Structural Form

Assume an economy with two variables: X_1 and X_2; and two structural shocks: s_1 and s_2. Law of motions of X_1 and X_2 has the following process:

$$X_t = B(L)X_{t-1} + As_t$$

which is

$$\begin{cases} X_{1,t} = B_{1,1}(L)X_{1,t-1} + B_{1,2}(L)X_{2,t-1} + A_{1,1}s_{1,t} + A_{1,2}s_{2,t} \\ X_{2,t} = B_{2,1}(L)X_{1,t-1} + B_{2,2}(L)X_{2,t-1} + A_{2,1}s_{1,t} - A_{2,2}s_{2,t} \end{cases}$$

I assume to know exactly the values of matrix $B(L)$ and to only know signs of each element of matrix A. Without loss of generality I will assume that each element of A is strictly positive. Objective is to recover from the reduced-form VAR,

$$\begin{cases} X_{1,t} = B_{1,1}(L)X_{1,t-1} + B_{1,2}(L)X_{2,t-1} + i_{1,t} \\ X_{2,t} = B_{2,1}(L)X_{1,t-1} + B_{2,2}(L)X_{2,t-1} + i_{2,t} \end{cases}$$

structural matrix A. In particular, I want to be able to solve the system

$$As_t = i_t$$

which is

$$\begin{cases} A_{1,1}s_{1,t} + A_{1,2}s_{2,t} = i_{1,t} \\ A_{2,1}s_{1,t} - A_{2,2}s_{2,t} = i_{2,t} \end{cases}$$

Since by assumption, $s_t's_t = I_2$, we have that $A'A = i_t'i_t$ which is

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}^2 & \sigma_{1,2}^2 \\ \sigma_{1,2}^2 & \sigma_{2,2}^2 \end{pmatrix}$$

which is

$$\begin{pmatrix} A_{1,1}^2 + A_{1,2}^2 & A_{1,1}A_{2,1} + A_{1,2}A_{2,2} \\ A_{1,1}A_{2,1} + A_{1,2}A_{2,2} & A_{2,1}^2 + A_{2,2}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}^2 & \sigma_{1,2}^2 \\ \sigma_{2,1}^2 & \sigma_{2,2}^2 \end{pmatrix}$$

which boils down to the following system,

$$\begin{cases} A_{1,1}^2 + A_{1,2}^2 = \sigma_{1,1}^2 \\ A_{1,1}A_{2,1} + A_{1,2}A_{2,2} = \sigma_{1,2}^2 \\ A_{2,1}^2 + A_{2,2}^2 = \sigma_{2,2}^2 \end{cases}$$

Assume you solve the system by using a standard Cholesky identification where $A_{1,2} = 0$,

$$chol(A) = \begin{pmatrix} \sigma_{1,1} & 0 \\ \frac{\sigma_{1,2}^2}{\sigma_{1,1}} & \sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & 0 \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

Now, define an orthogonal matrix D which will be used to identify the structural shocks,

$$D = \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix}$$

where $D'D = I$ which implies $\Gamma_1'\Gamma_1 = 1$, $\Gamma_2'\Gamma_2 = 1$, and $\Gamma_1'\Gamma_2 = 0$. Moreover, impact matrix is now,

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1}\gamma_{1,1} & c_{1,1}\gamma_{1,2} \\ c_{2,1}\gamma_{1,1} + c_{2,1}\gamma_{2,1} & c_{2,1}\gamma_{1,2} + c_{2,1}\gamma_{2,2} \end{pmatrix}$$