma_w = 0.0000023$	$\nu_\sigma = 0.0378$
time discount	$\delta = 0.998$	$\rho = 0.002$

BKY 2007		
Name	Discrete Time	Continuous Time
mean growth rate	$\mu = 0.0015$	$\overline{\mu} = 0.0015$
mean volatility	$\sigma = 0.0072$	$\nu_D = 0.0072$
growth persistence	$\rho = 0.975$	$\kappa_\mu = 0.0253$
volatility persistence	$\nu_1 = 0.999$	$\kappa_\sigma = 0.001$
growth rate volatility	$\phi_e = 0.038$	$\nu_\mu = 0.00274$
volatility volatility	$\sigma_w = 0.00000283$	$\nu_\sigma = 0.05401$
time discount	$\delta = 0.9989$	$\rho = 0.0011$