Discussion 2: Differential and Difference Equations - Examples

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Introduction

- We left off with some basic theory of difference and differential equations
- Today: examples related to the Solow Growth Model.
- Reference: Acemoglu, Introduction to Modern Economic Growth
- If you find any typos in these or future slides, please send me an email so I can correct them

Outline

The Solow Growth Model, discrete time

The Solow Growth Model, continuous time

Solow with two Types of Capital

The Solow Growth Model

- I will solve this model step by step at the whiteboard.
- Assumption 1: Continuity, Differentiability, Positive and Diminishing Marginal Products, and CRS of the Production Function
- Assumption 2: Inada Conditions.
- Law of motion of capital

$$K_{t+1} = sF(K_t, A_t L_t) + (1-\delta)K_t$$

- Population grows at rate n, technology at rate g. Then

$$k_{t+1} = \frac{1}{(1+n)(1+g)}[sf(k_t) + (1-\delta)k_t]$$

  where $k = K/AL$ and $f(k) = F(k, 1)$.
- In steady state $\frac{f(\bar{k})}{\bar{k}} = \frac{(1+n)(1+g)-(1-\delta)}{s}$.

The Solow Growth Model, discrete time

The Solow Growth Model, continuous time

Solow with two Types of Capital

The Solow Growth Model

- Same assumptions as previous slide
- Assume also exponential population growth. The law of motion of capital becomes

$$\dot{k}(t) = sf(k(t)) - (n+\delta)k(t)$$

- A steady state involves $k(t)$ remaining constant at some level k^* for any t. It is unique and such that:

$$\frac{f(k^*)}{k^*} = \frac{n+\delta}{s}$$

- In steady state, the amount of investment is used to replenish the capital.

The Law of Motion of Capital

- **Theorem**: Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, suppose there exists a unique x^* such that $g(x^*) = 0$. Moreover, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the non-linear differential equation $\dot{x}(t) = g(x(t))$, x^*, is globally asymptotically stable, i.e., starting with any $x(0), x(t) \to x^*$.
- **Proposition**: Suppose Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \to k^*$.
- Graph on the board.

Dynamics with the Cobb Douglas Production Function

- As an example, consider the production function

$$F[K, L, A] = AK^{\alpha}L^{1-\alpha}$$

- Per capita production function is $f(k) = Ak^{\alpha}$, and the law of motion is

$$\dot{k(t)} = sAk(t)^{\alpha} - (n + \delta)k(t)$$

and in the steady state

$$k^* = \left(\frac{sA}{n + \delta}\right)^{\frac{1}{1-\alpha}}$$

Dynamics with the Cobb Douglas Production Function

- To solve the law of motion of capital, let $x(t) = k(t)^{1-\alpha}$, so as to write the law of motion as

$$\dot{x}(t) = (1-\alpha)sA - (1-\alpha)(n+\delta)x(t)$$

- which is a liner differential equation, with a general solution

$$x(t) = \frac{sA}{n+\delta} + \left[x(0) - \frac{sA}{n+\delta}\right]\exp(-(1-\alpha)(n+\delta)t)$$

- and changing the variable back

$$k(t) = \left\{\frac{sA}{n+\delta} + \left[k(0)^{1-\alpha} - \frac{sA}{n+\delta}\right]\exp(-(1-\alpha)(n+\delta)t)\right\}^{\frac{1}{1-\alpha}}$$

- Note stability: starting at any $k(0)$ capital will converge to its steady state value.

The Solow Growth Model, discrete time

The Solow Growth Model, continuous time

Solow with two Types of Capital

Solow with Two types of Capital

- Consider a Cobb Douglas production function using two types of capital: *equipment* K_e and *structures* K_s

$$Y(t) = K_e(t)^{\alpha} K_s(t)^{\beta} (A(t)L(t))^{1-\alpha-\beta}$$

- Assume constant population growth n and constant rate of labor-augmenting technological progress g.
- Assumption 1 (on production function) and 2 (Inada) hold true for both types of capital.
- Define effective capital ratios as $k_e = K_e/AL$ and $k_s = K_s/AL$.
- Model from Acemoglu, Physical and Human Capital

Laws of Motion of Capital

- Then the laws of motion of capital are

$$\dot{k_e}(t) = s_{K_e} f(k_e(t), k_s(t)) - (\delta_{K_e} + g + n)k_e(t) \qquad (1)$$

$$\dot{k_s}(t) = s_{K_s} f(k_e(t), k_s(t)) - (\delta_{K_s} + g + n)k_s(t) \qquad (2)$$

where $f(k_e, k_s) = k_e^\alpha k_s^\beta$, $\alpha + \beta < 1$

- This is a system of differential equations: the two types of capital are state variables.

- A steady state is now defined in term of a couple (k_e^*, k_s^*) which satisfies the following two equations

$$s_{K_e} f(k_e^*, k_s^*) - (\delta_{K_e} + g + n)k_e^* = 0 \qquad (3)$$

$$s_{K_s} f(k_e^*, k_s^*) - (\delta_{K_s} + g + n)k_s^* = 0 \qquad (4)$$

where (3) is the locus $\dot{k_e}(t) = 0$ and (4) is the locus $\dot{k_s}(t) = 0$.

Stability - Graphical Representation

- The steady state (k_e^*, k_s^*) is

$$k_e^* = \left[\left(\frac{s_{K_e}}{n+g+\delta_{K_e}} \right)^{1-\beta} \left(\frac{s_{K_s}}{n+g+\delta_{K_s}} \right)^{\beta} \right]^{\frac{1}{1-\alpha-\beta}}$$

$$k_s^* = \left[\left(\frac{s_{K_e}}{n+g+\delta_{K_e}} \right)^{\alpha} \left(\frac{s_{K_s}}{n+g+\delta_{K_s}} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha-\beta}}$$

- In the $(k_e(t), k_s(t))$ space, the two loci - (3) and (4) - are upward sloping and intersect only once (not here, but can show by total differentiation, using concavity of the production function and Inada)
- Also, the steady state is globally stable. Diagrammatic proof at the board.

Stability - Analytical Derivation

- Recall the non-linear system of differential equations

$$\dot{k_e} = s_{K_e} f(k_e, k_s) - (\delta_{K_e} + g + n) k_e$$

$$\dot{k_s} = s_{K_s} f(k_e, k_s) - (\delta_{K_s} + g + n) k_s$$

- The linearized version around the steady state (f_{k_e} and f_{k_s} evaluated at the steady state)

$$\dot{k_e} = [s_{K_e} f_{k_e} - (\delta_{K_e} + g + n)](k_e - k_e^*) + s_{K_e} f_{k_s}(k_s - k_s^*)$$

$$\dot{k_s} = s_{K_s} f_{k_e}(k_e - k_e^*) + [s_{K_s} f_{k_s} - (\delta_{K_s} + g + n)](k_s - k_s^*)$$

- Now the system is linear and we can apply what we have learnt in discussion 1

Stability - Analytical Derivation

- The matrix of this system is

$$M = \begin{bmatrix} s_{K_e} f_{k_e} - (\delta_{K_e} + g + n) & s_{K_s} f_{k_s} \\ s_{K_s} f_{k_e} & s_{K_s} f_{k_s} - (\delta_{K_s} + g + n) \end{bmatrix}$$

- **Lemma**: For a $2by2$ matrix A, all eigenvalues must have negative real part if $trace(A) < 0$ and $determinant(A) > 0$.

$$Trace(M) = s_{K_e} f_{k_e} + s_{K_s} f_{k_s} - (\delta_{K_e} + g + n) - (\delta_{K_s} + g + n)$$

$$\begin{aligned} Det(M) = &(\delta_{K_e} + g + n)(\delta_{K_s} + g + n) - \\ &- s_{K_e} f_{k_e} (\delta_{K_s} + g + n) - s_{K_s} f_{k_s} (\delta_{K_e} + g + n) \end{aligned}$$

Stability - Analytical Derivation

- Note that since $f(k_e, k_s)$ is concave, then $f(k_e, k_s) > k_e f_{k_e} + k_s f_{k_s}$. Then in steady state

$$f_{k_e} < \frac{f(k_e^*, k_s^*)}{k_e^*} = \frac{\delta_{K_e} + g + n}{s_{K_e}}$$

  and so

$$s_{K_e} f_{k_e} < \delta_{K_e} + g + n$$

  Similarly

$$s_{K_s} f_{k_s} < \delta_{K_s} + g + n$$

- The previous two inequalities show that $Trace(M) < 0$.

Stability - Analytical Derivation

- Let's now show the determinant is positive.

$$Det(M) = (\delta_{K_e} + g + n)(\delta_{K_s} + g + n) - \\ - s_{K_e} f_{k_e}(\delta_{K_s} + g + n) - s_{K_s} f_{k_s}(\delta_{K_e} + g + n)$$

or

$$Det(M) = \frac{s_{K_e} f}{k_e^*} \frac{s_{K_s} f}{k_s^*} - s_{K_e} f_{k_e} \frac{s_{K_s} f}{k_s^*} - s_{K_s} f_{k_s} \frac{s_{K_e} f}{k_e^*}$$

or

$$Det(M) = \frac{s_{K_e} s_{K_s}}{k_e^* k_s^*} f[f - k_e^* f_{k_e} - k_s^* f_{k_s}]$$

- Hence $det(M) > 0$, and we get global stability.