On a Two-Sector Model of Economic Growth
Author(s): Hirofumi Uzawa
Source: *The Review of Economic Studies*, Vol. 29, No. 1 (Oct., 1961), pp. 40-47
Published by: Oxford University Press
Stable URL: http://www.jstor.org/stable/2296180

On a Two-Sector Model of Economic Growth[1]

1. In the present paper we are interested in the growth process in a two-sector model of capital accumulation and show that balanced growth equilibria are globally stable under the neoclassical hypotheses.

The neoclassical model of economic growth, as it has been developed by Solow [5] and Swan [6], is formulated in terms of the aggregate production function. The aggregate production function specifies the relationship between output and factors of production, and output is assumed to be composed of homogeneous quantities identical with capital, or at least price ratios between output and capital are assumed constant. The economy we are concerned with in this paper, on the other hand, consists of two types of goods, investment-goods and consumption-goods, to be produced by two factors of production, capital and labor; prices of investment-goods and consumption-goods are determined so as to satisfy the demand requirements.[2] It will be assumed that capital depreciates at a fixed rate, the rate of growth in labor is constant and exogenously determined, capitalists' income is solely spent on investment-goods, that of laborers on consumption-goods, and production is subject to the neoclassical conditions. Under such hypotheses, then, it will be shown that the state of steady growth exists and the growth process, starting at an arbitrary capital and labor composition, approaches some steady growth. If the consumption-goods sector is always more capital-intensive than the investment-goods sector, then the steady growth is uniquely determined and it is stable in the small as well as in the large.

2. We consider an economic system consisting of investment-goods and consumption-goods sectors, labelled 1 and 2, respectively. It is assumed that in both sectors production is subject to constant returns to scale, marginal rates of substitution are positive and diminishing, and there exist neither joint products nor external (dis-)economies.

The production processes in each sector are summarized by specifying each sector's production function; let $F_1(K_1,L_1)$ be the production function for the investment-goods sector, and $F_2(K_2,L_2)$ for the consumption-goods sector. $F_1(K_1,L_1)$ represents the quantity of the investment-goods, Y_1, produced by employing capital and labor by the quantities K_1 and L_1; and similarly for the consumption-goods sector's production function, $F_2(K_2,L_2)$.

In terms of production functions, the assumptions indicated above may be formulated as:

(1) $\quad F_i(\lambda K_i, \lambda L_i) = \lambda F_i(K_i, L_i),\ F_i(K_i, L_i) > 0$, for all $K_i, L_i > 0$, and $\lambda > 0$;

[1] This work was in part supported by the Office of Naval Research under Task NR-047-004. I owe much to Professor Robert M. Solow and the referees for their valuable comments and suggestions.

[2] Shinkai [4] has investigated the structure of growth equilibria in a two-sector model of growth in which technical coefficients are all constant. Our two-sector model presented here is a neoclassical version of Shinkai's model.

(2) $F_i(K_i, L_i)$ is twice continuously differentiable;

(3) $\partial F_i/\partial K_i > 0,\ \partial F_i/\partial L_i > 0,\ \partial^2 F_i/\partial K_i^2 < 0,\ \partial^2 F_i/\partial L_i^2 < 0$ for all $K_i, L_i > 0$.

In view of the constant-returns-to-scale hypothesis (1), the output-labor ratio y_i is a function of the capital-labor ratio k_i:

(4) $y_i = f_i(k_i)$,

where

$$y_i = Y_i/L_i,\ k_i = K_i/L_i,\ f_i(k_i) = F_i(k_i, 1), \quad i = 1, 2.$$

The assumptions (2-3) are then equivalent to:

(5) $f_i(k_i)$ is twice continuously differentiable;

(6) $f_i(k_i) > 0,\ f_i'(k_i) > 0,\ f_i''(k_i) < 0$, for all $k_i > 0$.

3. Let K and L be the aggregate quantities of capital and labor at time t; these quantities of the two factors of production are allocated competitively among the two sectors, and prices of goods are determined so as to satisfy the demand conditions. In what follows, we assume that both capital and labor are always fully employed and both goods are produced in positive quantities.

Let K_i and L_i be the quantities of capital and labor allocated to the i-th sector, P_1 and P_2 the price of investment-goods and of consumption goods, and r and w the returns to capital and the wage rate, respectively. Then we have,

(7) $Y_i = F_i(K_i, L_i)$,

(8) $$P_i \frac{\partial F_i}{\partial K_i} = r,\quad P_i \frac{\partial F_i}{\partial L_i} = w, \quad i = 1, 2.$$

(9) $K_1 + K_2 = K,\quad L_1 + L_2 = L$,

(10) $P_1 Y_1 = rK, \qquad P_2 Y_2 = wL.$

The condition (8) is familiar marginal productivity conditions, and (10) formulates the hypothesis that labor does not save and capital does not consume.

Let

$$\begin{aligned} k &= K/L \\ k_i &= K_i/L_i,\ y_i = Y_i/L_i, \quad \rho_i = L_i/L, \quad i = 1, 2. \\ \omega &= w/r. \end{aligned}$$

Then conditions (7-10) may be reduced to:

(11) $y_i = f_i(k_i), \quad i = 1, 2.$

(12) $$\omega = \frac{f_i(k_i)}{f_i'(k_i)} - k_i, \quad i = 1, 2.$$

(13) $$\rho_1 k_1 + \rho_2 k_2 = k,$$

(14) $$\rho_1 + \rho_2 = 1,$$

(15) $$\rho_1 f_1(k_1) = f_1'(k_1)k.$$

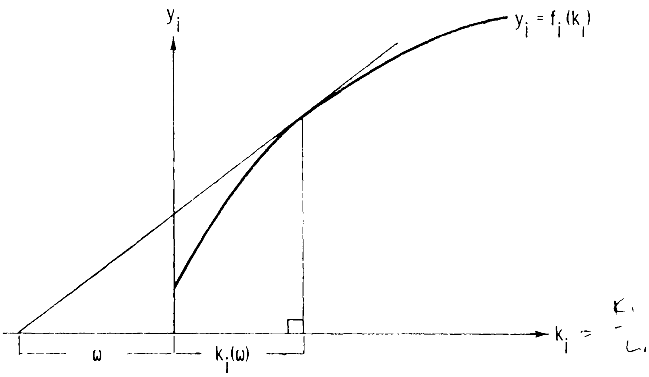
Differentiating (12) with respect to k_i, we have:

(16) $$\frac{d\omega}{dk_i} = \frac{-f(k_i) f_i''(k_i)}{[f_i'(k_i)]^2}$$

which is always positive in view of (6). Hence: *For any wage-rentals ratio* ω, *the optimum capital-labor ratio* k_i *in each sector is uniquely determined by the relation* (12), *provided*:

(17) $$\underline{\omega}_i = \lim_{k_i \to 0} \left[\frac{f_i(k_i)}{f_i'(k_i)} - k_i\right] < \omega \quad \bar{\omega}_i = \lim_{k_i \to \infty} \left[\frac{f_i(k_i)}{f_i'(k_i)} - k_i\right]$$

The optimum capital-labor ratio k_i corresponding to the wage-rentals ratio ω, uniquely determined by (12), will then be denoted by $k_i = k_i(\omega)$, $i = 1, 2$. The determination of the optimum capital-labor ratio $k_i(\omega)$ may be illustrated by the diagram:



We have from (16) that:

(18) $$\frac{dk_i}{d\omega} = \frac{[f_i'(k_i)]^2}{-f_i(k_i)f_i''(k_i)} > 0,$$
with $k_i = k_i(\omega)$, $i = 1, 2$.

In view of (12), the labor allocation ρ_1 to the investment-goods sector, determined by (15), may be written as:

(19) $$\rho_1 = \frac{k}{\omega + k_1(\omega)}$$

Substituting (14) and (19) into (13) and rearranging, we have

$$k = \frac{\omega + k_1(\omega)}{\omega + k_2(\omega)} k_2(\omega) \tag{20}$$

The equilibrium wage-rentals ratio ω is obtained by solving the equation (20).

4. Let the rate of growth in labor be a positive constant, say λ, and μ be the instantaneous rate of depreciation in capital. Then the growth process in the two-sector model we have described is formulated by the following differential equations:

$$\frac{\dot{K}}{K} = \frac{r}{P_1} - \mu, \tag{21}$$

and

$$\frac{\dot{L}}{L} = \lambda, \tag{22}$$

where r is the equilibrium return to capital and P_1 the equilibrium price of the investment-goods, both at time t.

The equations (21) and (22), together with the equilibrium condition (8), imply that:

$$\frac{\dot{k}}{k} = f_1'(k_1) - \lambda - \mu, \tag{23}$$

where $k_1 = k_1(\omega)$ and ω is an equilibrium wage-rentals ratio corresponding to the aggregate capital-labor ratio.

An aggregate capital-labor ratio k^* may be termed a *balanced capital-labor ratio* if

$$f_1'(k_1^*) = \lambda + \mu, \tag{24}$$

where $k_1^* = k_1(\omega^*)$ and ω^* is an equilibrium wage-rentals ratio corresponding to the aggregate capital-ratio k^*.

It is easily shown that at the growth process starting at a balanced capital-labor ratio k^*, the aggregate capital-labor ratio $k(t)$ and equilibrium wage-rentals ratio $\omega(t)$ both remain constant.

5. Suppose that the consumption-goods sector is always more capital-intensive than the investment-goods sector;[1] namely,

$$k_1(\omega) < k_2(\omega), \text{ for all } \omega \text{ such that } \max[\underline{\omega}_1, \underline{\omega}_2] < \omega < \min[\bar{\omega}_1, \bar{\omega}_2]. \tag{25}$$

Let

$$\Psi(\omega) = \frac{\omega + k_1(\omega)}{\omega + k_2(\omega)} k_2(\omega).$$

[1] The concept of relative factor intensities was introduced by Samuelson in the context of international trade theory; see, e.g., [3], p.9.

Differentiating $\log \Psi(\omega)$ with respect to ω, we have

$$\frac{1}{\Psi(\omega)} \frac{d\Psi}{d\omega} = \frac{1 + \frac{dk_1}{d\omega}}{\omega + k_1(\omega)} - \frac{1 + \frac{dk_2}{d\omega}}{\omega + k_2(\omega)} + \frac{\frac{dk_2}{d\omega}}{k_2(\omega)}$$

$$= \left[\frac{1}{\omega + k_1(\omega)} - \frac{1}{\omega + k_2(\omega)}\right] + \frac{\frac{dk_1}{d\omega}}{\omega + k_1(\omega)}$$

$$+ \frac{dk_2}{d\omega}\left[\frac{1}{k_2(\omega)} - \frac{1}{\omega + k_2(\omega)}\right],$$

which, by (18) and (25), is always *positive*.

Therefore, we have

(26) $\frac{d\Psi}{d\omega} > 0$, for all ω satisfying $\max\,[\underline{\omega}_1, \underline{\omega}_2] < \omega < \min\,[\bar{\omega}_1, \bar{\omega}_2]$.

The equation (20) has a positive solution ω if and only if:

(27) $\Psi(0) < k < \Psi(\infty)$,

and the solution ω is uniquely determined by k. The equilibrium factor-price ratio ω may be denoted by $\omega = \omega(k)$.

From (20) and (26), we have

(28) $\frac{d\omega}{dk} > 0$ for all k satisfying (27).

In view of conditions (6), (18), and (28), the function

$$f_1'[k_1(\omega(k))]$$

is a strictly decreasing function of the aggregate capital-labor ratio k.

Hence, the balanced capital-labor ratio k^* always exists and is uniquely determined if the following condition is satisfied :

(29) $\lim\limits_{k_1 \to 0} f_1'(k_1) > \lambda + \mu > \lim\limits_{k_1 \to \infty} f_1'(k_1)$.

It is easily shown that, for the growth process starting at an arbitrary initial capital-labor composition, the capital-labor ratio $k(t)$ approaches the balanced capital-labor ratio k^*.[1]

The results in this section may be summarized as:

Existence Theorem: Let the consumption-goods sector be more capital-intensive than the investment-goods sector for all relevant factor-price ratios ω. *Then for any given aggregate capital-labor ratio* k, *the equilibrium factor-price ratio* $\omega = \omega(k)$, *the optimum capital-labor ratios* $k_1 = k_1(\omega)$ *and* $k_2 = k_2(\omega)$ *in both sectors, and the equilibrium outputs per head for investment-goods and consumption-goods,* $y_1 = y_1(k)$ *and* $y_2 = y_2(k)$, *are all uniquely determined, provided the aggregate capital-labor ratio* k *satisfies the relation* (29).

Stability Theorem: Let λ *and* μ *be respectively the growth rate in labor and the instantaneous depreciation in capital; and the balanced capital-labor ratio* k^* *exist. Then, for the growth process starting at an arbitrary initial position, the capital-labor ratio* $k(t)$ *approaches the balanced capital-labor ratio* k^* *as* t *tend to infinity.*

6. The uniqueness of the balanced capital-labor ratio and its stability crucially hinge on the hypothesis that the consumption-goods sector be more capital-intensive than the investment-goods sector. In this section, we shall construct an example of the two-sector growth model in which the capital-intensity hypothesis above is not satisfied and there is an unstable balanced capital-labor ratio.

Let the production functions be:

$$y_1 = f_1(k_1) = \tfrac{1}{1000}\,(k_1^{-3} + 7^{-4})^{\frac{1}{3}},\ y_2 = f_2(k_2) = (k_1^{-3} + 1)^{\frac{1}{3}}.$$

The optimum capital-labor ratios are then given by:

$$k_1 = k_1(\omega) = 7\omega^{\frac{1}{4}},\ k_2 = k_2(\omega) = \omega^{\frac{1}{4}};$$

hence,

$$k_1(\omega) > k_2(\omega), \text{ for all } \omega > 0.$$

For aggregate capital-labor ratio k the equilibrium factor-price ratio ω is determined by:

$$k = \frac{\omega + 7\omega^{\frac{1}{4}}}{\omega + \omega^{\frac{1}{4}}}\,\omega^{\frac{1}{4}};$$

hence,

$$\frac{1}{k}\frac{d\omega}{dk} = \frac{1 + \frac{7}{4}\omega^{-\frac{3}{4}}}{\omega + 7\omega^{\frac{1}{4}}} - \frac{1 + \frac{7}{4}\omega^{-\frac{3}{4}}}{\omega + \omega^{\frac{1}{4}}} + \frac{\omega}{\frac{1}{4}}.$$

Let us consider the case in which the sum $\lambda + \mu$ of the rate of growth in labor and the rate of depreciation is

$$\frac{7\sqrt[3]{7}}{1600} \doteqdot .8\%.$$

[1] See, e.g. Arrow and Hurwicz [2], p. 540.

Then $k^* = 4$ is a balanced capital-labor ratio and $\omega^* = 1$ is the corresponding wage-rentals ratio.

But

$$\left(\frac{1}{k} \; \frac{dk}{d\omega}\right)_{\omega=1} = -\tfrac{1}{32} < 0;$$

hence, the balanced capital-labor ratio $k^* = 4$ is not stable.

7.[1] Let us now consider the general case in which the capital-intensity hypothesis is not necessarily satisfied. In this case, the balanced capital-labor ratio may be no longer uniquely determined for given rates of labor growth and of depreciation; hence, there may exist unstable balanced capital-labor ratios, as was discussed in the previous section.

If, however, the conditions (29) and

$$(30) \quad f_1(0) = 0, \; f_2(0) = 0,$$

are satisfied, then it is possible to show that the growth process represented by (21) and (22) is *globally stable* in the sense introduced by Arrow, Block and Hurwicz ([2], p. 85); namely, given any initial condition, the aggregate capital-labor ratio $k(t)$ converges to some balanced capital-labor ratio.

To see the global stability of the process (23), it suffices to show that[2]

$$(31) \quad \lim_{k \to \infty} [f_1'(k_1) - \lambda - \mu] < 0,$$

$$(32) \quad \lim_{k \to 0} [f_1'(k_1) - \lambda - \mu] < 0.$$

The relation (31) may be seen from the assumption (29) and the inequality:

$$k < \omega + k_1$$

which is derived from (20). On the other hand, to see the relation (32), let k tend to zero. Then the corresponding wage-rentals ratio ω converges to zero also; otherwise, the relation (20) would imply

$$0 = \frac{\bar{\omega} + k_1(\bar{\omega})}{\bar{\omega} + k_2(\bar{\omega})} \, k_2(\bar{\omega})$$

for some positive wage-rentals ratio $\bar{\omega}$, contradicting the assumption (30). Hence the corresponding capital-labor ratio $k_1 = k_1(\omega)$ converges to zero, again in view of (30). The relation (32) is then implied by the condition (29).

[1] This section has been written after I have read Professor Solow's note which suggests that the stability property of the growth equilibrium as discussed in the present paper may not depend on the capital-intensity hypothesis.

[2] See, e.g., Arrow and Hurwicz [2], p. 540.

We may summarize our results as:

Let the growth rate in labor λ and the depreciation rate in capital μ satisfy the conditions (29) *and* (30). *Then there exists at least one balanced capital-labor ratio, and, for the growth process starting an arbitrary initial capital-labor composition, the aggregate capital-labor ratio $k(t)$ converges to some balanced capital-labor ratio.*

Stanford. HIROFUMI UZAWA.

REFERENCES

[1] Arrow, K. J., H. D. Block, and L. Hurwicz. " On the Stability of the Competitive Equilibrium, II ", *Econometrica*, Vol. 27 (1959), pp. 82-109.

[2] Arrow, K. J., and L. Hurwicz. " On the Stability of the Competitive Equilibrium, I ", *Econometrica*, Vol. 27 (1958), pp. 522-552.

[3] Samuelson, P. A. " Prices of Factors and Goods in General Equilibrium," *Review of Economic Studies*, Vol. 21 (1953-54), pp. 1-20.

[4] Shinkai, Y. " On the Equilibrium Growth of Capital and Labor," *International Economic Review*, Vol. 1 (1960), pp. 107-111.

[5] Solow, R. M. " A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 70 (1956), pp. 65-94.

[6] Swan, T. W. " Economic Growth and Capital Accumulation," *Economic Record* Vol. 32 (1956), pp. 334-361.