Bayesian Time Series Methods: Introductory

Computer Tutorial 2: Bayesian Analysis of the Regression Model

Exercise 1: The Normal Linear Regression Model with Conjugate Prior (Analytical Results)

- How can we do Bayesian inference in normal linear regression model with natural conjugate priors in MATLAB?

- We are interested in estimating the regression model of the form

$$y_i = \beta_1 + x_{i2}\beta_2 + x_{i3}\beta_3 + x_{i4}\beta_4 + x_{i5}\beta_5 + \varepsilon_i .$$

- We use four explanatory variables with 546 number of observations. We calibrate the informative priors of the parameters as

$$\underline{\beta} = \begin{bmatrix} 0.0 \\ 10 \\ 5000 \\ 10000 \\ 10000 \end{bmatrix}.$$

➢ Exercise 1 (cont.): The Normal Linear Regression Model with Conjugate Prior (Analytical Results)

- Assuming Normal-Gamma implies

$$\text{var}(\beta) = \frac{\underline{v}\underline{s}^2}{\underline{v}-2}\underline{V}.$$

- With the diagonal elements of var(β_j) for j = 1, . . . , 5 are set

$$\text{var}(\beta) = \underline{V} = \begin{bmatrix} 2.4 & 0 & 0 & 0 & 0 \\ 0 & 6.0\times10^{-7} & 0 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0 & 0.60 & 0 \\ 0 & 0 & 0 & 0 & 0.60 \end{bmatrix},$$

- and the off-diagonal elements (the covariance priors) are set to zero.

➢ Exercise 1 (cont.): The Normal Linear Regression Model with Conjugate Prior (Analytical Results)

Type: >> n=size(hprice,1);

- size(hprice,1), returns the number of rows in house price data set, which is the number of observations.

>> y=hprice(:, 1);

- Returns 1st column of house prices data set, which is the dependent variable.

>> x=hprice(:, 2:5);

- The colon operator can also be used to extract a sub-matrix from a matrix. In this example, we extract the last four columns of house price data.

>> x=[ones(n, 1) x];

- Here, in addition to the last four of columns of the house price data, we add ones column to the *x* matrix.

Exercise 1 (cont.): The Normal Linear Regression Model with Conjugate Prior (Analytical Results)

- Calibrating the hyperparameters for natural conjugate priors in MATLAB

```
v0=5;
b0=0*ones(k,1);
b0(2,1)=10;
b0(3,1)=5000;
b0(4,1)=10000;
b0(5,1)=10000;
s02=1/4.0e-8;
capv0=2.4*eye(k);
capv0(2,2)=6e-7;
capv0(3,3)=.15;
capv0(4,4)=.6;
capv0(5,5)=.6;
capv0inv=inv(capv0);
```

Exercise 1 (cont.): The Normal Linear Regression Model with Conjugate Prior (Analytical Results)

- Obtain the OLS quantities

```
bols = inv(x'*x)*x'*y;
s2 = (y-x*bols)'*(y-x*bols)/(n-k);
bolscov = s2*inv(x'*x);
bolssd=zeros(k,1);
for i = 1:k
        bolssd(i,1)=sqrt(bolscov(i,i));
end
v=n-k;
```

Exercise 1 (cont.): The Normal Linear Regression Model with Conjugate Prior (Analytical Results)

- Write the posterior hyperparmeters for Normal-Gamma

```
xsquare=x'*x;
v1=v0+n;
capv1inv = capv0inv+ xsquare;
capv1=inv(capv1inv);
b1 = capv1*(capv0inv*b0 + xsquare*bols);
if det(capv0inv)>0
   v1s12 = v0*s02 + v*s2 + (bols-b0)'*inv(capv0 + inv(xsquare))*(bols-b0);
else
   v1s12 = v0*s02 + v*s2;
end
s12 = v1s12/v1;
bcov = capv1*v1s12/(v1-2);
bsd=zeros(k,1);
for i = 1:k
bsd(i,1)=sqrt(bcov(i,i));
end
```

➢ Exercise 2: The Normal Linear Regression Model with Conjugate Prior (Monte Carlo Integration)

- Consider the same regression model as in exercise 1 but perform the estimation with Monte Carlo integration.

- We can implement the Monte Carlo integration if the posterior density has a known form.

- Obtain random draws from a multivariate t distribution for β_2 of the form

$$\beta_2 \mid y \sim t(\overline{\beta}_2, \overline{s}^2 \overline{V}, \overline{v}),$$

- and then average them to get the estimates of

$$E(\beta_2 \mid y) = \overline{\beta}_2, \text{ and } \operatorname{var}(\beta_2 \mid y) = \frac{\overline{v}\,\overline{s}^2}{\overline{v} - 2} \overline{V}.$$

Exercise 2 (cont.): The Normal Linear Regression Model with Conjugate Prior (Monte Carlo Integration)

- Write the hyperparameters for the natural conjugate priors as in exercise 1.
- Get OLS quantities.
- Obtain the posterior hyperparameters for the Normal-Gamma.
- Start Monte Carlo Integration from $\beta_2 \mid y \sim t(\overline{\beta}_2, \bar{s}^2\overline{V}, \bar{v})$.
- Set number of replications to s = 10000;

➢ Exercise 2 (cont.): The Normal Linear Regression Model with Conjugate Prior (Monte Carlo Integration)

- Draw a $t(v_1)$ then transform to yield draw of β

```
for i = 1:s
   bdraw=b1 + vchol*trnd(v1,k,1);
   b2mean=b2mean+bdraw;
   b2square=b2square+bdraw.^2;
end
```

- Average them

```
b2mean=b2mean./s;
b2square=b2square./s;
b2var=b2square - b2mean.^2;
b2sd=sqrt(b2var);
```

Exercise 3: The Normal Linear Regression Model with Independent Norma-Gamma prior (Gibbs Sampling)

- Set the hyperparameter priors for β as in exercise 1 and 2 with

$$\underline{\nu} = 5 \text{ and } \underline{s}^{-2} = 4.0 \times 10^{-8}.$$

- Assuming independent Normal-Gamma priors, the prior variance of β

$$\mathrm{var}(\beta) = \underline{V} = \begin{bmatrix} 10000^2 & 0 & 0 & 0 & 0 \\ 0 & 5^2 & 0 & 0 & 0 \\ 0 & 0 & 2500^2 & 0 & 0 \\ 0 & 0 & 0 & 5000^2 & 0 \\ 0 & 0 & 0 & 0 & 5000^2 \end{bmatrix}.$$

- and by the natural conjugate prior we have

$$\mathrm{var}(\beta) = \frac{\underline{\nu}\underline{s}^2}{\underline{\nu} - 2}\underline{V}.$$

➢ Exercise 3 (cont.): The Normal Linear Regression Model with Independent Norma-Gamma prior (Gibbs Sampling)

- Use the Normal linear regression model.
- The assumption of independent Norma-Gamma prior requires using the Gibbs sampling algorithm.
- Obtain the OLS quantities.

```
%Calculate the following quantities outside the loop for later use
xsquare=x'*x;
v1=v0+n;
v0s02=v0*s02;
```

➢ Exercise 3 (cont.): The Normal Linear Regression Model with Independent Norma-Gamma prior (Gibbs Sampling)

- Start the Gibbs sampling algorithm

```
        %Initialize all draws
b_=[];
h_=[];

        %number of burnin replications
s0=1000;
        %number of retained replications
s1=10000;
s=s0+s1;

        %choose a starting value for h
hdraw=1/s2
```

Exercise 3 (cont.): The Normal Linear Regression Model with Independent Norma-Gamma prior (Gibbs Sampling)

```
hdraw=1/s2;                                            % Choose a starting value for h

for i = 1:s
    capv1inv = capv0inv+ hdraw*xsquare;                % Draw from p(β | y, h)
    capv1=inv(capv1inv);
    b1 = capv1*(capv0inv*b0 + hdraw*xsquare*bols);
    bdraw=b1 + norm_rnd(capv1);

    s12 = ((y-x*bdraw)'*(y-x*bdraw)+v0s02)/v1;         % Draw from p(h | y, β)
    hdraw=gamrnd(.5*v1,2/(v1*s12));

    if i>s0
        b_ = [b_ bdraw];
        h_ = [h_ hdraw];

    end
end
```

➢ Exercise 4 : The AR(p) Model as a Regression Model

- The model follows an AR(3) process with y_t being the log real GDP

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \beta_3 y_{t-3} + \varepsilon_t,\ \ \varepsilon_t \text{ is i.i.d. } N(0, h^{-1}).$$

- Properties of y_t depend on the roots of polynomial

$$B(z) = 1 - \sum_{i=1}^{3} \beta_i z^i .$$

- The roots denoted as r_1, r_2, and r_3 for i = 1, 2, 3.

- We are interested in two features of y_t:

1. y_t exhibits an explosive response to a shock when $C = \{\beta$: Two of r_i are complex$\}$.

2. y_t exhibits an oscillatory response to a shock when $D = \{\beta: \min |r_i| < 1\}$.

➢ Exercise 4 (cont.) : The AR(p) Model as a Regression Model

- We want to compute the posterior distribution mean and standard deviations of β and $\min|r_i|$.

- First of all, write the model as a linear equation in a compact way

$$\underset{(T\times 1)}{y} = \underset{(T-3)\times 4}{X} \; \underset{(k\times 1)}{\beta} + \underset{(T\times 1)}{\varepsilon}$$

- where $x_{t1} = 1$ for $t = 1, \ldots, T$ and $\beta = (\beta_0, \beta_1, \beta_2, \beta_3)'$.

Exercise 4 (cont.): The AR(*p*) Model as a Regression Model

How to translate our model into MATLAB? We work with quarterly U.S. real GDP data set for the period between 1947q1 and 2005q2. The program associated with this problem set performs Monte Carlo integration that runs for 10,000 replications.

```
            %Specify order of AR
p=3;
            %Log and lag raw data to set things up for AR(p) model in logs
            %Chop off p initial conditions and make matrices of lagged variables
y=log(yraw(p+1:traw,1));
ylag=zeros(traw-p,p);
for ii = 1:p
   ylag(:,ii)= log(yraw(p+1-ii:traw-ii,1));
end
t=size(y,1);
x = [ones(t,1) ylag];
k=size(x,2);
```

➢ Exercise 4 (cont.): The AR(p) Model as a Regression Model

- We assume a prior of the form

$$p(\beta_0, \ldots \beta_1, h) \propto h^{-1}.$$

- The posterior distribution for β and h can be written as

$$\beta, h \mid \mathrm{y} \sim NG\left(\overline{\beta}, \overline{V}, \overline{s}^{-2}, \overline{v}\right).$$

- where the parameters of the distribution are OLS quantities specified as

$$\begin{aligned}
\overline{\beta} &= (X'X)^{-1} X'y, \\
\overline{V} &= (X'X)^{-1} \\
\overline{v} &= T - 4, \\
s^2 &= \overline{v}^{-1}(y - X\hat{\beta})'(y - X\hat{\beta}).
\end{aligned}$$

Exercise 4 (cont.): The AR(*p*) Model as a Regression Model

```
%Posterior under noninformative prior is t(bmean,vscale,v)
v=t-k;
xtxinv = inv(x'*x);
bmean=xtxinv*x'*y;
s2 = (y-x*bmean)'*(y-x*bmean)/v;
vscale=s2*xtxinv;
vchol=chol(vscale);
vchol=vchol';
```

➢ Exercise 4 (a, b): AR(p) Model as a Regression Model

- a) We want to compute the posterior distribution mean and standard deviations of β and $\min|r_i|$ numerically.

Parameters	Mean	Standard Deviations
β_0	0.010	0.004
β_1	1.299	0.066
β_2	-0.196	0.108
β_3	-0.104	0.066
$\min\|r_i\|$	1.002	0.002

- b) We want to compute the posterior distribution of $p(\beta \in D \mid y)$ and $p(\beta \in D \mid y)$.

Probability that the series is explosive: 0.135
Probability that the series is oscillatory: 0.024

References

- Geweke, J. (1988), The Secular and Cyclical Behaviour of Real GDP in 19 OECD Countries, 1957-1983, Journal of Business and Economic Statistics, 6, 479–486.

- Koop, G. (2003), Bayesian econometrics, Wiley.

- Koop, G., Poirier, D. and Tobias, J. (2007), Bayesian econometric methods. Cambridge: Cambridge University Press.