Chapter 18

Multi-country (Large Open Economy) OG Model

In this chapter, we model an economy in which two countries interact that each look like the the S-period-lived agent economies from Chapter 3. In contrast to a small open economy model, changes within a country in the large open economy model can affect equilibrium interest rates. We call one country the Home country and the other country the Foreign country, where *Home* and *Foreign* are not relative terms but are fixed country names.

We assume that capital is mobile and that labor is not mobile. That is, Home country household savings can be invested in either the Home country intermediate goods production or in the Foreign country intermediate goods production. However, Home country household labor supply can only be hired by Home country firms. The techniques in this chapter generalize to an N-country model.

18.1 Households

18.1.1 Home household's lifetime problem

A unit measure of Home households is born each period and lives for S periods. These households supply labor inelastically according to the following equation.

$$n_{h,s} = \begin{cases} 1.0 & \text{if} \quad s < round\left(\frac{9}{16}\right) S \\ 0.2 & \text{if} \quad s \geq round\left(\frac{9}{16}\right) S \end{cases} \tag{18.1}$$

Home households choose how much of the Home final good to consume $c_{h,s,t}$ and how much to save in each country's production $b^h_{h,s,t}$ and $b^f_{h,s,t}$. Let the age-s Home household's budget constraint in each period t be the following,

$$c_{h,s,t} + b^h_{h,s+1,t+1} + b^f_{h,s+1,t+1} = (1 + r^h_{h,t})b^h_{h,s,t} + \left(1 + \frac{r^h_{f,t}}{q_t}\right) b^f_{h,s,t} + w_{h,t}n_{h,s} \quad \forall s, t \tag{18.2}$$

where $b^h_{h,s,t}$ is the amount of Home savings invested in the Home country intermediate goods production, $b^f_{h,s,t}$ is the amount of Home savings invested in Foreign country intermediate goods production, $r^h_{h,t}$ is the interest rate paid by Home country intermediate goods producers to Home country investors, $r^h_{f,t}$ is the interest rate paid by Foreign country intermediate goods producers to Home country investors, q_t is the real exchange rate (in units of Foreign country final goods per one unit of Home country final goods), and $w_{h,t}$ is the Home country wage. Because all the Home household's income is in units of Home country consumption good, the price of Home country consumption in (18.2) is arbitrarily set to one.

$$\max_{\{c_{h,s,t}, b^h_{h,s+1,t+1}, b^f_{h,s+1,t+1}\}} \sum_{s=1}^{S} \beta^{s-1} u\left(c_{h,s,t+s-1}\right) \tag{18.3}$$

$$\text{s.t.} \quad c_{h,s,t} + b^h_{h,s+1,t+1} + b^f_{h,s+1,t+1} = (1 + r^h_{h,t})b^h_{h,s,t} + \left(1 + \frac{r^h_{f,t}}{q_t}\right) b^f_{h,s,t} + w_{h,t}n_{h,s} \quad (18.2)$$

$$\text{where} \quad u\left(c_{h,s,t}\right) \equiv \frac{\left(c_{h,s,t}\right)^{1-\sigma} - 1}{1 - \sigma} \tag{18.4}$$

The Home household's savings decisions are characterized by the following two sets of $S-1$ first order conditions.

$$(c_{h,s,t})^{-\sigma} = \beta(1 + r^h_{h,t+1})\left(c_{h,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.5}$$

$$(c_{h,s,t})^{-\sigma} = \beta\left(1 + \frac{r^h_{f,t+1}}{q_{t+1}}\right)\left(c_{h,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.6}$$

Note that the optimization problem (18.3) and the two first order conditions (18.5) and (18.6) for an interior solution imply that if the household is going to invest their savings in both Home and Foreign intermediate goods production, then the return on both investment options must be equivalent. This is an equilibrium argument for the interest rate parity condition.

$$r^h_{h,t} = \frac{r^h_{f,t}}{q_t} \quad \forall t \tag{18.7}$$

Because the Home household is indifferent between the two investment options due to the equivalence in rates of return, we can simplify the household's problem and notation. Let $r_{h,t}$ represent the return on Home country savings in terms of Home country final goods production.

$$r_{h,t} \equiv r^h_{h,t} = \frac{r^h_{f,t}}{q_t} \quad \forall t \tag{18.8}$$

We can now rewrite the Home country budget constraint in terms of this unified rate of return $r_{h,t}$ defined in (18.8) and a single total amount of savings $b_{h,s,t}$.

$$c_{h,s,t} + b_{h,s+1,t+1} = (1 + r_{h,t})b_{h,s,t} + w_{h,t}n_{h,s} \quad \forall s,t \tag{18.9}$$

Demand for capital by intermediate goods producers in both the Home and Foreign countries will determine the percent $\Delta_{h,t}$ of total Home household savings that goes to Home country intermediate goods producers and the percent $(1-\Delta_{h,t})$ that goes to Foreign intermediate goods producers.

The Home household's choice of how much to save is characterized by the following $S-1$ Euler equations.

$$(c_{h,s,t})^{-\sigma} = \beta(1 + r_{h,t+1})\left(c_{h,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.10}$$

Note that these $S-1$ Home household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_{h,u}, r_{h,u}\}_{u=t}^{t+S-1}$. As will be shown in the following sections, those prices depend on the full distribution of Home and Foreign savings in equilibrium. Let the full distribution of Home and Foreign savings in period t be defined as $\mathbf{\Gamma}_t$.

$$\mathbf{\Gamma}_t \equiv \left\{b_{h,s,t}, b_{f,s,t}\right\}_{s=2}^{S} \quad \forall t \tag{18.11}$$

Then we can characterize general household beliefs (either Home or Foreign) about the evolution of the full distribution of savings as the following,

$$\mathbf{\Gamma}_{t+u}^e = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{18.12}$$

where $\mathbf{\Gamma}_{t+u}^e$ need not be correct, in general. The assumption of correct beliefs is the rational expectations assumption in equilibrium Definition 18.2.

18.1.2 Foreign household's lifetime problem

A unit measure of Foreign households is born each period and lives for S periods. These households also supply labor inelastically according to the following equation.

$$n_{f,s} = \begin{cases} 1.0 & \text{if} \quad s < round\left(\frac{9}{16}\right)S \\ 0.2 & \text{if} \quad s \geq round\left(\frac{9}{16}\right)S \end{cases} \tag{18.13}$$

Foreign households choose how much of the Foreign final good to consume $c_{f,s,t}$ and how much to save in each country's production $b_{f,s,t}^f$ and $b_{f,s,t}^h$. Let the age-s Foreign household's budget constraint in each period t be the following,

$$c_{f,s,t} + b^f_{f,s+1,t+1} + b^h_{f,s+1,t+1} = (1+r^f_{f,t})b^f_{f,s,t} + (1+q_t r^f_{h,t})b^h_{f,s,t} + w_{f,t}n_{f,s} \quad \forall s,t \tag{18.14}$$

where $b^f_{f,s,t}$ is the amount of Foreign savings invested in the Foreign country intermediate goods production, $b^h_{f,s,t}$ is the amount of Foreign savings invested in Home country intermediate goods production, $r^f_{f,t}$ is the interest rate paid by Foreign country intermediate goods producers to Foreign country investors, $r^f_{h,t}$ is the interest rate paid by Home country intermediate goods producers to Foreign country investors, q_t is the real exchange rate (in units of Foreign country final goods per one unit of Home country final goods), and $w_{f,t}$ is the Foreign country wage. Because all the Foreign household's income is in units of Foreign country consumption good, the price of Foreign country consumption in (18.14) is arbitrarily set to one.

$$\max_{\{c_{f,s,t},b^f_{f,s+1,t+1},b^h_{f,s+1,t+1}\}} \sum_{s=1}^{S} \beta^{s-1} u\left(c_{h,s,t+s-1}\right) \tag{18.15}$$

$$\text{s.t.} \quad c_{f,s,t} + b^f_{f,s+1,t+1} + b^h_{f,s+1,t+1} = (1+r^f_{f,t})b^f_{f,s,t} + (1+q_t r^f_{h,t})b^f_{h,s,t} + w_{f,t}n_{f,s} \quad (18.14)$$

$$\text{where} \quad u\left(c_{f,s,t}\right) \equiv \frac{\left(c_{f,s,t}\right)^{1-\sigma}-1}{1-\sigma} \tag{18.16}$$

The Foreign household's savings decisions are characterized by the following two sets of $S-1$ first order conditions.

$$\left(c_{f,s,t}\right)^{-\sigma} = \beta(1+r^f_{f,t+1})\left(c_{f,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.17}$$

$$\left(c_{f,s,t}\right)^{-\sigma} = \beta(1+q_{t+1}r^f_{h,t+1})\left(c_{f,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.18}$$

Note that the optimization problem (18.15) and the two first order conditions (18.17) and

(18.18) for an interior solution imply that if the household is going to invest their savings in both Foreign and Home intermediate goods production, then the return on both investment options must be equivalent. This is an equilibrium argument for the interest rate parity condition.

$$r^f_{f,t} = q_t r^f_{h,t} \quad \forall t \tag{18.19}$$

Because the Foreign household is indifferent between the two investment options due to the equivalence in rates of return, we can simplify the household's problem and notation. Let $r_{f,t}$ represent the return on Foreign country savings in terms of Foreign country final goods production.

$$r_{f,t} \equiv r^f_{f,t} = q_t r^f_{h,t} \quad \forall t \tag{18.20}$$

We can now rewrite the Foreign country budget constraint in terms of this unified rate of return $r_{f,t}$ defined in (18.20) and a single total amount of savings $b_{f,s,t}$.

$$c_{f,s,t} + b_{f,s+1,t+1} = (1 + r_{f,t})b_{f,s,t} + w_{f,t}n_{f,s} \quad \forall s, t \tag{18.21}$$

Demand for capital by intermediate goods producers in both the Foreign and Home countries will determine the percent $\Delta_{f,t}$ of total Foreign household savings that goes to Foreign country intermediate goods producers and the percent $(1 - \Delta_{f,t})$ that goes to Home intermediate goods producers.

The Foreign household's choice of how much to save is characterized by the following $S-1$ Euler equations.

$$(c_{f,s,t})^{-\sigma} = \beta(1 + r_{f,t+1})\left(c_{f,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.22}$$

Note that these $S-1$ Foreign household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_{f,u}, r_{f,u}\}_{u=t}^{t+S-1}$. As will be shown in the following sections, those prices depend on the full distribution of Home and Foreign savings in equilibrium. Let the full distribution of Home and Foreign savings in period t be defined as

$\mathbf{\Gamma}_t$ as in (18.11) with beliefs about the evolution of that distribution $\mathbf{\Gamma}^e_{t+u}$ defined in (18.12).

18.2 Firms

18.2.1 Home country intermediate goods producers

The Home country is populated by a unit measure of perfectly competitive intermediate goods producers that rent capital from households of both countries and transform those two inputs of household savings into an aggregated capital stock $K_{h,t}$ specific to the Home country. We assume that this is done through a constant elasticity of substitution (CES) production function in which the elasticity of substitution between Home capital and Foreign capital in the Home intermediate goods producer's production function is constant $\phi_h \geq 1$,

$$K_{h,t} \equiv \left[(1-\alpha_h)^{\frac{1}{\phi_h}}\left(K^h_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}} + (\alpha_h)^{\frac{1}{\phi_h}}\left(K^f_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}}\right]^{\frac{\phi_h}{\phi_h-1}} \quad \forall t \tag{18.23}$$

where α_h is related to the Foreign country capital share in Home country aggregated capital $K_{h,t}$. The two capital inputs on the right-hand-side of (18.23) $K^h_{h,t}$ and $K^f_{h,t}$ represent the total amount of country-specific household savings that was used in the Home country intermediate goods production function.

$$K^h_{h,t} \equiv \Delta_{h,t}\sum_{s=2}^{S} b_{h,s,t} \tag{18.24}$$

$$K^f_{h,t} \equiv (1-\Delta_{f,t})\sum_{s=2}^{S} b_{f,s,t} \tag{18.25}$$

The variables $\Delta_{h,t}$ and $\Delta_{f,t}$ are endogenous variables and represent the percent of total Home household savings that goes to Home production and the percent of total Foreign household savings that goes to Foreign production, respectively. The output of the Home country intermediate goods producer is a new type of aggregated capital that can be used by the Home country final goods producer.

An intuitive analogy for the role these intermediate goods producers fill is a domestic banking sector. The intermediate goods producer in the Home country takes household savings from the Home country $K^h_{h,t}$ and household savings from the Foreign country $K^f_{h,t}$ and bundles that savings together as an output for use by the final goods producer described in Section 18.2.3. The CES production function describes the technology with which these two different types of capital are combined in the production process and provides demand functions based on the relative price of the two forms of capital.

We can solve for the Home country intermediate goods producer's demand for Home and Foreign savings, respectively, by solving the expenditure minimization problem subject to a given level of aggregated capital.

$$\min_{K^h_{h,t},K^f_{h,t}} \; r^h_{h,t}K^h_{h,t} + r^f_{h,t}K^f_{h,t} \quad \text{s.t.} \quad K_{h,t} \leq \left[(1-\alpha_h)^{\frac{1}{\phi_h}}\left(K^h_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}} + (\alpha_h)^{\frac{1}{\phi_h}}\left(K^f_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}}\right]^{\frac{\phi_h}{\phi_h-1}} \tag{18.26}$$

where $r^h_{h,t}$ is the interest rate on Home country savings in Home output units from (18.2) and $r^f_{h,t}$ is the interest rate on Foreign country savings in Home output units from (18.14).

The Lagrangian for this problem is the following,

$$\mathcal{L} = r^h_{h,t}K^h_{h,t} + r^f_{h,t}K^f_{h,t} + r_t\left(K_{h,t} - \left[(1-\alpha_h)^{\frac{1}{\phi_h}}\left(K^h_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}} + (\alpha_h)^{\frac{1}{\phi_h}}\left(K^f_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}}\right]^{\frac{\phi_h}{\phi_h-1}}\right) \tag{18.27}$$

where r_t is the multiplier on the constraint and represents the marginal cost of an extra unit of aggregated Home capital. So r_t is interpreted as the price or interest rate of aggregated Home capital. The first order conditions are the following:

$$r^h_{h,t} = r_t\left[\frac{(1-\alpha_h)K_{h,t}}{K^h_{h,t}}\right]^{\frac{1}{\phi_h}} \quad \Rightarrow \quad r_{h,t} = r_t\left[\frac{(1-\alpha_h)K_{h,t}}{K^h_{h,t}}\right]^{\frac{1}{\phi_h}} \quad \forall t \tag{18.28}$$

$$r^f_{h,t} = r_t\left[\frac{\alpha_h K_{h,t}}{K^f_{h,t}}\right]^{\frac{1}{\phi_h}} \quad \Rightarrow \quad \frac{r_{f,t}}{q_t} = r_t\left[\frac{\alpha_h K_{h,t}}{K^f_{h,t}}\right]^{\frac{1}{\phi_h}} \quad \forall t \tag{18.29}$$

$$K_{h,t} = \left[(1-\alpha_h)^{\frac{1}{\phi_h}}\left(K^h_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}} + (\alpha_h)^{\frac{1}{\phi_h}}\left(K^f_{h,t}\right)^{\frac{\phi_h-1}{\phi_h}}\right]^{\frac{\phi_h}{\phi_h-1}} \quad \forall t \tag{18.23}$$

Dividing (18.28) by (18.29) gives the following relationship.

$$\frac{q_t r_{h,t}}{r_{f,t}}\left(\frac{K^h_{h,t}}{K^f_{h,t}}\right)^{\frac{1}{\phi_h}} = \left(\frac{1-\alpha_h}{\alpha_h}\right)^{\frac{1}{\phi_h}} \quad \forall t \tag{18.30}$$

Notice that in the Cobb-Douglas case when $\phi_h = 1$, the ratio of Home savings expenditure to Foreign savings expenditure is a constant. Also, note that solving (18.28) and (18.29) for $K^h_{h,t}$ and $K^f_{h,t}$, respectively, gives Home demand equations for consumption of Home savings and Foreign savings.

$$K^h_{h,t} = (1-\alpha_h)\left(\frac{r_{h,t}}{r_t}\right)^{-\phi_h} K_{h,t} \quad \forall t \tag{18.31}$$

$$K^f_{h,t} = \alpha_h\left(\frac{r_{f,t}}{q_t r_t}\right)^{-\phi_h} K_{h,t} \quad \forall t \tag{18.32}$$

The expression for the interest rate r_t on Home country aggregated capital $K_{h,t}$ is found by substituting the demand equations (18.31) and (18.32) into (18.23).

$$r_t = \left[(1-\alpha_h)\left(r_{h,t}\right)^{1-\phi_h} + \alpha_h\left(\frac{r_{f,t}}{q_t}\right)^{1-\phi_h}\right]^{\frac{1}{1-\phi_h}} \quad \forall t \tag{18.33}$$

In the case of Cobb-Douglas aggregation over Home household savings and Foreign household savings ($\phi_h = 1$), the expression for the interest rate on Home aggregated capital is the following.

$$r_t = \frac{1}{(1-\alpha_h)^{1-\alpha_h}(\alpha_h)^{\alpha_h}}\left(r_{h,t}\right)^{1-\alpha_h}\left(\frac{r_{f,t}}{q_t}\right)^{\alpha_h} \quad \forall t \tag{18.34}$$

18.2.2 Foreign country intermediate goods producers

The Foreign country is also populated by a unit measure of perfectly competitive intermediate goods producers that rent capital from households of both countries and transform those two inputs of household savings into an aggregated capital stock $K_{f,t}$ specific to the Foreign country. We assume that this is done through a constant elasticity of substitution (CES) production function in which the elasticity of substitution between Foreign capital and Home capital in the Foreign intermediate goods producer's production function is constant $\phi_f \geq 1$,

$$K_{f,t} \equiv \left[(1-\alpha_f)^{\frac{1}{\phi_f}}\left(K^f_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}} + (\alpha_f)^{\frac{1}{\phi_f}}\left(K^h_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}}\right]^{\frac{\phi_f}{\phi_f-1}} \quad \forall t \tag{18.35}$$

where α_f is related to the Home country capital share in Foreign country aggregated capital $K_{f,t}$. The two capital inputs on the right-hand-side of (18.35) $K^f_{f,t}$ and $K^h_{f,t}$ represent the total amount of country-specific household savings that was used in the Foreign country intermediate goods production function.

$$K^f_{f,t} \equiv \Delta_{f,t}\sum_{s=2}^{S} b_{f,s,t} \tag{18.36}$$

$$K^h_{f,t} \equiv (1-\Delta_{h,t})\sum_{s=2}^{S} b_{h,s,t} \tag{18.37}$$

The variables $\Delta_{f,t}$ and $\Delta_{h,t}$ are endogenous variables and represent the percent of total Foreign household savings that goes to Foreign production and the percent of total Home household savings that goes to Home production, respectively. The output of the Foreign country intermediate goods producer is a new type of aggregated capital that can be used by the Foreign country final goods producer.

We can solve for the Foreign country intermediate goods producer's demand for Foreign and Home savings, respectively, by solving the expenditure minimization problem subject to a given level of aggregated capital.

$$\min_{K^f_{f,t},K^h_{f,t}} \; r^f_{f,t}K^f_{f,t} + r^h_{f,t}K^h_{f,t} \quad \text{s.t.} \quad K_{f,t} \leq \left[(1-\alpha_f)^{\frac{1}{\phi_f}}\left(K^f_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}} + (\alpha_f)^{\frac{1}{\phi_f}}\left(K^h_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}}\right]^{\frac{\phi_f}{\phi_f-1}} \tag{18.38}$$

where $r^f_{f,t}$ is the interest rate on Foreign country savings in Foreign output units from (18.14) and $r^h_{f,t}$ is the interest rate on Home country savings in Foreign output units from (18.2).

The Lagrangian for this problem is the following,

$$\mathcal{L} = r^f_{f,t}K^f_{f,t} + r^h_{f,t}K^h_{f,t} + r^*_t\left(K_{f,t} - \left[(1-\alpha_f)^{\frac{1}{\phi_f}}\left(K^f_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}} + (\alpha_f)^{\frac{1}{\phi_f}}\left(K^h_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}}\right]^{\frac{\phi_f}{\phi_f-1}}\right) \tag{18.39}$$

where r^*_t is the multiplier on the constraint and represents the marginal cost of an extra unit of aggregated Foreign capital. So r^*_t is interpreted as the price or interest rate of aggregated Foreign capital. The first order conditions are the following:

$$r^f_{f,t} = r^*_t\left[\frac{(1-\alpha_f)K_{f,t}}{K^f_{f,t}}\right]^{\frac{1}{\phi_f}} \quad \Rightarrow \quad r_{f,t} = r^*_t\left[\frac{(1-\alpha_f)K_{f,t}}{K^f_{f,t}}\right]^{\frac{1}{\phi_f}} \quad \forall t \tag{18.40}$$

$$r^h_{f,t} = r^*_t\left[\frac{\alpha_f K_{f,t}}{K^h_{f,t}}\right]^{\frac{1}{\phi_f}} \quad \Rightarrow \quad q_t r_{h,t} = r^*_t\left[\frac{\alpha_f K_{f,t}}{K^h_{f,t}}\right]^{\frac{1}{\phi_f}} \quad \forall t \tag{18.41}$$

$$K_{f,t} = \left[(1-\alpha_f)^{\frac{1}{\phi_f}}\left(K^f_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}} + (\alpha_f)^{\frac{1}{\phi_f}}\left(K^h_{f,t}\right)^{\frac{\phi_f-1}{\phi_f}}\right]^{\frac{\phi_f}{\phi_f-1}} \quad \forall t \tag{18.35}$$

Dividing (18.40) by (18.41) gives the following relationship.

$$\frac{r_{f,t}}{q_t r_{h,t}}\left(\frac{K^f_{f,t}}{K^h_{f,t}}\right)^{\frac{1}{\phi_f}} = \left(\frac{1-\alpha_f}{\alpha_f}\right)^{\frac{1}{\phi_f}} \quad \forall t \tag{18.42}$$

Notice that in the Cobb-Douglas case when $\phi_f = 1$, the ratio of Foreign savings expenditure to Home savings expenditure is a constant. Also, note that solving (18.40) and (18.41) for $K^f_{f,t}$ and $K^h_{f,t}$, respectively, gives Foreign demand equations for consumption of Foreign savings and Home savings.

$$K^f_{f,t} = (1-\alpha_f)\left(\frac{r_{f,t}}{r^*_t}\right)^{-\phi_f} K_{f,t} \quad \forall t \tag{18.43}$$

$$K^h_{f,t} = \alpha_f\left(\frac{q_t r_{h,t}}{r^*_t}\right)^{-\phi_f} K_{f,t} \quad \forall t \tag{18.44}$$

The expression for the interest rate r^*_t on Foreign country aggregated capital $K_{f,t}$ is found by substituting the demand equations (18.43) and (18.44) into (18.35).

$$r_t^* = \left[(1-\alpha_f)\left(r_{f,t}\right)^{1-\phi_f} + \alpha_f\left(q_t r_{h,t}\right)^{1-\phi_f}\right]^{\frac{1}{1-\phi_f}} \quad \forall t \tag{18.45}$$

In the case of Cobb-Douglas aggregation over Foreign household savings and Home household savings ($\phi_f = 1$), the expression for the interest rate on Foreign aggregated capital is the following.

$$r_t^* = \frac{1}{(1-\alpha_f)^{1-\alpha_f}(\alpha_f)^{\alpha_f}}\left(r_{f,t}\right)^{1-\alpha_f}\left(q_t r_{h,t}\right)^{\alpha_f} \quad \forall t \tag{18.46}$$

18.2.3 Home country final goods producers

A unit measure of perfectly competitive final goods producing firms produce final goods output in the Home country $Y_{h,t}$ by renting aggregated capital $K_{h,t}$ from the representative intermediate goods firm and hiring labor $L_{h,t}$

$$Y_{h,t} = Z_h\left(K_{h,t}\right)^{\gamma_h}\left(L_{h,t}\right)^{1-\gamma_h} \quad \forall t \tag{18.47}$$

where Z_h is Home-country total factor productivity, $K_{h,t}$ is Home country aggregated capital demand in period t, and $L_{h,t}$ is Home country labor demand in period t.

The profit function of the representative Home country firm is the following,

$$Pr_{h,t} = Z_h\left(K_{h,t}\right)^{\gamma_h}\left(L_{h,t}\right)^{1-\gamma_h} - w_{h,t}L_{h,t} - (r_t + \delta_h)K_{h,t} \quad \forall t \tag{18.48}$$

where δ_h is the depreciation rate of aggregated capital in the Home country. The two first order conditions for the representative Home country firm are the following.

$$r_t = \gamma_h Z_h\left(\frac{L_{h,t}}{K_{h,t}}\right)^{1-\gamma_h} - \delta_h \tag{18.49}$$

$$w_{h,t} = (1-\gamma_h) Z_h\left(\frac{K_{h,t}}{L_{h,t}}\right)^{\gamma_h} \tag{18.50}$$

18.2.4 Foreign country final goods producers

A unit measure of perfectly competitive final goods producing firms also produce output in the Foreign country $Y_{f,t}$ by renting aggregated capital $K_{f,t}$ from the representative intermediate goods firm and hiring labor $L_{f,t}$

$$Y_{f,t} = Z_f \left(K_{f,t}\right)^{\gamma_f} \left(L_{f,t}\right)^{1-\gamma_f} \quad \forall t \tag{18.51}$$

where Z_f is Foreign-country total factor productivity, $K_{f,t}$ is Foreign country aggregated capital demand in period t, and $L_{f,t}$ is Foreign country labor demand in period t.

The profit function of the representative Foreign country firm is the following,

$$Pr_{f,t} = Z_f \left(K_{f,t}\right)^{\gamma_f} \left(L_{f,t}\right)^{1-\gamma_f} - w_{f,t}L_{f,t} - (r_t^* + \delta_f)K_{f,t} \quad \forall t \tag{18.52}$$

where δ_f is the depreciation rate in the Foreign country. The two first order conditions for the representative Foreign country firm are the following.

$$r_t^* = \gamma_f Z_f \left(\frac{L_{f,t}}{K_{f,t}}\right)^{1-\gamma_f} - \delta_f \tag{18.53}$$

$$w_{f,t} = (1-\gamma_f) Z_f \left(\frac{K_{f,t}}{L_{f,t}}\right)^{\gamma_f} \tag{18.54}$$

18.3 Market Clearing

$$L_{h,t} = \sum_{s=1}^{S} n_{h,s} \quad \forall t \tag{18.55}$$

$$L_{f,t} = \sum_{s=1}^{S} n_{f,s} \quad \forall t \tag{18.56}$$

$$K_{h,t}^h + K_{f,t}^h = \sum_{s=2}^{S} b_{h,s,t} \quad \forall t \tag{18.57}$$

$$K_{f,t}^f + K_{h,t}^f = \sum_{s=2}^{S} b_{f,s,t} \quad \forall t \tag{18.58}$$

Note the connection between the two capital market clearing conditions (18.57) and (18.58) and the definitions for $K^h_{h,t}$, $K^f_{h,t}$, $K^f_{f,t}$, and $K^h_{f,t}$ from (18.24), (18.25), (18.36), and (18.37).

The two goods market clearing conditions are the following resource constraints. Both of these constraints are redundant by Walras' Law.[1]

$$Y_{h,t} = C_{h,t} + K_{h,t+1} - (1-\delta_h)K_{h,t} \quad \forall t \tag{18.59}$$

$$Y_{f,t} = C_{f,t} + K_{f,t+1} - (1-\delta_f)K_{f,t} \quad \forall t \tag{18.60}$$

Note that the goods market clearing conditions (resource constraints) above include net exports in the aggregated capital terms $K_{h,t}$ and $K_{f,t}$.

This final market clearing equation says that the real exchange rate makes the value of all exports from the Home country to the Foreign country equal to the value of all imports from the Foreign country to the Home country.

$$q_t\big(K^h_{f,t+1} - K^h_{f,t} + r^f_{h,t}K^f_{h,t}\big) = K^f_{h,t+1} - K^f_{h,t} + r^h_{f,t}K^h_{f,t} \quad \forall t \tag{18.61}$$

The terms in parentheses are the net flows of capital, which are in terms of their respective country's final goods. And the terms multiplied by interest rates are net factor payments.

Substituting in the renamed interest rate variables in (18.8) and (18.20) from the interest rate parity conditions (18.7) and (18.19) gives the following simplification of the capital exchange market clearing condition (18.61).

$$q_t\Big[K^h_{f,t+1} - \big(1 + r_{h,t}\big)K^h_{f,t}\Big] = K^f_{h,t+1} - \big(1 + r_{f,t}\big)K^f_{h,t} \quad \forall t \tag{18.62}$$

18.4 Equilibrium

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

[1] Technically, these two constraints can be combined into one global resource constraint. And the global resource constraint is redundant.

Definition 18.1 (**Steady-state equilibrium**). A non-autarkic steady-state equilibrium in the two-country perfect foresight overlapping generations model with S-period lived agents and exogenous labor supply is defined as constant allocations of Home and Foreign consumption $\{\bar{c}_{h,s},\bar{c}_{f,s}\}_{s=1}^S$ and Home and Foreign savings $\{\bar{b}_{h,s},\bar{b}_{f,s}\}_{s=2}^S$, Home and Foreign real wages $\bar{w}_h$ and $\bar{w}_f$, Home and foreign real interest rates $\bar{r}_h$, $\bar{r}$, $\bar{r}_f$, and $\bar{r}^*$, and real exchange rate $\bar{q}$ such that:

i. Home and Foreign households optimize according to equations (18.10) and (18.22), respectively, subject to budget constraints (18.9) and (18.21),

ii. Home and Foreign intermediate goods producers optimize according to (18.31), (18.32), (18.43), and (18.44),

iii. Home and Foreign final goods producers optimize according to (18.49), (18.50), (18.53), and (18.54), and

iv. markets clear according to labor market clearing (18.55) and (18.56), capital market clearing (18.57) and (18.58), goods market clearing (18.59) and (18.60), and capital exchange market clearing (18.62).

Let the equilibrium policy functions for Home and Foreign savings be represented by $b_{h,s+1,t+1} = \psi_{h,s}(\mathbf{\Gamma}_t)$ and $b_{f,s+1,t+1} = \psi_{f,s}(\mathbf{\Gamma}_t)$. The arguments of the functions (the state) may change overtime causing the savings levels to change over time, but the function of the arguments is constant (stationary) across time.

Definition 18.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight two-country overlapping generations model with S-period lived agents and exogenous labor supply is defined as stationary allocation functions of the state $\left\{b_{h,s+1,t+1} = \psi_{h,s}\big(\mathbf{\Gamma}_t\big)\right\}_{s=1}^{S-1}$, $\left\{b_{f,s+1,t+1} = \psi_{f,s}\big(\mathbf{\Gamma}_t\big)\right\}_{s=1}^{S-1}$ and stationary price functions $w_h(\mathbf{\Gamma}_t)$, $w_f(\mathbf{\Gamma}_t)$, $r_h(\mathbf{\Gamma}_t)$ and $r_f(\mathbf{\Gamma}_t)$ such that:

i. Home and Foreign households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (18.12), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. Home and Foreign households optimize according to equations (18.10) and (18.22), respectively, subject to budget constraints (18.9) and (18.21),

iii. Home and Foreign intermediate goods producers optimize according to (18.31), (18.32), (18.43), and (18.44),

iv. Home and Foreign final goods producers optimize according to (18.49), (18.50), (18.53), and (18.54), and

v. markets clear according to labor market clearing (18.55) and (18.56), capital market clearing (18.57) and (18.58), goods market clearing (18.59) and (18.60), and capital exchange market clearing (18.62).

18.5 Solution Method

18.5.1 Steady-state solution method

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 18.1. Our suggested approach for solving the steady-state equilibrium is similar to our approach to solving for the transition path equilibrium. We divide the algorithm into an inner loop that solves the steady-state household problem given guesses for steady-state prices and an outer loop in which these prices are updated to satisfy aggregate conditions. Step (i) is the inner loop, and step (ii) is the outer loop.

i. Make a guess for the steady-state Home interest rate $\bar{r}^i_h$, Foreign interest rate $\bar{r}^i_h$, and real exchange rate $\bar{q}^i$.

   (a) A guess for $\bar{r}^i_h$, $\bar{r}^i_f$, and $\bar{q}^i$ implies values for the aggregated Home and Foreign interest rate indices $\bar{r}$ and $\bar{r}^*$ from (18.33) and (18.45).

   (b) A value for $\bar{r}$ implies a Home capital-labor ratio $\left(\bar{K}_h/\bar{L}_h\right)$ and a steady-state Home wage $\bar{w}_h$ through equations (18.49) and (18.50).

   (c) A value for $\bar{r}^*$ implies a Foreign capital-labor ratio $\left(\bar{K}_f/\bar{L}_f\right)$ and a steady-state Foreign wage $\bar{w}_f$ through equations (18.53) and (18.54).

   (d) Values for $\bar{r}^i_h$, $\bar{r}^i_f$, $\bar{w}_h$, and $\bar{w}_f$ allow us to solve for all Home and Foreign household solutions $\left\{\bar{c}_{h,s},\bar{b}_{h,s+1}\right\}_{s=1}^S$ and $\left\{\bar{c}_{f,s},\bar{b}_{f,s+1}\right\}_{s=1}^S$ using (18.9), (18.10), (18.21), and (18.22).

   (e) Home and Foreign aggregate labor supply $\bar{L}_h$ and $\bar{L}_f$ are trivially computed from exogenous household labor supply $n_{h,s}$ and $n_{f,s}$ and the labor market clearing conditions (18.55) and (18.55).

   (f) Values for $\bar{L}_h$ and $\bar{L}_f$ along with the capital-labor ratios solved for in steps (b) and (c) allow us to solve for aggregate final goods capital stocks $\bar{K}_h$ and $\bar{K}_f$.

(g) Values for $\bar{K}_h$, $\bar{L}_h$, $\bar{K}_f$, and $\bar{L}_f$ allow us to compute aggregate Home and Foreign final goods output $\bar{Y}_h$ and $\bar{Y}_f$, respectively, using the Final goods production functions (18.47) and (18.51).

(h) Values for $\bar{r}_h^i$, $\bar{r}_f^i$, $\bar{r}$, $\bar{r}^*$, $\bar{K}_h$, and $\bar{K}_f$ allow us to compute $\bar{K}_h^h$ and $\bar{K}_f^f$ using (18.31) and (18.43), respectively.

(i) Values of $\bar{K}_h^h$ and $\bar{K}_f^f$ from (h), and $\{\bar{b}_{h,s}, \bar{b}_{f,s}\}_{s=2}^S$ from (d) allow us to compute $\bar{K}_f^h$ and $\bar{K}_h^f$, respectively, using (18.57) and (18.58).

ii. Given stead-state solutions to the model based on guesses for $\bar{r}_h^i$, $\bar{r}_f^i$, and $\bar{q}^i$, check whether remaining Home and Foreign intermediate producer first order conditions (18.28) and (18.40) and capital exchange market clearing condition (18.62) are satisfied. Update guesses $\bar{r}_h^i$, $\bar{r}_f^i$, and $\bar{q}^i$ until these equations are satisfied.

(a) We can use the steady-state versions of (18.28), (18.40), and (18.62) to get implied updated values of the two interest rates and the real exchange rate $\bar{r}_h^{i'}$, $\bar{r}_f^{i'}$, and $\bar{q}^{i'}$ given our initial guesses for those variables $\bar{r}_h^i$, $\bar{r}_f^i$, and $\bar{q}^i$. Notice that all variables on the right-hand-side of the following three updated equations are based on initial guess i values, including $\bar{r}_h^i$, $\bar{r}_f^i$, and $\bar{q}^i$.

$$\begin{aligned}
\bar{r}_h^{i'} &= \frac{\bar{r}^*}{\bar{q}^i}\left[\frac{\alpha_f \bar{K}_f}{\bar{K}_f^h}\right]^{\frac{1}{\phi_f}} \\
\bar{r}_f^{i'} &= \bar{q}^i \bar{r}\left[\frac{\alpha_h \bar{K}_h}{\bar{K}_h^f}\right]^{\frac{1}{\phi_h}} \\
\bar{q}^{i'} &= \frac{\bar{r}_f^i \bar{K}_h^f}{\bar{r}_h^i \bar{K}_f^h}
\end{aligned}$$

(b) Evaluate a distance measure (norm) of the vector of implied prices $\left[\bar{r}_h^{i'}, \bar{r}_f^{i'}, \bar{q}^{i'}\right]$ from the vector of initial guess prices $\left[\bar{r}_h^i, \bar{r}_f^i, \bar{q}^i\right]$.

$$dist_{ss} = \left\|\left[\bar{r}_h^{i'}, \bar{r}_f^{i'}, \bar{q}^{i'}\right] - \left[\bar{r}_h^i, \bar{r}_f^i, \bar{q}^i\right]\right\|$$

(c) If the distance measure is less than a predetermined tolerance $dist_{ss} < toler_{ss}$, then all equations that characterize the steady-state equilibrium from Definition 18.1 are satisfied and the steady-state has been solved.

(d) If the distance measure is greater-than-or-equal to the tolerance $dist_{ss} \geq toler_{ss}$, update initial guesses for the interest rates and real exchange rate $\left[\bar{r}_h^{i+1}, \bar{r}_f^{i+1}, \bar{q}^{i+1}\right]$ as a convex combination ξ percent of the way between the original guess values $\left[\bar{r}_h^{i}, \bar{r}_f^{i}, \bar{q}^{i}\right]$ and the implied values $\left[\bar{r}_h^{i'}, \bar{r}_f^{i'}, \bar{q}^{i'}\right]$ and repeat steps (i) and (ii).

$$\left[\bar{r}_h^{i+1}, \bar{r}_f^{i+1}, \bar{q}^{i+1}\right] = \xi\left[\bar{r}_h^{i'}, \bar{r}_f^{i'}, \bar{q}^{i'}\right] + (1-\xi)\left[\bar{r}_h^{i}, \bar{r}_f^{i}, \bar{q}^{i}\right] \quad \text{for} \quad \xi \in (0,1]$$

18.5.2 Transition path equilibrium solution method

This section outlines the steps for computing the solution to the non-stead-state transition path equilibrium in Definition 18.2.

18.6 Calibration

Put calibration here.

18.7 Exercises

Put exercises here.