Chapter 18

Two-country (Large Open Economy) OG Model

In this chapter, we take model an economy in which two countries that each look like the the S-period-lived agent economies from Chapter 3. In this large-open-economy environment, changes within the country can affect equilibrium interest rates. We call one country the Home country and the other country the Foreign country, where *Home* and *Foreign* are not relative terms but are fixed country names.

We assume that capital is mobile and that labor is not mobile. That is, Home country household savings can be invested in either the Home country production or in the Foreign country production. However, Home country household labor supply can only be hired by Home country firms. The techniques in this chapter generalize to an N-country model.

18.1 Households

18.1.1 Home household demand for country-specific consumption

In the large open economy model, households value total consumption which is comprised of both Home country output and of Foreign country country output. We define $c_{h,s,t}$ as the total consumption of Home household, age-s, period-t as a constant elasticity of substitution (CES) aggregator of Home country age-s household consumption of Home output $c^h_{h,s,t}$ and of Home country age-s household consumption of foreign output $c^f_{h,s,t}$.

$$c_{h,s,t} \equiv \left[(1-\theta_h)^{\frac{1}{\varepsilon_h}}\left(c^h_{h,s,t}\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}} + (\theta_h)^{\frac{1}{\varepsilon_h}}\left(c^f_{h,s,t}\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}}\right]^{\frac{\varepsilon_h}{\varepsilon_h-1}} \quad \forall s,t \tag{18.1}$$

The constant elasticity is parameterized by ε_h, the the expenditure share on Foreign goods is given by θ_h. A nice property of the CES aggregator is that it nests the case of perfect substitutes ($\varepsilon = \infty$) in which total Home consumption is just the sum of Home consumption of Home output and Home consumption of Foreign output. It also nests the unit elastic Cobb-Douglas case ($\varepsilon = 1$) and the case of perfect compliments ($\varepsilon = 0$). However, we will assume that the elasticity of substitution between Home and Foreign consumption is strictly between zero and infinity $\varepsilon \in (0,\infty)$.

We can solve for the Home household's demand for Home and Foreign consumption, respectively, by solving the expenditure minimization problem subject to a given level of total consumption,

$$\min_{c^h_{h,s,t},c^f_{h,s,t}} \; P^h_t c^h_{h,s,t} + \frac{P^f_t}{e_t}c^f_{h,s,t} \quad \text{s.t.} \quad c_{h,s,t} \leq \left[(1-\theta_h)^{\frac{1}{\varepsilon_h}}\left(c^h_{h,s,t}\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}} + (\theta_h)^{\frac{1}{\varepsilon_h}}\left(c^f_{h,s,t}\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}}\right]^{\frac{\varepsilon_h}{\varepsilon_h-1}} \tag{18.2}$$

where P^h_t is the price of Home country consumption in Home output units, P^f_t is the price of Foreign country consumption in Foreign output units, and e_t is the exchange rate representing the number of Foreign output units per one Home output unit. The Lagrangian for this problem is the following,

$$\mathcal{L} = P^h_t c^h_{h,s,t} + \frac{P^f_t}{e_t}c^f_{h,s,t} + P_t\left(c_{h,s,t} - \left[(1-\theta_h)^{\frac{1}{\varepsilon_h}}\left(c^h_{h,s,t}\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}} + (\theta_h)^{\frac{1}{\varepsilon_h}}\left(c^f_{h,s,t}\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}}\right]^{\frac{\varepsilon_h}{\varepsilon_h-1}}\right) \tag{18.3}$$

where P_t is the multiplier on the constraint and represents the marginal cost of an extra unit of aggregate consumption. So P_t is interpreted as the price of total consumption. The first order conditions are the following:

$$P^h_t = P_t\left[\frac{(1-\theta_h)c_{h,s,t}}{c^h_{h,s,t}}\right]^{\frac{1}{\varepsilon_h}} \quad \forall s,t \tag{18.4}$$

$$\frac{P_t^f}{e_t} = P_t \left[\frac{\theta_h c_{h,s,t}}{c_{h,s,t}^f}\right]^{\frac{1}{\varepsilon_h}} \quad \forall s,t \tag{18.5}$$

$$c_{h,s,t} = \left[(1-\theta_h)^{\frac{1}{\varepsilon_h}}\left(c_{h,s,t}^h\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}} + (\theta_h)^{\frac{1}{\varepsilon_h}}\left(c_{h,s,t}^f\right)^{\frac{\varepsilon_h-1}{\varepsilon_h}}\right]^{\frac{\varepsilon_h}{\varepsilon_h-1}} \quad \forall s,t \tag{18.1}$$

Dividing (18.4) by (18.5) gives the following relationship:

$$\frac{e_t P_t^h}{P_t^f}\left(\frac{c_{h,s,t}^h}{c_{h,s,t}^f}\right)^{\frac{1}{\varepsilon_h}} = \left(\frac{1-\theta_h}{\theta_h}\right)^{\frac{1}{\varepsilon_h}} \quad \forall s,t \tag{18.6}$$

Notice that in the Cobb-Douglas case when $\varepsilon_h = 1$, the ratio of Home consumption expenditure to Foreign consumption expenditure is a constant. Also, note that solving (18.4) and (18.5) for $c_{h,s,t}^h$ and $c_{h,s,t}^f$, respectively, gives Home demand equations for consumption of Home goods and Foreign goods.

$$c_{h,s,t}^h = (1-\theta_h)\left(\frac{P_t^h}{P_t}\right)^{-\varepsilon_h} c_{h,s,t} \quad \forall s,t \tag{18.7}$$

$$c_{h,s,t}^f = \theta_h\left(\frac{P_t^f}{e_t P_t}\right)^{-\varepsilon_h} c_{h,s,t} \quad \forall s,t \tag{18.8}$$

The expression for the aggregate price index P_t of the Home consumption over Home and Foreign consumption is found by substituting the demand equations (18.7) and (18.8) into (18.1).

$$P_t = \left[(1-\theta_h)\left(P_t^h\right)^{1-\varepsilon_h} + \theta_h\left(\frac{P_t^f}{e_t}\right)^{1-\varepsilon_h}\right]^{\frac{1}{1-\varepsilon_h}} \quad \forall t \tag{18.9}$$

Note that this expression is the Home country consumer price index. In the case of Cobb-Douglas aggregation over Home consumption and Foreign consumption ($\varepsilon_h = 1$), the expression for aggregate price is the following.

$$P_t = \frac{1}{(1-\theta_h)^{1-\theta_h}(\theta_h)^{\theta_h}}\left(P_t^h\right)^{1-\theta_h}\left(\frac{P_t^f}{e_t}\right)^{\theta_h} \quad \forall t \tag{18.10}$$

18.1.2 Foreign household demand for country-specific consumption

We define $c_{f,s,t}$ as the total consumption of Foreign household, age-s, period-t as a constant elasticity of substitution (CES) aggregator of Foreign country age-s household consumption of Foreign output $c^f_{f,s,t}$ and of Foreign country age-s household consumption of Home output $c^h_{f,s,t}$.

$$c_{f,s,t} \equiv \left[(1-\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^f_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}} + (\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^h_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}}\right]^{\frac{\varepsilon_f}{\varepsilon_f-1}} \quad \forall s,t \tag{18.11}$$

The constant elasticity is parameterized by ε_f, the the expenditure share on Home goods is given by θ_f.

We can solve for the Foreign household's demand for Foreign and Home consumption, respectively, by solving the expenditure minimization problem subject to a given level of total consumption.

$$\min_{c^f_{f,s,t},c^h_{f,s,t}} \quad P^f_t c^f_{f,s,t} + e_t P^h_t c^h_{f,s,t} \quad \text{s.t.} \quad c_{f,s,t} \leq \left[(1-\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^f_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}} + (\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^h_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}}\right]^{\frac{\varepsilon_f}{\varepsilon_f-1}} \tag{18.12}$$

The Lagrangian for this problem is the following,

$$\mathcal{L} = P^f_t c^f_{f,s,t} + e_t P^h_t c^h_{f,s,t} + P^*_t\left(c_{f,s,t} - \left[(1-\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^f_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}} + (\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^h_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}}\right]^{\frac{\varepsilon_f}{\varepsilon_f-1}}\right) \tag{18.13}$$

where P^*_t is the multiplier on the constraint and represents the marginal cost of an extra unit of aggregate consumption. So P^*_t is interpreted as the Foreign price of total consumption. The first order conditions are the following.

$$P^f_t = P^*_t\left[\frac{(1-\theta_f)c_{f,s,t}}{c^f_{f,s,t}}\right]^{\frac{1}{\varepsilon_f}} \quad \forall s,t \tag{18.14}$$

$$e_t P^h_t = P^*_t\left[\frac{\theta_f c_{f,s,t}}{c^h_{f,s,t}}\right]^{\frac{1}{\varepsilon_f}} \quad \forall s,t \tag{18.15}$$

$$c_{f,s,t} = \left[(1-\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^f_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}} + (\theta_f)^{\frac{1}{\varepsilon_f}}\left(c^h_{f,s,t}\right)^{\frac{\varepsilon_f-1}{\varepsilon_f}}\right]^{\frac{\varepsilon_f}{\varepsilon_f-1}} \quad \forall s,t \qquad (18.11)$$

Dividing (18.14) by (18.15) gives the following relationship:

$$\frac{P^f_t}{e_t P^h_t}\left(\frac{c^f_{f,s,t}}{c^h_{f,s,t}}\right)^{\frac{1}{\varepsilon_f}} = \left(\frac{1-\theta_f}{\theta_f}\right)^{\frac{1}{\varepsilon_f}} \quad \forall s,t \qquad (18.16)$$

Notice that in the Cobb-Douglas case when $\varepsilon_f = 1$, the ratio of Home consumption expenditure to Foreign consumption expenditure is a constant. Also, note that solving (18.14) and (18.15) for $c^f_{f,s,t}$ and $c^h_{f,s,t}$, respectively, gives Home demand equations for consumption of Home goods and Foreign goods.

$$c^f_{f,s,t} = (1-\theta_f)\left(\frac{P^f_t}{P^*_t}\right)^{-\varepsilon_f} c_{f,s,t} \quad \forall s,t \qquad (18.17)$$

$$c^h_{f,s,t} = \theta_f\left(\frac{e_t P^h_t}{P^*_t}\right)^{-\varepsilon_f} c_{f,s,t} \quad \forall s,t \qquad (18.18)$$

The expression for the aggregate price index P^*_t of the Foreign consumption over Foreign and Home consumption is found by substituting the demand equations (18.17) and (18.18) into (18.11).

$$P^*_t = \left[(1-\theta_f)\left(P^f_t\right)^{1-\varepsilon_f} + \theta_f\left(e_t P^h_t\right)^{1-\varepsilon_f}\right]^{\frac{1}{1-\varepsilon_f}} \quad \forall t \qquad (18.19)$$

Note that this expression is the Foreign country consumer price index. In the case of Cobb-Douglas aggregation over Foreign consumption and Home consumption ($\varepsilon_f = 1$), the expression for aggregate price is the following.

$$P^*_t = \frac{1}{(1-\theta_f)^{1-\theta_f}(\theta_f)^{\theta_f}}\left(P^f_t\right)^{1-\theta_f}\left(e_t P^h_t\right)^{\theta_f} \quad \forall t \qquad (18.20)$$

18.1.3 Home household's lifetime problem

A unit measure of Home households is born each period and lives for S periods. These households supply labor inelastically according to the following equation.

$$n_{h,s} = \begin{cases} 1.0 & \text{if} \quad s < round\left(\frac{9}{16}\right)S \\ 0.2 & \text{if} \quad s \geq round\left(\frac{9}{16}\right)S \end{cases} \tag{18.21}$$

Because we can specify the Home household's demand for Home consumption (18.7) and for Foreign consumption (18.8) as a percent of total Home consumption $c_{h,s,t}$, we can simply solve the Home household's problem each period for total Home consumption $c_{h,s,t}$, given Home prices P^h_t, Foreign prices P^f_t, and the exchange rate e_t, as they are summarized by the Home CPI P_t (18.9).

Let the age-s Home household's budget constraint in each period t be the following,

$$P_t c_{h,s,t} + b_{h,s+1,t+1} = (1+r_t)b_{h,s,t} + w_{h,t}n_{h,s} \quad \forall s,t \tag{18.22}$$

$$\max_{\{c_{h,s,t},b_{h,s+1,t+1}\}} \sum_{s=1}^{S} \beta^{s-1} u\left(c_{h,s,t+s-1}\right) \tag{18.23}$$

$$\text{s.t.} \quad P_t c_{h,s,t} + b_{h,s+1,t+1} = (1+r_t)b_{h,s,t} + w_{h,t}n_{h,s} \tag{18.22}$$

$$\text{where} \quad u\left(c_{h,s,t}\right) \equiv \frac{\left(c_{h,s,t}\right)^{1-\sigma} - 1}{1-\sigma} \tag{18.24}$$

$$\frac{P_{t+1}}{P_t}\left(c_{h,s,t}\right)^{-\sigma} = \beta(1+r_{t+1})\left(c_{h,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.25}$$

18.1.4 Foreign household's lifetime problem

$$n_{f,s} = \begin{cases} 1.0 & \text{if} \quad s < round\left(\frac{9}{16}\right)S \\ 0.2 & \text{if} \quad s \geq round\left(\frac{9}{16}\right)S \end{cases} \tag{18.26}$$

$$P_t^* c_{f,s,t} + b_{f,s+1,t+1} = (1+r_t)b_{f,s,t} + w_{f,t}n_{f,s} \quad \forall s,t \tag{18.27}$$

$$\max_{\{c_{f,s,t},b_{f,s+1,t+1}\}} \sum_{s=1}^{S} \beta^{s-1} u\left(c_{f,s,t+s-1}\right) \tag{18.28}$$

$$\text{s.t.} \quad P_t^* c_{f,s,t} + b_{f,s+1,t+1} = (1+r_t)b_{f,s,t} + w_{f,t}n_{f,s} \tag{18.27}$$

$$\text{where} \quad u\left(c_{f,s,t}\right) \equiv \frac{\left(c_{f,s,t}\right)^{1-\sigma} - 1}{1-\sigma} \tag{18.29}$$

$$\frac{P_{t+1}^*}{P_t^*}\left(c_{f,s,t}\right)^{-\sigma} = \beta(1+r_{t+1})\left(c_{f,s+1,t+1}\right)^{-\sigma} \quad \text{for} \quad 1 \leq s \leq S-1, \quad \text{and} \quad \forall t \tag{18.30}$$

18.2 Firms

A unit measure of perfectly competitive firms produce output in the Home country $Y_{h,t}$ by renting capital $K_{h,t}$ and hiring labor $L_{h,t}$

$$Y_{h,t} = Z_h\left(K_{h,t}\right)^{\gamma_h}\left(L_{h,t}\right)^{1-\gamma_h} \quad \forall t \tag{18.31}$$

where Z_h is Home-country total factor productivity, $K_{h,t}$ is Home country capital demand in period t, and $L_{h,t}$ is Home country labor demand in period t.

The profit function of the representative Home country firm is the following,

$$Pr_{h,t} = P_t^h Z_h\left(K_{h,t}\right)^{\gamma_h}\left(L_{h,t}\right)^{1-\gamma_h} - w_{h,t}L_{h,t} - (r_t + \delta_h)K_{h,t} \quad \forall t \tag{18.32}$$

where $w_{h,t}$ is the wage in the Home country, r_t is the world interest rate on capital,[1] and δ_h is the depreciation rate in the Home country. The two first order conditions for the

[1] The interest rate r_t is a world interest rate and not a country-specific interest rate because we assume that capital is mobile. For this reason, a Bertrand equilibrium argument requires that the interest rate be the same across countries.

representative Home country firm are the following.

$$r_t = \gamma_h P_t^h Z_h \left(\frac{L_{h,t}}{K_{h,t}}\right)^{1-\gamma_h} - \delta_h \tag{18.33}$$

$$w_{h,t} = (1-\gamma_h) P_t^h Z_h \left(\frac{K_{h,t}}{L_{h,t}}\right)^{\gamma_h} \tag{18.34}$$

A unit measure of perfectly competitive firms also produce output in the Foreign country $Y_{f,t}$ by renting capital $K_{f,t}$ and hiring labor $L_{f,t}$

$$Y_{f,t} = Z_f \left(K_{f,t}\right)^{\gamma_f} \left(L_{f,t}\right)^{1-\gamma_f} \quad \forall t \tag{18.35}$$

where Z_f is Foreign-country total factor productivity, $K_{f,t}$ is Foreign country capital demand in period t, and $L_{f,t}$ is Foreign country labor demand in period t.

The profit function of the representative Home country firm is the following,

$$Pr_{f,t} = P_t^f Z_f \left(K_{f,t}\right)^{\gamma_f} \left(L_{f,t}\right)^{1-\gamma_f} - w_{f,t}L_{f,t} - (r_t + \delta_f)K_{f,t} \quad \forall t \tag{18.36}$$

where $w_{f,t}$ is the wage in the Foreign country and δ_f is the depreciation rate in the Foreign country. The two first order conditions for the representative Foreign country firm are the following.

$$r_t = \gamma_f P_t^f Z_f \left(\frac{L_{f,t}}{K_{f,t}}\right)^{1-\gamma_f} - \delta_f \tag{18.37}$$

$$w_{f,t} = (1-\gamma_f) P_t^f Z_f \left(\frac{K_{f,t}}{L_{f,t}}\right)^{\gamma_f} \tag{18.38}$$

18.3 Market Clearing

$$L_{h,t} = \sum_{s=1}^{S} n_{h,s} \quad \forall t \tag{18.39}$$

$$L_{f,t} = \sum_{s=1}^{S} n_{f,s} \quad \forall t \tag{18.40}$$

$$K_{h,t} + K_{f,t} = \sum_{s=2}^{S} \left(b_{h,s,t} + b_{f,s,t}\right) \quad \forall t \tag{18.41}$$

I am not sure about these two resource constraints. In particular, they should be something like $Y = C + I + NX$. But my intuition is that net exports have to be zero in this model because the value of imports must equal the value of exports

$$Y_{h,t} = C^h_{h,t} + C^h_{f,t} + K_{h,t+1} - (1-\delta_h)K_{h,t} \quad \forall t \tag{18.42}$$

$$Y_{f,t} = C^f_{f,t} + C^f_{h,t} + K_{f,t+1} - (1-\delta_f)K_{f,t} \quad \forall t \tag{18.43}$$

This final market clearing equation says that the exchange rate makes the value of exports from the Home country to the foreign country equal to the value of exports from the Foreign country to the Home country. This condition makes net exports in both countries equal to zero (balanced trade).

$$e_t P^h_t \sum_{s=1}^{S} c^h_{f,s,t} = P^f_t \sum_{s=1}^{S} c^f_{h,s,t} \tag{18.44}$$

18.4 Solution Method

18.5 Steady-state solution method

i. Guess values for $\bar{P}^h$, $\bar{P}^f$, $\bar{e}$, and $\bar{r}$.

- These imply a value for $\bar{w}_h$ if the Home country production function is Cobb-Douglas ($\varepsilon_h = 1$) by combining (18.33) and (18.34). Otherwise, you have to also make a guess for $\bar{w}_h$ in the outer loop.
- These imply a value for $\bar{w}_f$ if the Foreign country production function is Cobb-Douglas ($\varepsilon_f = 1$) by combining (18.37) and (18.38). Otherwise, you have to also make a guess for $\bar{w}_f$ in the outer loop.
- These imply a value for $\bar{P}$ through (18.9).
- These imply a value for $\bar{P}^*$ through (18.19).

ii. Given $\left(\bar{P}^h, \bar{P}^f, \bar{e}, \bar{P}, \bar{P}^*, \bar{r}, \bar{w}_h, \bar{w}_f\right)$, solve each household's lifetime decisions in each country $\{\bar{c}_{h,s}, \bar{b}_{h,s+1}\}_{s=1}^S$ and $\{\bar{c}_{f,s}, \bar{b}_{f,s+1}\}_{s=1}^S$ using $S-1$ Euler equations from the Home country (18.25) and $S-1$ Euler equations from the Foreign country (18.30).

iii. Solve for implied values of $\left(\bar{P}^h, \bar{P}^f, \bar{e}, \bar{r}\right)$ from household optimized values $\{\bar{c}_{h,s}, \bar{b}_{h,s+1}\}_{s=1}^S$ and $\{\bar{c}_{f,s}, \bar{b}_{f,s+1}\}_{s=1}^S$.

- $\bar{L}_h$ and $\bar{L}_f$ are essentially exogenous through (18.39) and (18.40).
- Capital market clearing (18.41) and the two firm capital demand equations (18.33) and (18.37) determine $\bar{K}_h$, $\bar{K}_f$ and the new value of $\bar{r}$.
- Solve for new $\bar{P}^h$ using (18.4)
- Solve for new $\bar{P}^f$ using (18.14)
- Solve for new $\bar{e}$ using (18.44)

iv. Iterate until find fixed point.