Structural Form

Assume an economy with two variables: X_1 and X_2; and two structural shocks: s_1 and s_2. Law of motions of X_1 and X_2 has the following process:

$$X_t = B(L)X_{t-1} + As_t \tag{1}$$

which is

$$\begin{cases} X_{1,t} = B_{1,1}(L)X_{1,t-1} + B_{1,2}(L)X_{2,t-1} + A_{1,1}s_{1,t} + A_{1,2}s_{2,t} \\ X_{2,t} = B_{2,1}(L)X_{1,t-1} + B_{2,2}(L)X_{2,t-1} + A_{2,1}s_{1,t} - A_{2,2}s_{2,t} \end{cases} \tag{2}$$

I assume to know exactly the values of matrix $B(L)$ and to only know signs of each element of matrix A. Without loss of generality I will assume that each element of A is strictly positive. Objective is to recover from the reduced-form VAR,

$$\begin{cases} X_{1,t} = B_{1,1}(L)X_{1,t-1} + B_{1,2}(L)X_{2,t-1} + i_{1,t} \\ X_{2,t} = B_{2,1}(L)X_{1,t-1} + B_{2,2}(L)X_{2,t-1} + i_{2,t} \end{cases} \tag{3}$$

structural matrix A. In particular, I want to be able to solve the system

$$As_t = i_t \tag{4}$$

which is

$$\begin{cases} A_{1,1}s_{1,t} + A_{1,2}s_{2,t} = i_{1,t} \\ A_{2,1}s_{1,t} - A_{2,2}s_{2,t} = i_{2,t} \end{cases} \tag{5}$$

Since by assumption, $s_t's_t = I_2$, we have that $A'A = i_t'i_t$ which is

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{2,1} \\ A_{1,2} & A_{2,2} \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}^2 & \sigma_{1,2}^2 \\ \sigma_{1,2}^2 & \sigma_{2,2}^2 \end{pmatrix} \tag{6}$$

which is

$$\begin{pmatrix} A_{1,1}^2 + A_{1,2}^2 & A_{1,1}A_{2,1} + A_{1,2}A_{2,2} \\ A_{1,1}A_{2,1} + A_{1,2}A_{2,2} & A_{2,1}^2 + A_{2,2}^2 \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}^2 & \sigma_{1,2}^2 \\ \sigma_{2,1}^2 & \sigma_{2,2}^2 \end{pmatrix} \tag{7}$$

which boils down to the following system,

$$\begin{cases} A_{1,1}^2 + A_{1,2}^2 = \sigma_{1,1}^2 \\ A_{1,1}A_{2,1} + A_{1,2}A_{2,2} = \sigma_{1,2}^2 \\ A_{2,1}^2 + A_{2,2}^2 = \sigma_{2,2}^2 \end{cases} \tag{8}$$

Assume you solve the system by using a standard Cholesky identification where $A_{1,2} = 0$,

$$chol(A) = \begin{pmatrix} \sigma_{1,1} & 0 \\ \frac{\sigma_{1,2}^2}{\sigma_{1,1}} & \sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & 0 \\ c_{2,1} & c_{2,2} \end{pmatrix} \tag{9}$$

Now, define an orthogonal matrix D which will be used to identify the structural shocks,

$$D = \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \tag{10}$$

where $D'D = I$. Moreover, impact matrix is now,

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1}\gamma_{1,1} & c_{1,1}\gamma_{1,2} \\ c_{2,1}\gamma_{1,1} + c_{2,2}\gamma_{2,1} & c_{2,1}\gamma_{1,2} + c_{2,2}\gamma_{2,2} \end{pmatrix} \tag{11}$$

which is

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} \sigma_{1,1}\gamma_{1,1} & \sigma_{1,1}\gamma_{1,2} \\ \frac{\sigma_{1,2}^2}{\sigma_{1,1}}\gamma_{1,1} + \sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2}\gamma_{2,1} & \frac{\sigma_{1,2}^2}{\sigma_{1,1}}\gamma_{1,2} + \sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2}\gamma_{2,2} \end{pmatrix} \tag{12}$$

Identification of $s_{1,t}$ is

$$\begin{aligned} \max_{\gamma_{1,1},\gamma_{2,1}} \quad & \sigma_{1,1}\gamma_{1,1} + \delta\left[\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\gamma_{1,1} + \sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2}\gamma_{2,1}\right] \\ \text{s.t.} \quad & 1 \geq \gamma_{1,1}^2 + \gamma_{2,1}^2 \end{aligned} \tag{13}$$

Resetting the problem with Lagrangian function is

$$\begin{aligned} \max_{\gamma_{1,1},\gamma_{2,1}} L(\gamma_{1,1}, \gamma_{2,1}, \lambda) \;=\; & \sigma_{1,1}\gamma_{1,1} + \delta\left[\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\gamma_{1,1} + \sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2}\gamma_{2,1}\right] \\ & + \;\lambda\big(1 - \gamma_{1,1}^2 - \gamma_{2,1}^2\big) \end{aligned} \tag{14}$$

First order conditions are

$$\frac{\partial L(\gamma_{1,1}, \gamma_{2,1}, \lambda)}{\partial \gamma_{1,1}} = \sigma_{1,1} + \delta\frac{\sigma_{1,2}^2}{\sigma_{1,1}} - 2\lambda\big(\gamma_{1,1}^*\big)^2 = 0 \quad \Rightarrow \quad \gamma_{1,1}^* = \frac{1}{2\lambda}\left[\sigma_{1,1} + \delta\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right] \tag{15}$$

$$\frac{\partial L(\gamma_{1,1}, \gamma_{2,1}, \lambda)}{\partial \gamma_{2,1}} = \delta\sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2} - 2\lambda\big(\gamma_{2,1}^*\big)^2 = 0 \quad \Rightarrow \quad \gamma_{2,1}^* = \frac{1}{2\lambda}\delta\sqrt{\sigma_{2,2}^2 - \left(\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)^2} \tag{16}$$

$$\lambda\frac{\partial L(\gamma_{1,1}, \gamma_{2,1}, \lambda)}{\partial \lambda} = \lambda\big[1 - \big(\gamma_{1,1}^*\big)^2 - \big(\gamma_{2,1}^*\big)^2\big] = 0 \tag{17}$$

where $\lambda \geq 0$.

- Equation 15 implies that
    1. $\gamma_{1,1}^* \geq 0$ for all $\delta \geq 0$ if $\sigma_{1,2}^2 \geq 0$.
    2. It exists $\bar{\delta}$ such that $\gamma_{1,1}^* \geq 0$ for all $0 \leq \delta \leq \bar{\delta}$ if $\sigma_{1,2}^2 \leq 0$.
- Equation 16 implies $\gamma_{2,1}^* \geq 0$ for all $\delta \geq 0$.

- Dividing 15 over 16 yields

$$\frac{\gamma^*_{1,1}}{\gamma^*_{2,1}} = \frac{\sigma_{1,1} + \delta\frac{\sigma^2_{1,2}}{\sigma_{1,1}}}{\delta\sqrt{\sigma_{2,2} - \left(\frac{\sigma^2_{1,1}}{\sigma_1 1}\right)^2}}$$

Taking first derivative with respect to δ implies

$$\frac{\partial\frac{\gamma^*_{1,1}}{\gamma^*_{2,1}}}{\partial\delta} = \frac{\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\delta\sqrt{\sigma_{2,2} - \left(\frac{\sigma^2_{1,1}}{\sigma_1 1}\right)^2} - \left(\sigma_{1,1} + \delta\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\right)\sqrt{\sigma_{2,2} - \left(\frac{\sigma^2_{1,1}}{\sigma_1 1}\right)^2}}{\delta^2\left(\sigma_{2,2} - \left(\frac{\sigma^2_{1,1}}{\sigma_1 1}\right)^2\right)}$$

which is

$$\frac{\partial\frac{\gamma^*_{1,1}}{\gamma^*_{2,1}}}{\partial\delta} = -\frac{\sigma_{1,1}\sqrt{\sigma_{2,2} - \left(\frac{\sigma^2_{1,1}}{\sigma_1 1}\right)^2}}{\delta^2\left(\sigma_{2,2} - \left(\frac{\sigma^2_{1,1}}{\sigma_1 1}\right)^2\right)} < 0.$$

Identification of $s_{2,t}$ is

$$\begin{aligned} \max_{\gamma_{1,2},\gamma_{2,2}} \quad & \sigma_{1,1}\gamma_{1,2} - \delta\left[\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\gamma_{1,2} + \sqrt{\sigma^2_{2,2} - \left(\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\right)^2}\gamma_{2,2}\right] \\ \text{s.t.} \quad & 1 \geq \gamma^2_{1,2} + \gamma^2_{2,2} \end{aligned} \tag{18}$$

Resetting the problem with Lagrangian function is

$$\begin{aligned} \max_{\gamma_{1,2},\gamma_{2,2}} L(\gamma_{1,2},\gamma_{2,2},\lambda) \;=\; & \sigma_{1,1}\gamma_{1,2} - \delta\left[\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\gamma_{1,2} + \sqrt{\sigma^2_{2,2} - \left(\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\right)^2}\gamma_{2,2}\right] \\ & + \;\lambda\left(1 - \gamma^2_{1,2} - \gamma^2_{2,2}\right) \end{aligned} \tag{19}$$

First order conditions are

$$\frac{\partial L(\gamma_{1,2},\gamma_{2,2},\lambda)}{\partial\gamma_{1,2}} = \sigma_{1,1} - \delta\frac{\sigma^2_{1,2}}{\sigma_{1,1}} - 2\lambda\left(\gamma^*_{1,2}\right)^2 = 0 \quad\Rightarrow\quad \gamma^*_{1,2} = \frac{1}{2\lambda}\left[\sigma_{1,1} - \delta\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\right] \tag{20}$$

$$\frac{\partial L(\gamma_{1,2},\gamma_{2,2},\lambda)}{\partial\gamma_{2,2}} = -\delta\sqrt{\sigma^2_{2,2} - \left(\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\right)^2} - 2\lambda\left(\gamma^*_{2,2}\right)^2 = 0 \quad\Rightarrow\quad \gamma^*_{2,2} = -\frac{1}{2\lambda}\delta\sqrt{\sigma^2_{2,2} - \left(\frac{\sigma^2_{1,2}}{\sigma_{1,1}}\right)^2} \tag{21}$$

$$\lambda\frac{\partial L(\gamma_{1,2},\gamma_{2,2},\lambda)}{\partial\lambda} = \lambda\left[1 - \left(\gamma^*_{1,2}\right)^2 - \left(\gamma^*_{2,2}\right)^2\right] = 0 \tag{22}$$

where $\lambda \geq 0$.

- Equation 20 implies that

1. $\gamma_{1,2}^* \geq 0$ for all $\delta \geq 0$ if $\sigma_{1,2}^2 \leq 0$.
2. It exists $\bar{\delta}$ such that $\gamma_{1,2}^* \geq 0$ for all $0 \leq \delta \leq \bar{\delta}$ if $\sigma_{1,2}^2 \geq 0$.

- Equation 21 implies $\gamma_{2,2}^* \leq 0$ for all $\delta \geq 0$.
- Dividing 20 over 21 yields

$$\frac{\gamma_{1,2}^*}{\gamma_{2,2}^*} = -\frac{\sigma_{1,1} - \delta\frac{\sigma_{1,2}^2}{\sigma_{1,1}}}{\delta\sqrt{\sigma_{2,2} - \left(\frac{\sigma_{1,1}^2}{\sigma_1 1}\right)^2}}$$

Taking first derivative with respect to δ implies

$$\frac{\partial\frac{\gamma_{1,2}^*}{\gamma_{2,2}^*}}{\partial\delta} = -\frac{-\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\delta\sqrt{\sigma_{2,2} - \left(\frac{\sigma_{1,1}^2}{\sigma_1 1}\right)^2} - \left(\sigma_{1,1} - \delta\frac{\sigma_{1,2}^2}{\sigma_{1,1}}\right)\sqrt{\sigma_{2,2} - \left(\frac{\sigma_{1,1}^2}{\sigma_1 1}\right)^2}}{\delta^2\left(\sigma_{2,2} - \left(\frac{\sigma_{1,1}^2}{\sigma_1 1}\right)^2\right)}$$

which is

$$\frac{\partial\frac{\gamma_{1,2}^*}{\gamma_{2,2}^*}}{\partial\delta} = \frac{\sigma_{1,1}\sqrt{\sigma_{2,2} - \left(\frac{\sigma_{1,1}^2}{\sigma_1 1}\right)^2}}{\delta^2\left(\sigma_{2,2} - \left(\frac{\sigma_{1,1}^2}{\sigma_1 1}\right)^2\right)} > 0$$

Proposition 1. *It exists a unique δ^* such that $\gamma_{1,1}^*\gamma_{1,2}^* + \gamma_{2,1}^*\gamma_{2,2}^* = 0$.*

Proof. There exist two possible cases to focus on: 1. $\delta \leq \bar{\delta}$ and 2. $\delta > \bar{\delta}$. I am going to show that Case 1. has a unique solution and Case 2. has no solutions. Moreover, since the problem is symmetric over Γ_1 and Γ_2 is irrelevant whether I focus on $\sigma_{1,2} \geq 0$ or $\sigma_{1,2} \leq 0$. Proof would hold symmetrically in either cases. For simplicity I am going to assume $\sigma_{1,2} \geq 0$.

1. When $\delta \leq \bar{\delta}$, at least a solution always exists since for $\delta = 0$,

   $$\gamma_{1,1}^*\gamma_{1,2}^* + \gamma_{2,1}^*\gamma_{2,2}^* > 0$$

   since $\gamma_{1,1}^* = \gamma_{1,2}^* = 1$, and $\gamma_{2,1}^* = \gamma_{2,2}^* = 0$. Moreover, for $\delta = \bar{\delta}$,

   $$\gamma_{1,1}^*\gamma_{1,2}^* + \gamma_{2,1}^*\gamma_{2,2}^* < 0$$

   since $\gamma_{1,2} = 0$, $\gamma_{2,1}^* > 0$, and $\gamma_{2,2}^* > 0$.[1]

   Thus, in order to show that solution is unique, I need to prove that $\gamma_{1,1}^*\gamma_{1,2}^* + \gamma_{2,1}^*\gamma_{2,2}^*$ is monotonically decreasing in δ.

   Since both $\gamma_{1,1}^*$ and $\gamma_{2,1}^*$ are positive, and since $\frac{\gamma_{1,1}^*}{\gamma_{2,1}^*}$ is decreasing in δ then it must be the case that $\gamma_{1,1}^*$ is decreasing in δ and $\gamma_{2,1}^*$ is increasing in δ.

   Since $\gamma_{1,2}^*$ is positive and $\gamma_{2,2}^*$ is negative, and since $\frac{\gamma_{1,2}^*}{\gamma_{2,2}^*}$ is increasing in δ then it must be the case that $\gamma_{1,2}^*$ is decreasing in δ and $|\gamma_{2,2}^*|$ is increasing in δ.

   As a result, we have $(\downarrow \gamma_{1,1}^*)(\downarrow \gamma_{1,2}^*) - (\uparrow \gamma_{2,1}^*)(\uparrow |\gamma_{2,2}^*|)$ which implies that when $\delta \leq \bar{\delta}$, then $\gamma_{1,1}^*\gamma_{1,2}^* + \gamma_{2,1}^*\gamma_{2,2}^*$ is monotonically decreasing in δ which implies that solution in this area is unique.

[1] I am implicitly using the result that $\gamma_{1,1}^*\gamma_{1,2}^* + \gamma_{2,1}^*\gamma_{2,2}^*$ is a continuous function of δ.

2. When $\delta > \bar{\delta}$, $\gamma^*_{1,1}\gamma^*_{1,2} + \gamma^*_{2,1}\gamma^*_{2,2}$ is never equal to zero. This happens because when $\delta > \bar{\delta}$, $\gamma_{1,1} > 0$, $\gamma_{2,1} > 0$, $\gamma_{1,2} < 0$, and $\gamma_{2,2} < 0$. As a result,

$$\gamma^*_{1,1}\gamma^*_{1,2} + \gamma^*_{2,1}\gamma^*_{2,2} < 0 \ \ \forall\, \delta > \bar{\delta}.$$

As a result, solution cannot be reached when $\delta > \bar{\delta}$.

This complete the proof. It always exists a unique δ^* such that $\gamma^*_{1,1}\gamma^*_{1,2} + \gamma^*_{2,1}\gamma^*_{2,2} = 0$ for $\delta \geq 0$. $\square$