

# A consumption-based model of the term structure of interest rates<sup>☆</sup>

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## Abstract

This paper proposes a consumption-based model that accounts for many features of the nominal term structure of interest rates. The driving force behind the model is a time-varying price of risk generated by external habit. Nominal bonds depend on past consumption growth through habit and on expected inflation. When calibrated to data on consumption, inflation, and the aggregate market, the model produces realistic means and volatilities of bond yields and accounts for the expectations puzzle. The model also captures the high equity premium and excess stock market volatility.

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## 1. Introduction

The expectations puzzle, documented by [Campbell and Shiller \(1991\)](#) and [Fama and Bliss \(1987\)](#), has long posed a challenge for general equilibrium models of the term structure. [Backus et al. \(1989\)](#) show that a model assuming power utility preferences and time-varying expected consumption growth cannot account for this puzzle. Although [Dai and Singleton \(2002\)](#) show that a statistical model of the stochastic discount factor can fit the puzzle, this only raises the question of what economic mechanism is at work.

This paper proposes a consumption-based model that captures key aspects of the empirical results of [Campbell and Shiller \(1991\)](#) and [Fama and Bliss \(1987\)](#). Campbell and Shiller run the regression

$$y_{n-1,t+1}^s - y_{nt}^s = \text{constant} + \beta_n \frac{1}{n-1} (y_{nt}^s - y_{1t}^s) + \text{error},$$

where  $y_{nt}^s = -(1/n) \ln P_{nt}^s$ , and  $P_{nt}^s$  is the price of a nominal bond with maturity  $n$ . According to the expectations hypothesis, excess returns on bonds are unpredictable, and all the variation in yield spreads is due to variation in future short-term interest rates. In terms of the regression above, this means  $\beta_n = 1$  for all  $n$ . But Campbell and Shiller show, on the contrary, that  $\beta_n$  is less than one and decreasing in  $n$ . The model in this paper reproduces these findings. The model also generates an upward-sloping average yield curve (as found in the data) and realistic bond yield volatility.

Two ingredients enable the model to capture these findings. The first is external habit persistence from [Campbell and Cochrane \(1999\)](#). Habit persistence generates time variation in investor preferences. After periods of unusually low consumption growth, the volatility of investors' marginal utility rises, causing them to demand greater premia on risky assets. As a result, the risk premium on the aggregate stock market varies countercyclically.

Habit utility preferences are clearly not enough: In the model of [Campbell and Cochrane \(1999\)](#), the risk-free rate is constant and the term structure is trivial. The second ingredient is thus a model for the short-term interest rate that makes long-term bonds risky in the first place. Without this ingredient, it is impossible for long-term bonds to have positive, countercyclical risk premia.

In this paper, the short-term real interest rate varies with surplus consumption, the ratio between current consumption minus a slow-moving weighted average of past consumption, and current consumption. The estimated model implies that surplus consumption and the real risk-free rate are negatively correlated; when past consumption growth is relatively low, investors borrow to give habit a chance to catch up to consumption. However, an increase in precautionary savings mitigates the effect, keeping the interest-rate volatility low. The negative correlation between surplus consumption and the risk-free rate leads to positive risk premia on real bonds, and an upward-sloping yield curve.

In order to speak to the empirical findings in the term structure, it is necessary to model nominal as well as real bonds. This paper assumes an exogenous affine process for the price level. The affine assumption allows for a tractable solution to the

nominal bond pricing problem. Nominal bonds are influenced by expected inflation as well as by surplus consumption growth. Expected inflation is calibrated purely to match inflation data. Thus, the factors driving interest rates and bond returns in this model are based in macroeconomics, rather than on asset prices.<sup>1</sup>

Besides the empirical literature on the expectations hypothesis, this paper draws on the earlier literature on habit formation (e.g., Abel, 1990; Chapman, 1998; Constantinides, 1990; Dybvig, 1995; Ferson and Constantinides, 1991; Heaton, 1995; and Sundaresan, 1989). Constantinides (1990) and Sundaresan (1989) show that habit formation models can be used to explain a high equity premium with low values of risk aversion. Like these models, the model proposed here assumes that the agent evaluates today's consumption relative to a reference point that increases with past consumption. Following Campbell and Cochrane (1999), this paper departs from earlier work by assuming that habit is external to the agent, namely that the agent does not take into account future habit when deciding on today's consumption. Abel (1990) also assumes external habit formation, but in his specification, agents care about the ratio of consumption to habit, rather than the difference. As a result, risk aversion is constant and risk premia do not vary through time.<sup>2</sup> Motivated by habit formation models, Li (2001) examines the ability of past consumption growth to predict excess returns on stocks. However, Li does not look at the predictive ability of consumption for short or long-term interest rates, nor does he consider the implications of habit formation for the expectations hypothesis.

An intriguing feature of the model in this paper is the link it produces between asset returns and underlying macroeconomic variables. When actual consumption and inflation data are fed through the model, the implied nominal risk-free rate and yield spread capture many of the short and long-run fluctuations of their data counterparts. This is in spite of the fact that the yields implied by the model are driven only by consumption growth and inflation. Finally, the model preserves the advantages of the original Campbell and Cochrane (1999) framework. It successfully captures excess volatility, the predictability of excess stock returns, and the high equity premium.

The outline of the paper is as follows. Section 2 describes the assumptions on the endowment, preferences, and the price level, and how the model is solved. Section 3 describes the estimation of the inflation process. Section 4 describes the calibration and the implications for the population moments of asset returns, and for the time series of asset returns in the post-war data.

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<sup>1</sup>Ang and Piazzesi (2003) also investigate the role of macroeconomic variables in the term structure. They consider an affine term structure model with output and inflation among the factors. Evans and Marshall (2003) consider the extent to which macroeconomic shocks can explain changes in yields, where the macroeconomic shocks are inferred using restrictions from general equilibrium models.

<sup>2</sup>Lately there has been increased interest in the empirical properties of habit formation models. Bekaert et al. (2004) consider a model where the investor's reference point is imperfectly correlated with past consumption. Brandt and Wang (2003) study habit preferences over inflation. Chen and Ludvigson (2003) evaluate habit specifications using nonparametric methods. Dai (2000) links the Constantinides model to a model for labor income. Menzly et al. (2004) calibrate an external habit model to explain returns on industry portfolios.

## 2. The model

### 2.1. Preferences

Assume that identical investors have utility over consumption relative to a reference point  $X_t$ :

$$E \sum_{t=0}^{\infty} \delta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}. \quad (1)$$

Habit,  $X_t$ , is defined indirectly through surplus consumption  $S_t$ , where

$$S_t \equiv \frac{C_t - X_t}{C_t}.$$

To ensure that  $X_t$  never falls below  $C_t$ , a process is specified for  $s_t = \ln S_t$ . This process is given by

$$s_{t+1} = (1 - \phi)\bar{s} + \phi s_t + \lambda(s_t)(\Delta c_{t+1} - E(\Delta c_{t+1})). \quad (2)$$

The process for  $s_t$  is heteroskedastic and perfectly correlated with innovations in consumption growth. The sensitivity function  $\lambda(s_t)$  will be described below.

The investor's habit is *external*: the investor does not take into account the effect that today's consumption decisions have on  $X_t$  in the future. Formally,  $X_t$  can be considered as aggregate habit and the agent as evaluating consumption relative to aggregate habit. All agents are identical so individual consumption and habit and aggregate consumption and habit can be treated interchangeably. Because habit is external, the investor's intertemporal marginal rate of substitution is given by

$$M_{t+1} = \delta \left( \frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma}. \quad (3)$$

Let  $R_{t+1}$  be the return on a traded asset. Then  $R_{t+1}$  satisfies the Euler equation

$$E_t[M_{t+1} R_{t+1}] = 1. \quad (4)$$

Following Campbell and Cochrane (1999), consumption is parametrized as a random walk:

$$\Delta c_{t+1} = g + v_{t+1}, \quad (5)$$

where  $v_{t+1}$  is a  $N(0, 1)$  shock that is independent across time. As shown in Campbell and Cochrane, this specification implies that  $x_t$  is approximately a weighted average of past consumption growth, as would be expected from an external habit formation model.

From the Euler Eq. (4), it follows that the real risk-free rate equals

$$\begin{aligned} r_{f,t+1} &= \ln(1/E_t[M_{t+1}]) \\ &= -\ln \delta + \gamma g + \gamma(1 - \phi)(\bar{s} - s_t) - \frac{\gamma^2 \sigma_v^2}{2}(1 + \lambda(s_t))^2. \end{aligned} \quad (6)$$

This risk-free rate has some familiar terms from the power utility case and others that are new to habit formation. As in the power utility model, positive expected consumption growth leads investors to borrow from the future to smooth consumption. This is reflected in the term  $\gamma g$  (however,  $\gamma$  is not equal to risk aversion as it is under power utility). The second term, proportional to  $\bar{s} - s_t$ , implies that as surplus consumption falls relative to its long-term mean, investors borrow more. This is due to the mean-reverting nature of surplus consumption: investors borrow against future periods when habit has had time to adjust and surplus consumption is higher. The last term reflects precautionary savings. A higher  $\lambda(s_t)$  implies that surplus consumption, and therefore marginal utility, is more volatile. Investors increase saving, and  $r_f$  falls.

The function  $\lambda(s_t)$  is chosen so that the intertemporal substitution and precautionary savings effects offset each other, and so that the model has intuitive properties of habit formation. Campbell and Cochrane (1999) choose the function so that these effects are completely offset and the risk-free rate is constant. In contrast, this paper allows the data to determine the net effect of  $s_t$  on the risk-free rate. For simplicity,  $\lambda(s_t)$  is restricted so that  $r_{f,t+1}$  is linear in  $s_t$ . In addition,  $\lambda(s_t)$  is chosen so that for  $s_t \approx \bar{s}$ ,  $x_t$  is a deterministic function of past consumption. These considerations imply that

$$\lambda(s_t) = (1/\bar{S})\sqrt{1 - 2(s_t - \bar{s})} - 1, \quad (7)$$

$$\bar{S} = \sigma_v \sqrt{\frac{\gamma}{1 - \phi - b/\gamma}}. \quad (8)$$

In order that the quantity within the square root remain positive,  $\lambda(s_t)$  is set to 0 when  $s_t > s_{\max}$ , for

$$s_{\max} = \bar{s} + \frac{1}{2}(1 - \bar{S}^2). \quad (9)$$

The variable  $s_t$  ventures above  $s_{\max}$  sufficiently rarely that this feature does not affect the behavior of the model. More details can be found in Appendix A. Substituting these equations into Eq. (6) reduces the risk-free rate equation to

$$r_{f,t+1} = -\ln \delta + \gamma g - \frac{\gamma(1 - \phi) - b}{2} + b(\bar{s} - s_t), \quad (10)$$

where  $b$  is a free preference parameter that will be estimated from the data.

Eqs. (6) and (10) indicate that the parameter  $b$  has an economic interpretation. If  $b > 0$ , the intertemporal smoothing effect wins out, and an increase in surplus consumption  $s_t$  drives down the interest rate. If  $b < 0$ , the precautionary savings effect wins out. An increase in surplus consumption  $s_t$  decreases the sensitivity  $\lambda(s_t)$  and drives up the interest rate. Setting  $b = 0$  results in a constant real interest rate, and gives the model of Campbell and Cochrane (1999).<sup>3</sup>

<sup>3</sup>Campbell and Cochrane briefly consider the case of  $b \neq 0$  in the working paper version of their model, Campbell and Cochrane (1995), but examine only the real term structure, and do not discuss implications for nominal bonds, long-rate regressions, or the time series of interest rates and risk premia, which are the focus of this paper.

While the functional form of  $\lambda(s_t)$  is chosen to match the behavior of the risk-free rate, it has important implication for returns on risky assets. It follows from the Euler equation (4) that

$$\frac{E_t(R_{t+1} - R_{f,t+1})}{\sigma_t(R_{t+1})} = -\rho_t(M_{t+1}, R_{t+1}) \frac{\sigma_t(M_{t+1})}{E_t(M_{t+1})},$$

where  $R_{t+1}$  is the return on some risky asset. As a consequence

$$\frac{E_t(R_{t+1} - R_{f,t+1})}{\sigma_t(R_{t+1})} \approx -\rho_t(M_{t+1}, R_{t+1}) \gamma \sigma_v (1 + \lambda(s_t)), \quad (11)$$

which follows from the lognormality of  $M_{t+1}$  conditional on time- $t$  information. Because  $\lambda(s_t)$  is decreasing in  $s_t$ , the ratio of the volatility of the stochastic discount factor to its mean varies countercyclically. This provides a mechanism by which Sharpe ratios, and hence risk premia, vary countercyclically over time.<sup>4</sup>

In the model of [Campbell and Cochrane \(1999\)](#), the mechanism in Eq. (11) does not create time-varying risk premia on bonds for the simple reason that bond returns are constant, and equal to the risk-free rate at all maturities. In terms of Eq. (11), the Campbell and Cochrane model implies that  $\rho_t(M_{t+1}, R_{t+1}) = 0$ , when  $R_{t+1}$  is the return on a bond. However, the model in this paper generates a time-varying risk-free rate. Therefore  $\rho_t(M_{t+1}, R_{t+1})$  is nonzero, and Eq. (11) provides a mechanism for risk premia on real bonds, as well as risk premia on stocks, to vary through time. Of course, this observation alone does not solve the expectations puzzle. The sign of bond premia, and the magnitude of time variation will depend on the results of the parameter estimation.

## 2.2. Inflation

To model nominal bonds, it is necessary to introduce a process for inflation. For simplicity, I follow [Boudoukh \(1993\)](#) and [Cox et al. \(1985\)](#) and model inflation as an exogenous process.<sup>5</sup> Let  $\Pi_t$  denote the exogenous price level and  $\pi_t = \ln \Pi_t$ . It is assumed that log inflation follows the process

$$\Delta \pi_{t+1} = \eta_0 + \eta Z_t + \sigma_\pi \varepsilon_{t+1}. \quad (12)$$

Here  $Z_t$  is an  $m \times 1$  vector of state variables that follow a vector-autoregressive process:

$$Z_{t+1} = \mu + \Phi Z_t + \Sigma \varepsilon_{t+1}, \quad (13)$$

where  $\Phi$  is an  $m \times m$  matrix and  $\mu$  is an  $m \times 1$  vector. The correlation between  $\Delta \pi_{t+1}$ ,  $Z_t$ , and  $\Delta c_{t+1}$  can be modeled in a parsimonious way by writing the consumption

<sup>4</sup>Harvey (1989) provides direct evidence that the risk-return tradeoff varies countercyclically.

<sup>5</sup>Buraschi and Jiltsov (2003) study a related model that puts the money supply directly in the utility function; they focus on the dynamics of inflation and the inflation risk premium, rather than the link between the term structure and consumption, which is the focus here.

growth shock  $v_{t+1}$  as

$$v_{t+1} = \sigma_c \varepsilon_{t+1}.$$

Here,  $\varepsilon_{t+1}$  is an  $(m+2) \times 1$  vector of independent  $N(0, 1)$  random variables,  $\sigma_c$  and  $\sigma_\pi$  are  $1 \times (m+2)$ , and  $\Sigma$  is  $m \times (m+2)$ .

This structure allows for an arbitrary number of state variables and cross-correlations. In addition, the state variables can be correlated with consumption growth or changes in the price level. Multiple lags can be accommodated by increasing the dimension of  $Z_t$ .

### 2.3. Model solution

This section calculates the prices of long-term bonds and stocks. To compute prices on nominal bonds, techniques from affine bond pricing<sup>6</sup> are combined with numerical methods. Introducing affine bond pricing techniques improves the efficiency of the calculation and provides insight into the model.

#### 2.3.1. Bond prices

This paper solves for prices of both real bonds (bonds whose payment is fixed in terms of units of the consumption good) and nominal bonds (bonds whose payoff is fixed in terms of units of the price level). As shown below, the assumption that expected inflation follows a multivariate auto-regressive process with Gaussian errors implies that bond yields are exponential affine in expected inflation. Following Campbell and Viceira (2001), let  $P_{n,t}$  denote the real price of a real bond maturing in  $n$  periods, and  $P_{n,t}^S$  the nominal price of a nominal bond. The real return on an  $n$ -period real bond is given by

$$R_{n,t} = \frac{P_{n-1,t+1}}{P_{n,t}}$$

with  $r_{n,t} = \ln R_{n,t}$ . The nominal return on an  $n$ -period nominal bond is

$$R_{n,t}^S = \frac{P_{n-1,t+1}^S}{P_{n,t}^S}$$

with  $r_{n,t}^S = \ln R_{n,t}^S$ . Finally, let

$$y_{n,t} = -\frac{1}{n} \ln P_{n,t}$$

and

$$y_{n,t}^S = -\frac{1}{n} \ln P_{n,t}^S$$

<sup>6</sup>See Backus et al. (2001) and Sun (1992) for illustrations of this approach in discrete-time and Duffie and Kan (1996) for an illustration in continuous time. Bakshi and Chen (1996), Bekaert et al. (2004), and Brennan et al. (2004) apply this approach to the pricing of bonds and equities.

denote the real yield on the real bond and the nominal yield on the nominal bond, respectively.

Bond prices are determined recursively by the investor's Euler equation. For real bonds, this translates into

$$P_{n,t} = E_t \left[ \delta \left( \frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma} P_{n-1,t+1} \right]. \quad (14)$$

When  $n = 0$ , the bond is worth one unit of the consumption good. This implies the boundary condition

$$P_{0,t} = 1.$$

For nominal bonds, the Euler equation implies that

$$P_{n,t}^S = E_t \left[ \delta \left( \frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma} \frac{\Pi_t}{\Pi_{t+1}} P_{n-1,t+1}^S \right] \quad (15)$$

with

$$P_{0,t}^S = 1.$$

The equations for nominal bond prices follow from the fact that the Euler Equation (4) must hold for real prices of nominal bonds. Therefore

$$\frac{P_{n,t}^S}{\Pi_t} = E_t \left[ M_{t+1} \frac{P_{n-1,t+1}^S}{\Pi_{t+1}} \right].$$

In real terms, the nominal bond maturing today is worth

$$\frac{P_{0,t}^S}{\Pi_t} = \frac{1}{\Pi_t}.$$

Note that  $r_{f,t+1} = r_{1,t+1} = y_{1,t}$ , and  $r_{f,t+1}^S = r_{1,t+1}^S = y_{1,t}^S$ .

Because the distribution of future consumption and surplus consumption depends only on the state variable  $s_t$ , Eq. (14) implies that real bond prices are functions of  $s_t$  alone:

$$P_{n,t} = F_n(s_t)$$

with  $F_0(s_t) = 1$ , and

$$F_n(s_t) = E_t \left[ \delta \left( \frac{S_{t+1}}{S_t} \frac{C_{t+1}}{C_t} \right)^{-\gamma} F_{n-1}(s_{t+1}) \right] \quad (16)$$

$$= E_t [\exp \{ \ln \delta - \gamma g - \gamma(1 - \phi)(\bar{s} - s_t) - \gamma(\lambda(s_t) + 1)\sigma_c \varepsilon_{t+1} \} F_{n-1}(s_{t+1})]. \quad (17)$$

Eq. (17) can be solved using numerical integration on a grid of values for  $s_t$ . For this problem, numerical integration is superior to calculating the expectation by Monte Carlo because the sensitivity of asset prices to rare events makes simulation unreliable.



Eq. (15) indicates that, unlike real bond prices, nominal bond prices are functions of the state variable  $Z_t$  as well as  $s_t$ . This potentially complicates the solution for nominal bond prices, because time-varying expected inflation introduces, at the least, one more state variable. Fortunately, a simple trick can be used to reduce computation time to what it would be for a single state variable. Appendix B shows that using the law of iterated expectations and conditioning on realizations of the shock  $v_{t+1} = \sigma_c \varepsilon_{t+1}$ , implies that nominal bond prices take the following form:

$$P_{n,t}^S = F_n^S(s_t) \exp\{A_n + B_n Z_t\}. \quad (18)$$

The functions  $F_n^S$  can be solved by one-dimensional numerical integration:

$$\begin{aligned} F_n^S(s_t) &= E_t[M_{t+1} \exp\{\xi_n \sigma_c \varepsilon_{t+1}\} F_{n-1}^S(s_{t+1})] \\ &= E_t[\exp\{\ln \delta - \gamma g - \gamma(1 - \phi)(\bar{s} - s_t) \\ &\quad + (\xi_n - \gamma(\lambda(s_t) + 1))\sigma_c \varepsilon_{t+1}\} F_{n-1}^S(s_{t+1})] \end{aligned}$$

while  $A_n$  and  $B_n$  are defined recursively by

$$A_n = A_{n-1} - \eta_0 + B_{n-1}\mu + \frac{1}{2}(B_{n-1}\Sigma - \sigma_\pi)[I - \sigma'_c(\sigma_c \sigma'_c)^{-1}\sigma_c](B_{n-1}\Sigma - \sigma_\pi)', \quad (19)$$

$$B_n = B_{n-1}\Phi - \eta \quad (20)$$

and

$$\xi_n = (B_{n-1}\Sigma - \sigma_\pi)\sigma'_c(\sigma_c \sigma'_c)^{-1}. \quad (21)$$

The boundary conditions are  $F_0^S(s_t) = 1$ ,  $A_0 = 0$ , and  $B_0 = 0_{1 \times m}$ . The last term in Eq. (19) follows from Jensen's inequality: because inflation is log-normally distributed, the volatility of inflation works to decrease bond yields at long maturities. These formulas can also be used to gain insight into the model, as explained in Section 2.4.

### 2.3.2. Aggregate wealth

In this economy, the market portfolio is equivalent to aggregate wealth, and the dividend equals aggregate consumption. The price–consumption ratio and the return on the market can be calculated using methods similar to those above, with a small but important modification. Analogously to the previous section, let  $P_{n,t}^e$  denote the price of an asset that pays the endowment  $C_{t+n}$  in  $n$  periods. The  $e$  superscript denotes equity. Because these assets pay no coupons, they have the same recursive pricing relation as bonds Eq. (17). Of course the prices are different, and this is because there is a different boundary condition:

$$P_{0,t}^e = C_t.$$

Unlike the case for bonds,  $P_{n,t}^e$  is not simply a function of  $s_t$ . It is a function of consumption  $C_t$  as well. To avoid introducing an additional variable into the problem, the equations for equity are rewritten in terms of price–consumption ratios, rather than simply prices:

$$\frac{P_{n,t}^e}{C_t} = E_t \left[ \delta \left( \frac{S_{t+1}}{S_t} \right)^{-\gamma} \left( \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \frac{P_{n-1,t+1}^e}{C_{t+1}} \right]. \quad (22)$$

The boundary condition can be rewritten as  $P_{0,t}^e/C_t = 1$ . Now the problem is analogous to that for bond pricing. The ratio of the price of zero-coupon equity to aggregate consumption can be written as a function  $F_n^e$  of  $s_t$ , where

$$F_n^e(s_t) = E_t[\exp\{\ln \delta + (1 - \gamma)g - \gamma(1 - \phi)(\bar{s} - s_t) + (1 - \gamma(\lambda(s_t) + 1))\sigma_c \varepsilon_{t+1}\} F_{n-1}^e(s_{t+1})]$$

with boundary condition  $F_n^e(s_t) = 1$ . This formula can be solved recursively using one-dimensional quadrature.

Finally, the price–consumption ratio of the market equals the sum of the price–consumption ratio on these zero-coupon securities:

$$\frac{P_t}{C_t} = \sum_{n=1}^{\infty} \frac{P_{n,t}^e}{C_t}. \quad (23)$$

This way of calculating the price–consumption ratio is equivalent to the more traditional fixed-point method used by [Campbell and Cochrane \(1999\)](#). In this endowment economy, Eq. (23) also represents the price–dividend ratio.

#### 2.4. Implications for bond risk premia

The nominal return on the one-period nominal bond (the nominal risk-free rate) can be determined using these equations, or directly from Eq. (15):

$$\begin{aligned} r_{f,t+1}^{\$} &= -\ln \delta + \gamma g - \frac{\gamma(1 - \phi) - b}{2} + b(\bar{s} - s_t) + (\eta_0 + \eta Z_t) \\ &\quad - \sigma_{\pi} \sigma'_c \gamma(\lambda(s_t) + 1) - \frac{1}{2} \sigma_{\pi} \sigma'_{\pi} \\ &= r_{f,t+1} + E_t[\Delta \pi_{t+1}] - \frac{1}{2} \sigma_{\pi} \sigma'_{\pi} - \sigma_{\pi} \sigma'_c \gamma(\lambda(s_t) + 1). \end{aligned} \quad (24)$$

Of interest is the risk premium on the nominal risk-free asset. Subtracting the real risk-free rate from the expected real return on the one-period nominal bond produces

$$E_t[r_{f,t+1}^{\$} - \Delta \pi_{t+1}] - r_{f,t+1} = -\sigma_{\pi} \sigma'_c \gamma(\lambda(s_t) + 1) - \frac{1}{2} \sigma_{\pi} \sigma'_{\pi}.$$

The term  $\frac{1}{2} \sigma_{\pi} \sigma'_{\pi}$  is an adjustment for Jensen's inequality. If  $\sigma_{\pi} \sigma_c < 0$ , the one-period nominal bond has a positive risk premium relative to the one-period real bond. Intuitively, this is because  $\sigma_{\pi} \sigma'_c < 0$  implies that inflation and consumption growth are negatively correlated. Because higher inflation lowers the return on the nominal risk-free bond, a negative correlation between inflation and consumption implies that the nominal bond pays off when investors need the money least. Therefore, the one-period nominal bond carries a risk premium relative to the one-period real bond.

The formulas derived in Section 2.3 can be used to show that nominal risk premia depend only on  $S_t$ . It follows from Eq. (18) that

$$\begin{aligned} E_t[r_{n,t+1}^S] &= E_t[\ln F_{n-1}^S(s_{t+1}) - \ln F_n^S(s_t) + A_{n-1} - A_n + B_{n-1}Z_{t+1} - B_nZ_t] \\ &= \text{constant} + E_t[\ln F_{n-1}^S(s_{t+1})] - \ln F_n^S(s_t) + (B_{n-1}\Phi - B_n)Z_t \\ &= \text{constant} + E_t[\ln F_{n-1}^S(s_{t+1})] - \ln F_n^S(s_t) + \eta Z_t. \end{aligned}$$

Moreover,

$$r_{1,t+1}^S = \text{constant} + b(\bar{s} - s_t) + \eta Z_t - \sigma_\pi \sigma'_c \gamma (\lambda(s_t) + 1)$$

(recall that  $r_{1,t+1}^S = r_{f,t+1}^S$ ). Therefore nominal risk premia depend only on  $s_t$ :

$$\begin{aligned} E_t[r_{n,t+1}^S - r_{1,t+1}^S] &= \text{constant} + E_t[\ln F_{n-1}^S(s_{t+1})] - \ln F_n^S(s_t) \\ &\quad - b(\bar{s} - s_t) + \sigma_\pi \sigma'_c \gamma (\lambda(s_t) + 1). \end{aligned} \quad (25)$$

In general, there is no closed form expression for nominal or real bond prices with maturity greater than one period. These can be determined in some special cases, as described below.

#### 2.4.1. Special cases

Suppose first that  $b = 0$ . Then the real risk-free rate is constant:

$$r_{f,t+1} = r_f.$$

Moreover, it follows from Eq. (14) that

$$P_{n,t} = \exp\{-nr_f\}. \quad (26)$$

Eq. (26) can be shown using induction. If  $P_{n-1,t} = \exp\{-(n-1)r_f\}$ , then

$$P_{n,t} = E_t[M_{t+1} \exp\{-(n-1)r_f\}] = E_t[M_{t+1}] \exp\{-(n-1)r_f\} = \exp\{-nr_f\}.$$

Moreover, risk premia are zero in this case.

Nominal bonds are a different story. As long as expected inflation varies, the nominal risk-free rate also varies. Even if  $b = 0$ , correlation between expected and unexpected inflation and consumption growth creates risk premia on nominal bonds. These risk premia vary with  $s_t$ , and it is again not possible to solve for bond prices in closed form. Suppose however that  $\Sigma \sigma'_c = 0$  and  $\sigma_\pi \sigma'_c = 0$ . Then inflation risk is not priced, and the same reasoning as above shows that

$$P_{n,t}^S = \exp\{-nr_f\} \exp\{A_n + B_n Z_t\}.$$

Substituting in from Eqs. (13), (19), and (20), it follows that

$$\begin{aligned} E_t[r_{n,t+1}^S] &= r_f + \eta_0 + \eta Z_t - \frac{1}{2}(B_{n-1} - \sigma_\pi)(B_{n-1}\Sigma - \sigma_\pi)' \\ &= r_{f,t+1}^S + \frac{1}{2}\sigma_\pi \sigma'_\pi - \frac{1}{2}(B_{n-1} - \sigma_\pi)(B_{n-1}\Sigma - \sigma_\pi)'. \end{aligned}$$

Thus, risk premia on nominal bonds are zero except for a constant Jensen's inequality term.

### 3. Estimation

The results of the previous section suggest that the process assumed for expected inflation will be an important determinant of yields and returns on nominal bonds. This section focuses on estimating this process.

A special case of the model presented in Section 2.2 is considered. I assume that expected inflation follows an AR(1) process, namely that  $Z_t$  is univariate. This is equivalent to assuming that realized inflation follows an ARMA(1,1) process. The advantage of this approach is that estimation via maximum likelihood is straightforward, and, as shown below, the resulting expected inflation series appears to capture much of the variation in realized inflation.

Model calibration requires not only the parameters of the inflation process, but also mean consumption growth, the variance of consumption growth, and the correlation between consumption and inflation. For simplicity, aggregate consumption growth is assumed to be independent and identically distributed across time. However, the literature has identified a number of reasons why *measured* consumption growth could exhibit temporal dependence (e.g., Christiano et al., 1991; Ferson and Harvey, 1992; Heaton, 1993). To account for this dependence in the estimation, inflation and consumption growth are each assumed to follow an ARMA(1,1) with correlated errors. That is, I estimate

$$\Delta c_{t+1} = (1 - \psi_1)g + \psi_1 \Delta c_t + \theta_1 v_{1,t} + v_{1,t+1}, \quad (27)$$

$$\Delta \pi_{t+1} = (1 - \psi_2)\bar{\pi} + \psi_2 \Delta \pi_t + \theta_2 v_{2,t} + v_{2,t+1}, \quad (28)$$

where

$$\begin{bmatrix} v_{1,t+1} \\ v_{2,t+1} \end{bmatrix} \sim N \left( 0, \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{bmatrix} \right). \quad (29)$$

Here,  $\psi_1$  is the autoregressive coefficient for consumption growth, while  $\theta_1$  is the moving-average coefficient. Similarly,  $\psi_2$  is the autoregressive coefficient of inflation, while  $\theta_2$  is inflation's moving-average coefficient. The parameter  $\rho$  represents the correlation between innovations to consumption growth and innovations to inflation. Eqs. (27)–(29) imply an exact likelihood function, derived in Appendix C. Section 4.1 describes the mapping from the parameters of this section to those of Section 2.2.

Eqs. (27)–(29) are estimated via maximum likelihood using quarterly data on inflation and consumption from 1952 to the second quarter of 2004. Data on real per capita consumption of nondurables and services come from the Bureau of Economic Analysis. Quarterly data on the consumer price index (CPI) are from the Center for Research in Security Prices (CRSP). Because CPI data end in the second quarter of 2004, and because CRSP zero-coupon Treasury data begin in the second quarter of 1952, this is the data range for the estimation. Table 1 shows the results of the estimation. The left column reports the parameter estimate and the right column reports the standard error. All parameters are in quarterly units, and means and standard deviations are in percentages. Mean quarterly consumption growth ( $g$ ) over

Table 1

Estimates of the model:

$$\Delta c_{t+1} = (1 - \psi_1)g + \psi_1 \Delta c_t + \theta_1 v_{1,t} + v_{1,t+1},$$

$$\Delta \pi_{t+1} = (1 - \psi_2)\bar{\pi} + \psi_2 \Delta \pi_t + \theta_2 v_{2,t} + v_{2,t+1}$$

using maximum likelihood and quarterly data on log consumption growth ( $\Delta c$ ) and log inflation ( $\Delta \pi$ ). Estimates are in natural units, except where otherwise indicated. AR denotes autoregressive coefficients, MA denotes moving-average coefficients, and  $\rho$  denotes the correlation between the shocks  $v_1$  and  $v_2$ . Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004.

Parameter	Estimate	Standard error
Mean cons. growth $g$ (%)	0.549	0.060
Mean infl. $\bar{\pi}$ (%)	0.920	0.230
AR term for cons. $\psi_1$	0.677	0.110
AR term for infl. $\psi_2$	0.941	0.022
MA term for cons. $\theta_1$	−0.352	0.134
MA term for infl. $\theta_2$	−0.644	0.055
Stand. dev. for cons. $\sigma_1$ (%)	0.431	0.022
Stand. dev. for infl. $\sigma_2$ (%)	0.588	0.030
Correlation $\rho$	−0.205	0.070

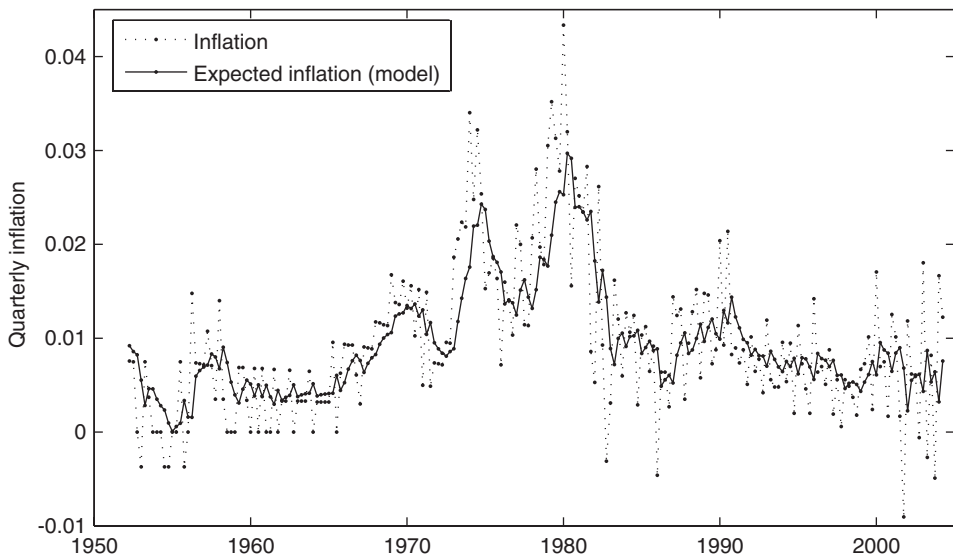


Fig. 1. Expected and realized inflation. The dotted line shows quarterly changes in log CPI. The solid line shows expected inflation, conditional on past realized inflation, implied by the estimation of Section 3 and the maximum likelihood estimates given in Table 1. Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004.

this period is 0.55%, while mean inflation ( $\bar{\pi}$ ) is 0.92%. The estimates indicate that expected inflation is highly persistent, with an autoregressive coefficient of 0.94. The correlation between innovations to consumption and innovations to inflation is −0.21.

Fig. 1 plots the time series of quarterly realized inflation together with the time series of expected inflation implied by Eqs. (27)–(29) and the estimates in Table 1. As described in Appendix C, this series is constructed recursively using past inflation data. Fig. 1 shows that the expected inflation series captures many of the lower-frequency fluctuations in realized inflation. Indeed, the expected inflation series implied by this process explains 47% of the variance of realized inflation.

The next section combines the estimation results of this section with the formulas of Section 2 to determine the implications of the model for the nominal term structure.

#### 4. Implications for asset returns

This section describes the implications of the model for returns on bonds and stocks. Section 4.1 describes the calibration of the parameters, and the data used to calculate moments of nominal bonds for comparison. Section 4.2 characterizes the price–dividend ratio and the yield spread on real and nominal bonds as functions of the underlying state variables  $s_t$  and expected inflation. Section 4.3 evaluates the model by simulating 100,000 quarters of returns on stocks and nominal and real bonds and compares the simulated moments implied by the model to those on stocks and nominal bonds in the data. Lastly, Section 4.4 shows the implications of the model for the time series of the short-term interest rate and the yield spread.

##### 4.1. Calibration

The processes for consumption and inflation are calibrated using the estimation of Section 3, while the preference parameters are calibrated using bond and stock returns. This calibration strategy is similar to that used in Boudoukh (1993), who investigates a term structure model where investors have power utility and consumption and inflation follow a vector-autoregression with heteroskedastic errors. The parameterization in Section 2.2 is under-identified, so certain parameters must be fixed. First, note that

$$\begin{bmatrix} \sigma_c \\ \sigma_\pi \end{bmatrix} [\sigma'_c \ \sigma'_\pi] = \begin{bmatrix} \sigma_1^2 & \sigma_1 \sigma_2 \rho \\ \sigma_1 \sigma_2 \rho & \sigma_2^2 \end{bmatrix}. \quad (30)$$

In order to identify  $\sigma_c$  and  $\sigma_\pi$ , assume that the matrix  $\begin{bmatrix} \sigma_c \\ \sigma_\pi \end{bmatrix}$  is lower triangular. Then  $\sigma_c$  and  $\sigma_\pi$  can be found by taking the Cholesky decomposition of the right-hand side of Eq. (30).

Assume  $\mu = 0$  and  $\eta = 1$ . Then the remaining parameters can be identified as follows:

$$\eta_0 = \bar{\pi}, \quad (31)$$

$$\Phi = \psi_2, \quad (32)$$

$$\Sigma = (\psi_2 + \theta_2)\sigma_\Pi. \quad (33)$$

The resulting process for inflation is identical to Eq. (28). This follows from solving for  $Z_t$  in Eq. (12) and substituting the resulting expression into Eq. (13). From Eqs. (31)–(33) and setting  $\mu = 0$  and  $\eta = 1$ , it follows that

$$\Delta\pi_{t+1} - \bar{\pi} - \sigma_\pi \varepsilon_{t+1} = \psi(\Delta\pi_t - \bar{\pi} - \sigma_\pi \varepsilon_t) + (\psi_2 + \theta_2)\sigma_\pi \varepsilon_t.$$

Solving for  $\Delta\pi_{t+1}$  and applying Eq. (30) produces (28).

Note that under this specification, expected inflation and realized inflation are assumed to be perfectly positively correlated. This assumption allows expected inflation to be identified from inflation data alone. As explained in Section 3, the ARMA parameters for consumption growth are set equal to zero. This is in part for simplicity and in part because these parameters capture predictability due to data construction, rather than predictability in underlying consumption growth itself.

Once consumption and inflation are determined, there remain four parameters of the investor's utility function that need to be identified. These are the discount rate  $\delta$ , the utility curvature  $\gamma$ , the persistence of habit  $\phi$ , and the loading of the interest rate on the negative of surplus consumption,  $b$ . The last parameter can be given an interpretation in terms of the utility function, as it determines the tradeoff between the precautionary savings and intertemporal smoothing effects of  $s_t$  on the risk-free rate.

From Eqs. (10) and (24), it follows that the parameter  $\delta$  has a one-to-one correspondence with the level of the risk-free rate. For this reason,  $\delta$  is set so that, in population, the mean of the nominal risk-free rate matches (approximately) that in the data. Given the other parameters, and an estimate of the mean of the nominal risk-free rate in the data  $\bar{r}^S$ , this is accomplished by setting

$$\delta = \exp \left\{ -\bar{r}^S + \gamma g - \frac{\gamma(1-\phi)-b}{2} + \bar{\pi} - \sigma_\pi \sigma'_c \gamma(\lambda(\bar{s}) + 1) - \frac{1}{2} \sigma_\pi \sigma'_\pi \right\}.$$

This implies that when the nominal risk-free rate in the model is evaluated at  $\bar{s}$ , it equals the yield on the three-month bond. Because  $\lambda(s_t)$  is a nonlinear function of  $s_t$ , the mean in population will not exactly equal that from the data. However, the simulation results in Section 4.3 show that the difference is small.

Because the purpose of this paper is to determine the implications for bond returns of a model that is intended to capture features of equity returns, these parameters are determined, as far as possible, by equity return data. This is possible for  $\gamma$  and  $\phi$ , but  $b$  has very little impact on equity returns. Therefore,  $b$  is set so that the model delivers reasonable implications for the quantities of interest: means and standard deviations of yields, and the magnitude of the failure of the expectations hypothesis. In order to generate an upward-sloping yield curve, it is necessary that  $b > 0$ , i.e., that the risk-free rate loads negatively on  $b$  (and that the intertemporal substitution effect dominates the precautionary savings effect). If  $b > 0$ , the real risk-free rate is negatively correlated with surplus consumption. This implies that bond returns will be positively correlated with surplus consumption, and thus that bond returns, both real and nominal, will have positive risk premia. Note also that the correlation between inflation and consumption is estimated to be negative. This implies that the

Table 2

Assumptions on the parameters of the investor's utility function

The first panel gives the independent parameters. The second panel gives the derived parameters. The long-run mean of log surplus consumption,  $\bar{s} = \ln \bar{S}$  is set equal to  $\ln(\sigma_v \sqrt{\gamma/(1-\phi-b/\gamma)})$ , where  $\sigma_v$  is the standard deviation of shocks to consumption. The discount rate  $\delta$  is determined so that, at  $s = \bar{s}$ , the nominal risk-free rate equals the risk-free rate in the data. The maximum surplus consumption ratio,  $s_{\max} = \bar{s} + \frac{1}{2}(1 - \bar{S}^2)$ . The model is simulated at a quarterly frequency.

	Value
<i>Parameters</i>	
Utility curvature $\gamma$	2.00
Coefficient on $-s_t$ in the riskfree rate $b$	0.011
Habit persistence $\phi$	0.97
<i>Derived parameters</i>	
Discount rate $\delta$	0.98
Long-run mean of log surplus consumption $\bar{s}$	-3.25
Maximum value of log surplus consumption $s_{\max}$	-2.75

risk premium due to inflation is positive, and further increases the premium on nominal bonds. For the numbers estimated here, however, this effect is small.

Simulation results show that the parameter  $\phi$  determines the first-order autocorrelation of the price–dividend ratio. This is reasonable given that  $P/D$  is a function of  $s_t$  alone. Therefore  $\phi$  is set to equal 0.97, the first-order autocorrelation of the price–dividend ratio in the data. Finally,  $\gamma$  is set so that the unconditional Sharpe ratio of equity returns is equal to the Sharpe ratio in the data. The parameter value choices are summarized in Table 2.

Relative to Campbell and Cochrane (1999), the free parameter in this model is  $b$ , the loading of the interest rate on the negative of surplus consumption. This parameter is fit to the cross-section of bond yields. However,  $b$  has time-series implications as well as cross-sectional ones. A value of  $b > 0$  implies that surplus consumption influences the real risk-free rate with a negative sign. As a brief investigation of these time-series implications, the ex post real interest rate is regressed on a surplus consumption proxy,  $\sum_{j=1}^{40} \phi^j \Delta c_{t-j}$ , which is approximately equal to  $s_t$ . While  $s_t$  is, in theory, influenced by surplus consumption going back to infinity it is necessary in practice to make a choice as to where to cut off past consumption. To capture the nature of  $s_t$  as a long-run variable, ten years is chosen as the cut-off point. The regression is therefore

$$r_{f,t+1}^s - \Delta\pi_{t+1} = a_0 + a_1 \sum_{j=1}^{40} \phi^j \Delta c_{t-j} + \varepsilon_{t+1}.$$

The results of this regression lend support for the choice of  $b > 0$ . The parameter  $a_1$  is found to be negative and statistically significant, with a point estimate of  $-0.110$ , and a standard error, adjusted for serial correlation and heteroskedasticity, of



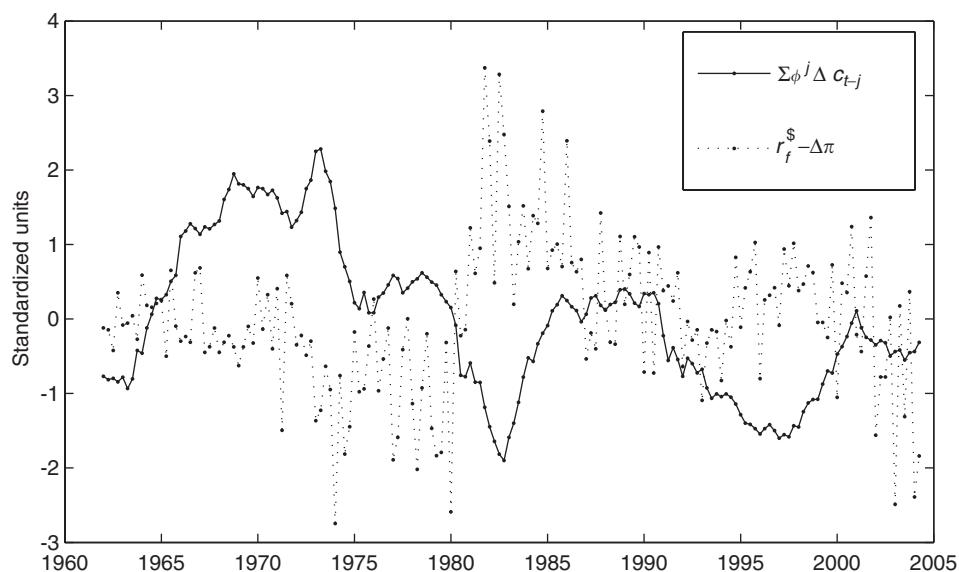


Fig. 2. Past consumption growth and the short-term interest rate. This figure plots the history of average past (inflation-adjusted) consumption growth ( $\sum_{j=0}^{40} \phi^j \Delta c_{t-j}$ ) and the continuously compounded rate of return on the 90-day Treasury bill adjusted for inflation ( $r_f^S - \Delta\pi$ ). The parameter  $\phi = 0.97$ . Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004. Variables are de-meaned and standardized.

0.033.<sup>7</sup> Fig. 2 shows the history of average past consumption ( $\sum_{j=1}^{40} \phi^j \Delta c_{t-j}$ ) and  $r_{f,t+1}^S - \Delta\pi_{t+1}$ . The negative relation between past consumption and the ex post real risk-free rate is apparent throughout the sample period.

Calibrating the parameters as described above, and comparing returns in the model to those in the data, requires data on nominal bond yields and on equity returns. The bond data, available from CRSP, consist of monthly observations on the three-month U.S. government bond yield, and interpolated zero-coupon bond yields for maturities of one, two, three, four, and five years. These data are available from the second quarter of 1952 until the third quarter of 2004. Following Campbell and Viceira (2001), I use only quarterly observations to eliminate the high-frequency fluctuations that would seem difficult to explain using a model with macro-based variables. Monthly observations on returns on a value-weighted index of stocks traded on the NYSE and Amex are taken from CRSP. These are used to compute quarterly returns and quarterly observations on the ratio of price to annual dividends.

<sup>7</sup> A potential concern with this regression is the relatively high degree of persistence in the surplus consumption ratio. A Monte Carlo exercise designed to correct for this persistence yields a 5% critical value (based on a two-tailed test) of  $-0.077$ , implying that the value of  $-0.110$  remains significant.

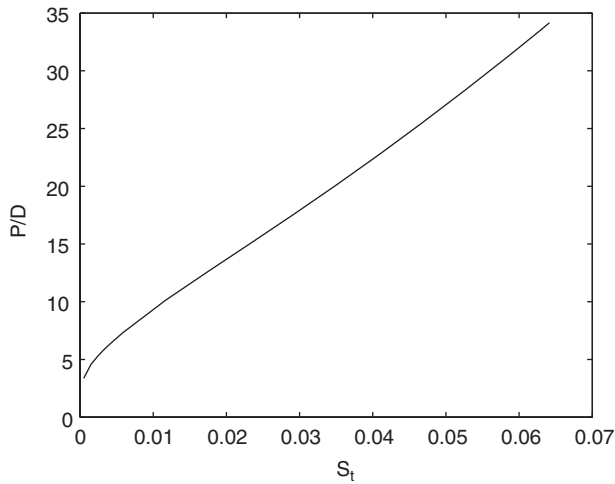


Fig. 3. The price–dividend ratio as a function of surplus consumption  $S_t$  implied by the parameters in Tables 1 and 2. The price–dividend ratio is the ratio of aggregate wealth to aggregate consumption (divided by four to annualize).

#### 4.2. Characterizing the solution

As shown in Fig. 3, the price–dividend ratio increases with surplus consumption  $S_t$ . As the price–dividend ratio is often taken to be a measure of the business cycle (e.g., Lettau and Ludvigson, 2001), this confirms the intuition that  $S_t$  is a procyclical variable.

Fig. 4 plots the yields on nominal and real bonds for maturities of three months and five years. Expected inflation is set equal to its long-run mean. Both nominal and real yields decrease with  $S_t$ , but the long yields are more sensitive to  $S_t$  than the short yields. Thus, the spread between the long and short yields is decreasing in  $S_t$  for both nominal and real yields. Fig. 4 also shows that the long-term yields generally lie above the short-term yields, and that nominal yields lie above real yields. The first of these effects follow from the fact that  $b > 0$ , i.e., that the interest rate loads negatively on  $S_t$ , while the second effect follows primarily from the fact that expected inflation growth is positive. For values of  $S_t$  that are very high, the five-year yield lies slightly below the three-month yield. This arises because the risk premium is very low for these values of  $S_t$ , and is dominated by the Jensen's inequality term in Eq. (19).

Fig. 5 plots the yields on nominal bonds as functions of surplus consumption  $S_t$  and expected inflation. Expected inflation is set equal to its long-run mean of 0.92%, and varied by plus and minus two unconditional standard deviations. Both long and short-term yields are increasing in expected inflation. However, the effect of expected inflation on short-term yields is greater than on long-term yields. This plot shows that two factors drive yields in the model. Expected inflation is more important at the short end of the yield curve, while surplus consumption dominates at the long end of the yield curve.

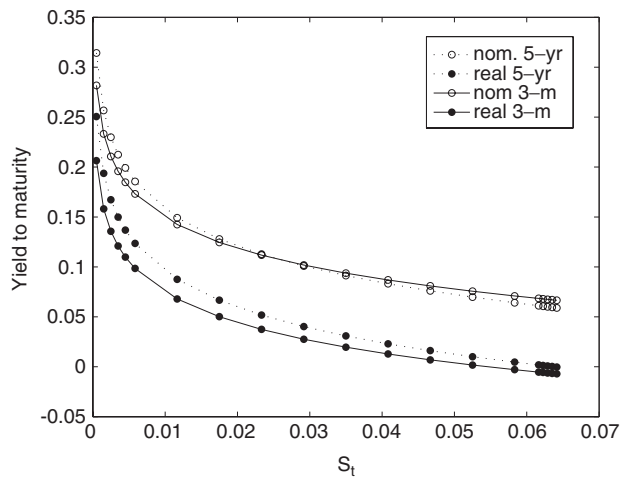


Fig. 4. Continuously compounded yields on real and nominal bonds as a function of surplus consumption implied by the parameters in Tables 1 and 2. Dotted lines denote yields on five-year nominal and real bonds; solid lines denote yields on three-month nominal and real bonds. Open circles denote nominal yields on nominal bonds; closed circles denote real yields on real bonds. For the nominal yields, expected inflation is set equal to its unconditional mean of 1% per quarter. Yields are in annual terms.

### 4.3. Simulation

To evaluate the predictions of the model for asset returns, 100,000 quarters of data are simulated. Prices of the claim on aggregate consumption (equity) and of real and nominal bonds are calculated numerically, using the method described in Section 2.3.

#### 4.3.1. Returns on the aggregate market

Table 3 shows the implications of this model for equity returns. Despite the difference in the parameter  $b$ , the implications of the present model for equity returns are nearly identical to those of Campbell and Cochrane (1999). The model fits the mean and standard deviation of equity returns, even though it was calibrated only to match the ratio. Thus, the model can fit the equity premium puzzle of Mehra and Prescott (1985). The persistence  $\phi$  is chosen so that the model fits the correlation of the price–dividend ratio by construction. However, the model can also reproduce the high volatility of the price–dividend ratio, demonstrating that the model fits the volatility puzzle described by Shiller (1981). Stock returns and price–dividend ratios are highly volatile even though the dividend process is calibrated to the extremely smooth post-war consumption data. In addition, results available from the author show that price–dividend ratios have the ability to predict excess returns on equities, just as in the data (Campbell and Shiller, 1988; Fama and French, 1989), and that declines in the price–dividend ratio predict higher volatility (Black, 1976; Schwert, 1989; Nelson, 1991). Given that the consequences for equity returns are so similar to those of Campbell and Cochrane (1999), the sections that follow focus on the

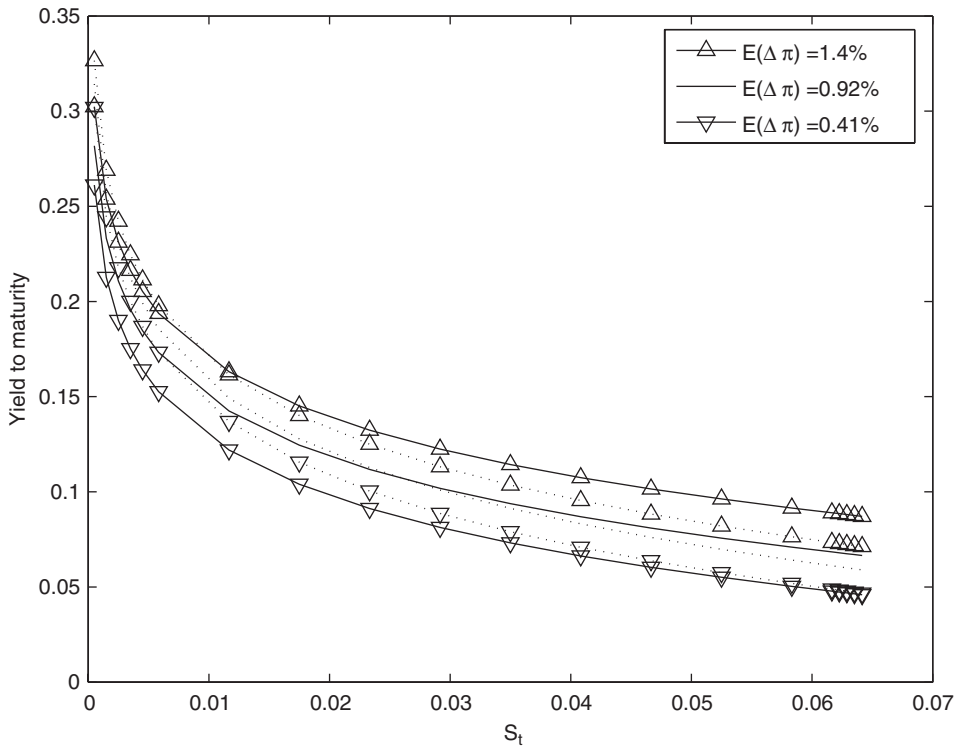


Fig. 5. Nominal continuously compounded bond yields as a function of surplus consumption and expected inflation implied by the parameters in Tables 1 and 2. Solid lines denote the three-month yield, dotted lines the five-year yield. Yields are plotted for expected inflation at its unconditional mean ( $E(\Delta\pi)$ ), at two unconditional standard deviations below the unconditional mean (upside-down triangles), and at two unconditional standard deviations above the unconditional mean (upright triangles). Yields are in annual terms. The unconditional standard deviation is calculated as  $\Sigma\Sigma'/(1-\phi^2)$ .

properties of bond returns. These sections demonstrate the model's ability to explain features of the bond data.

#### 4.3.2. Bond yields

Table 4 shows the implications of the model for means and standard deviations of real and nominal bond yields. Data moments for bond yields are provided for comparison. As shown in the first row, the model-implied nominal one-period yield and its standard deviation are well matched to the moments in the data. The low mean and volatility of the short-term interest rate follows from the fact that the  $\gamma$  required to fit the Sharpe ratio is very low, unlike in the traditional power utility model.

Table 4 also demonstrates that the average yield curve on real and nominal bonds is upward sloping. The average yield on the five-year nominal bond in the model is equal to 6.3%, similar to the data mean of 6.2%. The average yield of the

Table 3

Statistics for the aggregate market from actual and simulated quarterly data

In the data,  $r_m - r_f$  is the return on the NYSE-Amex index minus the return on the three-month Treasury bill. In the model,  $r_m - r_f$  is the excess return on the consumption claim minus the one-period real risk-free rate. Returns are continuously compounded, and the mean and standard deviation of returns are in annualized percentages (multiplied by four and two, respectively). The Sharpe ratio is the mean excess return divided by the standard deviation. The price–dividend ratio  $P/D$  is annualized (formed by summing dividends over the previous year in the data and dividing by four in the model).  $\text{Corr}(p - d)$  denotes the first-order autocorrelation of the log price–dividend ratio. Data are quarterly, begin in the second quarter of 1952, and end in the third quarter of 2004. \*denotes a moment matched by construction.

Statistic	Model	Data
$E(r_m - r_f)$ (%)	5.37	5.21
$\sigma(r_m - r_f)$ (%)	16.16	15.93
Sharpe*	0.33	0.33
$E(P/D)$	23.63	33.40
$\sigma(p - d)$	0.31	0.33
$\text{Corr}(p - d)^*$	0.97	0.97

Table 4

Means and standard deviations of continuously compounded zero-coupon bond yields in the model and in the data

Columns marked “Real” give statistics for real yields on real bonds (bonds that pay off in units of aggregate consumption) in the model; columns marked “Nominal” give statistics for nominal yields on nominal bonds in the model; columns marked “Data” give statistics for nominal yields on nominal bonds in the data. Yields are in annual percentages. Maturity is in quarters. Data are quarterly, begin in the second quarter of 1952, and end in the third quarter of 2004.

Maturity	Mean			Stand. dev.		
	Real	Nominal	Data	Real	Nominal	Data
1	1.46	5.17	5.22	1.91	2.35	2.93
4	1.62	5.34	5.60	1.96	2.35	2.93
8	1.83	5.58	5.81	2.03	2.37	2.89
12	2.05	5.83	5.98	2.10	2.40	2.82
16	2.28	6.07	6.11	2.17	2.44	2.79
20	2.51	6.32	6.19	2.24	2.48	2.74

three-month bond is 5.2%, about its mean in the data.<sup>8</sup> Note that the calibration procedure implies that these will be close but not exact. As explained above,  $b$  is set in part so that the model generates an upward-sloping yield curve. In the language of Section 2, a positive  $b$  implies that the intertemporal smoothing effect dominates the precautionary savings effect. An implication of this model is that bond term premia

<sup>8</sup>Longstaff (2000) notes, however, that the upward slope of the yield curve in the data may be overstated because of a liquidity premium in Treasury bill rates.

are increasing in maturity, a finding which Boudoukh et al. (1999) show has support in the data.<sup>9</sup>

The link between  $b$  and the slope of the yield curve can be understood in terms of the covariance form of the investor's Euler equation. For example, for real bonds,

$$E(R_{n,t} - R_{1,t}) = -\text{Cov}(R_{n,t} - R_{1,t}, M_t) \frac{\sigma(M_t)}{E(M_t)}, \quad (34)$$

where  $R_{n,t}$  is the return on a real bond maturing in  $n$  periods and  $M_t$  is the intertemporal marginal rate of substitution. A positive  $b$  implies that the short-term interest rate covaries negatively with  $s_t$ . Because bond returns move in the opposite direction from the short-term interest rate, a positive  $b$  implies that bonds have a positive covariance with  $s_t$ . This means that bonds have high returns in good times and poor returns in bad. Investors demand a risk premium to hold them. Because long-term bonds have higher expected returns than if there were no risk premia, they must have higher yields.

Table 4 also shows that the model generates higher yields for nominal bonds than for real bonds at all maturities. This is mostly due to the impact of expected inflation (3.7% per annum) on nominal yields. However, nominal yields also incorporate a positive risk premium due to inflation. For example, from Eq. (24), it follows that for the three-month yield the premium from inflation is equal to

$$-\sigma_\pi \sigma'_c \gamma E[\lambda(s_t) + 1] = 0.012\%,$$

or 0.048% in annual terms. Note that the premium due to inflation is positive because, as Table 1 implies, innovations to inflation and innovations to consumption growth are negatively correlated. Because bond prices are negatively correlated with inflation, nominal bonds pay off when consumption growth (and hence surplus consumption growth) is high. This contributes to the risk premium, and hence the yield, on nominal bonds. The model produces average nominal yields that are very similar to those in the data for bonds with maturities between three months and five years.

Finally, Table 4 shows that the model produces reasonable values for the standard deviation of bond yields. For example, the model implies that the standard deviation for the three-month nominal yield is 2.35%. In the data, it is 2.93%. For the five-year yield, the standard deviation implied by the model is 2.48%, while in the data it is 2.74%. It is important to note that the model does not match the standard deviations by construction. The parameter values were chosen to fit data on inflation, consumption, the three-month yield, and the aggregate market. The remaining parameter  $b$  determines both the mean and the standard deviation of bond yields; Table 4 shows that the model can simultaneously deliver reasonable fits to both of these aspects of the data.

<sup>9</sup>It is possible to calibrate the model so that the model exactly fits the average five-year yield. This requires lowering the parameter  $b$  from 0.011 to 0.0095. This also lowers the volatility of yields (the resulting volatility of the five-year yield is 2.22%). The coefficients from a regression of changes in yield on the scaled yield spread are slightly less negative than when  $b = 0.011$ , ranging from  $-0.22$  for the one-year bond to  $-1.11$  on the five-year bond.

The previous discussion shows that interest rate risk leads both real and nominal bonds to have positive risk premia. Because of these positive risk premia, there is a feedback effect that further raises the risk, and therefore the premium, on bonds. As shown below, risk premia on bonds vary. Variation in the risk premium itself induces price fluctuations, much like “excess volatility” in the stock market. This excess volatility makes expected returns on bonds larger than they otherwise would be.

This feedback effect helps in understanding why bonds command risk premia at all. After all, these bonds pay off a fixed amount. Why is it that investors simply do not wait until maturity to sell the bond, when the return is fixed? The power utility model of [Backus et al. \(1989\)](#) implies that bonds have negative excess returns that are very small in magnitude.<sup>10</sup> In the present model, by contrast, bonds are risky because their prices fall during periods of low surplus consumption, namely during recessions. These are the times when investor’s marginal utility is the highest, and when, as a result, they most want to increase their consumption. Long-term bonds thus command a premium not only because of their dependence on the time-varying risk-free rate, but because they do badly in recessions.

#### 4.3.3. Time-varying bond risk premia

The previous section pointed to time variation in risk premia as a source of variation in long-term bond prices. This section shows that risk premia are indeed time-varying, and explains why.<sup>11</sup>

Fig. 6 shows the outcome of regressions

$$y_{n-1,t+1}^{\$} - y_{nt}^{\$} = \text{constant} + \beta_n \frac{1}{n-1} (y_{nt}^{\$} - y_{1t}^{\$}) + \text{error} \quad (35)$$

in the data and in the model. These “long-rate” regressions were performed by [Campbell and Shiller \(1991\)](#) to test the hypothesis of constant risk premia on bonds, also known as the generalized expectations hypothesis. If risk premia are constant,  $\beta_n$  should be equal to one. Instead, Campbell and Shiller find a coefficient that is negative at all maturities, and significantly different from one. Moreover, the higher the maturity, the lower is  $\beta_n$ .

Fig. 6 depicts the coefficients  $\beta_n$  when the regression Equation (35) is run on the sample described in Section 4.1 and on simulated data from the model, as a function of maturity  $n$ . The lines with plus signs indicate the coefficients from the data; as in previous studies these coefficients are negative and downward sloping as a function of maturity. The lines with circles indicate coefficients when the regression Equation (35) is run on simulated data from the model. The resulting coefficients are below zero for all maturities, and are close in magnitude to the coefficients in the data.

<sup>10</sup>[Bansal and Coleman \(1996\)](#) develop a model where agents have power utility, but demand for liquidity generates an upward-sloping yield curve. [Boudoukh \(1993\)](#) explains the upward sloping yield curve in a model where agents have power utility and inflation and consumption growth are heteroskedastic.

<sup>11</sup>Independently and concurrently with this paper, [Seppala \(2003\)](#) shows that a model in which risk sharing is limited because of risk of default also can exhibit an upward-sloping yield curve and time-varying risk premia on inflation-indexed bonds.

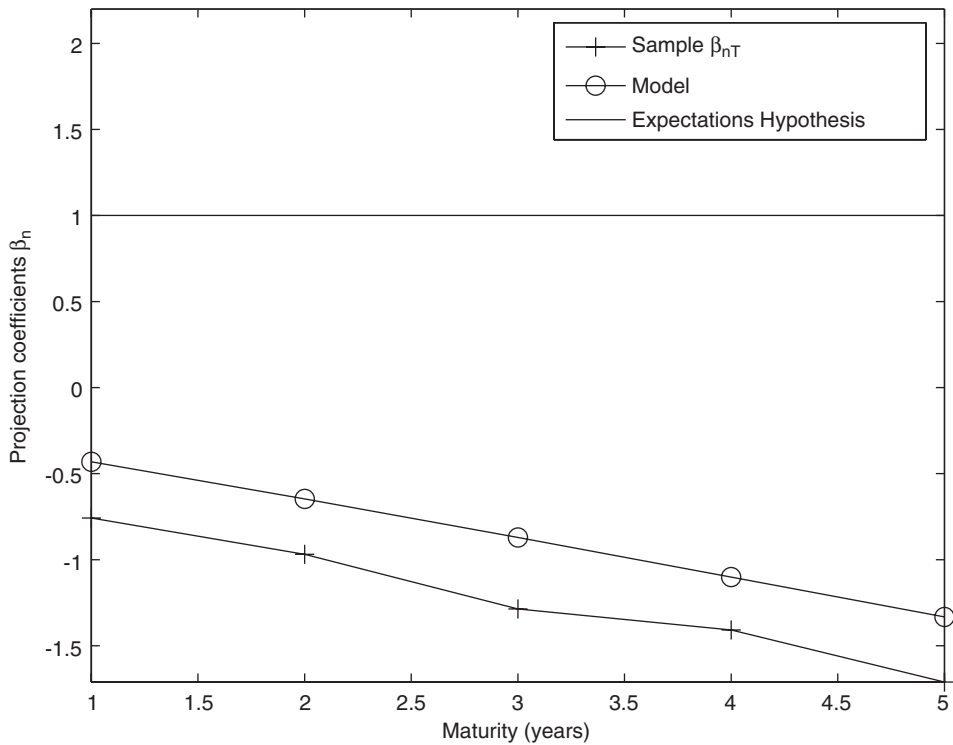


Fig. 6. Long-rate regressions. Coefficients  $\beta_n$  from the regression

$$y_{n-1,t+1}^s - y_{nt}^s = \alpha_n + \beta_n \frac{1}{n-1} (y_{nt}^s - y_{1t}^s) + \text{error}$$

using simulated and actual data on bond yields. Lines with circles denote coefficients implied by simulated data; lines with plus signs denote coefficients implied by actual data. Parameter values are as in Tables 1 and 2. The solid line denotes the coefficients if the expectations hypothesis were to hold. Data are quarterly, begin in the second quarter of 1952, and end in the third quarter of 2004.

Thus, the model can quantitatively match the failure of the expectations hypothesis.<sup>12</sup>

What drives the model's ability to explain the failure of the expectations hypothesis? The expectations hypothesis implies that  $\beta_n = 1$ , which is equivalent to the statement that excess returns on long-term bonds are unpredictable.<sup>13</sup> It follows

<sup>12</sup>The literature has identified several problems with this regression that could bias the coefficients upward or downward. Bekaert et al. (1997) show that the bias noted in Stambaugh (1999) implies that these regressions can understate the failure of the expectations hypothesis. Bekaert and Hodrick (2001) argue that standard tests tend to reject the expectations hypothesis even when it is true. They find, however, that the data remain inconsistent with the expectations hypothesis, even after adjusting for small-sample properties.

<sup>13</sup>Cochrane and Piazzesi (2005) provide direct evidence that bond returns are predictable. Moreover, they show that excess returns move together; a single linear combination of forward rates predicts excess



from the definition of yields and returns that

$$r_{n,t+1}^S = y_{nt}^S - (n-1)(y_{n-1,t+1}^S - y_{nt}^S).$$

Rearranging, and taking expectations,

$$E_t[y_{n-1,t+1}^S - y_{nt}^S] = \frac{1}{n-1}(y_{nt}^S - y_{1t}^S) - \frac{1}{n-1}E_t[r_{n,t+1}^S - y_{1t}^S]. \quad (36)$$

Thus, the coefficient of a regression of changes in yields on the scaled yield spread produces a coefficient of one only if risk premia on bonds are constant. In this model, risk premia are not constant. During recessions, the volatility of investor's marginal utility rises, as shown in Eq. (11). In [Campbell and Cochrane \(1999\)](#), this mechanism produces a time-varying risk premium on the aggregate market. Here, the same mechanism produces time-varying risk premia on bonds.<sup>14</sup>

It is instructive to compare the performance of this model to affine term structure models that are fit using term structure data alone. [Dai and Singleton \(2002\)](#) study three-factor term structure models in the essentially affine class of [Duffee \(2002\)](#) (see also [Fisher, 1998](#)). Each model has potentially three latent variables influencing risk premia. The models are distinguished by the number of factors that exhibit time-varying volatility. Dai and Singleton find that only the completely homoskedastic model can match the downward slope of the coefficients found in the data. The model with one factor influencing volatility produces coefficients that are smaller in magnitude and upward sloping, while the models with two or three factors influencing volatility produce coefficients very close to one. Therefore, time-varying risk premia are not sufficient to match the pattern and magnitude of the failure of the expectations hypothesis. This holds even in models that are fit to the term structure of interest rates and where the factors are linear combinations of bond yields, rather than driven by macro-variables as in the model in this paper.<sup>15</sup>

[Buraschi and Jiltsov \(2005\)](#) adopt an alternative approach to capturing the failure of the expectations hypothesis. In their model, marginal productivity of capital follows a square-root process, and the inflation process is endogenized through a money-in-the-utility function specification and a monetary authority policy rule. Buraschi and Jiltsov show that a specification of their model with three latent factors can deliver slope coefficients similar to those in the data, like the model in this paper. While both models are capable of accounting for the failure of the expectations

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(footnote continued)

returns on bonds at all maturities. This finding supports a feature of the habit model, namely that one variable,  $s_t$ , drives most of the time variation in bond premia.

<sup>14</sup>[Brandt and Wang \(2003\)](#) show that a model where risk aversion is driven by inflation uncertainty also implies that bond risk premia are positive and time-varying.

<sup>15</sup>However, using the generalized method of moments approach employed by [Gibbons and Ramaswamy \(1993\)](#), [Brandt and Chapman \(2002\)](#) show that when the parameters of the models with stochastic volatility are chosen so that the model fits the expectations hypothesis regressions, the models come closer to matching the patterns found in the data. This also occurs with the quadratic models of [Ahn et al. \(2002\)](#). [Bansal and Zhou \(2002\)](#) study a model with regime switches, and conclude that this type of model can also explain the expectations puzzle.

hypothesis, the focus of the papers is quite different. Buraschi and Jiltsov focus on extending the [Cox et al. \(1985\)](#) framework to a model with an endogenous inflation process. This results in a functional form relating yields to latent factors that is better able to capture the failure of the expectations hypothesis than completely affine models. They do not explore the implications for equity returns, nor do they directly link changes in yields in their model to changes in aggregate consumption. In contrast, the focus of this paper is on building a model for both equity returns and the term structure where the underlying factors can be tied directly to consumption and inflation.

To summarize, this section has shown that the population moments of the model are close to those in the data, both for the aggregate market and for bond yields. In addition, when changes in yields are projected onto the scaled yield spread, the resulting coefficient is negative and decreasing in maturity, quantitatively matching the failure of the expectations hypothesis in the data.

#### 4.4. Implications for the time series

The previous section shows the implications of the model for the population values of aggregate market moments, bond yields, and [Campbell and Shiller \(1991\)](#) regression coefficients. This section discusses the implications of the model for the

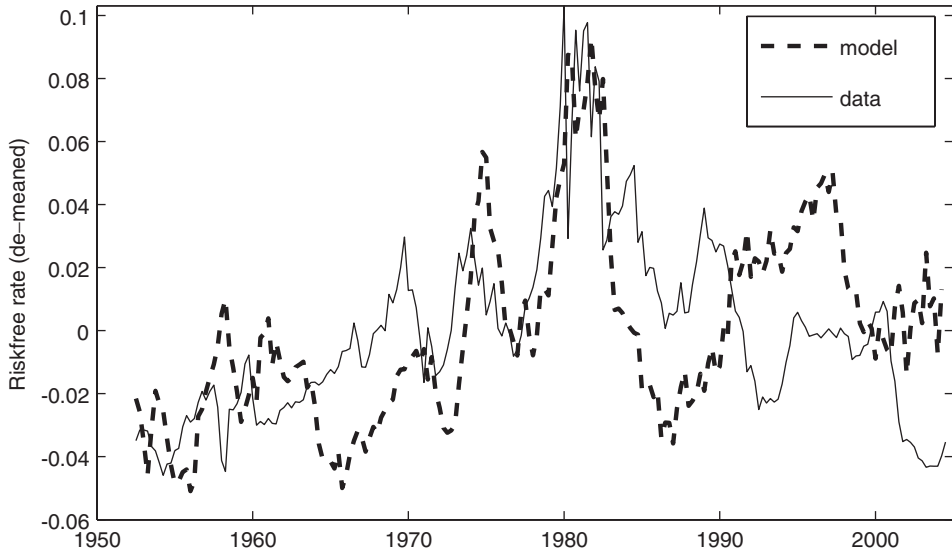


Fig. 7. Time series of the three-month yield in the data and predicted by the model. The solid line shows the time series of the nominal three-month yield in quarterly data. The dashed line shows the implied time series when quarterly data on consumption and the price level are fed into the model. Expected inflation is taken to be its mean conditional on past inflation data, given the maximum likelihood estimates in [Table 1](#). Using Eq. (2), surplus consumption is generated from actual consumption. Both series are de-meaned. Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004.

post-war time series of the short-term interest rate, the yield spread, and risk premia on bonds.

Fig. 7 shows the time series of the nominal yield on the three-month bond implied by the model (dashed lines), and the nominal three-month yield in the data (solid lines). To construct the nominal yield implied by the model, a time series of the state variables  $s_t$  and  $Z_t$  are constructed;  $s_t$  is constructed using Eq. (2) and data on quarterly consumption growth and  $Z_t$  is constructed using the maximum likelihood procedure described in Appendix C. Note that expected inflation growth cannot be observed in the data. The procedure in Appendix C constructs expected inflation growth given past inflation; however this series converges to  $Z_t$  as the number of data points grows.<sup>16</sup> Note that this series is identical to that shown in Fig. 1.

Given a series  $s_t$  and a series proxying for  $Z_t$ , it is possible to calculate the model's implications for the time series of nominal yields. Eq. (18) shows that bond yields are affine functions of  $Z_t$  multiplied by a function  $F_n^S(s_t)$  that is not available in closed form (for the three-month nominal yield, Eq. (24) is an approximate closed-form expression). Values for  $F_n^S(s_t)$  corresponding to the time series are interpolated on a grid of values for  $s_t$ . The resulting model-implied three-month yield has a sample mean of 5.8%, higher than the data mean of 5.2%. This difference arises because the sample mean of the model-implied three-month yield (computed using 210 quarters of actual data) does not equal the population mean of the model-implied three-month yield (computed using 100,000 quarters of simulated data). As discussed in Section 4.1, the parameters of the model are chosen so that the population mean, rather than the sample mean, matches the average three-month yield in the data.<sup>17</sup>

The resulting series for the three-month yield is plotted in Fig. 7, along with the de-meaned series from the data. Fig. 7 shows that the model captures many of the short-run and long-run fluctuations in the nominal risk-free rate. In the 1990s, the relation breaks down somewhat, and the model is unable to capture the very low interest rates after 2000. Overall, the correlation between the series implied by the model and the series in the data is 0.50, even though the series implied by the model is constructed using inflation and consumption data alone.

Fig. 8 repeats the procedure, this time plotting the de-meaned yield spread on the five-year nominal bond over the three-month bond implied by the model, and the same series from the data. Again, the model matches many of the short- and long-run fluctuations in the nominal yield spread from the data. The model does predict a

<sup>16</sup>This convergence result follows because Eq. (C.8) in Appendix A.3 is satisfied by  $\sigma_z^2$ . Therefore, the expected value of  $\pi_{t+1}$  given data on inflation up to  $t$  converges to  $\psi_2 \Delta \pi_t + \theta_2 v_{2t}$ . The argument in Section 4.1 shows that this series is equal to  $Z_t$ .

<sup>17</sup>This raises the question of why the sample means of the model-implied yields differ from the population means. The answer lies in the nonlinear relation between consumption growth and yields. The model is calibrated based on sample means of consumption growth and inflation, so, by construction, the sample means of consumption growth and inflation are equal to their population means assumed in the model. However, yields are nonlinear functions of consumption growth, so it does not follow that the sample means of implied yields will equal the corresponding population means (though in sufficiently large samples, the two must be equal).

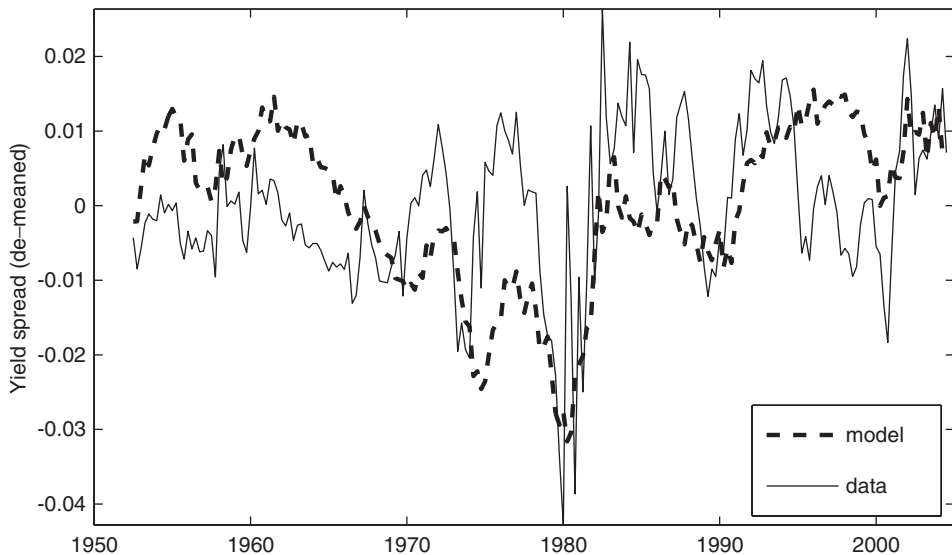


Fig. 8. Time series of the yield spread in the data and predicted by the model. The yield spread is the difference in yields between the five-year nominal bond and the three-month bond. The solid line shows the time series of the yield spread between bonds in the data. The dashed line shows the implied time series when quarterly data on consumption and the price level are fed into the model. Expected inflation is taken to be its mean conditional on past inflation data, given the maximum likelihood estimates in Table 1. Using Eq. (2), surplus consumption is generated from actual consumption. Both series are de-meanned. Data are quarterly, begin in the second quarter of 1952, and end in the second quarter of 2004.

lower yield spread in the 1970s than actually occurred. A reason for the discrepancy could be that the simple, constant volatility model for inflation misses the increase in the inflation level and volatility during this period. Indeed, Fig. 1 shows that the errors for the inflation model are unusually large in the 1970s. In addition, the model misses the decline in the yield spread in the latter part of the 1990s. However, in this case the failure of the nominal side of the model is less apparent, and the reason for the discrepancy could lie in the real side of the model: the unusually low long-run consumption growth preceding the period did not translate into higher risk premia for bonds. In the most recent period, however, the relation between the model and the data is restored. Overall, the correlation between the yield spread implied by the model and that in the data is 0.34.

Dai and Singleton (2002) propose another metric by which to judge the time series implications of the model. Rearranging Eq. (36) produces

$$E_t[y_{n-1,t+1}^s - y_{nt}^s] + \frac{1}{n-1} E_t[r_{n,t+1}^s - y_{1t}^s] = \frac{1}{n-1} (y_{nt}^s - y_{1t}^s).$$

This relation is a consequence of the present-value identity for yields, and thus holds in any term structure model. Based on this equation, Dai and Singleton propose

running the following regression on actual data:

$$y_{n-1,t+1}^s - y_{nt}^s + \frac{1}{n-1} \hat{E}_t[r_{n,t+1}^s - y_{1t}^s] = \text{constant} + \beta_n^R \frac{1}{n-1} (y_{nt}^s - y_{1t}^s) + \text{error}, \quad (37)$$

where  $\hat{E}_t[r_{n,t+1}^s - y_{1t}^s]$  is the risk premium on nominal bond yields implied by the model. If adding implied risk premia to the left-hand side brings  $\beta_n$  closer to one, then the model helps to resolve the expectations puzzle.

This model diagnostic differs from the one performed in the previous section, summarized in Eq. (35), in a number of respects. The regression (35) is run using simulated data and the results are compared to the results when (35) is run using actual data. In contrast, it does not make sense to run (37) on simulated data, because by definition  $\beta_n \equiv 1$  in population. Instead, (37) is run using the actual time series of data for bond yields  $y_{n-1,t+1}^s$ ,  $y_{nt}^s$ , and  $y_{1t}^s$ . For the models considered by Dai and Singleton (2002) and the model in this paper, the conditional risk premium  $\hat{E}_t[r_{n,t+1}^s - y_{1t}^s]$  on the  $n$ -period nominal bond is a function of the state variables at time  $t$ . This function of the state variables is scaled by  $1/(n-1)$  and added to the change in yield on the left-hand side. Thus, the diagnostic demonstrates the degree to which variation in the implied risk premium matches variation in the actual risk premium in the time series.

For the model in this paper, risk premia are not available in closed form. Nonetheless, they can be easily computed using Eq. (25), derived in Section 2.4. This computation is simplified by the fact that, for (37), it is only necessary to know risk premia up to a (maturity-dependent) constant. Moreover, as shown in Section 2.4,

Table 5

The second column shows coefficients  $\beta_n$  from the regression

$$y_{n-1,t+1}^s - y_{nt}^s = \alpha_n + \beta_n \frac{1}{n-1} (y_{nt}^s - y_{1t}^s) + \text{error}$$

in quarterly data, where  $y_{nt}^s$  denotes the time- $t$  continuously-compounded nominal yield on a nominal bond maturing in  $n$  quarters. The third column shows coefficients  $\beta_n^R$  from the regression

$$y_{n-1,t+1}^s - y_{nt}^s + \frac{1}{n-1} \hat{E}_t[r_{n,t+1}^s - y_{1t}^s] = \alpha_n^R + \beta_n^R \frac{1}{n-1} (y_{nt}^s - y_{1t}^s) + \text{error},$$

where  $\hat{E}_t[r_{n,t+1}^s - y_{1t}^s]$  is the premium on the bond with maturity  $n$  implied by the model, given the level of surplus consumption and expected inflation. Maturity is in quarters. Data begin in the second quarter of 1952, and end in the third quarter of 2004.

Maturity	Data coefficient	Adjusted coefficient
4	-0.76	-0.73
8	-0.97	-0.70
12	-1.29	-0.84
16	-1.41	-0.81
20	-1.71	-1.01

risk premia are only functions of surplus consumption, not of expected inflation. To obtain the time series of risk premia for use in Eq. (37), a series for surplus consumption using actual consumption data is produced from (2), and then values for (25) are interpolated.

Table 5 shows the coefficients  $\beta_n^R$  from the regression (37), along with the coefficients  $\beta_n$  from (35) found in the data. As described above, the coefficients from the data are negative and decreasing with maturity. However, the risk-adjusted coefficients  $\beta_n^R$  decline more slowly with maturity, and are always higher than  $\beta_n$ . Therefore,  $\hat{E}_t[r_{n,t+1}^S - y_{1t}^S]$ , computed based on surplus consumption, helps to capture some of the time variation in risk premia. Not surprisingly, the model cannot capture all of the time variation, as  $\hat{E}_t[r_{n,t+1}^S - y_{1t}^S]$  is calculated based on a single factor derived from aggregate consumption rather than from prices.

To summarize, this section has shown that the model captures features of the time series of short and long-term interest rates. This was shown in two ways. First, the series of the implied three-month nominal yield in the model, and the series of the implied spread on the five-year yield over the three-month yield, were compared to those in the data. The correlation between the data and the model was 0.50 in the case of the short-term yield, and 0.34 in the case of the yield spread. Time-series plots show that the model captures many of the short- and long-term fluctuations in the data. Second, when regressions of yield changes on the yield spread are adjusted by the time series of bond risk premia implied by the model, as proposed by Dai and Singleton (2002), the projection coefficients come closer to what would be found if the expectations hypothesis were to hold.

## 5. Conclusion

This paper offers a theory of the nominal term structure based on the preferences of a representative agent. By generalizing a model already known to fit stylized facts about the aggregate stock market, that of Campbell and Cochrane (1999), this paper is able to parsimoniously model both bond and stock returns. This paper departs from the model of Campbell and Cochrane by exploring the implications of allowing surplus consumption to affect the risk-free rate, and by introducing a process for inflation. The first extension is accomplished by introducing a preference parameter that represents a tradeoff between the intertemporal substitution effect and the precautionary savings effect. The second is accomplished by specifying an exogenous process for inflation, and estimating this process using inflation data.

The new preference parameter is set to deliver a reasonable fit to key features of nominal term structure data. As argued in the paper, positive risk premia on real bonds (and risk premia on nominal bonds that are large enough to match those in the data) imply that the intertemporal substitution effect dominates the precautionary savings effect. This parameter can be chosen so that the average slope of the yield curve is close to that in the data, while at the same time producing reasonable volatilities of bond yields. The remaining preference parameters are set exactly as in

Campbell and Cochrane (1999) to match the average risk-free rate, the Sharpe ratio on equity returns, and the autocorrelation of the price–dividend ratio. This ensures the model can quantitatively fit the equity data.

A second question is whether the model offers a realistic account of changes in yields in the post-war data. In general this is not a challenge for term structure models as the latent variables in these models are linear combinations of prices. However, in this model, the latent factors are based on consumption and inflation. Nonetheless, the implied three-month and five-year nominal yields in the model are shown to capture many of the short- and long-term fluctuations of their counterparts in the data.

Finally, the model is able to replicate the expectations puzzle found in the data. When changes in bond yields are regressed on the scaled yield spread, as in Campbell and Shiller (1991), the resulting coefficients are negative and decreasing in maturity, and about the same magnitude as in the data. In summary, the model is able to capture many of the properties of moments of bond returns in the data, and to explain much of the time-series variation in short- and long-term bond yields. Thus, the model has the potential to unify stock and bond pricing, and to connect them both to underlying macroeconomic behavior.

#### Appendix A. Deriving the sensitivity function $\lambda(s_t)$

The sensitivity function  $\lambda(s_t)$  is specified to produce a real risk-free rate that is linear in  $s_t$ . Setting the equation for the real risk-free rate (6) equal to the linear expression (10) produces the following general form for  $\lambda$ :

$$\lambda(s_t) = \frac{\sqrt{2}}{\gamma\sigma_v} \left( \gamma(1-\phi)(\bar{s} - s_t) - b(s_t - \bar{s}) + \frac{\gamma(1-\phi) - b}{2} \right)^{1/2} - 1. \quad (\text{A.1})$$

Campbell and Cochrane (1999) further impose the conditions

$$\lambda(\bar{s}) = \frac{1}{\bar{S}} - 1, \quad (\text{A.2})$$

$$\lambda'(\bar{s}) = -\frac{1}{\bar{S}}. \quad (\text{A.3})$$

They show that these conditions are equivalent to requiring that for  $s_t \approx \bar{s}$ ,  $x_t$  is approximately a deterministic function of past consumption. Eqs. (A.1)–(A.3) lead to the expressions for  $\bar{S}$  in the text.

#### Appendix B. Nominal bond pricing

The equations for nominal bond prices are derived using induction. Assume that Eq. (18) holds for the bond with  $n - 1$  periods to maturity. From the Euler (4), it

follows that

$$\begin{aligned} P_{n,t}^{\$} &= E_t \left[ M_{t+1} \frac{\Pi_t}{\Pi_{t+1}} \exp\{A_{n-1} + B_{n-1} Z_{t+1}\} F_{n-1}^{\$}(s_{t+1}) \right] \\ &= \exp\{A_{n-1} - \eta_0 + B_{n-1} \mu + (B_{n-1} \Phi - \eta) Z_t\} \\ &\quad \times E_t[M_{t+1} F_{n-1}^{\$}(s_{t+1}) E[e^{(B_{n-1} \Sigma - \sigma_{\pi}) \varepsilon_{t+1}} | \sigma_c \varepsilon_{t+1}]]. \end{aligned}$$

The second equality follows from the law of iterated expectations. By the properties of the multivariate normal distribution,

$$\begin{aligned} (B_{n-1} \Sigma - \sigma_{\pi}) \varepsilon_{t+1} | \sigma_c \varepsilon_{t+1} &\sim N(\xi_n \sigma_c \varepsilon_{t+1}, (B_{n-1} \Sigma - \sigma_{\pi}) \\ &\quad \times (I - \sigma'_c (\sigma_c \sigma'_c)^{-1} \sigma_c) (B_{n-1} \Sigma - \sigma_{\pi})'), \end{aligned}$$

where  $\xi_n$  is defined as in Eq. (21). Therefore,

$$\begin{aligned} P_{n,t}^{\$} &= \exp\{A_{n-1} - \eta_0 + B_{n-1} \mu + \frac{1}{2} (B_{n-1} \Sigma - \sigma_{\pi}) [I - \sigma'_c (\sigma_c - \sigma'_c)^{-1} \sigma_c] \\ &\quad \times (B_{n-1} \Sigma - \sigma_{\pi})' (B_{n-1} \Phi - \eta) Z_t\} E_t[M_{t+1} e^{\xi_n \sigma_c \varepsilon_{t+1}} F_{n-1}^{\$}(s_{t+1})]. \end{aligned}$$

Therefore, Eq. (18) is satisfied with

$$\begin{aligned} A_n &= A_{n-1} - \eta_0 + B_{n-1} \mu + \frac{1}{2} (B_{n-1} \Sigma - \sigma_{\pi}) [I - \sigma'_c (\sigma_c \sigma'_c)^{-1} \sigma_c] (B_{n-1} \Sigma - \sigma_{\pi})', \\ B_n &= B_{n-1} \Phi - \eta, \\ F_n^{\$}(s_t) &= E_t[M_{t+1} e^{\xi_n \sigma_c \varepsilon_{t+1}} F_{n-1}^{\$}(s_{t+1})]. \end{aligned}$$

### Appendix C. Likelihood function

This section derives the likelihood function estimated in Section 3. Let  $h_t = [\Delta c_t \ \Delta \pi_t]'$ ,  $\bar{h} = [g, \bar{\pi}]'$  and

$$\begin{aligned} \Psi &= \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix}, \\ \Theta &= \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}. \end{aligned}$$

Then

$$h_{t+1} = (I - \Psi) \bar{h} + \Psi h_t + \Theta v_t + v_{t+1}, \quad (\text{C.4})$$

where

$$v_t = \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix} \sim N(0, \Sigma_v),$$

and  $v_t$  is independent of  $v_{t-1}, \dots, v_0$  and  $h_{t-1}, \dots, h_0$ . The following proposition describes the likelihood function for the process in Eq. (C.4), conditional on observables:



**Proposition 1.**

$$h_{t+1}|h_t, \dots, h_0 \sim N(\hat{h}_t, \hat{\Sigma}_t), \quad (\text{C.5})$$

where

$$\hat{h}_t = (I - \Psi)\bar{h} + \Psi h_t + \Theta \Sigma_v \hat{\Sigma}_{t-1}^{-1} (h_t - \hat{h}_{t-1}), \quad (\text{C.6})$$

$$\hat{h}_0 = \bar{h} \quad (\text{C.7})$$

and

$$\hat{\Sigma}_t = \Sigma_v + \Theta \Sigma_v \Theta' - \Theta \Sigma_v \hat{\Sigma}_{t-1}^{-1} \Sigma_v' \Theta', \quad (\text{C.8})$$

$$(\hat{\Sigma}_0)_{ij} = \frac{(\theta_i \theta_j + \psi_i \theta_j + \psi_j \theta_i + 1)(\Sigma_v)_{ij}}{1 - \psi_i \psi_j}. \quad (\text{C.9})$$

**Proof.** The proof is by induction. Eq. (C.7) follows from taking unconditional expectations on both sides of Eq. (C.4):

$$\hat{h}_0 = (I - \Psi)\bar{h} + \Psi \hat{h}_0.$$

Subtracting  $\Psi \hat{h}_0$  from both sides and inverting  $I - \Psi$  shows that  $\hat{h}_0 = \bar{h}$ .

Note that

$$\text{Cov}(h_t, v_t) = E(h_t v_t) = EE_{t-1}(h_t v_t) = EE_{t-1}(v_t v_t) = E[\Sigma_v] = \Sigma_v. \quad (\text{C.10})$$

Taking the unconditional variance on both sides of Eq. (C.4) produces

$$\hat{\Sigma}_0 = \Psi \hat{\Sigma}_0 \Psi' + \Theta \Sigma_v \Theta' + \Psi \Sigma_v \Theta' + \Theta \Sigma_v \Psi' + \Sigma_v.$$

In the case of diagonal  $\Psi$ , this can be inverted element-by-element to produce (C.9).

Now assume by induction that

$$h_t|h_{t-1}, \dots, h_0 \sim N(\hat{h}_{t-1}, \hat{\Sigma}_{t-1}). \quad (\text{C.11})$$

It follows from Eq. (C.10) that

$$\begin{bmatrix} h_t \\ \varepsilon_t \end{bmatrix} | h_{t-1}, \dots, y_0 \sim N \left( \begin{bmatrix} \hat{h}_{t-1} \\ 0 \end{bmatrix}, \begin{bmatrix} \hat{\Sigma}_{t-1} & \Sigma_v \\ \Sigma_v & \Sigma_v \end{bmatrix} \right).$$

By the properties of the normal distribution,

$$v_t|h_t, h_{t-1}, \dots, h_0 \sim N(\Sigma_v \hat{\Sigma}_{t-1}^{-1} (h_t - \hat{h}_{t-1}), \Sigma_v - \Sigma_v \hat{\Sigma}_{t-1}^{-1} \Sigma_v').$$

It follows from Eq. (C.4) that  $h_{t+1}$  is conditionally normally distributed, and that

$$E[h_{t+1} | h_t, \dots, h_0] = (I - \Psi)\bar{h} + \Psi h_t + \Theta \Sigma_v \hat{\Sigma}_{t-1}^{-1} (h_t - \hat{h}_{t-1}),$$

$$\text{Var}[h_{t+1} | h_t, \dots, h_0] = \Theta \Sigma_v \Theta' - \Theta \Sigma_v \hat{\Sigma}_{t-1}^{-1} \Sigma_v' \Theta' + \Sigma_v. \quad \square$$

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