Solving and Simulating Unbalanced Growth Models using Linearization about the Current State: Technical Appendices

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A-1 Derivation of the Euler equations

$$V(k_t, x_{t-1}, z_t) = \max_{k_{t+1}, h_t} \frac{(c_t - \psi h_t^\theta c_t^\gamma x_{t-1}^{1-\gamma})^{1-\sigma}}{1-\sigma} + \beta E_t \left[V(k_{t+1}, c_t^\gamma x_{t-1}^{1-\gamma}, z_{t+1}) \right]$$

$$c_t = w_t h_t + (1 + r_t - \delta) k_t - k_{t+1}$$

Define the following:

$$\Gamma_t \equiv (c_t - \psi h_t^\theta c_t^\gamma x_{t-1}^{1-\gamma})^{-\sigma} \tag{A.1.1}$$

$$\Delta_t \equiv 1 - \psi \gamma h_t^\theta c_t^{\gamma-1} x_{t-1}^{1-\gamma} \tag{A.1.2}$$

$$\Phi_t \equiv \psi (1-\gamma) h_t^\theta c_t^\gamma x_{t-1}^{-\gamma} \tag{A.1.3}$$

$$\Pi_t \equiv \psi \theta h_t^{\theta-1} c_t^\gamma x_{t-1}^{1-\gamma} \tag{A.1.4}$$

$$\Lambda_t \equiv \gamma c_t^{\gamma-1} x_{t-1}^{1-\gamma} \tag{A.1.5}$$

The first order conditions from household's problem are given below.

$$\Gamma_t \Delta_t (-1) + \beta E_t \left\{ V_k(k_{t+1}, x_t, z_{t+1}) + V_h(k_{t+1}, x_t, z_{t+1}) \Lambda_t (-1) \right\} = 0 \tag{A.1.6}$$

$$\Gamma_t (\Delta_t w_t - \Pi_t) + \beta E_t \left\{ V_h(k_{t+1}, x_t, z_{t+1}) \Lambda_t w_t \right\} = 0 \tag{A.1.7}$$

The envelope conditions are given by:

$$V_k(k_t, x_{t-1}, z_t) = \Gamma_t \Delta_t (1 + r_t - \delta) \tag{A.1.8}$$

$$V_x(k_t, x_{t-1}, z_t) = \Gamma_t \Phi_t \tag{A.1.9}$$

Substituting and simplifying:

$$\Gamma_t \Delta_t = \beta E_t \left\{ \Lambda_t \Phi_{t+1} + \Gamma_{t+1} \Delta_{t+1} (1 + r_{t+1} - \delta) \right\} \tag{A.1.10}$$

$$\Gamma_t \Delta_t w_t = \Gamma_t \Pi_t + \beta E_t \left\{ \Lambda_t w_t \Gamma_{t+1} \Phi_{t+1} \right\} \tag{A.1.11}$$

A-2 Transformed Model When There is Balanced Growth

$$z_t = \rho z_{t-1} + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.(0, \omega^2) \tag{A.2.1}$$

$$\tilde{y}_t = \tilde{k}_t^{\alpha}(h_t e^{z_t})^{1-\alpha} \tag{A.2.2}$$

$$r_t = \alpha \frac{\tilde{y}_t}{\tilde{k}_t} \tag{A.2.3}$$

$$\tilde{w}_t = (1-\alpha)\frac{\tilde{y}_t}{h_t} \tag{A.2.4}$$

$$\tilde{c}_t = \tilde{w}_t h_t + (1 + r_t - \delta)\tilde{k}_t - \tilde{k}_{t+1}e^{g} \tag{A.2.5}$$

$$\tilde{x}_t = \tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.6}$$

$$\tilde{\Gamma}_t \equiv (\tilde{c}_t - \psi h_t^{\theta}\tilde{c}_t^{\gamma}\tilde{(}x_{t-1}e^{-g})^{1-\gamma})^{-\sigma} \tag{A.2.7}$$

$$\Delta_t \equiv 1 - \psi\gamma h_t^{\theta}\tilde{c}_t^{\gamma-1}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.8}$$

$$\Phi_t \equiv \psi(1-\gamma)h_t^{\theta}\tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{-\gamma} \tag{A.2.9}$$

$$\tilde{\Pi}_t \equiv \psi\theta h_t^{\theta-1}\tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.10}$$

$$\Lambda_t \equiv \gamma\tilde{c}_t^{\gamma-1}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.11}$$

$$\tilde{\Gamma}_t\Delta_t = \beta E_t\left\{\Lambda_t\Phi_{t+1} + \tilde{\Gamma}_{t+1}e^{g}\Delta_{t+1}(1 + r_{t+1} - \delta)\right\} \tag{A.2.12}$$

$$\tilde{\Gamma}_t\Delta_t\tilde{w}_t = \tilde{\Gamma}_t\tilde{\Pi}_t + \beta E_t\left\{\Lambda_t\tilde{w}_t\tilde{\Gamma}_{t+1}e^{g}\Phi_{t+1}\right\} \tag{A.2.13}$$

$$\tilde{x}_t = \tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.14}$$

A-3 Model with GHH Preferences and Unbalanced Growth

$$z_t = \rho z_{t-1} + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.(0, \omega^2) \tag{A.3.1}$$

$$s_t = s_{t-1} + 1 \tag{A.3.2}$$

$$y_t = k_t^{\alpha}(h_t e^{gt+z_t})^{1-\alpha} \tag{A.3.3}$$

$$r_t = \alpha \frac{y_t}{k_t} \tag{A.3.4}$$

$$w_t = (1-\alpha)\frac{y_t}{h_t} \tag{A.3.5}$$

$$c_t = w_t h_t + (1 + r_t - \delta)k_t - k_{t+1} \tag{A.3.6}$$

$$\Gamma_t = (c_t - \psi h_t^{\theta} c_t)^{-\sigma} \tag{A.3.7}$$

$$\Delta_t = 1 - \psi h_t^{\theta} \tag{A.3.8}$$

$$\Pi_t = \psi \theta h_t^{\theta-1} c_t \tag{A.3.9}$$

$$\Gamma_t \Delta_t = \beta E_t \left\{ \Gamma_{t+1} \Delta_{t+1} (1 + r_{t+1} - \delta) \right\} \tag{A.3.10}$$

$$\Delta_t w_t = \Pi_t \tag{A.3.11}$$