1 Derivation

HJB is

$$0 = \rho\theta V_t(\frac{C_t^{\frac{1-\gamma}{\theta}}}{((1-\gamma)V_t)^{\frac{1}{\theta}}} - 1) + \frac{E[dV_t]}{dt}$$

Let's define G_t such that

$$V_t = \frac{C_t^{1-\gamma}}{1-\gamma} G_t$$

By Ito,

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t E \frac{\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt} + E\frac{dG_t}{dt} + E\frac{dG_t \frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}}}{dt}$$

Denoting μ_C and σ_C the geometric drift of C_t, we have

$$\frac{dC_t^{1-\gamma}}{C_t^{1-\gamma}} = ((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C^2)dt + (1-\gamma)\sigma_C dW_t$$

Injecting this expression into HJB and denoting μ_G, σ_G the arithmetic drift and volatility of G_t

$$0 = \rho\theta(G_t^{1-\frac{1}{\theta}} - G_t) + G_t((1-\gamma)\mu_C - \frac{1}{2}(1-\gamma)\gamma\sigma_C'\sigma_C) + \mu_G + \sigma_G'(1-\gamma)\sigma_C$$

2 Long run risk model

2.1 Derivation

We now assume that the evolution of consumption is driven by two state variables μ_t and σ_t:

$$\begin{aligned}
\frac{dC_t}{C_t} &= \mu_t dt + \nu_D\sqrt{\sigma_t}dZ_t \\
d\mu_t &= \kappa_\mu(\bar{\mu} - \mu_t)dt + \nu_\mu\sqrt{\sigma_t}dZ_t^\mu \\
d\sigma_t &= \kappa_\sigma(1-\sigma_t)dt + \nu_\sigma\sqrt{\sigma_t}dZ_t^\sigma
\end{aligned}$$

We write $G_t = G(\mu, \sigma)$ and we get the PDE

$$\begin{aligned}
0 = &\rho\theta[G^{1-\frac{1}{\theta}} - G] + G((1-\gamma)\mu - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma) \\
&+ \kappa_\mu(\bar{\mu} - \mu)\frac{\partial G}{\partial \mu} + \kappa_\sigma(1-\sigma)\frac{\partial G}{\partial \sigma} \\
&+ \frac{1}{2}\nu_\mu^2\sigma\frac{\partial^2 G}{\partial \mu^2} + \frac{1}{2}\nu_\sigma^2\sigma\frac{\partial^2 G}{\partial \sigma^2}
\end{aligned}$$

2.1.1 Log linearization method

We guess $G = e^{A_0+A_1\mu_t+A_2\sigma_t}$ and linearize the non linear term in G

$$G^{-\frac{1}{\theta}} \approx e^{-\frac{1}{\theta}(A_0+A_1\overline{\mu}+A_2)}(1-\frac{1}{\theta}(A_1(\mu_t-\overline{\mu})+A_2(\sigma_t-1))$$

We can then solve for A_0, A_1, A_2 by setting the constant term, the term in μ_t and the term in σ_t to zero in the linearized PDE.

2.1.2 Finite Difference Method

We can discretize this PDE on a grid using a Finite Difference Scheme. We assume that the processes associated with state variables have reflecting boundaries at the borders (except at the LHS border for σ). This gives a supplementary condition of the form $\partial G = 0$ at the borders[1]

To handle boundary conditions, we upwind the first derivative. This means that we approximate ∂G by a forward difference when the drift is positive and a backward difference when the drift is negative. The second derivative at the border involves points outside the grid. We find the value of G at these points by applying the reflecting boundary condition at these points. [2]

To sum up the scheme is

$$\begin{aligned}
0 &= \rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}] + G_{ij}((1-\gamma)\mu_i - \frac{1}{2}(1-\gamma)\gamma\nu_D^2\sigma_j) \\
&+ (\kappa_{\mu_i}(\bar{\mu}-\mu_i))^+\frac{G_{i+1,j}-G_{i,j}}{\Delta\mu} + (\kappa_{\mu_i}(\bar{\mu}-\mu_i))^-\frac{G_{i,j}-G_{i-1,j}}{\Delta\mu} \\
&+ (\kappa_{\sigma_j}(1-\sigma_j))^+\frac{G_{i,j+1}-G_{i,j}}{\Delta\sigma} + (\kappa_{\sigma_j}(1-\sigma_j))^-\frac{G_{i,j}-G_{i,j-1}}{\Delta\sigma} \\
&+ \frac{1}{2}\nu_{\mu_i}^2\sigma_j\frac{G_{i+1,j}-2G_{i,j}+G_{i-1,j}}{(\Delta\mu)^2} + \frac{1}{2}\nu_{\sigma_j}^2\sigma_j\frac{G_{i,j+1}-2G_{i,j}+G_{i,j-1}}{(\Delta\sigma)^2}
\end{aligned}$$

Denote Y the vector of $(G_{ij})_{1\leq i,j\leq n}$. The scheme defines a function F such that $F(Y) = 0$. We can solve for Y using one of these methods:

1. Use a non linear solver for the system $F(Y) = 0$. These algorithms start with an initial guess, and update based on the Jacobian of F. In some PDE (not this one), you need a good initial guess. A technique is to solve the PDE for $\theta = 1$, and use the solution at an initial guess for other values of θ.

2. Use an ODE solver for the system $F(Y) = \dot{Y}$. The solution when $T \to +\infty$ is the solution of the PDE. This method is called "the method of lines".

[1] At the frontier we have

$$dG_t = G'(x)\sigma(x)\sqrt{dt} + G'(x)\mu(x)dt + \frac{1}{2}G''(x)\sigma^2(x)dt + o(dt)$$

Since dG_t is $O(dt)$, we must have $G'(x) = 0$, at least when $\sigma(x) \neq 0$.

[2] At the points $j = 0$, the volatility is zero. Although we don't have the border condition $\partial_\sigma G = 0$ or $\partial_\mu G = 0$ anymore, the term involving the second derivative disappears naturally

Both methods require the jacobian of F, which can generally be automatically computed using automatic or numerical differenciation.

A solution that would *not* work is to solve for G by iterating over time

$$\frac{G_{n+1} - G_n}{\Delta t} = F(G_n)$$

This method can be seen as a special case of a non linear solver (fixed point method) or as a special case of an ODE solver (in this context, it is called the Euler method). The criterion for the convergence of this method is that F is monotonous in G (Barles Souganadis theorem). This is not the case here, due to the non linear term $\rho\theta[(G_{ij})^{1-\frac{1}{\theta}} - G_{ij}]$.

2.1.3 Spectral Method (= collocation method)

We can also solve the PDE by looking at solutions of the form

$$V(\mu, \sigma) = \sum_{kl} a_{kl}\phi_k(\mu)\psi_l(\sigma)$$

where ϕ, ψ denote any basis of functions (splines or chebyshev polynomials). The value function is characterized by its coordinates a_{kl} on this basis. Writing the PDE on a grid gives a non linear system in term of the coordinates, which we can solve. We use the same border condition and initial guess as the Finite Difference method.

2.2 Comparison

Name	BY04	BY04	This paper	This paper	Link
mean growth rate	μ	0.0015	$\bar{\mu}$	0.0015	$\mu = \bar{\mu}$
mean volatility	σ^2	0.00006084	ν_D	0.0078	$\sqrt{\sigma^2} = \nu_D$
growth persistence	ρ	0.979	κ_μ	0.0212	$-\log(\rho) = \kappa_\mu$
volatility persistence	ν_1	0.987	κ_σ	0.0131	$-\log(\nu_1) = \kappa_\sigma$
growth rate volatility	φ_e	0.044	ν_μ	0.0003432	$\varphi_e \times \sqrt{\sigma^2} = \nu_\mu$
volatility volatility	σ_w	0.0000023	ν_σ	0.0378	$\sigma_w/\sigma^2 = \nu_\sigma$
time discount	δ	0.998	ρ	0.002	$-\log(\delta) = \rho$
RRA	$1-\gamma$(RRA)	7.5 or 10	$1-\gamma$	-6.5 or -9	$1 - \mathrm{RRA} = 1 - \gamma$
IES	ψ	1.5	ψ	1.5	$\psi = \psi$

Also, $\theta = (1-\gamma)/(1-1/\psi) =$ -19.50 or -27. Let's express the wealth to consumption ratio K_t in term of state variables.

$$V = G_t K_t^{\gamma-1}\frac{W^{1-\gamma}}{(1-\gamma)}$$

FOC for consumption can be written

$$K_t^{-1} = \rho^\psi K_t^{\psi-1} G_t^{\frac{1-\psi}{1-\gamma}}$$

We conclude

$$K_t = \rho^{-1} G_t^{1/\theta}$$

Bansal Yaron find

$$\log K_t \propto A_1 \mu_t + A_2 \nu_D^2 \sigma_t$$
$$A1 = \frac{1 - \frac{1}{\psi}}{1 - 0.997 e^{-\kappa_\mu}}$$
$$A2 = 0.5\theta \frac{(1 - \frac{1}{\psi})^2 + (A_1 0.997 \frac{\nu_\mu}{\nu_D})^2}{1 - 0.997 e^{-\kappa_\sigma}}$$