Discussion 1: Differential and Difference Equations - Theory

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Introduction

- Economists need to know basic theory of difference and differential equations as they describe law of motions of variables of interest
- Differential equations describe movements in continuous time, difference equations in discrete time
- Reference on differential equations: Acemoglu, Introduction to Modern Economic Growth. Appendix B
- Reference on difference equations: Galor, Dynamical Systems
- If you find any typos in these or future slides, please send me an email so I can correct them

Outline

Differential Equations and Stability

Difference Equations and Stability

Some Terminology

- An explicit first order differential equation (DE) is

$$\frac{dx(t)}{dt} \equiv \dot{x}(t) = g(x(t), t) \tag{1}$$

- An implicit first order DE is

$$H(\dot{x}(t), x(t), t) = 0$$

- We will deal only with first-order explicit DEs.
- A DE is *autonomous* if it can be written as

$$\dot{x}(t) = g(x(t))$$

Some Terminology

- An nth-order DE is

$$\frac{d^n x(t)}{dt^n} = g\left(\frac{d^{n-1}x(t)}{dt^{n-1}}, \ldots, \frac{dx(t)}{dt}, x(t), t\right)$$

- Higher order DE can always be transformed to systems of first-order DEs

- A linear first-order DE takes the form

$$\dot{x(t)} = a(t)x(t) + b(t)$$

- *Homogenous equation*: $b(t) = 0$.

- *Constant coefficient equation*: $a(t) = a$, $b(t) = b$.

Initial Value Problem

- In economics, a *state variable* is a variable which is backward looking (e.g. a stock of capital). In this case, a differential equation as in (1) is specified together with an initial condition $x(0) = x_0$.

- **Example**: Assume capital k evolves each period by a factor $a > 1$, and consider the initial condition $k(0) = k_0$.

$$\dot{k(t)} = ak(t)$$

  Divide both sides by $k(t)$ and integrate wrt t to get

$$\ln |k(t)| + c_0 = at + c_1$$

  Take exponent on both sides to get the *general solution*

$$k(t) = c \exp(at)$$

  Plug initial condition to get $k(t) = k_0 \exp(at)$.

Terminal condition

- A *co-state variable* is a variable which instead depends on the future (e.g. prices, value functions, marginal utilities). In this case, boundary conditions are specified by "transversality conditions": the terminal value of $x(t)$ is specified at some $T < \infty$ or $T = \infty$.
- Will soon see examples of transversality condition in this course.

General versus Particular Solution

- Go back to

$$\dot{x}(t) = g(x(t), t)$$

- A family of functions that satisfies the previous equation is called a "general solution"
- Furthermore, if a boundary condition is specified, then the function that solves the differential equation and satisfies the boundary condition is called "particular solution".
- The boundary condition pins down the constant in the solution. See previous slides.

Solving a Non-Homogenous Linear DE with Constant Coefficients

- Start from
$$\dot{x(t)} = ax(t) + b \tag{2}$$
- Let $y(t) = x(t) + b/a$. Then rewrite (2) as
$$\dot{y(t)} = ay(t)$$
- Solve it as previous case to get $y(t) = c\exp(at)$, then change the variable back
$$x(t) = -b/a + c\exp(at)$$
- With initial condition $x(0) = x_0$: $c = x_0 + b/a$.

Solving a Non-Homogenous Linear DE with Constant Coefficients

- Therefore the particular solution is

$$x(t) = -b/a + (x_0 + b/a)\exp(at)$$

- Let's discuss **stability**. Steady state, where $\dot{x(t)} = 0$, is

$$x(t) = x^* = -b/a$$

- Two cases as t increases
  - $a < 0$: $x(t)$ apporaches x^*
  - $a > 0$: $x(t)$ diverges

Systems of Linear Differential Equations

- Consider a system of first-order DEs with constant coefficients
$$\dot{x}(t) = Ax(t) \tag{3}$$
- This system always has a unique solution
- **Theorem**: Suppose A has n distinct real eigenvalues $\xi_1, \ldots, \xi_n$. Then the unique solution to the system (3) takes the form
$$x(t) = \sum_{j=1}^{n} c_j \exp(\xi_j t) v_{\xi_j}$$
where $v_{\xi_1}, \ldots, v_{\xi_n}$ denote the eigenvectors corresponding to the eigenvalues $\xi_1, \ldots, \xi_n$, and $c_1, \ldots, c_n$ denote the constants of integration
- Note: the constant of integration are determined by the boundary conditions.

Stability

- The stability properties of the system depend on the signs of its eigenvalues.
  - If all eigenvalues are positive, the system is unstable
  - If $m < n$ eigenvalues are negative, then there exists an m-dimensional subspace such that the solution tends to the steady state only starting with an initial value on this subspace (*saddle path stability*)
  - If eigenvalues are complex, the system entails oscillating behaviour
  - The system is stable only when all eigenvalues are negative

Differential Equations and Stability

Difference Equations and Stability

Linear Systems and Solution

- One-dimensional, first-order, autonomous, linear difference equation
$$y_{t+1} = ay_t + b \tag{4}$$
- A solution to (4) is a trajectory of the state variable $(y_t)_{t=0}^{\infty}$ that satisfies the law of motion at any point in time.
- The derivation may follow several methods, we'll use "method of iterations".
- Given an initial value y_0, start at time 1 $y_1 = ay_0 + b$ and substitute recursively to get
$$y_t = a^t y_0 + b \sum_{i=0}^{t-1} a^i$$

Linear Systems and Solution

- $\sum_{i=0}^{t-1} a^i$ is a geometric series with factor a, so

$$\sum_{i=0}^{t-1} a^i = \begin{cases} \frac{1-a^t}{1-a} & \text{if} \quad a \neq 1 \\ t & \text{if} \quad a = 1 \end{cases}$$

  and therefore

$$y_t = \begin{cases} a^t y_0 + b\frac{1-a^t}{1-a} & \text{if} \quad a \neq 1 \\ y_0 + bt & \text{if} \quad a = 1 \end{cases}$$

  or alternatively

$$y_t = \begin{cases} [y_0 - \frac{b}{1-a}]a^t + \frac{b}{1-a} & \text{if} \quad a \neq 1 \\ y_0 + bt & \text{if} \quad a = 1 \end{cases}$$

- Thus the entire trajectory of the state variable is uniqued determined given the initial condition

Existence and Uniqueness of Steady-State Equilibria

- A steady state equilibrium (SS) of the difference equation $y_{t+1} = ay_t + b$ is $\bar{y}$ such that $\bar{y} = a\bar{y} + b$

- A SS exists if and only if $a \neq 1$ or $\{a = 1$ and $b = 0\}$:

$$\bar{y} = \begin{cases} \frac{b}{1-a} & \text{if} \quad a \neq 1 \\ y_0 & \text{if} \quad a = 1 \quad \text{and} \quad b = 0 \end{cases}$$

- Finally, the SS is unique if and only if $a \neq 1$ (the second case entail a continuum of SS reflecting the entire set of feasible initial conditions)

- Therefore substituting the value of $\bar{y}$ into the solution previously found

$$y_t = \begin{cases} [y_0 - \bar{y}]a^t + \bar{y} & \text{if} \quad a \neq 1 \\ y_0 + bt & \text{if} \quad a = 1 \end{cases}$$

Stability of Steady-State Equilibria

- A steady state equilibrium $\bar{y}$ of the difference equation $y_{t+1} = ay_t + b$ is
  - globally (asymptotically) stable if $\lim_{t\to\infty} y_t = \bar{y} \;\; \forall y_0$
  - locally (asymptotically) stable if $\lim_{t\to\infty} y_t = \bar{y} \;\; \forall y_0$ such that $|y_0 - \bar{y}| < \epsilon$ for some $\epsilon > 0$
- A steady state equilibrium $\bar{y}$ of the difference equation $y_{t+1} = ay_t + b$ is globally asymptotically stable only if it is unique
  - While global stability requires global uniqueness of the SS, local stability necessitates the local uniqueness of the SS

Stability of Steady-State Equilibria

- As follows from the definition of stability, we have to examine the system as time approaches infinity. This leads to

$$\lim_{t\to\infty} |y_t| = \begin{cases} |\bar{y}| & \text{if } \; |a| < 1 \; \text{ or } \; \{|a| > 1 \text{ and } y_0 = \bar{y}\} \\ |y_0| & \text{if } \; \{a = 1 \text{ and } b = 0\} \\ |y_0| & \text{for even t if } \; a = -1 \\ |b - y_0| & \text{for odd t if } \; a = -1 \\ \infty & \text{otherwise} \end{cases}$$

- I'll only present graphs for the interesting cases

Unique, globally stable SS

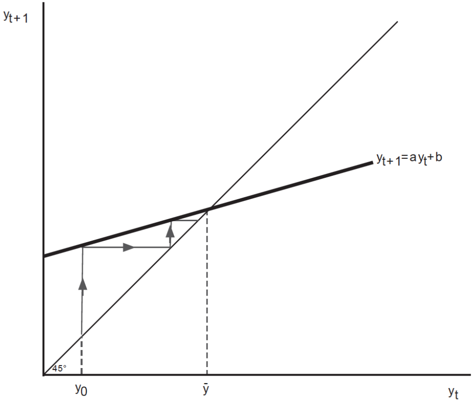

Figure 1.1

a ε (0,1)

Unique, Globally Stable, Steady-State Equilibrium

(Monotonic Convergence)

Unique, globally stable SS

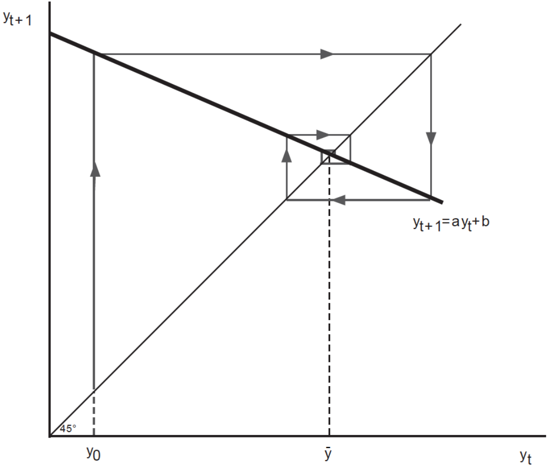

Figure 1.2

a ε (-1,0)

Unique, Globally Stable, Steady-State Equilibrium

(Oscillatory Convergence)

Continuum of unstable SS

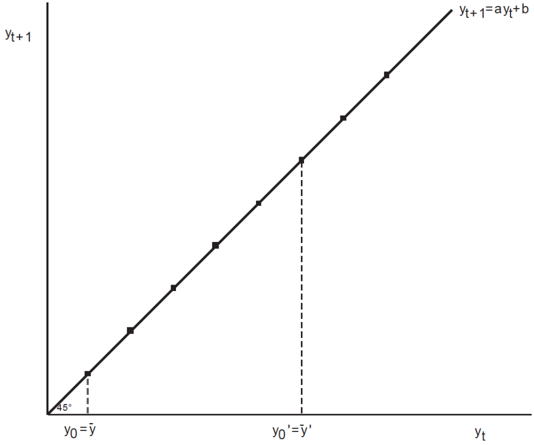

Figure 1.3

a = 1 & b = 0

Continuum of Unstable Steady-State Equilibria

Non-existence of a SS

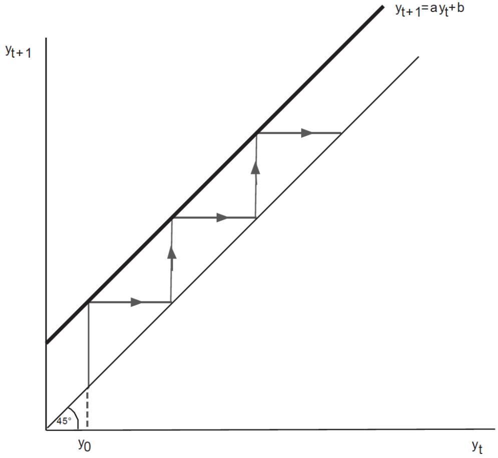

Figure 1.4

$a = 1$ & $b \neq 0$

Continuum of Unstable Steady-State Equilibria

Unique unstable SS

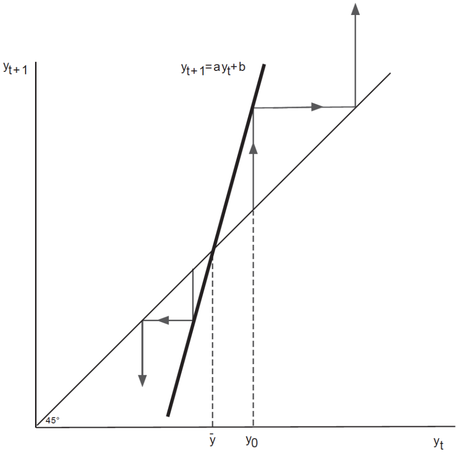

Figure 1.6

a > 1

Unique and Unstable Steady-State Equilibrium

(Monotonic Divergence)

Necessary and Sufficient Condition for Global Stability

- Hence the whole discussion leads us to the main stability result:
- A steady-state equilibrium of the difference equation $y_{t+1} = ay_t + b$ is **globally** stable **if and only if** $|a| < 1$.

Non-linear systems and solution

- We will study the evolution of the state variable *in the proximity* of a SS. Then give sufficient conditions for stability. Consider the one-dimensional first-order state variable

$$y_{t+1} = f(y_t)$$

- Again we use the method of iterations to describe the trajectory of the state variable

$$y_1 = f(y_0) \implies y_2 = f(f(y_0)) \equiv f^2(y_0) \implies y_t = f^t(y_0)$$

- Unlike the solution to the linear system, the solution for the nonlinear system is not very informative about the factors of convergence

- Hence additional methods of analysis are required (linear approximation around the SS).

Steady-State and Linearization

- A SS of the difference equation $y_{t+1} = f(y_t)$ is a level of $\bar{y}$ such that $\bar{y} = f(\bar{y})$. Generically, a non-linear may be characterized by unique SS, multiple SS, chaotic behavior, or non-existence of SS

- Consider a first orderTaylor expansion of $y_{t+1} = f(y_t)$ around the SS

  $$y_{t+1} = f(\bar{y}) + f'(\bar{y})(y_t - \bar{y}) = f'(\bar{y})y_t + f(\bar{y}) - f'(\bar{y})\bar{y} = ay_t + b$$

  where $a = f'(\bar{y})$ and $b = f(\bar{y}) - f'(\bar{y})\bar{y}$ are given constants.

- Now, simply apply the main result we have learnt for linear systems! However we go for global to local because of the linearization

- The SS $\bar{y}$ is **locally** stable iff $|f'(\bar{y})| < 1$.
  - This is the result used in the Solow model, plus fundamental theorem of calculus to obtain global stability. See proof of proposition 2.5 in Acemoglu

Unique globally stable SS

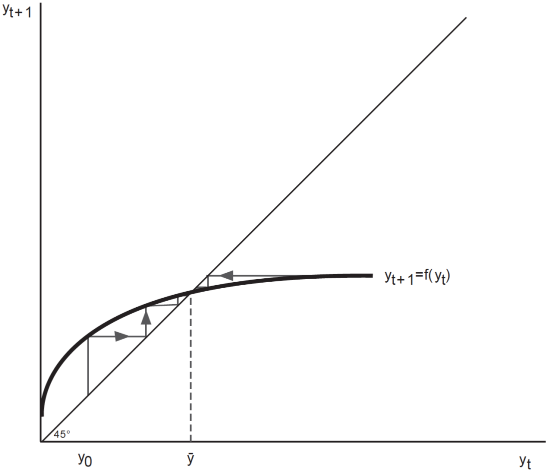

Figure 1.8

Unique and Globally Stable Steady-State Equilibrium

Multiple SS

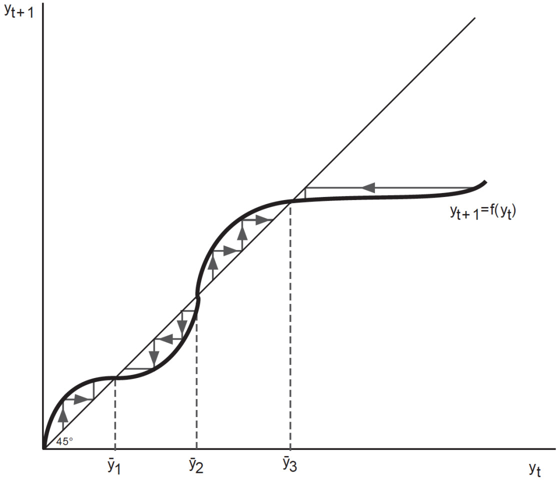

Figure 1.9

Multiple Locally Stable Steady-State Equilibria

Higher-order Systems (Acemoglu page 44-45)

- **Stability for Systems of Linear Difference Equations**: Consider $x_{t+1} = Ax_t + b$, where x and b are vectors and A a matrix. Suppose that all of the eigenvalues of A are strictly *inside the unit circle* in the complex plane. Then the SS of the difference equation is globally asymptotically stable.
- **Local Stability for Systems of Non-Linear Difference Equations**: Consider $x_{t+1} = G(x_t)$, where x is a vector and G a non-linear function differentiable at x^*. Define $A \equiv DG(x^*)$, where DG denotes the matrix of partial derivatives of G. Suppose that all of the eigenvalues of A are strictly *inside the unit circle*. Then the SS of the difference equation is locally stable (there exists an open ball where the trajectory converges).