14.451 Lecture Notes 1

Bruno Strulovici[*]

Fall 2011

1 An example: reaching the center

Consider the following static problem: an agent is located at some point $x_0 \in \mathbb{R}$. The agent wants to reach point 0 but traveling is costly: traveling a distance d costs d^2. After traveling, the agent pays cost D^2 if the distance from point 0 is D. Therefore, the problem is to choose the location x that minimizes $d^2 + D^2$ with

$$\begin{aligned} d &= |x - x_0|, \\ D &= |x|. \end{aligned}$$

Standard quadratic problem

$$\min_{x \in \mathbb{R}} (x - x_0)^2 + x^2.$$

The first order condition is

$$2(x - x_0) + 2x = 0,$$

and the minimizer is

$$x = \frac{1}{2} x_0.$$

Suppose now the problem is repeated for two periods: $t = 0, 1$. The agent discounts second period payoff with discount factor β.

$$\min (x_1 - x_0)^2 + x_1^2 + \beta \left[(x_2 - x_1)^2 + x_2^2 \right]$$

Recursive approach: use the agent location at the end of the first period x_1 as the initial condition for the problem in the second period. That problem is the same as above (just

[*]All lecture notes of this class, except those on stochastic control, are based on Guido Lorenzoni's Lecture Notes

with x_1 in place of x_0). We know that the optimal thing to do for the agent starting at $x_1 \in [-1, 1]$ at $t = 1$ is to choose

$$x = \frac{1}{2} x_1$$

and obtain payoff

$$\frac{1}{2} x_1^2.$$

But then, going back to $t = 0$, we can solve problem

$$\min_{x \in \mathbb{R}} (x - x_0)^2 + x^2 + \beta \frac{1}{2} x^2$$

and get

$$2 (x - x_0) + 2x + \beta x = 0,$$

$$x = \frac{2}{4 + \beta} x_0.$$

We can then do it for T periods, with geometric discounting. Let $x_t \in \mathbb{R}$ denote the agent location at the beginning of the period. The problem is to minimize

$$\sum_{t=0}^{T-1} \beta^t \left((x_t - x_{t+1})^2 + x_{t+1}^2 \right)$$

with x_0 given.

Start from

$$V_0 (x) = \frac{1}{2} x^2$$

and then derive V_T by iterating:

$$V_T (x) = \min_{y \in \mathbb{R}} (x - y)^2 + y^2 + \beta V_{T-1} (y) . \tag{1}$$

Claim V_T takes the form

$$V_T (x) = A_T x^2,$$

for some constant A_T.

We prove this claim by induction. We have shown that it is true for $T = 1$. Suppose that it is true for $T - 1$. We have

$$V_T (x) = \min_{y \in [-1,1]} (x - y)^2 + y^2 + \beta A_{T-1} y^2,$$

which yields

$$y = \frac{1}{2 + \beta A_{T-1}} x \tag{2}$$

$$x - y = \frac{1 + \beta A_{T-1}}{2 + \beta A_{T-1}} x \tag{3}$$

so

$$V_T(x) = \frac{1+\beta A_{T-1}}{2+\beta A_{T-1}} x^2$$

and

$$A_T = \frac{1+\beta A_{T-1}}{2+\beta A_{T-1}}.$$

If instead the horizon is infinite, the problem is the same in all periods (stationarity), so we can define $V(x)$ as the optimal value of the infinite horizon problem and maybe setup something similar to (1). That is,

$$V(x) = \min_{y\in\mathbb{R}} (x-y)^2 + y^2 + \beta V(y).$$

Technical problem (to be solved): we don't have a starting point for our induction. Maybe we can conjecture that $V(x) = Ax^2$ and end up with one equation in one unknown:

$$A = \frac{1+\beta A}{2+\beta A}.$$

We can also ask if the constant A_T obtained from the finite horizon problem converges to A as T goes to infinity.

We then can derive an optimal rule that tells us how the location of the agent evolves over time:

$$x_{t+1} = \frac{1}{2+\beta A} x_t$$

and characterize properties of the optimal path (e.g., show that $x_t \to 0$ for all x_0).

So our objectives are to

- develop tools to check that this approach works;
- use this approach to characterize the solution, i.e., the optimal path.

We will pursue these objectives in a general environment.

2 A general recursive problem

We will work first on a general *deterministic* problem.

Ingredients:

- Infinite horizon
- Discrete time
- Discounting at rate $\beta \in (0, 1)$
- State variable x_t in some set X
- A constraint correspondence $\Gamma : X \to X$

$$x_{t+1} \in \Gamma(x_t)$$

- Payoff function

$$F(x_t, x_{t+1})$$

  defined for all $x_t \in X$ and all $x_{t+1} \in \Gamma(x_t)$. In other words, with $A \equiv \{(x, y) \in X \times X : y \in \Gamma(x)\}$, we have a function $F : A \to \mathbb{R}$.

A *plan* is a sequence $\{x_t\}_{t=0}^{\infty}$.

A *feasible plan from* x_0, is a plan with $x_{t+1} \in \Gamma(x_t)$. Set of feasible plans denoted $\Pi(x_0)$.

Objective is to find a sequence $\{x_t\}_{t=0}^{\infty}$ that maximizes the discounted sum of payoffs

$$\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

This needs to be well defined for all feasible plans.

An *optimal plan from* x_0 is a plan that achieves the maximum.

Assumption 1. $\Gamma(x)$ is non-empty $\forall x \in X$

Assumption 2.

$$\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) \text{ exists}$$

for all feasible plans from any initial $x_0 \in X$.

3 Principle of optimality

The *sequence problem* (SP):is defined by

$$V^*(x_0) = \sup \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$$

subject to

$$x_{t+1} \in \Gamma(x_t) \qquad t = 0, 1, ...$$

and x_0 given.

The function V^* is the called the *value function* of the problem.

The *Bellman equation* corresponding to this problem is

$$V(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\},$$

which is a functional equation (FE). Solving this problem is finding a V that satisfies this equation.

Define also the policy correspondence

$$G(x) \equiv \arg \max_{x \in \Gamma(x)} \{F(x, y) + \beta V^*(y)\}$$

We will usually make sufficient assumptions for G to be nonempty.

The principle of optimality establishes a relationship between these two problems, about the relationship between v solving FE and V^*defined by SP. It is also about the relationship between optimal plans x^* for SP and plans generated using the policy correspondence G.

Theorem 1 *The value function $V^*(x)$ satisfies the Bellman equation*

$$V^*(x) = \sup_{y \in \Gamma(x)} \{F(x, y) + \beta V^*(y)\}.$$

Moreover, any maximizing sequence $\{x_t^*\}_{t \geq 1}$ satisfies $x_{t+1}^* \in G(x_t)$.

Proof. For simplicity, we assume the existence of a maximizing sequence $\{x_t^*\}_{t \geq 1}$ Take any $x_0 \in X$. Let $\{x_t^*\}_{t=0}^{\infty}$ be an optimal plan from x_0. By definition,

$$V^*(x_0) = F(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq F(x_0, x_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1})$$

for all plans $\{x_t\}_{t=0}^{\infty}$ which are feasible from x_0. Taking any $x_1 \in \Gamma(x_0)$ let $\{x_t\}_{t=1}^{\infty}$ be an optimal plan from x_1 so

$$V^*(x_1) = \sum_{t=1}^{\infty} \beta^{t-1} F(x_t, x_{t+1}).$$

Then we have

$$F(x_0, x_1^*) + \beta \sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq F(x_0, x_1) + \beta V^*(x_1)$$

for all $x_1 \in \Gamma(x_0)$. Using this at $x_1 = x_1^*$ we have

$$\sum_{t=1}^{\infty} \beta^{t-1} F(x_t^*, x_{t+1}^*) \geq V^*(x_1^*),$$

but since $\{x_t^*\}_{t=1}^{\infty}$ is a feasible plan starting at x_1^* this must hold as an equality. So we have

$$V^*(x_0) = F(x_0, x_1^*) + \beta V^*(x_1^*) \geq F(x_0, x_1) + \beta V^*(x_1) \text{ for all } x_1 \in \Gamma(x_0).$$

■

Note: The proof in Stokey-Lucas-Prescott (SLP) does not assume the existence of a maximizing sequence. The argument is more subtle but similar.

This theorem implies that an optimal plan satisfies $x_{t+1}^* \in G(x_t^*)$. Is the reverse always true? Is a plan generated by the policy function necessarily optimal? No, there are counterexamples as the next example illustrates.

Example (from SLP). Take

$$F(x,y) = x - \beta y$$
$$X = \mathbb{R}_+$$
$$\Gamma(x) = [0, \beta^{-1}x].$$

Economically, this corresponds to a savings problem for a consumer with initial wealth x_0 that has linear utility over consumption $c_t = x_t - \beta x_{t+1}$ and faces a market gross interest rate $R = \beta^{-1}$. Intuitively, the consumer is indifferent to many plans for consumption. In particular, consuming immediately ($x_t = 0$ for $t = 1, 2, ...$) is optimal and $V^*(x_0) = x_0$. Consuming everything in the second period is also optimal ($x_1 = \beta^{-1}x_0$ and $x_t = 0$ for $t = 2, 3, ...$) which involves maximal savings in the very first period. Many other plans are optimal. This multiplicity of solutions is also reflected in policy correspondence which is

$$G(x) = \arg\max_{y \in \Gamma(x)} \{x - \beta y + \beta y\} = \Gamma(x).$$

However, the path $x_t = \beta^{-t}x_0$ that is generated from G by setting $x_{t+1} = \beta^{-1}x_t$ for all $t = 0, 1, ...$is clearly not optimal since then $c_t = F(x_t, x_{t+1}) = 0$ for all $t = 0, 1, ...$and $u(x) = 0 < V^*(x_0) = x_0$. The consumer is willing to postpone for any number of periods, but not forever.

We need an extra condition to rule out plans that essentially never deliver. The condition turns out to be a limiting condition that is much like a "No-Ponzi" condition for values $V^*(x_t)$ along the proposed path.

We now strengthen the previous result with such a condition and provide the converse.

Theorem 2 *Suppose that v solves the Bellman equation and that the following transversality condition holds:*

$$\lim_{t\to\infty} \beta^t v(x_t) = 0$$

for all feasible plan $\{x_t\}_{t\geq 1}$. Then, v is the value function of the problem.

Proof. For first show that $v(x) \geq V(x)$ for any x. For any arbitrary sequence $(x_1, x_2, \ldots)$, the Bellman equation implies that

$$\begin{aligned} v(x_0) &\geq F(x_0, x_1) + \beta v(x_1) \\ &\geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 v(x_2) \\ &\geq \ldots \geq \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} v(x_{T+1}) \end{aligned}$$

Taking $T \to \infty$ and recalling the transversality condition, we obtain

$$v(x_0) \geq \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),$$

and, hence, that

$$v(x_0) \geq V(x_0).$$

For the reverse, define the control sequence $\{x_t^*\}$ so that $x_0^* = x_0$ and $x_{t+1}^* \in G(x_t^*)$ for all t.[1] By construction, the inequalities in the previous computation are replaced by equalities, which implies that

$$v(x) = \sum_{t\geq 0} \beta^t F(x_t^*, x_{t+1}^*),$$

and, therefore, that $v(x) \leq V(x)$. Hence, we proved that $v(x) = V(x)$ ■

The following example illustrates the necessity of the transversality condition.

Example. Take

$$F(x, y) = x - \beta y$$

$$X = \mathbb{R}$$

and set $\Gamma(x) = \{-\beta^{-1}x\}$ if $x > 0$ and $\Gamma(x) = \{\beta^{-1}x\}$ otherwise. Thus, there is only one feasible plan for any x_0 and $V^*(x_0) = \max\{2x, 0\}$. Note that $\beta^t V^*(x_t) \to 0$ for the only feasible path. But what about the Bellman equation? There is another solution with $v(x) = x$. Note that for this solution it is not true that $\lim \beta^t v(x_t) = 0$ for all feasible paths.

[1] If the argmax is empty, the argument should be extended by taking a control sequence arbitrarily close to satisfying the equation.