Solving and Simulating an Unbalanced Growth Model using Linearization about the Current State *

Kerk L. Phillips†

September 2016
(version 16.09.a)

Abstract

This paper presents an adjustment to commonly used linear approximation methods for dynamic stochastic general equilibrium (DSGE) models. Policy functions approximated around the steady state will be inaccurate away from the steady state. In some cases the model may not have a well-defined steady state, or the nature of the steady state may be at odds with its off-steady-state dynamics. We show how to simulate a DSGE model with no well-defined steady state by approximating about the current state. Our method minimizes the error associated with a finite-order Taylor-series expansion of the models characterizing equations. This method is easily implemented and has the advantage of mimicking highly non-linear behavior. However, it also requires choosing N out of $2N$ possible roots from a matrix quadratic equations and only in the univariate case is the choice of this root obvious.

keywords: dynamic stochastic general equilibrium, linearization methods, numerical simulation, computational techniques, simulation modeling, unbalanced growth.

JEL classifications: C63, C68, E37

*This research benefited from the computing resources of the BYU Macroeconomics and Computational Laboratory. Thanks to Chase Coleman, Mary Li, Yulong Li and Jake Orchard for their excellent research assistance and to Richard W. Evans for insightful comments and support.

†Brigham Young University, Department of Economics, 166 FOB, Provo, Utah 84602, kerk_phillips@byu.edu.

1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are an important class of macroeconomic modeling that have been in use now for several decades. They are increasingly used in policy contexts to simulate the effects of policy changes on the macroeconomy.[1]

Usually these models are too complex to find closed-form solutions for dynamic policy functions that map the current state of the economy into the values for next period's endogenous state variables. Instead, these models must be solved and simulated using some approximation method. The most widelt used techniques include the linearization methods in Uhlig (1999) and Christiano (2002) who employ a method of undetermined coefficients to solve the state-space representation outlined in Blanchard and Kahn. (1980). Higher-order polynomial approximations developed by Judd (1992), Guu and Judd (2001), Collard and Juillard (2001) and Schmitt-Grohe and Uribe (2004) are increasingly widely used. This is the approach taken with the poplar DSGE software package Dynare, for example.[2]

This paper presents an easy adjustment to linear and higher-order approximation methods. Since approximation is almost always done about the model's steady state, the linear policy functions can be inaccurate if the simulation is often away from the steady state. In some cases, this leads to only small errors. In other cases, however, the model may not have a well-defined steady state, or the nature of the steady state may be at odds with its off-steady-state dynamics.

Approximating about the steady state requires the existence of a steady state. If one does not exist, the models variables must be redefined so that they are stationary. Some models of interest to researchers cannot be easily transformed in this manner, however. These include multi-sector models with unbalanced growth and models where the parameters are time-varying.

We show how to simulate a DSGE model by approximating about the current state, rather than the steady state. This method is easily implemented and has the

[1] see for example Smets and Wouters (2007) and Christiano et al. (2005).

[2] see Adjemian et al. (2014) for details.

advantage of accurately mimicking non-linear behavior.

We proceed similarly to Uhlig (1999) and Christiano (2002) who linearize their dynamic equations about the steady state. They hypothesize linear policy functions and use the method of undetermined coefficients to solve for the coefficients of the policy function. We propose an alternative strategy, which we call "current state linearizarion" (CSL), where one approximates the policy function for each period of the simulation about the current state of the economy. This is computationally more intensive as it requires linearization each period, rather than only once. However, this method has the advantage of being much more accurate when the economy is far from the steady state. This method can easily replicate the behavior of highly nonlinear policy functions. This is because the Taylor-series approximation is highly accurate in the neighborhood of the point about which a function is approximated. Since we are always approximating about the current state, our linear policy function will be very close to the true policy function for that state.

One major hurdle in polynomial approximations is choosing the appropriate roots from a matrix quadratic equation. In the steady state this is relatively simple since the system is generally stable about the steady state. Most macroeconomic models yield a unique set of roots that imply stability in that neighborhood. The same is not true away from the steady state. We show that for our simple unbalanced growth model, the criterion used for steady state stability works well, generating very small Euler errors during simulation.

While we illustrate our method using linear approximations, the concepts and method apply to higher-order polynomial approximations as well.

2 Derivation of an Approximation about the Current State

Consider a set of nonlinear expectational functions, in our case from a dynamic general equilibrium model. The state variables are grouped into two categories: exogenous state variables are grouped into the $S \times 1$ column vector, Z_t, while endogenous ones are placed in the $K \times 1$ column vector, X_t. There are R equations and they can be represented as in equation (2.1).

$$E_t\{\Gamma(X_{t+2}, X_{t+1}, X_t, Z_{t+1}, Z_t\} = 0 \tag{2.1}$$

This system of equations can be approximated by taking a first-order Taylor series expansion about an arbitrary point in the state space. We choose the current value for the state variables, $\theta_t = \{X_t, Z_t\}$. This transformation is given in equation (2.2).

$$E_t\{T_t + F_t\tilde{X}_{t+2} + G_t\tilde{X}_{t+1} + H_t\tilde{X}_t + L_t\tilde{Z}_{t+1} + M_t\tilde{Z}_t\} = 0 \tag{2.2}$$

In the above equation, F_t, G_t and H_t are $R \times S$ matrices, L_t and M_t are $R \times K$ matrices, and T_t is an $R \times 1$ vector. All these will depend on which point is chosen for the linearization. Tildes denote absolute deviations from θ_t values. Note that if we choose to linearize about the steady state, $\bar{\theta} = \{\bar{X}, \bar{Z}\}$ the value of T_t is zero. While this is true of the steady state, it will not be true generally.

The law of motion for the exogenous state variables is assumed to be a first-order vector autoregression of the form in equation (2.3).

$$Z_{t+1} = (I - N)\bar{Z} + NZ_t + E_{t+1} \tag{2.3}$$

Since we are allowing for linearization around any value of Z, we proceed to transform (2.3) into (2.4).

$$E_t\{\tilde{Z}_{t+1}\} = Z_t - \bar{Z} \tag{2.4}$$

As with standard linearization techniques, our goal is to find a linear approximation to the policy function, (2.5).

$$\tilde{X}_{t+1} = U_t + P_t\tilde{X}_t + Q_t\tilde{Z}_t \tag{2.5}$$

Where U_t is an $S \times 1$ column vector, P_t is an $S \times S$ matrix and Q_t is $S \times K$.

The major differences between (2.5) and the standard linear policy function are: First, the inclusion of the constant term, U_t, which makes it possible for the endogenous state variables to drift away from the current state. And second, the time-varying nature of the parameters P_t, Q_t and U_t. Iterative substitution of (2.4) and (2.5) into (2.2) yields the follwong three conditions which define P_t, Q_t & U_t.

$$F_t P_t^2 + G_t P_t + H_t = 0 \tag{2.6}$$

$$(F_t Q_t + L_t)N + (F_t P_t + G_t)Q_t + M_t = 0 \tag{2.7}$$

$$T_t + [F_t U_t + F_t P_t U_t) + G_t U_t + (F_t Q_t + L_t)(N - I)(Z_t - \bar{Z}) = 0 \tag{2.8}$$

The CSL method allows us to solve for each periods coefficients in isolation, without having to refer to next periods actual values. It has the advantage of not needing to solve for a benchmark time path via some other method. Indeed, it is not even necessary to solve for the steady state. Instead, we generate the time path as we solve and simulate each period. A disadvantage is that we must recalculate the unique values of P_t, Q_t & U_t each period in each simulation.

Choosing the roots from equation (2.6) is often straightforward when linearizing about the steady state as there is usually a unique set of roots that implies stability of the system. This need not be the case for CSL, however. We show in our unbalanced growth example below, that this criterion works very well for CSL for this particular model. Unfortunately, this need not generally be the case.

3 Applying this Method to an Unbalanced Growth Model

In this section we consider a model with a labor leisure decision and technical progress. Our formulation of utility leads to a model with unbalanced growth, since utility from consumption rises without bound, but utility from leisure does not.

The household's problem in this case is shown below. We adopt a constant Frisch elasticity formulation of labor disutility.

$$V(k_t, z_t) = \max_{\ell_t, k_{t+1}} \frac{(c_t - 1)^{1-\gamma}}{1-\gamma} - \chi \frac{(1-\ell_t)^{\theta+1}}{\theta+1} + \beta E_t \left[V(k_{t+1}, z_{t+1})\right]$$
$$c_t = w_t(1-\ell_t) + (1 + r_t - \delta)k_t - k_{t+1} \tag{3.1}$$

The first order conditions from this problem yield the following two Euler equations. The first governs the choice of leisure vs consumption goods, while the second is the standard intertemporal Euler equation.

$$c_t^{-\gamma} w_t = \chi(1-\ell_t)^\theta \tag{3.2}$$
$$c_t^{-\gamma} = \beta E_t \left\{ c_{t+1}^{-\gamma}(1 + r_{t+1} - \delta) \right\} \tag{3.3}$$

Note that (3.2) illustrates how our model is not one with balanced growth, since w_t and c_t are growing without bound while ℓ_t is not.

Production is given by a Cobb-Douglas production function with exogenous labor-augmenting technical progress at the rate g per period.

$$Y_t = K_t^\alpha (L_t e^{gt+z_t})^{1-\alpha} \tag{3.4}$$

The first order conditions from profit maximization yield the following conditions which define wage and interest rates.

$$r_t = \alpha \frac{Y_t}{K_t} \tag{3.5}$$
$$w_t = (1-\alpha) \frac{Y_t}{L_t} \tag{3.6}$$

Market clearing gives the following two equations.

$$1 - \ell_t = L_t \tag{3.7}$$
$$k_t = K_t \tag{3.8}$$

It is useful for the simulation algorithm if we define a time counter s_t which follows the following trivial law of motion.

$$s_{t+1} = s_t + 1 \tag{3.9}$$

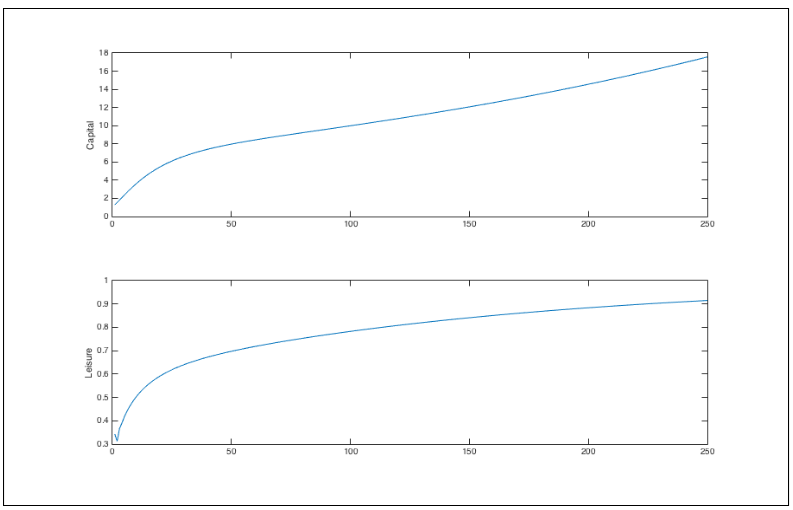
Given $\{k_{t+1}, \ell_t, s_t\}$ for periods $t+1$, t and $t-1$; and z_t for periods $t+1$ and t use (3.4), (3.5), (3.6), (3.1) as definitions to get $\{Y_t, r_t, w_t, c_t\}$. We then use these to evaluate (3.2), (3.3) and (3.9) which implicitly define k_{t+1}, ℓ_t and s_t over time.

This system cannot be approximated about a steady state, because no steady state exists. Hence, we solve and simulate this using the CSL method.

For our simulations we use baseline parameterization of: $\alpha = .35$, $\beta = .98$, $\gamma = 2.5$, $\delta = .035$, $g = .01$, $\chi = 5$, $\theta = 1.5$ and $\sigma = 0.13$. We use a starting value of $K_0 = 10$. If we set the variance of the innovations to technology (σ) to zero, we generate the time plots shown in Figure 1.

We have no other solution method to compare against. However, the behavior of our simulation is exactly as we would expect. That is, the capital stock rises rapidly and then settles into a long-run growth path where it increases at a constant rate over time. Leisure, however is bounded above by one. As technology rises and consumption increases, the household gradually increases consumption of leisure over time so that leisure asymptotically approaches one in the limit.

Figure 1: Unbalanced Growth $K_0 = 1, \sigma = 0$



We simulate this model for 1000 periods using the baseline and several other parameterizations. We report the Euler errors and execution time in Table 1.

Table 1: Euler Errors for Unbalanced Growth Model
1000 periods per simulation

	RMSEE	MAEE	Time
Baseline	$4.14e-05$	$5.99e-04$	2.43
$\beta = .995$	$6.31e-05$	$2.54e-04$	2.40
$\gamma = 1$	$3.86e-06$	$2.15e-05$	2.37
$\delta = .1$	$4.32e-05$	$1.27e-04$	2.35
$\rho = 0$	$4.00e-05$	$5.86e-04$	2.42
$g = .001$	$1.07e-05$	$5.95e-05$	2.40
$\theta = 3$	$4.98e-04$	$1.67e-04$	2.41

RMSEE is the root mean squared Euler error, MAEE is the maximum absolute Euler error, and Time us the execution time in seconds.

Table 1 shows that the Euler errors are small and well within the range reported for other simulation methods as reported in Evans and Phillips (2015). That paper finds values for RMSEE and MAEE in the range of $1.00e-03$ to $1.00e-07$ for stable models using 1st-order approximations, 2nd-order approximations and value function iteration over grids.

4 Conclusion

This paper has proposed and analyzed a solution method for DSGE models which we dub "current state linearization" or CSL. We show that this method is valuable is in its ability to simulate an unbalanced model with no well-defined steady state . Our method is quite accurate as measured by Euler errors and executes quickly.

References

Adjemian, Stephane, Houtan Bastani, Frederic Karame, Michel Juillard, Junior Maih, Ferhat Mihoubi, George Perendia, Johannes Pfeifer, Marco Ratto, and Sebastien Villemot, "Dynare Reference Manual, version 4.4.3," Technical Report, Dynare Team 2014.

Blanchard, Olivier Jean and Charles M. Kahn., "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 1980, *48*, 1305–1311.

Christiano, Lawrence J., "Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients," *Computational Economics*, 2002, *20*, 21–55.

_, Martin Eichenbaum, and Charles L. Evans, "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy.," *Journal of Political Economy*, 2005, *113*, 1–45.

Collard, Fabrice and Michel Juillard, "A Higher-Order Taylor Expansion Approach to , Simulation of Stochastic Forward-Looking Models with an Application to a Nonlinear Phillips Curve Model," *Computational Economics*, 2001, *17*, 125–139.

Evans, Richard W. and Kerk L. Phillips, "Linearization about the Current State: A Computational Method for Approximating Nonlinear Policy Functions during Simulation," Technical Report 2015-02, Brigham Young University Macroeconomics and Computational Lab 2015.

Guu, Sy-Ming and Kenneth L. Judd, "Asymptotic methods for asset market equilibrium analysis," *Economic Theory*, 2001, *18*, 127–157.

Judd, K L, "Projection Methods for Solving Aggregate Growth Models," *Journal of Economic Theory*, 1992, *58*, 410–452.

Schmitt-Grohe, Stephanie and Martin Uribe, "Solving dynamic general equilibrium models using a second-order approximation to the policy function," *Journal of Economic Dynamics and Control.*, 2004, *28*, 755–775.

Smets, Frank and Rafael Wouters, "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, 2007, *97*, 586–606.

Uhlig, Harald., *Computational Methods for the Study of Dynamic Economies*, Oxford University Press,