Uncertainty and Disagreement in Equilibrium Models

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Leading equilibrium concepts require agents' beliefs to coincide with the model's true probabilities and thus be free of systematic errors. This implicitly assumes a criterion that tests beliefs against the observed outcomes generated by the model. We formalize this requirement in stationary environments. We show that there is a tension between requiring that beliefs can be tested against systematic errors and allowing agents to disagree or be uncertain about the long-run fundamentals. We discuss the application of our analysis to asset pricing, Markov perfect equilibria, and dynamic games.

I. Introduction

Leading equilibrium concepts such as Bayesian Nash equilibrium and rational expectations equilibrium incorporate the idea that agents' beliefs about future outcomes coincide with the model's true probabilities. A natural motivation for this requirement is that competitive pressures in economic and strategic interactions create strong incentives to avoid systematic forecast errors.

Requiring beliefs to be free of systematic errors implicitly assumes that there is a criterion or test to verify that they are consistent with observed outcomes. This paper formalizes this requirement and examines its im-

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plications for the sort of equilibria one might expect to arise. Roughly, an equilibrium is *testable* if every nonequilibrium belief can be rejected with positive probability by a test that compares that belief with observed outcomes. Testability is motivated by, and is a stylized representation of, statistical tests that outside observers, such as an econometrician, might use to take theoretical models to the data. Our focus, however, is on the economic and strategic implications of testability rather than the empirics of testing per se.

We first take the perspective of an outside observer who models an infinite-horizon economy as a stochastic process P. This process describes the assumed evolution of exogenous variables as well as the optimizing behavior of agents and the resulting equilibrium outcomes. Leading equilibrium concepts require agents' beliefs to agree with P. We start with the case in which the outside observer assumes that the system is already in equilibrium, without explicitly modeling the agents' optimization. Our central assumption is that P is stationary. Stationarity is a natural assumption, widely used in dynamic stochastic models for its tractability and conceptual appeal. It is also necessary for the application of many standard empirical methods. We illustrate the role and limitations of stationarity in a number of important contexts, such as consumption-based asset pricing, Markov perfect equilibria, and Bayesian Nash equilibrium in Markovian environments.

Our main result states that requiring an equilibrium to be testable is equivalent to any one of the following three properties. First, an equilibrium P is testable if and only if it precludes disagreement: any alternative belief Q that is consistent with observed outcomes with probability one must be equal to P. Second, we consider uncertainty about long-run fundamentals, or "structural uncertainty" for short.[1] We show that an equilibrium is testable if and only if agents' beliefs preclude such uncertainty, in the sense that no realization of past observations can change their opinions about the model's long-run properties. Intuitively, the absence of structural uncertainty means that agents have nothing further to learn from data about structural parameters. Learning may still occur, but only for predicting short-run outcomes. Third, testability is related to the properties of moment conditions commonly used in the econometrics of nonlinear dynamic stochastic models. We show that an equilibrium is testable if and only if the empirical moment conditions identify the model's true parameters asymptotically.[2]

[1] See our companion paper (Al-Najjar and Shmaya 2012) for a formal discussion of learnability and "long-run fundamentals." Essentially, these are the finest properties of the environment that can be asymptotically learned.

[2] We focus on the generalized method of moments (GMM) introduced by Hansen (1982). See Sec. III.C for a detailed formal discussion.

We interpret these findings as highlighting a connection (and potential tension) between compelling desiderata of dynamic stochastic models. Testing agents' equilibrium beliefs against actual observations reflects the view that the concept of equilibrium should be judged on the basis of its observable implications rather than taken as an article of faith. A nontestable equilibrium is nonfalsifiable in the following sense: there exists a different theory about the process, with possibly different implications for behavior, such that the equilibrium theory cannot be rejected with positive probability, regardless of the amount of data. Testability is also naturally related to empirical estimation of equilibrium models, a link formalized by its equivalence to the consistency of empirical moment conditions. Given the wide use of moment conditions in empirical work, we view this consistency as another desirable property a model should have.

Our main theorem says that testability precludes disagreement and uncertainty about long-run fundamentals, two properties that are increasingly important in modeling economic and financial phenomena. For example, the absence of disagreement, in the form of common or concordant priors, implies no-trade theorems that are widely viewed as inconsistent with observed investors' behavior and volume of trade. A natural response to this conflict between empirical evidence and theoretical predictions is to weaken the common prior assumption so agents can disagree on how to interpret information.[3] Relatedly, structural uncertainty and its gradual resolution through learning are increasingly viewed as relevant to understanding a number of empirical puzzles. In their survey of the subject, Pástor and Veronesi (2009, 362) suggest that "many facts that appear baffling at first sight seem less puzzling once we recognize that parameters are uncertain and subject to learning."[4] Our theorem indicates a potential tension between disagreement and structural uncertainty on the one hand and the usual conceptual justification for equilibrium and its empirical estimation on the other.

A stochastic process describing a dynamic economic system in equilibrium must take into account that agents' decisions are optimal given their beliefs. Section V focuses on how learning and equilibrium relate to testability and the other properties discussed above. In the case of passive learning, proposition 3 shows, under mild additional assumptions, that structural uncertainty about exogenous variables leads to nonstationary endogenous responses. Intuitively, uncertainty about long-run funda-

[3] There is a large and growing literature on the role of disagreement in explaining the trade volume and other puzzles in financial markets. See Hong and Stein (2007) for an excellent survey. Ross (1989, 94), for instance, notes that "it seems clear that the only way to explain the volume of trade is with a model that is at one and the same time appealingly rational and yet permits divergent and changing opinions in a fashion that is other than ad hoc."

[4] Brav and Heaton (2002), Lewellen and Shanken (2002), and Weitzman (2007) are early examples of works that introduce learning about fundamentals to explain equity premia, risk-free rates, excess volatility, and predictability of returns. See Pástor and Veronesi's (2009) survey of the literature.

mentals implies that an agent's beliefs will be nonstationary as uncertainty decreases through learning. This result is in the spirit of Nachbar's (2005) theorem on the tension between learning and optimization in repeated games. We discuss Nachbar's work in Section VI.

Although leading equilibrium concepts agree in requiring that beliefs coincide with the true probabilities, they differ in how they implement this idea. In a Bayesian Nash equilibrium (with a common prior), agents are required to only have a subjective belief about how nature selects the true parameters of the model. A more demanding approach is rational expectations in which beliefs are required to coincide with the model's observed empirical frequencies. As we show below, beliefs are testable in a rational expectations equilibrium, but not in a Bayesian Nash equilibrium that accommodates structural uncertainty and disagreement. The ideal would be to "have it both ways," that is, maintain the flexibility of the Bayesian Nash equilibrium without sacrificing testability or identification via moment conditions. Our results point out the difficulties in reaching this goal.

We should emphasize that the message of this paper is not that disagreement and structural uncertainty lack empirical content. In fact, the appeal of these concepts lies in their potential to account for important empirical anomalies.[5] Rather, our point is to explain why it may be difficult to reconcile disagreement and structural uncertainty with the standard practice of postulating a "true model" shared by agents and outside observers.[6]

The paper proceeds as follows. In Section II, we introduce the central concept of testability. We also discuss models of asset pricing and game theory to motivate the main concepts. Section III introduces the properties of disagreement, structural uncertainty, and the consistency of empirical moment conditions. Section IV states our main theorem, while Section V connects the theorem to learning and equilibrium. Finally, Section VI provides additional connections to the literature.

II. Testing Beliefs

A. Preliminaries

We consider an infinite-horizon model, with time periods denoted $n = 0, 1, \ldots$. In period n, an outcome s_n in a finite set S is realized.[7] Let $H = S \times S \times \cdots$ denote the set of infinite histories. The set of histories of

[5] Section V.C points out settings in which disagreement and structural uncertainty have empirically measurable implications.

[6] According to Sargent (2008), "a rational expectations equilibrium asserts that the same model is shared by (1) all of the agents within the model, (2) the econometrician estimating the model, and (3) nature, also known as the data generating mechanism."

[7] The finiteness of S is convenient to avoid inessential technical complications.

length n is denoted H_n, and the (finite) algebra of events generated by these histories is $\mathcal{H}_n$. Stochastic processes induce distributions on the set of infinite histories endowed with the σ-algebra $\mathcal{H}$ generated by finite histories $\cup_{n=1}^{\infty} H_n$.

A probability distribution P on H will stand for either the true (exogenous or equilibrium) process or an agent's belief about that process. Which of these interpretations is being referred to will be clear from the context. Our focus is on stationary distributions. Recall that a probability distribution P is stationary if, for every k, its marginal distribution on the algebra $\mathcal{H}_l^{l+k}$ generated by coordinates $l, \ldots, l+k$ does not depend on l (see, e.g., Gray 2009). Let $\mathcal{P}$ denote the set of stationary distributions over H.

B. Tests

Equilibrium concepts, such as Nash equilibrium and rational expectations equilibrium, postulate a true equilibrium distribution and require agents' beliefs to coincide with it. Consider an outside observer (such as a modeler or an econometrician) who assumes that agents hold equilibrium beliefs and observes a realization of the process. Verifying that agents' beliefs coincide with the model's probabilities requires an objective criterion, or a mechanism that compares beliefs with observed outcomes. Statistical testing is a natural language to formalize this requirement.

Fix a stationary stochastic process P, which we interpret as the "equilibrium" process. This process describes the evolution of the model's exogenous variables (e.g., income and technology shocks) as well as endogenous variables (e.g., strategies, consumption levels, or asset prices). A statistical test is any function

$$T : \mathcal{P} \times H \to \{0, 1\}$$

that takes as input a distribution P and a history h and returns a yes/no answer. Interpret an outcome $T(P, h) = 1$ to mean "history h is consistent with the process P"; otherwise, h is inconsistent with P. The set

$$T_P \equiv \{h : T(P, h) = 1\}$$

is the set of all observations viewed as consistent with P under this test. We may interpret T_P as the empirical predictions of P relative to T: if P is correct, then observations must be in T_P.

Tests in this paper depend on the entire infinite realization of the process, while real-world statistical tests use finite observations.[8] On the

[8] Formally, a finite test must be measurable with respect to $\mathcal{H}_n$ for some finite n. The set of finite tests is obviously a subset of the set of all tests. Section F in the Appendix discusses finite tests.

other hand, asymptotic testing makes testability an easier hurdle to overcome: if an equilibrium theory is not testable in our sense, then it is certainly not testable when restricted to finite observations. Failure of testability in our model is not caused by scarcity of data.

The *type I error* of a test T at a process P is the number $1 - P(T_P)$ representing the probability that the test rejects P when it is true. Statistical models usually require tests with low type I error. In our asymptotic testing setting, it is natural to consider tests that are free of type I error, that is, ones for which $P(T_P) = 1$ for every P. This makes for a sharper statement and interpretation of the results. For tests based on finite observations, small but positive type I errors are more natural. See the discussion in Section F of the Appendix.

Definition 1. A stationary distribution P is *testable* if for every stationary $Q \neq P$ there is a test T such that

1. T is free of type I error and
2. $Q(T_P) < 1$.

The restriction to stationarity in the definition of testability is important. See Section IV.C for detailed discussion. We can interpret the definition in two ways. First, think of P as the theory held by an outside observer, such as a theorist or an econometrician, of the data-generating process implied by an economic model. The theory P is testable if, when P is incorrect and the correct alternative Q is presented, T can be proven wrong with some probability. In statistical language, testability requires that for any such Q, there is a test T for P against Q that has some power. That is, the type II error of this test is not 100 percent.

We may alternatively interpret P as the belief of an agent within the model. Equilibrium requires agents to have unquestioning faith in their equilibrium beliefs. Such agents have no use for tests that check whether their beliefs are right or wrong (since they are convinced they know the truth). On the other hand, agents less certain about their knowledge may view the testability of their beliefs as a desirable criterion.

Our notion of testability is meant to shed light on the implications of equilibrium rather than as a guide for designing practical testing procedures. Actual empirical tests are finite and must therefore tolerate some type I error. Requiring asymptotic tests to be free of such errors, as we do, is conceptually simpler and, in some cases, results in stronger theorems. In summary, testability retains some essential features of real-world empirical tests but abstracts from others in order to focus on theoretical questions about equilibrium beliefs.

Before turning to testability in economic models, we provide a stylized mathematical example.[9]

[9] A similar example appears in Al-Najjar and Weinstein (2015).

EXAMPLE 1. For $0 \leq p \leq 1$, let μ_p denote the distribution of independent and identically distributed (i.i.d.) coin tosses, with probability p of success. We will prove later that $P = \mu_p$ is testable. In fact, the test can be chosen independently of the alternative Q. For example, the test for $\mu_{1/2}$ checks that every block of k consecutive outcomes appears in the realized sequence at frequency $1/2^k$.

As an example of a nontestable belief, let $P = 1/2\mu_{2/3} + 1/2\mu_{1/3}$. Under this distribution the observations are i.i.d. but with unknown mean; the agent believes the coin is $\mu_{2/3}$ or $\mu_{1/3}$ with equal probability. This belief is nontestable: any alternative belief $Q = \alpha\mu_{2/3} + (1-\alpha)\mu_{1/3}$ for some $\alpha \in (0, 1)$ cannot be separated from P in the sense of definition 1.

C. *Example: Asset Pricing*

As a first illustration, consider the canonical consumption-based asset pricing model. The primitive is a stochastic process $\{c_n\}$, interpreted as the consumption of a representative agent with a time-separable utility $\sum_{n=1}^{\infty} \delta^n u(c_n)$. Here, u is a (differentiable) period utility and $\delta \in [0, 1)$ is the discount factor. Assume a finite outcome space rich enough to model the consumption process and any collection of asset returns or other variables of interest.[10]

The marginal rate of substitution between consumption in periods $n+1$ and n is the random variable

$$m_n \equiv \delta \frac{u'(c_{n+1})}{u'(c_n)}, \tag{1}$$

also known as the stochastic discount factor.

We assume that m_n is stationary with distribution P.[11] In a consumption-based asset pricing model, the process $\{m_n\}$ determines the equilibrium rates of returns of all assets. Specifically, an asset is represented by a stochastic process $\{R_n\}$ (not necessarily stationary) giving the gross rate of return at time n of a dollar invested in that asset at time $n-1$. Equilibrium requires $\{R_n\}$ to satisfy the Euler equation $E_P[m_{n+1}R_{n+1}|h^n] = 1$, for all positive probability histories h^n.

The usual practice, as in Lucas (1978) style endowment economies, is to take the consumption process as exogenous, presumably being the outcome of an unmodeled intertemporal optimization.[12] Once consump-

[10] We assumed, for technical convenience, that the outcome space is finite. In asset pricing theory, it is more natural to consider continuous outcome spaces. We can either extend the model to continuous spaces or assume S to be some appropriately fine grid.

[11] The stationarity of the discount factor can be derived from primitive assumptions about the consumption process and the utility function. Here we assume stationarity directly.

[12] See Cochrane (2005) for a textbook exposition. What is considered endogenous and exogenous is, obviously, model dependent. We follow the common practice in asset pricing

tion and utility are given, equilibrium requires that assets be priced according to the Euler equation.

The model postulates P as the true process governing how $\{m_n\}$ evolves. Our formal notion of testability attempts to give an operational meaning to the statement "P is the true process." Suppose that the modeler, or the representative agent, is confronted by an arbitrageur with an alternative theory $Q \neq P$ of the stock market. Testability captures the intuition that the modeler's theory P must have empirical content in the sense that if Q were the correct theory of asset pricing, then there is a way to "prove P wrong."[13]

Since even a wrong theory can give, by accident, more accurate predictions than the true theory, the statement "proving P wrong" must be qualified.[14] The notion of testability accomplishes this by requiring the modeler (or the agent) to produce, for any competing theory $Q \neq P$, an objective criterion that does not fail P but has some power against Q.

D. *Example: Markov Perfect Equilibria*

Our next example is a class of dynamic games with a Markovian state variable. These models are extensively used in fields such as industrial organization and political economy, among others. There is a finite set of states, X, and players, I. The set of action profiles is $A = A^1 \times \cdots \times A^I$, where A^i is player i's set of actions. In accordance with our notation, we write $S = X \times A$ to denote the outcome of the game in any given round.

In each period $n = 0, 1, \ldots$, players choose action profile $a_n \in A$; then a state x_n is realized. The distribution of x_n given period $n - 1$'s state x_{n-1} and period n's action profile a_n is determined by a known time-invariant transition function $\pi : X \times A \to \Delta(X)$.[15] We assume for simplicity that $\pi(x, a)$ has full support for every outcome (x, a). The initial outcome (x_0, a_0) is chosen according to some distribution p on S. A strategy for player i specifies his mixed action at every stage as a function of past states and action profiles. A strategy is Markovian if players' actions depend only on the current state. Thus, a Markovian strategy for player i has the form $\sigma^i : X \to \Delta(A)$.

A Markovian strategy profile σ and the transition function π induce a unique stationary Markovian distribution P over $\Delta(H)$ that describes

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theory by assuming that consumption is exogenous, although a richer model would treat it as endogenously determined from more primitive processes.

[13] Another way to describe this is to say that P is falsifiable, although the term *falsifiability* may be better suited for deterministic theories. We use the term testability to emphasize the connection with statistical practice.

[14] A theory that predicts a fair coin to land on heads in a given toss is more accurate than the true theory (that heads has probability .50) about half of the time!

[15] Note that players choose actions before the state is realized. This timing is convenient for our passive learning example in Sec. V.A.

the steady state of the play. The steady-state distribution does not depend on the initial choice of (x_0, a_0).

Agent i maximizes discounted expected utility, with period utility $u^i(x, a)$ and discount factor δ. A profile σ is a Markov perfect equilibrium (MPE) if, for each player i, σ^i is optimal against the profile of strategies of his opponents, σ^{-i}.[16] The MPE model is widely used in applied work to model, among other things, industry dynamics following the seminal work of Ericson and Pakes (1995). An attractive feature of this model is that it lends itself naturally to empirical implementation, an issue we shall return to below.[17]

We will show later that an MPE distribution P on S^∞ is testable: given an alternative stationary (not necessarily Markovian) theory $Q \neq P$ of how the game evolves, we can construct a type I error-free statistical test for P against Q with some power, that is, such that P is rejected with positive probability under Q. We cannot in general conclude that P will necessarily be rejected with high probability. For example, if Q assigns high probability $1 - \epsilon$ to P and probability ϵ to some $P' \neq P$, then P is accepted most of the time.

A variant of the above model is the hidden Markov model discussed below in which we assume that an analyst observes the players' actions and outcome X' that depends on the state via a function $f : X \to X'$.[18] For example, in empirical models it is common to assume that players condition their actions on disturbances that are unobservable by the analyst. Every MPE distribution induces a belief of the analyst over $(A \times X')^{\mathbb{N}}$. The analyst's belief is typically not a Markov process but is still stationary. We will show later that it is also testable.

III. Disagreement and Structural Uncertainty

We introduce three properties a dynamic economic model may have: the possibility of disagreement, the absence of structural uncertainty, and the feasibility of empirical estimation using moment conditions. Our main theorem will show that each of the three properties is equivalent to testability.

A. *Disagreement*

A standard assumption in economic models is that agents share a common prior (Aumann 1976, 1987) and, as a result, a common interpretation of information. This assumption has been questioned on a number

[16] As usual, all deviations are allowed, and not just Markovian ones.

[17] In empirical models, strategies usually depend also on unobservable disturbances.

[18] In particular, if f is constant, then the analyst sees only the players' actions but not the state of nature.

of grounds. First, that agents may disagree about how to interpret information seems both intuitive and consistent with the basic axioms of rationality. Second, the absence of disagreement leads to paradoxical theoretical conclusions, such as the no-trade theorems (Milgrom and Stokey 1982), that are difficult to reconcile with reality. Third, as noted in the introduction, there is large empirical evidence that seems difficult to reconcile with agents having a common interpretation of information.[19]

A natural way to introduce disagreement is to assume that agents have different prior beliefs about the underlying uncertainty. Heterogeneous prior models are used to generate realistic trade volumes and to account for other asset pricing anomalies. See Hong and Stein (2007) and Pástor and Veronesi (2009) for surveys.

An agent's uncertainty about the true process P is formally represented by a distribution $\mathcal{Q} \in \Delta(\Delta(H))$ on the set of stochastic processes. To any belief $\mathcal{Q}$ corresponds a process $Q \in \Delta(H)$ with identical distribution on sample paths.[20] Since there is no observable difference between $\mathcal{Q}$ and the process Q, we use the latter to describe beliefs.

DEFINITION 2. Two beliefs Q_1, Q_2 are *compatible* if for every event B, $Q_1(B) = 1$ if and only if $Q_2(B) = 1$.

Compatibility of beliefs is the property known as mutual absolute continuity, a condition that appears in the seminal paper by Blackwell and Dubins (1962) and was introduced to the study of learning in repeated games by Kalai and Lehrer (1993). Blackwell and Dubins showed that compatible beliefs "merge," in the sense of generating the same predictions in the limit as data increase.[21]

Compatibility is a common requirement in heterogeneous belief models. It has the interpretation that, while beliefs may initially disagree, their disagreement must eventually vanish. Incompatible beliefs must continue to disagree even in the limit, with infinite data. In Section VI.A, we elaborate on the appeal and limitations of compatibility.

DEFINITION 3. A stationary belief P *precludes disagreement* if for every stationary belief Q compatible with P, we have $Q = P$.

As the name suggests, such a process P leaves no room for disagreement with a compatible belief. The following stylized example illustrates the mathematical structure of the definition.

EXAMPLE 2. Consider again the belief $P = \mu_{1/2}$ in example 1, where outcomes are generated by the independent tossing of a fair coin. This

[19] There are motives for trading under common priors (unanticipated liquidity and rebalancing needs). Hong and Stein (2007) suggest that these motives are far too small to explain the \$51 trillion volume of trade in equity in 2005, say.

[20] Define $Q(A) \equiv \int_{\Delta(H)} P(A)\mathcal{Q}(dP)$. Note that since the set of stationary distributions is closed and convex, if a belief $\mathcal{Q}$ is concentrated on stationary distributions, then its reduction Q is also stationary.

[21] See Blackwell and Dubins (1962) or Kalai and Lehrer (1993) for a formal statement of this well-known result.

P precludes disagreement: every belief Q that is compatible with P must assign probability one to the sequences of observations in which each block of length k appears with long-run frequency $1/2^k$. The only stationary belief Q that satisfies this condition is $Q = P$.

Consider next $P = 1/2\mu_{2/3} + 1/2\mu_{1/3}$ introduced in example 1. This belief does not preclude disagreement: If $Q = 3/4\mu_{2/3} + 1/4\mu_{1/3}$, then P and Q are compatible and represent the beliefs of two agents who will disagree forever about how the initial choice of the coin was made.

B. *Structural Uncertainty*

An intuitive distinction can be made between situations in which agents understand the structure of their environment and situations of "structural uncertainty" in which the long-run fundamentals are unknown. How should such uncertainty be defined? Uncertainty about fundamentals suggests that an agent does not know all the long-run properties of the process and that, consequently, additional learning is possible. As a simple example, suppose that there are just two candidate processes, P_1 and P_2, each of which is as in examples 1 and 2. For an agent who knows that the true process is P_1, say, no number of new observations can change his beliefs about the probability of future outcomes. In this case, there is no structural uncertainty, and new information has no predictive value. On the other hand, it is intuitive to think of an agent who believes that the true process is P_1 with probability α and P_2 with probability $1 - \alpha$ as someone who is uncertain about the structure. For such an agent, additional observations will potentially change his predictions.

This simple intuition breaks down in more general settings. Suppose that P_1 and P_2 are, instead, two non-i.i.d. Markov processes. If $\alpha \in (0, 1)$, the agent is uncertain about the true process and information is as valuable as before. The problem is that information is valuable even if the agent knows that the true process is P_1, say. The reason is that with a Markovian process, the outcome of one period is informative about outcomes of future periods even when the process is known.

Intuitively, "structural uncertainty" suggests a lack of knowledge about the long-run properties of the process rather than outcomes in the "near" future. In the Markovian example above, for an agent who knows P_1, additional observations are informative about the near future but have no impact on his beliefs about the probability of outcomes in the distant future.

To make this distinction formal, fix a function $f: S^k \to \mathbb{R}^m$, with finite k, and define $f_n \equiv f(s_{n-k+1}, \ldots, s_n)$, $n = k, k + 1, \ldots$. We can think of f_k, $f_{k+1}, \ldots$ as a payoff stream in which, in each period n, a payment f_n is made that depends on the realization of the past k outcomes according to the (stationary) formula f.[22] For example, f_k may represent the history-dependent dividend paid by an asset in period k and $f_k, f_{k+1}, \ldots$ is the stream of such payments. Another example is that f takes values in a finite set of actions $\{a_1, \ldots, a_m\}$ in some vector space and represents the actions of an opponent who uses some Markovian strategy.

For a sequence $(x_k, x_{k+1}, \ldots)$ of elements in $\mathbb{R}^m$, define the limiting average:

$$V(x_k,\ x_{k+1}, \ldots) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=k}^{k+n-1} x_i$$

whenever the limit exists. We shall abuse notation and refer to $f_k, f_{k+1}, \ldots$ as f. Let $V(f)$ be the random vector whose value $V(f)(h)$ at a history h is the limiting average of $f_k, f_{k+1}, \ldots$ generated by f at h. This random variable is well defined with probability one for any stationary P and function f (proposition A1).

As an illustration, consider the dynamic game setting of Section II.D with I players and strategy profiles A. Let $\Delta(A)$ denote the set of probability distributions on A viewed as a subset of $\mathbb{R}^A$ and let $f: A \to \Delta(A)$ be given by $f(a) = \delta_a$ for every $a \in A$. Then $V(f)(h)$ would simply represent the limiting empirical frequency of actions played along a history h. More complex dynamic features of a history h can be captured with suitable choices of f.

Definition 4. P displays no *structural uncertainty* if for every function $f: S^k\mathbb{R}^m$ and finite history h^{t-1} with $P(h^{t-1}) > 0$,

$$E_{P(\cdot|h^{t-1})} V(f) = E_P V(f).$$

When f represents payoffs, the limit of averages criterion can be replaced with discounted payoffs, provided that the discount increases to one. The limit of averages criterion simplifies the statement of the property and the results.

C. *Empirical Identification*

In empirical studies, the implications of an economic model are usually summarized by *moment conditions.*

Definition 5. A *moment condition* is a bounded continuous function

$$f : Z \times S^k \to \mathbb{R}^q, \tag{2}$$

where k and q are positive integers, and $Z \subset \mathbb{R}^m$ is a compact set. We say that f *identifies* P if there is a unique $\bar{z} = \bar{z}(P, f)$ such that

[22] To allow history dependence of the payoff stream, payments start in period k.

$$E_P f(\bar{z}, s_1, \ldots, s_k) = 0.$$

The identification property in definition 5 is assumption (iii) in theorem 2.1 in Hansen (1982).

EXAMPLE 3. Let $S = \{0, 1\}$, $Z = [0, 1]$, and $k = q = 1$. Let $f : Z \times S \to \mathbb{R}$ be the moment condition $f(z, s) \equiv z - s$. For every stationary distribution P, we have $E_P f(z, s) = 0$ precisely at $\bar{z} = E_P s_n$, the expectation of s_n under P (which does not depend on n because of stationarity).

In our terminology, this moment condition identifies P. More accurately, it identifies some feature of P, in this case the mean. Intuitively, a moment condition represents a finite-horizon feature of P. The moment condition in example 3 does not capture intertemporal aspects of P. For example, if P_1 is i.i.d. 0.5 while P_2 is the process that puts equal weights on 0, 1, 0, . . . and 1, 0, 1, . . . , then these two processes are indistinguishable from the perspective of the moment condition f since $E_{P_1} f(z, s) = E_{P_2} f(z, s) = 0$ at $\bar{z} = 1/2$.

Moment conditions underlie many statistical techniques used in econometric practice, such as the GMM, introduced in Hansen (1982), which generalizes many standard techniques. The idea is to estimate the true $\bar{z}$ by the element $\hat{z} \in Z$ that minimizes the empirical estimate of $E_P f$. Formally, given f and length of data $n \geq k$, define the empirical average

$$F_n(z, h) = \frac{1}{n - k} \sum_{t=k+1}^{n} f(z, s_{t-k}, \ldots, s_t),$$

viewed as a $q \times 1$ column vector, with transpose denoted $F_n^{\top}$. The GMM estimator is the random variable[23]

$$\hat{z}_n(h) \in \arg\min_{z \in Z} F_n(z, h)^{\top} F_n(z, h). \tag{3}$$

DEFINITION 6. A stationary distribution P can be *empirically identified* if $\hat{z}_n \xrightarrow{P} \bar{z}(f, P)$ for every moment condition that identifies P.[24]

The property that P can be empirically identified from data means that all of the implications of P can be recovered, via moment conditions, from observing the evolution of the process. Alternatively, if P' cannot be empirically identified, then there must be an implication of P', in the sense that $E_{P'} f(\bar{z}, s_1, \ldots, s_k) = 0$ such that $\bar{z}$ cannot be recovered, even asymptotically as data grow without bound.

EXAMPLE 4. Consider again the moment condition $f(z, s) \equiv z - s$ in example 3, which identifies the mean of every stationary P. Let $P =$

[23] The GMM estimator looks at the quadratic form $F(z, h)^{\top} K F(z, h)$, where K is an appropriately chosen $q \times q$ matrix. Hansen (1982) shows how to select K optimally. We focus here on the baseline case in which K is the identity for simplicity.

[24] That is, $P\{h : |\hat{z}_n(h) - \bar{z}(f, P)| > \alpha\} \to 0$ for every α as $n \to \infty$.

$1/2\mu_{1/3} + 1/2\mu_{2/3}$, where μ_p is the belief that outcomes are i.i.d. p. Then $\bar{z}(f, P) = 1/2$. However, the GMM estimator will not converge to this $\bar{z}$, but to a random variable that gets the values 1/3 and 2/3 with probability one-half.

The term "data-generating process" is sometimes used to describe a stochastic process that is estimated from the data. For empirically identified beliefs P, the process that is estimated from the data is P itself. In example 4 what we will observe in the data is not P itself but the true data-generating process, which is either $\mu_{1/3}$ or $\mu_{2/3}$.

IV. Characterization of Testable Beliefs

A. Main Result

Our main result relates the four concepts introduced earlier.

THEOREM 1. For any stationary process P, the following four statements are equivalent:

1. P is testable;
2. P precludes disagreement;
3. P precludes structural uncertainty;
4. P can be empirically identified.

Theorem 1 highlights the tension between desirable properties. Testability is compelling conceptually as a minimal condition for equilibrium, while moments-based empirical methods are important for linking models to data. These reasons explain in part why the processes used in dynamic stochastic equilibrium models usually fall into the testable category. The theorem says that a commitment to testable models makes it difficult to incorporate disagreement and structural uncertainty. As discussed in the introduction, empirical and theoretical work increasingly point to disagreement and structural uncertainty as important sources of observed anomalies.

B. Asset Pricing and MPE Revisited

Many familiar models in the literature correspond to the testable case.

1. Irreducible and Hidden Markov Models

Our stylized examples 1 and 2 emphasized i.i.d. processes for their simplicity. More substantive economic models usually build on distributions with complex intertemporal structures. We begin by considering classes of processes that are important in applications.

The following are standard definitions. A transition function $\pi : S \to \Delta(S)$ is *irreducible* if for every s', $s \in S$, there exists some n such that $\pi^n(s'|s) > 0$, where π^n is the nth power of π. Every irreducible π defines a unique invariant distribution $p \in \Delta(S)$ such that

$$p(t) = \sum_s p(s)\pi(t \mid s).$$

We say that P is a (stationary, irreducible) Markov process if for every n,

$$P(s_0, s_1, \dots, s_n) = \pi(s_n \mid s_{n-1}) \cdots \pi(s_1 \mid s_0)p(s_0).$$

The definition is standard and extends naturally to processes with memory k. We use memory k Markov processes below without formally defining them. These models often appear in the form of vector autoregressions and can display complex intertemporal correlations.[25]

Another important class of processes is *hidden Markov models*. These processes model situations in which we observe a function $f : X \to X'$ of the underlying unobservable Markov process. The distribution induced over H is stationary but not necessarily Markovian of finite memory. Hansen (2007) proposes the hidden Markov model as a framework for uncertainty and learning in asset pricing.

The following fact will be useful in discussing the applications below.

PROPOSITION 1. The conditions of theorem 1 are satisfied for every irreducible memory k Markov process and every hidden Markov process in which the underlying Markov process is irreducible.

2. Asset Pricing

Assets in the consumption-based asset pricing model are priced according to the Euler equation $E_P[m_{n+1}R_{n+1} \mid h^n] = 1$, where m_{n+1} is the stochastic discount factor defined in (1) and evolving under a stationary distribution P.

One approach to model asset markets is to assume that the primitives, such as consumption, follow an exogenous stochastic structure of a specific form. In the classic Lucas asset pricing model, dividends, and therefore consumption, follow a Markov process. More generally, processes with a memory k Markov structure are testable by proposition 1. The theorem implies that an asset pricing model with testable $\{m_n\}$ must preclude disagreement and structural uncertainty.

An alternative approach is to impose no exogenous stochastic structure, but to estimate it from the data. A standard empirical practice is to

[25] Autoregressive processes assume a continuous state space while we assume that S is finite. See our comment on this issue in fn. 10.

estimate moment conditions in nonlinear models using GMM. Suppose that we are given a stationary model that is empirically identified for every moment condition, in the sense of definition 6. We can again apply the theorem to conclude that the proposed model must preclude disagreement and structural uncertainty.

3. Markov Perfect Equilibrium

Consider next the MPE model introduced in Section II.D. The model generates equilibrium distributions that are irreducible Markov processes whenever the transition function has full support. By proposition 1 and theorem 1, the model precludes disagreement and structural uncertainty. Knowledge of the long-run fundamentals (as we define them) is a strong assumption in an industry in which firms expect (or fear) disruptive regulatory, technological, or demand changes. In such environments, firms are likely to believe that these changes could fundamentally affect profitability but may disagree on the nature of this impact.

C. *Stationarity and the Need for Structure*

A key restriction we impose is that processes must be stationary. An obvious motivation for stationarity is the role it plays in economic and statistical models. Here we offer more subtle motivation: the absence of stationarity may lead to significant conceptual difficulties in defining what it means to test beliefs.

The next proposition considers what happens if we remove stationarity completely from the definition of testability.

PROPOSITION 2. Extend definition 1 by removing the stationarity assumption on P and Q. Then a distribution P is testable under this extended definition if and only if it is a Dirac measure that puts unit mass on a single sample path.

Proof. Dirac's atomic measures on realizations are the extreme point of $\Delta(H)$. Therefore, if P is not Dirac's atomic measure, then it can be written as $P = \lambda P' + (1 - \lambda)P''$ for some beliefs P, P'' such that $P' \neq P''$ and some $0 < \lambda < 1$. Therefore, every type I error-free test must have the property that $P'(T_P) = P''(T_P) = 1$. If $Q = \mu P' + (1 - \mu)P''$ for some $0 < \mu < 1$ and $\mu \neq \lambda$, then $Q \neq P$ but

$$Q(T_P) = \mu P'(T_P) + (1 - \mu)P''(T_P) = 1$$

for every such test. QED

If no restriction is imposed on the set of processes, then by proposition 2, requiring beliefs to be testable implies that they must be deterministic. This is obviously unsatisfactory in modeling phenomena that are stochastic in nature, such as asset markets and games. Models with nontrivial testable beliefs require some structure to restrict the processes allowed.

Stationarity is one way to introduce such structure, but it is obviously not the only one. The question of the "right" stochastic structure in economic and statistical models is subtle and lies outside the scope of this paper. Jackson, Kalai, and Smorodinsky (1999) propose that a stochastic structure should be coarse enough to be learnable but fine enough so that no learnable pattern remains unlearned in the long run. They point out that the class of all distributions is too fine to be learnable, so a more restrictive structure is needed. In our companion paper (Al-Najjar and Shmaya 2012), we show that the class of stationary distributions is learnable, providing some foundation for its use in economic models. We refer the reader to these papers for elaboration on these points.

V. Learning and Equilibrium

The main theorem divides stationary models into testable and nontestable ones. The theorem implies that in a nontestable process there is always something to learn about the long-run properties of their environment. As agents learn, their beliefs about these properties change, possibly leading to nonstationary decisions. This, in turn, undermines the stationarity of the process itself.

This informal intuition suggests a tension between stationarity and optimization. To better understand this tension, it will be convenient to assume, as in the MPE model of Section II.D, that S can be written as a product $S = X \times A$ of action profiles A and exogenous variables X that are not affected by these actions.

The evolution of the system is described by a stochastic process P on $H = (X \times A)^\infty$. Let P_X and P_A denote the marginals of P on X^∞ and A^∞. We will require only P_X to be stationary (rather than the entire process P). Given this weaker assumption, we can apply theorem 1 to beliefs about the exogenous variables, making it possible to explore the implications of testability even when agents' behavior is nonstationary.

A. *Passive Learning*

Consider a variation on the MPE setting of Section II.D in which agents are uncertain about the transition π. Pakes and Ericson (1998) study a model along these lines to empirically test the implications of passive learning on industry dynamics.

We follow closely that section except that agents may be uncertain about the transition. Formally, agents have beliefs μ_i, $i \in I$, with a finite and common support $\Pi = \{\pi_1, \dots, \pi_L\}$, $L \geq 1$, representing their uncertainty about the transition. Agents learn about the process from data, but learning is passive in the sense that their decisions do not influence the evolution of the exogenous variables. Formally, we require that each $\pi_l \in \Pi$ takes the form $\pi_l(x_n)$ and so depends only on x_n.

Let $P_X(\pi_l)$ be the stationary distribution on X^∞ induced by the transition $\pi_l \in \Pi$ as described in Section IV.B.1. Agent i believes that the exogenous variables evolve according to the stationary distribution

$$P_X(\mu_i) = \mu_i(\pi_1)P_X(\pi_1) + \cdots + \mu_i(\pi_l)P_X(\pi_l),$$

which no agent can influence. We distinguish three cases.

Case 1. $L = 1$, so agents know the true transition.

Case 2. $L > 1$ and agents share a common prior ($\mu_i = \mu$ for all i).

Case 3. $L > 1$ and agents have different priors ($\mu_i \neq \mu_j$ for some i, j).

Case 1 is a special case of the MPE introduced earlier.[26] The marginal process P_X is Markovian and testable and displays no structural uncertainty. The same holds for the full equilibrium process P under an MPE.

Turning to case 2, we model decisions using Bayesian Nash equilibrium as a solution concept. Fix one such equilibrium and let P denote the distribution it generates. In any such equilibrium, the marginal process P_X on exogenous variables is the stationary process $P_X(\mu)$. Applying theorem 1 to P_X, it is easy to verify that it is nontestable. For instance, agents are uncertain about the long-run ergodic distribution, so condition 3 in the theorem fails.

The fact that the marginal process P_X is stationary and nontestable makes it "likely" that the equilibrium process P as a whole is not stationary. Intuitively, when agents learn about the long-run fundamentals (the true π), their beliefs and decisions will change in a nonstationary fashion. The next result formalizes this intuition.

Proposition 3. Suppose that agents have a common prior μ with support $\{\pi_1, \dots, \pi_L\}$, $L > 1$. Then there exist action sets and utilities, A^i, u^i, for $i = 1, \dots, I$, such that the resulting game admits no Markov equilibrium profile.

When players know the Markov transition, their beliefs about the next state depend only on their current state. Under uncertainty, however, the current state is no longer sufficient to determine agents' beliefs

[26] Special because we require that agents' actions do not influence the state.

about the next outcome, and behavior may depend on the entire history of past observations. Early observations can have a persistent effect on agents' decisions. (Making this argument formal requires that changes in beliefs have an impact on actions. This is why the theorem is stated in terms of some specification of payoffs.) Pakes and Ericson (1998) exploit this fact to construct a statistical test of whether observed industry dynamics is consistent with firms learning about their environment.

We conclude by noting a connection between testability and disagreement. In case 2, the process P_X is nontestable yet agents agree on it. As argued earlier, justifying agreement that μ is the true process is difficult when there can be no statistical test that, even in principle and with infinite data, can disprove μ against an alternative belief $\mu' \neq \mu$. If one views testability as a (necessary) condition for agreement, then case 2 seems artificial, and case 3, where agents disagree, is the more compelling alternative.

B. *Bayesian Nash Equilibrium*

Next we consider learning in a repeated game with incomplete information using the setup of Kalai and Lehrer (1993). For simplicity, assume that there are just two players, each with a finite number of actions. The type of a player is his payoff matrix. The set of types of player i is denoted by $T^i = \{D^i_1, \ldots, D^i_m\}$.[27] A type profile is realized from common prior distribution μ on $T^1 \times T^2$. Type profiles are drawn and each player observes his own type. The standard solution concept for these games is that of a Bayesian Nash equilibrium. Kalai and Lehrer's theorem 2.1 shows, roughly, that conditional on (D^1, D^2) being the true type profile, the play of the game will eventually be close to that of a Nash equilibrium of the true game.

We consider an outside analyst who observes the actions of the players but not the realized types. We illustrate the relationship between testability, stationarity, and learning in three examples chosen for their tractability. In example 5 types are completely correlated, so players have no uncertainty.

Example 5. Assume that μ is uniform on $\{(D^1_j, D^2_j)\}_{j=1,\ldots,m}$. Then players know the type profile, but an outside observer knows only that the profile is drawn according to μ. Consider the following Bayesian Nash equilibrium: when the type profile is (D^1_j, D^2_j), players repeatedly play some Nash equilibrium of that game.

The process induced by the equilibrium profile in this example is stationary (in fact, exchangeable) but will typically not be testable. The

[27] Note that the number of possible types, m, is the same for both players for simplicity.

reason is that there will typically be uncertainty about the long-run distribution of play unless the support of μ is concentrated on type profiles with the same Nash equilibrium.

In the next example, players are uncertain about the game, so they learn something about their opponents' type as they observe the evolution of play. However, the payoffs are such that learning does not affect the players' behavior. Say that a type D of player i is committed to action a if playing a each period is a dominant strategy for player i in the repeated game when his payoff matrix is D.

Example 6. Assume that μ has full support over $T^1 \times T^2$ and that all types are committed (to different actions). In a Bayesian Nash equilibrium, each player always plays his commitment strategy.

The equilibrium profile in example 6 is, trivially, stationary, but it is not testable when some types of a player are committed to different actions.

In our final example, players have uncertainty, and the example is such that learning must affect behavior. In this case, no Bayesian Nash equilibrium induces a stationary play.

Example 7. Assume the following:

1. μ has full support over $T^1 \times T^2$.
2. There exist types D, $D' \in T^1$ of player 1 that are committed to different actions a, a'.
3. There exists type D'' of player 2 such that the set $B(a)$ of best responses of player 2 under D'' to a and the $B(a')$ of best responses of player 2 under D'' to a' are disjoint.

We have the following result.

Proposition 4. A Bayesian Nash equilibrium of the game described in example 7 cannot be stationary.

The proof is in the Appendix. Roughly speaking, this follows from the fact that in every equilibrium in the game with payoff matrices (D, D''), the players' action profile is in $\{a\} \times B(a)$ and that in every equilibrium in the game with payoff matrices (D, D''), the players' action profile is in $\{a'\} \times B(a')$. Kalai and Lehrer's theorem implies that players learn to play an equilibrium of the game with the realized types, so there is a positive probability that from some period onward the action profile played is always in $\{a\} \times B(a)$ and a positive probability that from some period onward the action profile played is always in $\{a'\} \times B(a')$. Because of stationarity, this should be the case already for the actions at period 0, and because these actions are independent given the types, it follows that at day 0 there is a positive probability that the action will be in $\{a\} \times B(a')$, which is a contradiction.

C. Testing Beliefs versus Testing Behavior

Disagreement and structural uncertainty can have observable implications on agents' behavior and market outcomes. This is not in contradiction with our main theorem. That theorem concerns the difficulties associated with objectively testing whether beliefs are right or wrong. The fact that it is difficult to test whether an agent "knows the true" model makes it more likely that his choices will reflect his model uncertainty.

To make this more formal, apply our main theorem to the process P_X governing the evolution of the exogenous variables as discussed earlier. If P_X is not testable, then the theorem implies that agents have structural uncertainty. While it is not possible to test whether beliefs about P_X are correct, agents' uncertainty about it can have empirically measurable implications on their behavior.

The details will obviously depend on the model. Pakes and Ericson (1998) develop econometric tests in a Markovian model of industry dynamics to detect firms' learning. Firms' beliefs are not directly observable. On the other hand, subjective uncertainty has the implication that early observations have a persistent impact on posterior beliefs and, therefore, on future choices. Evidence for learning may be found by looking for nonstationary behavior, such as a correlation between early observations and the long-run evolution of the model.

Another example can be found in the context of asset pricing. Brav and Heaton (2002) and Lewellen and Shanken (2002) consider the impact of investors' learning on asset pricing. Using stylized settings in which dividends are i.i.d. with unknown parameters, they show that learning can cause returns to appear predictable and excessively volatile even though prices react efficiently to information. These authors note that learning may confound empirical testing of asset pricing models.

VI. Discussion and Related Literature

A. Compatibility and Disagreement

The requirement of compatibility of beliefs is essential for the parts of our theorem related to disagreement. Although this assumption is common in models of heterogeneous beliefs, it is not universal. For instance, compatibility is central to Kalai and Lehrer's (1993) model of learning in games, but beliefs are not compatible in Harrison and Kreps's (1978) classic example of speculative trading. Morris (1996) provides a model of speculative trade with compatible beliefs. Given these modeling differences, a more detailed discussion seems warranted.

When beliefs are not compatible, it is possible that agents never reach agreement even in the limit, after being exposed to identical unbounded amounts of data. In Harrison and Kreps's (1978) speculative trade example, two classes of risk-neutral investors differ in their beliefs about future dividends. The belief of each class is represented by a transition matrix with commonly known coefficients. As Harrison and Kreps note, "each class [of investors] is positive that it knows the actual transition matrix, so that no amount of transition frequency data will alter their assessments" (327).

Many may feel that this dramatic failure of learning is artificial. We make this precise by showing that compatibility obtains if initial beliefs are slightly perturbed. Suppose that we are given two (possibly noncompatible) beliefs $P_i \in \Delta(H)$, $\alpha_i \in (0, 1)$, $i = 1, 2$. For example, consider the beliefs P_1, P_2 corresponding to the distinct transition matrices in the Harrison and Kreps example. These beliefs are mutually singular and thus fail compatibility in a dramatic way: there are two disjoint events A_1, A_2 such that $P_i(A_i) = 1$ and $P_i(A_j) = 0$ for $i \neq j$.

Starting with P_1, P_2 as initial beliefs, suppose that we require each individual to entertain a small doubt that the other's theory may be correct. Beliefs after this perturbation are denoted by $\mu_1, \mu_2 \in \Delta(H)$ and are required to satisfy, for some $\alpha_1, \alpha_2 \in (0, 1)$,

$$\mu_i = \alpha_i P_i + (1 - \alpha_i)\mu_j, \quad i \neq j.$$

When α_1, α_2 are close to one, each individual is nearly convinced that P_i is the true process but is willing to concede that there is a small chance the other is right.[28]

The new beliefs μ_1, μ_2 defined above are obviously compatible. This suggests that the restriction to compatible beliefs in definition 3 is not as strong as it might initially appear: any pair of beliefs can be slightly perturbed to make them compatible.[29]

B. *Testing Strategic Experts*

A recent literature examines the problem of testing an expert's knowledge of the underlying process. An uninformed strategic expert may

[28] It is easy to verify that such perturbations are consistent: substituting in μ_j, we have

$$\mu_i = \alpha_i P_i + (1 - \alpha_i)[\alpha_j P_j + (1 - \alpha_j)\mu_i].$$

Simplifying, we obtain

$$\mu_i = \frac{\alpha_i P_i + \alpha_j(1 - \alpha_i)P_j}{1 - (1 - \alpha_i)(1 - \alpha_j)}.$$

[29] Note that this does not say that a belief can be perturbed to make it simultaneously compatible with all other stationary beliefs.

attempt to manipulate the test with a randomized forecast. A number of papers, including Sandroni (2003), Olszewski and Sandroni (2008), and Shmaya (2008), provide general conditions under which a strategic expert can manipulate any test. Results establishing the existence of nonmanipulable tests appear in Dekel and Feinberg (2006), Shmaya (2008), Olszewski and Sandroni (2009), and Al-Najjar et al. (2010), among others. See Olszewski (2015) for a survey.

In our paper, agents do not strategically randomize their forecasts to manipulate the test. Rather, they hold subjective beliefs about their environment and choose optimally given these beliefs. Our focus is on the implications of requiring that erroneous beliefs can be rejected by data rather than on agents trying to defeat statistical tests.

C. *Bayesian Learning in Games*

Section V.B illustrates the connection with the literature on learning in games. The concepts of testability, structural uncertainty, and stationarity are related to Nachbar's (2005) results for learning in games. In his motivating example (459–60), two players play an infinitely repeated game, where the stage game has just two actions A, B for each player. Player $i = 1, 2$ believes that player $j \neq i$ follows an i.i.d. strategy in which A is played with probability q_i each period. Player i's prior belief about the strategy of the opponent is a probability distribution μ_i over $[0, 1]$. Nachbar's starting point is the observation that even ϵ-optimization will lead to the unraveling of the assumed stationary equilibrium.

In our terminology, each player faces a stationary environment described by his belief about the strategy of his opponent. In this example, there is structural uncertainty if and only if μ does not put unit mass on one value of q_i. Proposition 3 shows that when players update their beliefs, their behavior will not in general be i.i.d. and is therefore not consistent with the players' priors about their opponents' strategies.

Nachbar obtains a general result for infinitely repeated game strategies in which the assumption of stationarity is not natural. Our arguments, by contrast, rely heavily on stationarity. The concepts of testability and empirical identification also have no counterpart in Nachbar's work.

D. *Self-Confirming Equilibrium*

The process P in this paper describes the evolution of all variables in the model, including agents' decisions. Beliefs are in equilibrium if their predictions about future outcomes coincide with P. In our analysis, the process P puts no restrictions on agents' predictions at counterfactual events, that is, events that have zero probability under P. This is important in games in which players' decisions depend on their beliefs about what their opponents will do at all events, including those not observed in equilibrium. So, although the process describes everything that an outside observer can hope to see, P may be insufficient to explain what players do. This distinction is captured by the notion of self-confirming equilibrium introduced by Fudenberg and Levine (1993) in which players are assumed to know the true distribution on observed outcomes but may have incorrect off-path conjectures.

The concept of self-confirming equilibrium is often motivated as the outcome of a steady-state learning process. Related to our discussion of disagreement, Dekel, Fudenberg, and Levine (2004) argue that this steady-state interpretation of self-confirming equilibrium is difficult to reconcile with heterogeneous beliefs about the state of nature.

VII. Concluding Remarks

The argument in this paper can be recast in terms of two traditions of reasoning about uncertainty in economics. The subjectivist tradition, using as a foundation Savage's theory of subjective expected utility, postulates a framework in which there is no objective "data-generating process." Beliefs of economic agents are subjective "states of mind," with no limits on the extent to which they can disagree. The other tradition, associated with the idea of rational expectations, postulates an objective data-generating process about which all economic agents and outside observers agree.[30] Disagreement in such a framework is difficult to reconcile with rationality.

The ideal is an equilibrium theory that combines the strengths of the two traditions: a theory in which subjectivity (say, some forms of disagreement and structural uncertainty) coexists with objective equilibrium outcomes that can be tested and empirically identified. This paper contributes to understanding the power (and limitations) of both traditions by highlighting the trade-offs and challenges one is likely to face in combining them into a subjectivist equilibrium theory.

Appendix

A. *Mathematical Preliminaries*

A function $f: H \to \mathbb{R}$ is *finitely based* if there is an integer k such that for any history h, f depends only on the first k coordinates of h (i.e., f is $\mathcal{H}^k$-measurable). Abusing notation, we express finitely based functions in the form $f: S^k \to \mathbb{R}$. We use the following lemma.

[30] See the quotation from Sargent (2008) in fn. 6.

Lemma A1. For any pair of distributions $P, Q \in \Delta(H)$, $P \neq Q$, there exists a finitely based function f such that $E_P f \neq E_Q f$.

Given a stationary P and any (Borel) function $g: H \to \mathbb{R}$, the ergodic theorem states that the limit

$$\tilde{g} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(s_i,\ s_{i+1}, \ldots) \tag{A1}$$

exists P-a.s. (almost surely) and that $E_P\tilde{g} = E_P g$. If the limit $\tilde{g}$ is constant for every g (i.e., $\tilde{g} = E_P g$-a.s.), then P is called *ergodic*. It is sufficient for ergodicity that the limit be constant for every finitely based function g.[31]

The following proposition states the implication of the ergodic theorem for finitely based functions. In the terminology of Section III, it states that the (random) limiting average payoff of a finitely based f is well defined.

Proposition A1. Let P be stationary and $f: S^k \to \mathbb{R}$ be finitely based. Then

$$V(f)(h) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=k}^{k+n-1} f(s_{i-k}, \ldots, s_{i-1})$$

exists for P-a.e. (almost every) history $h = (s_0, s_1, \ldots)$, and $E_P V(f) = E_P f$. Moreover, if P is ergodic, then $V(f(h))$ is P-a.s. the constant $E_P f$.

The set of all stationary distributions P over H is convex and weak *-compact. We denote by $\mathcal{E} \subset \mathcal{P}$ the set of all ergodic distributions over H. The *ergodic decomposition theorem* states that there exists a Borel function $\varepsilon: H \to \mathcal{E}$ such that $\mu(\varepsilon^{-1}(\mu)) = 1$ for every ergodic μ and such that

$$P(E) = \int \mu(E)\bar{P}(d\mu) \tag{A2}$$

for every event $E \subseteq H$, where $\bar{P} \in \Delta(\mathcal{E})$ is the push-forward of P under ε (i.e., $\bar{P}(E) = P(\varepsilon^{-1}(E))$ for every Borel subset E of $\mathcal{E}$). It follows from the ergodic decomposition theorem that the extreme points of this set are the ergodic distributions. That is, P is ergodic if and only if there exist no stationary distributions R', R'' such that $R' \neq R''$ and $P = \lambda R' + (1-\lambda)R''$ for some $0 < \lambda < 1$.

B. *Proof of Theorem 1*

We will show that the properties in the theorem are equivalent to P being ergodic.

1. P Is Ergodic $\Rightarrow$ P Displays No Structural Uncertainty

Assume that P is ergodic and let $f: S^k \to \mathbb{R}$ be bounded. By proposition A1, $V(f)$ is a.s. the constant $E_P V(f)$. It follows that $E_P(V(f)|E) = E_P V(f)$ for every event E with $P(E) > 0$. In particular, $E_{P(\cdot|h^{t-1})} V(f) = E_P V(f)$ for every finite history h^{t-1} with $P(h^{t-1}) > 0$.

[31] For a textbook exposition of the parts of ergodic theory relevant to this paper, see Gray (2009).

2. P Displays No Structural Uncertainty ⇒ P Is Testable

Assume that P displays no structural uncertainty. Let $f: S^k \to \mathbb{R}$ be bounded. From the martingale convergence theorem it follows that

$$E_{P(\cdot|h^{t-1})} V(f) \underset{t\to\infty}{\to} V(f)(h)$$

for P-a.e. h. If P displays no structural uncertainty, then

$$E_{P(\cdot|h^{t-1})} V(f) = E_P V(f) = E_P f,$$

where the second equality follows from proposition A1. Therefore, $V(f)(h) = E_P f$ for P-a.e. h.

Let T_f be a test such that $T_f(P) = \{h \mid V(f)(h) = E_P f\}$ and $T_f(Q) = \Omega$ for every stationary $Q \neq P$. By the argument above it follows that T is type I error-free. Moreover, if $Q(T_f(P)) = 1$ for some stationary distribution Q, then $V(f)(h) = E_P f$ for Q-a.e. h, and therefore, $E_Q f = E_Q V(f) = E_P f$, where the first equality follows from proposition A1.

If $Q \neq P$ is any stationary distribution, then by proposition A1 there exists a bounded function $f: S^k \to \mathbb{R}$ such that $E_P f \neq E_Q f$. Then it follows that T_f is type I error-free and that $Q(T_f(P)) < 1$, as desired.

3. P Is Testable ⇒ P Precludes Disagreement

Let P be testable and let Q be any stationary belief such that $Q \neq P$. We claim that Q is not compatible with P. Indeed, let T be a type I error-free test such that $Q(T(P)) < 1$. Since T is type I error-free, we get that $P(T(P)) = 1$. Therefore, P and Q are not compatible, as desired.

4. P Precludes Disagreement ⇒ P Is Ergodic

Assume by contradiction that P is not ergodic so that $P = \lambda R' + (1 - \lambda)R''$ for some stationary beliefs $R' \neq R''$ and $0 < \lambda < 1$. Since $P(B) = \lambda R'(B) + (1 - \lambda)R''(B)$ for every event B, it follows that $P(B) = 1$ if and only if $R'(B) = R''(B) = 1$. Let $0 < \mu < 1$ such that $\mu \neq \lambda$ and let $Q = \mu R + (1 - \mu)R''$. Then $Q(B) = 1$ if and only if $R'(B) = R''(B) = 1$, and so $Q(B) = 1$ if and only if $P(B) = 1$. Thus P and Q are compatible, and since $R \neq R''$ and $\lambda \neq \mu$, it follows that $P \neq Q$, contradicting the assumption that P precludes disagreement.

5. P Is Ergodic ⇒ P Can Be Empirically Identified

Hansen (1982) shows that the consistency of the GMM estimator obtains for any ergodic P. For this argument, it is enough to consider $k = 1$; however, it is not in general possible to prove the converse without using higher k's. We begin by restating Hansen's result in our notation.

PROPOSITION A2. Assume that some ergodic distribution P is identified by a moment condition (2) with $k = 1$. Let $\hat{z}_n$ be the GMM estimator given by (3). Then $\hat{z}_n \xrightarrow{P} \bar{z}$.

We need to adapt this result to arbitrarily finite k. Let P be ergodic and let f be a moment condition given by (2):

$$f : Z \times S^k \to \mathbb{R}^q.$$

Consider the new set of outcomes $\bar{S} = S^k$. Define the probability distribution $\bar{P}$ over $\bar{H}$ as the push-forward of P under the map

$$(s_0,\ s_1,\ s_2, \ldots) \in H \mapsto (\bar{s}_0,\ \bar{s}_1, \ldots),$$

where

$$\bar{s}_n = (s_n,\ s_{n+1}, \ldots,\ s_{n+k-1}).$$

Then $\bar{P}$ is ergodic, the moment conditions f on P translate to a moment condition on $\bar{P}$ with $k = 1$, and the GMM estimator for $\bar{P}$ under $\bar{f}$ is the same as for P under f. Thus, by proposition A2, it follows that the GMM estimator is consistent.

6. P Can Be Empirically Identified $\Rightarrow$ P Is Ergodic

Let $P \in \mathcal{P}$ be stationary and empirically identified. Fix an integer k, and let $Z = \Delta(S^k)$. Define the moment condition $f : Z \times S^k \to \mathbb{R}^{S^k}$ by

$$f(z,\ h)(h') = z(h') - \delta_{h,h'}$$

for every $z \in \Delta(S^k)$ and $h' \in S^k$, where

$$\delta_{h,h'} = \begin{cases} 1 & \text{if } h = h' \\ 0 & \text{otherwise.} \end{cases}$$

.

If μ is a stationary distribution, then f identifies μ since the unique $\bar{z}$ that satisfies $E_\mu f(\bar{z}, h) = 0$ is given by $\bar{z} = \Pi^k(\mu)$, where $\Pi^k : \mathcal{P} \to \Delta(S^k)$ is such that $\Pi^k(\mu)$ is the distribution on k-tuples induced by μ for every stationary distribution μ.

It follows from Section B.5 that for every ergodic $\mu \in \mathcal{E}$,

$$\hat{z}_n \xrightarrow{\mu} \Pi^k(\mu) \tag{A3}$$

under μ, where z_n is the GMM estimator.

We claim that

$$\hat{z}_n \xrightarrow{P} \Pi^k \circ \varepsilon, \tag{A4}$$

where $\varepsilon : H \to \mathcal{E}$ is the ergodic decomposition function introduced earlier satisfying (A2). Indeed, for every $\alpha > 0$,

$$\begin{aligned} &P(\{\omega \in H \mid | \hat{z}_n(\omega) - \Pi^k \circ \varepsilon(\omega) \mid\ > \alpha\}) \\ = &\int \bar{P}(d\mu)\mu(\{\omega \in H \mid | \hat{z}_n(\omega) - \Pi^k(\mu) \mid\ > \alpha\}) \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

where the equality follows from the definition of ε and the limit from the bounded convergence theorem and (A3).

On the other hand,

$$\hat{z}_n \xrightarrow{P} \Pi^k(P) \tag{A5}$$

since P can be empirically identified. It follows from (A4) and (A5) that $\Pi^k \circ \varepsilon = \Pi^k(P)$, P-a.s. Since this is true for every k, it follows that ε is constant P-a.s., that is, that P is ergodic.

C. *Proof of Proposition 1*

We proved that the conditions of theorem 1 are satisfied for every ergodic process. Every irreducible Markov process is ergodic (Durrett 2010, example 7.1.7). Also, every function of an ergodic process is ergodic (Durrett's theorem 7.1.3). In particular, every hidden Markov process is ergodic when the underlying (unobservable) Markov process is irreducible.

D. *Proof of Proposition 3*

Since $\pi_1, \ldots, \pi_L$ are not identical, there exists some $s^* \in S$ such that $\pi_1(s^*) \neq \pi_2(s^*)$. Therefore, there exists some $u \in \mathbb{R}^S$ and $\alpha \in \mathbb{R}$ such that

$$\pi_1(s^*) \cdot u < \alpha < \pi_2(s^*) \cdot u,$$

where $\cdot$ is the inner product in $\mathbb{R}^S$.

Assume that at every period n each player has two actions: a safe action that gives payoff α regardless of the next state and a risky action that gives payoff $u(s_{n+1})$. Thus, a player's payoff depends only on her own action and not on the opponent's actions. In this game, every equilibrium strategy σ is such that after every history h the player chooses the safe action if the conditional expectation of $u(s_{n+1})$ is smaller than α and the risky action if the conditional expectation of $u(s_{n+1})$ is greater than α. Since after sufficiently long periods the player learns the ergodic transition π, it follows that there are arbitrary long histories $(s_0, s_1, \ldots, s_n)$ with $s_n = s^*$ after which the player must play the safe action and arbitrary long histories after which the player must play the risky action. Thus, the equilibrium cannot be Markovian.

E. *Proof of Proposition 4*

The proof uses the following lemma, which follows immediately from the ergodic theorem.

LEMMA A2. Let S be a finite set of outcomes, and let P be a stationary distribution over H. Then P-a.e. realization $\omega = (s_0, s_1, \ldots) \in H$ has the property that s_0 appears infinitely often in ω.

Proof. By the ergodic decomposition we can assume without loss of generality that P is ergodic. Let $p \in \Delta(S)$ be the marginal one-period distribution of P, that is, $p(s) = P(\{\omega = (s_0, s_1, \ldots) \mid s_0 = s\})$ for every $s \in S$. Then it follows from the ergodic theorem that, for P-a.e. realization ω, every $s \in S$ appears in ω with frequency $p(s)$. In particular, if $p(s) > 0$, then s appears infinitely often in ω. This implies that s_0 appears infinitely often in ω for almost every realization $\omega = (s_0, s_1, \ldots) \in H$, as desired. QED

To prove the proposition, let $\epsilon > 0$ be small enough such that the following properties hold:

1. The only subgame perfect ϵ-equilibrium in the repeated game with incomplete information (D', D'') is such that player 1 always plays a and player 2 always plays an action from $B(a)$.
2. The only subgame perfect ϵ-equilibrium in the repeated game with incomplete information (D', D'') is such that player 1 always plays a' and player 2 always plays an action from $B(a')$.

Let $\eta > 0$ be sufficiently small. It follows from Kalai and Lehrer's theorem that in every equilibrium of the game with incomplete information, if the types are (D, D''), there is a time from which, with probability at least $1 - \eta$, player 2 always plays an action from $B(a)$. It follows from lemma A2 that according to the equilibrium profile, conditioned on the types being (D, D''), player 2 plays an action in $B(a)$ with probability at least $1 - \eta$ (since almost every realization of the play path on which he does not play an action in $B(a)$ cannot have the property that he plays an action from $B(a)$ from some point onward). This implies that, according to his equilibrium strategy, player 2 must play an action from $B(a)$ with probability $1 - \eta$ at day 0 if his type is D''. Similarly, player 2 must play an action from $B(a')$ with probability $1 - \eta$ at day 0 if his type is D''. Since $B(a) \cap B(a'') = \varnothing$, we get a contradiction.

F. Finite Horizon Testing

The properties in theorem 1 are formulated in terms of infinite outcome sequences. Although the definitions have natural analogues for finite horizons, we find the asymptotic approach to provide a clearer picture of the implications of testability.

Call a test T *finite* if there exists an integer n such that T depends only on the outcome of the first n periods.[32] Tests used in statistical practice are obviously finite. It is easy to show the following.

Claim. If P is testable, then for every stationary Q and every $\alpha > 0$, there exists a finite horizon test T such that

1. the type I error of T is smaller than α;
2. $Q(T(P)) < 1 - \alpha$.

Formulating our result in a finite-horizon setting would require close attention to the amount of data available to the test, an issue that does not arise in asymptotic tests. For finite tests, the horizon needed will in general depend on the complexity of the intertemporal structure of the processes being tested. To illustrate, consider the following simple example.

[32] Formally, T is measurable with respect to $\mathcal{H}_n$.

Example 8. Let $S = \{0, 1\}$ and let P be the steady-state distribution of a Markov process with transition

$$\pi(0 \mid 0) = \pi(1 \mid 1) = 1 - \epsilon \text{ and}$$
$$\pi(1 \mid 0) = \pi(0 \mid 1) = \epsilon$$

for a small $\epsilon < 1/n$. Since P is an irreducible Markov chain, it follows from proposition 1 above that P is testable. Let Q be the process that puts unit mass on the sequence $(0, 0, \ldots)$. Then for every test T of horizon n and type I error smaller than $\alpha = 0.10$, we have $Q(T(P)) = 1$.

The problem in this example is that the process P is highly persistent. If we are given a very short horizon, this persistence can easily confound P with the constant process Q.

Testability with such a short horizon has little bite. This should not be surprising: no statistical technique based on such limited data can distinguish the process P from Q. Relatedly, the GMM estimator of the one-dimensional marginal of P will fail to converge to its correct value over such a small horizon.

In summary, finite data are a limiting factor to statistical methods in general, and not specifically our model. To apply the notion of testability in a finite horizon, either the class of processes has to be restricted or the horizon extended (or both). In our asymptotic testing framework, we impose no restrictions beyond stationarity but require the horizon to be infinite. Alternatively, we can limit n to be finite, but then we must restrict the class of processes to ensure a fast enough rate of convergence to make statistical testing feasible.[33]

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[33] A natural way to proceed is to require the process to be mixing with a uniform mixing rate. This, indeed, is the common assumption made in practice. Vector autoregressive processes, e.g., belong to this class. See also our discussion of Pakes and Ericson (1998).

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