Stationarized System with Spillover

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$$y_{c,t} = \Gamma_{c,t} k_{i,t}^{\gamma} h_{1,t}^{1-a-b} k_{c,1,t}^{a} k_{i,1,t}^{b} \tag{1}$$

$$y_{i,t} = \Gamma_{i,t} k_{i,t}^{\gamma} h_{2,t}^{1-a-b} k_{c,2,t}^{a} k_{i,2,t}^{b} \tag{2}$$

$$k_{c,t+1} \exp(g_c) = (1-\delta_c) k_{c,t} + i_{c,t} \tag{3}$$

$$k_{i,t+1} \exp(g_i) = (1-\delta_i) k_{i,t} + i_{i,t} \tag{4}$$

$$y_{c,t} = c_t + i_{c,t} \tag{5}$$

$$y_{i,t} = i_{i,t} \tag{6}$$

$$w_t = \chi c_t \tag{7}$$

$$k_{c,t} = k_{c,1,t} + k_{c,2,t} \tag{8}$$

$$k_{i,t} = k_{i,1,t} + k_{i,2,t} \tag{9}$$

$$h_t = h_{1,t} + h_{2,t} \tag{10}$$

$$1 = \beta\, \mathbb{E}\left[(\exp(g_c))^{-1} \frac{c_t}{c_{t+1}} \left(r_{c,t+1} + 1 - \delta_c \right) \right] \tag{11}$$

$$1 = \beta\, \mathbb{E}\left[(\exp(g_i))^{-1} \frac{c_t}{c_{t+1}} \left(\frac{r_{i,t+1}}{p_t} + 1 - \delta_i \right) \right] \tag{12}$$

$$w_t = (1-a-b) \Gamma_{c,t} k_{i,t}^{\gamma} h_{1,t}^{-a-b} k_{c,1,t}^{a} k_{i,1,t}^{b} \tag{13}$$

$$r_{c,t} = a \Gamma_{c,t} k_{i,t}^{\gamma} h_{1,t}^{1-a-b} k_{c,1,t}^{a-1} k_{i,1,t}^{b} \tag{14}$$

$$r_{i,t} = b \Gamma_{c,t} k_{i,t}^{\gamma} h_{1,t}^{1-a-b} k_{c,1,t}^{a} k_{i,1,t}^{b-1} \tag{15}$$

$$\frac{w_t}{p_t} = (1-a-b)\Gamma_{i,t} k_{i,t}^{\gamma} h_{2,t}^{-a-b} k_{c,2,t}^{a} k_{i,2,t}^{b} \tag{16}$$

$$\frac{r_{c,t}}{p_t} = a\Gamma_{i,t} k_{i,t}^{\gamma} h_{2,t}^{1-a-b} k_{c,2,t}^{a-1} k_{i,2,t}^{b} \tag{17}$$

$$\frac{r_{i,t}}{p_t} = b\Gamma_{i,t} k_{i,t}^{\gamma} h_{2,t}^{1-a-b} k_{c,2,t}^{a} k_{i,2,t}^{b-1} \tag{18}$$

Steady-State Procedure

Step 0

Divide 13 over 16 and obtain

$$p = \frac{\Gamma_c}{\Gamma_i}. \tag{19}$$

Solve for r_c in 11 and obtain

$$r_c = 1/\beta \exp(g_c) - (1-\delta_c) \tag{20}$$

where $\exp(g_c) = \Gamma_c^{\frac{1-b-\gamma}{1-a-b-\gamma}} \Gamma_i^{\frac{b+\gamma}{1-a-b-\gamma}}$.

Solve for r_i in 12 and obtain

$$r_i = 1/\beta \exp(g_i) - (1-\delta_i) \tag{21}$$

where $\exp(g_i) = \Gamma_c^{\frac{a}{1-a-b-\gamma}} \Gamma_i^{\frac{1-a}{1-a-b-\gamma}}$.

Step 1

Solve 13, 14, and 15 for k_i, $\bar{k}_c = \frac{k_{c,1}}{h,1}$, and $\bar{k}_i = \frac{k_{i,1}}{h,1}$ in function of unknown wage w. It yields

$$\bar{k}_c = \frac{a}{1-a-b} \frac{w}{r_c}, \tag{22}$$

$$\bar{k}_i = \frac{b}{1-a-b} \frac{w}{r_i}, \tag{23}$$

and

$$k_i = \left[\frac{r_c}{a\Gamma_c} \bar{k}_c^{1-a} \bar{k}_i^{-b} \right]^{\frac{1}{\gamma}} \tag{24}$$

Moreover, divide 14 over 15 and obtain

$$\frac{\bar{k}_i}{\bar{k}_c} = \frac{b}{a}\frac{r_c}{r_i} \tag{25}$$

Finally, using 13-18 notice that

$$\bar{k}_c = \frac{k_{c,1}}{h_1} = \frac{k_{c,2}}{h_2} = \frac{k_c}{h},$$

$$\bar{k}_i = \frac{k_{i,1}}{h_1} = \frac{k_{i,2}}{h_2} = \frac{k_i}{h},$$

and

$$\frac{\bar{k}_i}{\bar{k}_c} = \frac{k_i}{k_c} = \frac{k_{i,1}}{k_{c,1}} = \frac{k_{i,2}}{k_{c,2}}.$$

Step 2

Obtain from 18

$$i_i = G_i k_i \tag{26}$$

where $G_i = \exp(g_i) - (1 - \delta_i)$.

Substitute 26 into 6 and then into 2 to obtain

$$G_i k_i = \Gamma_i k_i^{\gamma} h_2^{1-a-b} k_{c,2}^a k_{i,2}^b$$

which is

$$G_i k_i = \Gamma_i k_i^{\gamma} h_2 \bar{k}_{c,2}^a \bar{k}_{i,2}^b$$

Isolate over h_2 and get

$$h_2 = \frac{G_i}{\Gamma_i} k_i^{1-\gamma} \bar{k}_{c,2}^{-a} \bar{k}_{i,2}^{-b} \tag{27}$$

which yields

$$k_{i,2} = \bar{k}_i h_2 \quad \text{and} \quad k_{c,2} = \bar{k}_c h_2.$$

Step 3

Obtain from 15

$$i_c = G_c k_c \tag{28}$$

where $G_c = \exp(g_c) - (1 - \delta_c)$.

Substitute 7 and 28 into 5 and then into 1 to obtain

$$\frac{w}{\chi} + G_c k_c = \Gamma_c k_i^{\gamma} h_1^{1-a-b} k_{c,1}^{a} k_{i,1}^{b}$$

which is

$$\frac{w}{\chi} + G_c k_c = \Gamma_c k_i^{\gamma} h_1 \bar{k}_{c,1}^{a} \bar{k}_{i,1}^{b}$$

adjust as follows

$$\frac{w}{\chi} + G_c \left(\frac{k_i}{k_c}\right)^{-1} k_i = \Gamma_c k_i^{\gamma} h_1 \bar{k}_{c,1}^{a} \bar{k}_{i,1}^{b}$$

Isolate over h_1 and get

$$h_1 = \left[\frac{w}{\chi} + G_c \left(\frac{k_i}{k_c}\right)^{-1} k_i\right] \frac{1}{\Gamma_c} k_i^{-\gamma} \bar{k}_c^{-a} \bar{k}_i^{-b} \tag{29}$$

which yields

$$k_{i,1} = \bar{k}_i h_1 \quad \text{and} \quad k_{c,1} = \bar{k}_c h_1.$$

Step 4

Complete the missing variables such as

$$h = h_1 + h_2,$$

$$k_c = k_{c,1} + k_{c,2},$$

$$y_c = \Gamma_c k_i^{\gamma} h_1^{1-a-b} k_{c,1}^{a} k_{i,1}^{b},$$

$$y_i = \Gamma_i k_i^{\gamma} h_2^{1-a-b} k_{c,2}^{a} k_{i,2}^{b},$$

$$i_i = G_i k_i,$$

$$i_c = G_c k_c,$$

$$c = y_c - I_c.$$

Step 5

Solve either numerically or analytically

$$f(w) = w^* - \chi c(w^*) = 0$$

Step 6

Get back to Step 1 but know $w = w^*$ is known.