Kalman Filter for Uncertainty - Noise Representation

Setup

Observable equation is described by

$$y_t = x_{t+1} + \eta_t \tag{1}$$

where y_t can be observed at time t and $\eta_t \sim N(0, \sigma_\eta^2)$ is a noise shock which prevent to correctly observe future state x_{t+1}.

State transition equation is described by

$$x_t = x_{t-1} + \varepsilon_t \tag{2}$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ is a structural shock which affects the transition from x_{t-1} to x_t.

Procedure

Goal is to optimize the forecast of x_{t+1} using available information at time t, i.e. y_t and x_t. Given the initial value for $x_{0,0}$ and $Var(x_{1,0} - x_{0,0}) = P_{1,0}$. Procedure can be summarized as follows,

1. Forecast x_{t+1} using information at time t and evaluate the error variance of this prediction.
2. Find the steady state variance $P_{t,t}$.

As a generalization,

1. $y_t = x_{t+1,t}$ and the forecast error variance is $\Omega_t^y = Var(x_{t+1,t}) + \sigma_\eta^2 = P_{t,t-1} + \sigma_\eta^2$.
2. We want to forecast $x_{t+1,t}$ using all the available information up to time t. As a simplification we try to forecast $x_{t+1} - x_t$ using $y_t - x_t$. Coefficient β^{KG} is derived as follows

$$\begin{aligned}
\beta^{KG} &= \frac{Cov(x_{t+1} - x_t, y_t - x_t)}{Var(y_t - x_t)} \\
&= \frac{Cov(x_{t+1} - x_t, x_t + \eta_t - x_t)}{Var(x_{t+1} + \eta_t - x_t)} \\
&= \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}
\end{aligned} \tag{3}$$

This implies that

$$x_{t+1} - x_t = \beta^{KG}(y_t - x_t) \tag{4}$$

which is

$$
\begin{aligned}
x_{t+1} &= x_t + \beta^{KG}(y_t - x_t) \\
&= (1 - \beta^{KG})x_t + \beta^{KG} y_t \\
&= \left(1 - \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}\right) x_t + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} y_t \\
&= \frac{\sigma_\eta^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} x_t + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} y_t \\
&= \frac{\frac{1}{\sigma_\varepsilon^2}}{\frac{1}{\sigma_\eta^2} + \frac{1}{\sigma_\varepsilon^2}} x_t + \frac{\frac{1}{\sigma_\eta^2}}{\frac{1}{\sigma_\eta^2} + \frac{1}{\sigma_\varepsilon^2}} y_t
\end{aligned}
\tag{5}
$$

and this is the reason why β^{KG} is called Kalman gain, i.e. it defines how much to weight signal y_t to infer x_{t+1}.

Now we need to figure out the forecast error variance of $x_{t+1} - x_{t+1,t}$, i.e. $P_{t+1,t} =$

$$
\begin{aligned}
&Var(x_{t+1} - x_{t+1,t}) \\
&\quad P_{t+1,t} = Var\left[x_{t+1} - x_t - \beta^{KG}(y_t - x_t)\right] \\
&\quad = Var\left[x_{t+1} - x_t - \beta^{KG}(x_{t+1} + \eta_t - x_t)\right] \\
&\quad = Var\left[x_{t+1} - x_t - \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}(x_{t+1} + \eta_t - x_t)\right] \\
&\quad = E\left\{\left[x_{t+1} - x_t - \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}(x_{t+1} + \eta_t - x_t)\right]^2\right\} \\
&\quad = E\left\{\left[x_{t+1} - x_t\right]^2\right\} - 2E\left\{\left(x_{t+1} - x_t\right)\left(\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}(x_{t+1} + \eta_t - x_t)\right)\right\} \\
&\quad + E\left\{\left[\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}(x_{t+1} + \eta_t - x_t)\right]^2\right\} \\
&\quad = \sigma_\varepsilon^2 - 2\sigma_\varepsilon^2 \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} + \left(\frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}\right)^2 (\sigma_\varepsilon^2 + \sigma_\eta^2) \\
&\quad = \sigma_\varepsilon^2 - 2\sigma_\varepsilon^2 \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} + \sigma_\varepsilon^2 \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} \\
&\quad = \sigma_\varepsilon^2 - \sigma_\varepsilon^2 \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} \\
&\quad = \sigma_\varepsilon^2 \left(1 - \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_\eta^2}\right) \\
&\quad = \frac{\sigma_\varepsilon^2 \sigma_\eta^2}{\sigma_\varepsilon^2 + \sigma_\eta^2} \\
&\quad = \frac{1}{\frac{1}{\sigma_\varepsilon^2} + \frac{1}{\sigma_\eta^2}}
\end{aligned}
\tag{6}
$$