Sunspots, Coordination, and Innovation Cycles

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Abstract

I propose a theoretical mechanism that offers a possible explanation for the post-Great Recession stagnation experience observed in many industrialized countries. The theory is based on failures in the coordination of the beliefs of productive and innovative agents. In the model, an innovative sector determines endogenously the long-run growth rate of the economy, and research exhibits positive spillovers. Failing to internalize the general equilibrium effects of their innovation decisions, agents can coordinate into pervasive equilibria in which no innovation occurs, which leads the economy to growing through lower- and older-quality production for as long as pessimism persists. Through simulations, I show that the model can generate patterns that are qualitatively similar to those experienced by the U.S. output and private R&D investment since the 2008 financial crisis. In particular, output drops on impact due to the endogenously generated, belief-driven recession, and the subsequent recovery is sluggish and settles around a trend that is both lower and flatter than its pre-crisis level.

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1 Introduction

In the recent years following the 2008 financial crisis, both the U.S. and most economies in the Euro area have experienced recoveries that have been characterized by low rates of potential output growth and persistently slow convergence to pre-crisis trend levels. The upper panel of Figure 1 plots the time path for real quarterly GDP in the U.S. for the period 1970:Q1 – 2013:Q4 and illustrates the historically unique nature of the Great Recession in at least three dimensions: an exceptionally large drop below long-run levels on impact, a slow recovery, and a perduring deviation from trend (a phenomenon sometimes called "secular stagnation", after Summers (2013) and Krugman (2013)).

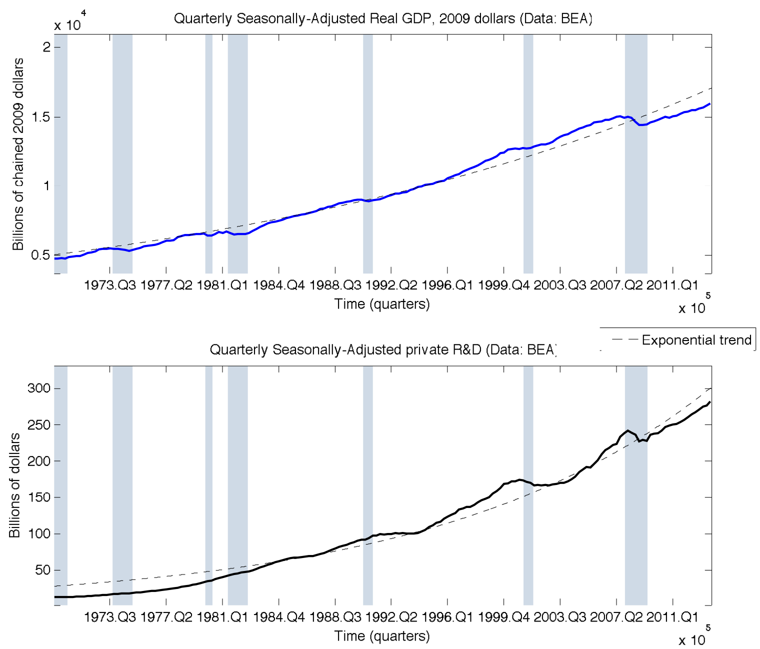

Figure 1: Real quarterly GDP (upper panel) and private R&D investment (lower panel) for 1970:Q1 – 2013:Q4, with their respective exponential trends. Shaded columns represent NBER recessions. Source: Bureau of Economic Analysis (BEA), National Income and Product Accounts.

During this slow recovery, and due to the increase in external financing needs in a context of tight financial constraints, firms failed to coordinate their efforts into conducting growth-enhancing activities and, as a result, the innovative sector, one of the main engines of economic growth, remained depressed below trend levels as well. The lower panel of Figure 1 depicts private Research and Development (R&D) real investment spending in the U.S. at the quarterly frequency for the same time period. Firstly, R&D spending co-moves with the cycle: in recessions, firms are discouraged from investing into making new discoveries, for these activities typically take time and have highly uncertain returns. This in turn slows down innovation and further depresses output growth. When negative shocks are strong enough to make pessimism persist, the pervading effects can be amplified through time. In this sense, and secondly, a persistent deviation from trend has also occurred in the market of ideas in the recent years, as clearly apparent from the figure. During this process, innovators' pessimism about future economic outcomes eroded investment incentives, making the research sector fail to internalize the spillover gains from innovation, lowering potential output growth and creating a feedback loop between low growth and low expectations. To the extent that innovative capacities suffer from depreciation over this process, the miscoordination problem would have led the economy to the low-productivity-based recovery and the ensuing convergence to a trend that, relative to its pre-crisis counterpart, has both shifted downward and become flatter.

Arguably, the expectational channel was particularly important during the last recession. As shown by several measures of consumer confidence, e.g. the Index of Consumer Sentiment[1], the aftermath of the 2008 financial crisis was characterized by notoriously depressed expectations about future economic outcomes and a slow-moving long-term recovery of confidence that, to date, still remains below pre-crisis levels (see Figure 2). In this sense, the sentiment dimension of the Great Recession and its undoing is of unprecedented magnitude relative to previous recessions in the U.S.

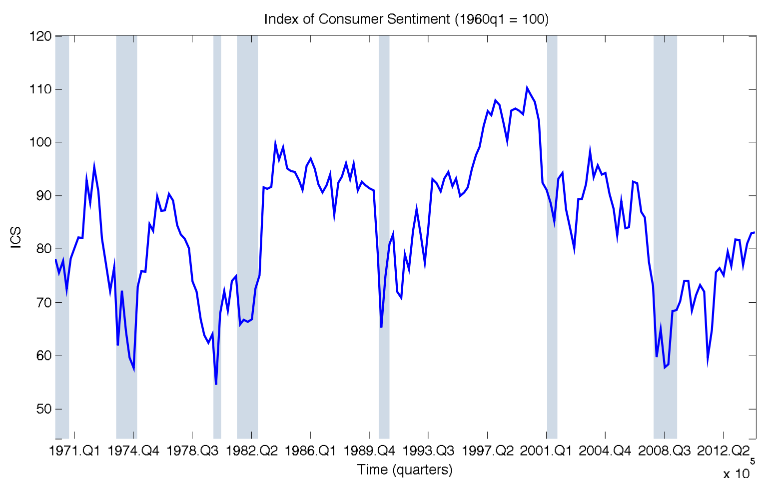

Figure 2: University of Michigan's Index of Consumer Sentiment (quarterly).
Source: http://data.sca.isr.umich.edu/tables.php.

Accordingly, this paper proposes a belief-based theoretical explanation for the large drop, the slow convergence and the non-trend-reversion features behind the recovery process in the post-crisis era. The argument hinges on the coordinative nature of economic activity. In particular, because economies grow endogenously through their innovative capacity, I focus on coordination failures that originate in the innovation sector. In the model, overlapping generations of monopolistically competitive producers of different idiosyncratic productivities and potentially facing binding credit constraints can invest in R&D to improve the quality of the capital with which they operate. Investing in R&D generates a probability of successfully implementing an idea. Implementation then

[1]The Index of Consumer Sentiment (ICS) is a monthly index constructed by the University of Michigan from a representative survey based on telephonic household interviews made throughout continental United States. It seeks to reflect how consumer view their own financial situation, and both the short-term and long-run prospects of the economy.

advances quality through a ladder as in the classic models of Grossman and Helpman (1991) and Aghion and Howitt (1992). A point of departure from these models is the detailed exploration of how positive aggregate spillovers in the research sector can stand as a source of strategic complementarity that opens the door to self-fulfilling equilibria and, in turn, to equilibrium selection over the business cycle.

In particular, there exist two types of equilibria. In one type of equilibrium, innovators are optimistic and coordinate into collectively employing specialized labor, which can give successful innovators access to more productive technologies. This boosts output productivity and ultimately renders firms' expectations self-fulfilling. Due to non-linearities in the innovation technology, there are potentially more than one such situations that can be sustained in rational expectations. In another type of equilibrium, which is unique, innovators' prior expectations are pessimistic and agents choose to produce at current capital-quality levels. In this perverse equilibrium, innovators fail to benefit from each others' ideas, and collectively neglect the spillover effects of their own research on the growth rate of output. This contemporaneously leads the economy to a low equilibrium, in line with their prior expectations. When this occurs, and for as long as these agents keep being pessimistic about real economic outcomes, the quality of the products with which they can produce is eroded (e.g. due to obsolescence or to other damaging conditions within the depressed economy), which also lowers the output's potential long-run growth levels. This long-run effect gets aggravated the longer prospects remain depressed, and is reinforced by the fact that market frictions (in the form of monopolistic power and collateral constraints) create misallocation of productive factors and prevent the economy from reverting back quickly. When the coordination failure is eventually resolved and research can resume, the resources with which agents produce have fallen down through the ladder, and as the recovery unfolds, innovative efforts and consequently output levels remain below the pre-crisis trend for a potentially long period of time. Moreover, because agents can potentially coordinate into different optimistic equilibria, the post-crisis experience may differ in terms of potential output growth, as well, and steady-state trends can change in slope.

Through the lens of this stylized theory, the three unique features of the Great Recession and the ensuing deviation that have been listed above can be thought of as being the result of a particularly negative and pervasive shock to firms' confidence. While not aiming at offering a quantitative evaluation of the output costs of the crisis, the model can qualitatively replicate its patterns. On the one hand, the initial drop in output is caused by a direct loss of confidence, whether or not it is in response to a negative exogenous shock in the productivity of the innovative sector. Secondly, the prolonged existence of pessimism among innovators accounts for the slow convergence observed in the data. Finally, the ultimate long-run deviation from pre-shock trends is accounted for by the fact that, during the period of the slump, low growth in productivity is insufficient to reverse the depreciation of qualities across different products, which has fallen down the ladder. This may in turn make agents more prone to coordinate on lower-growth optimistic-equilibrium paths, and affect steady-state growth rates as well.

The remainder of the paper is organized as follows. Section 2 offers a review of related literatures and details the dimensions in which the present paper extends them. Section 3 presents the model and derives the multiplicity result. Section 4 focuses on the potential sources of market inefficiency and shows that the economy is isomorphic to a representative-agent growth model with market frictions and endogenous TFP. Section 5 provides results for simulation experiments in the sprit of qualitatively replicating some of the patterns visible in the data during the aftermath of the Great Recession, and in particular we examine closely the interaction between the persistence of beliefs and the persistence of crises, both in terms of short-run responses and long-run trends. Section 6 concludes by proposing avenues for future research. Appendices at the end includes all proofs, additional results, and some additional figures.

2 Literature review

Procyclicality of R&D and losses during recessions This paper argues that the loss in potential output during the Great Recession can be viewed as a feedback loop between the erosion in the incentives to innovate during the economic downturn and the downturn itself, together with a depreciation of the intangible capital in the economy. There is evidence, empirical as well as theoretical, in both these dimensions.

First, the endogenous growth literature has now widely recognized that R&D moves procyclically in the business cycle[2]. The early theories of endogenous growth by Schumpeter (1934) stressed that recessions prepare the ground for new technological improvements, as there is competition among innovators in search of new ideas that will render their rivals' obsolete, a process known as "creative destruction". According to this view, innovation should be highest in recessions for those are the periods in which the opportunity costs of foregone production are lowest. However, since the formalization of the endogenous growth literature in the early nineties[3], multiple theories have cast this view into doubt. The contributions of Barlevy (2007) and Comin and Gertler (2006), among others, argue that R&D expenditure goes up in booms because the strongly procyclical private rents associated to the implementation of ideas create a dynamic externality through which innovators prefer to implement their discoveries while they are still profitable. Matsuyama (1999) uses a similar insight to show that short-lived monopolistic rents generate cycles through lower factor prices that open up a stock of idle innovations.

Some theories, starting with Aghion and Saint-Paul (1993) and Stiglitz (1993), suggest instead that R&D spending is lower in recessions because those are times in which financial frictions are tightest. In the paper by Aghion, Angeletos, Banerjee, and Manova (2010), long-term investments (such as R&D) are countercyclical because they are exposed to future liquidity shocks, which are more frequent in downturns. If liquidity risk is high enough, this effect can overturn the opportunity cost effect mentioned above, and break Schumpeter's countercyclicality prediction. Moreover, albeit through different mechanisms, Garcia-Macia (2014) and Schmitz (2015) both show that the increased cost of external financing in recessions acts as an amplification mechanism

[2]For excellent surveys on endogenous growth theory, and particularly on neo-Schumpeterian models, see Acemoglu (2009) and Aghion and Howitt (1998, 2009).

[3]Most importantly, Romer (1986, 1990), Grossman and Helpman (1991), Aghion and Howitt (1992), and Jones (1995).

for innovative firms during times of financial distress.

A different strand of the literature views procyclicality as stemming from the gains of delaying implementation. In the paper by Shleifer (1986), innovative firms create ideas at different points in time but wait for an expansion to implement en-masse, which allows them to take advantage of the boom in aggregate demand. Whereas this clustering occurs in his framework because imitation limits the longevity of monopoly profits, in the theory by Francois and Lloyd-Ellis (2003) entrepreneurs implement when the economy is in a non-depress state in order to delay the dissemination of knowledge and maximize their reign of incumbency. This induces a negative relationship between long-run growth and output volatility, as observed in the empirical results by Ramey and Ramey (1995).

In my theory, R&D is lower in recessions because those are times in which entrepreneurial confidence is lowest. The combined impact of depressed expectations and negative shocks to productivity can induce permanents long-run deviations in growth. With the exception of Francois and Lloyd-Ellis (2003), none of the aforementioned papers studies the possibility of obtaining cycles from Keynesian "animal spirits" that originate within the research sector, a dimension that the present paper focuses on in depth.

Moreover, in my model, financial frictions are used to create tighter credit conditions in crises, as these are likely to have fed back into the vicious loop between low growth and depressed expectations that generated persistence during the recovery. There is evidence that innovative firms are constrained by the financial market conditions according to which they operate. For instance, Brown and Petersen (2011) and Brown, Martinsson, and Petersen (2012) document for the U.S. and for Europe that it is costly for innovative firms to adjust their R&D investment in response to transitory shocks, that these firms tend to prefer financing through liquid equity and internal debt rather than by external funds, and that the impact of financial factors appears to be more relevant among younger and smaller firms, as these firms rely more on external debt. For a panel of French companies, Aghion, Askenazy, Berman, Cette, and Eymard (2012) show that the relation between credit constraints and sales is asymmetric over the business cycle: in more credit constrained firms, R&D investment share plummets during recessions but does not increase proportionally during upturns. Moreover, these firms appear to be unable to catch up during expansions, which would suggest that tighter credit might indeed decrease average productivity growth levels, and would serve as a possible explanation for the overall departure from trend in the recent years in this and other industrialized countries.

There also exists abundant evidence on the cyclical properties of the depreciation of intangible capital, a fact that my theory will exploit. A long tradition starting with the extensive work by Zvi Griliches since the sixties has studied the empirical connexion between research capital and growth, and has attempted to quantify the social value of research as well as the depreciation rate of intangible capital[4]. More generally, though specifically during the recent crisis in the U.S., several studies, including Christiano, Eichenbaum, and Trabandt (2014), Hall (2011), and Ball

[4] Although later studies like Hall (2007) show that estimates vary greatly depending on the methodology, an estimate that is commonly used for the depreciation rate is 15%, which can be regarded as a lower bound.

(2014), demonstrate quantitatively that there indeed have been profound losses in potential output and very persistent trend-deviations in TFP. Fernald (2014) argues that the inception of this severe slowdown in productivity growth in fact pre-dates the onset of the financial crisis itself, and that it originated in industries that produced or used information technology (IT) intensively. This, he argues, can be seen as the flip side of the contribution of these sectors to TFP in the early 2000s. Overall, these observations are consistent with the qualitative predictions of my model, as we will see, which can account for permanent TFP losses in downturns through the combined action of misallocation in capital due to imperfect competition and a financial friction in the credit market, and coordination failures in an environment of depressed real conditions.

Other related literature This paper is related to the aforementioned works in that it proposes a take on the slow recovery of R&D investment that is based on the impact that depressed aggregate demand has in a context of tight credit frictions and pessimistic beliefs. In leaving room for self-fulfilling prophecies, my work also relates to the literature on indeterminacy, increasing returns and business cycles driven by confidence shocks and sunspots, including the early contributions by Diamond (1982) and Benhabib and Farmer (1994, 1996), and more recently Angeletos and La'O (2013), Angeletos, Collard, and Dellas (2014) and Benhabib, Wang, and Wen (2015). The papers of Matsuyama (2007), Benhabib and Wang (2013), and Liu and Wang (2014) show, for instance, that binding credit constraints can themselves be a source of indeterminacy, and therefore not only amplify shocks as in the financial accelerator literature but also act as a fundamental source of endogenous volatility and generate, for instance, countercyclical markups[5]. Finally, Benhabib and Perli (1994), Benhabib, Perli, and Xie (1994), and Benhabib (2014) show that the textbook endogenous growth models of Lucas (1988), Romer (1990), and Aghion and Howitt (1992), respectively, can exhibit both local and global indeterminacy due to complementarities between intermediate inputs, under mild parametric conditions. My model uses similar ideas, but the multiplicity emerges due to direct spillovers in the returns of the production of ideas.

Two papers with a close theoretical interpretation of stagnation crises to the present paper are Schaal and Taschereau-Dumouchel (2015) and Benigno and Fornaro (2015). In the former paper, an aggregate demand externality pushes firms to coordinate the capacity at which they desire to operate, and non-convexities in technology in the form of a fixed cost create two steady-state equilibria. Each steady state is a powerful basis of attraction, and only shocks of enough size and duration (like the ones that were likely at play at the onset of the Great Recession) can settle the economy around a long-lasting recession. In the second paper, sunspot-driven expectations play a role in the duration of crises: the equilibrium can get stuck into stagnations traps (a feedback loop between a growth trap and a liquidity trap), in which weak aggregate demand limits profits

[5] In my model, a parametric condition on the degree of financial frictions vis-à-vis the degree of imperfect competition will ensure that, potentially, a non-zero measure of firms is effectively credit-constrained. Therefore, in principle, the indeterminacy result coming from binding constraints, of special emphasis in Benhabib and Wang (2013) and Liu and Wang (2014), is also present in my economy. However, because I want to focus on spillovers in the research sector as the sole channel of equilibrium multiplicity, I will follow the bulk of the financial accelerator literature and assume that the parameters lie in the space of non-indeterminacy. The interested reader is referred to said papers for further detail.

of innovative firms, depresses growth, and this in turn pushes the interest rate toward its zero lower-bound through a deflationary pressure, leaving small room for monetary policy, and thereby aggravating the effects of the slump.

Another strand of literature the present paper touches upon studies how market frictions cause misallocation of capital across sectors and affect TFP endogenously. The present model allows for both imperfect competition and agency problems in the form of credit constraints. While the former introduces dispersion in production in and on itself due to a standard taste-for-variety motive[6], the existence of a financial friction implies that, in equilibrium, resources flow to unproductive sectors, which in turn yields productivity losses with respect to the first-best in which only the most productive firm is operational. Both layers of inefficiency (imperfect competition and agency problems) hence allow us to obtain a non-zero measure of firms that actively innovate in equilibrium, and creates room for strategic coordination between them, a key aspect of the model. That financial frictions create capital misallocation is not a new insight: Hsieh and Klenow (2009) use labor and capital misallocation to generate TFP losses through wedges in marginal productivities across plants, and Restuccia and Rogerson (2008) show that policies which create dispersion in prices faced by heterogeneous producers can lead to sizable drops in output and TFP. In related work, Moll (2014) uses a similar mechanism to show that, in an otherwise standard growth model, contracting frictions not only can create misallocation and TFP losses, but these are also steeply decreasing in the persistence of productivity shocks. My model nests these theories and extends them by allowing beliefs to play a crucial role in the determination and the evolution of these losses.

3 Model

In this section I describe the baseline model. The environment combines the quality-ladder model of the standard neo-Schumpeterian theory à la Grossman and Helpman (1991) with sunspot-driven fluctuations and a simple capital-accumulation overlapping generations model in the spirit of Diamond (1965). As mentioned above, in the baseline model two dimensions will force resources to flow to unproductive entrepreneurs in equilibrium: financial frictions and market power arising from imperfect competition[7]. Accordingly, in Section 4 we will analyze the role of each of these two inefficiencies separately.

Demographics and Environment Time is discrete and infinite, and is indexed by $t = 1, 2, 3, \ldots$ The economy is populated by two distinct groups of agents: an infinitely-lived representative firm, and overlapping generations of a time-invariant measure-one continuum of two-period lived, risk-neutral agents. Each cohort has within-generation heterogeneity, and each agent is in-

[6]In particular, a final-good firm will purchase intermediate goods, which are supplied by a set of monopolistic competitors. Because the final-good firm will demand goods of any variety, even low-productivity intermediate producers will face a non-zero inverse demand function. Adding financial frictions on top of this formulation gives another layer of inefficiency in the economy. The contribution of each will then have to be studied in detail.

[7]Dynamic inefficiency, standard in OLG models, will be parametrically assumed away for simplicity.

dexed by $z \in [0,1]$.

The representative firm hires labor and purchases intermediate goods (also called "inputs") provided in that period to produce final goods in order to maximize the infinite stream of discounted profits. On the other hand, members of each generation of agents extract no utility from early consumption and simply seek to maximize old-age consumption. In particular, each agent z from any generation (say, generation $t-1$) is endowed with one unit of labor when born, which is supplied inelastically during young age. Labor can be used as a productive factor in both the final-good and the intermediate-good sectors, in a way that will be specified below. Young workers earn a wage rate of w_{t-1} for their work, common to all types within a generation. There is no storage, but young agents may lend their wage in the credit market to old agents from the previous generation at a lending rate r_t. The proceedings, $r_t w_{t-1}$, can only be consumed when young lenders become old. Additionally, young agents may choose a borrowing level $k_t(z)$, and transact these funds when old. Borrowing is paid back at the lending rate r_t so there are no arbitrage opportunities. The total net idiosyncratic returns from financial markets are therefore $r_t \cdot (w_{t-1} - k_t(z))$, which are collected only in old age[8].

Taking r_t as given, old agents can then utilize this wealth to open an intermediate-good firm that uses capital $k_t(z)$ for production and sells inputs of the corresponding variety to the final-good firm. Specifically, besides serving as the agent's identity, z also stands for the intermediate's variety type, as well as for the part of productivity that is idiosyncratic to the firm (further details are given below). Firms can then choose whether to be innovative by hiring skilled labor. Moreover, to model tight financial conditions in a simple manner, we will assume that all agents are allowed to participate in the credit market, but there is an agency cost limiting the ability to raise credit.

We next describe each one of the features of the market structure in detail.

3.1 Final Good Sector

The representative firm (or continuum of identical firms operating competitively) produces output according to the following technology:

$$Y_t = L_t^{\alpha} X_t^{1-\alpha}$$

where $\alpha \in (0,1)$, L_t is labor and X_t is a Dixit-Stiglitz aggregate of intermediate-good inputs in the economy, given by

$$X_t \equiv \left[\int_0^1 \left(x_t(z)\right)^{\rho} dz \right]^{\frac{1}{\rho}}$$

where $x_t(z)$ is the quantity of input of variety $z \in [0,1]$ produced in period t and sold by agent z of generation $t-1$ at a price of $p_t(z)$, and $\rho \in (0,1)$ is the degree of substitutability between different input varieties, with $\rho \to 1$ and $\rho \to 0$ being the limit cases of perfect substitutes and

[8] We will use the convention that if $w_{t-1} - k_t(z) < 0$ (or > 0), agent z of generation $t-1$ is a net debtor (or creditor).

perfect complements, respectively.

Taking wages w_t and input prices $\{p_t(z)\}_{z\in[0,1]}$ as given, the representative firm chooses an allocation of labor L_t and intermediate inputs $\{x_t(z)\}_{z\in[0,1]}$ purchased from old-age intermediate-good producers in order to maximize its infinite stream of discounted profits. Given the overlapping generations nature of the demography, intermediate-good producers are replaced period after period, and the final good firm effectively faces the following static program:

$$\Pi_t \equiv \max_{L,(x(z))_{z\in[0,1]}} \left\{ L^\alpha \left[\left(\int_0^1 \left(x_t(z)\right)^\rho dz \right)^{\frac{1}{\rho}} \right]^{1-\alpha} - w_t L - \int_0^1 p_t(z)x(z)dz \right\} \tag{1}$$

where Π_t represents period-t profits. Due to the competitive market nature of this sector, we will impose that $\Pi_t = 0$.

3.2 Intermediate Good Sector

Production technology An agent from generation $t-1$ is endowed with an index $z \in [0,1]$ when born. Indices are uniformly distributed in $[0,1]$, are known by all agents, and do not change over the lifetime. Each agent z from generation $t-1$ enters period t (old age) with net earnings $r_t \cdot (w_{t-1} - k_t(z))$ from young age. At the beginning of the period, the agent is then given the opportunity to open a firm of size $k_t(z)$ producing intermediate good of type z. For simplicity, all physical capital depreciates fully in production within the period.

Different varieties are produced in a monopolistically competitive environment and are sold to the final-good firm at a price of $p_t(z)$. In particular, each agent z can produce $x_t(z)$ units of variety z using the following linear technology:

$$x_t(z) = z \cdot q_t(z) \cdot k_t(z)$$

Firstly, note that z indexes both a variety type and the capital-specific productivity with which this variety is produced[9], so that the effective capital stock in possession of agent z is $z \cdot k_t(z)$. Secondly, productivity is augmented by $q_t(z)$, which in the tradition of the endogenous growth literature represents the quality (or vintage) of the sector-specific leading-edge technology with which effective capital is transformed into final output[10]. The endogenous component of growth in the sector will be given by advances in this variable through a process described in what follows.

[9]The latter can be understood as a proxy for entrepreneurial ability, the know-how technology given the current level of capital efficiency, or a specific idiosyncratic demand shock that realizes once-and-for-all at the beginning of life.

[10]The variable q_t is flexible enough to be interpreted in different ways. In one broad interpretation, it proxies embodied technological change. In another interpretation, it stands for the aggregate component of sectorial productivity, which is endogenous to the R&D decisions of the whole industry. More precisely, it can be viewed as the efficiency of intangible capital (or knowledge), such as the human-capital skills embedded into the technology of the production process, the level of technical progress in the employment of physical capital in production, or process-specific technology. Finally, q_t can also be viewed as the quality of the sector-specific variety produced in that period. Henceforth, we will mostly stick to the latter interpretation.

R&D and innovation technology The engine of growth in the economy are process innovations leading to quality improvements across sectors. In equilibrium, these will amount to advances in productivity at the aggregate level. In the spirit of Grossman and Helpman (1991) and Aghion and Howitt (1992), we assume that, upon operation of a firm, manager z must choose whether or not to employ resources that would facilitate the advance of the current quality, $q_t(z)$, by a step factor. In particular, the agent can hire skilled labor ("researchers") in the competitive labor market to generate creation of ideas and a non-zero probability of a technological change (i.e., an "implementation") within the sector. A realized technological change advances quality forward in the sector to the next step in the ladder. Critically, however, the step sizes of this ladder are themselves an equilibrium outcome of the economy, and depend crucially upon the prevailing degree of economic spillovers.

At the beginning of every period t, every sector z starts with the same baseline quality, denoted by q_{t-1}. Upon observation of q_{t-1}, but prior to production choices, new managers must choose whether or not to conduct productivity-enhancing activities, which we label as R&D.

The innovation technology is as follows. If the agent employs $l_t^{RD}(z)$ researchers from the pool of young workers, it generates a probability $\phi(l_t^{RD}(z))$ of implementing a technical change into the production function, where $\phi : \mathbb{R}_+ \to [0,1]$, with $\phi(0) = 0$ and $\phi(1) = \overline{\phi} \in (0,1]$. We further assume that ϕ is strictly increasing and concave, and that the Inada condition

$$\lim_{\ell \to 0} \phi'(\ell) = +\infty \tag{2}$$

is satisfied. Labor specialized in research in period t earns the competitive wage w_t, just like (unskilled) workers in the final good firm. Finally, an innovation that is successfully implemented in the production process leads to an increment in quality of a step the size of which increases with the total amount of aggregate research in the economy. Specifically, we make the following assumption:

Assumption 1 *Define the aggregate labor employed in R&D activities in the economy by* $L_t^{RD} \equiv \int_0^1 l_t^{RD}(z)dz$*. Then, the quality level* $q_t(z)$ *with which any firm* $z \in [0,1]$ *operates in period* t *is*

$$q_t(z) = \begin{cases} \Gamma(L_t^{RD}) \cdot q_{t-1} & \text{with probability } \phi(l_t^{RD}(z)) \\ q_{t-1} & \text{otherwise} \end{cases}$$

where $\Gamma : \mathbb{R}_+ \to [1, +\infty)$*, with* $\Gamma(0) = 1$ *and* $\Gamma(1) = \overline{\Gamma} > 1$*. Moreover,* Γ *is strictly increasing and strictly concave.*

Several aspects of this formulation are worth discussing at this point. First, specialized labor is an essential factor in research, as $\phi(0) = 0$, and there is no memory in the development of ideas (current innovative efficiency is not explicitly affected by past skilled labor) other than through that which is contained in the state q_{t-1}. Secondly, the assumption that the step size $\Gamma(\cdot)$ is an increasing function of total research means that there exist positive and aggregate spillovers in innovation investment[11]. Importantly, this is the source of strategic complementarity, and eventually of multiplicity of equilibria, in this economy, and it serves as a simple mechanism through which positive externalities between innovating sectors can create volatility in outcomes. Because agents are atomistic, their individual research is ineffective almost certainly unless a sufficiently large mass of other sectors is expected to undertake R&D as well. This is a complementarity in the best-response function that critically relies on each agent's prior expectations about other agents' actions. Consequently, if the market is pessimistic with strictly interior probability, an equilibrium mixed strategy is played in which each individual agent assigns belief-consistent probabilities to positive aggregate R&D expenditure. To the extent that any distribution of beliefs can rationalize this behavior, multiplicity will arise and a sunspot variable will select the equilibrium outcome. We relegate the details of this discussion to the next section.

Finally, concavity in $\Gamma(\cdot)$ implies that strategic complementarity forces are stronger for lower levels of basic research, thereby leaving a wider room for coordination. Intuitively, ideas become marginally less valuable as they become more abundant, and the degree at which they can disseminate through the economy is smaller when a large measure of sectors invest in the production of new ones. Additionally, the assumption $\Gamma(0) = 1$ is merely a normalization that assumes no exogenous spillover effects from unsuccessful research, but it is unimportant for our results below. The assumption $\Gamma(1) = \overline{\Gamma} > 1$ places a cap on the potential output growth of the economy, and it is a non-binding assumption as concavity in $\phi(\cdot)$ will ensure that $L_t^{RD} < 1$ in equilibrium.

The individual problem The intra-period problem of producer z involves two steps, which are taken in sequentiality. In the first step, consistent with the optimal choice of capital, the agent chooses the quality level target at which production is expected to take place. The outcome of this stage problem is upgrading quality with a certain probability and potentially earning higher profits. The second step is making production choices at the quality level that results from research, if any. In what follows we describe the problem of a typical agent z from generation $t-1$ that has observed the outcome of his research. Moreover, we suppose that all agents have already coordinated into a prior belief through the sunspot process described below. This allows us to drop all expectation operators.

Starting with the second stage, suppose manager z of generation $t-1$ has access to capital of vintage $q(z) \in \{\Gamma q, q\}$, where $q \equiv q_{t-1}$. To study the general equilibrium effects of tight financial

[11] The formalization that research can exhibit positive externalities goes back to the beginnings of the endogenous growth literature. In the spirit of Aghion and Howitt (1992) and other creative destruction models of product innovation, an alternative formulation is to assume a constant step size $\Gamma \geq 1$ and an individual-specific probability of innovation that depends on both individual and aggregate investment, so that $\phi(l_t^{RD}(z), L_t^{RD})$ with, for example, a Cobb Douglas specification. Similarly in expanding-variety models, Romer (1990) and Jones (1995) assume, in a continuous-time set-up, that the growth rate of TFP, say Z_t, is given by $\frac{\dot{Z}_t}{Z_t} = \delta L_Z^{\lambda} Z_t^{\phi-1}$, where L_Z is labor employed in technology, $\lambda > 0$ and $\delta > 0$ are technological parameters, and $\phi > 0$ measures the degree of spillovers in the research sector. Note ϕ controls whether the growth rate accelerates (if $\phi > 1$) or decelerates (if $\phi < 1$) over time. These papers, however, do not focus on the strategic complementarities arising from these kind of externalities, and are silent about the possibility of equilibrium indeterminacy. More recent papers do recognize the possibility of multiplicity from positive demand externalities, but opt to conduct the analysis on the subset of the state space for which uniqueness holds (see, e.g., Acemoglu and Cao (2015)).

market conditions that were present during the Great Recession, we assume that a simple credit friction limits ex-ante the extend to which of agents can raise funds from lenders. In particular, we follow the standard approach in the literature and suppose that the agent cannot credibly commit to repay more than a fraction $\lambda \in (0, 1)$ of revenues, $p(z)x(z, q(z))$, where $p(z)$ is the monopoly price at which good z is sold to the final-good firm. In anticipation, the lender would never agree to sign a contract in which the incurred debt were greater than the maximum repayment threshold[12]. This gives rise to:

$$r_t(k(z) - w_{t-1}) \leq \lambda \cdot p(z)x(z, q(z)) \tag{3}$$

In this simple formulation, higher financial frictions are associated with a reduction in the λ parameter. The monopolist z's problem is:

$$\pi(z, q(z)) = \max_{k(z),p(z)} \left\{ p(z)x(z, q(z)) - r_t(k(z) - w_{t-1}) \right\} \tag{4}$$

subject to technology $x(z, q(z)) = zq(z)k(z)$, to the financial constraint (3), and to an inverse demand function from final-good producers.

In the previous stage, and in anticipation of the value of $\pi(z, q(z))$ for each $q(z) \in \{\Gamma_t q_{t-1}, q_{t-1}\}$, agent z makes innovation decisions, that is, chooses to hire $l_t^{RD}(z)$ researchers to solve the following ex-ante problem:

$$c_t^{t-1}(z) \equiv \max_{\ell^{RD} \geq 0} \left\{ \phi(\ell^{RD}) \cdot \pi(z, \Gamma q) + (1 - \phi(\ell^{RD})) \cdot \pi(z, q) - w_t \cdot \ell^{RD} \right\} \tag{5}$$

The final level of consumption for any agent z will therefore be the outcome of the maximization in (4) for the case in which $q(z)$ is the ex-post outcome of problem (5). Here, we denote $c_t^{t-1}(z)$ as the ex-ante level of consumption of agent z from generation $t - 1$, which is the relevant objective function with which agents make ex-ante decisions.

Patents and exogenous shocks We assume that innovators that succeed in implementing ideas can protect their discovery with a patent that expires at death. In this sense, once old managers of a new generation take over the sectors left by agents that die, the leading-edge technology becomes public knowledge and, since its patent dies with its owner, it can be costlessly adopted by all new sectors starting their operations. With this structure, the assumption that all sectors start with the same quality is without loss of generality.

[12]The constraint admits other interpretations as well. It can be seen as: (i) a collateral constraint, if written as $w_{t-1} \geq k(z) - \lambda \frac{p(z)x(z,q(z))}{r_t}$, where the right-hand side is the downpayment or collateral requirement for the lending transaction to take place; (ii) a leverage constraint, if written as $\frac{r_t(k(z)-w_t)}{p(z)x(z,q(z))} \leq \lambda$, which says that leverage (the ratio of debt to project revenues) cannot exceed a number $\lambda < 1$; (iii) agency costs, if borrowers strategically default whenever the repayment obligations $k(z) - w_{t-1}$ exceed the default costs, which are proportional to the project revenue because, in case of default, a fraction $(1 - \lambda)$ of firm z's revenue is assumed to be lost forever and the remainder to be absorbed by the lender; and (iv) a limited contract enforcement problem, if contracts are enforced imperfectly and the firm can default on its debt repayment obligations and be caught with probability λ, in which case the entirety of its revenues are ceased.

We assume further that, due to obsolescence or other market frictions that are exogenous to the agents' decisions, there is possibly an end-of-period exogenous shock to quality determining the quality level that the new generation will start out with. To be specific, the law of motion of the leading-edge quality $q_t \equiv \max_{z\in[0,1]}\{q_t(z)\}$, which becomes available to all agents from generation t starting their productive activity in period $t+1$, is given by

$$q_t = \left[A_t \cdot \Gamma(L_t^{RD}) - \delta_q\right] \cdot q_{t-1} \tag{6}$$

where $\delta_q \in (0,1)$ is the depreciation rate of quality, and $A_t \sim \mathcal{F}_A$ is a stochastic process exogenous to agents' decisions[13]. The variable A_t is the only aggregate shock in the economy. It proxies shocks to endogenous capital efficiency, and serves as an exogenous source of growth in capital factor productivity. In this sense, it is observationally similar to a TFP shock. It is only realized after all period-t choices have been made and markets have cleared, but before the beginning of period $t+1$. The history of shocks is common knowledge across agents.

Timing To close the environment and recap, the timing within a period t is as follows:

1. Upon the observation of the state history $\mathbf{s}^{t-1}$, a sunspot $\xi_t \equiv \xi(\mathbf{s}^{t-1})$ chooses agents' prior beliefs.
2. Old agents from generation $t-1$ observe q_{t-1} and make innovation decisions, young skilled workers are employed in the production of ideas (if needed), and $q_t(z)$ is determined stochastically for each z according to $\phi(z)$.
3. Given $q_t(z)$, each agent z chooses production $x_t(z)$ and sells it to the FG firms at price $p_t(z)$.
4. The final good firm employes labor L_t, assembles intermediates into the composite X_t and produces output Y_t.
5. Prices w_t, r_t and $\{p_t(z, q_t(z))\}_{z\in[0,1]}$ clear labor, credit and final good markets.
6. The shock $A_t \sim \mathcal{F}_A$ is realized and the new aggregate quality is given by $q_t = (A_t\Gamma_t - \delta_q)q_{t-1}$.
7. Period $t+1$ starts with the state $\mathbf{s}_t = (A_t, \xi_t)$.

Notice that, once again, we have here introduced a sunspot in the initial sub-stage of the period. In the next section we show precisely how this mechanism works and why it is needed in the present framework.

Market inefficiencies There exist three sources of market inefficiency in this environment: (i) capital overaccumulation due to a dynamic inefficiency problem, standard in all OLG models; (ii) agency costs in the form of collateral constraints (the assumption $\lambda < 1$); and (iii) market power arising from imperfect competition in the intermediate-good sector ($\rho < 1$). The economy is dynamically efficient if the parameter α is not "too high" (as we will show in the discussion of Section

[13] The specific process governing the evolution of A_t is irrelevant for now. Further assumptions will be made in Section 5, when we present simulation results.

4), an assumption that will be kept throughout. Of more importance to the focus of this paper is the presence of financial frictions and monopolistic competitors, both of which give rise to capital misallocation and aggregate productivity losses. This, consequently, opens the door to strategic complementarity and equilibrium multiplicity. Accordingly, we will analyze the separate effects of each one of these inefficiencies in Section 4, and examine output responses in crisis environments under different market frictions in Section 5.

3.3 Equilibrium

In this section, we explore the general characterization of the equilibrium and then show how multiplicity can arise from the quality-choice stage.

3.3.1 Characterization

The demand functions for labor L_t and intermediate goods $\{x_t(z)\}_{z\in[0,1]}$ can be readily obtained from the problem of the final good firm in equation (1). Our first lemma shows the solution:

Lemma 1 (Final good producers) *Define the function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ by $x \mapsto \alpha \left(\frac{1-\alpha}{x}\right)^{\frac{1-\alpha}{\alpha}}$. The demands for intermediate-good and labor inputs in the final good sector are given by:*

$$x_t(z) = \left[\frac{p_t(z)}{P_t}\right]^{-\sigma} X_t; \qquad \forall z \in [0,1] \tag{7}$$

$$L_t \begin{cases} = +\infty & \text{if } \Psi(P_t) > w_t \\ \in \mathbb{R}_+ & \text{if } \Psi(P_t) = w_t \\ = 0 & \text{if } \Psi(P_t) < w_t \end{cases} \tag{8}$$

respectively, where

$$P_t \equiv \left[\int_0^1 \big(p_t(z)\big)^{1-\sigma} dz\right]^{\frac{1}{1-\sigma}}$$

is the Dixit-Stiglitz composite index of intermediate good prices, and $\sigma \equiv \frac{1}{1-\rho} \in (1, +\infty)$ is the elasticity of substitution between different varieties.

Proof. See Appendix A.1.

Lemma 1 shows a familiar result: in this setting, we obtain the standard iso-elastic inverse demand for intermediates, where the price-elasticity of demand is equal to the elasticity of substitution between varieties. Moreover, firm optimality yields a perfectly elastic demand for labor due to the fact that profits are linear in labor. To see this, Appendix A.1 shows that, in the optimal allocation,

$$X_t = \left(\frac{1-\alpha}{P_t}\right)^{\frac{1}{\alpha}} \cdot L_t \tag{9}$$

which allows us to write profits of the representative firm as $\Pi_t = (\Psi(P_t) - w_t) \cdot L_t$, hence yielding demand (8). Moreover, final good output can be written as $Y_t = \frac{\Psi(P_t)}{\alpha} \cdot L_t$ and total intermediate goods expenditures as $\int_0^1 p_t(z)x_t(z)dz = (1-\alpha)Y_t$. Additionally, if $\Pi_t = 0$, then $w_t L_t = \alpha Y_t$, a standard result for a Cobb-Douglas technology. In equilibrium, market clearing will require an interior solution (see details below), and therefore $w_t = \Psi(P_t)$ and $\Pi_t = 0$. Combined with a perfectly inelastic supply, the result that demand is perfectly elastic implies that the intra-period allocation of labor between intermediate- and final-good sectors operates only through an extensive margin, for a fixed wage.

Intermediate good sectors Let us now describe the two-stage problem of the intermediate producers from generation $t-1$. As before, we start by describing the second stage for a given first-stage realization of quality $q_t(z) \in \{\Gamma_t q_{t-1}, q_{t-1}\}$, and describe later how the choice of quality is made in anticipation of production profits for each quality realization.

The following lemma describes the solution to this problem in relation to whether or not the firm is effectively constrained by the agency problem:

Lemma 2 (Intermediate good producers: Production stage) *In sector z, the monopoly price for intermediate goods of variety z and quality $q(z)$ is given by*

$$p_t(z, q(z)) = \mathcal{M}_t(z, q(z)) \cdot \left(\frac{\sigma}{\sigma - 1}\right) \frac{r_t}{zq(z)} \tag{10}$$

where

$$\mathcal{M}_t(z, q(z)) \equiv \frac{1 + \mu_t(z, q(z))}{1 + \lambda \cdot \mu_t(z, q(z))} \geq 1$$

and $\mu_t(z, q(z)) \geq 0$ is the Lagrange multiplier on the financial constraint (3). The demand for capital is given by

$$k_t(z, q(z)) = \begin{cases} \left(\frac{p_t(z,q(z))}{P_t}\right)^{-\sigma} \frac{X_t}{zq(z)} & \text{if } \mu_t(z, q(z)) = 0 \\ \overline{k}_t(z, q(z)) & \text{if } \mu_t(z, q(z)) > 0 \end{cases}$$

where

$$\overline{k}_t(z, q(z)) \equiv w_{t-1} \cdot \left[1 - \lambda \cdot z \cdot \frac{p_t(z, q(z)) \cdot q(z)}{r_t}\right]^{-1}$$

is the level of capital at which the financial constraint is binding.

Proof. See Appendix A.2.

Lemma 2 distinguishes between at most two types of agents that can exist in equilibrium: *financially unconstrained* and *financially constrained* agents. Unconstrained agents, which are particularly unproductive (low z), are such that $\mu_t(z, q(z)) = 0$ and $\mathcal{M}_t(z, q(z)) = 1$, and set the standard monopoly price

$$p_t^U(z, q(z)) = \left(\frac{\sigma}{\sigma - 1}\right) \frac{r_t}{zq(z)}$$

that is, a markup over the marginal cost, which is here the cost of funds per unit of capital efficiency, $\frac{r_t}{zq(z)}$. This markup arises due to market power coming from imperfect competition. As usual in models of Dixit-Stiglitz competition, it is time-invariant and decreasing in the price-elasticity of demand, σ, and the degree of substitutability, ρ. Sector z's production decision satisfies the inverse demand function $x_t(z, q(z)) = (p_t(z, q(z))/P_t)^{-\sigma} X_t$, which implies a demand for capital that satisfies $k_t(z, q(z)) = x_t(z, q(z))/(zq(z)) < \overline{k}_t(z, q(z))$. Here, $\overline{k}_t(z, q(z))$ is defined as the level of capital for which the financial constraint (3) holds with equality. In other words, in spite of the tight credit frictions, the borrowing constraint faced by unconstrained firms is slack because they are particularly inefficient in producing intermediates, conditional on quality $q(z)$.

The most productive agents (high z's), however, may borrow up to their limit capacity and demand $k_t(z, q(z)) = \overline{k}_t(z, q(z))$, the maximum debt they can credibly commit to repay. Since this does not satisfy their needs, these agents are forced to charge an *additional* markup $\mathcal{M}_t(z, q(z)) > 1$ over the unconstrained monopoly price. This second markup, which arises solely due to the binding financial constraint, is decreasing in the degree of financial development (lower λ).

In particular, in Appendix A.2 we show two important results. First, the price set by a constrained agent z of quality $q(z)$ is equal to the implicit solution p to the following non-linear equation:

$$p^{-\sigma} \left(\frac{r_t}{zq(z)} - \lambda p\right) = \frac{r_t w_{t-1}}{X_t P_t^{\sigma}} \tag{11}$$

which solves into the unconstrained price $p_t^U(z, q(z))$ when $\mu_t(z, q(z)) = 0$, and otherwise $p_t^C \in \left(p_t^U, \frac{p_t^U}{\lambda}\right)$. In the frictionless limit $\lambda \to 1$, this last equation shows that $p_t^C \to p_t^U$, as to be expected. It is therefore the presence of frictions ($\lambda < 1$) which directly accounts for the possibility of a share of productive firms in the economy to be constrained. Equation (11) shows the specific solution in such a case. Due to its non-linearity, it is not possible to find closed-form solutions for price unless we focus on special limit cases, a task that we examine in Section 4.

Second, in equilibrium, the following is both a necessary and sufficient condition on parameters to have a non-zero measure of constrained firms in the economy:

Assumption 2 *$\lambda \leq \rho$.*

Recall that $\rho = \frac{\sigma-1}{\sigma}$, and thus ρ is the inverse of the price mark-up. Assumption 2 then states that there is enough substitutability between goods compared to the degree of financial frictions, or that mark-ups are not "too high"[14]. That is, we require a combination of low enough λ (high agency costs) and high enough ρ (high substitutability and low market power) in order for some agents to be constrained in equilibrium[15].

Equilibrium productivity cutoffs Using monotonicity and noting that, for every pair $(z, q(z))$, there exists a unique solution to (11), we next show the existence of a marginally constrained firm, or a threshold productivity level below which firms are unconstrained and above which the financial friction is always binding, conditional on the realization of $q(z)$ (namely, there is one such threshold for each possible quality level).

This is summarized in our following lemma:

Lemma 3 (Cutoffs) *Suppose $q(z) = q_{t-1}$. Then, there exists $\overline{z}_t \in [0,1]$ such that $\mu_t(z', q(z')) > 0$, $\forall z' > \overline{z}_t$ with $q(z') = q_{t-1}$, and $\mu_t(z', q(z')) = 0$, $\forall z' < \overline{z}_t$ with $q(z') = q_{t-1}$. In particular,*

$$\overline{z}_t = \frac{\Phi(\sigma,\lambda)}{q_{t-1}} \left[\frac{w_{t-1} r_t^{\sigma}}{X_t P_t^{\sigma}}\right]^{\frac{1}{\sigma-1}} \tag{12}$$

where $\Phi(\sigma, \lambda) \equiv \frac{1}{\sigma-1}\left[\frac{\sigma^{\sigma}}{\sigma(1-\lambda)-1}\right]^{\frac{1}{\sigma-1}} \geq 0$. If instead $q(z) = \Gamma_t q_{t-1}$, where $\Gamma_t \equiv \Gamma(L_t^{RD}) \geq 1$, the relevant threshold is $\underline{z}_t \equiv \overline{z}_t / \Gamma_t$.

Proof. See Appendix A.3.

Lemma 3 establishes the existence of two thresholds, $\underline{z}_t$ and $\overline{z}_t \geq \underline{z}_t$, with the following properties. First, the two thresholds are one and the same if $L_t^{RD} = 0$, that is, if there is no research in the economy. Second, the thresholds are interior if $\overline{z}_t = \Gamma_t \underline{z}_t \leq 1$, an equilibrium condition that we will check in equilibrium. Third, and most important, these cutoffs classify firms in the $(z, q(z))$-space in different groups. Partition the space into $\mathcal{Z}_t^L \equiv [0, \underline{z}_t)$, $\mathcal{Z}_t^M \equiv [\underline{z}_t, \overline{z}_t)$ and $\mathcal{Z}_t^H \equiv [\overline{z}_t, 1]$. Then,

1. If $z \in \mathcal{Z}_t^L$, firm z is unconstrained, whether or not it innovates in the first stage.

[14] This parametric condition is rather standard in models with imperfect competition in which binding financial frictions are an outcome of the equilibrium. For example, Benhabib and Wang (2013) obtain an almost identical condition.

[15] The intuition for this result is the following. Note that we can write profits of firm z as

$$\pi(z) = PX^{1-\rho}x(z)^{\rho} - \frac{r}{zq(z)}x(z) + rw$$

On the one hand, if $\rho > \frac{r}{zq(z)p(z)}$, then firms have an incentive to increase their production as marginal revenues (which increase at a power of ρ) respond more strongly to output than marginal costs (which increase one-for one). If it were the case that $\frac{r}{zq(z)p(z)} \leq \lambda\rho$, in the event of default lenders would recover more than the cost on the marginal unit, but then the original allocation would not have been optimal as firms would have increased their production and profits. Thus, for a constrained z, we must have that $\frac{r}{zq(z)p(z)} \geq \lambda\rho$, so that the firm cannot increase production further due to the binding constraint (marginally increasing output would not be enough to cover the unit cost). Using $k(z) = \overline{k}(z)$ when z is constrained, then the lender is willing to enter the contract with positive borrowing if the marginal unit of revenue recovered by the lender in case of default (λ) is not too high compared to the marginal unit of revenue absorbed by the borrower (ρ) in case the equilibrium case of no-default.

2. If $z \in \mathcal{Z}_t^M$, firm z is unconstrained when it does not innovate in the first stage, but is constrained when it successfully implements an innovation.

3. If $z \in \mathcal{Z}_t^H$, firm z is constrained, whether or not it innovates in the first stage.

Intuitively, some relatively unproductive firms can have access to higher price markups if they conduct R&D and they successfully implement their ideas. Note, however, that due to the strategic complementarities in this economy (though the Γ function), this is only possible if other agents are also behaving in this manner. Generally, however, it is always true that successful innovation is profitable from the production-stage ex-post perspective[16].

In the first stage, where each agent can anticipate the profits that would accrue from production decisions at each quality level, firms solve the ex-ante problem of choosing their R&D spending optimally. Our next lemma describes the optimal innovation decision for each group $n = L, M, H$ of firms defined above:

Lemma 4 (Intermediate good producers: Innovation stage) *Taking prices w_t and r_t, and aggregates X_t and P_t as given, sector $z \in \mathcal{Z}_t^n$, for each $n = L, M, H$, chooses the number $l_t^{RD}(z)$ of workers employed in research so that*

$$w_t = \Delta_t^n(z) \cdot \phi'(l_t^{RD}(z)) \tag{13}$$

where $\Delta_t^n(z) \geq 0$ is the change in profits from advancing quality to $\Gamma_t q_{t-1}$, defined by

$$\Delta_t^n(z) \equiv \begin{cases} \pi_t^U(z, \Gamma_t q_{t-1}) - \pi_t^U(z, q_{t-1}) & \text{for } n = L \\ \pi_t^C(z, \Gamma_t q_{t-1}) - \pi_t^U(z, q_{t-1}) & \text{for } n = M \\ \pi_t^C(z, \Gamma_t q_{t-1}) - \pi_t^C(z, q_{t-1}) & \text{for } n = H \end{cases}$$

Proof. See Appendix A.4.

Intuitively, a given sector in the economy chooses the number of specialized workers to maximize the expected payoff from an increase in quality. Since the probability with which quality advances is, conditional on an innovating equilibrium, itself increasing in the share of specialized labor, the

[16]To see this, note that second-stage profits, given by $\pi(z, q(z)) = [p - r/(zq(z))] \cdot x(z) + rw$, can be written as

$$\begin{aligned} \pi^U(z) &= \left(\frac{1}{\sigma - 1}\right) \frac{r}{zq(z)} x(z) + rw \\ \pi^C(z) &= \left(p^C(z) - \frac{r}{zq(z)}\right) \frac{zq(z)w}{1 - \lambda z \frac{p(z)q(z)}{r}} + rw \end{aligned}$$

for low-productivity (unconstrained) and high-productivity (constrained) agents, respectively. Unconstrained profits increase with quality because a higher quality increases the demand for capital: using the demand function and the unconstrained monopoly price from Lemma 2, note $k(z) = XP^\sigma \left(\frac{\sigma-1}{\sigma}\right)^\sigma \frac{(zq(z))^{\sigma-1}}{r^\sigma}$. Since $\sigma > 1$, then clearly $k(z, \Gamma q) \geq k(z, q)$ and $\pi^U(z, \Gamma_t q) \geq \pi_t^U(z, q)$. Constrained profits increase with quality, as well, because even though setting a higher quality implies charging a lower price, the reduction in marginal costs is larger due to the price markup.

sector will hire workers until the cost of the marginal researcher, which equals the wage w_t that he is hired for, is equalized to the marginal benefit added by his research, which is given by the advance in profit $\Delta(z)$ in case of success, times the marginal probability of success, $\phi'(l_t^{RD}(z))$.

Market clearing We can finally describe how factor prices r_t and w_t and goods prices P_t will adjust to clear the markets for credit, labor, and output.

First, credit (or capital) market clearing requires that savings by young lenders are funds for the old borrowers, namely $w_{t-1} = \int_0^1 k_t(z)dz$. Using law of large numbers, we know that $\phi(\ell)$ is the fraction of agents choosing innovation intensity $\ell \geq 0$. Where $q \equiv q_{t-1}$, then:

$$w_{t-1} = \sum_{n\in\{L,M,H\}} \int_{\mathcal{Z}_t^n} \left[\frac{x_t(z,q)}{zq} + \phi(l_t^{RD}(z)) \cdot \left(\frac{x_t(z,\Gamma_t q)}{z\Gamma_t q} - \frac{x_t(z,q)}{zq} \right) \right] dz \tag{14}$$

where, in this equation, $x_t(z,\cdot) = (p_t(z,\cdot))^{-\sigma} X_t P_t^{\sigma}$, and we can use the optimality condition $\phi(l_n^{RD}(z)) = \phi\left((\phi')^{-1}\left(\frac{w_t}{\Delta_t^n(z)}\right)\right)$ for each $n \in \{L,M,H\}$ from Lemma 4.

Second, labor market clearing requires that the inelastic supply of labor from young agents is equated to the perfectly elastic demand coming from the final good firm (L_t) and the demand for skilled labor from innovative intermediate good producers (L_t^{RD}). Hence, $L_t + L_t^{RD} = 1$, or

$$1 - L_t = \sum_{n\in\{L,M,H\}} \int_{\mathcal{Z}_t^n} (\phi')^{-1}\left(\frac{w_t}{\Delta_t^n(z)}\right) dz \tag{15}$$

Finally, the final goods market clearing condition requires, from the labor demand from the final good producers in (8), that $w_t \geq \Psi(P_t)$. Else, labor demand by final good firms is infinite and feasibility is violated. In particular, if $w_t > \Psi(P_t)$, then we encounter a corner equilibrium in which $(L_t, L_t^{RD}) = (0,1)$ and $\Gamma(L^{RD}) = \overline{\Gamma}$, a situation that will henceforth be ignored as it yields $Y_t = 0$. In the interior equilibrium, therefore,

$$w_t = \Psi(P_t) \tag{16}$$

so $L_t \in (0,1)$ and $L_t^{RD} \in (0,1)$, implying that $\Gamma(L^{RD}) < \overline{\Gamma}$ (that is, the economy never exhausts its potential research frontier).

Market-clearing equations (14), (15) and (16) provide equilibrium maps $P_t = \Psi^{-1}(w_t)$, $w_{t-1} = T_w\big(w_t, r_t\big)$, and $w_{t-1} = T_r\big(w_t, r_t\big)$, which embed the full equilibrium characterization of the economy. Critically, as we argue in the next section, the solution of the system is not unique for each given set of parameters.

3.3.2 Multiplicity of equilibria

Due to the strong form of strategic complementarities in this economy, two situations can be sustained as self-fulfilling equilibria.

First, there is a (unique) low self-fulfilling equilibrium in which $L_t^{RD} = 0$ (no aggregate research), which we call *pessimistic* equilibrium. To see that this equilibrium exists, suppose that agents coordinate their beliefs to expect ex-ante that $L^{RD} = 0$ will be the mass of specialized labor in the aggregate economy at the end of period t. Then, firms expect $\Gamma_t = 1$ and $\underline{z}_t = \overline{z}_t$ (i.e, $\mathcal{Z}_t^M = \emptyset$), and they do not expect any gains from research even if they unilaterally invest in R&D, as $\Delta_t^n(z) = 0$, $\forall n \in \{L, H\}$. From the innovation-demand functions (13) and the Inada condition (2) on $\phi(\cdot)$, then $l^{RD}(z) = 0$ for all $z \in [0, 1]$. Aggregating, $L^{RD} \equiv \int_0^1 l^{RD}(z)dz = 0$, which confirms the prior conjecture and renders agents' expectations self-fulfilling. Intuitively, since firms' research is ineffective almost-surely unless skilled workers are employed in other sectors of the economy (due to economies of scale, non-pecuniary externalities or other forms of technological spillovers), there is a pessimistic outcome in which agents' beliefs coordinate in a way that the individual best-response is not to hire specialized labor because other sectors are not expected to spend on R&D, either. Since agents are atomistic, they do not anticipate that this individual prospect yields low growth in the aggregate economy, and effectively lowers their consumption. Like in a public good problem, the pessimistic outcome is a free-rider equilibrium.

However, agents can also tacitly coordinate into being optimistic. In particular, there exist self-fulfilling equilibria in which $L_t^{RD} > 0$ (positive aggregate research), which we label as *high* equilibria. To prove the existence of such type of equilibrium, suppose that agents expect ex-ante that $L^{RD} > 0$. Then, firms expect $\Gamma_t > 1$ and $\underline{z}_t > \overline{z}_t$ (i.e, $\mathcal{Z}_t^M \neq \emptyset$). Therefore, for sufficiently productive sub-sectors $n \in \{L, M, H\}$, an individual firm $z \in \mathcal{Z}_t^n$ can expect to have gains from research given by an increase in quality from q_{t-1} to $\Gamma_t q_{t-1}$ with probability $\phi(l_t^{RD}(z)) > 0$. By symmetry, all sectors within group n will invest into R&D, or $l_t^{RD}(z) > 0$, $\forall z \in \mathcal{Z}_t^n$, $\forall n$. Moreover, by law of large numbers, a non-zero fraction of the economy will succeed into implementing technological progress. As before, agents' prior expectations will have been fulfilled in the equilibrium.

In both equilibria, as it is standard in models of equilibrium multiplicity, beliefs must be consistent in the sense that the equilibrium mass of skilled labor expected by private agents L_t^{RD} is the solution to the fixed point problem between those agents' beliefs about implied prices and their actual distribution under the law of motion describing the true generating process of the economy. This belief consistency criterion is the sense in which agents' expectations are rational. Particularly, an equilibrium is a fixed point L_t^{RD} to the labor market clearing equation (15) that is consistent with capital market clearing (14) and goods market clearing (16). Clearly, for given prices, $L_t^{RD} = 0$ is one solution of (15). Moreover, due to non-linearities in $\phi(\cdot)$, there is potentially more than one $L_t^{RD} > 0$ that solves (15).

All in all, we have arrived at the main result of this paper:

Theorem 1 (Multiplicity) *There exist two types of equilibria, labeled "optimistic" and "pessimistic". On the one hand, the pessimistic equilibrium is unique and satisfies market-clearing equations (14), (15) and (16) with $L_t^{RD} = 0$. On the other hand, there is potentially more than one (and at most a discrete number of) optimistic equilibria. An optimistic equilibrium is a solution $L_t^{RD} \in (0, 1]$ of (15), and prices that are consistent with (14) and (16).*

Proof. In the main text.

We have just shown that expectations about low and high equilibria can both be self-confirming in any region of the parameter space. An issue that arises is how to discipline the selection both between and within the two types of equilibria. As announced above, we follow a familiar route in the literature and assume that there is a sunspot selecting the ex-ante beliefs of agents[17]. Accordingly, we let the space of beliefs be denoted by $\Xi \equiv \{O, P\}$. The sunspot's realization is denoted $\xi_t = \xi(\mathbf{s}^{t-1}) \in \Xi$, so that if $\xi_t = P$, the pessimistic equilibrium is played, and otherwise if $\xi_t = O$. The ξ_t realization arrives each period before any decisions are made, and is a function of the history of states $\mathbf{s}^{t-1} = (A^{t-1}, \xi^{t-1})$. This means that the sunspot is allowed to possess a certain degree of cyclicality (correlation to the real capital-efficiency shock A_t), as well as a certain degree of serial autocorrelation (belief persistence). The formulation is thus general enough to allow for sluggish adjustment of expectations as well as co-movement to real shocks, both of which are features that were pointed out from Figure 2 in the Introduction.

3.3.3 Definition of equilibrium

We are now ready to define an equilibrium in this economy:

Definition 1 (Equilibrium) *Given assumptions 1-2 and a history $\boldsymbol{s}^{t-1} = (A^{t-1}, \xi^{t-1})$, an equilibrium are sunspot $\xi_t = \xi(\boldsymbol{s}^{t-1})$ and capital-efficiency A_t processes, and streams of prices $\{p(\xi_t)\}_{t=0}^{+\infty}$, consumption levels $\{(c(z,\xi_t))_{z\in[0,1]}\}_{t=0}^{+\infty}$, labor choices $\{L(\xi_t)\}_{t=0}^{+\infty}$ and $\{(l^{RD}(z,\xi_t))_{z\in[0,1]}\}_{t=0}^{+\infty}$, and qualities $\{q(\xi_t)\}_{t=0}^{+\infty}$, such that, for any period $t \in \mathbb{Z}_+$:*

1. *Given ξ_t and prices, the ex-ante allocation of consumption solves the intermediate-good firms' problem:*

$$
\begin{aligned}
c_{t-1}^{t-1}(z, \xi_{t-1}) &= 0 \\
c_t^{t-1}(z, \xi_t) &= \mathbb{E}_\xi\left[\pi_t(z, q) + \phi(l^{RD}(z)) \cdot \Delta(z) - w_t l^{RD}(z) \middle| \boldsymbol{s}^{t-1}\right]; \qquad \forall z, \xi_t, \boldsymbol{s}^{t-1}
\end{aligned}
$$

2. *Given ξ_t and prices, the final-good representative firm chooses production optimally, i.e. demands (7), (8), and (9) hold.*
3. *Capital, labor and goods markets clear, that is, wages, interest rates and final good prices solve equations (14), (15), and (16).*

[17]For classic models of indeterminacy and sunspots, see Cass and Shell (1983) and Benhabib and Farmer (1994). An alternative approach is selection through a global game in the spirit of Carlsson and van Damme (1993) and Morris and Shin (1998) (see Schaal and Taschereau-Dumouchel (2015) for an application to business cycles). According to this approach, there is incomplete information about certain fundamental processes of the economy and agents base their choices on informative Gaussian signals. A complete review of this method and their applications in macroeconomics is available from the author upon request. Another route to explore multiplicity further is the use of "correlated equilibria", a generalized Nash equilibrium notion first formulated by Aumann (1974) and later applied in Maskin and Tirole (1987). Using this approach, agents would coordinate their strategies after the observation of the value of a public signal which, in the current set-up, could be the A_t shock. This tool has few applications in macroeconomics so far, with Angeletos and La'O (2013) and Benhabib, Wang, and Wen (2015) being relevant examples

4. *The law of motion of quality is* $q(\xi_t) = \left[A_t \cdot \Gamma\left(\int l^{RD}(z, \xi_t)dz\right) - \delta_q\right] \cdot q(\xi_{t-1})$.
5. *Belief are consistent in the sense that* $\mathbb{E}_\xi[m_t|\boldsymbol{s}^{t-1}] = m_t$ *for any equilibrium variable* $m_t \equiv m(p(\xi_t); \boldsymbol{s}^{t-1})$, $\forall \boldsymbol{s}^{t-1}$.

We use a standard rational-expectations equilibrium definition: agents' decisions are optimal given their budget sets, prices clear all markets, and beliefs are consistent in the sense that the sunspot induces self-confirming choices, so that the subjective law of motion coincides with the equilibrium process implied by the actual distribution of prices.

4 Aggregation and Efficiency

Even though we have originally stated the economy in a heterogeneous-agent environment with collateral constraints, the decentralized allocation turns out to be equivalent to the following representative-agent allocation:

Proposition 1 (Aggregation) *For any equilibrium variable, denote* m_t *to be short-hand for* $m(\xi_t)$. *Define by* $K_t \equiv \int_0^1 k_t(z, q(z))dz$ *the aggregate stock of capital, and by*

$$Q_t \equiv \left[\int_0^1 \left(\frac{zq_t(z)}{\mathcal{M}_t(z, q(z))}\right)^{\sigma-1} dz\right]^{\frac{1}{\sigma-1}}$$

the composite index of endogenous aggregate capital efficiency, normalized by the price markup $\mathcal{M} \geq 1$ *coming from financial frictions. Then, for a given realization of the sunspot* $\xi_t \in \Xi$ *and any* $t \in \mathbb{Z}_+$*:*

1. *Aggregate quantities are given by:*

$$\begin{aligned} Y_t &= Z_t \cdot L_t^{\alpha} K_t^{1-\alpha} \\ K_{t+1} &= \alpha \cdot Z_t \cdot \left(\frac{L_t}{K_t}\right)^{\alpha-1} \end{aligned} \tag{17}$$

with $X_t = Z_t^{\frac{1}{1-\alpha}} \cdot K_t$ *and* $L_t = 1 - L_t^{RD}$, *where* $L_t^{RD} \geq 0$ *is a fixed point of equation (15), and* Z_t *is measured TFP, given by*

$$Z_t \equiv \left[\frac{Q_t^{\sigma}}{\int_0^1 \frac{1}{zq(z)}\left(\frac{zq(z)}{\mathcal{M}_t(z,q(z))}\right)^{\sigma} dz}\right]^{1-\alpha} \tag{18}$$

2. *Equilibrium prices are given by:*

$$P_t = \frac{\partial Y_t}{\partial X_t} = (1-\alpha)\left(\frac{L_t}{X_t}\right)^{\alpha}; \qquad w_t = \frac{\partial Y_t}{\partial L_t} = \alpha Z_t \left(\frac{L_t}{K_t}\right)^{\alpha-1}; \qquad r_t = \zeta_t \cdot \frac{\partial Y_t}{\partial K_t} = \zeta_t \cdot (1-\alpha) Z_t \left(\frac{L_t}{K_t}\right)^{\alpha}$$

where $\zeta_t \equiv \rho \cdot Q_t \cdot Z_t^{-\frac{1}{1-\alpha}}$.

Proof. See Appendix A.5.

The result states that our environment, in spite of being populated by heterogeneous agents with borrowing constraints and market power, is isomorphic to a representative-agent economy whose dynamics are similar to those of the standard Solow (1956) model: total final output is a Cobb-Douglas composite of total unskilled labor and physical capital inputs with measured TFP[18] equal to Z_t, and total capital investment is equal to the marginal product of labor[19].

There are three important differences with respect to the Solow model, however: firstly, imperfect competition (the assumption $\rho < 1$) and credit frictions (the assumption $\lambda < 1$) generate price markups and create capital misallocation; as a consequence, and secondly, rather than an exogenous process, measured TFP is endogenously given by innovative equilibrium decisions (which determine $q(z)$), as well as the degree of financial development λ, via the measure of constraintness μ_t determining the size of the idiosyncratic markup $\mathcal{M}_t(z, q(z))$, and the degree of imperfect competition, via the measure of input substitutability ρ determining the aggregate markup $\frac{\sigma}{\sigma-1}$; thirdly, rather than having a unique equilibrium path, the economy possesses two steady-state equilibria and sunspot-driven cyclicality between the two.

Since prices are marked up with respect to marginal costs due to both frictions, there may exist time-varying wedges in factor income shares. In particular, labor markets are frictionless and thus wages earn their marginal product, so that the labor income share is constant and equal to $\frac{w_t L_t}{Y_t} = \alpha$. Similarly, the goods market is also frictionless, and final-good firms purchase intermediates until the marginal product of these inputs is equalized to their composite unit price, i.e. $P_t = \frac{\partial Y_t}{\partial X_t}$. Thus, the intermediate-input income share is also constant, as $\frac{P_t X_t}{Y_t} = 1 - \alpha$. However, frictions in the capital market place a time-varying wedge ζ_t on the marginal product, so that $\frac{r_t K_t}{Y_t} = (1-\alpha) \cdot \zeta_t$. The size of the ζ_t wedge (i.e, the number $|\zeta_t - 1|$) increases in both price markups, that is, with both lower ρ and lower λ, and it is typically smaller than 1[20]. Wedges in marginal products are a standard result in the misallocation literature (see Hsieh and Klenow (2009) and Moll (2014)), a

[18]Throughout, I will follow the literature and define measured TFP by using a Solow residual: $TFP_t = \frac{Y_t}{L_t^\alpha K_t^{1-\alpha}}$.

[19]In equilibrium, investment is equal to savings. While the latter is given by last periods' wages, the former is the total physical capital stock available for next period (since there is full depreciation by assumption), and therefore $K_{t+1} = w_t$. Equation (17) then states the familiar result that $w_t = MPL_t \equiv \frac{\partial Y_t}{\partial L_t}$.

[20]From the definition of ζ_t and Z_t, note

$$\zeta_t = \rho \cdot \frac{\int_0^1 \mathcal{M}_t(z, q(z)) \left(\frac{zq(z)}{\mathcal{M}_t(z,q(z))} \right)^{\sigma-1} dz}{\int_0^1 \left(\frac{zq(z)}{\mathcal{M}_t(z,q(z))} \right)^{\sigma-1} dz}$$

In the frictionless limit with no borrowing constraints, $\lambda \to 1$ and $\mathcal{M}_t(z, q(z)) = 1$, $\forall z$, so $\zeta_t \to \rho$. Additionally, in perfect competition, when $\rho \to 1$, the wedge vanishes, as $\zeta_t \to 1$. The two frictions ($\lambda < 1$ and $\rho < 1$) therefore increase the size of the wedge. More market power in the intermediate-good sector (lower ρ) implies a lower ζ_t. Higher financial frictions (lower λ) means that more z's are constrained, so $\mathcal{M}(z, q(z)) > 1$ for a bigger measure of entrepreneurs. Assumption 2 implies that the effect coming from imperfect competition is stronger, so that typically $\zeta_t < 1$.

point that we will come back to later for our special cases of $\rho \to 1$ and $\lambda \to 1$.

Aggregate profits in the productive stage of the intermediate-good sector, $\Pi_t^{IG} \equiv \int_0^1 \pi_t(z, q(z))dz$, are strictly positive and equal to $\Pi_t^{IG} = \frac{r_t K_t}{\zeta_t}$. The profit share of output is $\frac{\Pi_t^{IG}}{Y_t} = 1 - \alpha$, so $\Pi_t^{IG} = P_t X_t$, as to be expected. In the innovative stage, profits (which are equal to ex-ante expected consumption by entrepreneurs, our measure of welfare in the economy[21]) are given by $C_t = \Pi_t^{IG} - K_{t+1} L_t^{RD}$. This states that, in the low equilibrium ($L_t^{RD} = 0$), all second-stage profits are absorbed in the first stage as $C_t = \Pi_t^{IG}$, while in any high equilibrium ($L_t^{RD} > 0$), second-stage profits are not fully consumed due to the costs of research in the first-stage, $K_{t+1} L_t^{RD} > 0$. However, final consumption C_t is higher in high equilibria because the advance in capital efficiency, which reflects on a greater build-up of capital stock, more than compensates for these costs[22].

Measured TFP is endogenously given by the level of aggregate quality in the economy Q_t, normalized by the heterogeneous price markup arising from the collateral constraint, both of which fluctuate due to the sunspot shock. If beliefs are optimistic, agents accumulate knowledge and build aggregate productivity. In particular, capital follows the law of motion given by $K_{t+1} = s_t Y_t + (1 - \delta) K_t$, where $\delta = 1$ by initial assumption and the savings rate is defined by $s_t \equiv \frac{\alpha}{L_t}$. Notice, then, that the standard Solow result of a constant savings rate given by $s_t = \alpha$ is nested here when the sunspot chooses the low equilibrium. In the high equilibrium, however, $L_t < 1$ and $s_t > \alpha$, which shows that when agents are optimistic, their marginal propensity to consume is lower but they take the gamble that with some probability their own individual consumption will increase by a step factor. Thanks to $\alpha < 1$, the capital evolution map is increasing and concave so that for any given initial level $K_0 \in \mathbb{R}_+$ and any given path $\{\xi_t\}_{t=0}^{+\infty}$, K_t cyclically converges to a system of two locally stable steady state levels, defined by

$$K_\xi = (Z_\xi \cdot s_\xi)^{\frac{1}{\alpha}} \cdot L_\xi \tag{19}$$

where the notation K_ξ corresponds to the level toward which capital converges when $\xi_{t+1} = \xi_t = \xi$, $\forall t > \tau$ and some $\tau \geq 0$. Here, $L_\xi \leq 1$ with strict equality if $\xi = O$, Z_ξ is defined in equation (18), and $s_\xi = \alpha / L_\xi$.

Along the transition, since $Y_t = Z_t \cdot L_t^\alpha K_t^{1-\alpha}$, the growth rate of output is given by $g_t^Y = g_t^Z + \alpha \cdot g_t^L + (1 - \alpha) \cdot g_t^K$ where, for any variable m_t, we define $g_t^m \equiv \ln(m_t / m_{t-1})$. Moreover, since $K_{t+1} = s_t Y_t$, we have that $g_t^K = g_t^s + g_t^Y$, and since $s_t = \alpha / L_t$, then $g_t^s = -g_t^L$, which allows us to write the growth rate of the economy as

$$g_t^Y = \frac{g_t^Z}{\alpha} - \left(\frac{1 - 2\alpha}{\alpha}\right) \cdot g_t^L \tag{20}$$

Along the low equilibrium, $g^L = 0$ and therefore $g_t^Y = \frac{g_t^Z}{\alpha}$, where g_t^Z embeds only exogenously-driven growth as there is no endogenous growth because $\Gamma_t = 1$, $\forall t \in \{\tau : \xi_\tau = P\}$. However, when

[21]Recall $c_t^t(z) = 0$ and $c_t^{t-1}(z) = \pi_t(z, q) + \phi(l_t^{RD}(z)) \cdot \Delta_t(z)$, and that an agent's utility is equal to his old-age consumption. Thus, choosing an equally-weighted utilitarian measure W_t of welfare, $W_t = C_t \equiv \int_0^1 [c_t^{t-1}(z) + c_t^t(z)]dz$.

[22]Notice also that $C_t = \Pi_t^{IG} - K_{t+1} L_t^{RD}$, $\Pi_t^{IG} = P_t X_t = (1 - \alpha) Y_t$, and $K_{t+1} L_t^{RD} = K_{t+1} - w_t L_t = K_{t+1} - \alpha Y_t$, we get $C_t + K_{t+1} = (1 - \alpha) Y_t + \alpha Y_t = Y_t$, and thus the income identity holds in this setting.

beliefs switch from pessimism (in period $t-1$) to optimism (in period t), then $g^L_t < 0$ due to the growth in skilled labor, and vice versa if there is an optimism-to-pessimism switch. The impact of such regime switches on the growth rate of output then relies critically on the sign of the multiple $\frac{1-2\alpha}{\alpha}$ and, in turn, upon whether the economy is in a dynamically inefficient region. A sufficient condition for *dynamic efficiency* is that α is low enough, namely, that the marginal propensity to save out of income is not "too high"[23]:

Assumption 3 *The economy is dynamically efficient, so that* $\alpha \leq \frac{L_\xi}{1+L_\xi} \in \left(0, \frac{1}{2}\right]$.

If this is the case, steady-state consumption lies strict below its golden-rule level, meaning that disinvestment leading to a reduction in the steady-state stock of physical capital cannot increase long-run consumption levels. Assumption 3 then gives that P-to-O transitions have an unambiguously positive impact on the growth rate of output from two different channels: a higher growth of TFP (from the endogenous scaling factor $\Gamma_t > 1$), and a higher growth of skilled labor, because $\frac{1-2\alpha}{\alpha} \geq 0$ under Assumption 3. Intuitively, a surge in skilled labor translates into capital accumulation which, by definition of dynamic efficiency, increases consumption (and welfare) as well[24].

Beyond this inefficiency, which is standard in overlapping generations models, our focus is to understand how both equilibrium dynamics and multiplicity results, and thus welfare, are affected by the composition of the credit market and competitive structure frictions. To study the effects of each of these two frictions, we next explore the equilibrium allocation when either one or both are not at play.

4.1 Unconstrained planner allocation

The multiplicity result vanishes if *both* market frictions are shut down, namely $(\rho, \lambda) \to (1, 1)$ (an entirely frictionless economy). In particular, since spillovers are critical to generate incentives to conduct research, only the bad equilibrium survives. We then reach the following special case of Proposition 1:

[23]To derive the result stated in Assumption 3, fix $\xi \in \Xi$ and recall from the income identity that the steady-state level of aggregate consumption is $C_\xi = Y_\xi - K_\xi$. Hence, C_ξ is maximized for $\frac{\partial Y_\xi}{\partial K_\xi} = 1$, that is, using that $r_t = \zeta_t \cdot \frac{\partial Y_t}{\partial K_t}$ in equilibrium, for $r^{gold}_\xi = \zeta_\xi$. The steady-state interest rate r^{gold}_ξ is the so-called "golden rule" rate. Using that the steady-state rate of return to capital is $r_\xi = \zeta_\xi \cdot \frac{1-\alpha}{s_\xi}$, the economy is *dynamically inefficient* if $s_\xi \geq 1-\alpha$. In this case, $r_\xi \leq r^{gold}_\xi$ and there is capital overaccumulation, or $K_\xi \geq K^{gold}_\xi$. The equilibrium condition under which the economy is *dynamically efficient* is $L_\xi \geq \frac{\alpha}{1-\alpha}$. Namely, $\alpha \leq \frac{L_\xi}{1+L_\xi}$. In the pessimistic equilibrium, where $L_\xi = 1$, this reduces to $\alpha \leq \frac{1}{2}$. In the optimistic equilibrium, where $L_\xi < 1$, the condition is even more stringent.

[24]An additional concern in OLG economies with binding credit constraints is that pure rational bubbles may emerge even in the dynamically efficient region of the parameter space. This is because bubbles help relax these constraints and raise investment, making both borrowers and lenders willing to trade them in the first place. The are not exploring this channel here. For references, see Caballero and Krishnamurthy (2006), Farhi and Tirole (2012), and Martin and Ventura (2012).

Proposition 2 (Aggregation with no credit frictions and perfect competition) *In the unconstrained planner economy, the equilibrium is unique, and Proposition 1 holds for the following specialization:*

1. *Measured TFP is given by $Z_t = q_{t-1}^{1-\alpha}$.*
2. *There is no wedge between the marginal product of capital and the rental rate, or $\zeta_t = 1$, $\forall t$.*

Proof. In the frictionless economy all resources flow to $z = 1$, who then remains the only producer of the industry[25]. Capital demand is $k(1, q(1)) = \kappa_t \cdot \mathbf{1}_{[P_t \geq r_t/q(1)]}$ for some $\kappa_t \in \mathbb{R}_+$, which yields profits of $\pi_t(1, q(1)) = (P_t - r_t/q(1)) \cdot \kappa_t + r_t w_{t-1}$. Since $L_t^{RD} = l_t^{RD}(1)$, there exist no strategic complementarities in the research sector and the equilibrium is unique. Moreover, using $X_t = x_t(1, q(1))$ and the final-good producer solution, we get $\kappa_t = (1/q(1)) \cdot (1-\alpha)^{\frac{1}{\alpha}} P_t^{-\frac{1}{\alpha}} (1 - l_t^{RD}(z))$, and $P_t = \Psi(w_t) = r_t/q(1)$, or else demand would be either zero or unbounded, a violation of feasibility. This means that $\pi_t(1, \Gamma_t q_{t-1}) = \pi_t(1, q_{t-1})$, and therefore $\Delta_t(1) = 0$, $l_t^{RD}(1) = 0$, and $q(1) = q_{t-1}$, so $L_t = 1$. Output is $Y_t = Z_t \cdot K_t^{1-\alpha}$, where $K_t \equiv \int_0^1 k_t(z, q(z))dz = \kappa_t$ is the total capital stock, and the law of motion of capital reads $K_{t+1} = \alpha \cdot Z_t \cdot K_t^{1-\alpha}$, where $Z_t \equiv q_{t-1}^{1-\alpha}$ is measured TFP. □

In the unconstrained planner allocation, therefore, the savings rate is $s_t = s = \alpha$, and the model boils down to the textbook Solow model. Final aggregate consumption (or welfare) is given by $C_t^P = r_t w_{t-1} = (1-\alpha)(q_{t-1}\kappa_t)^{1-\alpha}$, and the equilibrium growth rate of the economy, written in (20), collapses to $g_t^Y = g_{t-1}^q/\alpha$, an exogenous rate of growth, where $g_{t-1}^q = \sum_{\tau=0}^{t-1} g_\tau^A$.

In sum, the unconstrained-efficient allocation collapses to a representative-agent economy in which there exist no strategic complementarities in innovation across firms. Because the positive externality from research is internalized by the planner, the economy cannot settle around the research-intensive equilibrium and it grows exogenously through changes in q_{t-1}, without endogenously advancing through the productivity ladder. Introducing either market power or agency costs (or both) ensures that capital is allocated to lower-productivity entrepreneurs as well, thereby opening the door to the existence of equilibria with positive externalities and permitting multiplicity. Let us examine, separately, the contribution of each of these market frictions.

4.2 Frictionless credit markets with imperfect competition

Assume first that credit markets are frictionless, so that $\lambda \to 1$. Suppose however that each agent z enjoys a certain degree of market power, so $\rho < 1$ (i.e, $\sigma < +\infty$). Then the equilibrium allocation is equivalent to that of the following aggregate economy:

Proposition 3 (Aggregation with no credit frictions) *For a given a realization of the sunspot $\xi_t \in \Xi$ and any $t \in \mathbb{Z}_+$, Proposition 1 holds for the following specialization:*

[25] The proof included in Appendix A.6 will show this as a special case when $\lambda = 1$.

1. *Measured TFP is given by* $Z_t = Q_t^{1-\alpha}$, *and endogenous aggregate quality is*

$$Q_t = \left[\int_0^1 (zq_t(z))^{\sigma-1} dz\right]^{\frac{1}{\sigma-1}} = q_{t-1} \cdot \left[\int_0^1 z^{\sigma-1}\left[1 + \phi(l_t^{RD}(z)) \cdot (\Gamma(L_t^{RD})^{\sigma-1} - 1)\right] dz\right]^{\frac{1}{\sigma-1}}$$

2. *The wedge between the marginal product of capital and the rental rate is constant and given by*

$$\zeta_t = \rho < 1$$

Proof. Follows directly from using $\lambda = 1$ (so that $\mathcal{M}_t(z, q(z)) = 1$, $\forall z \in [0,1]$) into the expressions of Proposition 1. See the addendum at the end of Appendix A.5 for additional results on this special case. □

In the monopolistic economy with no credit frictions, all firms set the unconstrained price because markups coming from financial frictions are absent, or $\mathcal{M}_t(z, q(z)) = 1$. The analysis becomes very tractable. First, credit markets are frictionless, but the presence of market power still carries a wedge $\zeta < 1$ between the marginal product of capital and its return: all else equal, capital increases the final-good firm's total intermediate input endowment less than one-for-one because, given that this firm has love-for-variety preferences, it matters which entrepreneur is choosing to employ it. In spite of this, an immediate result is that, when there is coordination, welfare is higher than the unconstrained planner allocation of Section 4.1, as $C_t > r_t w_{t-1} = C_t^P$ if $\xi_t = O$. This is because the constrained planner takes into account the positive externalities embedded into innovation, and is able to exploit them by coordinating beliefs into conducting research. As a result, output growth is higher: under Assumption 3, g_t^Y overshoots due to the surge in research at the belief-switching point (when $g_t^L < 0$), and stays high for as long as TFP can grow endogenously over and above its exogenously-driven path (as $g_t^Q > g_{t-1}^q$).

The flip side is that, while in the good equilibrium innovation can push quality up through the ladder and create positive capital accumulation and growth as described above, the bad sunspot shock tends to force the economy into *deaccumulation* of capital in the long run. Importantly, this is true even when there are no exogenous forces of growth, namely when A_t shocks are absent and there is no depreciation of quality, $\delta_q = 0$. The reason why is because $K_{t+1} = \alpha Z_t K_t^{1-\alpha}$ when there is a coordination failure in period t, and TFP $Z_t = Q_t^{1-\sigma}$ is decreasing along the no-research equilibrium since

$$Q_t = q_{t-1} \cdot \left[\int_0^1 z^{\sigma-1} dz\right]^{\frac{1}{\sigma-1}} = q_{t-1} \cdot \sigma^{\frac{1}{1-\sigma}} < q_{t-1}$$

when $\xi_t = P$, and additionally $Q_t < Q_{t-1}$ when $(\xi_t, \xi_{t-1}) = (P, P)$. Similarly, repeated miscoordination causes a loss in output in the long run, as $Y_t = Z_t K_t^{1-\alpha}$ along the low equilibrium path, and both capital stock and TFP are declining in this case. In terms of growth rates in equation

(20), $g_t^Z < 0$ and $g_t^L = 0$ if $(\xi_t, \xi_{t-1}) = (P, P)$, so $g_t^Y < 0$ as well. Note that this force pushes the economy in the opposite direction to that of the standard accumulation force coming from diminishing returns (the assumption $\alpha < 1$), and it may eventually surpass it if pessimism persists for long enough. Moreover, the less substitutable inputs are, the stronger the quality effect is[26], the earlier it will overcome the capital accumulation effect, and the faster capital will converge to its steady state which, under everlasting pessimism, is zero. In this sense, pessimistic beliefs can, on their own, create crises in real outcomes. To illustrate this point, Figure 3 plots the path for output in an example economy in which $\xi_t = P$, $\forall t$, for different levels of ρ. Section 5 will present more detailed simulation exercises.

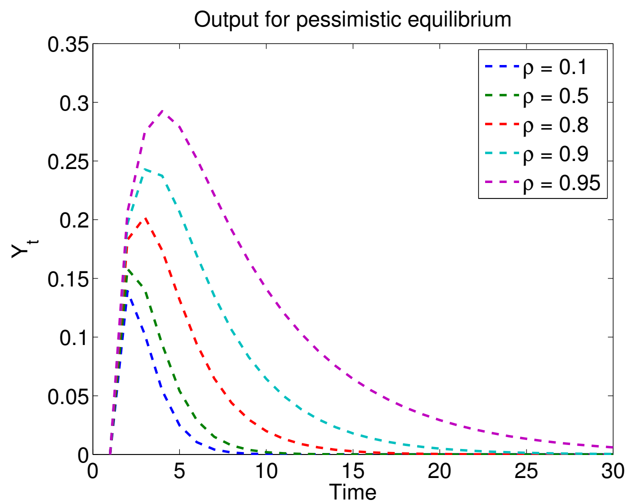

Figure 3: Example for $\alpha = 0.49$, $q_0 = 1$, $k_0 = 0.01$, $\delta_q = 0$, $\xi_t = P$, $A_t = 1$, $\forall t \in \mathbb{Z}_+$, and $\rho \in \{0.1, 0.5, 0.8, 0.9, 0.95\}$. Higher ρ (more substitutability) translates into outward shifts in the capital accumulation line.

In sum, with respect to the unconstrained planner, the presence of market power ($\rho < 1$) both creates an additional (and virtuous) equilibrium, and aggravates the one that already existed. The former is because the distribution of borrowing is non-degenerate under imperfect competition. As for the latter, intuitively, since the representative final good firm has preferences for different varieties, unproductive agents get to sell their inputs, and since varieties monopolistically compete, the efficiency with which capital is employed is eroded through inefficiently high pricing unless agents coordinate into advancing their technologies (a deflationary pressure).

4.3 Financial frictions with perfect competition

Suppose now that all frictions emerge solely from the agency problem. Accordingly, we assume that the market for intermediates is perfectly competitive ($\rho \to 1$) but intermediate borrowers face a credit constraint. The following lemma describes demands for this special case:

[26] And vice versa. Indeed, note that $\lim_{\sigma \to +\infty} \sigma^{\frac{1}{1-\sigma}} = 1$.

Lemma 5 *For the special case of perfect substitute varieties in the intermediate good sector ($\rho \to 1$), the demand for capital for any sector z of quality $q(z) = q_{t-1}$ is given by*

$$k_t(z, q_{t-1}) = \begin{cases} 0 & \text{if } z < \overline{z}_t \\ w_{t-1} \cdot (1 - \lambda \cdot (z/\overline{z}_t))^{-1} & \text{if } z \in [\overline{z}_t, \overline{z}_t/\lambda) \\ +\infty & \text{if } z \geq \overline{z}_t/\lambda \end{cases} \tag{21}$$

where $\overline{z}_t \equiv \frac{r_t}{q_{t-1}P_t}$. If $q(z) = \Gamma_t q_{t-1}$, demand is similarly defined, for the threshold $\underline{z}_t \equiv \frac{\overline{z}_t}{\Gamma_t} \leq \overline{z}_t$.

We can now state what aggregation looks like in this special case:

Proposition 4 (Aggregation with no imperfect competition) *For a given a realization of the sunspot $\xi_t \in \Xi$ and any $t \in \mathbb{Z}_+$, Proposition 1 holds for the following specialization:*

1. *Measured TFP is given by $Z_t = Q_t^{1-\alpha}$, and endogenous aggregate quality is*

$$Q_t = \int_{\overline{z}_t/\Gamma_t}^{1} zq(z)dz = \mathbb{E}_t\left[zq(z)|z \geq \underline{z}_t\right] \tag{22}$$

2. *The wedge between the marginal product of capital and the rental rate is given by*

$$\zeta_t \equiv \frac{\underline{z}_t}{\mathbb{E}_t\left[zq(z)|z \geq \underline{z}_t\right]^\alpha} \cdot q_{t-1}$$

Proof. See Appendix A.6 for both proofs, and additional results discussed in the main text.

Intuitively, in the perfect substitutes case, all firms are price-takers and, given the linearity in payoffs, a positive measure of them may choose not to produce if individual productivity is not high enough to make up for the cost of borrowing[27]. In particular, a share $\overline{z}_t$ of the market remains idle and simply consumes revenues from lending, while the remainder measure of entrepreneurs borrows up to the constraint, produces and sells intermediates at the market-clearing (homogenous) price $P_t = \Psi^{-1}(w_t)$.

Aggregation still holds in this environment and the results are similar to those of Proposition 1. A critical point of departure is the determination of endogenous TFP. Specifically, Appendix A.6 shows that closed-form solutions can also be obtained for this special case. Firstly, in the low equilibrium (no coordination), where $L_t^{RD} = 0$ and all labor is non-specialized, the threshold $\overline{z}_t$ is constant along the no-research equilibrium, or $\overline{z}_t = \overline{z}(\lambda)$, $\forall t \in \{\tau : \xi_\tau = P\}$. Namely, because all firms face the same price and operate with the same capital quality, the share of productive firms in the low equilibrium is fixed in time. Moreover, $\overline{z}(\lambda)$ is shown to be strictly increasing and strictly concave in λ, with $\overline{z}(0) = 0$ and $\overline{z}(1) = 1$. These properties say that when credit markets are completely shut down ($\lambda = 0$), all agents produce but cannot borrow and the average

[27]Indeed, note that Assumption 2 ensures the existence of constrained firms in equilibrium for any $\lambda < 1$.

productivity in the economy is lowest; and in the perfectly frictionless environment ($\lambda = 1$) only the most productive agent is active, and the average productivity in the economy is highest.

One can additionally show that measured TFP in the low equilibrium is given by:

$$Z_t = \left(\frac{\overline{z}(\lambda)^2}{\lambda} \cdot q_{t-1} \right)^{1-\alpha} \tag{23}$$

which is such that $\frac{\partial Z_t}{\partial \lambda} > 0$ and $\frac{\partial^2 Z_t}{\partial \lambda^2} < 0$. That is, higher financial frictions (lower λ) decrease TFP, and this occurs at an increasing rate the less financially developed the economy is. That aggregate TFP is endogenously affected by the measure of active entrepreneurs through the degree of financial frictions is in line with the insights of Moll (2014) and others. In fact, the formula for TFP in (22) nests the one derived in that paper, in spite of the differences in set-ups. In addition to his insights, in my model endogenous growth through quality q_{t-1} has an impact on aggregate productivity of capital as well, a dimension that was absent in Moll's work[28]. In fact, in the low equilibrium, all changes in TFP are accounted for by changes in quality. For example, if the low equilibrium persists in time due to a sequence of negative shocks, TFP will decrease while expectations are low due to the continued depreciation of intangible capital[29]. These intuitions carry through in the high equilibrium as well, but the analysis is more cumbersome (see Appendix A.6 for further details and discussion).

Comparing TFP losses Figure 4 compares pessimistic TFP levels Z_t in the specialized models of Section 4.2 (given by $Z_t = \sigma^{\frac{1}{1-\sigma}} \cdot q_{t-1}$) and 4.3 (given by equation (23)), plotted against ρ and λ, respectively. Here, $q_{t-1} = 1$ is normalized for simplicity. Note that in the frictionless economy we have $Z_t \to q_{t-1}^{1-\alpha}$ in both cases, in accordance to the results of the unconstrained planner of Section 4.1. Any degree of frictions misallocates capital and lowers TFP, so that $Z_t < q_{t-1}^{1-\alpha}$ in both cases for $\lambda, \rho < 1$. Interestingly, the loss in TFP is greater due to imperfect competition compared to imperfect credit markets. This means that coordination failures, i.e coordination into pessimistic outcomes, imply a deeper loss in productivity due to monopolistic pricing compared to collateral constraints. This result will be critical in explaining the impulse response to belief crises for each one of these models, an exercise that we develop next.

[28] Indeed, Moll (2014) obtains a remarkably similar result. In his paper, z also stands for idiosyncratic entrepreneurial productivity. The exact formula for TFP is written in Proposition 1 of the paper: $Z_t = \mathbb{E}_t[z|z \geq \underline{z}_t]^{1-\alpha}$, where the cutoff $\underline{z}_t$ is such that any agent $z \leq \underline{z}_t$ would make negative profits if she stayed active. The result in equation (22) differs in that productivity is augmented by the endogenous variable $q(z)$.

[29] Another similarity to Moll's result is that, like in his model, financial frictions do not asymptotically matter for aggregate TFP. In his model, this is because financial frictions have no direct effects on savings (other than through TFP itself) and because, if productivities are fixed, in the limit of time only the most productive entrepreneur, who is unconstrained, will accumulate all wealth. In my model, financial frictions are irrelevant for as long as there is coordination into the low equilibrium because the depreciation of quality ensures convergence to a non-production state in the long-run (conditional on repeated miscoordination), for any level of financial development.

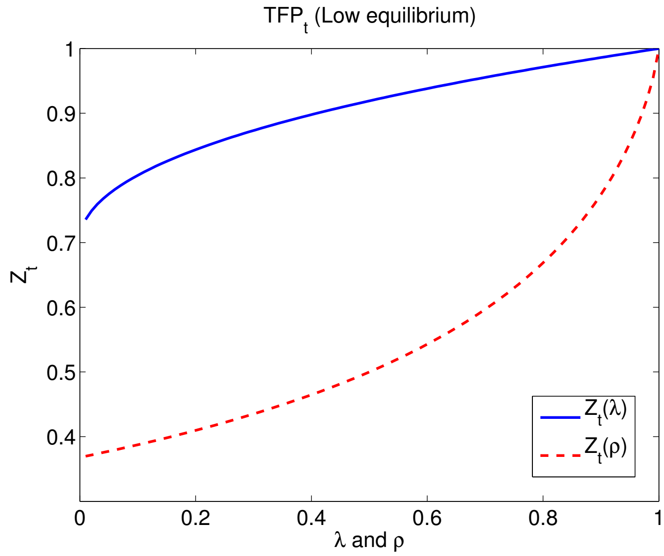

Figure 4: Measured TFP Z_t in low equilibrium plotted as a function of ρ (for special case of Section 4.2) and λ (for special case of Section 4.3). Parameters: $q_{t-1} = 1$ and $\alpha = 0.49$.

5 Crisis experiments

This section presents simulation results for a reasonable parametrization of the economy described above that aim at qualitatively replicating the three salient cyclical features of the Great Recession stated in the Introduction (the output drop, the slow recovery and the trend deviation). The simulations allow us to compare the contribution of each source of inefficiency studied above to assess which friction was more likely to be at play during the recovery[30]. Appendix B.1 provides details on the parametrization of the model, including how A_t and ξ_t shocks correlate, and the way in which the sunspot is assumed to persist in time. For simplicity, A_t is assumed to take two values: A_H (boom) and A_L (bust).

5.1 The baseline experiment

Our first experiment aims at isolating the effects of belief switches from those of real shocks on the path of the economy. To this purpose, we simulate an economy in which, for simplicity, there is repeated coordination on the high equilibrium and a high exogenous rate of growth ($A_t = A_H$) until the economy converges to a high steady state, at which point a transitory shock is implemented. In the baseline experiment, we shock beliefs without a shock to the exogenous rate of growth, which remains high for all time periods. Moreover, the belief shock is assumed to be short-lived (it lasts for only one periods), after which optimism returns forever. We use this exercise to assess the way in which changes in belief regimes can, in and on themselves, generate persistent real crises without

[30]Due to its stylized nature, and specially its two-period overlapping generations structure, the present model is not particularly apt for quantitative evaluations, which are relegated to future work. However, it offers a clean and unambiguous way of interpreting the proposed mechanism the rationalizes the highlighted patterns of the Great Recession. This section aims at providing a convincing depiction that accounts for these patterns in a qualitative dimension.

the need of exogenous real shocks in the economy.

Figure 5 shows the time series in logs of final output and TFP[31] (northern panels), as well as the growth rate of output (south-west panel) for the models with both perfect capital markets ($\lambda = 1$) and perfect competition ($\rho = 1$) of Section 4.1 (red dashed line), with perfect capital markets ($\lambda = 1$) and imperfect competition ($\rho < 1$) of Section 4.2 (blue solid line), and with imperfect capital markets ($\lambda < 1$) and perfect competition ($\rho = 1$) of Section 4.3 (black dash-dotted line). The south-east panel shows the A_t and ξ_t processes, where a black cross "$\times$" denotes coordination into optimism (with a blank space being a pessimistic realization), and a red circle "$\circ$" denotes that $A_t = A_H$ (with a blank space denoting $A_t = A_L$).

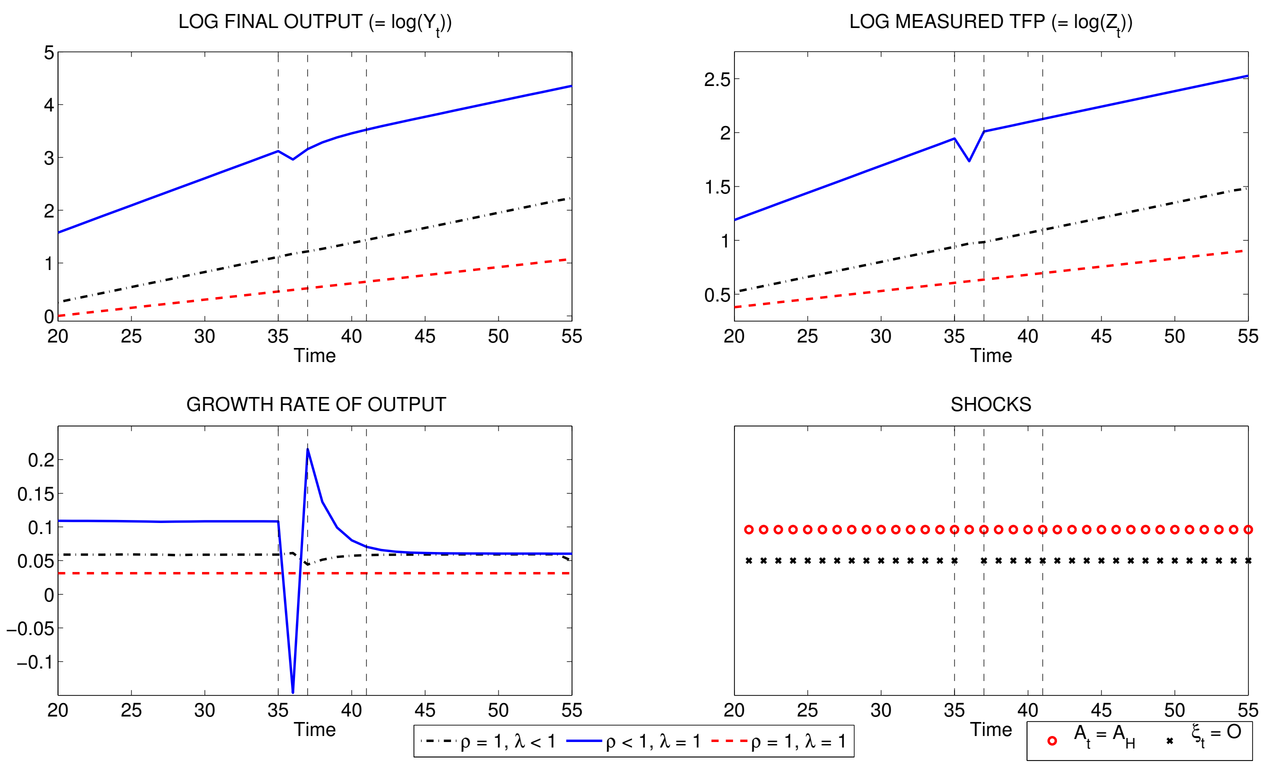

Figure 5: Baseline experiment: An isolated belief shock.

The simulation results show that, without the need of a real A_t shock, the belief-driven optimal response is able to qualitatively account for the three main distinctive features of the Great Recession listed in the Introduction: the output drop in the periods immediately following the shock, the slow medium-term recovery and the longer-run convergence to a both lower and flatter trend. The dashed vertical lines in all panels of Figure 5 accordingly split the evolution of the equilibrium variables into four stages: the pre-crisis regime ($t < 35$), the crisis stage ($t \in \{35, 36\}$), the slow recovery stage ($t \in \{37, \ldots, 41\}$), and the trend-deviation stage ($t \geq 42$), which we describe in detail in the following.

[31] Once again, TFP is computed as the Solow residual, which is endogenous in our environment: $TFP_t = \frac{Y_t}{K_t^{1-\alpha} L_t^{\alpha}}$, where $K_{t+1} = w_t = MPL_t$.

The pre-crisis regime Before the crisis in beliefs hits, the economy coordinates into employing workers in the look for new ideas, quality advances through a ladder, and output grows at a fixed and positive rate (a balanced-growth path equilibrium). The unconstrained planner economy grows slower than both the perfect capital markets equilibrium (blue solid line) and the perfect competition equilibrium (black dash-dotted line) because, unlike in those, positive externalities are not being exploited by the planner[32].

The crisis stage A negative shock to beliefs implies a slowdown in output that may place the economy in a recession. For as long as pessimism persists (a single period in this example), agents coordinate into producing at low quality levels and the economy allocates no labor to the specialized production of ideas. While this occurs, generations earn lower wages, capital is misallocated into unproductive sectors, and TFP slows down as well. Moreover, sectors operate at decelerated levels of quality, which may even "fall down through the ladder", for the depreciation rate δ_q is not offset neither by an exogenous force nor through endogenously generated advances in technology.

The depth of the crisis varies depending on which friction is at play. In the unconstrained planner, no crisis takes place for in this special case there is no coordination problem. If there is no monopoly power but capital markets are imperfect, output does not drop and it only slows down, as reflected by a decrease in the growth rate. However, when firms can place markups over their prices, output drops and the growth rate becomes negative. Behind these differences in the impulse responses is the result reflected in Figure 4: TFP losses are much greater when they arise from competitive frictions than if they are coming from collateral constraints. If the economy has accumulated enough wealth, coordination to pessimism can have a much deeper negative impact in the former economy.

In either case, output growth remains depressed during the crisis periods because of the slowdown in TFP and because, since the fact that labor supply is inelastic means that there cannot be a compensatory equilibrium price adjustment, the scarcity of the capital stock translates into a loss of production. Notice, however, that in both benchmarks the growth rate recovers some of this loss for the negative impact of quality is highest at the onset of the crisis. Finally, though not shown here, interest rates are increasing during this period because there is a decrease in savings, and capital becomes relatively cheaper.

The slow recovery stage As soon as the economy exists the recession (as marked by the return of optimism), economic growth resumes because of market-wide coordination to allocate some young workers into specialized research. The endogenous recovery is led by two forces on growth that agree in sign but differ in their implications for the speed of convergence. On the one hand, diminishing returns to both physical (or *tangible*) capital (the assumption that $\alpha > 0$) push

[32]The *levels* of the perfect-competition and the perfect-capital-market benchmarks are not easily comparable to one another. The reason for this is that the specialized models of Sections 4.2 and 4.3 introduce two separate distortions, $\rho < 1$ and $\lambda < 1$ respectively, that are difficult to relate to one another in terms of welfare. Thus, we will henceforth focus solely on responses on the time dimension, rather than in levels for each period, when making cross comparisons.

for a rapid, wealth accumulation-based recovery, as in the standard Solow growth model. On the other hand, increasing returns to quality[33] (the fact that $\phi'(\cdot) > 0$) push for a productivity-based recovery of *intangible* capital accumulation, as in the standard neo-Schumpeterian growth model

This is particularly the case for the benchmark of Section 4.2 (blue solid line), in which the recovery is faster at the beginning with a gradual slowdown. In the periods immediately following the end of the crisis, capital is scarce and very productive, so decreasing returns are strong and, in particular, more than offset the recovery based on increasing returns. This accounts for the rapid initial recovery in output, apparent in Figure 5, and the fact that the growth rate of output overshoots, as we already made clear in our discussion in the previous section about equation (20). The explanation works through a surge in research: as wages remain depressed, the research sector is highly productive as it comes out of the crisis (during which it was left unused), and it experiences a spike in the inflow of workers. Through spillovers, this boom translates into bigger advances in both magnitude (bigger step sizes, Γ_t) and likelihood (higher fraction of innovators, ϕ_t). In the periods that immediately follow, there is a steep recovery in wages and output. As *tangible* capital can resume its accumulation, however, decreasing returns to this factor gradually fade away in favor of research-based accumulation of *intangible* capital and, as the interest rate falls back toward pre-recession levels, output growth settles down to a constant growth rate.

In the benchmark of Section 4.3 (black dash-dotted line), the growth rate does not overshoot during the recovery because TFP losses are not as severe, output does not drop on impact, and there is less room for the capital-accumulation channel described above to cast a role in the growth pattern. As a result, when optimism returns, research is highly profitable coming out the recession and the growth rate recovers, faster at the beginning until it slows down to its pre-recession levels.

The trend-deviation stage In longer horizons ($t \geq 41$), output continues to increase through a balanced growth path (BGP), but may never revert to its pre-recession trend. Once again, this is mostly apparent in the case of $\rho < 1$ and $\lambda = 1$: while before the crisis agents could align their expectations into a highly research-intensive optimistic equilibrium (with a growth rate of about 10%), the loss in confidence during the recession ultimately drives agents to failing to go back to such an equilibrium, and they end up coordinating along a *worse* optimistic steady state. As a consequence, the growth rate of output becomes constant but *lower* than it was prior to the shock (about 6%). There is therefore a permanent deviation of output, TFP and capital accumulation from trend, which has *flattened* as a result of the sunspot-belief shock, and a convergence around a lower BGP. All in all, the economy is unable to fluctuate around the trend levels that would have prevailed had expectations not failed to coordinate in the first place.

In sum, the aftermath of the recession is characterized by a recovery based on rapid tangible capital accumulation in the short run and research-based productivity growth in the long run[34].

[33]As seen in Proposition 1, $Y_t = L_t^\alpha (Q_t K_t)^{1-\alpha}$, where Q_t scales with L_t^{RD} to a power that is proportional to a constant $\gamma \in (0,1)$ (see Appendix B.1 for details). Therefore, we can write output as a function of three factors, $Y_t = L_t^\alpha Q(K_t^I)^\vartheta K_t^{1-\alpha}$, where $Q(\cdot)$ embeds intangible capital K_t^I, and $\vartheta > 0$ is some constant, which shows that the production function can be though of implicitly containing increasing returns.

[34]In the context of a different model, a similar result is obtained by Matsuyama (1999), who finds that growth is based

Failure to catch up to pre-crisis trend levels in longer horizons is accounted for by the destruction of confidence, which leads agents to coordinating into lower-productivity optimistic equilibria (thereby flattening the trend).

5.2 Extensions

The role of A shocks While transitory belief shocks can qualitatively account for permanent responses in the growth rate of the economy and flatter trends, adding A shocks to the story can additionally account for permanent *downward shifts* in the trend, as there is a destruction of intangible capital stock ($\delta_q > 0$) whose reversion is impeded during the crisis stage by the very persistence of pessimism that contributes to it. This permanently lowers quality by pulling the whole quality ladder down[35].

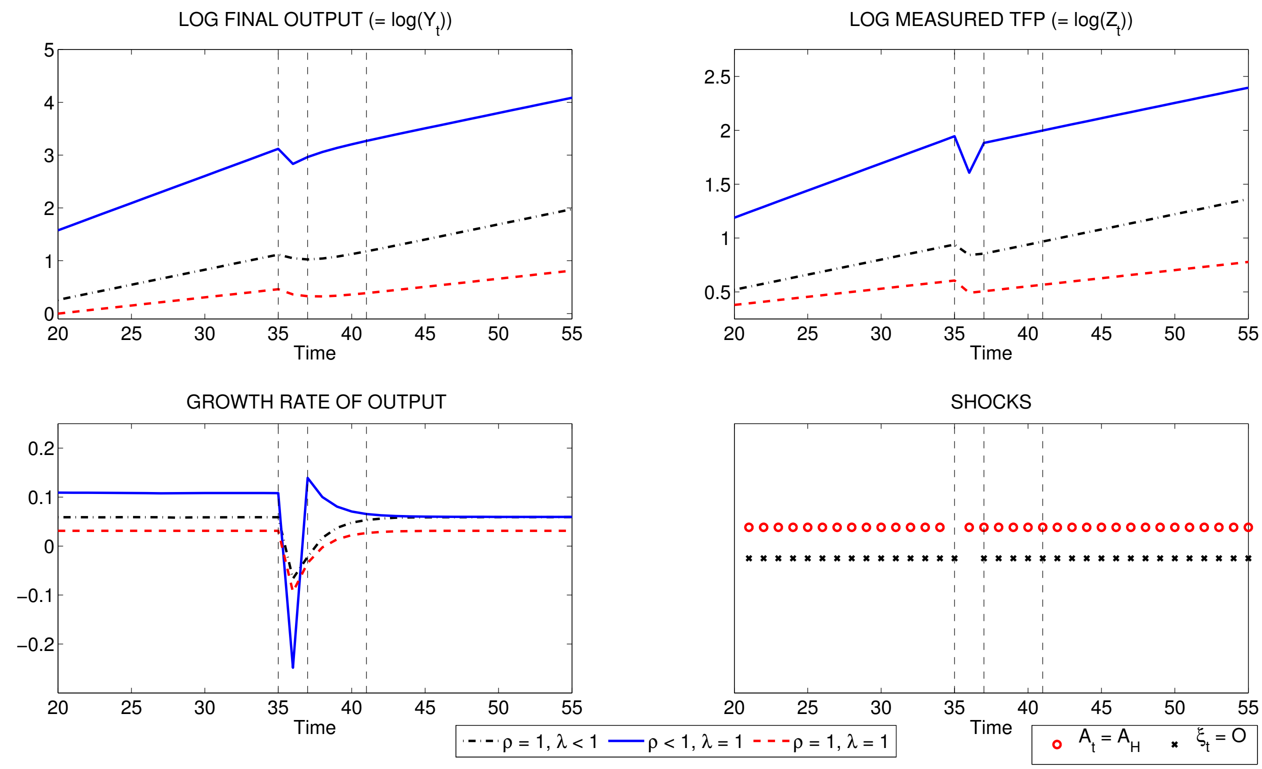

Figure 6: Extension 1: A belief shock in response to an A shock.

Figure 6 shows simulation results for an identical economy to that of Figure 5 with a single exception: the belief shock in period $t = 36$ is now in response to an A shock in period $t = 35$, as opposed to being isolated[36]. The A shock now has a permanent effect for all three benchmarks in

on a Solow regime for low levels of capital and on a Schumepetrian-Romerian regime for higher levels of capital. His mechanism, however, is based on the short-term nature of monopoly profits that accrue from innovative activities, and not on the cyclical evolution of beliefs.

[35] To draw a maybe clarifying analogy, it is as if there were many ladders, one on top of the other and aligned in a parallel fashion, and negative A shocks made the economy jump down to lower ladders.

[36] Note that sunspots respond with a one-period lag because A shocks are only realized at the end of each period.

that it shifts trends down in a parallel manner (relative to the counterfactual trends of Figure 5 shown above). During the crisis period in which there is no endogenous growth, the exogenous rate of growth is depressed and, in particular, cannot compensate the loss in quality coming from the depreciation of intangible capital. This shifts the entire ladder down so that, when the recovery unfolds, the economy converges to a lower steady state both in terms of potential long-run output levels and potential growth rates. Moreover, in the cases of the unconstrained planner and perfect competition with imperfect capital markets there is now, unlike before, an actual drop in output in levels.

Differences in persistence One can further experiment with other compositions of shocks in order to assess the importance of the persistence in responses. According to our initial motivation, a unique feature of the Great Recession and its aftermath is the fact that agents have remained pessimistic about their economic prospects for an unusually long period of time (recall Figure 2). To illustrate this point in our framework, we briefly consider three minimal departures from our experiment above.

In the first extension, agents keep being pessimistic long after the A shock has reverted back to boom levels (see Figure 7 in Appendix B.2). The experience of the economy is here similar to our last simulation, but the persistence in beliefs deepens the severity of the recession, which lasts longer and causes a larger drop in output. The crisis stage lasts for as long as pessimism exists, and not for as long as exogenous growth is depressed. This shows the importance of miscoordination is the duration of recessions, even if the shocks that originated them recover fast.

In the second extension, agents are *never* pessimistic even though a real shock A_L hits the economy (see Figure 8 in Appendix B.2). In this case, the economy may re-emerge from the crisis stronger than before, with coordination into a higher optimistic equilibrium than its pre-recession counterpart. This case qualitatively rationalizes episodes in which high increasing returns after the destruction of capital has taken place spur coordination into higher-growth potentials.

Lastly, in our final extension, agents do respond pessimistically to a negative and persistent A shock, but they regain optimism before the crises in A ends (see Figure 9 in Appendix B.2). The response here exhibits interesting non-monotonicities. In particular, two consecutive recession-recovery episodes may occur (see blue solid line): depressed beliefs and low exogenous growth cause a first recession as described above, but a second recession emerges endogenously shortly after. The reason why this happens is that, after the period of rapid growth based on capital accumulation is over, the endogenous force of growth stemming from optimistic investment into innovation is unable to compensate for a relatively large depreciation of quality, which has kept falling down the ladder as the real shock is still in a low state. Reversion back to the A_H regime marks the end of the second recession, innovation takes over depreciation and quality starts to advance up the ladder again. From then on, the trend-deviation stage unfolds as explained above[37].

[37] In general, the non-monotonic recovery stage on top of the long-run non-reversion give a possible intuition for the experience of certain industrialized countries in the Euro area, who suffered relatively medium-sized replicas some quarters after the initial drop in output. This phenomenon is sometimes referred to as a double-dip (or "W-shaped") recovery. A

To sum up, through the lens of the theory, one can view these examples as offering an interpretation of the Great Recession and Great Deviation experiences in recent years. First, belief switches can cause a crisis without the need of a real shock to take place. Second, the short-run response is tied to the recovery of confidence. Third, the degree of coordination out of a crisis determines whether the new trend is steeper or flatter than its pre-recession counterpart. If crises are grave enough, trend slope-deviations may occur. Fourth, adding real shocks can account for shift-deviations in trend as well, and permanent losses in aggregate efficiency coming from depreciation in quality during the crisis periods that cannot be reverted by growth in the research sector. Finally, the timing of the recovery in real shocks vis-à-vis belief shocks determines the shape of the recovery, and non-monotonicities with potential crisis replicas may emerge endogenously.

6 Conclusion

In this paper, I have proposed a stylized theory of crises that are followed by slow recoveries and stagnations around lower pre-crisis trend levels (or economic stagnations). In the model, coordination failures in innovative-intensive sectors can lead the economy into a path of low growth, destruction of intangible capital (or knowledge) and depressed expectations regarding future aggregate demand. I have shown that, in response to aggregate shocks to confidence, the model can generate some of the most salient qualitative features of the Great Recession (namely the large drop, the slow convergence and the stagnation around lower and flatter trend levels). A critical dimension in the equilibrium response is the exposure of the economy to pessimistic beliefs and, in particular, to their duration in the crisis state with respect to real exogenous shocks to growth: if pessimisms pervades, the recovery is based on capital accumulation in the short-run and productivity growth in the long run. This can create non-monotonic responses that resemble some of the recent experiences in certain industrialized economies.

A desirable avenue for future research is to study further the interaction between belief formation and the equilibrium outcomes that, under those beliefs, agents coordinate upon. In my model, a sunspot chooses the equilibrium with independence of the realization of the aggregate shock. In simulations, we correlated both these processes in a meaningful yet exogenous manner. A way of internalizing the feedback loop would be to discipline the selection of equilibria with a global game in which agents would receive informative yet noisy signals about unobserved economic fundamentals, such as A_t. Another way would be to endogenize the information sets of agents by making uncertainty in the economy be an equilibrium outcome of their very own decisions.

Finally, in attempting to replicate some of the more quantitative patterns in the data, a promising line of research would be to frame the main mechanisms of the present model into a quantitative endogenous growth model, for instance in the spirit of models of multi-product firms à la Klette

typical example is the 1980s recession in the U.S. The economy fell into recession from January 1980 to July 1980, and it shrunk at an 8% annual rate from April 1980 to June 1980. The country experienced then a short period of growth, and from January to March of 1981 grew at an 8.4% annual rate.

and Kortum (2004), such as Akcigit and Kerr (2010) and Acemoglu and Cao (2015). Taking into account heterogeneities in the portfolio of products and being able to distinguish between small and large firms in a more meaningful way could add new insights into the theoretical mechanisms developed above. These extensions remain in the agenda for future work.

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A Proofs and additional results

A.1 Proof of Lemma 1

We seek to solve problem (1). Taking first-order conditions with respect to L_t and $x_t(z)$ gives

$$\begin{aligned} \alpha L_t^{\alpha-1} X_t^{1-\alpha} &= w_t \\ \left(\frac{1-\alpha}{\rho}\right) L_t^{\alpha} X_t^{1-\alpha-\rho} \rho x_t(z)^{\rho-1} &= p_t(z) \end{aligned}$$

Defining the elasticity of substitution between varieties as $\sigma \equiv -\frac{\partial \ln x_t(z)/x_t(z')}{\partial \ln p_t(z)/p_t(z')}$, $\forall (z,z') \in [0,1]^2$, note $\sigma = \frac{1}{1-\rho}$. We can rewrite the first-order condition for $x_t(z)$ as

$$x_t(z) = \left(\frac{(1-\alpha)L_t^{\alpha}}{p_t(z)}\right)^{\sigma} X_t^{1-\alpha\sigma} \tag{24}$$

which, inside the definition of X_t, gives

$$X_t = \left(\int_0^1 \left(x_t(z)\right)^{\frac{\sigma-1}{\sigma}} dz\right)^{\frac{\sigma}{\sigma-1}} = \left[(1-\alpha)L_t^{\alpha}\right]^{\sigma} \left(X_t\right)^{1-\alpha\sigma} \left(P_t\right)^{-\sigma}$$

where P_t is the composite price index defined in Lemma 1. Solving for X_t, we obtain the relationship from equation (9). Using the latter into (24) then yields the iso-elastic inverse demand for intermediates in equation (7).

As for the demand for labor in equation (8), first notice that one can use the definitions for X_t and P_t and the optimality conditions above to write aggregate expenditure in the simple reduced form $\int_0^1 p_t(z)x_t(z)dz = P_t X_t$. Next, consider using condition (9) to restate problem (1). Then, we have

$$\begin{aligned} \Pi_t &\equiv \max_{L \geq 0, (x(z))_{z\in[0,1]}} \left\{ L^{\alpha} X_t^{1-\alpha} - w_t L - P_t X_t \right\} \\ &= \max_{L\geq 0} \left\{ \left(\frac{1-\alpha}{P_t}\right)^{\frac{1-\alpha}{\alpha}} L - w_t L - P_t \left(\frac{1-\alpha}{P_t}\right)^{\frac{1}{\alpha}} L \right\} \\ &= \max_{L\geq 0} \left\{ \left[\left(\frac{1-\alpha}{P_t}\right)^{\frac{1-\alpha}{\alpha}} - P_t \left(\frac{1-\alpha}{P_t}\right)^{\frac{1}{\alpha}} \right] - w_t \right\} L \\ &= \max_{L\geq 0} \left\{ \left(\Psi(P_t) - w_t\right) L \right\} \end{aligned}$$

where $\Psi(x) = \alpha \left(\frac{1-\alpha}{x}\right)^{\frac{1-\alpha}{\alpha}}$. The demand function for labor in equation (8) then follows immediately. $\square$

A.2 Proof of Lemma 2, and additional results

Given a decision for $q(z) \in \{\Gamma(L_t^{RD})q_{t-1}, q_{t-1}\}$ and taking prices r_t and w_t, and aggregates X_t and P_t as given, monopolist z from generation $t-1$ and youth earnings of w_{t-1} solves the problem

$$\pi_t(z, q(z)) = \max_{k(z), p(z)} \left\{ p(z) x(z, q(z)) - r_t (k(z) - w_{t-1}) \right\}$$

subject to the production function $x(z,q(z)) = zq(z)k(z)$, the inverse demand function $x(z) = \left(\frac{p(z)}{P_t}\right)^{-\sigma} X_t$ and the borrowing constraint $r_t(k(z) - w_{t-1}) \leq \lambda \cdot p(z)x(z,q(z))$.

Plugging the technology and the inverse demand for intermediates into the objective, and defining $\mu_t(z,q(z)) \geq 0$ as the multiplier of agent z on the financial constraint, the Lagrangian equation is:

$$\mathcal{L}_t(z,q(z)) = \max_{p(z)} \left\{ p(z)^{1-\sigma} P_t^{\sigma} X_t - \frac{r_t}{zq(z)} \left(\frac{p(z)}{P_t}\right)^{-\sigma} X_t + r_t w_{t-1} + \mu_t(z,q(z)) \cdot \left[\left(\frac{p(z)}{P_t}\right)^{-\sigma} X_t \left(\lambda p(z) - \frac{r_t}{zq(z)}\right) + r_t w_{t-1} \right] \right\}$$

The FOC reads:

$$\begin{aligned} 0 \quad = \quad & (1-\sigma)p(z)^{-\sigma} P_t^{\sigma} X_t + \frac{r_t}{zq(z)} \sigma p(z)^{-(1+\sigma)} P_t^{\sigma} X_t \\ & + \mu_t(z,q(z)) \cdot \left[(1-\sigma)\lambda p(z)^{-\sigma} P_t^{\sigma} X_t + \frac{r_t}{zq(z)} \sigma p(z)^{-(1+\sigma)} P_t^{\sigma} X_t \right] \end{aligned}$$

By the complementary slackness condition, we require:

$$\mu_t(z,q(z)) \cdot \left[\left(\frac{p_t(z)}{P_t}\right)^{-\sigma} X_t \left(\lambda p_t(z) - \frac{r_t}{zq(z)}\right) + r_t w_{t-1} \right] = 0 \tag{25}$$

From the FOC, we can obtain a solution for price and for the multiplier, for each agent z selling goods of quality $q(z)$:

$$\begin{aligned} p_t(z,q(z)) &= \left(\frac{\sigma}{\sigma-1}\right) \frac{r_t}{zq(z)} \frac{1+\mu_t(z,q(z))}{1+\lambda \cdot \mu_t(z,q(z))} \\ \mu_t(z,q(z)) &= \frac{\frac{r_t}{zq(z)} - \left(\frac{\sigma-1}{\sigma}\right) p_t(z,q(z))}{\lambda \left(\frac{\sigma-1}{\sigma}\right) p_t(z,q(z)) - \frac{r_t}{zq(z)}} \end{aligned} \tag{26}$$

In the first equation, which we wrote in (10) in Lemma 2, we observe that the function

$$p_t^U(z,q(z)) \equiv \left(\frac{\sigma}{\sigma-1}\right) \frac{r_t}{zq(z)}$$

is the equilibrium price when $\mu_t(z,q(z)) = 0$. From the complementary slackness condition, this is the case when the borrowing constraint is slack, and the agent is unconstrained.

Constrained agents are such that $\mu_t(z,q(z)) > 0$. By the FOC, this forces them to set a price that is higher than the unconstrained one, or $p_t^C > p_t^U$. Their demand for capital is given by $k_t(z,q(z)) = \overline{k}_t(z,q(z))$, where $\overline{k}_t(z,q(z))$ is the solution to $r_t(\overline{k}_t(z,q(z)) - w_{t-1}) = \lambda p_t^C(z,q(z))\overline{x}_t(z,q(z))$, that is

$$\overline{k}_t(z,q(z)) = \frac{w_{t-1}}{1 - \lambda z \frac{p_t^C(z,q(z))q(z)}{r_t}} = \frac{w_{t-1}}{1 - \lambda \left(\frac{\sigma}{\sigma-1}\right) \frac{1+\mu_t(z,q(z))}{1+\lambda \cdot \mu_t(z,q(z))}}$$

and $\overline{x}_t(z,q(z))$ satisfies both the inverse demand function $\overline{x}_t(z,q(z)) = \left(p_t^C(z)/P_t\right)^{-\sigma} X_t$, and the technological requirement that $\overline{x}_t(z,q(z)) = zq(z)\overline{k}_t(z,q(z))$.

Using that $\overline{k}_t(z,q(z)) = \left(\frac{p_t^C(z,q(z))}{P_t}\right)^{-\sigma} \frac{X_t}{zq(z)}$, we obtain that the price set by agent z is implicitly defined by the solution p_t to

$$(p_t)^{-\sigma}\left(\frac{r_t}{zq(z)} - \lambda p_t\right) = \frac{r_t w_{t-1}}{X_t P_t^{\sigma}}$$

which shows how to obtain equation (11).

Finally, we show why Assumption 2 is sufficient and necessary for a non-zero measure share of the industry to be constrained in equilibrium. From equation (26) we note that the only instance in which $\mu_t(z, q(z))$ can be strictly positive is if the following condition holds

$$\frac{r_t}{zq(z)p_t^C(z, q(z))} \in \left(\lambda\left(\frac{\sigma-1}{\sigma}\right), \frac{\sigma-1}{\sigma}\right)$$

From this, we have that the feasibility condition $\overline{k}(z, q(z)) \geq 0$ holds if, and only if, $\lambda \leq \rho$. We also see that $\overline{k}_t(z, q(z)) > w_{t-1}$, so that the agent is effectively borrowing, if $\lambda < 1$, which is always true. Incidentally, it is precisely this condition that shows a basic relation between constrained and unconstrained prices:

$$p^C(z, q(z)) \in \left(p^U(z, q(z)), \frac{p^U(z, q(z))}{\lambda}\right)$$

for all $z \in [0, 1]$ and $q(z) \in \{\Gamma q, q\}$. □

A.3 Proof of Lemma 3

Existence of the threshold follows from the fact that there is a unique solution to (11). As for its characterization, if $q(z) = q_{t-1}$ (low quality), the marginal agent at period t has identity $z = \overline{z}_t$ such that $\mu_t(\overline{z}_t, q_{t-1}) = 0$ and yet $k_t(\overline{z}_t, q_{t-1}) = \overline{k}_t(\overline{z}_t, q_{t-1})$. Using these identities in the solutions found in Appendix A.2 will, after some algebra, give the solution in (12). Finally, if instead $q(z) = \Gamma_t q_{t-1}$ where $\Gamma_t \equiv \Gamma(L_t^{RD})$ (high quality), the proof is identical except that the relevant threshold becomes $\underline{z}_t \equiv \overline{z}_t/\Gamma_t \leq \overline{z}_t$. □

A.4 Proof of Lemma 4

In anticipation of the production-stage profits at given $q \equiv q_{t-1}$, agents make ex-ante decisions on innovation. Any agent z solves

$$\max_{l^{RD} \geq 0}\left\{\phi(l^{RD}) \cdot \pi_t^I(z, q) + (1 - \phi(l^{RD})) \cdot \pi_t^{NI}(z, q) - w_t l^{RD}\right\}$$

where π^I and π^{NI} stand for profits after innovation ($q(z) = \Gamma_t q$) and after no innovation ($q(z) = q$), respectively. In particular,

$$\pi_t^I(z, q) \equiv \begin{cases} \pi_t^U(z, \Gamma q) & \text{if } z \in \mathcal{Z}_t^L \\ \pi_t^C(z, \Gamma_t q) & \text{if } z \in \mathcal{Z}_t^M \\ \pi_t^C(z, \Gamma_t q) & \text{if } z \in \mathcal{Z}_t^H \end{cases} \qquad \text{and} \qquad \pi_t^{NI}(z, q) \equiv \begin{cases} \pi_t^U(z, q) & \text{if } z \in \mathcal{Z}_t^L \\ \pi_t^U(z, q) & \text{if } z \in \mathcal{Z}_t^M \\ \pi_t^C(z, q) & \text{if } z \in \mathcal{Z}_t^H \end{cases}$$

where π^C and π^U have been defined in the main text. Taking first order conditions, we see that

$$
\begin{aligned}
w_t &= \phi'(l_t^{RD}(z)) \cdot \Big[\underbrace{\pi_t^U(z,\Gamma q) - \pi_t^U(z,q)}_{\equiv \Delta_t^L(z)}\Big] \\
w_t &= \phi'(l_t^{RD}(z)) \cdot \Big[\underbrace{\pi_t^C(z,\Gamma q) - \pi_t^U(z,q)}_{\equiv \Delta_t^M(z)}\Big] \\
w_t &= \phi'(l_t^{RD}(z)) \cdot \Big[\underbrace{\pi_t^C(z,\Gamma q) - \pi_t^C(z,q)}_{\equiv \Delta_t^H(z)}\Big]
\end{aligned}
$$

from which the demand function in Lemma 4 is immediate. □

A.5 Proof of Proposition 1, and additional results

Since the general baseline model does not admit explicit solutions (in particular, under Assumption 2, some firms set prices which are the solution to a non-linear equation), we derive all aggregate equilibrium expressions in terms of generic Lagrange multipliers. In particular, recall that

$$\mathcal{M}_t(z,q(z)) = \frac{1+\mu_t(z,q(z))}{1+\lambda \cdot \mu_t(z,q(z))} \geq 1$$

is the price markup coming from financial frictions, where $\mu_t(z,q(z)) \geq 0$ is the Lagrange multiplier on the financial constraint (3). For brevity, throughout we use $q \equiv q_{t-1}$ and $q(z) \in \{q, \Gamma_t q\}$. From Lemma 2, we know that each agent z of quality $q(z)$ sets the price:

$$p_t(z,q(z)) = \mathcal{M}_t(z,q(z)) \left(\frac{\sigma}{\sigma-1}\right) \frac{r_t}{zq(z)}$$

Capital demand is $k_t(z,q(z)) = \left(\frac{p_t(z,q(z))}{P_t}\right)^{-\sigma} \frac{X_t}{zq(z)}$, namely

$$k_t(z,q(z)) = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \frac{X_t P_t^{\sigma}}{r_t^{\sigma}(zq(z))^{1-\sigma}} \mathcal{M}_t(z,q(z))^{-\sigma} \tag{27}$$

and profits from the production stage are

$$
\begin{aligned}
\pi_t(z,q(z)) &= \left[\left(\frac{\sigma}{\sigma-1}\right)\mathcal{M}_t(z,q(z)) - 1\right] \cdot r_t \cdot k_t(z,q(z)) + r_t w_{t-1} \\
&= \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} \frac{X_t P_t^{\sigma}}{r_t^{\sigma-1}} \frac{1}{zq(z)} \left(\frac{zq(z)}{\mathcal{M}_t(z,q(z))}\right)^{\sigma} \left[\left(\frac{\sigma}{\sigma-1}\right)\mathcal{M}_t(z,q(z)) - 1\right] + r_t w_{t-1}
\end{aligned}
$$

From this last equation, we make the following observations. First, the aggregate profits in the productive stage of the intermediate-good sector are given by

$$
\begin{aligned}
\Pi_t^{IG} \equiv \int_0^1 \pi_t(z,q(z))dz &= \left(\frac{\sigma}{\sigma-1}\right) r_t \int_0^1 k_t(z,q(z))\mathcal{M}_t(z,q(z))dz \\
&= \left[\left(\frac{\sigma-1}{\sigma}\right)\frac{Q_t}{r_t}\right]^{\sigma-1} X_t P_t^{\sigma}
\end{aligned}
$$

where $K_t \equiv \int_0^1 k_t(z, q(z))dz$ is the aggregate capital stock, $K_t = w_{t-1}$ by market clearing, and

$$Q_t \equiv \left[\int_0^1 \left(\frac{zq_t(z)}{\mathcal{M}_t(z, q(z))} \right)^{\sigma-1} dz \right]^{\frac{1}{\sigma-1}}$$

is the composite index of endogenous capital efficiency, normalized for the price markup coming from financial frictions. In turn, Q_t is endogenously given by whether or not there is innovation in equilibrium. In the bad equilibrium, $q(z) = q_{t-1}$, $\forall z \in [0, 1]$, and therefore $Q_t = q_{t-1} \cdot (\int_0^1 (z/\mathcal{M}(z, q))^{\sigma-1} dz)^{\frac{1}{\sigma-1}}$. In the good equilibrium, $q(z) = \Gamma_t q_{t-1}$ for $z \in \mathbb{I}_t$, and $q(z) = q_{t-1}$ otherwise, where $\mathbb{I}_t \subseteq [0, 1]$ is the set of agents who successfully implement ideas in their technologies. Since an innovative agent is successful with probability $\phi(\cdot)$, we have $Q_t = \{\int_0^1 [(zq/\mathcal{M}(z, q))^{\sigma-1} + \phi(l_t^{RD}(z)) \cdot ((z\Gamma_t q/\mathcal{M}(z, \Gamma q))^{\sigma-1} - (zq/\mathcal{M}(z, q))^{\sigma-1}) dz]\}^{\frac{1}{\sigma-1}}$. Using that $\Gamma_t = 1$ in the bad equilibrium, we can encompass both cases into

$$Q_t = q_{t-1} \cdot \left[\int_0^1 z^{\sigma-1} \left[\mathcal{M}_t(z, q)^{1-\sigma} + \phi(l_t^{RD}(z)) \cdot \left(\left(\frac{\Gamma_t}{\mathcal{M}_t(z, \Gamma_t q)} \right)^{\sigma-1} - \mathcal{M}_t(z, q)^{1-\sigma} \right) \right] dz \right]^{\frac{1}{\sigma-1}}$$

Second, marginal profits out of innovation, which we defined as $\Delta_t(z) \equiv \pi_t(z, \Gamma_t q) - \pi_t(z, q)$, are given by

$$\begin{aligned} \Delta_t(z) \quad &= \quad \left(\frac{\sigma - 1}{\sigma} \right)^{\sigma} \frac{X_t P_t^{\sigma}}{r_t^{\sigma-1}} (zq)^{\sigma-1} \Bigg\{ \Gamma_t^{\sigma-1} \mathcal{M}_t(z, \Gamma_t q)^{-\sigma} \left[\left(\frac{\sigma}{\sigma - 1} \right) \mathcal{M}_t(z, \Gamma_t q) - 1 \right] \\ &\quad -\mathcal{M}_t(z, q)^{-\sigma} \left[\left(\frac{\sigma}{\sigma - 1} \right) \mathcal{M}_t(z, q) - 1 \right] \Bigg\} \end{aligned}$$

Third, final consumption for each sector entrepreneur z is given by $c_t^{t-1}(z) = \pi_t(z, q) + \phi(l_t^{RD}(z)) \cdot \Delta_t(z) - w_t l_t^{RD}(z)$, so that aggregate consumption in period t, or $C_t \equiv \int_0^1 [c_t^{t-1}(z) + c_t^t(z)] dz$, is

$$\begin{aligned} C_t \quad &= \quad \left(\frac{\sigma - 1}{\sigma} \right)^{\sigma} \frac{X_t P_t^{\sigma}}{r_t^{\sigma-1}} \int_0^1 (zq)^{\sigma-1} \Bigg\{ \mathcal{M}_t(z, q)^{-\sigma} \left[\left(\frac{\sigma}{\sigma - 1} \right) \mathcal{M}_t(z, q) - 1 \right] \\ &\quad +\phi(l_t^{RD}(z)) \cdot \left(\Gamma_t^{\sigma-1} \mathcal{M}_t(z, \Gamma_t q)^{-\sigma} \left[\left(\frac{\sigma}{\sigma - 1} \right) \mathcal{M}_t(z, \Gamma_t q) - 1 \right] - \mathcal{M}_t(z, q)^{-\sigma} \left[\left(\frac{\sigma}{\sigma - 1} \right) \mathcal{M}_t(z, q) - 1 \right] \right) \Bigg\} dz \\ &\quad +r_t K_t - w_t L_t^{RD} \end{aligned}$$

where we have used capital market clearing, i.e. $w_{t-1} = K_t$, in the last line. Using capital demand (27), the market clearing condition $w_{t-1} = K_t = \int_0^1 k_t(z, q(z))dz$ reads

$$K_t = \left[\left(\frac{\sigma - 1}{\sigma} \right) \frac{P_t}{r_t} \right]^{\sigma} X_t \int_0^1 \frac{1}{zq(z)} \left(\frac{zq(z)}{\mathcal{M}_t(z, q(z))} \right)^{\sigma} dz$$

Moreover, the aggregate price index can be written as follows:

$$P_t = \left[\int_0^1 \Big(p_t(z, q(z)) \Big)^{1-\sigma} dz \right]^{\frac{1}{1-\sigma}} = \left(\frac{\sigma}{\sigma - 1} \right) \frac{r_t}{Q_t}$$

Thus, $r_t = \rho P_t Q_t$. Plugging this back into market clearing, we get that

$$\frac{K_t}{X_t} = Q_t^{-\sigma} \int_0^1 \frac{1}{zq(z)} \left(\frac{zq(z)}{\mathcal{M}_t(z, q(z))} \right)^{\sigma} dz$$

Using that $Y_t = L_t^{\alpha} X_t^{1-\alpha}$, then output can be written in terms of capital and labor as $Y_t = Z_t \cdot L_t^{\alpha} K_t^{1-\alpha}$. This shows that measured TFP, defined as $Z_t \equiv \frac{Y_t}{L_t^{\alpha} K_t^{1-\alpha}}$, is given by equation (18). Good market clearing requires that $w_t = \Psi(P_t)$, where $w_t = K_{t+1}$ from credit market clearing, and therefore we can write

$$K_{t+1} = \Psi(P_t) = \Psi \left[\left(\frac{\sigma}{\sigma - 1} \right) \frac{r_t}{Q_t} \right] = \Psi \left((1 - \alpha) \frac{Y_t}{Q_t K_t} \right) = \alpha Z_t \left(\frac{L_t}{K_t} \right)^{\alpha - 1}$$

a law of motion for capital, where the last equality uses that $P_t X_t = (1 - \alpha) Y_t$ from the final good firm's solution. Moreover, using the results inside our formula for Π_t^{IG} gives that $\Pi_t^{IG} = P_t X_t$. In turn, using P_t, X_t and Q_t back into our formula for C_t, and exploiting the fact that $w_t = K_{t+1}$, some algebra shows that $C_t = \Pi_t^{IG} - K_{t+1} L_t^{RD}$. To show that the income identity $Y_t = C_t + K_{t+1}$ holds, note

$$C_t + K_{t+1} = \Pi_t^{IG} - K_{t+1} L_t^{RD} + K_{t+1} = P_t X_t + w_t L_t = (1 - \alpha) Y_t + \alpha Y_t = Y_t$$

where we have used $K_{t+1} = w_t$, $\Pi_t^{IG} = P_t X_t$, and $\frac{P_t X_t}{Y_t} = 1 - \alpha$ and $\frac{w_t L_t}{Y_t} = \alpha$ from the final-good producer's solution. Finally, L_t^{RD} is a fixed point solving the labor market clearing equation, $L_t^{RD} = \int_0^1 (\phi')^{-1} \left(\frac{w_t}{\Delta_t(z)} \right) dz$. $\square$

Addendum. **Special case with $\rho < 1$ and $\lambda \to 1$:** Specializing the above results to $\lambda \to 1$ we have that the capital demanded by agent z is

$$k_t(z, q(z)) = \left(\frac{\sigma - 1}{\sigma} \right)^{\sigma} \frac{X_t P_t^{\sigma}}{r_t^{\sigma} (zq(z))^{1-\sigma}}$$

and profits are

$$\pi_t(z, q(z)) = \frac{(\sigma - 1)^{\sigma - 1}}{\sigma^{\sigma}} \frac{X_t P_t^{\sigma}}{r_t^{\sigma - 1} (zq(z))^{1-\sigma}} + r_t w_{t-1}$$

so that marginal profits out of innovation are given by

$$\Delta_t(z) = \frac{(\sigma - 1)^{\sigma - 1}}{\sigma^{\sigma}} \frac{X_t P_t^{\sigma}}{r_t^{\sigma - 1}} (zq)^{\sigma - 1} \left[\Gamma_t^{\sigma - 1} - 1 \right]$$

Final consumption for each sector entrepreneur z is given by

$$c_t^{t-1}(z) = \frac{(\sigma - 1)^{\sigma - 1}}{\sigma^{\sigma}} \frac{X_t P_t^{\sigma}}{r_t^{\sigma - 1}} (zq)^{\sigma - 1} \left[1 + \phi(l_t^{RD}(z)) \cdot (\Gamma_t^{\sigma - 1} - 1) \right] + r_t w_{t-1} - w_t l_t^{RD}(z)$$

so that aggregate consumption in period t, or $C_t \equiv \int_0^1 [c_t^{t-1}(z) + c_t^t(z)] dz$, is

$$C_t = \frac{(\sigma - 1)^{\sigma - 1}}{\sigma^{\sigma}} \frac{X_t P_t^{\sigma}}{r_t^{\sigma - 1}} \int_0^1 (zq)^{\sigma - 1} \left[1 + \phi(l_t^{RD}(z)) \cdot (\Gamma_t^{\sigma - 1} - 1) \right] dz + r_t K_t - w_t L_t^{RD}$$

Capital market clearing then simply reads $w_{t-1} = \left[\left(\frac{\sigma - 1}{\sigma} \right) \frac{P_t}{r_t} \right]^{\sigma} Q_t^{\sigma - 1} X_t$, where $Q_t \equiv \left[\int_0^1 (zq(z))^{\sigma - 1} dz \right]^{\frac{1}{\sigma - 1}}$ is a composite index of endogenous aggregate capital efficiency. Moreover, the aggregate price index can be written $P_t = \left[\int_0^1 \left(p_t(z, q(z)) \right)^{1-\sigma} dz \right]^{\frac{1}{1-\sigma}} = \left(\frac{\sigma}{\sigma - 1} \right) \frac{r_t}{Q_t}$, which back into market clearing delivers $X_t = Q_t \cdot K_t$.

Using that $Y_t = L_t^{\alpha} X_t^{1-\alpha}$, then output can be written in terms of capital and labor as $Y_t = L_t^{\alpha}\left(Q_t K_t\right)^{1-\alpha}$. Therefore, measured TFP is given by $Z_t = Q_t^{1-\alpha}$.

In turn, Q_t is endogenously given by whether or not there is innovation in equilibrium. In the bad equilibrium, $q(z) = q_{t-1}$, $\forall z \in [0,1]$, and therefore $Q_t = q_{t-1} \cdot (\int_0^1 z^{\sigma-1} dz)^{\frac{1}{\sigma-1}} = q_{t-1} \cdot \sigma^{\frac{1}{1-\sigma}}$. In the good equilibrium, $q(z) = \Gamma_t q_{t-1}$ for $z \in \mathbb{I}_t$, and $q(z) = q_{t-1}$ otherwise. Thus, for any $t \in \mathbb{Z}_+$,

$$Q_t = q_{t-1} \cdot \left[\int_0^1 z^{\sigma-1}\left[1 + \phi(l_t^{RD}(z)) \cdot (\Gamma(L_t^{RD})^{\sigma-1} - 1)\right] dz\right]^{\frac{1}{\sigma-1}}$$

as written in equation (21). Finally, L_t^{RD} is a fixed point solving the labor market clearing equation, $L_t^{RD} = \int_0^1 (\phi')^{-1}\left(\frac{w_t}{\Delta_t(z)}\right) dz$. Due to the assumptions placed on $\phi(\cdot)$, it is not hard to show that this equation has at least one solution $L_t^{RD} > 0$, which corresponds to the case of $\xi_t = O$. In particular, using $w_t = K_{t+1} = MPL_t$ and the formula for $\Delta_t(z)$, one can show that $L_t^{RD} > 0$ is a fixed point of

$$L_t^{RD} = \int_0^1 (\phi')^{-1}\left\{\frac{\Psi(P_t)}{P_t \Psi(P_{t-1})} \frac{\sigma Q_t^{\sigma-2}}{(zq_{t-1})^{\sigma-1}[\Gamma(L_t^{RD})^{\sigma-1} - 1]}\right\} dz$$

which, due to non-linearities in $(\phi')^{-1}(\cdot)$, may admit more than one solution. ■

A.6 Proofs of Lemma 5 and Proposition 4

In the case $\rho \to 1$ (that is, $\sigma \to +\infty$), we know that $p_t(z, q(z)) = P_t \equiv \int p_t(z, q(z))dz$ from the final good producer's problem, as perfect substitutability means price-taking behavior through Bertrand limit price-competition. From the intermediate good producer problem, taking P_t as given, we can write profits for a firm z operating at quality $q(z)$ as

$$\pi_t(z, q(z)) = r_t w_{t-1} + \max_{x \geq 0}\left\{\left(P_t - \frac{r_t}{zq(z)}\right) \cdot x, \text{ s.t. } \left(\frac{r_t}{zq(z)} - \lambda P_t\right) \cdot x \leq r_t w_{t-1}\right\}$$

Due to the linearity of both the objective and the constraint, we get the following bang-bang solution:

$$x(z, q(z)) = \begin{cases} 0 & \text{if } z < \frac{r}{q(z)P} \\ zq(z) \cdot \frac{r_t w_{t-1}}{r_t - \lambda \cdot zq(z)P_t} & \text{if } z \in \left[\frac{r_t}{q(z)P_t}, \frac{1}{\lambda}\frac{r_t}{q(z)P_t}\right) \\ +\infty & \text{if } z \geq \frac{1}{\lambda}\frac{r_t}{q(z)P_t} \end{cases}$$

where $q(z) \in \{q_{t-1}, \Gamma_t q_{t-1}\}$, from which equation (21) follows.

By feasibility, we need to check that λ is so that $\frac{1}{\lambda}\frac{r_t}{q(z)P_t} \geq z$, $\forall z$. Otherwise, a measure of firms would be demanding infinite capital, and markets could not clear. A sufficient condition is $\Gamma_t \lambda \leq \overline{z}_t$, to be checked in equilibrium. Under this condition,

1. If $z < \underline{z}_t$, then firm z does not produce, whether or not it innovates.
2. If $z \in [\underline{z}_t, \overline{z}_t)$, then firm z produces up until the constraint binds only if it innovates, and otherwise it does not produce at all.
3. If $z \geq \overline{z}_t$, then firm z produces up until the constraint binds, whether or not it innovates.

Then, for a firm of generic quality $q(z)$, profits of being constrained or unconstrained will be

$$\pi_t^C(z,q(z)) = \begin{cases} r_t w_{t-1}\cdot\left[\frac{(1-\lambda)\cdot(z/\overline{z}_t)}{1-\lambda\cdot(z/\overline{z}_t)}\right] & \text{if } q(z)=q_{t-1} \\ r_t w_{t-1}\cdot\left[\frac{(1-\lambda)\cdot(z/\underline{z}_t)}{1-\lambda\cdot(z/\underline{z}_t)}\right] & \text{if } q(z)=\Gamma_t q_{t-1}\end{cases} \qquad \text{and} \qquad \pi_t^U(z,q(z)) = r_t w_{t-1};\quad \forall q(z)$$

respectively. Note that (i) $\frac{\partial \pi^C(z,q)}{\partial z} > 0$, $\frac{\partial^2 \pi^C(z,q)}{\partial z^2} > 0$, and (ii) since z is constrained if $z > \overline{z}_t$, then $\pi_t^C \geq r_t w_{t-1} = \pi_t^U$, with $\pi_t^C(\overline{z}_t) = \pi_t^U$ and $\pi_t^C(1) = r_t w_{t-1}\left(\frac{(1-\lambda)\overline{z}_t}{\overline{z}_t-\lambda}\right)$. Moreover, $\pi_t^C(z,\Gamma_t q_{t-1}) \geq \pi_t^C(z,q_{t-1})$ but $\pi_t^U(z,\Gamma_t q_{t-1}) = \pi_t^U(z,q_{t-1})$, namely innovation is profitable only for those who are constrained either prior or after the innovation takes place. In terms of our notation in the main text, $\Delta_t^L(z)=0$ and $\Delta_t^n(z)>0$, $\forall n\in\{M,H\}$. In particular, straightforward algebra shows

$$\begin{aligned}
\Delta_t^L(z) &= 0\\
\Delta_t^M(z) &= r_t w_{t-1}\left(\frac{\Gamma_t z/\overline{z}_t - 1}{1-\lambda(\Gamma_t z/\overline{z}_t)}\right)\\
\Delta_t^H(z) &= r_t w_{t-1}\left[\frac{(\Gamma_t-1)z}{\overline{z}_t}\left(\frac{1-\lambda}{(1-\lambda(\Gamma_t z/\overline{z}_t))(1-\lambda(z/\overline{z}_t))}\right)\right]
\end{aligned}$$

which means that the innovation decision is

$$l_t^{RD}(z) = \begin{cases} 0 & \text{if } z\in[0,\underline{z}_t) \\ (\phi')^{-1}\left[\frac{w_t}{r_t w_{t-1}}\left(\frac{1-\lambda(\Gamma z/\overline{z}_t)}{\Gamma_t z/\overline{z}_t-1}\right)\right] & \text{if } z\in[\underline{z}_t,\overline{z}_t) \\ (\phi')^{-1}\left[\frac{w_t}{r_t w_{t-1}}\frac{\overline{z}_t}{(\Gamma_t-1)z}\left(\frac{(1-\lambda(\Gamma_t z/\overline{z}_t))\cdot(1-\lambda(z/\overline{z}_t))}{1-\lambda}\right)\right] & \text{if } z\in[\overline{z}_t,1]\end{cases}$$

We can use these expressions to characterize closed-form equilibrium expressions. Using law of large numbers, $\phi(\ell)$ is the fraction of agents who successfully innovate at innovation intensity ℓ. Denoting current quality level by $q\equiv q_{t-1}$, recall that $\phi(l_n^{RD}(z)) = \phi\left((\phi')^{-1}\left(w_t/\Delta_n(z)\right)\right)$ for each $n\in\{L,M,H\}$. Moreover, we assume the following functional forms for the probability and the step size of a successful innovation:

$$\begin{aligned}
\Gamma(L) &= 1+(\overline{\Gamma}-1)\cdot L^{1-\gamma}\\
\phi(\ell) &= \overline{\phi}\cdot\ell^{1-\beta}
\end{aligned}$$

where $(\overline{\phi},\beta,\gamma)\in(0,1]^3$. Then, optimal R&D investment is

$$l_t^{RD}(z) = \begin{cases} 0 & \text{if } z\in[0,\overline{z}_t) \\ \left[\phi(1-\beta)\frac{rw_{t-1}}{w_t}\left(\frac{\Gamma z/\overline{z}_t-1}{1-\lambda(\Gamma_t z/\overline{z}_t)}\right)\right]^{\frac{1}{\beta}} & \text{if } z\in[\underline{z}_t,\overline{z}_t) \\ \left[\phi(1-\beta)\frac{rw_{t-1}}{w_t}\frac{(\Gamma_t-1)z}{\overline{z}_t}\left(\frac{1-\lambda}{(1-\lambda(\Gamma_t z/\overline{z}_t))(1-\lambda(z/\overline{z}_t))}\right)\right]^{\frac{1}{\beta}} & \text{if } z\in[\overline{z}_t,1]\end{cases}$$

which implies probabilities of innovation of

$$\phi(l_t^{RD}(z)) = \begin{cases} 0 & \text{if } z\in[0,\overline{z}_t) \\ \phi\left[\phi(1-\beta)\frac{rw_{t-1}}{w_t}\left(\frac{\Gamma_t z/\overline{z}_t-1}{1-\lambda(\Gamma_t z/\overline{z}_t)}\right)\right]^{\frac{1-\beta}{\beta}} & \text{if } z\in[\underline{z}_t,\overline{z}_t) \\ \phi\left[\phi(1-\beta)\frac{rw_{t-1}}{w_t}\frac{(\Gamma_t-1)z}{\overline{z}_t}\left(\frac{1-\lambda}{(1-\lambda(\Gamma_t z/\overline{z}_t))(1-\lambda(z/\overline{z}_t))}\right)\right]^{\frac{1-\beta}{\beta}} & \text{if } z\in[\overline{z}_t,1]\end{cases}$$

Then, credit market clearing (14) reads:

$$
\begin{aligned}
1 \quad = \quad & \int_{\overline{z}/\Gamma}^{\overline{z}} \left(\frac{\phi}{1-\lambda\Gamma(z/\overline{z})}\right)^{\frac{1}{\beta}} \left[(1-\beta)\frac{rw_{t-1}}{w_t}\left(\Gamma z/\overline{z}-1\right)\right]^{\frac{1-\beta}{\beta}} dz \\
& + \int_{\overline{z}}^{1} \left[\frac{1}{1-\lambda(z/\overline{z})} + \lambda\left(\frac{\phi(\Gamma-1)(z/\overline{z})}{(1-\lambda(\Gamma z/\overline{z}))(1-\lambda(z/\overline{z}))}\right)^{\frac{1}{\beta}} \left[(1-\beta)\frac{rw_{t-1}}{w_t}(1-\lambda)\right]^{\frac{1-\beta}{\beta}}\right] dz
\end{aligned}
$$

Using $l_n^{RD}(z) = (\phi')^{-1}(w_t/\Delta_n(z))$ for each $n \in \{L, M, H\}$, labor market clearing reads $1 - L_t = L_t^{RD}$ as in equation (15), that is,

$$
\begin{aligned}
1 - L_t \quad = \quad & \int_{\overline{z}/\Gamma}^{\overline{z}} \left[\phi(1-\beta)\frac{rw_{t-1}}{w_t}\left(\frac{\Gamma z/\overline{z}-1}{1-\lambda(\Gamma z/\overline{z})}\right)\right]^{\frac{1}{\beta}} dz \\
+ \quad & \int_{\overline{z}}^{1} \left[\phi(1-\beta)\frac{rw_{t-1}}{w_t}\frac{(\Gamma-1)z}{\overline{z}}\left(\frac{1-\lambda}{(1-\lambda(\Gamma z/\overline{z}))(1-\lambda(z/\overline{z}))}\right)\right]^{\frac{1}{\beta}} dz
\end{aligned}
$$

As in the general case, in equilibrium wages must adjust so that $w_t = \Psi(P_t)$. If $w_t > \Psi(P_t)$, then $L_t = 0$ and $Y_t = 0$, and the equilibrium is trivial. If $w_t < \Psi(P_t)$, then final good firms demand infinite labor, and markets cannot clear. The solution to the high equilibrium aggregate specialized labor L_t^{RD} is then the typical fixed point which makes agents' prior expectations self-fulfilling.

Finally, we need to check that, in equilibrium, good market clearing, represented by condition (9), holds. That is,

$$
P_t = (1-\alpha)\left(\frac{1-L_t^{RD}}{X_t}\right)^{\alpha} \tag{28}
$$

where, using that $X = \int_0^1 x(z, q(z))dz$,

$$
\begin{aligned}
X_t \quad = \quad & \int_{\overline{z}/\Gamma}^{\overline{z}} z\Gamma q\left(\frac{\phi w_{t-1}}{1-\lambda\Gamma(z/\overline{z})}\right)^{\frac{1}{\beta}} \left[(1-\beta)\frac{r}{w_t}\left(\Gamma z/\overline{z}-1\right)\right]^{\frac{1-\beta}{\beta}} dz \\
& + \int_{\overline{z}}^{1} \left[\frac{zqw_{t-1}}{1-\lambda(z/\overline{z})} + \left(\frac{\phi(\Gamma-1)zw_{t-1}}{(1-\lambda(\Gamma z/\overline{z}))(1-\lambda(z/\overline{z}))}\right)^{\frac{1}{\beta}} \left[(1-\beta)\frac{r}{w_t}\left(\frac{1-\lambda}{\overline{z}}\right)\right]^{\frac{1-\beta}{\beta}}\right] dz
\end{aligned}
$$

It is possible to find exact solutions to these integrals under the assumed functional forms for $\phi(\cdot)$ and $\Gamma(\cdot)$. For brevity, we start by defining the following short-cut notation. First, define the functions $H_1 : [0,1]^2 \to \mathbb{R}_+$, $H_2 : [0,1] \to \mathbb{R}_+$, and $H_3 : [0,1] \to \mathbb{R}_+$ by:

$$
\begin{aligned}
H_1(x,y) \quad & \equiv \quad F_1\left(\frac{1+\beta}{\beta}; \frac{1}{\beta}, \frac{1}{\beta}; \frac{1+2\beta}{\beta}; x, y\right) \\
H_2(z) \quad & \equiv \quad {}_2F_1\left(-\frac{1-\beta}{\beta}, -\frac{1-\beta}{\beta}; \frac{1-2\beta}{\beta}; z\right) \\
H_3(z) \quad & \equiv \quad {}_2F_1\left(-\frac{1}{\beta}, -\frac{1-\beta}{\beta}; -\frac{1-2\beta}{\beta}; z\right)
\end{aligned}
$$

The notation $F_1(a;b_1,b_2;c;x,y)$ stands for an Appell hypergeometric series, defined by

$$F_1(a;b_1,b_2;c;x,y) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n m!n!} x^m y^n$$

where we use the Pochhammer symbol $(x)_n \equiv \prod_{i=0}^{n-1}(x+i)$, for $n \geq 1$. Similarly, ${}_2F_1(a,b;c;z)$ is a Gaussian or ordinary hypergeometric function, defined by

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

where $|z| \leq 1$. Both these functions are numerically approximated with finite sums of order 10 in the simulations shown in Section 5. Next, define the following objects:

$$\begin{aligned}
A &\equiv \lambda\frac{\beta}{1+\beta}\left(\frac{(1-\beta)(1-\lambda)r}{w_t}\right)^{\frac{1-\beta}{\beta}}\Big[(\Gamma-1)\phi\Big]^{\frac{1}{\beta}}\left\{\overline{z}^{-\frac{1}{\beta}}H_1\left(\frac{\lambda}{\overline{z}},\frac{\Gamma\lambda}{\overline{z}}\right)-\overline{z}H_1\left(\lambda,\Gamma\lambda\right)\right\}\\
B &\equiv \frac{\overline{z}}{\lambda}\ln\left(\frac{\overline{z}(1-\lambda)}{\overline{z}-\lambda}\right)\\
C &\equiv \left[\frac{1-\lambda}{1-\Gamma\lambda}\right]^{\frac{1-\beta}{\beta}} H_2\left(\frac{1-\Gamma\lambda}{1-\lambda}\right)-H_2(1)\\
D &\equiv \left[\frac{1-\lambda}{1-\Gamma\lambda}\right]^{\frac{1-\beta}{\beta}} H_3\left(\frac{1-\Gamma\lambda}{1-\lambda}\right)-H_3(1)\\
E &\equiv \frac{\beta\overline{z}}{(1-\beta)^2\Gamma}\left[\phi(1-\beta)\frac{1}{\lambda}\right]^{\frac{1}{\beta}}\left[\frac{r}{w_t}\right]^{\frac{1-\beta}{\beta}}
\end{aligned}$$

Then, the exact solutions for goods, capital, and labor market clearing conditions are, respectively,

$$\frac{X_t}{w_{t-1}} = \frac{q_{t-1}\overline{z}}{\lambda}\cdot\left[w_{t-1}^{\frac{1-\beta}{\beta}}\cdot\left(E\cdot(\lambda\cdot C+(1-\lambda)\cdot D)+\frac{A}{q}\right)+B-(1-\overline{z})\right] \tag{29}$$

$$1 = w_{t-1}^{\frac{1-\beta}{\beta}}\cdot\Big(E\cdot C+A\Big)+B \tag{30}$$

$$1-L_t = \frac{(1-\beta)(1-\lambda)}{\lambda}\frac{rw_{t-1}}{w_t}\Big[w_{t-1}^{\frac{1-\beta}{\beta}}\cdot E\cdot D+A\Big] \tag{31}$$

From the second equation we get that $w_{t-1}^{\frac{1-\beta}{\beta}} = \frac{1-B}{E\cdot C+A}$. Using $w_{t-1}=K_t$, we can then write output, in any equilibrium, as $Y_t = Z_t\cdot L_t^{\alpha}K_t^{1-\alpha}$, where measured TFP is

$$Z_t \equiv \left\{\frac{q_{t-1}\overline{z}}{\lambda}\cdot\left[B+(1-B)\cdot\frac{E\cdot(\lambda\cdot C+(1-\lambda)\cdot D)+\frac{A}{q}}{E\cdot C+A}-(1-\overline{z})\right]\right\}^{1-\alpha}$$

For a given sunspot ξ_t selecting the equilibrium play, the high-equilibrium prices are solutions P_t, $\overline{z}_t$ and L_t^{RD} to the system of equations (29), (30) and (31). This is a highly non-linear system that must be solved numerically and may have more than one solution.

In the low equilibrium, however, one can obtain insightful closed-form solutions. Where $L_t^{RD}=0$ and

$\Gamma_t = 1$, the expression for capital market clearing (30) can be shown to reduce to[38]

$$e^{-\lambda/\overline{z}_t} = \frac{\overline{z}_t - \lambda}{\overline{z}_t(1-\lambda)} \tag{32}$$

meaning that $\overline{z}_t = \overline{z}(\lambda)$, $\forall t \in \{\tau : \xi_\tau = P\}$. Solving,

$$\overline{z}(\lambda) = \frac{\lambda}{1 + W\left(-\frac{1-\lambda}{e}\right)}$$

where $W(\cdot)$ is the Lambert W function, namely the principal branch of the inverse relation of the function $f(W) = We^W$. One can then show that $\overline{z}(\lambda)$ is strictly increasing and strictly concave in λ. Moreover, $\bar{z}(0) = 0$ and $\overline{z}(1) = 1$. Interest rates are then given by $r_t = \overline{z}(\lambda) \cdot q_{t-1}P_t$. Next, using $X_t = \int_0^1 x_t(z, q(z))dz$ with $q(z) = q_{t-1}$, we obtain[39]

$$\frac{X_t}{w_{t-1}} = \frac{1}{w_{t-1}}\int_0^1 x_t(z)dz = \int_{\overline{z}(\lambda)}^1 \frac{zq}{1-\lambda\frac{z}{\overline{z}(\lambda)}}dz = -\frac{q\overline{z}(\lambda)}{\lambda}\left[1 - \overline{z}(\lambda) + \frac{\overline{z}(\lambda)}{\lambda}\ln\left(\frac{\overline{z}(\lambda)-\lambda}{(1-\lambda)\cdot\overline{z}(\lambda)}\right)\right] = \frac{q\overline{z}(\lambda)^2}{\lambda}$$

where the last equality uses equation (32). Therefore, $X_t = \frac{\overline{z}(\lambda)^2}{\lambda}q_{t-1}K_t$, and we can write output as $Y_t = Z_t \cdot L_t^\alpha K_t^{1-\alpha}$, where measured TFP is given by

$$Z_t \equiv \left(\frac{\overline{z}(\lambda)^2}{\lambda} \cdot q_{t-1}\right)^{1-\alpha}$$

which, as shown by Figure 4, is such that $\frac{\partial Z_t}{\partial \lambda} > 0$ and $\frac{\partial^2 Z_t}{\partial \lambda^2} < 0$, with $\lim_{\lambda \nearrow 1} Z_t = q_{t-1}^{1-\alpha}$ and $\lim_{\lambda \searrow 0} Z_t = 0$.

Finally, for the general case, we have seen that the aggregation result of Proposition 1 holds in this environment. In particular, we know that $\mathcal{M}_t(z, q(z)) = 1$, $\forall z$ (as agents do not have price-setting power). Moreover, we have shown that only a measure $1 - \underline{z}_t$ of entrepreneurs are active in equilibrium. Therefore, using equation (18), we have $Z_t = Q_t^{1-\alpha}$ and

$$Q_t = \int_{\overline{z}_t/\Gamma_t}^1 zq(z)dz = \mathbb{E}_t\left[zq(z)|z \geq \underline{z}_t\right]$$

as stated in the proposition. □

[38]To obtain this result, impose $\Gamma = 1$ directly into the system (29)-(31). In particular, when $\Gamma \to 1$ we have $A, C, D \to 0$, so $B \to -1$ after using l'Hôpital's rule in equation (30). Another way of arriving at the same condition is to see that, in the low equilibrium, capital market clearing is simply given by

$$1 = \int_{\overline{z}_t}^1 \left(1 - \lambda\frac{z}{\overline{z}_t}\right)^{-1} dz \quad \Rightarrow \quad -\frac{\overline{z}_t}{\lambda}\ln\left(\frac{\overline{z}_t - \lambda}{\overline{z}_t(1-\lambda)}\right) = 1$$

In both cases, result (32) follows immediately.

[39]Once again, this result can also be obtained by imposing $A, C, D \to 0$ into equation (29).

B Simulation Appendices

B.1 Parametrization

We use the following functional forms for the innovation technologies of intermediate good producers:

$$\begin{aligned} \Gamma(L) &= 1 + \left(\overline{\Gamma} - 1\right) \cdot L^{1-\gamma} \\ \phi(\ell) &= \overline{\phi} \cdot \ell^{1-\beta} \end{aligned}$$

with $(\overline{\phi}, \beta, \gamma) \in (0,1]^3$. Note that these functional forms are in line with Assumption 1.

To allow for persistence in beliefs, the sunspot is assumed to be a two state Markov process with a 2×2 transition matrix $\mathbf{P}_\xi$ with typical element $p_{ij}^{\xi} \equiv Pr[\xi_t = \xi_j | \xi_{t-1} = \xi_i] \in (0,1)$ for $i, j \in \{O, P\}$, which denotes both the probability of an i-to-j transition as well as the probability defining the equilibrium mixed strategy played by agents. Moreover, for simplicity, we assume that A_t is also a two-state Markov process taking values in $\mathbb{A} = \{A_L, A_H\}$, where $A_H > A_L$ (boom versus bust). The transition matrix is denoted by $\mathbf{P}_A$ and has typical element $p_{ij}^{A} \equiv Pr[A_t = A_j | A_{t-1} = A_i]$, for $i, j \in \{L, H\}$. Because of its Markovian structure, agents from generation $t-1$ producing in period t condition their expectations only on A_{t-1}, a number they observe at the end of their youth. The observation of A_{t-1} will then critically determine how overall expectations are aligned within the period and the equilibrium selection mechanism.

In practice, we look at impulse response functions by letting A_t take a high value for all periods until a crisis unfolds, in which $A_t = A_L$ for at least one period. We then look at the different responses of equilibrium variables for different sunspot shocks in response to the real shock.

Moreover, to create a non-trivial role for these shocks, we make the following parametric assumption:

Assumption 4 *The values A_L and A_H are such that* $0 < \frac{1+\delta_q}{\overline{\Gamma}} < A_L \leq 1 + \delta_q \leq A_H < \overline{\Gamma}$.

The following explains why these assumptions are sensible. Given an initial $q_0 > 0$,

- The assumption that $A_t > 0$, $\forall t$ (first inequality) is necessary for $q_t > 0$, $\forall t$, which is required by feasibility.
- The second and third inequalities imply that, in the context of $A_{t-1} = A_L$ (a bust), $q_t > q_{t-1}$ as $L_t^{RD} \nearrow 1$ and $q_t < q_{t-1}$ when $L_t^{RD} \searrow 0$, respectively. These two restrictions together guarantee that $\exists L \in (0,1)$ such that $q_t = q_{t-1}$ as $L_t^{RD} \to L$. In equilibrium, however, only the third inequality will be of relevance: in the bad equilibrium, in which the research sector is completely idle ($L^{RD} = 0$), a negative shock to quality would effectively pull productivity down the ladder. This is the critical channel through which we can obtain non-reversion to pre-crisis trend levels in the presence of negative A shocks, absent ξ shocks.
- The fourth inequality says that a high regime $A_{t-1} = A_H$ can on its own reverse the fallback and re-initiate positive growth in qualities back up through the ladder, even if agents remain pessimistic about economic outcomes. This latter assumption implies that the trend-deviation problem is uniquely stemming from the combination of persistence in pessimism and persistence of low regimes, a feedback loop that was likely to occur in the actual recovery of industrialized economies in the post-2008 era.

- Finally, the last inequality guarantees that exogenous growth does not outgrow endogenous growth when the economy is producing ideas at maximum capacity ($L_t^{RD} \nearrow 1$) and is experiencing an expansion ($A_{t-1} = A_H$). This last inequality is effectively insubstantial as such a situation never occurs in equilibrium.

Finally, the model is run for $T = 70$ periods, and throughout we use the following values for the parameters: $\lambda = 0.466$, $\rho = 0.9$, $\beta = 0.6$, $\overline{\phi} = 0.95$, $\gamma = 0.9$, $\overline{\Gamma} = 1.5$, $\delta_q = 0.2$, $w_0 = 2$, $q_0 = 1$, $A_L = 1$, and $A_H = 1.03 + \delta_q = 1.05$. In accordance with Assumption 3, we set $\alpha = 0.49$ and check ex-post that the results of the simulation are such that $\alpha < \frac{L_t}{1+L_t}$, $\forall t \in [T]$. We use the algebraic solutions provided in Appendices A.5 and A.6, so the depicted equilibrium paths are not approximated but rather exact paths.

B.2 Additional figures

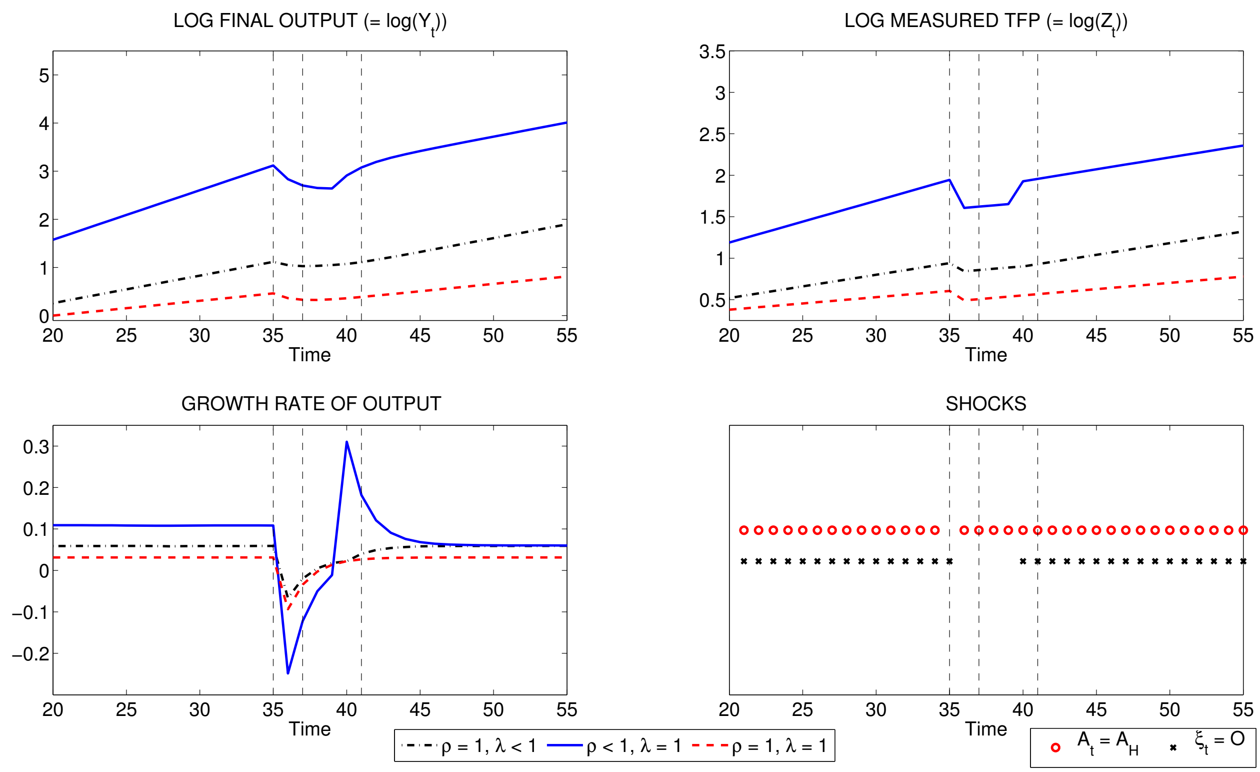

Figure 7: Persistent pessimistic belief response to a negative A shock.

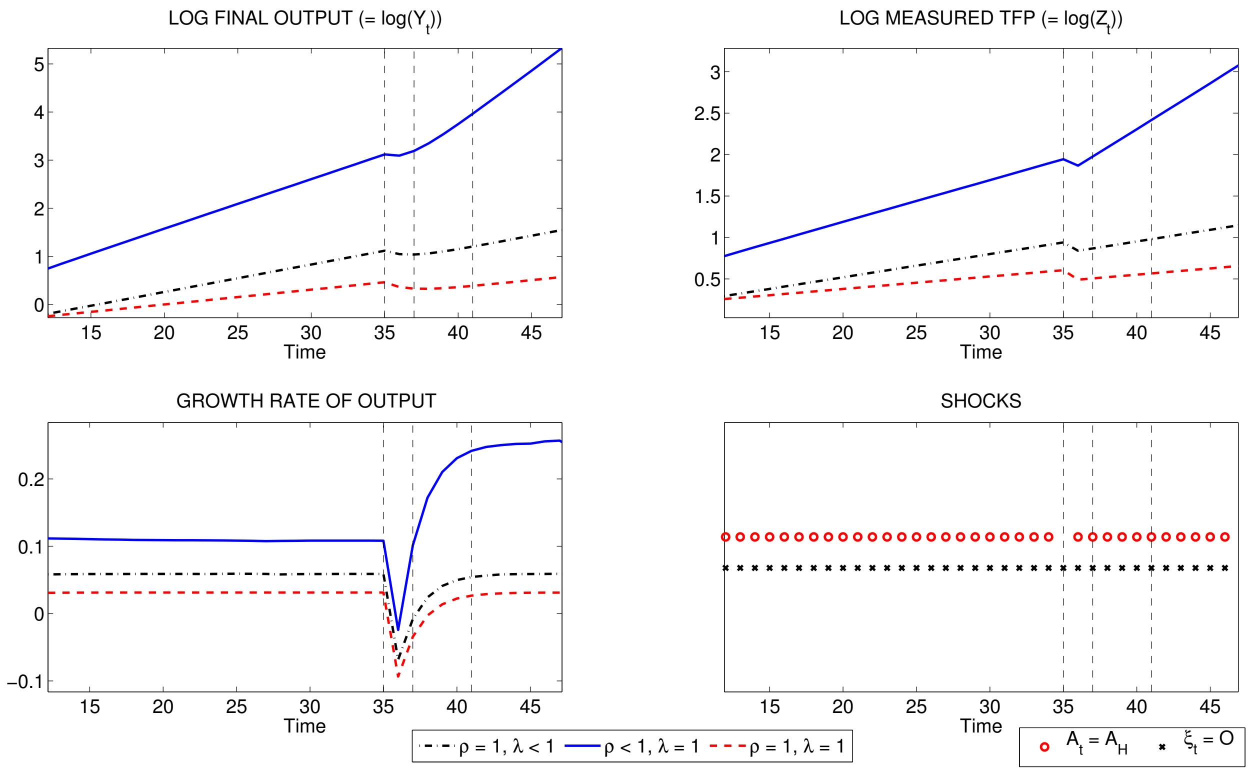

Figure 8: Negative A shock with no belief response.

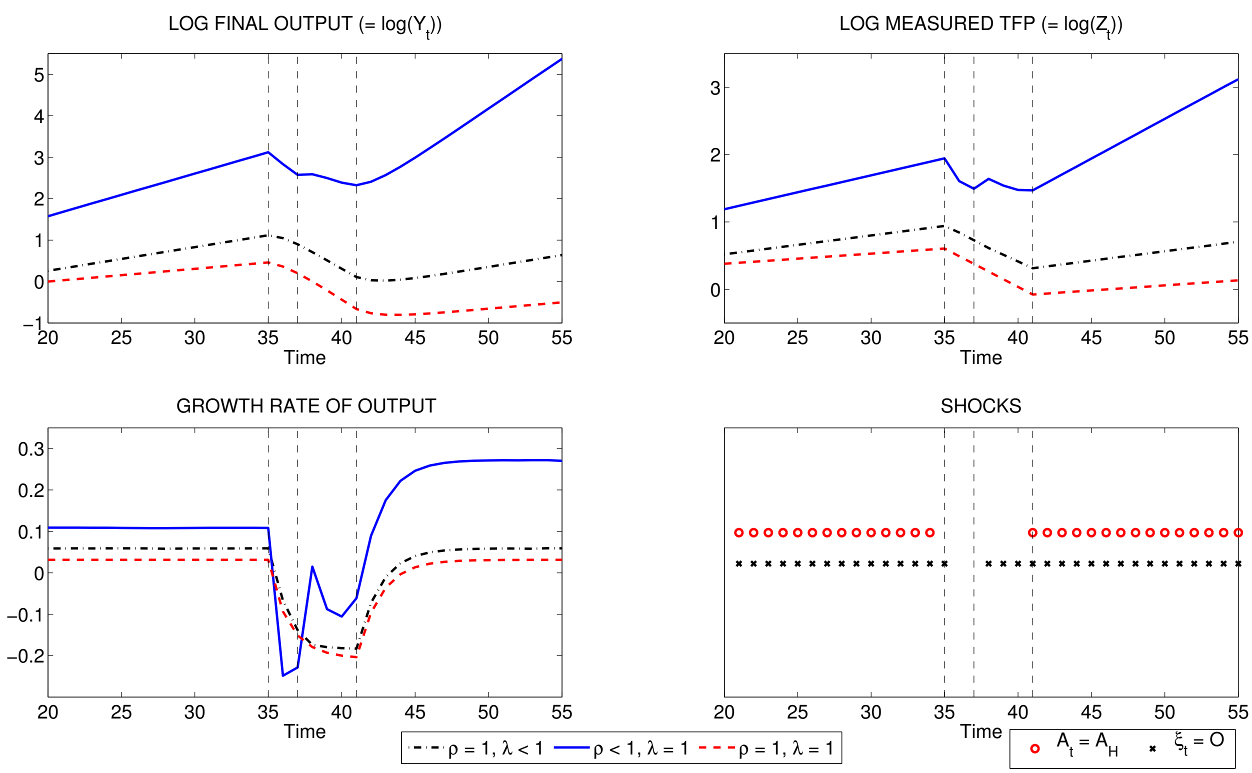

Figure 9: Optimism recovers earlier than the A shock does.