

Capital Supply Uncertainty, Cash Holdings, and Investment

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We develop a dynamic model of investment, financing, and cash management decisions in which investment is lumpy and firms face capital supply uncertainty. We characterize optimal policies explicitly, demonstrate that smooth-pasting conditions may not guarantee optimality, and show that firms may not follow standard single barrier policies. In the model, firms with high investment costs differ in their behaviors from firms with low investment costs, financing policy does not follow a strict pecking order, and the optimal payout policy may feature several regions with both incremental and lumpy dividend payments. (*JEL* D83, G24, G31, G32, G35)

Following Modigliani and Miller (1958), standard models of investment under uncertainty assume that capital markets are frictionless so that firms are always able to secure funding for positive net present value projects and cash reserves are irrelevant.¹ This traditional view recently has been called into question by a large number of empirical studies.² These studies show that firms often face

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¹ See, for example, the early papers of McDonald and Siegel (1986), Dixit (1989) or the recent contributions of Manso (2008), Grenadier and Malenko (2010), or Carlson, Fisher, and Giammarino (2010). Dixit and Pindyck (1994) and Stokey (2009) provide excellent surveys of this literature.

² See, for example, Gan (2007), Becker (2007), Massa, Yasuda, and Zhang (2008), Duchin, Ozbas, and Sensoy (2010), Campello, Graham, and Harvey (2010), and Lemmon and Roberts (2010).

uncertainty regarding their future access to capital markets. They also reveal that the resulting liquidity risk leads firms to accumulate large cash balances, with an average cash-to-assets ratio for U.S. industrial firms that has increased from 10.5% in 1980 to 23.2% in 2006 (see Bates, Kahle, and Stulz 2009; BKS).

While it may be clear to most economists that capital supply frictions can affect corporate policies, it is much less clear exactly how they do so. In this paper, we develop a dynamic model of cash management, financing, and investment decisions in which the Modigliani and Miller assumption of infinitely elastic supply of capital is relaxed and firms have to search for investors when in need of funds. With this model, we seek to understand when and how capital supply uncertainty affects real investment. We are also interested in determining the effects of capital market frictions on firms' financing and cash management policies, that is, on the decision to pay out or retain earnings and the decision to issue securities.

To aid in the intuition of the model, consider the following two settings in which capital supply uncertainty and search frictions are likely to be especially important:³

1. **Public equity offerings and capital injections for private firms:** Firms first sell their equity to the public through an initial public offering (IPO). One of the main features of IPOs is the book-building process, whereby the lead underwriter and firm management search for investors until it is unlikely that the issue will fail. Yet, the risk of failure is often not eliminated and a number of IPOs are withdrawn every year. For example, Busaba, Benveniste, and Guo (2001) show that between the mid-1980s and mid-1990s almost one in five IPOs was withdrawn. Evidence from more recent periods suggests that this fraction has increased in some years to over one in two (see Dunbar and Foerster 2008). Search frictions are also important for firms that remain private but need new capital injections. Indeed, when a private company decides to raise equity capital, it must search for investors such as angel investors, venture capital firms, or institutional investors. Even when initial investors are found, the firm will need to search for new investors in every subsequent financing round.
2. **Financial crises and economic downturns:** Search frictions are also important for large, publicly traded firms when capital becomes scarce, for example during a financial crisis or an economic downturn. The recent global financial crisis has provided a crisp illustration of the potential effects of capital supply dry-ups on corporate behavior, with a number of studies (e.g., Duchin, Ozbas, and Sensoy 2010) documenting a large decline in investment following the onset of the crisis (controlling for time variation in investment opportunities). A survey of 1,050

³ We thank Darrell Duffie for suggesting these applications.

CFOs by Campello, Graham, and Harvey (2010) also indicates that the contraction in capital supply during the crisis led firms to burn through more cash to fund their operations and to bypass attractive projects.

To illustrate the effects of capital supply frictions on corporate behavior, we consider a firm with assets in place that generate stochastic cash flows and a finite number of opportunities to expand operations (i.e., growth options). In the model, the firm faces two types of frictions: capital supply frictions and lumpiness in investment. Notably, we consider that the firm has to search for investors when in need of capital. Therefore, it faces uncertainty regarding its ability to raise funds. In addition, the firm has to pay a lump-sum cost when investing. Indeed, as noted by Caballero and Engel (1999), minor upgrades and repairs aside, investment projects are intermittent and lumpy rather than smooth (see also Doms and Dunne 1998; Cooper, Haltiwanger, and Power 1999). The firm maximizes its value by making three interrelated decisions: how much cash to retain and pay out, when to invest, and whether to finance investment with internal or external funds. Using this model, we first show that capital supply frictions lead firms to value financial slack. Second, and more importantly, we demonstrate that the interplay between lumpiness in investment and capital supply frictions implies that barrier policies for investment, payout, and financing decisions may not be optimal. We then use these results to shed light on existing empirical facts and to generate a rich set of testable predictions.

To understand these effects, consider the case of a firm with assets in place but no growth option. If capital markets are frictionless, the firm can instantly raise as much capital as it wants and there is no need to safeguard against future liquidity needs by hoarding cash. With capital supply frictions, cash holdings serve to cover potential operating losses and, thus, avoid inefficient closure. Holding cash, however, is costly because of the lower return of liquid assets inside the firm. Based on this trade-off, the paper provides an explicit characterization of the value-maximizing payout and financing policies and shows that, absent growth options, optimal policies are always of barrier type. Specifically, we show that there exists a target level for cash holdings (the barrier) such that the optimal payout policy is to distribute dividends to maintain cash holdings at or below the target and to retain earnings and search for investors when cash holdings are below the target.

Next consider a firm with both assets in place and growth options. For such a firm, cash holdings generally serve two purposes: reducing the risk of inefficient closure and financing investment. Our analysis demonstrates that when the cost of investment is low (i.e., for growth firms), it is again optimal to follow a barrier strategy whereby the firm retains earnings and invests if cash reserves reach some target level or upon obtaining outside funds. We show, however, that when the cost of investment is high (i.e., for value firms), barrier strategies are no longer optimal. Notably, when cash reserves are high, the firm optimally retains earnings and invests if cash reserves reach some target level or upon obtaining

outside funds. However, if cash reserves fall to a critical level following a series of losses, the firm abandons the option of financing investment internally as it becomes too costly to accumulate enough cash to invest. At this point, the marginal value of cash drops to one and it is optimal to make a lump-sum payment to shareholders. After making this payment, the firm retains earnings again but finances investment exclusively with outside funds.

Our model therefore shows that, when capital supply is uncertain, the optimal policy choices of the firm are in stark contrast with the theoretical predictions of canonical inventory models. In particular, a striking feature of the model is that it may not be optimal for firms to follow the standard Miller and Orr (1966) barrier policy. We also demonstrate in the paper that the smooth-pasting conditions used to characterize optimal policies in prior contributions are necessary, but may not be sufficient, for an optimum in our model.

Building on these results, we show that when barrier policies are suboptimal, firms follow band strategies whose policy implications differ significantly from those of barrier strategies. Notably, firms may optimally raise funds before exhausting internal resources and the optimal payout policy may feature several payout regions, with both incremental and lumpy dividend payments. We also demonstrate that firms with low cash reserves do not finance investment internally and may decide to pay dividends early. By contrast, firms with high cash reserves may finance investment internally and optimally retain earnings. Therefore, in our model, investment and payout do not always increase with slack, challenging the use of investment-cash-flow sensitivities and payout ratios as measures of financing constraints.

A second set of implications relates to the use of inside and outside cash by financially constrained firms. We show that the choice between internal and external funds for financing investment does not follow a strict pecking order, in that any firm can use both internal and external funds to finance investment. We find however that firms usually wait until external financing arrives before investing, consistent with Opler et al. (1999; OPSW), BKS (2009), and Lins, Servaes, and Tufano (2010). We also find that the probability of investment with internal funds increases with asset tangibility and agency costs and decreases with cash-flow volatility and market depth, and that when financing investment with external funds, firms should increase their cash reserves, consistent with Kim and Weisbach (2008) and McLean (2011). Finally, we show that negative capital supply shocks should hamper investment even if firms have enough slack to finance investment, consistent with Gan (2007), Becker (2007), and Lemmon and Roberts (2010).

In the model, the suboptimality of barrier strategies is implied by a local convexity in the value function. We therefore investigate whether this convexity may lead to a gambling behavior, in which management engages in zero NPV investments with random returns in an attempt to improve firm value. We show that while it is theoretically possible to eliminate the convexity in firm value and restore the optimality of barrier strategies, this requires gambling strategies

that are infeasible or unrealistic. In our base-case environment, for example, eliminating band strategies requires taking positions in derivatives contracts that multiply cash-flow volatility by thirteen, a number inconsistent with the recent study of Duchin et al. (2014) on the composition of corporate cash reserves.

The present paper relates to several strands of literature. First, it relates to the literature on inventory models applied to liquid assets. Seminal contributions in this literature include Baumol (1952), Miller and Orr (1966), and Tobin (1968). Recent contributions include Anderson and Carverhill (2012), Décamps et al. (2011; DMRV), and Bolton, Chen, and Wang (2011; BCW). This literature generally assumes that liquid assets holdings have continuous paths and, therefore, does not allow for lumpy investment. Notable exceptions are Alvarez and Lippi (2009) and Bar-Ilan, Perry, and Stadje (2004).⁴ One important difference between our paper and these contributions is that jumps in our model correspond to endogenous investment or financing decisions. Another key difference is that these papers restrict their attention to barrier strategies. By contrast, our paper provides a complete characterization of optimal decision rules and shows that barrier strategies may not be optimal when investment is lumpy.

Second, our paper relates to the large literature on investment under uncertainty, in which it is generally assumed that firms can instantaneously tap capital markets at no cost to finance investment (see footnote 1 and the references therein). In these models, there is no role for cash holdings, investment is financed exclusively with outside funds, and firms may raise funds infinitely many times to cover temporary losses.

Third, our paper relates to the literature analyzing investment and financing decisions in the presence of financing constraints (see, e.g., DMRV 2011; Gryglewicz 2011). In this literature, firms can always access capital markets. Depending on whether the costs of external finance are high or low, firms either never raise funds or are never liquidated. In addition, when the cost of external finance is low, there is a strict inside/outside funds dominance in that firms only raise funds when their cash reserves are depleted. The papers that are most closely related to our analysis in this literature are those of BCW (2011, 2013). In these papers, the authors assume that investment is infinitely divisible and that the firm follows a barrier policy for investment and financing decisions. Our analysis considers instead an environment in which investment is lumpy and demonstrates that in this case the optimal policy may not take the form of a barrier policy. Another distinctive feature of our analysis is that in our paper firms raise external funds in discrete amounts on a regular basis and some firms can be liquidated even when issuance costs are low. Also, whereas firms can use

⁴ Another exception is the study of Décamps and Villeneuve (2007), who examine the dividend policy of a firm that owns a single growth option and has no access to outside funds.

both internal and external funds to finance investment in our model, investment is financed exclusively with internal funds in BCW.

Our model also shares some of its objectives with the discrete-time models of Riddick and Whited (2009), Eisfeldt and Muir (2013), and Falato, Kadhryzanova, and Sim (2013). These papers consider neoclassical investment models in which firms face adjustment costs that are proportional to firm size and derive implications for corporate behavior based on a numerical solution of the model. By contrast, our paper provides an analytical characterization of firms' policy choices when the cost of investment is fixed and investment is lumpy. Another difference is that our paper incorporates capital supply frictions and shows that these frictions can have first order effects on corporate behavior.

Lastly, our paper relates to the studies of DeMarzo, Livdan, and Tchistyi (2014) and Makarov and Plantin (Forthcoming), who investigate management's incentives to increase firm risk by engaging in gambling strategies.

1. Model and Assumptions

Throughout the paper, agents are risk neutral and discount cash flows at a constant rate $\rho > 0$. Time is continuous and uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with the filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions.

We consider a firm with assets in place and a growth option to expand operations.⁵ Before investment, assets in place generate a continuous stream of cash flows dX_t satisfying

$$dX_t = \mu_0 dt + \sigma dB_t,$$

where the process B_t is a Brownian motion and (μ_0, σ) are constant parameters representing the mean and volatility of cash flows. The growth option allows the firm to increase cash flows to $dX_t + (\mu_1 - \mu_0)dt$, where $\mu_1 > \mu_0$.⁶ The firm has full flexibility over the timing of investment but needs to pay a lump sum cost $K > 0$ upon investment (see, e.g., Doms and Dunne 1998, Cooper, Haltiwanger, and Power 1999, or Caballero and Engel 1999 for evidence on lumpy investment). Investment is irreversible.⁷

Management acts in the best interest of shareholders and chooses not only the firm's investment policy but also its financing, payout, and liquidation policies.

⁵ We show in the Supplementary Appendix that our results naturally extend to the case in which the firm has a finite number of growth options and pays both issuance and search costs.

⁶ If mean cash flows increase and volatility decreases after investment, all our results and proofs go through. However, if investment increases volatility sufficiently strongly, the optimal strategy will typically fail to be of barrier type and there will be multiple disconnected intervals in which investment is optimal.

⁷ Bolton, Chen, and Wang (2011) consider instead an AK — model in which capital is infinitely divisible, firms can invest at a rate, and all frictions are proportional to the size of the firm (i.e., to its capital stock). They then solve numerically for the optimal financing and payout barrier strategies. We show below that the combination between financing constraints and lumpiness in investment produces interesting and unique empirical implications on corporate behavior and may render barrier strategies suboptimal.

Notably, we allow management to retain earnings inside the firm and denote by C_t the firm's cash holdings at any time $t \geq 0$ (we use indifferently the terms cash holdings and cash reserves). Cash holdings earn a constant rate of interest $r < \rho$ inside the firm and can be used to fund investment or to cover operating losses if other sources of funds are costly or unavailable. The wedge $\delta = \rho - r > 0$ represents a carry cost of cash.

The firm can increase its cash reserves either by retaining earnings or by raising funds in capital markets. A key difference between our setup and previous contributions is that we explicitly take into account capital supply frictions by considering that it takes time to secure outside funding and that capital supply is uncertain. Specifically, we assume that the firm needs to search for investors to raise funds and that, conditional on searching, it meets investors at the jump times of a Poisson process N_t with arrival rate $\lambda \geq 0$.⁸ Under these assumptions, the cash reserves of the firm evolve according to

$$dC_t = (rC_{t-} + \mu_0 + 1_{\{T \leq t\}}(\mu_1 - \mu_0))dt + \sigma dB_t + f_t dN_t - dD_t - 1_{\{t=T\}}K, \quad (1)$$

where T is a stopping time representing the time of investment, f_t is a nonnegative predictable process representing the funds raised upon finding investors, and D_t is a nondecreasing adapted process with $D_{0-} = 0$ representing the cumulative dividends paid to shareholders. Equation (1) shows that cash reserves grow with earnings, with outside financing, and with the interest earned on cash holdings, and decrease with payouts to shareholders and with the cost of investment. In our model, T , D , and f are endogenously determined.

As documented by a series of empirical studies, capital supply conditions are among the key determinants of firms' financing decisions, the level of cash holdings, and the level of corporate investment (see footnote 2). These studies also show that firms often face uncertainty regarding their access to capital markets and that this uncertainty has important feedback effects on their policy choices. Our model captures this important feature of capital markets with the parameter λ , which governs the arrival rate of investors.⁹ A comparison with some special cases to our setup illustrates how capital supply uncertainty affects corporate policies. When $\lambda = 0$, firms cannot raise funds in the capital markets and have to rely exclusively on internal funds to cover operating losses and to finance investment. This is the environment

⁸ A growing body of literature argues that assets prices are more sensitive to supply shocks than standard asset pricing theory predicts. Search theory plays a key role in the formulation of models capturing this idea (see Duffie, Garleanu, and Pedersen 2005; Vayanos and Weill 2008; Lagos and Rocheteau 2009).

⁹ The financial crisis of 2007–2008 has shown that the supply of external finance for nonfinancial firms could be subject to significant shocks. Our model can be extended to consider capital supply shocks, thereby allowing a study of the effects of capital supply uncertainty for large, publicly traded firms, for which access to outside funds may be limited only during a financial crisis or an economic downturn. Such shocks could be introduced by assuming that the arrival rate of investors is governed by a two-state Markov chain. One state would correspond to economic conditions in which capital is readily available and the cost of capital is low, while the other state would correspond to economic conditions in which capital is scarce and the cost of capital is high. The value of cash would then be highest in the state with low capital supply.

considered by Radner and Shepp (1996), Décamps and Villeneuve (2007), and Asvanunt, Broadie, and Sundaresan (2007). When $\lambda \rightarrow \infty$, capital markets are frictionless and firms can instantly raise funds from the financial markets. In that case, the firm has no need for cash reserves and finances both operating losses and investment by (costlessly) issuing new equity. This is the environment considered by Manso (2008), Tserlukevich (2008), Morellec and Schürhoff (2010), or Carlson, Fisher, and Giammarino (2010).

Because capital supply is uncertain, new investors may be able to capture part of the surplus generated at refinancing dates. That is, we consider that once management and investors meet, they bargain over the terms of the new issue to determine the cost of capital or, equivalently, the proceeds from the stock issue. We assume that the allocation of this surplus between incumbent shareholders and new investors results from Nash bargaining. Denoting the bargaining power of new investors by $\eta \in [0, 1]$, the amount π^* that new investors can extract when the firm raises $f \geq 0$ satisfies

$$\pi^* = \underset{\pi \geq 0}{\operatorname{argmax}} \pi^\eta [S_f V(c) - \pi]^{1-\eta} = \eta S_f V(c),$$

where $V(c)$ gives the value of the firm as a function of its cash holdings and

$$S_f V(c) = V(c + f) - f - V(c)$$

gives the financing surplus. Therefore, whenever $\eta \neq 0$, effective issuance costs are stochastic, time varying, and depend on the financial health of the firm. In the model, the bargaining power of investors can be related to the supply of funds in capital markets by assuming that $\eta = \frac{a}{a+\lambda}$ for some $a > 0$. In this case, the fraction of the surplus captured by investors and the cost of capital decrease with capital supply, consistent with Gompers and Lerner (2000).

The firm can be liquidated if its cash reserves reach zero following a series of negative cash-flow shocks. Alternatively, it can choose to abandon its assets at any time by distributing all of its cash reserves. We consider that the liquidation value of the assets of a firm with mean cash-flow rate μ_i is given by $\ell_i = \frac{\varphi \mu_i}{\rho}$ where $1 - \varphi \in [0, 1)$ represents a haircut related to the partial irreversibility of investment. In the analysis below, we denote by τ_0 the stochastic liquidation time of the firm, refer to φ as the tangibility of assets, and consider that $\varphi < 1$.

Because management can decide to pay a liquidating dividend at any point in time and capital supply is uncertain, liquidation is automatically triggered when cash reserves reach zero. The problem of management is therefore to maximize the present value of future dividends by choosing the firm's payout $(D_t)_{t \geq 0}$, financing $(f_t)_{t \geq 0}$, and investment (T) policies. That is, management solves:

$$V(c) = \sup_{(f, D, T)}$$

$$E_c \left[\int_0^{\tau_0} e^{-\rho t} (dD_t - (f_t + \eta S_{f_t} V(C_{t-})) dN_t) + e^{-\rho \tau_0} (\ell_0 + 1_{\{\tau_0 > T\}} (\ell_1 - \ell_0)) \right].$$

The first term in this expression represents the present value of payments to incumbent shareholders until the liquidation time τ_0 , net of the claim of new (outside) investors on future cash flows. The second term represents the present value of the cash flows to shareholders in liquidation, which depends on whether liquidation occurs before or after investment.¹⁰

Because firm value appears in the objective function via the surplus generated at refinancing dates, the above optimization problem is akin to a rational expectations problem: when bargaining over the terms of financing, outside investors have to correctly anticipate the strategy that the firm will use in the future. We show in the Supplementary Appendix that introducing bargaining in the model and solving the corresponding rational expectations equilibrium is equivalent to reducing the arrival rate of investors from λ to $\lambda^* \equiv \lambda(1 - \eta)$ in an otherwise similar model in which outside investors have no bargaining power.

Before proceeding, it should be noted that we do not allow the firm to finance investment with debt. Issuing debt at the time of investment would reduce the mean cash-flow rate to shareholders after investment and would increase the likelihood of inefficient liquidation. In addition, if interest payments were tax deductible, debt financing would reduce the effective cost of investment. As will be clear below, introducing debt financing does not have any qualitative effect on our results and predictions as long as credit supply is uncertain. In a related paper, Hugonnier, Malamud, and Morellec (2014) develop a dynamic capital structure model with search frictions in debt markets. In their model, firms can raise equity instantaneously and costlessly, so that cash holdings are irrelevant.

2. Model Solution

2.1 Value of the firm with no growth option

To facilitate the analysis of shareholders' optimization problem, we start by deriving the value $V_i(c)$ of a firm with mean cash-flow rate μ_i and no growth option. When $i = 1$, this function also gives the value of the firm after the exercise of the growth option.

When there is no growth option, management only needs to determine the firm's payout, liquidation, and financing policies. Because the marginal cost of cash holdings is constant and their marginal benefit is decreasing, we conjecture that there exists some level C_i^* for cash reserves where the marginal cost and benefit of cash holdings are equalized and it is optimal to start paying dividends. In addition, because there are no issuance costs other than those generated by

¹⁰ We do not allow the firm to spin off its new assets when its cash balances hit zero after investment. Such partial asset sales would allow the firm to reinvest the proceeds in its cash reserves and bring them back to their optimal level. The firm would then be liquidated the next time that its cash reserves reach zero. That is, the main effect of such asset sale would be to increase the life expectancy of the firm after investment.

the bargaining friction, we expect that below C_i^* it is optimal for the firm to search for new investors so as to increase its cash reserves back to the target. Finally, because the marginal value of cash is strictly higher than one below the target, our conjecture implies that the firm only liquidates if its cash holdings reach zero.

Denote by $v_i(c; b)$ the value of a firm that follows a barrier policy as above with target level $b \geq 0$. Standard arguments show that in the region $(0, b)$ in which the firm retains earnings and searches for investors, $v_i(c; b)$ satisfies

$$\rho v_i(c; b) = v_i'(c; b)(rc + \mu_i) + \frac{\sigma^2}{2} v_i''(c; b) + \lambda^* [v_i(b; b) - b + c - v_i(c; b)], \quad (2)$$

with $\lambda^* = \lambda(1 - \eta)$ and the boundary condition

$$v_i(0; b) = \ell_i \quad (3)$$

at the point at which the firm runs out of cash. The left-hand side of Equation (2) represents the required rate of return for investing in the firm. The first and second terms on the right-hand side capture the effects of cash savings and of cash-flow volatility on firm value. The third term reflects the effect of capital supply uncertainty. This last term is the product of the instantaneous probability of meeting investors and the fraction of the surplus that accrues to incumbent shareholders when raising cash reserves from c to the given target b .

Consider next the payout region $[b, \infty)$. In this region, the firm distributes all cash holdings above b (with a specially designated dividend or a share repurchase) and we have

$$v_i(c; b) = v_i(b; b) + c - b, \quad c \geq b. \quad (4)$$

Subtracting $v_i(b; b)$ from both sides, dividing by $c - b$, and taking the limit as c tends to b yields

$$\lim_{c \uparrow b} v_i'(c; b) = 1. \quad (5)$$

Denote by $C_{i,t}$ the uncontrolled cash reserves process associated with μ_i , let $\tau_{i,x}$ be the stopping time at which this process reaches the level $x \in \mathbb{R}$ for the first time, and define

$$L_i(c; b) = E_c \left[e^{-(\rho + \lambda^*)\tau_{i,0}} 1_{\{\tau_{i,0} \leq \tau_{i,b}\}} \right], \quad (6)$$

$$H_i(c; b) = E_c \left[e^{-(\rho + \lambda^*)\tau_{i,b}} 1_{\{\tau_{i,b} \leq \tau_{i,0}\}} \right], \quad (7)$$

for all $c \in [0, b]$. The function $L_i(c; b)$ gives the present value of one dollar to be paid in liquidation, should it occur before the uncontrolled cash reserves reach $b > 0$ or finding new investors. Similarly, $H_i(c; b)$ gives the present value of one dollar to be received when the uncontrolled cash reserves reach $b > 0$, should

this occur before liquidation or finding new investors. Closed-form expressions for these two functions are provided in the Appendix.

Using the above notation, together with basic properties of diffusion processes as found for example in Stokey (2009, chapter V), it can be shown that the unique solution to Equations (2), (3), and (4) satisfies

$$\begin{aligned} v_i(c; b) = & v_i(b; b)H_i(c; b) + \ell_i L_i(c; b) + \Pi_i(c; b) - \Pi_i(b; b)H_i(c; b) \\ & - \Pi_i(0; b)L_i(c; b) \end{aligned} \quad (8)$$

over the region $(0, b)$ with

$$\Pi_i(c; b) = \frac{\lambda^*}{\rho + \lambda^*} \left(v_i(b; b) - b + c + \frac{\mu_i + rc}{\rho + \lambda^* - r} \right).$$

To better understand this solution, recall that over $(0, b)$ the firm retains all earnings. This implies that over this region firm value is the sum of the present value of the payments that shareholders obtain if cash holdings reach either the payout trigger or the liquidation point before external financing can be secured (first two terms), plus the present value of the claim that they receive if the firm gets an opportunity to raise funds from outside investors before its cash holding reach either endpoints of the region (last three terms).

Equations (2), (3), (4), and (5) or equivalently Equations (5) and (8), characterize the value of a given barrier strategy. To determine the optimal target level C_i^* , we further impose the high-contact condition (see, e.g., Dumas 1991)

$$\lim_{c \uparrow C_i^*} v_i''(c; C_i^*) = 0 \quad (9)$$

at the dividend distribution boundary.¹¹ In the Appendix, we show that there exists a unique target level that solves Equation (9) and prove that the corresponding barrier strategy is optimal among all strategies. This leads to the following result.

Proposition 1 (Firm value without growth option). The value of a firm with mean cash-flow rate μ_i and no growth option is concave and given by $V_i(c) = v_i(c; C_i^*)$, where $v_i(c; b)$ is defined in Equations (4) and (8) and C_i^* is the unique solution to (9). The optimal target C_i^* increases with cash-flow volatility σ^2 , decreases with capital supply λ and asset tangibility φ , and is non-monotonic in the cash-flow mean μ_i .

¹¹ Using Equations (4), (5) and (8) in conjunction with the explicit expressions of the functions $H_i(c, b)$ and $L_i(c, b)$ (defined in Equations (6) and (7)) given in Appendix B.2, it can be shown that the first-order condition with respect to the dividend barrier level, that is

$$\left. \frac{\partial v}{\partial b}(c, b) \right|_{b=C_i^*} = 0, \quad c > 0,$$

is equivalent to the high contact condition (9). See the Appendix I for details.

Note that given (2) and (5), the high contact condition implies

$$\lim_{c \uparrow C_i^*} v_i(c; C_i^*) = \frac{\mu_i}{\rho} + C_i^* - \left(1 - \frac{r}{\rho}\right) C_i^*. \quad (10)$$

This equation shows that the value of a constrained firm at the target level of cash holdings is equal to the value of an unconstrained firm (first two terms on the right-hand side) minus the present value of the cost of keeping cash inside the firm (last term). In this equation, the volatility of cash flows and the arrival rate of investors only appear indirectly through their effects on the dividend threshold C_i^* .

2.2 Value of the firm with a growth option

We now turn to the analysis of corporate policies when the firm has the option to increase its mean cash-flow rate by paying a lump sum cost K . The growth option changes the firm's policy choices and value only if the project has positive net present value. The following proposition provides a necessary and sufficient condition for this to be the case.

Proposition 2. The option to invest has positive net present value if and only if the cost of investment is lower than K^* defined by:

$$\frac{\mu_1 - \mu_0}{\rho} = K^* + \left(1 - \frac{r}{\rho}\right) (C_1^* - C_0^*), \quad (11)$$

where C_i^* is the target cash level for a firm with mean cash-flow rate μ_i and no growth option.

The intuition for this result is clear. The left-hand side of Equation (11) represents the expected present value of the increase in cash flows following the exercise of the growth option. The right-hand side represents the total cost of investment, which incorporates both the direct cost of investment and the change in the carry cost of the optimal cash balance. In the following, we consider that the cost of investment is below K^* . If this was not the case, the firm would simply follow the barrier strategy described in proposition 1.

2.2.1 Optimality of a barrier strategy. Following the logic of the previous section, it is natural to conjecture that for a firm with a growth option there exists a cutoff level $C_U^* \geq K$ for cash reserves such that it is optimal to retain earnings and search for investors when $c < C_U^*$ and to invest when cash reserves reach C_U^* or upon obtaining outside funds. As before, the marginal value of cash is always above one in the retention region and the firm is only liquidated if a sequence of losses depletes its cash reserves.

To determine when such a barrier strategy is optimal, we start by deriving the value $u(c; b)$ of a firm that follows a barrier strategy with some investment

trigger $b \geq K$. Standard arguments show that in the region $(0, b)$ over which the firm retains earnings, $u(c; b)$ satisfies

$$\rho u(c; b) = u'(c; b)(rc + \mu_0) + \frac{\sigma^2}{2} u''(c; b) + \lambda^* [V_1(C_1^*) - C_1^* - K + c - u(c; b)], \quad (12)$$

subject to the value-matching conditions

$$u(0; b) = \ell_0, \quad (13)$$

$$u(c; b) = V_1(c - K), \quad c \geq b, \quad (14)$$

at the point at which the firm runs out of cash and in the region in which it uses its cash reserves to invest. This equation is similar to that derived in the previous section for the value of the firm after investment. The only difference is that the financing surplus is now given by $V_1(C_1^*) - C_1^* - K + c - u(c; b)$ because, upon obtaining outside funds, the firm invests and simultaneously readjusts its cash reserves to the level C_1^* that is, optimal after investment.

The unique solution to (12), (13), and (14) is given by

$$u(c; b) = V_1(b - K)H_0(c; b) + \ell_0 L_0(c; b) + \Phi(c) - \Phi(b)H_0(c; b) - \Phi(0)L_0(c; b), \quad (15)$$

where

$$\Phi(c) = \frac{\lambda^*}{\rho + \lambda^*} \left(V_1(C_1^*) - C_1^* - K + c + \frac{\mu_0 + rc}{\rho + \lambda^* - r} \right). \quad (16)$$

The interpretation of this solution is similar to that of (8). The first two terms give the present value of the payments that incumbent shareholders receive if the firm invests with internal funds or is liquidated before external funding can be secured. The other terms gives the present value of the payments that shareholders receive if the firm invests with external funds before its cash holdings reach either zero or the given investment trigger.

Equations (14) and (15) characterize firm value for a given barrier strategy. Because the decision to invest with internal funds is an optimal stopping problem, we naturally expect that the optimal trigger is determined by the smooth-pasting condition:

$$\lim_{c \uparrow C_U^*} u'(c; C_U^*) = V_1'(C_U^* - K). \quad (17)$$

We show in the Appendix that this is indeed the case if the investment cost is not too small in that $K > \underline{K}$, for some threshold $\underline{K} < K^*$ for which we provide an explicit characterization. If the investment cost lies below this threshold, then the smooth-pasting condition no longer characterizes the optimal investment trigger and we have the corner solution $C_U^* = K$. In this case, the growth option is so profitable that it becomes optimal to invest as soon as possible and liquidate immediately thereafter if the cost of investment is financed internally. In either case, the barrier policy associated with the investment trigger C_U^* is optimal in the class of barrier policies but it is not necessarily optimal among all strategies. Notably, we establish the following result in the Appendix.

Theorem 1 (Optimality of a barrier strategy). There exist two constants $\underline{K} \leq K^{**} < K^*$ such that a barrier strategy is optimal if and only if $K \leq K^{**}$. In this case, firm value is given by $V(c) = u(c; C_U^*)$, where $u(c; b)$ is defined in Equations (14) and (15) and C_U^* is the unique solution to (17) when $K > \underline{K}$ and $C_U^* = K$ otherwise.

Figure 1 represents the value of the firm as a function of its cash holdings when $K \leq K^{**}$. Firm value is equal to ℓ_0 when cash holdings reach zero and the firm liquidates. Investment occurs at C_U^* , where firm value before investment smooth pastes with the value of the firm after investment using internal funds $V_1(c - K)$. As shown by the figure, the value of the firm may not always be globally concave, providing incentives for management to gamble in financial markets (see Section 3.1). Finally, the marginal value of cash is one at the point $C_1^* + K$, where the firm starts paying dividends.

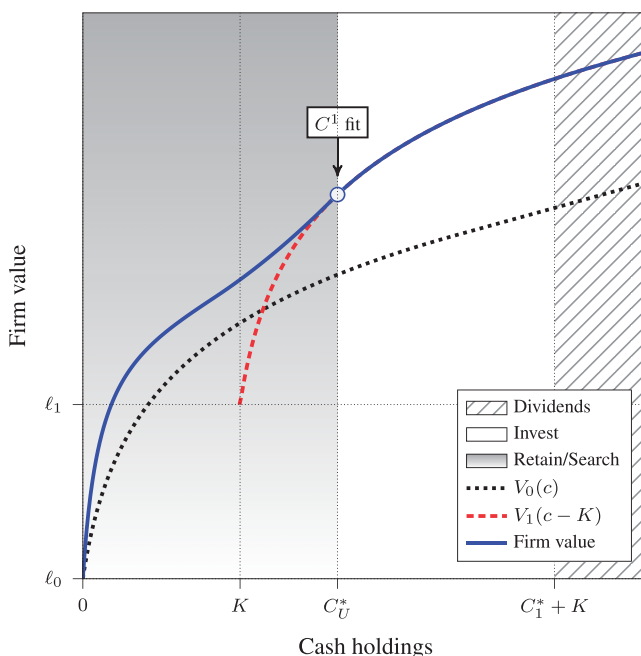


Figure 1
Value of the firm under the optimal barrier strategy

This figure represents the value of a firm with a growth option as a function of its cash holdings when $\underline{K} < K \leq K^{**}$. In the shaded area below the investment trigger C_U^* , the optimal policy is to retain earnings and search for investors. In the unshaded area between C_U^* and $C_1^* + K$, the optimal policy is to invest in the growth option from internal funds. In the hatched area above $C_1^* + K$, the optimal policy is to invest with internal funds and distribute dividends to decrease cash holdings to the target level C_1^* after investment.

2.2.2 Optimality of a band strategy. The function defined by $U(c)=u(c; C_U^*)$ gives the value of the firm under the optimal barrier policy and can be constructed for any investment cost. However, theorem 1 shows that this barrier strategy is suboptimal if the investment cost is high, in which case the smooth-pasting condition (17) is not sufficient to determine the globally optimal strategy. The intuition for this finding is that with a sufficiently high investment cost, it becomes too expensive for a firm with low cash holdings to accumulate the amount of cash necessary to invest. Specifically, we show in the Appendix that with a high investment cost the barrier strategy fails to be optimal because there exists a level below C_U^* where the marginal value of cash $U'(c)$ drops to one. At that point, incumbent shareholders would rather abandon the option of financing investment internally and receive dividends, than continue hoarding cash inside the firm. Figure 2 provides an illustration of the marginal value of cash associated with the optimal barrier strategy with high and low investment costs.

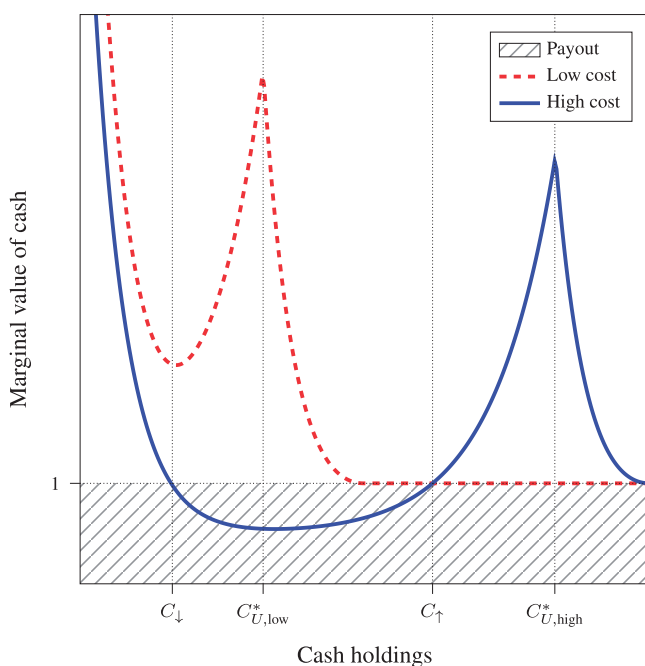


Figure 2

Marginal value of cash under the optimal barrier strategy

This figure plots the marginal value of cash $U'(c)$ under the optimal barrier strategy in an environment in which the investment cost is low (dashed line) and in an environment in which the costs of investment is high (solid line). In the latter case, the marginal value of the cash drops to one at the point $C_↓$ and remains below one over the interval $(C_↓, C_↑)$, indicating that shareholders would rather abandon the option of investing from internal funds and receive dividends than continue hoarding cash inside the firm.

Following this line of argument, we conjecture and later verify that, when the investment cost is high, the optimal strategy includes an intermediate payout region and can be described in terms of thresholds $C_W^* \leq C_L^* \leq C_H^*$ as follows: when cash holdings are in (C_L^*, C_H^*) , the firm retains earnings and invests either upon obtaining outside funds or when its cash holdings reach the level $C_H^* \geq K$. If cash holdings drop to the level C_L^* following a sequence of operating losses, the firm abandons the option of financing investment internally and makes a lump-sum payment $C_L^* - C_W^*$ to shareholders. Finally, if cash holdings are at or below the level C_W^* , the firm retains earnings, pays dividends to keep its cash reserves in $(0, C_W^*)$, and finances investment exclusively with outside funds. As in the case of a barrier strategy, the firm is liquidated only if its cash reserves reach zero. Importantly, because the abandonment of the option to invest from internal funds is irreversible, the optimal band policy has exactly three bands when the firm has one growth option.

To verify our conjecture, we start by constructing the value $v(c; b)$ of a firm that follows a strategy as above with thresholds $b = (b_1, b_2, b_3)$ for some arbitrary constants $b_1 \leq b_2 \leq b_3$ with $b_3 \geq K$. Standard arguments show that in the region $(0, b_1) \cup (b_2, b_3)$ over which the firm retains earnings and searches for investors, $v(c; b)$ satisfies:

$$\rho v(c; b) = v'(c; b)(rc + \mu_0) + \frac{\sigma^2}{2} v''(c; b) + \lambda^* [V_1(C_1^*) - C_1^* - K + c - v(c; b)], \quad (18)$$

subject to the value-matching conditions

$$v(0; b) = \ell_0, \quad (19)$$

$$v(c; b) = V_1(c - K), \quad c \geq b_3. \quad (20)$$

Equation (18) is the same as Equation (12), but the solutions we are looking for differ because the strategy now includes an intermediate payout region $[b_1, b_2]$ over which firm value does not satisfy (18). Instead, the value of the firm is given by

$$v(c; b) = v(b_1; b) + c - b_1, \quad b_1 \leq c \leq b_2. \quad (21)$$

over this intermediate payout region and the fact that, once below b_1 , the firm distributes dividends so as to maintain its cash holdings at or below this level implies that we have

$$\lim_{c \uparrow b_1} v'(c; b) = 1. \quad (22)$$

From these boundary conditions, it is clear that because $v(0; b) = U(0) = \ell_0$, we must necessarily have $v'(0; b) > U'(0)$ for our candidate strategy to dominate the optimal barrier strategy. We show in the Appendix that this condition is equivalent to the restriction $K > K^{**}$.

Proceeding as in the two previous cases, it is easily shown that the unique solution to (18), (19), (20), (21), and (22) satisfies

$$v(c; b) = v(b_1; b)H_0(c; b_1) + \ell_0 L_0(c; b_1) + \phi(c; b) - \Phi(0)L_0(c; b_1) - \Phi(b_1)H_0(c; b_1) \quad (23)$$

in the retention interval $(0, b_1)$, and

$$v(c; b) = V_1(b_3 - K)H_0(c; b_2, b_3) + v(b_2; b)L_0(c; b_2, b_3) + \phi(c; b) - \Phi(b_2)L_0(c; b_2, b_3) - \Phi(b_3)H_0(c; b_2, b_3) \quad (24)$$

in the retention interval (b_2, b_3) .

In these equations, the discount factors $H_0(c; b_2, b_3)$ and $L_0(c; b_2, b_3)$ are defined as in (6) and (7), but with the first hitting time of the level b_2 instead of the liquidation time, and $\Phi(c)$ is defined as in (16). The first two terms in Equation (23) give the present value of the payments that incumbent shareholders receive if cash reserves reach either the liquidation point or the payout trigger b_1 before external financing can be secured. The last three terms give the value of the payment that they receive if the firm finds new investors before its cash reserves reach either zero or b_1 . Similarly, the first line in (24) gives the present value of the payments that incumbent shareholders receive if cash reserves reach either the intermediate payout trigger b_2 or the internal investment trigger b_3 before external financing can be secured. The second line gives the present value of the claims that they receive if external funds are raised before cash reserves reach either b_2 or b_3 .

Equations (21), (23), and (24) characterize firm value for given thresholds $b = (b_1, b_2, b_3)$. It remains to determine the triple $C^* = (C_W^*, C_L^*, C_H^*)$ of optimal thresholds. As in the case without growth option, we determine the point C_W^* below which the firm invests exclusively with outside funds by imposing the high-contact condition

$$\lim_{c \uparrow C_W^*} v''(c; C^*) = 0 \quad (25)$$

at the upper end of the lower payout region. In addition, because the decision to invest with internal funds and the decision to make an intermediate dividend payment can be seen as a joint optimal stopping problem, it is natural to expect that the two remaining thresholds are determined by the smooth-pasting conditions

$$\lim_{c \downarrow C_L^*} v'(c; C^*) = 1, \quad (26)$$

$$\lim_{c \uparrow C_H^*} v'(c; C^*) = V'_1(C_H^* - K). \quad (27)$$

We show in the Appendix that these boundary conditions uniquely determine a triple of thresholds C^* and we prove that the corresponding band strategy is globally optimal when the investment cost is above K^{**} . The following theorem summarizes our findings.

Theorem 2 (Optimality of a band strategy). If the investment cost is such that $K \in (K^{**}, K^*)$, then the value of the firm with a growth option is given by $V(c) = v(c; C^*)$, where the thresholds $C^* = (C_W^*, C_L^*, C_H^*)$ are the unique solutions to (25), (26), and (27).

These result shows that when the investment cost is high and cash reserves are below C_W^* , the optimal strategy for shareholders is to finance investment exclusively with external funds and to use cash reserves only to cover operating losses. When cash holdings are between C_W^* and C_L^* , the firm abandons the option of investing with internal funds and pays a specially designated dividend to lower its cash holdings. Finally, when cash holdings are above C_L^* , the firm finances investment using either internal or external funds and the optimal policy is to retain earnings until the firm invests or its cash reserves drop to C_L^* . The shape and properties of the firm value under this band strategy are illustrated in Figure 3.

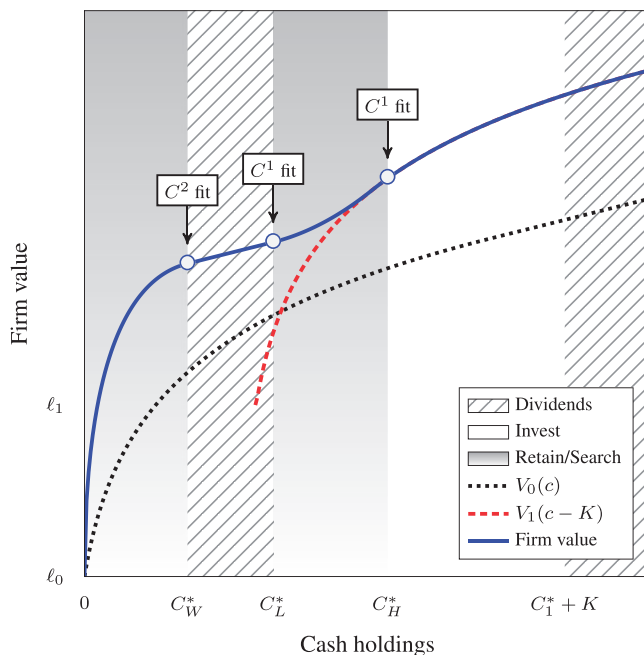


Figure 3
Value of the firm under the optimal strategy

This figure represents the value of a firm with a growth option as a function of its cash holdings when $K^{**} \leq K < K^*$. In the shaded areas, the optimal policy is to retain earnings and search for investors. In the first hatched area, the optimal policy is to distribute dividends to decrease the level of cash holdings to C_W^* . In the unshaded area between C_H^* and $C_1^* + K$, the optimal policy is to invest in the growth option with internal funds. In the second hatched area, the optimal policy is to invest with internal funds and distribute dividends to decrease cash holdings to the target level C_1^* after investment.

Interestingly, the change in the optimal financing and payout policies that occurs at C_L^* implies that $V'(C_L^*)=1$. Because the marginal value of cash is constantly greater or equal to one, it follows that firm value is never globally concave with a high investment cost. In fact, we show in the Appendix that the value of the firm is strictly convex over the whole interval (C_L^*, C_H^*) when the investment cost is high. To understand this feature, recall that in our model cash holdings generally serve two purposes: reducing the risk of inefficient closure and financing investment. However, when the investment cost is high and cash reserves are below C_L^* , it is optimal to invest exclusively with outside funds and the value of cash holdings only comes from their mitigating effect on liquidation risk. At the point C_L^* the marginal value of cash is equal to one and it is optimal to start paying dividends. Above this point, the possibility to finance investment internally introduces a second motive for holding cash that pushes the marginal value of cash above one, leading firm value to be convex. Figure 4 provides an illustration of the marginal value of cash under the globally optimal band strategy when the cost of investment is high.

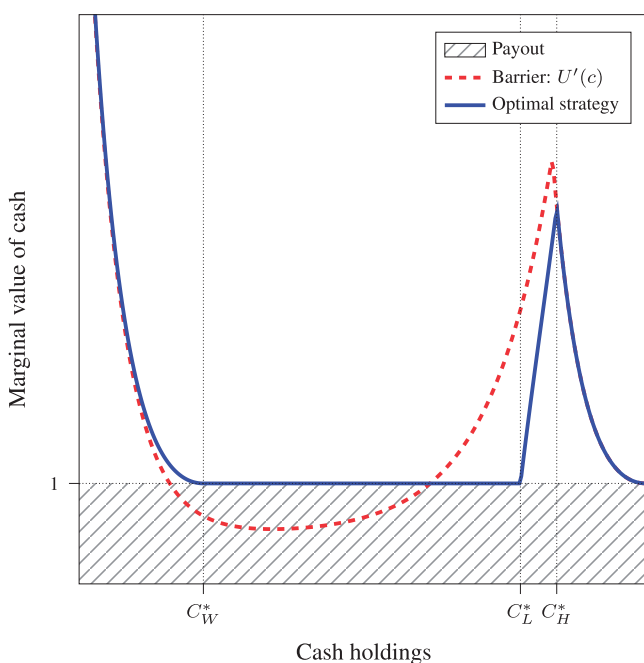


Figure 4

Marginal value of cash for a firm with a growth option

This figure plots the marginal value of cash $V'(c)$ under the globally optimal strategy (dashed line) and the marginal value of cash $U'(c)$ under the optimal barrier strategy (solid line) in an environment in which the investment cost is high. In the latter case, the failure of global optimality is due to the fact that the marginal value of the cash drops below one, indicating that shareholders would rather abandon the option of investing from internal funds and receive dividends than continue hoarding cash inside the firm.

While the nature of the optimal policy of the firm with high investment costs is unexpected, it is of universal nature for optimization problems with fixed costs and capital supply frictions. In the Supplementary Appendix, we solve a model with N growth options, issuance costs, and search costs, and show that this property of the model depends neither on the number of growth options available to the firm nor on the absence of search and issuance costs. We also show that for a firm with N growth options, there exist up to N dividend distribution intervals. Finally, we show that with fixed issuance costs, firms only raise funds when cash reserves are below some threshold C_F^* , where the net present value of raising funds equals 0.

3. Convexity and Gambling

An important result of Section 2 is that the interplay between lumpiness in investment and capital supply frictions can give rise to local convexity in firm value. This naturally raises the question of whether this convexity may lead to gambling behavior, in which management engages in zero NPV investments with random returns in an attempt to improve firm value. In this section, we investigate two alternative gambling specifications. In the first one, we follow BCW (2011) and assume that the firm has access to a futures contract and can alter the volatility of cash flows by investing in this contract. In the second one, we follow Makarov and Plantin (Forthcoming) or DeMarzo, Livdan, and Tchisty (2014) and assume that the firm can invest in lotteries at the jump times of a Poisson process.

3.1 Continuous gambling

As a clean and stylized way of incorporating gambling in the model, we first follow BCW (2011) and assume that the firm has access to a futures contract whose price is a Brownian motion Z_t , uncorrelated with the Brownian motion B_t driving the firm's cash flows. A position π_t in the futures contract thus implies that the firm's cash flows change from dX_t to $dX_t + \pi_t dZ_t$. Hedging positions are generally constrained by margin requirements. To capture these requirements and make our results easily interpretable, we consider that the size of the firm's futures position $|\pi_t|$ cannot exceed some fixed size G and study the effects of varying G on optimal policies and firm value.

Assuming frictionless trading in the futures contract, the value-maximizing gambling policy is always of a bang-bang type. Indeed, standard arguments show that in the region where the firm retains earnings and searches for investors, the value of the firm satisfies:

$$\begin{aligned} \rho V(c; G) = & V'(c; G)(rc + \mu_0) + \frac{1}{2} \sigma^2 V''(c; G) \\ & + \lambda^* [V_1(C_1^*) - C_1^* - K + c - V(c; G)] + \frac{1}{2} \max_{|\pi| \leq G} \pi^2 V''(c; G), \quad (28) \end{aligned}$$

where the last term captures the effects of gambling on firm value. In the region in which firm value is concave, we have $V''(c; G) \leq 0$, the firm does not gamble and $\pi = 0$. In the region where firm value is convex, we have $V''(c; G) > 0$ and it is optimal to choose the maximal possible position in futures contracts, so that $\pi = G$. Clearly, Because the function $V_1(c)$ is concave, gambling is not optimal after investment and, hence, neither $V_1(c; G)$ nor the critical investment cost K^* depend on G . This justifies the fact that the last term in (28) is the same as in the base case.

We start our analysis by determining when gambling may be optimal.

Theorem 3 (Optimality of gambling). For any $G \geq 0$, there exist two critical investment costs $0 \leq \underline{K}(G) \leq K^{**}(G) \leq K^*$ such that

- If $K \leq K^{**}(G)$, then there exists a threshold $C_U^*(G)$, with $C_U^*(G) = K$ if $K \leq \underline{K}(G)$, such that it is optimal to follow a barrier strategy as in theorem 1 but with threshold $C_U^*(G)$. In addition, when $U''(C_U^*) > 0$, there exists an additional threshold $C_g^*(G)$ such that it is optimal for the firm to gamble at the maximal rate on $[C_g^*(G), C_U^*(G)]$.
- If $K > K^{**}(G)$, then there exist two thresholds $C_L^*(G) \leq C_H^*(G)$ such that it is optimal for the firm to follow a band strategy as in theorem 2 but with thresholds $(C_W^*, C_L^*(G), C_H^*(G))$ and to gamble at the maximal rate on $[C_L^*(G), C_H^*(G)]$.

Theorem 3 shows that, if gambling is optimal, it happens at the maximal rate and only for large levels of cash holdings, all the way to the investment boundary. That is, the firm gambles to invest from internal funds. In the limit where G goes to infinity, the firm gambles in an all or nothing fashion: either go all the way to the exercise boundary $C_H^*(G)$, or lose cash, make a lump-sum dividend payment, and abandon the option to invest from internal funds (see the Supplementary Appendix for an analysis of this limiting case).

The following proposition characterizes the effect of gambling on optimal policies.

Proposition 3 (Optimal strategies with gambling).

- The investment triggers $C_U^*(G)$ and $C_H^*(G)$ are increasing in G while the lumpy dividend trigger $C_L^*(G)$ is decreasing in G .
- The critical investment cost $K^{**}(G)$ is monotone increasing in G and, for any given investment cost K , there exists a finite threshold $\tilde{G}(K)$ such that band strategies are suboptimal for all $G \geq \tilde{G}(K)$.
- The constant $G^* = \sup_{K \geq 0} \tilde{G}(K)$ is finite and satisfies $K^{**}(G^*) = K^*$, so that band strategies are suboptimal for all $G \geq G^*$ irrespective of the investment cost.

Proposition 3 delivers two main results. First, relaxing hedging constraints increases both cash-flow volatility and the threshold for investing from internal

funds. This may therefore increase or decrease investment rates from internal funds. Second, sufficiently weak hedging constraints eliminate band strategies. That is, when G is sufficiently large, the firm never abandons the option to invest from internal funds. In Section 4, we investigate the empirical predictions of the model and show that the minimum gambling \bar{G} required to eliminate band strategies is unrealistically large in most economic environments.

3.2 Lottery gambling

To study the robustness of our results, we now briefly investigate an alternative form of gambling that is based on discrete opportunities to participate in lotteries rather than on infinitely frequent bets on the increment of a Brownian motion.

Suppose that the firm has access to a full set of instantaneous lotteries at the jump times of a Poisson process with arrival rate Λ , and that these lotteries have a payoff x such that $E[x] \leq 0$. Each time that such a gambling opportunity arises, the firm solves

$$\max_{x: E[x] \leq 0} E[\tilde{V}(c+x; \Lambda) - \tilde{V}(c; \Lambda)],$$

where the function $\tilde{V}(c; \Lambda)$ denotes the value of the firm in this new environment. Following Makarov and Plantin (Forthcoming), it can be shown that the solution to this problem is given by a binary lottery that can be constructed as follows. Denote by $\mathcal{C}[\tilde{V}](c; \Lambda)$ the concave envelope of the function $\tilde{V}(c; \Lambda)$, that is, the smallest concave function dominating $\tilde{V}(c; \Lambda)$ from above, and denote by $C_{g,1}(c)$ (resp., $C_{g,2}(c)$) the largest point (resp. the smallest point) below c (resp., above c) at which these two functions are equal. Then the optimal gamble $x^*(c)$ is given by a binary random variable taking values $C_{g,1}(c) - c$ and $C_{g,2}(c) - c$ with probabilities calculated in such a way that

$$E[\tilde{V}(c+x^*(c); \Lambda) - \tilde{V}(c; \Lambda)] = \mathcal{C}[\tilde{V}](c; \Lambda) - \tilde{V}(c; \Lambda). \quad (29)$$

If the function $\tilde{V}(c; \Lambda)$ is globally concave then the optimal lottery $x^*(c) \equiv 0$ is degenerate for all $c \geq 0$, and gambling is never optimal. Notably, because the function $V_1(c)$ is globally concave, gambling is never optimal after investment, and it follows that neither the postinvestment firm value nor the critical investment cost K^* depend on the arrival rate of gambling opportunities.

Standard arguments together with Equation (29) show that in the region where the firm retains earnings and searches for investors, the value of the firm satisfies:

$$\begin{aligned} \rho \tilde{V}(c; \Lambda) = & V'(c; \Lambda)(rc + \mu_0) + \frac{\sigma^2}{2} \tilde{V}''(c; \Lambda) \\ & + \lambda^* [V_1(C_1^*) - C_1^* - K + c - \tilde{V}(c; \Lambda)] + \Lambda (\mathcal{C}[\tilde{V}](c; \Lambda) - \tilde{V}(c; \Lambda)). \end{aligned}$$

Comparing this equation with (28) shows that the arrival rate of gambling opportunities Λ plays in this specification the same role as the parameter G in

the continuous gambling specification. When the arrival rate is sufficiently small, the value of the firm remains close to the one that prevails absent gambling opportunities and, as a result, a band strategy may be optimal. On the contrary, when the arrival rate of gambling opportunities is sufficiently large, the marginal value of cash remains constantly above one so that a barrier strategy is always optimal. The following proposition confirms this intuition.

Proposition 4. If $U''(C_U^*) \leq 0$ then gambling is suboptimal for any $\Lambda \geq 0$. Otherwise, there exists $\Lambda^* > 0$ and three thresholds $C_{g,1}^*(\Lambda) \leq C_{U,\ell}^*(\Lambda) \leq C_{g,2}^*(\Lambda)$ such that, for all $\Lambda > \Lambda^*$, it is optimal for the firm to follow a barrier strategy as in Theorem 1, but with investment threshold $C_{U,\ell}^*(\Lambda)$, and to gamble using the binary lottery associated with $C_{g,1}^*(\Lambda)$ and $C_{g,2}^*(\Lambda)$ whenever an opportunity arises and $c \in [C_{g,1}^*(\Lambda), C_{U,\ell}^*(\Lambda)]$.

Proposition 4 and theorem 3 offer similar conclusions regarding the impact of gambling but there are some notable differences. First, while the continuous gambling of theorem 3 is optimal at a point if and only if the value of the firm is convex at that point, the lottery gambling of proposition 4 can be optimal at a point even if the firm value is locally concave around that point. Second, when the firm uses a lottery gamble, it does so to invest and always overshoots its target in that a successful gamble takes it to a level of cash holdings $C_{g,2}^*(\Lambda)$ that exceeds the investment threshold $C_{U,\ell}^*(\Lambda)$. That is, while the formal gambling interval is given by $[C_{g,1}^*(\Lambda), C_{g,2}^*(\Lambda)]$, the firm only gambles for $c \in [C_{g,1}^*(\Lambda), C_{U,\ell}^*(\Lambda)]$ (that is, before investment) because investment is irreversible.

The following proposition shows that these differences only exist for finite values of the parameters G and Λ that govern the extent of the firm's gambling, and that the two gambling specifications lead to the same value function, investment trigger, and gambling intervals in the limit at which either continuous gambling is unconstrained or lottery gambling is constantly available. Importantly, this proposition establishes that the value of the firm is robust to the specification of the gambling mechanism.¹²

Proposition 5. In the limit $\Lambda, G \rightarrow \infty$ the continuous and lottery gambling specifications coincide in the sense that

$$\begin{aligned}\lim_{\Lambda \rightarrow \infty} C_{g,1}^*(\Lambda) &= \lim_{G \rightarrow \infty} C_g^*(G), \\ \lim_{\Lambda \rightarrow \infty} C_{g,2}^*(\Lambda) &= \lim_{\Lambda \rightarrow \infty} C_{U,\ell}^*(\Lambda) = \lim_{G \rightarrow \infty} C_U^*(G), \\ \lim_{\Lambda \rightarrow \infty} \tilde{V}(c; \Lambda) &= \lim_{G \rightarrow \infty} V(c; G), \quad \forall c \geq 0.\end{aligned}$$

¹² Note that the function $V(c; \infty) = \tilde{V}(c; \infty)$ in proposition 5 is only defined in the limit and does not correspond to the value function of an actual control problem. This function is concave and lies above the value function $V(c) = V(c; 0) = \tilde{V}(c; 0)$ that prevails in the absence of gambling opportunities but it does not coincide with the concave envelope of this function.

4. Model Implications

4.1 General properties of the model

While our model pertains to the literature analyzing the effects of financing constraints on corporate policies, its predictions regarding payout and financing decisions differ significantly from those of prior contributions (e.g., BCW 2011; DMRV 2011). Indeed, in this literature firms *always* follow a barrier policy. In addition, firms either never raise external funds (when the cost of external finance is high) or are never liquidated (when the cost is low). Finally, when the cost of external finance is low, firms only raise funds when their cash reserves are completely depleted. That is, firms never simultaneously hold cash and raise external funds and they only tap capital market following a series of negative shocks.

Our paper considers an environment in which capital supply is uncertain and investment is lumpy and shows that firms only follow a barrier policy when the cost of investment is low or, equivalently, when the net present value of the project is high. In our model, firms may raise outside funds before exhausting internal resources, some firms may be liquidated even with low issuance costs, and the optimal payout policy may feature several payout regions, with both incremental and lumpy dividend payments. Our analysis also demonstrates that constrained firms with low cash holdings will not finance investment internally and may decide to pay dividends early. By contrast, constrained firms with high cash holdings may finance investment internally and will retain earnings. These results are in sharp contrast with those of standard models of financing constraints, imply that investment and payout do not always increase with slack, and challenge the use of investment-cash-flow sensitivities or payout ratios as measures of financing constraints.

Another prediction of our model is that, when financing investment with external funds, the optimal policy is to raise enough funds to finance both the capital expenditure and the potential gap between current cash holdings and the optimal level after investment, consistent with Kim and Weisbach (2008) and McLean (2011). Also, while in prior contributions the size of equity issues is constant, there exists some time series variation in our model. Because the amount raised depends on the cash reserves of the firm when it meets investors.

Lastly, another key contribution of our model is to show that, with capital supply uncertainty, the choice between internal and external funds for financing investment does not follow a strict pecking order, in that any firm can use both internal and external funds. This is in sharp contrast with the financing policy in prior contributions, in which firms exhaust internal funds before issuing securities and finance investment exclusively with internal funds.

4.2 Additional implications

4.2.1 Characterizing the firm's optimal strategy. This section provides additional results using numerical examples. Table 1 reports the parameter

Table 1
Benchmark parameters and implied variables

Benchmark parameters:		
Discount rate	ρ	0.06
Interest rate on cash	r	0.0055
Mean cash-flow rate	μ_0	0.05
Mean cash-flow rate after investment	μ_1	0.1
Volatility of cash flow	σ	0.1
Asset tangibility	φ	0.0
Bargaining power of incumbents	η	0.25
Arrival rate of investors	λ	2.0
Benchmark thresholds:		
Target cash holding before investment	C_0^*	0.2971
Target cash holding after investment	C_1^*	0.2794
Critical investment costs		
Minimal investment cost	\underline{K}	0.0
Switching point	K^{**}	0.3490
Zero NPV investment cost	K^*	0.8494
Low-cost environment (Growth firm)		
Investment cost	K	0.15
Investment threshold	C_U^*	0.3133
Share of the growth option in value	$1 - \frac{V_0(C_0^*)}{U(C_0^*)}$	44.4264%
High-cost environment (Value firm)		
Investment cost	K	0.75
Dividend distribution threshold	C_W^*	0.3037
Lumpsum dividend threshold	C_L^*	0.9177
Investment threshold	C_H^*	0.9372
Share of the growth option in value	$1 - \frac{V_0(C_0^*)}{V(C_0^*)}$	9.9720%
Gambling required to rule out the band strategy	\bar{G}/σ	12.4039
Gain from using a band strategy	$\max_s \frac{V(s)}{U(s)} - 1$	1.0432%

values used in our numerical analysis and the values of the implied variables, such as the investment and payout thresholds, the critical investment costs K^* and K^{**} , the fraction of firm value accounted for by the growth option, the minimum gambling \bar{G} required to eliminate band strategies, or the value gain from using a band strategy when $K > K^{**}$.

A key feature of our model is that firms only follow a barrier policy when the cost of investment is low (i.e. $K < K^{**}$). To better understand this feature, Figure 5, panel A, plots the zero- NPV threshold K^* and the threshold K^{**} that triggers a change in corporate policies, as functions of capital supply λ , asset tangibility φ , cash-flow volatility σ , and the carry cost of cash $\delta = \rho - r$. Figure 5, panel B, plots the internal rate of return of the project when $K = K^{**}$ (solid line) and $K = K^*$ (dashed line) defined as the solution to

$$\frac{\mu_1 - \mu_0}{R} = K + \left(1 - \frac{r}{R}\right)(C_1^*(R) - C_0^*(R))$$

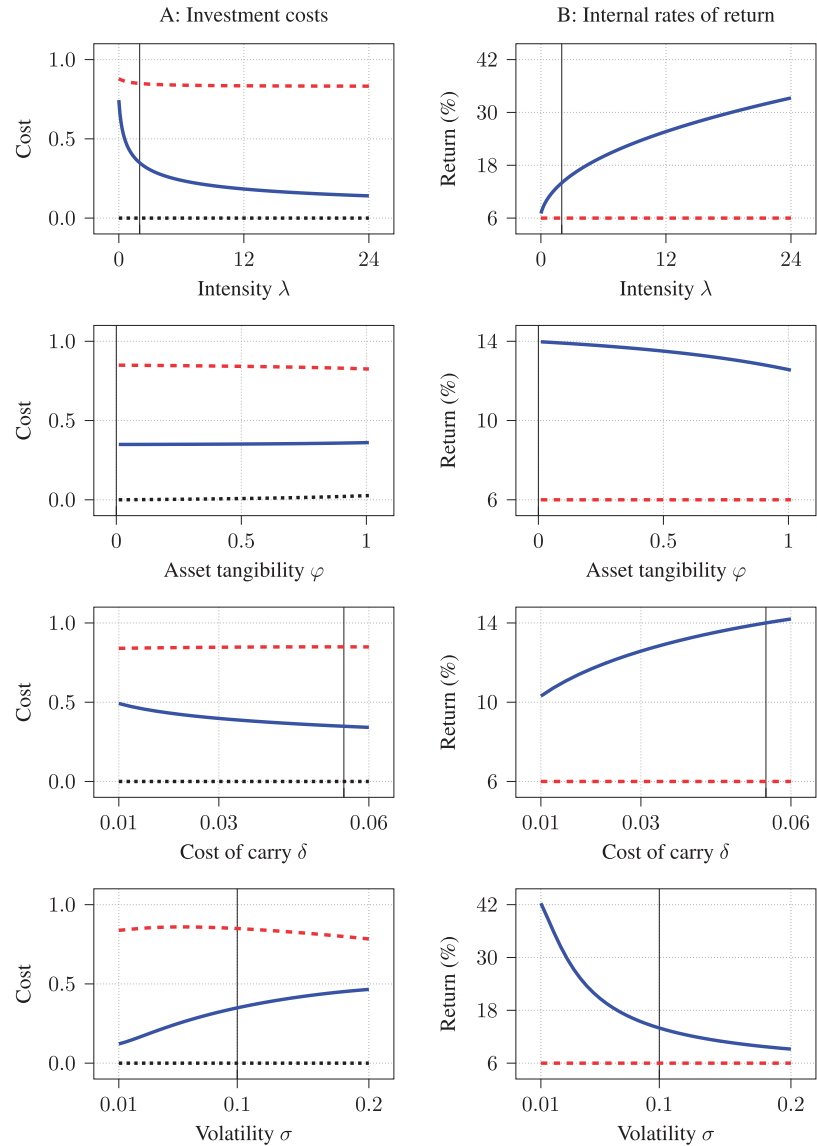


Figure 5
Critical investment costs and internal rates of return
Panel A plots the critical investment costs K (dotted), K^{**} (solid) and K^* (dashed) as functions of the arrival rate of investors λ , the tangibility of assets φ , the carry cost of cash δ , and the volatility of cash flows σ . Panel B plots the internal rate of returns associated with K^{**} (solid) and K^* (dashed) as functions of the same parameters. In each panel, the vertical line indicates the base value of the parameter.

Table 2
Characteristics of the optimal strategy

	Share of growth option in value (%) $1 - \frac{V_0(C_0^*)}{V(C_0^*)}$	Gain from the band strategy (%) $\max_s \frac{V(s)}{U(s)} - 1$	Gambling to rule out the band strategy \bar{G}/σ
BENCHMARK	9.9720	1.0432	12.4039
Meeting intensity (Benchmark: $\lambda = 2.00$)			
$\lambda = 0.50$	10.1757	2.5364	5.2090
$\lambda = 1.00$	10.2919	1.7076	8.0175
$\lambda = 3.00$	9.7091	0.7574	15.9484
Bargaining power of incumbents (Benchmark: $\eta = 0.25$)			
$\eta = 0.00$	9.7868	0.8387	14.8328
$\eta = 0.50$	10.1931	1.4054	9.6132
$\eta = 0.75$	10.2977	2.1836	6.2162
Asset tangibility (Benchmark: $\varphi = 0.00$)			
$\varphi = 0.25$	9.6899	1.0492	12.5850
$\varphi = 0.50$	9.3067	1.0567	12.8213
$\varphi = 0.75$	8.7391	1.0663	13.1421
Carry cost of cash (Benchmark: $\delta = 0.0545$)			
$\delta = 0.01$	7.0375	0.1869	5.2889
$\delta = 0.02$	7.8510	0.3092	7.5069
$\delta = 0.04$	9.1832	0.7095	10.5898
Volatility of cash flows (Benchmark: $\sigma = 0.10$)			
$\sigma = 0.08$	10.6686	1.1114	12.1385
$\sigma = 0.15$	7.2724	0.9132	14.5789
$\sigma = 0.20$	3.4994	0.8535	22.2102

where $C_i^*(R)$ is the optimal level of cash reserves when shareholders' discount rate is R . Firms with projects that fall below the solid line in panel B are firms for which $K > K^{**}$.

Figure 5 shows that only projects with very high internal rates of return (or very low cost of investment) will lead to a barrier policy that mirrors those in prior contributions. In our base case environment, in which the cost of capital is $\rho = 6\%$, only projects with an internal rate of return above 13.53% will induce the firm to follow barrier policies (i.e., will correspond to the low investment cost case). As shown by the figure, the cutoff level for the internal rate of return increases with capital supply and the carry cost of cash and decreases with cash-flow volatility and asset tangibility.

Table 2 reports the value gain associated with a switch from a barrier to a band strategy when $K > K^{**}$. In our base case, firm value increases by 1.04%. As shown by Table 2, this value gain increases with capital supply frictions, the bargaining power of investors, and the carry cost of cash and decreases with volatility. Table 2 also shows that during crisis times, that is, when λ is low and/or η is high, the gain in value can reach 2–3%.

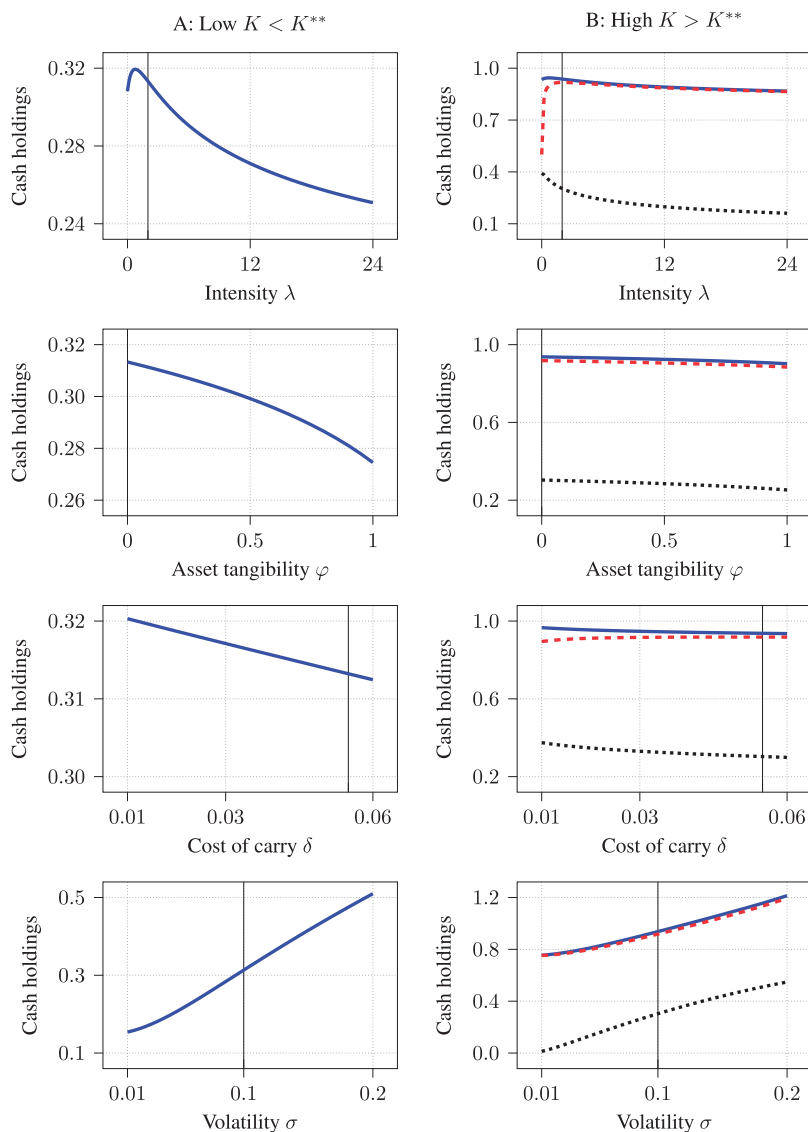
To understand whether band strategies can be eliminated with realistic futures positions, we also compute the value of the threshold \bar{G} that gives the minimum

exposure to the Brownian shock Z_t eliminating these strategies. In our base case environment, we find that the threshold \bar{G} is 12.4 larger than the cash-flow volatility. That is, restoring barrier strategies requires the firm to multiply cash-flow volatility by at least 13.4. To put this number in perspective, it should be noted that a recent study by Duchin et al. (2014) shows, using a firm-by-firm analysis of the composition of cash reserves for industrial firms in the S&P 500 index, that the value of risky securities is on average 26.8% of that of corporate cash holdings. In addition, they find that most of these risky investments are made in debt securities and that the average beta of these risky assets is 50% higher than the beta of the firms' productive assets. Therefore, our results imply that while it is theoretically possible to eliminate the convexity in firm value and make barrier strategies optimal, this may require gambling strategies that are infeasible or unrealistic.

4.2.2 Determinants of cash holdings. Proposition 1 provides an analytical characterization of the properties of target cash holdings after investment, C_1^* . This section examines instead the determinants of target cash holdings before investment. In our base case, we have $K^{**}=0.349$ and $K^*=0.849$. Therefore, we consider two cases: one in which the cost of investment is low in that $K=0.15$ and the firm follows a *barrier* strategy and one in which it is high in that $K=0.75$ and the firm follows a *band* strategy. Figure 6, panel A, plots target the cash holdings C_U^* of the low investment cost case as a function of capital supply λ , asset tangibility φ , cash-flow volatility σ , and agency costs δ . Panel B plots the target cash holdings C_W^* and C_H^* and the intermediate payout threshold C_L^* of the high investment cost case.

The figure shows that the three target cash buffers C_U^* , C_W^* , and C_H^* decrease with the arrival rate of investors, except for very low values of λ . That is, as λ increases, the likelihood of finding outside investors increases and the need to hoard cash within the firm decreases. This result is consistent with the evidence in OPSW (1999) and BKS (2009), who find that firms hold more cash when their access to external capital markets is more limited. For low λ , the firm benefits from increasing its cash holdings to have more time to raise funds from investors and thereby avoid inefficient closure. This effect fades off as λ increases.

Another prediction of the model is that target cash holdings should increase with cash-flow volatility (Because the risk of closure increases with σ), consistent with Harford (1999) and BKS (2009). Interpreting $\delta=\rho-r$ as an agency cost of free cash in the firm, the model also predicts that target cash holdings decrease with the severity of agency conflicts, consistent with Harford, Mansi, and Maxwell (2008). Lastly, the model predicts that cash holdings decrease with asset tangibility φ (Because liquidation becomes less inefficient as φ increases), consistent with Almeida and Campello (2007).

**Figure 6****Cash holdings for a firm with a growth option**

Panel A plots the investment threshold C_U^* for a firm with a low investment cost as a function of the arrival rate of investors λ , the tangibility of assets φ , the carry cost of cash δ and the volatility of cash flows σ . Panel B plots the investment threshold C_H^* (solid) and the payout thresholds C_L^* (dashed) and C_W^* (dotted) for a firm with a high investment cost as functions of the same parameters. In each panel, the vertical line indicates the base value of the parameter.

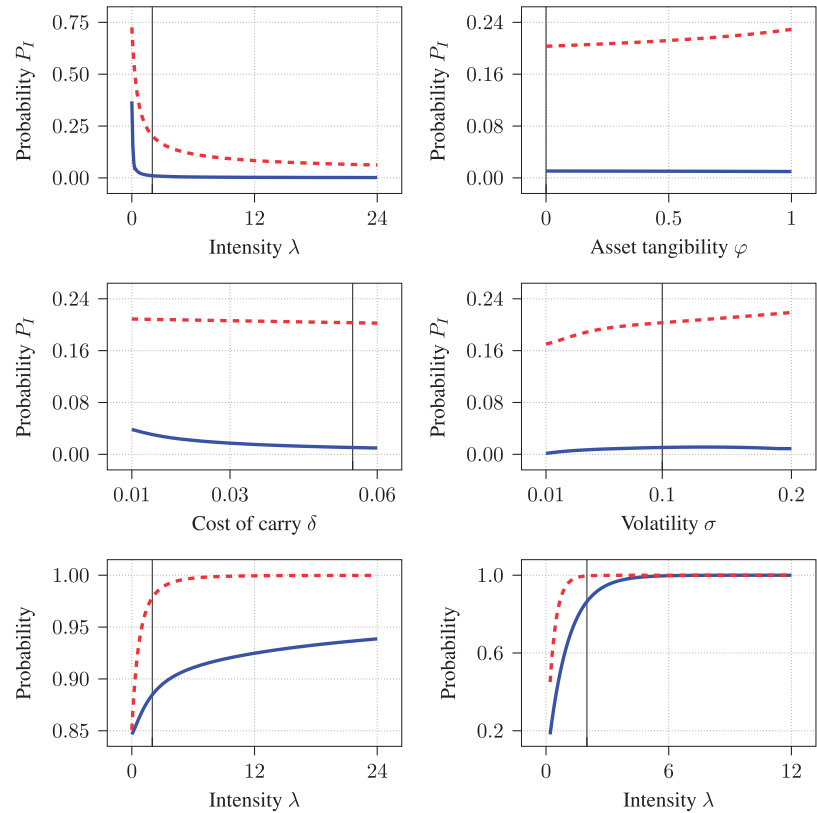


Figure 7
Probabilities of investment

The top four panels plot the average probability of investment with internal funds for a firm with a low investment cost (dashed line) and a high investment cost (solid line) as functions of the arrival rate of investors λ , the tangibility of assets φ , the carry cost of cash δ , and the volatility of cash flows σ . The lower panels plot the total probability of investment at an horizon of one year (solid line) and three years (dashed line) for a firm with cash holdings $C = K < K^{**}$ (left) and $C = K > K^{**}$ (right) as functions of the arrival rate of investors λ . In each panel, the vertical line indicates the base value of the parameter.

4.2.3 Financing investment. An important question is whether capital supply uncertainty affects investment and the source of funds used by firms when financing investment. To answer this question, we examine the determinants of the probability of investment with internal funds ($P_I(c)$). Appendix J shows how to compute this probability.

The top four panels of Figure 7 plot the average probability of investment with internal funds for a cross-section of firms with cash reserves uniformly distributed between 0 and C_U^* ($K < K^{**}$; dashed line) and between 0 and C_H^* ($K > K^{**}$; solid line) as a function of the arrival rate of investors λ , asset tangibility φ , cash-flow volatility σ , and the carry cost of cash δ . The two lower panels plot the total probability of investing over a one-year (solid line)

and over a three-year (dashed line) horizon as functions of the arrival rate of investors λ for a firm with low investment cost (left) and high investment cost (right).

Consider the probability of investment with internal funds (top four panels). In our base-case environment, the probability that the average firm invests with internal funds is 20.31% when the cost of investment is low and 1.06% when the cost of investment is high. This suggests that cash holdings are mostly used to cover operating losses and that firms will wait until external financing arrives before investing, consistent with the large sample studies by OPSW (1999) and BKS (2009) and the survey of Lins, Servaes, and Tufano (2010). Another property of the model illustrated by the figure is that the probability of financing investment with internal funds decreases with the arrival rate of investors and increases with asset tangibility. This last feature follows from the fact that C_U^* decreases with φ and implies that the investment-cash-flow sensitivity increases with the tangibility of assets, consistent with Almeida and Campello (2007).

Lastly, the bottom panels of Figure 7 show that the overall probability of investment before a fixed horizon decreases as λ decreases. In the figure, we assume that $c = K$ so that firms have enough cash to finance investment internally. Thus, our model predicts that a negative shock to the supply of capital may hamper investment even if firms have enough financial slack, consistent with Gan (2007), Becker (2007), Lemmon and Roberts (2007), and Campello, Graham, and Harvey (2010).

5. Concluding Remarks

We develop a model of investment, financing, and cash management decisions in which investment is lumpy and firms face uncertainty regarding their ability to raise funds in capital markets. We characterize optimal policies explicitly and demonstrate that the interplay between lumpiness in investment and capital supply frictions may lead firm value to be locally convex and barrier strategies to be suboptimal. We also investigate whether this convexity may lead management to gamble by engaging in zero NPV investments with random returns. We show that while it is theoretically possible to restore the optimality of barrier strategies, this may require gambling strategies that are infeasible or unrealistic.

Turning to the predictions of the model, we show that firms with high investment costs are qualitatively and quantitatively different in their behaviors from firms with low investment costs. In addition, we show that firms may raise outside funds before exhausting internal resources and that the optimal payout policy may feature several payout regions. The analysis in the paper also reveals that investment and payout do not always increase with slack and that the choice between internal and external funds does not follow a strict pecking order. Finally, the paper generates a number of new

predictions relating the use of inside and outside cash to capital supply and firm characteristics.

Appendix A. Bargaining with Outside Investors

In the Supplementary Appendix, we show that management's optimization problem in an environment with outside bargaining power $\eta > 0$ and meeting intensity λ is equivalent to an auxiliary problem in which there is no bargaining power but a reduced meeting intensity $\lambda^* = \lambda(1 - \eta)$.

On the basis of this result we will assume throughout the Appendixes that there is no bargaining so that we only need to consider the firm's optimization problem with $\eta = 0$.

Appendix B. Proofs of the Results in Section 2.1

B.1 Intuition and Road Map

To facilitate the proofs, we start by introducing some notation that will be of repeated use throughout the appendix. Let \mathcal{L}_i denote the differential operator defined by

$$\mathcal{L}_i \phi(c) := \phi'(c)(rc + \mu_i) + \frac{\sigma^2}{2} \phi''(c) - \rho \phi(c),$$

set

$$\mathcal{F} \phi(c) := \max_{f \geq 0} \lambda(\phi(c + f) - \phi(c) - f),$$

and denote by Θ the set of dividend and financing strategies such that

$$E_c \left[\int_0^{\tau_0} e^{-\rho s} (dD_s + f_s dN_s) \right] < \infty$$

for all $c \geq 0$, where τ_0 is the first time that the firm's cash holdings fall to zero and $E_c[\cdot]$ denotes an expectation conditional on the initial value $C_{0-} = c$.

Let $\hat{V}_i(c)$ denote the value of a firm with no growth option when the mean cash-flow rate is μ_i . To apply dynamic programming techniques, we will proceed in four steps.

1. Derive the Hamilton-Jacobi-Bellmann (HJB) equation.
2. Show that any smooth solution $\phi(c)$ to the HJB equation dominates the value function.
3. Conjecture an optimal policy and derive the corresponding firm value.
4. Show that the value of the firm associated with the conjectured optimal policy of step 3 is indeed a smooth solution to the HJB equation.

In accordance with the theory of singular stochastic control (see Fleming and Soner (1993)), the HJB equation for the value of a firm with no option is given by

$$\max\{\mathcal{L}_i \phi(c) + \mathcal{F} \phi(c), 1 - \phi'(c), \ell_i(c) - \phi(c)\} = 0, \quad (\text{B.1})$$

where $\ell_i(c) = \ell_i + c$ denotes the liquidation value of the firm. Thus, we are done with step 1. Lemma B.1 below accomplishes step 2. To proceed to step 3, we conjecture that the optimal policy is of a threshold form and show that the system defined by (2)–(5) and (9) has a unique smooth solution that is, given by $(V_i(c), C_i^*) = (v_i(c; C_i^*), C_i^*)$ for some $C_i^* > 0$. Finally, to complete step 4, we will show that this function solves (B.1), that is, that

- (a) $V_i'(c) \geq 1$ for all $c \geq 0$,
- (b) $\phi(c) = \ell_i(c)$ satisfies $\mathcal{L}_i \phi(c) + \mathcal{F} \phi(c) \leq 0$ for all $c \leq C_i^*$,
- (c) $V_i(c) \geq \ell_i(c)$ for all $c \geq 0$.

As we show below, $V_i(c)$ is concave and therefore items (a) and (c) easily follow.¹³ Item (b) follows by direct calculation. Finally, step 4 is proven in lemma B.9. Thus, it remains to implement step 3 and show that a solution exists and that it is concave. To this end, we will introduce another function $w(c; b)$ defined as the unique solution to (2) satisfying (5) and (9). As we note in the main text, (2) implies that (9) is equivalent to (10), that is,

$$w_i(b; b) = (rb + \mu_i) / \rho.$$

Thus, for any fixed b , Equation (2) turns into a standard ordinary differential equation that can be explicitly solved via special functions as we show below in lemma B.2. Lemma B.4 proves the concavity of $w_i(c; b)$. The value function $v_i(c; b)$ satisfies (4)–(5). Thus, to determine the optimal threshold, it remains to find b such that $w_i(0; b) = \ell_i(0)$. This is done in lemma B.8. Obviously, $w_i(c; C_i^*) = v_i(c; C_i^*)$, and the proof is complete.

B.2 Proofs

Lemma B.1. If $\phi \in C^2(0, \infty)$ is a solution to (B.1), then $\phi(c; b) \geq \hat{V}_i(c)$.

Proof. Let ϕ be as in the statement, fix a strategy $(D, f) \in \Theta$, and consider the process

$$Y_t := e^{-\rho t} \phi(C_t) + \int_{0+}^t e^{-\rho s} (dD_s - f_s dN_s).$$

Using the assumption of the statement in conjunction with Itô's formula for semimartingales (see Dellacherie and Meyer (1980), Theorem VIII.25), we get $dY_t = dM_t - e^{-\rho t} dA_t$ where the process M is a local martingale and

$$\begin{aligned} dA_t &= (\phi(C_{t-} + f_t) - \phi(C_{t-}) - f_t - \mathcal{F}\phi(C_{t-}))dt \\ &\quad + (\Delta D_t - \phi(C_{t-} - \Delta D_t) + \phi(C_{t-})) + (\phi'(C_{t-}) - 1)dD_t^c, \end{aligned}$$

where ΔD_t and D_t^c denote, respectively, the jump and the continuous components of the dividend policy under consideration, that is,

$$\Delta D_t = D_t - \lim_{s \uparrow t} D_s = D_t - D_{t-}$$

$$D_t^c = D_t - \sum_{s \leq t} \Delta D_s.$$

The definition of \mathcal{F} and the fact that $\phi' \geq 1$ then imply that A is nondecreasing and it follows that Y is a local supermartingale. The liquidation value being nonnegative, we have

$$Z_t := Y_{t \wedge \tau_0} \geq - \int_0^{\tau_0} e^{-\rho s} f_s dN_s$$

and because the random variable on the right-hand side is integrable by definition of the set Θ , we conclude that Z is a supermartingale. In particular,

$$\begin{aligned} \phi(C_{0-}) &= \phi(C_0) - \Delta\phi(C_0) = Z_0 - \Delta\phi(C_0) \geq E_c[Z_{\tau_0}] - \Delta\phi(C_0) \\ &= E_c \left[e^{-\rho\tau_0} \phi(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta\phi(C_0) \\ &= E_c \left[e^{-\rho\tau_0} \ell_i(0) + \int_0^{\tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta D_0 - \Delta\phi(C_0) \\ &\geq E_c \left[e^{-\rho\tau_0} \ell_i(0) + \int_0^{\tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right], \end{aligned} \tag{B.2}$$

¹³ Concavity is a rare and useful property for this class of models. As we show below, firm value is no longer globally concave when the firm has a growth option and, without this property, the verification of property (a) becomes a lot more difficult.

where the first inequality follows from the optional sampling theorem for supermartingales, the fourth equality follows from $C_{\tau_0}=0$, and the last inequality follows from

$$\Delta D_0 + \Delta \phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-} - \Delta D_0}^{C_{0-}} (1 - \phi'(c)) dc \leq 0$$

The desired result follows by taking supremum over $(D, f) \in \Theta$ on both sides of (B.2). ■

Lemma B.2. Let $b \geq 0$ be fixed. Then the unique continuously differentiable solution to

$$\mathcal{L}_i \phi(c; b) + \lambda(\phi(b) - b + c - \phi(c; b)) = 0, \quad c \leq b, \quad (\text{B.3})$$

$$\phi(c; b) - \phi(b) + b - c = 0, \quad c \geq b, \quad (\text{B.4})$$

satisfies

$$\begin{aligned} \phi(c; b) &= \phi(b) H_i(c; b) + \phi(0) L_i(c; b) \\ &\quad + \Pi_i(c; b) - \Pi_i(b; b) H_i(c; b) - \Pi_i(0; b) L_i(c; b). \end{aligned} \quad (\text{B.5})$$

In this equation, we have

$$\Pi_i(c; b) = \frac{\lambda}{\rho + \lambda} \left(\phi(b) - b + c + \frac{\mu_i + r c}{\rho + \lambda - r} \right),$$

and

$$\begin{aligned} L_i(c; b) &= E_c \left[e^{-(\rho + \lambda)\tau_{i,0}} 1_{\{\tau_{i,0} \leq \tau_{i,b}\}} \right] = \frac{G_i(b) F_i(c) - F_i(b) G_i(c)}{G_i(b) F_i(0) - F_i(b) G_i(0)}, \\ H_i(c; b) &= E_c \left[e^{-(\rho + \lambda)\tau_{i,b}} 1_{\{\tau_{i,b} \leq \tau_{i,0}\}} \right] = \frac{F_i(0) G_i(c) - G_i(0) F_i(c)}{G_i(b) F_i(0) - F_i(b) G_i(0)}, \end{aligned}$$

with

$$F_i(x) = M\left(v, 1/2; -(rx + \mu_i)^2 / (\sigma^2 r)\right), \quad (\text{B.6})$$

$$G_i(x) = \frac{rx + \mu_i}{\sigma \sqrt{r}} M\left(v + 1/2, 23/2; -(rx + \mu_i)^2 / (\sigma^2 r)\right), \quad (\text{B.7})$$

where $v = -(\rho + \lambda)/2r$ and $M(a, b; z)$ is the confluent hypergeometric function defined by (see Dixit and Pindyck 1994, 163):

$$M(a, b; z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} \frac{z^3}{3!} + \dots$$

The first claim of lemma B.2 (i.e., formula (B.5)) follows by standard arguments. The proof of the second claim (i.e., representation via the hypergeometric functions) is based on the following auxiliary result.

Lemma B.3. The general solution to the homogenous equation

$$\lambda \phi_i(c) = \mathcal{L}_i \phi_i(c) \quad (\text{B.8})$$

is explicitly given by $\phi_i(c) = \gamma_1 F_i(c) + \gamma_2 G_i(c)$ for some constants γ_1, γ_2 , where the functions F_i, G_i are defined as in Equations (B.6) and (B.7).

Proof. The change of variable $\phi(c; b) = g(-(rc + \mu_i)^2 / (r\sigma^2))$ transforms (B.8) for ϕ into Kummer's ODE for g and the conclusion now follows from standard results. ■

Let $w_i(c; b)$ be the unique continuously differentiable solution to (B.3) and (B.4) with the boundary conditions $w'_i(b; b) - 1 = w''_i(b; b) = 0$.¹⁴

Lemma B.4. The function $w_i(c; b)$ is increasing and concave with respect to $c \geq 0$ and strictly monotone decreasing with respect to b .

To prove lemma B.4, we will rely on the following three useful results.

Lemma B.5. Suppose that k is a solution to

$$\mathcal{L}_i k(c) + \phi(c; b) = 0 \quad (\text{B.9})$$

for some ϕ . Then k does not have negative local minima if $\phi(c; b) \geq 0$ and does not have positive local maxima if $\phi(c; b) \leq 0$.

Proof. At a local minimum we have $k'(c) = 0$, $k''(c) \geq 0$ and the claim follows from (B.9) and the nonnegativity of ϕ . The case of a nonpositive ϕ is analogous. ■

Lemma B.6. Suppose that k is a solution to (B.9) for some $\phi(c; b) \leq 0$ and that $k'(c_0) \leq 0$, $k(c_0) \geq 0$ and $|k(c_0)| + |k'(c_0)| + |\phi(c_0)| > 0$. Then $k(c) > 0$ and $k'(c) < 0$ for all $c < c_0$.

Proof. Suppose on the contrary that $k'(c)$ is not always negative for $c < c_0$ and let z be the largest value of $c < c_0$ at which $k'(c)$ changes sign. Then z is a positive local maximum and the claim follows from lemma B.5. ■

Lemma B.7. Suppose that k is a solution to (B.9) for some ϕ such that $\phi'(c) \leq 0$ and that $k'(c_0) \geq 0$, $k''(c_0) \leq 0$ and $|k'(c_0)| + |k''(c_0)| + |\phi'(c_0)| > 0$. Then $k'(c) > 0$ and $k''(c) < 0$ for all $c < c_0$. Furthermore, if $k''(c_0) = 0$, then $k''(c) > 0$ for $c > c_0$ and $k'(c_0) = \min_{c \geq 0} k'(c)$.

Proof. Differentiating (B.9) shows that $m = k'$ is a solution to $\mathcal{L}_i m(c) + rm(c) + \phi'(c) = 0$ and the conclusion follows from lemma B.6. The case $c > c_0$ is analogous. ■

Proof of lemma B.4 As is easily seen, the function

$$k(c) = w_i(c; b) - \frac{\lambda}{\lambda + \rho} \left(w_i(b; b) - b + \frac{(\rho + \lambda)c + \mu_i}{\lambda + \rho - r} \right)$$

is a solution to (B.8) and satisfies $k'(b) = 1 > 0$ and $k''(b) = 0$. Together with lemma B.7, this implies that $k(c)$ and hence also $w_i(c; b)$ is increasing and concave for $c \leq b$. To establish the required monotonicity, let $b_1 < b_2$ be fixed and consider the function $m(c) = w_i(c; b_1) - w_i(c; b_2)$. Using the first part of the proof it is easily seen that m solves

$$\mathcal{L}_i m(c) - \lambda m(c) - \lambda(1 - r/\rho)(b_1 - b_2) = 0$$

with the boundary conditions $m'(b_1) = 1 - w'_i(b_1; b_2) < 0$, $m''(b_1) = -w''_i(b_1; b_2) \geq 0$. Thus, it follows by a straightforward modification of lemma B.7 that m is monotone decreasing and it only remains to show that $m(b_1) > 0$. To this end, observe that

$$m(b_1) = w_i(b_1; b_1) - w_i(b_1; b_2)$$

¹⁴ Imposing these boundary conditions immediately pins down the value $w_i(b; b)$ at the boundary, and then existence and uniqueness follows from standard results for linear ODEs.

$$\begin{aligned}
 &= w_i(b_1; b_1) - w_i(b_2; b_2) + \int_{b_2}^{b_1} w'_i(c; b_2) dc \\
 &\geq w_i(b_1; b_1) - w_i(b_2; b_2) + b_2 - b_1 = (r/\rho - 1)(b_1 - b_2) > 0
 \end{aligned}$$

where the first inequality follows from $w'_i(b; b) = 1$ and the first part of the proof, and the last inequality follows from the fact that by assumption $\rho > r$. ■

Lemma B.8. There exists a unique solution C_i^* to $w_i(0; C_i^*) = \ell_i(0)$ and the function $V_i(c) = w_i(c; C_i^*) = v_i(c; C_i^*)$ is a twice continuously differentiable solution to (B.1).

Proof. By lemma B.2 we have that $V_i(c)$ is twice continuously differentiable and solves (2) subject to (4), (5), and (9), so we only need to show that

$$w_i(0; C_i^*) = \ell_i(0) \quad (\text{B.10})$$

has a unique solution C_i^* . By lemma B.4, we have that $w_i(0; b)$ is monotone decreasing in b . On the other hand, a direct calculation shows that $w_i(0; 0) = \mu_i / \rho > \ell_i(0)$, $w_i(0; \infty) < 0$ and it follows that (B.10) has a unique solution.

To complete the proof, it remains to show that V_i is a solution to the HJB equation. Using the concavity of $V_i(c) = w_i(c; C_i^*)$ in conjunction with the smooth-pasting condition, we obtain that $1 - V'_i(c)$ is negative below the threshold C_i^* and zero otherwise so that

$$\ell_i(c) - V_i(c) = \int_0^c (1 - V'_i(x)) dx \leq 0.$$

On the other hand, using the concavity of $V_i(c)$ in conjunction with lemma B.2 and the smooth-pasting condition, we obtain

$$\begin{aligned}
 (\mathcal{L}_i + \mathcal{F})V_i(c) &= \mathcal{L}_i V_i(c) + 1_{\{c < C_i^*\}} \lambda(V_i(C_i^*) - b + c - V_i(c)) = 1_{\{c \geq C_i^*\}} \mathcal{L}_i V_i(c) \\
 &= (r - \rho)(c - C_i^*)^+ \leq 0
 \end{aligned}$$

and combining the above results shows that $V_i(c)$ is a solution to (B.1). ■

Lemma B.9. We have $\hat{V}_i(c) \geq V_i(c)$ for all $c \geq 0$.

Proof. Combining the results of lemmas B.1 and B.8 shows that $V_i(c) \geq \hat{V}_i(c)$ for all $c \geq 0$. To establish the reverse inequality, consider the dividend and financing strategy defined by $D_t^* = L_t$ and $f_t^* = (C_t^* - C_{t-})^+$, where the process C evolves according to

$$dC_t = (rC_{t-} + \mu_i)dt + \sigma dB_t - dD_t^* + f_t^* dN_t$$

with initial condition $C_{0-} = c \geq 0$ and $L_t = \sup_{s \leq t} (b_t - C_s^*)^+$ where

$$db_t = (rb_{t-} + \mu_i)dt + \sigma dB_t + (C_t^* - b_{t-})^+ dN_t.$$

To show that the strategy (D^*, f^*) is admissible, we start by observing that

$$E_c \left[\int_0^\infty e^{-\rho t} f_t^* dN_t \right] \leq E_c \left[\int_0^\infty e^{-\rho t} C_t^* dN_t \right] = \frac{\lambda C_i^*}{\rho},$$

where the inequality follows from the definition of f^* . Using this bound in conjunction with Itô's lemma and the assumption that $r < \rho$, we then obtain

$$\begin{aligned} E_c \left[\int_0^t e^{-\rho s} dD_s^* \right] &= C_0 + E_c \left[\int_0^t e^{-\rho s} ((r - \rho)C_{s-} + \mu_i) ds + \int_0^t e^{-\rho s} f_s^* dN_s \right] \\ &\leq C_0 + E_c \left[\int_0^\infty e^{-\rho s} \mu_i ds + \int_0^\infty e^{-\rho s} f_s^* dN_s \right] \leq C_0 + \frac{1}{\rho} (\mu_i + \lambda C_i^*) \end{aligned}$$

for any $t < \infty$, and it now follows from Fatou's lemma that $(D^*, f^*) \in \Theta$. Applying Itô's formula for semimartingales to

$$Y_t = e^{-\rho(t \wedge \tau_0)} V_i(C_{t \wedge \tau_0}) + \int_{0+}^{t \wedge \tau_0} e^{-\rho s} (dD_s^* - f_s^* dN_s)$$

and using the definition of (D^*, f^*) in conjunction with the fact that $V_i(c)$ solves the HJB equation we obtain that Y is a local martingale. Now, using the fact that $C_t \in [0, C_i^*]$ for all $t \geq 0$ together with the increase of V_i , we deduce that

$$|Y_\theta| \leq |V_i(C_i^*)| + \int_0^\infty e^{-\rho t} (dD_t^* + f_{t-}^* dN_t)$$

for any stopping time θ and, because the right-hand side is integrable, we conclude that Y is a uniformly integrable martingale. In particular, we have

$$\begin{aligned} V_i(c) &= Y_{0-} = Y_0 - \Delta Y_0 = Y_0 + \Delta D_0^* = E_c[Y_{\tau_0}] + \Delta D_0^* \\ &= E_c \left[e^{-\rho \tau_0} V_i(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s} (dD_s^* - f_s^* dN_s) \right] + \Delta D_0^* \\ &= E_c \left[e^{-\rho \tau_0} \ell_i(0) + \int_0^{\tau_0} e^{-\rho s} (dD_s^* - f_s^* dN_s) \right] \end{aligned}$$

where the third equality follows from the definition of V_i and the fourth from the martingale property of Y . This shows that $V_i(c) \leq \hat{V}_i(c)$ and establishes the desired result. ■

Lemma B.10. The level of cash holdings C_i^* that is, optimal for a firm with no growth option is monotone decreasing in λ and φ and increasing in σ^2 .

Proof. Monotonicity in φ follows from the definition of C_i^* and the monotonicity of ℓ_i . To establish the required monotonicity in λ , it suffices to show that $w_i(0; b, \lambda)$ is monotone decreasing in λ . Indeed, in this case we have

$$\ell_i(0) = w_i(0; C_i^*(\lambda_1), \lambda_1) \leq w_i(0; C_i^*(\lambda_1); \lambda_2)$$

for all $\lambda_1 < \lambda_2$ and therefore $C_i^*(\lambda_2) \leq C_i^*(\lambda_1)$ because of the fact that $w_i(0; b, \lambda)$ is decreasing in b . To establish the required monotonicity observe that $w_i(b; b, \lambda) = \frac{rb + \mu_i}{\rho}$ does not depend on λ . As a result, it follows from lemma B.2 that the function

$$k(c) = w_i(c; b, \lambda_1) - w_i(c; b, \lambda_2)$$

for some $\lambda_1 < \lambda_2$ satisfies

$$k(b) = k'(b) = k''(b) = k^{(3)}(b) = k^{(4)}(b) = 0$$

and solves the ODE

$$\mathcal{L}_i k(c) - \lambda_1 k(c) = (\lambda_2 - \lambda_1)(w_i(b; b, \lambda_2) - w_i(c; b, \lambda_2) - (b - c)). \quad (\text{B.11})$$

Because, by lemma B.4, $w_i(c; b, \lambda_2)$ is concave in c and $w'_i(b; b, \lambda_2) = 1$, the right-hand side of (B.11) is nonnegative for all $c \leq b$ and it follows by a slight modification of lemma B.5 that $k(c)$ cannot have a positive local maximum. Because

$$k^{(5)}(b) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)w_i^{(3)}(b; b, \lambda_2) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)(\rho - r) < 0,$$

we conclude that k is decreasing in a small neighborhood of b . Therefore, it is decreasing for all $c \leq b$ and hence $k(c) > k(b) = 0$ for all $c \leq b$. Similarly, if $\sigma_1^2 > \sigma_2^2$, then $k(c) = w_i(c; b; \sigma_1^2) - w_i(c; b; \sigma_2^2)$ satisfies

$$\mathcal{L}_i(\sigma_1^2)k(c) - \lambda k(c) = 0.5(\sigma_2^2 - \sigma_1^2)w_i''(c; b; \sigma_2^2) > 0$$

for $c \leq b$ and the required monotonicity follows by the same arguments as above. ■

Appendix C. Proof of Proposition 2

The proof of proposition 2 will be based on a series of lemmas. To facilitate the presentation, let \hat{V} denote the value of the firm and Π denote the set of triples $\pi = (\tau, D, f)$, where τ is a stopping time and $(D, f) \in \Theta$ is an admissible dividend and financing strategy.

Our first result shows that for any fixed policy π , the value of the firm is equal to the present value of all dividends net of issuing costs, up to the time τ of investment, plus the present value of the value of the firm at the time of investment.

Lemma C.1. The value of the firm satisfies

$$\hat{V}(c) = \sup_{\pi \in \Pi} E_c \left[\int_0^{\tau \wedge \tau_0} e^{-\rho t} (dD_t - f_t dN_t) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_1(C_\tau) \right].$$

If $V_0(c) \geq V_1(c - K)$ for all $c \geq K$ then it is optimal to abandon the growth option.

Proof. The proof of the first part follows from standard dynamic programming arguments and therefore is omitted. To establish the second part assume that $V_0(c) \geq V_1(c - K)$ for all $c \geq K$ and observe that $\Delta C_\tau = -K + 1_{\{\tau \in \mathcal{N}\}} f_\tau$ where \mathcal{N} denotes the set of jump times of the Poisson process. Using this identity in conjunction with the first part, we obtain

$$V(c) \leq \sup_{\pi \in \Pi} E_c \left[\int_0^{\tau \wedge \tau_0} e^{-\rho t} (dD_t - f_t dN_t) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_0(C_{\tau-} + 1_{\{\tau \in \mathcal{N}\}} f_\tau) \right].$$

and the desired result follows because the right-hand side of this inequality is equal to $V_0(c)$ by standard dynamic programming arguments. ■

By lemma C.1, the option has a nonpositive net present value if and only if $V_0(c) \geq V_1(c - K)$ for all $c \geq 0$. Thus, to establish proposition 2 it now suffices to show that this condition is equivalent to the inequality $K \geq K^*$. This is the objective of the following:

Lemma C.2. The constant K^* is nonnegative and the following are equivalent:

- (a) $K \geq K^*$
- (b) $K \geq V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*)$
- (c) $V_0(c) \geq V_1(c - K)$ for all $c \geq K$.

Proof. The equivalence of (a) and (b) follows from the definition of K^* and the fact that

$$V_i(C_i^*) = (rC_i^* + \mu_i) / \rho.$$

To show that the constant K^* is nonnegative, we argue as follows: Because $\mu_0 < \mu_1$, the set of feasible strategies for V_0 is included in the set of feasible strategies for V_1 . It follows that $V_0 \leq V_1$ and combining this with the definition of C_i^* shows that

$$K^* = V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*) = \max_{C \geq 0} \{V_1(C) - C\} - \max_{C \geq 0} \{V_0(C) - C\} \geq 0.$$

To establish the implication (a) \Rightarrow (c) it suffices to show that under (a) we have $V_1(c - K^*) \leq V_0(c)$ for all $c \geq K^*$. Indeed, if that is, the case, then (b) also holds because

$$V_1(c - K^*) \leq V_1(c - K^*), \quad c \geq K^* \geq K^*,$$

because of the increase of the function V_1 . For $c \geq K^* \vee C_0^*$, the concavity of V_1 and the fact that V_0 is linear with slope one above the level C_0^* jointly imply that

$$V_1(c - K^*) \leq V_1(C_1^*) + (c - K^* - C_1^*) = V_0(c) + C_0^* - V_0(C_0^*) - K^* - C_1^* = V_0(c)$$

and it remains to prove the result for $c \in [K^*, C_0^*]$. Consider the function $k(c) = V_0(c) - V_1(c - K^*)$. Using lemma B.2 in conjunction with the fact that $C_0^* < C_1^* + K^*$ by lemma C.3 below we have that the function k is a solution to

$$\mathcal{L}_0 k(c) - \lambda k(c) + (\mu_0 - \mu_1 + rK^*)V_1'(c - K^*) = 0$$

on the interval $[K^*, C_0^*]$. Combining lemma C.3 below with the increase of V_1 shows that the last term on the left-hand side is positive and because

$$k(C_0^*) = V_0(C_0^*) - V_1(C_0^* - K^*) \geq V_0(C_0^*) - V_1(C^*) - (C_0^* - K^* - C_1^*) = 0,$$

$$k'(C_0^*) = V_0'(C_0^*) - V_1'(C_0^* - K^*) = 1 - V_1'(C_0^* - K^*) \leq 0$$

by the concavity of V_1 , we can apply lemma B.6 to conclude that $k(c) \geq 0$ for all $c \leq C_0^*$. Finally, the implication (c) \Rightarrow (b) follows by taking $c > C_0^* \vee (C_1^* + K^*)$. ■

The optimal cash levels before and after investment are given by C_0^* and $C_1^* + K$ respectively. By lemma C.2, we have that $V_0(c) \geq V_1(c - K^*)$ for all $c \geq K$. Therefore, it is natural to expect that the better-off firm will have a lower optimal cash level, that is, $C_0^* \leq C_1^* + K^*$. Similarly, it is natural to expect that the increase in the mean cash-flow rate net of cost is nonnegative for $K \leq K^*$, that is, $(\mu_1 - \mu_0)/r \leq K$. It turns out that both of these intuitive results are indeed true, as is shown by the following lemma.

Lemma C.3. We have $C_0^* < C_1^* + K^*$ and $\mu_1 - \mu_0 - rK^* > 0$.

Proof. The definition of K^* implies that the first inequality is equivalent to the second which is in turn equivalent to

$$r(C_1^* - C_0^*) > \mu_0 - \mu_1. \quad (\text{C.12})$$

Denote by $C^*(\mu)$ the optimal cash holdings for a firm with mean cash-flow rate μ and no growth option. To prove the validity of (C.12), it suffices to show that

$$\frac{1}{r} + \frac{\partial C^*(\mu)}{\partial \mu} > 0 \quad (\text{C.13})$$

Let $w(c; b, \mu)$ stand for the function $w_i(c; b)$ with $\mu_i = \mu$ and use a similar notation for the hypergeometric functions $F_i(c)$, $G_i(c)$ and the liquidation value $\ell_i(c)$. Combining the result of lemma B.2 with (10) and Abel's identity

$$G'(c; \mu)F(c; \mu) - F'(c; \mu)G(c; \mu) = e^{-\frac{(rc+\mu)^2}{r\sigma^2}} \frac{\sqrt{r}}{\sigma}$$

we get that

$$w(c; b, \mu) = q(c; b, \mu) + \frac{\lambda}{\rho + \lambda} \left(\frac{\mu + br}{\rho} - b + c + \frac{\mu + rc}{\rho + \lambda - r} \right),$$

where the function on the right-hand side is defined by

$$q(c; b, \mu) = \alpha(b; \mu)F(c; \mu) - \beta(b; \mu)G(c; \mu) \quad (\text{C.14})$$

with

$$\alpha(c; \mu) = \frac{(r - \rho)\sigma^3 G''(c; \mu)}{2\sqrt{r}(\rho + \lambda - r)(\rho + \lambda)e^{-(\sigma^2 r)^{-1}(rc + \mu)^2}},$$

$$\beta(c; \mu) = \frac{(r - \rho)\sigma^3 F''(c; \mu)}{2\sqrt{r}(\rho + \lambda - r)(\rho + \lambda)e^{-(\sigma^2 r)^{-1}(rc + \mu)^2}}.$$

In this notation, we have that the equation that defines the optimal level of cash holdings for a firm with no growth option is given by

$$q(0; C^*(\mu), \mu) + \frac{\lambda}{\lambda + \rho} \left(\frac{\mu + (r - \rho)C^*(\mu)}{\rho} + \frac{\mu}{\lambda + \rho - r} \right) = \ell(0).$$

Using Equations (B.6) and (B.7) in conjunction with the definition of the functions $\alpha(c; \mu)$ and $\beta(c; \mu)$ we obtain

$$rq_\mu(0; b, \mu) = q_c(0; b, \mu) + q_b(0; b, \mu),$$

where a subscript denotes a partial derivative, and it thus follows from the implicit function theorem that

$$\frac{\partial C^*(\mu)}{\partial \mu} = \frac{\varphi/\rho - q_b(0; C^*(\mu), \mu)/r - q_c(0; C^*(\mu), \mu)/r - B}{q_b(0; C^*(\mu), \mu) - A}$$

where we have set

$$A = \frac{\lambda}{\lambda + \rho} \left(1 - \frac{r}{\rho} \right), \quad B = \frac{\lambda}{\lambda + \rho} \left(\frac{1}{\rho} + \frac{1}{\lambda + \rho - r} \right).$$

By lemma B.4 we have that the function $q(c; b, \mu)$ is decreasing in b and because $A > 0$ it follows that the validity of inequality (C.13) is equivalent to

$$-q_b(0; C^*(\mu), \mu) - q_c(0; b; \mu) - r(B - \varphi/\rho) < A - q_b(0; C^*(\mu), \mu),$$

which in turn follows from

$$q_c(0; b; \mu) + r(B - 1/\rho) > 0. \quad (\text{C.15})$$

Because the difference $q - w$ is a linear function of c , we have from lemma B.4 that $q(c; b, \mu)$ is concave in c and it follows from the smooth-pasting condition that

$$q_c(0; C^*(\mu), \mu) \geq q_c(C^*(\mu); C^*(\mu), \mu)$$

$$= w_c(C^*(\mu); C^*(\mu), \mu) - \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r}.$$

Combining this with a straightforward calculation shows that (C.15) holds. ■

Appendix D. Optimality of Barrier Strategies

D.1 Intuition and Road Map

In this section we construct the function $u(c; C_U^*)$ that fits $V_1(c - K)$ at some cash level C_U^* . Whether this fit is smooth will depend on the size of the investment cost K .

Let us first consider the function $u(c; K)$ associated to the following policy: Search for external investors to raise funds and invest when $c < K$; otherwise invest from internal funds when $c = K$ and liquidate immediately thereafter to receive $\ell_1(c)$. When $c = K$ the firm will just invest and immediately liquidate, so that $u(c; K)$ satisfies the boundary conditions $u(0; K) = \ell_0(0)$ and $u(K; K) = \ell_1(0)$. This policy can be optimal even though the function $u(c; K)$ does not satisfy the smooth fit principle if the marginal value of cash is higher slightly below the investment cost than slightly above, that is, if $u'(K; K) > V_1'(0)$. lemma D.2 confirms this intuition by showing that $u'(K; K) \leq V_1'(0)$ is both necessary and sufficient for the existence and uniqueness of a solution to (12), (13) and (14) with the smooth-pasting condition (17). When $u'(K; K) > V_1'(0)$ we let $C_U^* = K$ so that $u(c; C_U^*) = u(c; K)$ and the fit to the postinvestment firm value is nonsmooth.

D.2 Proofs

Lemma D.1. Let F_0, G_0 be as in (B.6)-(B.7). Let q denote an arbitrary function and define \hat{q} implicitly through

$$q(c) = F_0(c)\hat{q}(Z(c)) = F_0(c)\hat{q}\left(\frac{G_0(c)}{F_0(c)}\right).$$

Then we have:

- (a) the function Z is monotone increasing and $\hat{q}(y) = q(Z^{-1}(y))/F_0(Z^{-1}(y))$,
- (b) the function q solves (B.8) if and only if the function \hat{q} is linear and
- (c) for an arbitrary $c \geq 0$,

$$\min\{\hat{q}'(y)(q(c)/F_0(c))', \hat{q}''(y)(\mathcal{L}_0 q(c) - \lambda q(c))\} \geq 0$$

with $y = Z(c)$.

Proof. The first two claims follow by direct calculation using the definition of \hat{q} , F_0 , and G_0 . The third claim is formula (6.2) in Dayanik and Karatzas (2003). ■

Lemma D.2. If the condition $u'(K; K) \leq V_1'(0)$ holds, then there exists a unique solution $(C_U^*, u(c; C_U^*))$ to (12), (13), (14) and (17), and this solution is such that $u(c; C_U^*) \geq V_1(c - K)$ for all $c \geq K$.

Proof. Consider the function defined by

$$v_1(c - K) = V_1(c - K) - \Phi(c)$$

with $\Phi(c)$ as in (16). To prove the result, we start by observing that thanks to lemma D.1, finding a solution to the system (12), (13), (14), (17) is equivalent to finding a linear function ϕ that is, tangent to the graph of the function \hat{v}_1 defined by¹⁵

$$v_1(c - K) = F_0(c)\hat{v}_1(Z(c)) = F_0(c)\hat{v}_1\left(\frac{G_0(c)}{F_0(c)}\right) \quad (\text{D.16})$$

¹⁵ Namely, lemma D.1 shows that the map $v \mapsto \hat{v}$ transforms solutions to the homogeneous ODE into linear functions that are solutions to the homogeneous ODE for the case in which the underlying stochastic process is a Brownian motion, and there is no time discounting.

and such that $\phi(Z(0))F_0(0)=\ell_0-\Phi(0)$. A direct calculation using the results of lemmas B.8 and C.3 shows that

$$\mathcal{L}_0 v_1(c-K)-\lambda v_1(c)=(r-\rho)(c-C_1^*-K)^++(\mu_0-\mu_1+rK)V_1'(c-K)\leq 0$$

for all $c\geq K$, and it now follows from lemma D.1 that $\hat{v}_1(y)$ is concave for all $y\geq Z(K)$. On the other hand, because V_1 is concave we obtain

$$v_1'(c-K)=V_1(c-K)-\frac{\lambda}{\lambda+\rho-r}\geq V_1'(C_1^*)-\frac{\lambda}{\lambda+\rho-r}=\frac{\rho-r}{\lambda+\rho-r}>0$$

and it follows $v_1(c-K)$ is positive for sufficiently large values of c . Because F_0 is nonnegative and decreasing, this implies that the ratio $v_1(c-K)/F_0(c)$ is increasing for large c and it now follows from lemma D.1 that $\hat{v}_1(y)$ is increasing for large values of y and is therefore globally increasing in $y\geq Z(K)$ because it is concave in that region.

Because by assumption $u'(K;K)\leq V_1'(0)$ we have that the line passing through the points

$$(Z(0),(\ell_0-\Phi(0))/F_0(0)) \quad \text{and} \quad (Z(K),(\ell_1-\Phi(0))/F_0(0))$$

has a higher slope at $y=Z(K)$ than \hat{v}_1 . Using the concavity and increase of \hat{v}_1 , it is then immediate that there exists a unique line passing through $(Z(0),(\ell_0-\Phi(0))/F_0(0))$ that is, tangent to \hat{v}_1 at some $y^*>Z(K)$. Setting $C_U^*=Z^{-1}(y^*)$ proves the existence of a unique solution to the value-matching and smooth-pasting conditions. Because \hat{v}_1 is concave, it lies below its tangent line at y^* and, transforming back to $V_1(c-K)$ and $u(c;C_U^*)$, we get $u(c;C_U^*)\geq V_1(c-K)$. ■

Lemma D.3. We have $u(c;C_U^*)\leq \hat{V}(c)$, where $\hat{V}(c)$ denotes the value function of the firm's optimization problem.

Proof. Consider the investment, dividend, and financing strategy π^U defined by $\tau=\tau_N\wedge\tau_U^*$, $D^U=0$ and

$$f_t^U=(C_1^*+K-C_{t-})^+ \tag{D.17}$$

where τ_N denotes the first jump time of the Poisson process and τ_U^* denotes the first time that the firm's cash reserves reach the level C_U^* . As is easily seen, we have

$$E_c\left[\int_0^{\tau_0}e^{-\rho t}(dD_t^U+f_t^U dN_t)\right]\leq E_c\left[\int_0^\infty e^{-\rho t}(C_1^*+K)dN_t\right]=\frac{\lambda}{\rho}(C_1^*+K)$$

and it follows that $\pi^U\in\Pi$. On the other hand, using an argument similar to that of the proof of proposition 1 it can be shown that

$$Y_t=e^{-\rho t\wedge\tau_0\wedge\tau_U^*}u(C_{t\wedge\tau_0\wedge\tau_U^*};C_U^*)+\int_0^{t\wedge\tau_0\wedge\tau_U^*}e^{-\rho t}(dD_s^U-f_s^U dN_s)$$

is a uniformly integrable martingale. An application of the optional sampling theorem at the finite stopping time τ_N then implies

$$\begin{aligned} u(c;C_U^*)=Y_0=E[Y_{\tau_N}]=E_c\left[e^{-\rho\tau\wedge\tau_0}u(C_{\tau\wedge\tau_0};C_U^*)+\int_0^{\tau\wedge\tau_0}e^{-\rho t}(dD_s^U-f_s^U dN_s)\right] \\ =E_c\left[1_{\{\tau<\tau_0\}}e^{-\rho\tau}V_1(C_\tau)+1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_0+\int_0^{\tau\wedge\tau_0}e^{-\rho t}(dD_s^U-f_s^U dN_s)\right] \end{aligned}$$

and the desired result now follows from lemma C.1. ■

Appendix E. Value of a Firm that Only Invests with External Funds

E.1 Intuition and Road Map

The goal of this section is to construct and derive useful properties of the value $W(c)$ of a firm that accumulates cash to an optimally determined target level $C_W^* = C_W^*(K)$ and invests exclusively from external funds as soon as financing can be secured. The formal existence and basic properties of this function are provided in lemma E.1.

By lemma C.2, the option has a positive net present value for $K < K^*$ in which case $V_0(c)$ is smaller than $V_1(c - K)$ for some values of $c > 0$. We expect the same to be true for the function $W(c)$. Clearly, we have that $V_1(c) > W(c)$ and so we expect that $W(c) < V_1(c - K)$ for high levels of cash holdings when the effect of the fixed investment cost is smaller. By definition, we have that $W(c) = W(C_W^*) + (C - C_W^*)$ and $V_1(c) = V_1(C_1^*) + (c - C_1^*)$ for $c \geq \max\{C_W^*, C_1^* + K\}$. Therefore, our conjecture can be confirmed by showing that $W(C_W^*) - C_W^* < V_1(C_1^*) - (C_1^* + K)$. This is accomplished by lemma E.3 whose proof is in turn based on lemma E.2. Having established this important result, we naturally expect that the inequality $V_1(c - K) > W(c)$ can be violated only at low cash holdings levels and this intuition is confirmed by lemma E.4.

E.2 Proofs

Denote the value of the firm by $\hat{V}(c)$ as before. The following lemma establishes existence of the function $W(c)$ and shows that it is concave. To state the result let $w(c; b)$ be the twice continuously differentiable solution to

$$0 = \mathcal{L}_0 w(c; b) + \lambda(V_1(C_1^*) - C_1^* - K + c - w(c; b)) = w'(b; b) - 1 = w''(b; b).$$

The same argument as for the function $V_i(c)$ implies that

$$w(b; b) = \frac{rb + \mu + \lambda(V_1(C_1^*) - C_1^* - K + b)}{\rho + \lambda},$$

and the optimal target level C_W^* for the firm's cash holdings can then be determined by imposing the value-matching condition

$$w(0; C_W^*) = \ell_0(0) \quad (\text{E.18})$$

at the point where the firm runs out of cash.

Lemma E.1. There exists a unique solution C_W^* to (E.18) and the function $W(c) = w(c; C_W^*)$ is increasing, concave, and satisfies $W(c) \leq \hat{V}(c)$.

Proof. The results follow by arguments similar to those we used in the proof of Proposition 1. We omit the details. The fact that $W(c) \leq \hat{V}(c)$ follows from the fact that the policy corresponding to the value function $W(c)$ is a priori suboptimal. ■

By lemma C.2, the constant K^* is the investment cost for which the net present value of the growth option is identically zero. Therefore, as K increases to K^* the firm following the strategy associated with $W(c)$ will gradually reduce its target cash holding level and for $K = K^*$ the difference between the functions $V_0(c)$ and $W(c)$ should vanish at which point we should have $C_W^*(K^*) = C_0^*$. The following lemma shows that this intuition is correct.

Lemma E.2. The threshold $C_W^* = C_W^*(K)$ is decreasing in K and satisfies $C_W^*(K^*) = C_0^*$.

Proof. Using the same arguments as in the case without growth options we have that that solving (E.18) is equivalent to solving for b in

$$w(0; b) = q(0; b; \mu_0) + \frac{\lambda}{\lambda + \rho} \left((r/\rho - 1)C_1^* - K + \frac{\mu_1}{\rho} + \frac{\mu_0}{\lambda + \rho - r} \right) = \ell_0(0)$$

where the function $q(c; b; \mu_0)$ is defined as in (C.14). As shown in the proof of lemma B.4 the function $q(c; b; \mu_0)$ is monotone decreasing in b , so that $w(0; b)$ is monotone decreasing. On the other hand, a direct calculation shows that $w(0; 0) = \mu_0/\rho > \ell_0(0)$ and $w(0; \infty) < 0$ and it follows that there exists a unique solution C_W^* to the value-matching condition.

Because the function $q(c; b; \mu_0)$ is decreasing in b the monotonicity of C_W^* with respect to K follows from the implicit function theorem. To show that the target C_W^* converges to C_0^* as the investment cost converges to K^* we argue as follows. By definition, we have

$$V_1(C_1^*) - C_1^* - K^* = V_0(C_0^*) - C_0^*.$$

Thus, it follows from lemma B.8 that the function $V_0(c)$ solves

$$0 = \mathcal{L}_0 V_0(c) + \lambda[V_0(C_0^*) - C_0^* + c - V_0(c)] = \mathcal{L}_0 V_0(c) + \lambda[V_1(C_1^*) - C_1^* - K^* + c - V_0(c)]$$

on the interval $[0, C_0^*]$ with the boundary conditions $V_0'(C_0^*) = 1$, $V''(C_0^*) = 0$ and the desired result follows from the uniqueness part of lemma E.1. ■

By definition, we have that $K < K^*$ if and only if $V_0(C_0^*) - C_0^* < V_1(C_1^*) - (C_1^* + K)$. The following lemma shows that a similar inequality holds true for the function $W(c)$.

Lemma E.3. The following are equivalent:

- (a) $K > K^*$,
- (b) $W(C_W^*(K)) - C_W^*(K) > V_1(C_1^*) - (C_1^* + K)$.

Proof. Evaluating the ODE

$$\mathcal{L}_0 W(c) + \lambda[V_1(C_1^*) - C_1^* - K + c - W(c)] = 0$$

at the point $c = C_W^*$ and using the definition of K^* , we obtain that

$$(\lambda + \rho)(W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K)) = \rho(K - K^*) + (\rho - r)(C_0^* - C_W^*)$$

and the desired equivalence now follows from lemma E.2. ■

Lemma E.4. The following statements hold:

- (a) If $K \geq K^*$, then $W(c) \geq V_1(c - K)$ for all $c \geq K$.
- (b) If $K < K^*$, then either $V_1(c - K) \geq W(c)$ for all $c \geq K$ or there exists a unique crossing point $K \leq \tilde{C} \leq C_1^* + K$ such that $V_1(c - K) < W(c)$ if and only if $c < \tilde{C}$.

Proof. We only prove part (b) as both claims follow from similar arguments. Because the function $W(c) = w(c; C_W^*)$ is concave by lemma E.1 and $W'(C_W^*) = 1$, we have

$$W(c) \leq W(C_W^*) + c - C_W^*$$

and it now follows from lemma E.3 that

$$k(c) = W(c) - V_1(c - K) \leq W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K) \leq 0.$$

for all $c \geq C_1^* + K$. To complete the proof of the first part we distinguish three cases depending on the location of the threshold C_W^* .

Case 1: $C_W^* \leq K$. In this case the function $W(c)$ is linear for $c \geq K$. Because $V_1(c)$ is concave, the functions $V_1(c - K)$ and $W(c)$ can have at most two crossing points. But because $V_1(c - K) > W(c)$ for large c as shown above there can be at most one crossing point.

Case 2: $C_W^* \geq C_1^* + K$. Suppose towards a contradiction that the function k has more than one zero and denote by $z_0 \leq z_1$ its two largest zeros in the interval $[K, C_1^* + K]$. Then $k(c) > 0$ for $c \in (z_0, z_1)$ because of the above inequality and it follows that the function k has a positive local maximum in the open interval (z_0, z_1) . Because $C_W^* \geq C_1^* + K$ it follows from lemmas B.8 and E.1 that the function $k(c)$ satisfies

$$\mathcal{L}_0 k(c) - \lambda k(c) + (\mu_0 - \mu_1 + rK) V_1'(c - K) = 0 \quad (\text{E.19})$$

in the interval $[0, C_1^* + K]$ and the required contradiction now follows from lemma B.5 and the fact that $\mu_1 - \mu_0 - rK > 0$ whenever $K \leq K^*$ as a result of lemma C.3.

Case 3: $K \leq C_W^* \leq C_1^* + K$. If $z_1 \leq C_W^*$, then the same argument as in case 2 still applies so assume that $k(c)$ has zeros in the interval $[C_W^*, C_1^* + K]$. Because $V_1(c - K)$ is concave and $k(C_1^* + K) \leq 0$ we know that it can have at most one zero there. Denote the location of this zero by \bar{z} so that $k(c) > 0$ for $c \in [C_W^*, \bar{z})$ and $k(c) \leq 0$ for $c \geq \bar{z}$. Because $k(c)$ solves (E.19) on $[0, C_W^*]$ and satisfies $k(C_W^*) > 0$ and $k'(C_W^*) = 1 - V_1'(C_W^* - K) < 0$ it follows from lemma B.6 that $k(c) > 0$ for all $c \leq C_W^*$. ■

Appendix F. Proof of Theorems 1 and 2

F.1 Intuition and Road Map

To prove theorems 1 and 2, we need to proceed with the following four steps.

1. The HJB equation now takes the form

$$\max\{\mathcal{L}_0 \phi(c; b) + \mathcal{F} \phi(c; b); 1 - \phi'(c); V_1(c - K) - \phi(c; b), \ell_0(c) - \phi(c; b)\} \leq 0. \quad (\text{F.20})$$

The only difference between this equation and the HJB Equation (B.1) without the option is that because the firm may invest as soon as it has enough cash the value function needs to satisfy the additional condition $\phi(c; b) \geq V_1(c - K)$ for all $c \geq K$.

2. This step consists in establishing a verification result for the HJB equation and is accomplished by lemma F.1 below.
3. To proceed with step 3 we need to conjecture the form of the optimal policy to (F.20). The first candidate for this is the barrier policy associated with $u(c; C_U^*)$. For this strategy to be optimal, it should be that its value dominates that of any other policy. In particular, it is necessary that $u(c; C_U^*) \geq W(c)$ for all $c \geq 0$ and, because $u(0; C_U^*) = W(0) = \ell_0(0)$ by construction it is also necessary that $u'(0; C_U^*) \geq W'(0)$. Lemma F.2 shows this condition is in fact both necessary and sufficient for the function $u(c; C_U^*)$ to be the value function and for the global optimality of the corresponding barrier policy.

Suppose now that $W'(0) > u'(0; C_U^*)$ and consider the function $\bar{W}(c)$ that solves Equation (2) and coincides with the function $W(c)$ for all $c \leq C_W^*$. Because the functions $U(c)$ and $\bar{W}(c)$ solve the same equation subject to the same initial value our assumption about the derivatives of these functions at the origin allows us to show that $\bar{W}(c) > u(c; C_U^*)$ for all $c > 0$. By construction, the function $\bar{W}(c)$ touches the linear part of the function $W(c)$ at the point C_W^* . Pick a cash level C_L^* that we will vary between C_W^* and the threshold \bar{C} of lemma E.4 and for each such level denote by $S(c; C_L^*)$ the unique solution to (2) satisfying value-matching and smooth-pasting with $W(c)$ at the point C_L^* . By continuity,

there exists a unique $C_L^* \in (C_W^*, \tilde{C})$ such that $S(c; C_L^*)$ touches the graph of the function $V_1(c - K)$ from above at some point C_H^* and we will take the function

$$V(c) = 1_{\{c \leq C_L^*\}} W(c) + 1_{\{C_L^* < c \leq C_H^*\}} S(c; C_L^*) + 1_{\{c > C_H^*\}} V_1(c - K)$$

as the value of our candidate optimal policy. A rigorous implementation of this construction is provided below.

4. step 4 consists in proving that the value of our candidate optimal strategy solves the HJB Equation (F.20) and is accomplished by lemma F.5.

Once these four steps are complete, it will remain to show that $u(c; C_U^*)$ is the value function if and only if the investment cost is below a threshold K^{**} and that $C_U^* = K$ if and only if K is below another threshold $\underline{K} < K^{**}$. This is done in lemmas F.6 and F.8.

F.2 Proofs

We start this Appendix with a standard verification result for the HJB equation associated with the firm's problem:

Lemma F.1. If $\phi(c)$ is a continuous and piecewise twice continuously differentiable function such that

$$\max\{\mathcal{L}_0\phi(c) + \mathcal{F}\phi(c); 1 - \phi'(c); V_1(c - K) - \phi(c), \ell_0(c) - \phi(c)\} \leq 0$$

and

$$\phi'(c_-) \geq \phi'(c_+)$$

at each point of discontinuity of $\phi'(c)$ then $\phi(c) \geq \hat{V}(c)$ for all $c \geq 0$.

Proof. Fix an arbitrary strategy $\pi \in \Pi$, denote by C_t the corresponding cash buffer process and consider the process

$$Y_t = e^{-\rho t \wedge \tau_0} \phi(C_{t \wedge \tau_0}) + \int_{0+}^{t \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s).$$

Using the Ito-Tanaka formula (see Karatzas and Shreve 1991, chapter 3.6) together with arguments similar to those of the proof of lemma B.1 it can be shown that Y_t is a local supermartingale and because

$$Y_t \geq - \int_0^{\tau_0} e^{-\rho s} f_s dN_s,$$

where the right-hand side is integrable by definition of Π , we conclude that Y_t is a supermartingale. In particular, for any stopping time τ , we have

$$\begin{aligned} \phi(c) &= \phi(C_0) - \Delta\phi(C_0) = Y_0 - \Delta\phi(C_0) \geq E_c[Y_\tau] - \Delta\phi(C_0) \\ &= E_c \left[e^{-\rho \tau \wedge \tau_0} \phi(C_{\tau \wedge \tau_0}) + \int_{0+}^{\tau \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta\phi(C_0) \\ &= E_c \left[e^{-\rho \tau \wedge \tau_0} \phi(C_{\tau \wedge \tau_0}) + \int_0^{\tau \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta D_0 - \Delta\phi(C_0) \\ &\geq E_c \left[1_{\{\tau_0 \leq \tau\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau_0 > \tau\}} e^{-\rho \tau} V_1(C_\tau) \right] \\ &\quad + E_c \left[\int_0^{\tau \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right] - \Delta D_0 - \Delta\phi(C_0) \\ &\geq E_c \left[1_{\{\tau_0 \leq \tau\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau_0 > \tau\}} e^{-\rho \tau} V_1(C_\tau) + \int_0^{\tau \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right], \quad (\text{F.21}) \end{aligned}$$

where the first inequality follows from the optional sampling theorem, the second follows from the assumptions of the statement, and the third follows from

$$\Delta D_0 + \Delta \phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-} - \Delta D_0}^{C_{0-}} (1 - \phi'(c)) dc \leq 0.$$

Taking the supremum over $\pi \in \Pi$ on both sides of (F.21) then gives

$$\phi(c) \geq \sup_{\pi \in \Pi} E_c \left[1_{\{\tau_0 \leq \tau\}} e^{-\rho \tau_0} \ell_i(0) + 1_{\{\tau_0 > \tau\}} e^{-\rho \tau} V_1(C_\tau) + \int_0^{\tau \wedge \tau_0} e^{-\rho s} (dD_s - f_s dN_s) \right]$$

and the desired result now follows from lemma C.1. \blacksquare

Lemma F.2. If the condition $u'(0; C_U^*) \geq W'(0)$ holds then the function $u(c; C_U^*)$ satisfies the conditions of lemma F.1, and we have $C_U^* \leq C_1^* + K$.

Proof. If $u'(K; K) \geq V_1'(0)$, then $C_U^* = K$, and we only need to show that $u'(c; K) \geq 1$ for $c \leq K$. To this end, let the function $\bar{W}(c)$ be the unique solution to

$$\mathcal{L}_0 \bar{W}(c) - \lambda \bar{W}(c) + \lambda (V_1(C_1^*) - C_1^* - K + c) = 0, \quad c \geq 0,$$

which coincides with $W(c)$ on the interval $[0, C_W^*]$. Because this function satisfies $\bar{W}'(C_W^*) = 1$ and $\bar{W}''(C_W^*) = 0$, it follows from lemma B.7 that $\bar{W}'(c) \geq \bar{W}'(C_W^*) = 1$ for all $c \geq 0$. Then the difference $m(c) = u(c; K) - \bar{W}(c)$ satisfies

$$\mathcal{L}_0 m(c) - \lambda m(c) = 0, \quad c \in [0, K].$$

Furthermore, we have $m(0) = 0$ and $m'(0) \geq 0$ because $u'(0; K) \geq W'(0)$ by assumption, and it now follows from the result of lemma B.6 that $m'(c) \geq 0$ or equivalently $u'(c; K) \geq \bar{W}'(c) \geq 1$, which is what had to be proved.

Now assume that $u'(K; K) < V_1'(0)$ so that $C_U^* > K$. To show that $u'(c; C_U^*) \geq 1$ for all $c \geq 0$, consider the function defined by

$$\phi(c) = u(c; C_U^*) - \bar{W}(c)$$

with the function $\bar{W}(c)$ as above. By lemmas E.1 and D.2, we know that $\phi(c)$ solves (F.2) and, because $\phi(0) = 0 < \phi'(0)$ by assumption, it follows from lemma B.5 that we have

$$u'(c; C_U^*) \geq \bar{W}'(c) \geq 1, \quad c \leq C_U^*. \quad (\text{F.22})$$

Using Equation (F.22) in conjunction with the definition of the liquidation value, it is immediate to show that the function $u(c; C_U^*)$ satisfies

$$u(c; C_U^*) = u(0; C_U^*) + \int_0^c u'(x; C_U^*) dx = \ell_0(0) + \int_0^c u'(x; C_U^*) dx \geq \ell_0(0) + c = \ell_0(c),$$

and, because the inequality $u(c; C_U^*) \geq V_1(c - K)$ follows from lemma D.2, the proof that $u(c; C_U^*)$ satisfies the conditions of lemma F.1 will be complete once we show that

$$\mathcal{L}_0 u(c; C_U^*) + \mathcal{F}u(c; C_U^*) \leq 0.$$

A direct calculation using the definition of the functions $u(c; C_U^*)$ and $V_1(c)$ together with the fact that, as shown below, $C_U^* \leq C_1^* + K$ gives

$$\mathcal{L}_0 u(c; C_U^*) + \mathcal{F}u(c; C_U^*) = \begin{cases} 0, & c \leq C_U^*, \\ (rK - \mu_1 + \mu_0)V_1'(c - K), & C_U^* \leq c \leq C_1^* + K, \\ (r - \rho)(c - (C_1^* + K)) + \mu_0 - \mu_1 + rK, & c \geq C_1^* + K. \end{cases}$$

and the desired result now follows from the increase of the function $V_1(c)$ and the fact that we have $\mu_0 - \mu_1 + rK < 0$ for all $K \leq K^*$ by lemma C.3.

To show that $C_U^* \leq C_1^* + K$ assume that $u'(K; K) < V_1'(0)$ for otherwise there is nothing to prove and suppose that the desired inequality does not hold. In this case we have that $u'(C_U^*; C_U^*) = 1$ and because $u(c; C_U^*) > V_1(c - K)$ for $c < C_U^*$ we get that $u(c; C_U^*)$ is strictly convex in a small neighborhood to the left of the point C_U^* . This implies that $u'(c; C_U^*) < u'(C_U^*; C_U^*) = 1$ in this small neighborhood, which is impossible by (F.22). \blacksquare

Having dealt with the case in which the firm uses exclusively the barrier strategy associated with the value function $u(c; C_U^*)$ we now turn to the case in which it combines this strategy with the strategy associated with the function $W(c)$. To state the result let

$$L_i(c; a, b) = E_c \left[e^{-(\rho+\lambda)\tau_{i,a}} 1_{\{\tau_{i,a} \leq \tau_{i,b}\}} \right]$$

$$H_i(c; a, b) = E_c \left[e^{-(\rho+\lambda)\tau_{i,a} \wedge \tau_{i,b}} \right] - L_i(c; a, b) = E_c \left[e^{-(\rho+\lambda)\tau_{i,a}} 1_{\{\tau_{i,b} \leq \tau_{i,a}\}} \right],$$

where the stopping time $\tau_{i,x}$ denotes the first time that the uncontrolled cash buffer process of a firm with mean cash-flow rate μ_i reaches the level x . Closed-form expressions for these two functions are provided in Appendix J.

Lemma F.3. Assume that $u'(0; C_U^*) < W'(0)$. Then the unique piecewise twice continuously differentiable solution to the free boundary problem (18)–(22) is given by

$$V(c) = \begin{cases} W(c), & c \leq C_L^*, \\ S(c), & C_L^* \leq c \leq C_H^*, \\ V_1(c - K), & c \geq C_H^*, \end{cases}$$

for some constants $C_W^* \leq C_L^* \leq C_H^*$ with $C_L^* > C_U^*$ where

$$S(c) = \Phi(c) + (W(C_L^*) - \Phi(C_L^*))L_0(c; C_L^*, C_H^*) + (V_1(C_H^* - K) - \Phi(C_H^*))L_0(c; C_L^*, C_H^*)$$

Furthermore, $\max\{W(c), V_1(c - K)\} \leq V(c)$ for all $c \geq 0$.

Proof. By lemma D.1, finding a solution to (18)–(22) is equivalent to finding a linear function that is, tangent to the graph of the functions $\hat{p}(c)$ and $\hat{v}_1(c)$ defined by

$$p(c) = W(c) - \phi(c; b) = F_0(c)\hat{p}(Z(c)) = F_0(c)\hat{p}\left(\frac{G_0(c)}{F_0(c)}\right).$$

and (D.16). A direct calculation using the results of lemma E.1 shows that

$$\mathcal{L}_0 p(c) - \lambda p(c) = (r - \rho)(c - C_W^*)^+$$

and it now follows from lemma D.1 that the function $\hat{p}(c)$ is linear for $y \leq Z(C_W^*)$ and strictly concave otherwise. Because $W(c)$ is concave by lemma E.1, we get

$$p'(c) = W'(c) - \frac{\lambda}{\lambda + \rho - r} \geq W'(C_W^*) - \frac{\lambda}{\lambda + \rho - r} = \frac{\rho - r}{\lambda + \rho - r} > 0.$$

Because F_0 is nonnegative and decreasing, the ratio $p(c)/F_0(c)$ is positive and strictly increasing for sufficiently large c . Therefore, lemma D.1 implies that \hat{w} is increasing for sufficiently large values of y , and because $\hat{p}(y)$ is concave, it is globally increasing.

Because $u(0; C_U^*) = W(0) = \ell_0$, we have that $\hat{w}(c)$ and the function

$$\hat{u}(y) = (u(Z^{-1}(y); C_U^*) - \Phi(Z^{-1}(y))) / F_0(Z^{-1}(y))$$

are both linear on $[Z(0), Z(C_W^*) \wedge Z(C_U^*)]$ and coincide at the point $Z(0)$. On the other hand, the inequality $u'(0; C_U^*) < W'(0)$ implies that $\hat{u}(y) \leq \hat{p}(y)$ for all $y \in [Z(0), Z(C_W^*) \wedge Z(C_U^*)]$. It follows that $C_W^* \leq \tilde{C} < C_U^*$ because \hat{p} is a linear function that crosses the graph of the concave function $\hat{v}_1(y)$ at $Z(\tilde{C})$ and by the definition of \tilde{C} , we have $\hat{p}(y) < \hat{v}_1(y)$ for all $y > Z(\tilde{C})$.

Because we know that $u(c; C_U^*) \geq V_1(c - K)$ for all $c \leq C_U^*$ by lemma D.2, we get that the linear function defined by

$$\hat{p}(y) = \frac{\hat{p}(Z(C_W^*)) - \hat{p}(Z(0))}{Z(C_W^*) - Z(0)} y + \frac{\hat{p}(Z(0))Z(C_W^*) - \hat{p}(Z(C_W^*))Z(0)}{Z(C_W^*) - Z(0)}$$

is tangent to the concave function $\hat{p}(y)$ and lies strictly above the concave function $\hat{v}_1(y)$ for all $y \geq Z(\tilde{C})$. On the other hand, because

$$\hat{v}_1(Z(\tilde{C})) = \hat{p}(Z(\tilde{C})) \quad \text{and} \quad \hat{v}_1'(Z(\tilde{C})) > \hat{p}'(Z(\tilde{C}))$$

as a result of lemma E.4, we have that the tangent line to \hat{p} at the point $y = Z(\tilde{C})$ lies strictly below \hat{v}_1 for $y > Z(\tilde{C})$. By continuity, this implies that there exists a unique $y_L^* \in (Z(C_W^*), Z(\tilde{C}))$ such that the tangent line to \hat{p} at y_L^* is also tangent to \hat{v}_1 at some $y_H^* > y_L^*$. Setting

$$C_W^* = Z^{-1}(Z(C_W^*)) \leq C_L^* = Z^{-1}(y_L^*) < Z^{-1}(y_H^*) = C_H^*$$

produces the unique solution to value-matching and smooth-pasting conditions. Because $\hat{p}(c)$ is increasing and concave its tangent line at the point y_L^* crosses the vertical axis above the level $\hat{p}(Z(0))$. Therefore, if the target C_U^* were greater or equal to C_H^* then this tangent would have to cross the vertical axis below $\hat{u}(Z(0)) = \hat{p}(Z(0))$ thus leading to a contradiction.

To complete the proof it now only remains to show that $V(c) \geq \max\{W(c), V_1(c - K)\}$ but this follows immediately from the fact that because the functions $\hat{p}(c)$ and $\hat{v}_1(c)$ are both concave we have $\hat{v}(c) \geq \max\{\hat{p}(c), \hat{v}_1(c)\}$ for all $c \geq 0$. ■

Lemma F.4. If $u'(0; C_U^*) < W'(0)$, then the function $V(c)$ of lemma F.3 satisfies the conditions of lemma F.1 and we have $C_H^* < C_1^* + K$.

Proof. To show that $V'(c) \geq 1$ for all $c \geq 0$ we start by observing that this inequality holds in the region $[0, C_L^*] \cup [C_H^*, \infty)$ as a result of the definition and lemmas B.8, E.1, D.2. On the other hand, because we know that $C_H^* \geq C_L^* \geq C_W^*$ we have $V'(C_L^*) = W'(C_L^*) = 1$ and

$$V(c) \geq W(c) = W(C_W^*) + c - C_W^*, \quad C_L^* \leq c \leq C_H^*.$$

This immediately implies that $V''(C_L^*) \geq 0$ and because $J(c) = V'(c)$ is a solution to

$$\mathcal{L}_0 J(c) + \lambda - (\lambda - r)J(c) = 0,$$

it follows from lemma B.5 that $J'(c) = V''(c)$ can have at most one zero in the interval $\mathcal{I} = [C_L^*, C_H^*]$. If no such zero exists then $V''(c) \geq 0$ in \mathcal{I} and consequently $V'(c) \geq V'(C_L^*) = 1$ for all $c \in \mathcal{I}$. If, on the contrary, $V''(c)$ has one zero located at some $c^* \in \mathcal{I}$ then we have that $V'(c)$ reaches a global maximum over \mathcal{I} at the point c^* and because $V'(C_H^*) = V_1'(C_H^* - K) \geq 1$ because of the concavity of $V_1(c)$, we conclude that $V'(c) \geq 1$ holds for all $c \in \mathcal{I}$.

Let us now show that $C_H^* < C_1^* + K$. If not then we have $V'(C_H^*) = 1$ and because $V(c) > V_1(c - K)$ for all $c \geq K$, we have that $V(c)$ is convex in a neighborhood of C_H^* . This in turn implies that $V'(c) < V'(C_H^*) = 1$ in this neighborhood, which is impossible. Using the fact that $V'(c) \geq 1$ in conjunction with the definition of the liquidation value, we obtain

$$V(c) = V(0) + \int_0^c V'(x) dx = \ell_0(0) + \int_0^c V'(x) dx \geq \ell_0(0) + c = \ell_0(c).$$

Finally, because $C_W^* \leq C_L^* \leq C_H^* \leq C_1^* + K$ by the first part of the proof it follows from the definition and concavity of the functions $W(c)$ and $V_1(c)$ that

$$\mathcal{L}_0 V(c) + \mathcal{F}V(c) = \begin{cases} 0, & c \leq C_W^*, \\ (r - \rho)(C - C_W^*), & C_W^* \leq c \leq C_L^*, \\ 0, & C_L^* \leq c \leq C_H^*, \\ A V_1'(c - K) + (r - \rho)(c - C_1^* - K)^+, & c \geq C_H^*, \end{cases}$$

where we have set $A = \mu_0 - \mu_1 + rK$. By lemma C.2, we know that $A < 0$ whenever $K \leq K^*$. Therefore, it follows from the increase of the function $V_1(c)$ that we have

$$\mathcal{L}_0 V(c) + \mathcal{F}V(c) \leq 0, \quad c \geq 0,$$

and the proof is complete. ■

Lemma F.5. We have $V(c) \leq \hat{V}(c)$ for all $c \geq 0$ where $\hat{V}(c)$ denotes the value function of the firm's optimization problem.

Proof. Let $\tau_L^* = \tau_{0, C_L^*}$ (resp., $\tau_H^* = \tau_{0, C_H^*}$) denote the first time that the uncontrolled cash buffer of a firm with mean cash-flow rate μ_0 falls below C_L^* (resp., increases above C_H^*). Using arguments similar to those of the proof of lemma F.2, it can be shown that

$$V(c) = E_c \left[1_{\{\tau_H^* < \tau_N \wedge \tau_L^*\}} e^{-\rho \tau_H^*} V_1(C_H^* - K) + 1_{\{\tau_L^* < \tau_N \wedge \tau_H^*\}} e^{-\rho \tau_L^*} W(C_L^*) + 1_{\{\tau_N < \tau_L^* \wedge \tau_H^*\}} e^{-\rho \tau_N} (V_1(C_1^*) - C_1^* - K + C_{\tau_N}^-) \right].$$

On the other hand, using arguments similar to those of the proof of Proposition 1, it can be shown that the function $W(c)$ satisfies

$$W(c) = E_c \left[1_{\{\tau_0 < \tau_N\}} e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq \tau_N\}} e^{-\rho \tau_N} V_1(C_1^*) + \int_0^{\tau_0 \wedge \tau_N} e^{-\rho s} (dL_s - f_s^U dN_s) \right],$$

where $L_t = \sup_{s \leq t} (b_t - C_W^*)^+$ with

$$db_t = (rb_{t-} + \mu_i)dt + \sigma dB_t + (C_W^* - b_{t-})^+ dN_t,$$

and the process f_t^U is defined as in Equation (D.17). Combining these two equalities and using the law of iterated expectations then gives

$$V(c) = E_c \left[1_{\{\tau_0 < \tau\}} e^{-\rho \tau_0} \ell_0 + 1_{\{\tau_0 \geq \tau\}} e^{-\rho \tau} V_1(C_\tau) + \int_0^{\tau_0 \wedge \tau} e^{-\rho s} (d\bar{D}_s - f_s^U dN_s) \right],$$

where we have set $\tau = \tau_N \wedge \tau_H^*$ and the cumulative dividend process is defined by

$$\bar{D}_t = \int_0^t 1_{\{C_s \leq C_L^*\}} dL_s.$$

The same arguments as in the proof of lemma D.3 then show that strategy (τ, \bar{D}, f^U) is admissible and the desired result now follows from the result of lemma C.1. ■

Lemma F.6. There exists a unique $K^{**} \in (0, K^*)$ such that $u'(0; C_U^*) > W'(0)$ if and only if $K < K^{**}$.

Proof. We will use the notation $P(c; K) = u(c; C_U^*(K); K)$ and $W(c; K)$ to show the dependence of these functions on the investment cost. From the proof of lemma F.8, we know that $P'(0; K) > W'(0; K)$ for sufficiently small values of K . Similarly, we know that the function $V_1(c - K^*)$ touches the function $W(c; K^*)$ from below at $C_1^* + K^*$ so that $C_L^*(K^*) = C_H^*(K^*) = C_1^* + K^*$ and $P'(0; K^*) < W'(0; K^*)$. Therefore, it suffices to show that there exists a unique critical investment cost such that we have

$$P'(0; K^{**}) = W'(0; K^{**}).$$

Assume towards a contradiction that this is not the case so that there exist $K_1 < K_2$ such that $P'(0; K_i) = W'(0; K_i)$. Let the function $\bar{W}_i(c)$ denote the unique solution to

$$\mathcal{L}_0 \bar{W}_i(c) - \lambda \bar{W}_i(c) + \lambda (V_1(C_1^*) - C_1^* - K_i + c) = 0, \quad c \geq 0,$$

which coincides with the function $W(c; K_i)$ on the interval $[0, C_W^*(K_i)]$. From the proof of lemmas F.2 and B.7, we know that this function is concave for $c \leq c_i^* = C_W^*(K_i)$ and convex for $c \geq c_i^*$ so that

$\bar{W}'_i(c) > \bar{W}'(c_i^*) = 1$ for all $c \neq c_i^*$. Because $P(0; K_i) = \bar{W}_i(0)$ by definition, the equality $U'(0; K_i) = W'(0; K_i)$ implies that the two functions coincide for $c \leq C_U^*(K_i)$. Consider the function defined by

$$m(c) = \bar{W}_i(c) - V_1(c - K_i)$$

and satisfying (E.19). If $C_U^*(K_i) > K_i$, then $m(C_U^*(K_i)) = m'(C_U^*(K_i)) = 0$. On the other hand, it follows from the proof of lemma F.2 that $m(c) \geq 0$ for all $c \geq K_i$. If $C_U^*(K_i) = K_i$ then $\bar{W}'_i(C_U^*(K_i)) \geq V'_1(0)$ implies that $m(C_U^*(K_i)) = 0$ and $m'(C_U^*(K_i)) \geq 0$ and therefore $m(c) \geq 0$ for $c \geq K_i$ by lemma B.6. Now consider the function

$$k(c) = \bar{W}'_2(c) - \bar{W}'_1(c),$$

which is a solution to

$$\mathcal{L}_0 k(c) - (\lambda - r)k(c) = 0, \quad c \geq 0,$$

and satisfies $k(c_2^*) < 0$, $k(c_1^*) > 0$. Because $k(c)$ cannot have negative local minima by lemma B.5, we have that there exists a unique point $c_* \in (c_2^*, c_1^*)$ such that $k(c_*) = 0$, $k'(c_*) > 0$ and $k(c) > 0$ for all $c > c_*$ and $k(c) < 0$ for $c < c_*$. That is, $\bar{W}_2(c) - \bar{W}_1(c)$ attains a global minimum at the point c_* and $(\bar{W}_2 - \bar{W}_1)'(c_*) > 0$. Evaluating the differential equation

$$\mathcal{L}_0(\bar{W}_2 - \bar{W}_1) - \lambda(\bar{W}_2 - \bar{W}_1)(c) + \lambda(K_1 - K_2) = 0$$

at the point $c = c_*$, we get

$$\bar{W}_2(c_*) - \bar{W}_1(c_*) > \frac{\lambda}{\rho + \lambda}(K_1 - K_2),$$

and therefore

$$\bar{W}_1(c) - \bar{W}_2(c) < \frac{\lambda}{\rho + \lambda}(K_2 - K_1)$$

for all $c \geq 0$. However, because $\bar{W}_1(c) \geq V_1(c - K_1)$ for $c \geq K_1$ and $V'_1(c) \geq 1$ for all $c \geq 0$, we finally conclude that

$$\begin{aligned} \frac{\lambda}{\rho + \lambda}(K_2 - K_1) &\geq W_1(C_H^*(K_2)) - W_2(C_H^*(K_2)) = W_1(C_H^*(K_2)) - V_1(C_H^*(K_2) - K_2) \\ &\geq V_1(C_H^*(K_2) - K_1) - V_1(C_H^* - K_2) \geq K_2 - K_1, \end{aligned}$$

which establishes a contradiction. ■

Lemma F.7. We have

$$\begin{aligned} u(c; K) = \Phi(c, K) + \frac{G_0(K)(\ell_0 - \Phi(0; K)) - G_0(0)(\ell_1 - \Phi(K; K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}F_0(c) \\ - \frac{F_0(K)(\ell_0 - \Phi(0; K)) - F_0(0)(\ell_1 - \Phi(K; K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}G_0(c) \end{aligned}$$

where the function $\Phi(c) = \Phi(c; K)$ is defined as in Equation (16).

Proof. The proof follows by direct calculation and thus is omitted. ■

Lemma F.8. There exists a unique $\underline{K} \in (0, K^{**})$ such that, for $K \in (0, K^{**})$, we have $u'(K, K) < V'_1(0)$ if and only if $K > \underline{K}$.

Proof. First of all, we claim that $\lim_{K \downarrow 0} u'(K; K) = \infty$. Indeed, up to a first-order approximation, we have that

$$G_0(K)F_0(0) - F_0(K)G_0(0) \approx (G'_0(0)F_0(0) - F'_0(0)G_0(0))K = \alpha K$$

with $\alpha > 0$. Therefore, it follows from lemma F.7 that

$$u'(K; K) \approx \frac{(G_0(0)(\ell_0 - \ell_1)F'_0(0) - F_0(0)(\ell_0 - \ell_1)G'_0(0))}{\alpha K} = \frac{\ell_1 - \ell_0}{K}$$

and the required assertion follows from the fact that $\ell_1 \geq \ell_0$. Because $u'(K; K)$ is continuous in K it remains to show that $u'(K; K) = V'_1(0)$ can have at most one solution. Suppose to the contrary that $K_1 < K_2 \leq K^{**}$ are two solutions, and let $g_i(c) = u(c; K_i)$ so that

$$\mathcal{L}_0 g_i(c) - \lambda g_i(c) + \lambda(V_1(C_1^*) - C_1^* - K_i + c) = 0, \quad c \geq 0.$$

Observe that because $K_i \leq K^{**}$, we have that $g'_i(c) \geq 1$ for all $c \leq K_i$ by lemma F.2. Now a direct calculation using the above differential equation shows that the functions $h_i(y) = g_i(y + K_i)$ satisfy

$$\frac{\sigma^2}{2} h''_i(y) + (ry + rK_i + \mu_0)h'_i(y) - (\rho + \lambda)h_i(y) + \lambda(V_1(C_1^*) - C_1^* + y) = 0,$$

and it follows that the function $m(c) = h_1(c) - h_2(c)$ solves

$$\frac{\sigma^2}{2} m''(y) + (ry + rK_1 + \mu_0)m'(y) - (\rho + \lambda)m(y) + r(K_1 - K_2)h'_2(y) = 0$$

subject to $m(0) = m'(0) = 0$. Because $h'_2(y) > 0$, we know from lemma B.6 that the function $m(c)$ is positive and monotone decreasing and it follows that $h_1(c) > h_2(c)$ for $c < 0$. Because the function $h_2(c)$ is monotone increasing, this in turn implies that we have

$$\ell_0 = h_1(-K_1) > h_2(-K_1) > h_2(-K_2) = \ell_0,$$

which provides the required contradiction. ■

Appendix G. Additional Proofs

The proofs of all the additional results are gathered in a supplementary appendix.

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