Solving and Simulating Unbalanced Growth Models using Linearization about the Current State *

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Abstract

This paper presents an adjustment to commonly used linear approximation methods for dynamic stochastic general equilibrium (DSGE) models. Policy functions approximated around the steady state will be inaccurate away from the steady state. In some cases the model may not have a well-defined steady state, or the nature of the steady state may be at odds with its off-steady-state dynamics. We show how to simulate a DSGE model with no well-defined steady state by approximating about the current state. Our method minimizes the error associated with a finite-order Taylor-series expansion of the models characterizing equations. This method is easily implemented and has the advantage of mimicking highly non-linear behavior. It also requires choosing N out of $2N$ possible roots from a matrix quadratic equations and the choice of this root not obvious away from the steady state. However, simulations show that using the same criteria as when linearizing about the steady state yield yield reasonable, well-fitting results.

keywords: dynamic stochastic general equilibrium, linearization methods, numerical simulation, computational techniques, simulation modeling, unbalanced growth.

JEL classifications: C63, C68, E37

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1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are an important class of macroeconomic modeling that have been in use now for several decades. They are increasingly used in policy contexts to simulate the effects of policy changes on the macroeconomy.[1]

Usually these models are too complex to find closed-form solutions for dynamic policy functions that map the current state of the economy into the values for next period's endogenous state variables. Instead, these models must be solved and simulated using some approximation method. The most widely used techniques include the linearization methods in Uhlig (1999) and Christiano (2002) who employ a method of undetermined coefficients to solve the state-space representation outlined in Blanchard and Kahn. (1980). Higher-order polynomial approximations developed by Judd (1992), Guu and Judd (2001), Collard and Juillard (2001) and Schmitt-Grohe and Uribe (2004) are increasingly widely used. This is the approach taken with the poplar DSGE software package Dynare, for example.[2]

This paper presents an easy adjustment to linear and higher-order approximation methods. Since approximation is almost always done about the model's steady state, the linear policy functions can be inaccurate if the simulation is often away from the steady state. In some cases, this leads to only small errors. In other cases, however, the model may not have a well-defined steady state, or the nature of the steady state may be at odds with its off-steady-state dynamics. Approximating about the steady state requires the existence of a steady state. If one does not exist, the models variables must be redefined so that they are stationary. We refer to this method as "steady state linearization" (SSL).

Some models of interest to researchers cannot be easily transformed in this manner, however. These include multi-sector models with unbalanced growth and models where the parameters are time-varying. We show how to simulate an unbalanced DSGE model by approximating about the current state, rather than the steady state.

[1] see for example Smets and Wouters (2007) and Christiano et al. (2005).
[2] see Adjemian et al. (2014) for details.

This method is easily implemented and has the advantage of accurately mimicking non-linear behavior. The ability to accurately solve and simulate models that have no steady state makes it possible to examine the behavior of a richer class of models that may be better able to mimic the real economy.

We proceed similarly to Uhlig (1999) and Christiano (2002) who use SSL. They hypothesize linear policy functions and use the method of undetermined coefficients to solve for the coefficients of the policy function. We propose an alternative strategy, which we call "current state linearizarion" (CSL), where one approximates the policy function for each period of the simulation about the current state of the economy. This is computationally more intensive as it requires linearization each period, rather than only once. However, this method has the advantage of being much more accurate when the economy is far from the steady state and being feasible when no steady state exists. CSL can easily replicate the behavior of highly nonlinear policy functions. This is because the Taylor-series approximation is highly accurate in the neighborhood of the point about which a function is approximated. Since we are always approximating about the current state, our linear policy function will be very close to the true policy function for that state.

One major hurdle in polynomial approximations is choosing the appropriate roots from a matrix quadratic equation. In the steady state this is relatively simple since the system is generally stable about the steady state. Most macroeconomic models yield a unique set of roots that imply stability in that neighborhood. The same is not true away from the steady state. However, we show that for our simple unbalanced growth model, the criterion used for steady state stability works well, generating very small Euler errors during simulation. There is no guarantee that the roots with the smallest modulus correspond to the policy functions associated with the saddle path. Fortunately, with the simple illustrative model in this paper that criterion performs well.

While we illustrate our method using linear approximations, the concepts and method apply to higher-order polynomial approximations as well.

2 Derivation of an Approximation about the Current State

Consider a set of nonlinear expectational functions, in our case from a dynamic general equilibrium model. The state variables are grouped into two categories: exogenous state variables are grouped into the $S \times 1$ column vector, Z_t, while endogenous ones are placed in the $K \times 1$ column vector, X_t. There are K equations and they can be represented as in equation (2.1).

$$E_t\{\Gamma(X_{t+2}, X_{t+1}, X_t, Z_{t+1}, Z_t\} = 0 \tag{2.1}$$

This system of equations can be approximated by taking a first-order Taylor series expansion about an arbitrary point in the state space. We choose the current value for the state variables, $\theta_t = \{X_t, Z_t\}$. This transformation is given in equation (2.2).

$$E_t\{T_t + F_t\tilde{X}_{t+2} + G_t\tilde{X}_{t+1} + H_t\tilde{X}_t + L_t\tilde{Z}_{t+1} + M_t\tilde{Z}_t\} = 0 \tag{2.2}$$

In the above equation, F_t, G_t and H_t are $K \times K$ matrices, L_t and M_t are $K \times S$ matrices, and T_t is an $K \times 1$ vector. All these will depend on which point is chosen for the linearization. Tildes denote absolute deviations from θ_t values. Note that if we choose to linearize about the steady state, $\bar{\theta} = \{\bar{X}, \bar{Z}\}$ the value of T_t is zero. While this is true of the steady state, it will not be true generally.

The law of motion for the exogenous state variables is assumed to be a first-order vector autoregression of the form in equation (2.3).

$$Z_{t+1} = (I - N)\bar{Z} + NZ_t + E_{t+1} \tag{2.3}$$

Since we are allowing for linearization around any value of Z, we proceed to transform (2.3) into (2.4).

$$E_t\{\tilde{Z}_{t+1}\} = Z_t - \bar{Z} \tag{2.4}$$

As with standard linearization techniques, our goal is to find a linear approximation to the policy function, (2.5).

$$\tilde{X}_{t+1} = U_t + P_t\tilde{X}_t + Q_t\tilde{Z}_t \tag{2.5}$$

Where U_t is an $K \times 1$ column vector, P_t is an $K \times K$ matrix and Q_t is $K \times S$.

The major differences between (2.5) and the standard linear policy function are: First, the inclusion of the constant term, U_t, which makes it possible for the endogenous state variables to drift away from the current state. And second, the time-varying nature of the parameters P_t, Q_t and U_t. Iterative substitution of (2.4) and (2.5) into (2.2) yields the following three conditions which define P_t, Q_t & U_t.

$$F_tP_t^2 + G_tP_t + H_t = 0 \tag{2.6}$$

$$(F_tQ_t + L_t)N + (F_tP_t + G_t)Q_t + M_t = 0 \tag{2.7}$$

$$T_t + [F_tU_t + F_tP_tU_t) + G_tU_t + (F_tQ_t + L_t)(N - I)(Z_t - \bar{Z}) = 0 \tag{2.8}$$

The CSL method allows us to solve for each periods coefficients in isolation, without having to refer to next periods actual values. It has the advantage of not needing to solve for a benchmark time path via some other method. Indeed, it is not even necessary to solve for the steady state. Instead, we generate the time path as we solve and simulate each period. A disadvantage is that we must recalculate the unique values of P_t, Q_t & U_t each period in each simulation.

Choosing the roots from equation (2.6) is often straightforward when linearizing about the steady state as there is usually a unique set of roots that implies stability of the system. This need not be the case for CSL, however. Stability of the endogenous state variables need not hold away from the steady state. The true roots may imply a path that would be unstable were one to use them for the entire simulation. However, since they are used only for one period the instability is purely temporary and part of the convergence process. We show in our unbalanced growth example below, that choosing he roots with the smallest modulus works very well for CSL for our particular

model. Unfortunately, this need not generally be the case.

3 Applying this Method to an Unbalanced Growth Model

In this section we consider a model with a labor leisure decision and technical progress. The household's problem is shown below. We adopt preferences as laid out in Jaimovich and Rebelo (2009).

$$V(k_t, x_{t-1}, z_t) = \max_{k_{t+1}, h_t} \frac{(c_t - \psi h_t^\theta x_t)^{1-\sigma}}{1-\sigma} + \beta E_t \left[V(k_{t+1}, x_t, z_{t+1})\right]$$

$$x_t = c_t^\gamma x_{t-1}^{1-\gamma} \tag{3.1}$$

$$c_t = w_t h_t + (1 + r_t - \delta)k_t - k_{t+1} \tag{3.2}$$

The first-order conditions from this problem yield the following two Euler equations. The first is the standard intertemporal Euler equation, while the second governs the choice of leisure vs consumption goods.

$$\Gamma_t \Delta_t = \beta E_t \left\{\Lambda_t \Phi_{t+1} + \Gamma_{t+1}\Delta_{t+1}(1 + r_{t+1} - \delta)\right\} \tag{3.3}$$

$$\Gamma_t \Delta_t w_t = \Gamma_t \Pi_t + \beta E_t \left\{\Lambda_t w_t \Gamma_{t+1} \Phi_{t+1}\right\} \tag{3.4}$$

where

$$\Gamma_t \equiv (c_t - \phi h_t^\theta c_t^\gamma x_{t-1}^{1-\gamma})^{-\sigma} \tag{3.5}$$

$$\Delta_t \equiv 1 - \psi\gamma h_t^\theta c_t^{\gamma-1} x_{t-1}^{1-\gamma} \tag{3.6}$$

$$\Phi_t \equiv \psi(1-\gamma) h_t^\theta c_t^\gamma x_{t-1}^{-\gamma} \tag{3.7}$$

$$\Pi_t \equiv \psi\theta h_t^{\theta-1} c_t^\gamma x_{t-1}^{1-\gamma} \tag{3.8}$$

$$\Lambda_t \equiv \gamma c_t^{\gamma-1} x_{t-1}^{1-\gamma} \tag{3.9}$$

Production is given by a Cobb-Douglas production function with exogenous labor-

augmenting technical progress at the rate g per period.

$$y_t = k_t^{\alpha}(h_t e^{gt+z_t})^{1-\alpha} \tag{3.10}$$

The first order conditions from profit maximization yield the following conditions which define wage and interest rates.

$$r_t = \alpha \frac{y_t}{k_t} \tag{3.11}$$

$$w_t = (1-\alpha)\frac{y_t}{h_t} \tag{3.12}$$

Technology deviations from trend follow the law of motion below.

$$z_t = \rho z_{t-1} + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.(0, \omega^2) \tag{3.13}$$

Market clearing is implied by the choice of notation; using k_t and h_t in both the household's problem and the firm's problem.

In the unbalanced growth case it is useful for the simulation algorithm if we define a time counter s_t which follows the following trivial law of motion.

$$s_{t+1} = s_t + 1 \tag{3.14}$$

Given $\{K_{t+1}, H_t, x_{t-1}, s_t\}$ for periods $t+1$, t and $t-1$; and z_t for periods $t+1$ and t we can use (3.10), (3.11), (3.12), (3.2) as definitions to get $\{Y_t, r_t, w_t, c_t\}$. We then use these to evaluate (3.3), (3.4), (3.1) and (3.14) which implicitly define k_{t+1}, h_t, x_t and s_t over time.

This system exhibits balanced growth only if $\gamma > 0$. In this case we can define stationary versions of the growing variables, $\{k_t, y_t, x_t, c_t, w_t, \Gamma_t, \Pi_t\}$ by dividing by e^{gt} to get $\{\tilde{k}_t, \tilde{y}_t, \tilde{x}_t, \tilde{c}_t, \tilde{w}_t, \tilde{\Gamma}_t, \tilde{\Pi}_t\}$. This transformed model can be solved and simulated using any number of techniques, including both SSL and CSL. If $\gamma = 0$ the preferences are those from Greenwood et al. (1988) and the model exhibits unbalanced growth. In this case there is no way to define stationary variables and the only solution method

is our proposed CSL algorithm.

For our balanced growth simulations we use baseline parameterization of: $\alpha = .35$, $\beta = .98$, $\delta = .035$, $g = .01$, $\psi = 2$, $\sigma = 2.5$, $\gamma = 1$ and $\theta = 1.4$. Setting $\gamma = 1$ yields the preferences from King et al. (1988). For our unbalanced growth simulations we use the same values, but set $\gamma = 0$. For the balanced growth case we solve and simulate the model using both SSL and CSL, for the unbalanced growth case we can only use CSL.

Table 1 shows the results for a baseline simulation of the balanced growth model. We choose $\rho = .95$ and $\omega = .013$ for the baseline case. We simulate this model for 10,000 periods and report the Euler errors and execution time. We formulate our Euler equations so that the errors are interpretable as percent errors. For the SSL and CSL simulations we use the same random draws of ε_t for shocks to the z process. We also report the results when ω is five time larger and five times smaller. As the table shows, CSL takes about 35 times as much time to execute as SSL[3]. However, the gain in accuracy is also substantial. Root-mean-squared Euler errors are almost 100 times smaller with CSL in the case with large variance in the shock process and even with very small variance are still almost 10 times smaller.

Given these promising results, we proceed to solve and simulate the unbalanced version of the model when $\gamma = 0$. We start with a value of $k_1 = .01$ and generate 10 simulations of 1050 observations each[4]. We discard the first 50 observations and use the remaining 10,000 observations to construct the data in Table 2. We use the same baseline parameters as with the balanced growth model above.

Table 2 shows that the Euler Errors are larger in this case as measured by the maximum absolute Euler errors and the root-mean-squared Euler error measures. However, the magnitudes are well in line with those from Table 1 and with those reported for a variety of other simulations methods on slightly different models reported in Evans and Phillips (2015).

[3]If we simulate with a sample size of 300, or 75 years of quarterly data, CSL takes about 17 times longer to execute.

[4]Simulations of long length raise issues associated with machine precision when variables are growing over time. We avoid this problem by limiting the length of a single simulation, but running multiple simulations to generate the same sample size as the balanced growth model

Table 1: Euler Errors for Balanced Growth Model
10,000 Periods per Simulation

$\omega = .013$	SSL	CSL	ratio
AvgEE	3.07E-04	-2.05E-05	-0.0668
MAEE	2.83E-03	1.92E-04	0.0677
RMSEE	5.57E-04	4.07E-05	0.0731
Time	0.5756	19.8033	34.4019
$\omega = .065$	SSL	CSL	ratio
AvgEE	2.30E-03	1.42E-04	0.0614
MAEE	3.27E+00	4.51E-03	0.0014
RMSEE	2.67E-02	2.87E-04	0.0108
Time	0.5813	20.4615	35.1993
$\omega = .0026$	SSL	CSL	ratio
AvgEE	5.62E-05	-6.18E-06	-0.1100
MAEE	4.08E-04	2.90E-05	0.0711
RMSEE	1.04E-04	1.09E-05	0.1050
Time	0.5677	20.5493	36.1950

AVGE is the average value of all Euler errors, RMSEE is the root mean squared Euler error, MAEE is the maximum absolute Euler error, and Time is the execution time in seconds.

Table 2: Euler Errors for Unbalanced Growth Model
1000 Periods per Simulation with 10 Simulations

	$\omega = .013$	$\omega = .065$	$\omega = .0026$
AvgEE	1.31E-04	6.86E-04	3.14E-05
MAEE	1.92E-03	7.28E-03	1.08E-03
RMSEE	6.06E-03	1.38E-02	3.25E-03
time	19.0936	20.2386	19.4912

AVGE is the average value of all Euler errors, RMSEE is the root mean squared Euler error, MAEE is the maximum absolute Euler error, and Time is the execution time in seconds.

4 Conclusion

This paper has proposed and analyzed a solution method for DSGE models which we dub "current state linearization" or CSL. We show that this method is valuable is in its ability to simulate an unbalanced model with no well-defined steady state . Our method is substantially more accurate as measured by Euler errors than standard linearization about the steady state, but still executes quickly. We show that for our simple model choosing the roots with smallest modulus to construct the coefficients of the linear policy function works well. However, we have no formal proof that this choice will always be the correct one.

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A-1 Derivation of the Euler equations

$$V(k_t, x_{t-1}, z_t) = \max_{k_{t+1}, h_t} \frac{(c_t - \psi h_t^\theta c_t^\gamma x_{t-1}^{1-\gamma})^{1-\sigma}}{1-\sigma} + \beta E_t \left[V(k_{t+1}, c_t^\gamma x_{t-1}^{1-\gamma}, z_{t+1}) \right]$$
$$c_t = w_t h_t + (1 + r_t - \delta) k_t - k_{t+1}$$

Define the following:

$$\Gamma_t \equiv (c_t - \psi h_t^\theta c_t^\gamma x_{t-1}^{1-\gamma})^{-\sigma} \tag{A.1.1}$$
$$\Delta_t \equiv 1 - \psi\gamma h_t^\theta c_t^{\gamma-1} x_{t-1}^{1-\gamma} \tag{A.1.2}$$
$$\Phi_t \equiv \psi(1-\gamma) h_t^\theta c_t^\gamma x_{t-1}^{-\gamma} \tag{A.1.3}$$
$$\Pi_t \equiv \psi\theta h_t^{\theta-1} c_t^\gamma x_{t-1}^{1-\gamma} \tag{A.1.4}$$
$$\Lambda_t \equiv \gamma c_t^{\gamma-1} x_{t-1}^{1-\gamma} \tag{A.1.5}$$

The first order conditions from household's problem are given below.

$$\Gamma_t \Delta_t(-1) + \beta E_t \left\{ V_k(k_{t+1}, x_t, z_{t+1}) + V_h(k_{t+1}, x_t, z_{t+1}) \Lambda_t(-1) \right\} = 0 \tag{A.1.6}$$
$$\Gamma_t(\Delta_t w_t - \Pi_t) + \beta E_t \left\{ V_h(k_{t+1}, x_t, z_{t+1}) \Lambda_t w_t \right\} = 0 \tag{A.1.7}$$

The envelope conditions are given by:

$$V_k(k_t, x_{t-1}, z_t) = \Gamma_t \Delta_t (1 + r_t - \delta) \tag{A.1.8}$$
$$V_x(k_t, x_{t-1}, z_t) = \Gamma_t \Phi_t \tag{A.1.9}$$

Substituting and simplifying:

$$\Gamma_t \Delta_t = \beta E_t \left\{ \Lambda_t \Phi_{t+1} + \Gamma_{t+1} \Delta_{t+1} (1 + r_{t+1} - \delta) \right\} \tag{A.1.10}$$
$$\Gamma_t \Delta_t w_t = \Gamma_t \Pi_t + \beta E_t \left\{ \Lambda_t w_t \Gamma_{t+1} \Phi_{t+1} \right\} \tag{A.1.11}$$

A-2 Transformed Model When There is Balanced Growth

$$z_t = \rho z_{t-1} + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.(0, \omega^2) \tag{A.2.1}$$

$$\tilde{y}_t = \tilde{k}_t^{\alpha}(h_t e^{z_t})^{1-\alpha} \tag{A.2.2}$$

$$r_t = \alpha \frac{\tilde{y}_t}{\tilde{k}_t} \tag{A.2.3}$$

$$\tilde{w}_t = (1-\alpha)\frac{\tilde{y}_t}{h_t} \tag{A.2.4}$$

$$\tilde{c}_t = \tilde{w}_t h_t + (1 + r_t - \delta)\tilde{k}_t - \tilde{k}_{t+1}e^{g} \tag{A.2.5}$$

$$\tilde{x}_t = \tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.6}$$

$$\tilde{\Gamma}_t \equiv (\tilde{c}_t - \psi h_t^{\theta}\tilde{c}_t^{\gamma}\tilde{(}x_{t-1}e^{-g})^{1-\gamma})^{-\sigma} \tag{A.2.7}$$

$$\Delta_t \equiv 1 - \psi\gamma h_t^{\theta}\tilde{c}_t^{\gamma-1}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.8}$$

$$\Phi_t \equiv \psi(1-\gamma)h_t^{\theta}\tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{-\gamma} \tag{A.2.9}$$

$$\tilde{\Pi}_t \equiv \psi\theta h_t^{\theta-1}\tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.10}$$

$$\Lambda_t \equiv \gamma\tilde{c}_t^{\gamma-1}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.11}$$

$$\tilde{\Gamma}_t\Delta_t = \beta E_t\left\{\Lambda_t\Phi_{t+1} + \tilde{\Gamma}_{t+1}e^{g}\Delta_{t+1}(1 + r_{t+1} - \delta)\right\} \tag{A.2.12}$$

$$\tilde{\Gamma}_t\Delta_t\tilde{w}_t = \tilde{\Gamma}_t\tilde{\Pi}_t + \beta E_t\left\{\Lambda_t\tilde{w}_t\tilde{\Gamma}_{t+1}e^{g}\Phi_{t+1}\right\} \tag{A.2.13}$$

$$\tilde{x}_t = \tilde{c}_t^{\gamma}(\tilde{x}_{t-1}e^{-g})^{1-\gamma} \tag{A.2.14}$$

A-3 Model with GHH Preferences and Unbalanced Growth

$$z_t = \rho z_{t-1} + \varepsilon_t; \quad \varepsilon_t \sim i.i.d.(0, \omega^2) \tag{A.3.1}$$
$$s_t = s_{t-1} + 1 \tag{A.3.2}$$
$$y_t = k_t^{\alpha}(h_t e^{gt+z_t})^{1-\alpha} \tag{A.3.3}$$
$$r_t = \alpha \frac{y_t}{k_t} \tag{A.3.4}$$
$$w_t = (1-\alpha)\frac{y_t}{h_t} \tag{A.3.5}$$
$$c_t = w_t h_t + (1 + r_t - \delta)k_t - k_{t+1} \tag{A.3.6}$$
$$\Gamma_t = (c_t - \psi h_t^{\theta} c_t)^{-\sigma} \tag{A.3.7}$$
$$\Delta_t = 1 - \psi h_t^{\theta} \tag{A.3.8}$$
$$\Pi_t = \psi\theta h_t^{\theta-1} c_t \tag{A.3.9}$$
$$\Gamma_t \Delta_t = \beta E_t \left\{\Gamma_{t+1}\Delta_{t+1}(1 + r_{t+1} - \delta)\right\} \tag{A.3.10}$$
$$\Delta_t w_t = \Pi_t \tag{A.3.11}$$