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Macroeconomic volatility and trade in OLG economies

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This paper analyzes the effect of free-trade integration on the dynamical properties of economies. We formulate a two-country, two-good, two-factor overlapping generations model where countries only differ with respect to their discount rate. The main contribution of this paper is to show that opening up to international trade may have a destabilizing effect. In particular, we prove that, under perfect mobility of labor and capital between countries, sunspot cycles can occur in the trade regime although one country is characterized by saddle-point stability in the autarky regime.

Key words two-sector OLG model, two-country, local indeterminacy, endogenous fluctuation, dynamic efficiency

JEL classification C62, E32, F11, F43, O41

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1 Introduction

The phenomenon of globalization has expanded rapidly in recent decades. One particular feature of globalization is the increasing development of international trade. For example, in the past three decades the value of the share of trade in the gross domestic product has increased by a factor of 2 for the main OECD countries (OECD 2010). Along with this globalization, the global crisis since 2008 has evidenced that business cycles in countries are increasingly interlinked. Recent empirical studies, using industry-level data to estimate the link between macroeconomic volatility and trade openness, show that countries more exposed to trade are more volatile (Kose *et al. 2003*). A growing interest has thus emerged in understanding the effect of international trade on the instability of trading economies.

How can trade affect the business cycles of trading countries? In this paper we attempt to address this question by analyzing the co-movement of cycles between countries in a two-good (consumption and investment), two-factor (capital and labor), two-sector model in which the two factors are

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internationally mobile and countries differ only with respect to their discount rate. In particular, our aim is to study the dynamic behavior of two countries through the occurrence of endogenous fluctuations. A large proportion of the literature considers that endogenous cycles occur through sunspot equilibria and are driven by changes in expectations about fundamentals. This change in expectations is based on the concept of sunspot equilibria defined in Shell (1977). As shown by Woodford (1986), the existence of sunspot equilibria is related to the indeterminacy of the equilibrium under perfect foresight, that is, the existence of a continuum of equilibrium paths converging towards one steady state from the same initial value of the state variable.

The literature demonstrates that opening up to international trade may have different impacts on the stability properties of trading countries. They may be classified into two subsets. In the first we find the contributions based on a two-country version of Benhabib and Nishimura (1998) who study the existence of local indeterminacy in a closed two-sector (consumption and investment) infinitely lived agent model with sector-specific externalities and social constant returns with international immobility of factors.[1] Nishimura and Shimomura (2002) consider a model where countries only differ with respect to their initial factor endowments. They show that international trade has no effect on the stability properties of the two countries.[2] By contrast, Sim and Ho (2007) consider that the technologies are different across countries.[3] They prove that the world economy is characterized by saddle-point stability even if before trade one country exhibits sunspot fluctuations. Finally, Hu and Mino (2013) consider different trade structures with lending and borrowing, and show that international trade produces endogenous cycles in both countries, even if before trade sunspot cycles do not emerge in the two countries. The second subset of models contains the contributions which deal with international capital mobility and international labor immobility. Nishimura *et al.* (2010) consider an infinitely lived agent model with asymmetric technologies across countries and sector-specific externalities. They show that trade creates a contagion of sunspot cycles from one country to another. In a subsequent paper, Nishimura *et al.* (2014) consider an infinitely lived agent model with asymmetric technologies across countries and Cobb–Douglas decreasing returns to scale technologies. They analyze the existence of flip bifurcation and deterministic cycles and prove that the destabilizing effect of international trade and international capital mobility arises under certain parameter configurations. In other words, the opening up of trade may create persistent endogenous fluctuations at the world level while the closed-economy equilibrium in each country is saddle-point stable.

This literature focuses on the infinitely lived agent model. In this framework, local indeterminacy necessarily requires the presence of market imperfection such as sector-specific externalities. It implies that any equilibrium is Pareto inefficient. By contrast, local indeterminacy together with dynamic efficiency can arise in overlapping generations (OLG) models without any market imperfection (Nourry and Venditti 2011). In OLG models, Pareto efficiency is associated with under-accumulation of capital stock with respect to the golden rule. Reichlin (1986) shows how the coexistence of Pareto efficiency and local indeterminacy in OLG models is an important question in terms of stabilization policies. If sunspot fluctuations occur under dynamic efficiency, a fiscal policy can simultaneously stabilize the economy and reach the Pareto optimal steady state.

[1] They show that sunspot fluctuations arise provided that the investment good sector is more capital intensive than the consumption good sector from the social perspective and less capital intensive from the private perspective, and that the elasticity of intertemporal substitution in consumption is large enough.

[2] Iwasa and Nishimura (2014) extend Nishimura and Shimomura (2002) by introducing a consumption capital good. They show that international trade can create sunspot fluctuations in the world economy.

[3] One country is characterized by sector-specific externality and the other country is not.

Most models that investigate the existence of sunspot fluctuations in the OLG framework consist of closed economies. Among the few exceptions which analyze the effects of international trade on the stability properties of trading countries, two papers deserve particular attention. Aloi and Lloyd-Braga (2010) show that when countries differ in their labor market regulations, one country's sunspot fluctuations can spread throughout the world once trade opens, even if the other country has determinacy under autarky. A second exception is the paper of Aloi *et al.* (2000) which studies a small open monetary economy without capital and shows that sunspot fluctuations can exist because households can substitute consumption of non-traded goods for consumption of traded goods.[4] However, two points have to be made. First, in contrast to the literature with an infinitely lived agent model, they do not consider at a same time the two-sector approach and the effect of international trade on the dynamical properties of economies. Second, these two contributions do not consider the efficiency property of the dynamic equilibrium, and as a result they do not mention the issue raised by Reichlin (1986).

The purpose of our paper is to investigate the impact of international trade on the stability properties in Pareto efficient economies. We consider a two-sector, two-country OLG version of Drugeon *et al.* (2010) with one consumption good and one investment good. Our formulation differs from Drugeon *et al.* (2010) in two dimensions. To this end, recall that Nourry and Venditti (2011) show that local indeterminacy is likely to occur under dynamic efficiency if the sectoral technologies are close enough to Leontief functions. We thus express our model with a CES technology in the consumption good sector and a Leontief technology in the investment good sector. Second, we assume decreasing returns to scale on the consumption good sector. This permits us to guarantee a non-degenerate social production function at the world level. We also consider a trade reform consisting of the following hypothesis. First, the two countries are in an autarky regime, meaning that goods and factors are traded only on their respective domestic market. Second, both countries are in a trade regime, implying that goods are traded on the international market with no transaction cost. In the trade regime, lending and borrowing are not permitted and the factors of production are internationally mobile.

In the autarky and trade regimes, we show the existence of endogenous fluctuations together with dynamic efficiency when the consumption good is capital intensive, the value of the elasticity of intertemporal substitution in consumption is intermediate and the degree of returns to scale is high enough. However, the conditions on these elasticities and shares to obtain sunspot fluctuations in the autarky and the trade regime differ. Our main focus is to analyze the effect of international trade while the elasticity of intertemporal substitution in consumption is made to vary. For this, we base our analysis on necessary conditions on technologies for local indeterminacy (in the autarky and trade regimes) and discuss only the effect of the elasticity of intertemporal substitution in consumption. We are then able to show that opening up to trade may create a contagion of sunspot cycles from one country to another.

This paper is organized as follows. Section 2 describes an economy in the autarky regime, while Section 3 introduces the analysis of the local dynamics of the closed economy. Section 4 provides the analysis of the two-country model, the pattern of trade and the stability properties. Section 5 shows the existence of the effect of international trade on the dynamical properties in countries with the help of a numerical example. Section 6 concludes. The proofs are gathered in the Appendix.

[4] In a two-sector, two-country model, Bajona and Kehoe (2008) and Mountford (1998) show the occurrence of local indeterminacy in an open economy but do not consider the effect of international trade on the dynamical properties of economies.

2 The autarky model

We consider a closed economy, North or South, that has two goods (consumption and investment), two factors (capital and labor), and two generations (young and old) in each period. In this section we extend the two-sector OLG model of Nourry and Venditti (2011) by considering non-increasing returns to scale in the consumption good sector. To simplify the exposition we do not use any superscripts for the North and South. However, from Section 4 on, when the two countries are considered at the same time, we will add superscripts $\{N, S\}$ to distinguish them.

2.1 Technology

Consider a competitive economy in which there are two sectors, one representative firm for each sector, and each firm producing one good. In this economy there exist two goods: one consumption good produced in quantity $Y_{0,t}$ and one investment good produced in quantity Y_t. The consumption good is taken as the numéraire. Each sector uses two factors, capital K_t and labor L_t, and both factors are mobile between sectors. Depreciation of capital is complete within one period:[5] $K_{t+1} = Y_t$, where K_{t+1} is the total amount of capital in period $t+1$. The consumption good $Y_{0,t}$ is assumed to be produced with a CES technology and the investment good Y_t is assumed to be produced with a Leontief technology:

$$
\begin{aligned}
Y_{0,t} &= F^0\left(K_t^0, L_t^0\right) = \Theta\left\{\mu\left(K_t^0\right)^{-\rho} + (1-\mu)\left(L_t^0\right)^{-\rho}\right\}^{-\frac{\nu}{\rho}}, \\
Y_t &= F^1\left(K_t^1, L_t^1\right) = \min\left\{\frac{K_t^1}{\eta}, L_t^1\right\},
\end{aligned} \tag{1}
$$

where the parameter $\mu \in (0, 1)$ reflects the capital intensity in production, $\sigma = 1/(1+\rho) > 0$ is the sectoral elasticity of capital–labor substitution in the consumption good sector (with $\rho > -1$), $\nu > 0$ is the degree of returns to scale in the consumption good sector, $\eta > 0$ is the capital intensity in the investment good sector, and $\Theta > 0$ a normalization constant. We assume non-increasing returns to scale in the consumption good sector (i.e., $\nu \leq 1$).

Labor is normalized to 1 and given by $L = L_t^0 + L_t^1 = 1$, and the capital stock is given by $K_t = K_t^0 + K_t^1$. The optimal allocation of factors between sectors is defined by the social production function $T(K_t, Y_t, L)$:

$$
\begin{aligned}
T(K_t, Y_t, L) = &\max_{K_t^j, L_t^j, j\in\{0,1\}} Y_{0,t} \\
\text{s.t.} Y_t \leq F^1&(K_t^1, L_t^1),\ K_t^0 + K_t^1 \leq K_t,\ L_t^0 + L_t^1 \leq L.
\end{aligned} \tag{2}
$$

The social production function is the frontier of the production possibility set and gives the maximal output of the consumption good. Using Equation (1) and the resource constraints, $K_t = K_t^0 + K_t^1$ and $L = L_t^0 + L_t^1$, we define the social production function as

$$
T(K_t, Y_t, L) = \Theta\left[\mu\left(K_t - \eta Y_t\right)^{-\rho} + (1-\mu)\left(L - Y_t\right)^{-\rho}\right]^{-\frac{\nu}{\rho}}. \tag{3}
$$

[5] In a two-period OLG model, full depreciation of capital is justified by the fact that one period is about 30 years.

Let us denote by r_t the rental rate of capital, p_t the price of the investment good, and w_t the wage rate, all in terms of the price of the consumption good. Using the envelope theorem, we obtain the following three relationships:[6]

$$r(K_t, Y_t, L) = T_1(K_t, Y_t, L), \;\; p(K_t, Y_t, L) = -T_2(K_t, Y_t, L), \;\; w(K_t, Y_t, L) = T_3(K_t, Y_t, L), \quad (4)$$

where $T_1 = \partial T/\partial K_t, T_2 = \partial T/\partial Y_t$ and $T_3 = \partial T/\partial L$. Using Equation (1) and the resource constraints, we obtain the relative capital intensity difference b_t and the capital intensity in the consumption good sector a_t:

$$b\,(K_t, Y_t, L) = \frac{L_t^1}{Y_t}\left(\frac{K_t^1}{L_t^1} - \frac{K_t^0}{L_t^0}\right) = \frac{\eta - K_t}{L - Y_t}, \quad a\,(K_t, Y_t, L) = \frac{K_t^0}{L_t^0} = \frac{K_t - \eta Y_t}{L - Y_t}. \quad (5)$$

The sign of b is positive (negative) if and only if the consumption good is labor (capital) intensive. The Stolper–Samuelson effect (dr/dp, dw/dp) and the Rybczynski effect (dY_0/dK, dY/dK) are determined respectively by the factor–price frontier and the two full-employment conditions:

$$\frac{dr}{dp} = \frac{dY}{dK} = b^{-1}, \quad \frac{dw}{dp} = \frac{dY_0}{dK} = -ab^{-1}. \quad (6)$$

Under a labor (capital) intensive consumption good the Stolper–Samuelson effect states that an increase (decrease) of the relative price decreases (increases) the rental rate of capital and raises (decreases) the wage rate, whereas the Rybczynski effect specifies that an increase (decrease) of the capital–labor ratio decreases (increases) the production of the consumption good and increases (decreases) the production of the investment good.

Since there may exist decreasing returns to scale in the consumption good sector, firms earn positive profit, π_c. From the first-order conditions, we derive $\pi_c\,(K_t, Y_t, L) = T\,(K_t, Y_t, L)\,(1 - \nu)$. In the following, we suppose that the owner of the firm spends all the profit by purchasing the consumption good, $\pi_c\,(K_t, Y_t, L_t) = A_t$. Defining the GDP function as $T\,(K_t, Y_t, L_t) + p\,(K_t, Y_t, L_t)\,Y_t = w\,(K_t, Y_t, L_t)\,L_t + r\,(K_t, Y_t, L_t)\,K_t + \pi_c\,(K_t, Y_t, L_t)$, we get the share of capital in the economy:

$$s\,(K_t, Y_t, L) = \frac{r\,(K_t, Y_t, L)\,K_t}{T\,(K_t, Y_t, L) + p\,(K_t, Y_t, L)\,Y_t - \pi_c\,(K_t, Y_t, L)} \in (0, 1). \quad (7)$$

2.2 Preferences

Consider an infinite-horizon, discrete-time economy that is populated by overlapping generations of agents who live for two periods: they are thus the young and the old. There is no population growth and the population is normalized to 1. In the first period, young agents inelastically supply one unit of labor and receive an income w_t. They assign this income between savings ϕ_t and first-period consumption C_t. In the second period, old agents are retired. The return on saving $R_{t+1}\phi_t$ provides their income which they spend entirely on second-period consumption D_{t+1}. An agent born in period t has preferences defined over consumption of C_t and D_{t+1}. Intertemporal preferences of

[6] See details in Appendix A1.

agents are described by the CES utility function

$$U\left(C_t, D_{t+1}\right) = \left[C_t^{\frac{\gamma-1}{\gamma}} + \delta\left(\frac{D_{t+1}}{\Gamma}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}}, \tag{8}$$

where δ is the discount factor, γ the elasticity of intertemporal substitution in consumption and $\Gamma > 0$ a scaling constant parameter. Under perfect foresight and perfect competition, w_t and R_{t+1} are considered as given. A young agent who is born in period t solves the dynamic program

$$\max_{C_t, \phi_t, D_{t+1}} \left\{U\left(C_t, D_{t+1}\right) \mid C_t + \phi_t = w_t, D_{t+1} = R_{t+1}\phi_t\right\}. \tag{9}$$

Solving the first-order conditions gives

$$C_t = \alpha\left(\frac{R_{t+1}}{\Gamma}\right) w_t, \quad \alpha\left(\frac{R_{t+1}}{\Gamma}\right) = \frac{1}{1 + \delta^{\gamma}\left(\frac{R_{t+1}}{\Gamma}\right)^{\gamma-1}}, \tag{10}$$

where $\alpha\left(R_{t+1}/\Gamma\right) \in (0, 1)$ is the propensity to consume of a young agent in period t. From the budget constraint (9), we obtain the saving function ϕ_t:

$$\phi_t = \left[1 - \alpha\left(\frac{R_{t+1}}{\Gamma}\right)\right] w_t. \tag{11}$$

We assume that saving increases with respect to the gross rate of return R_{t+1}.

Assumption 1 $\gamma > 1$.

This standard assumption states that the substitution effect following an increase in the gross rate of return R_{t+1} is greater than the income effect.

2.3 Dynamic equilibrium

In a dynamic competitive equilibrium, total savings equal the production of the investment good: $\phi_t = p_t Y_t$. A perfect-foresight competitive equilibrium of an economy in the autarky regime is defined as follows.

Definition 1 *A sequence $\{K_t, Y_t\}_{t=0}^{\infty}$, with $K_{t=0}$ given, is a perfect-foresight competitive equilibrium if:*
(i) producers and households are at their optimum, that is, the first-order condition (4) and (10)–(11) are satisfied and $R_{t+1} = r_{t+1}/p_t$;
(ii) the capital accumulation is determined by $p_t Y_t = \phi_t$ with $Y_t = K_{t+1}$;
(iii) the market clearing condition for the consumption good is given by $A_t + C_t + D_t = T\left(K_t, Y_t, 1\right)$.

We derive from Definition 1 that the dynamics of an economy in the autarky regime is described by the evolution of the capital stock,

$$p\left(K_t, K_{t+1}, 1\right) K_{t+1} - w\left(K_t, K_{t+1}, 1\right)\left\{1-\alpha\left[\frac{r\left(K_{t+1}, K_{t+2}, 1\right)}{\Gamma p\left(K_t, K_{t+1}, 1\right)}\right]\right\} = 0. \tag{12}$$

Since $K_{t+1} = Y_t$, the set of admissible (K_t, K_{t+1}) is defined as follows:

$$\Omega = \left\{(K_t, K_{t+1}) \in \mathbb{R}_+^2 \mid K_t \leq \overline{K},\ K_{t+1} \leq F^1\left(K_t, 1\right)\right\}, \tag{13}$$

where the maximum admissible value of $\overline{K}$ is solution of $K - F^1(K, 1) = 0$.

2.4 Steady-state and efficiency properties

A steady state $K_t = K_{t+1} = K_{t+2} = K^*$ is defined by

$$p\left(K^*, K^*, 1\right) K^* - w\left(K^*, K^*, 1\right)\left\{1-\alpha\left[\frac{r\left(K^*, K^*, 1\right)}{\Gamma p\left(K^*, K^*, 1\right)}\right]\right\} = 0. \tag{14}$$

In the following, we consider a set of economic systems parametrized by the elasticity of intertemporal substitution in consumption γ. We follow the same approach as Drugeon *et al.* (2010): building on the homogeneity property of the utility function (8), we exploit the scaling parameter Γ in order to give conditions for the existence of a normalized steady state (NSS) $K^* \in (0, \overline{K})$ which remain unaltered as γ is varied. However, we need also to ensure that the value of the propensity to consume of young agents $\alpha(R/\Gamma)$ when evaluated at the NSS does not depend on γ. This characteristic will be derived by choosing appropriately the value of δ. Let us express $\xi = R/\Gamma$. Under Assumption 1, $\alpha(\xi)$ is a monotone decreasing function with $\lim_{\xi\to 0}\alpha(\xi) = \alpha_{\sup}$, $\lim_{\xi\to +\infty}\alpha(\xi) = \alpha_{\inf}$ and $(\alpha_{\inf}, \alpha_{\sup}) \subseteq (0, 1)$. We define the inverse function of $\alpha(\xi)$ as

$$\Phi_{K^*} = 1 - \frac{K^* p\left(K^*, K^*, 1\right)}{w\left(K^*, K^*, 1\right)}. \tag{15}$$

By adopting a proper value for K^*, we may find a corresponding value for $\Phi_{K^*} \in (\alpha_{\inf}, \alpha_{\sup})$. Then, the following proposition holds.

Proposition 1 *Under Assumption 1, let $K^* \in (0, \overline{K})$ be such that $\Phi_{K^*} \in (\alpha_{\inf}, \alpha_{\sup})$. Then, there exists a unique value $\Gamma(K^*) > 0$ solving (14) such that K^* is a steady state if and only if $\Gamma = \Gamma(K^*)$.*

PROOF: See Appendix A2. □

In the remainder of the paper we make the following assumption so that the existence of an NSS K^* is ensured in the autarky regime.

Assumption 2 $\Gamma = \Gamma(K^*)$.

Then, for a given set of parameters describing the consumption and production behavior, we will be able to dissociate the role of γ on the local stability of competitive equilibria.

Let us consider the dynamic efficiency properties of the competitive equilibrium around the NSS K^*. In two-sector OLG models, the golden-rule level of capital $\hat{K}$ is characterized on the basis of the total stationary consumption $A + C + D = T(\hat{K}, \hat{K}, 1)$. Denoting $R^A(\hat{K}, \hat{K}, 1) = -T_1(\hat{K}, \hat{K}, 1)/T_2(\hat{K}, \hat{K}, 1)$, $\hat{K}$ satisfies $R^A(\hat{K}, \hat{K}, 1) = 1$. Using Equation (7), it follows that $w/rK = (1-s)/s$, and we derive from (14) that the stationary gross rate of return at the NSS K^* is

$$R^A = \frac{s}{(1-\alpha)(1-s)}. \tag{16}$$

If $R^A > 1$ ($R^A < 1$), the NSS K^* is lower (higher) than the golden-rule level, that is, the economy displays under- (over-)accumulation of capital. Using the golden-rule level $R^A = 1$ and Equation (16), we derive, following the same methodology as Drugeon *et al.* (2010), a condition on the propensity to consume of young agent α to obtain an NSS K^* lower than the golden-rule level and to ensure the dynamic efficiency of equilibria. Then the following lemma applies.

Lemma 1 *Under Assumptions 1–2, let* $\underline{\alpha} = (1-2s)/(1-s)$. *An intertemporal competitive equilibrium converging toward the NSS is dynamically efficient if* $\alpha \in (\underline{\alpha}, 1)$, *and dynamically inefficient if* $\alpha \in (0, \underline{\alpha})$.

Lemma 1 states that under-accumulation of capital occurs when labor income of agents is less than capital income, that is, $s \geqslant 1/2$, and a young agent does not have enough labor income to save a sufficient amount. Under-accumulation of capital can be attained provided the share of consumption of young agents is high enough ($\alpha > \underline{\alpha}$). In the remainder of this paper, the following assumption is made.

Assumption 3 $\alpha \in (\underline{\alpha}, 1/2)$, $s \in (1/3, 1/2)$, $b < 0$.

Assumption 3 states that we consider dynamically efficient paths, that we restrain the share of capital in the economy s in order to get a positive value for $\underline{\alpha}$, and that we concentrate on a capital-intensive consumption good configuration.[7] Using national accounting data on the most developed countries, Takahashi *et al.* (2012) show that the aggregate consumption good sector is more capital intensive than the investment good sector.

3 Local indeterminacy in the autarky regime

This section discusses the existence of endogenous fluctuations under dynamic efficiency derived from changes in expectations about fundamentals. Our model consists of one predetermined variable, the current capital stock, and one forward variable, the capital stock of the next period. Local indeterminacy occurs when there exists a continuum of equilibrium paths converging to one steady state from the same initial value of the capital stock, whereas local determinacy occurs when there is a unique converging equilibrium path for a given initial capital stock. In our setting, the existence of local indeterminacy occurs if the two characteristic roots associated with the linearization of the dynamical system (12) around the NSS have a modulus less than 1. Let us introduce the elasticity of

[7] Cecchi and García-Peñalosa (2010) show that over the period 1960–2003, OECD countries were characterized by a share of capital between 0.35 and 0.5.

the rental rate of capital,

$$\varepsilon_{rk} = -\frac{T_{11}\left(K^*, K^*, 1\right) K^*}{T_1\left(K^*, K^*, 1\right)} \in (0, +\infty). \tag{17}$$

To proceed through the analysis of the local stability of the NSS, we linearize the difference equation (12) (see Appendix). We follow the methodology of Grandmont *et al.* (1998) and study the variation of the trace $\mathscr{T}^A(\gamma)$ and the determinant $\mathscr{D}^A(\gamma)$ in the $(\mathscr{T}^A(\gamma), \mathscr{D}^A(\gamma))$ plane as one parameter of interest, γ, is made to vary continuously in its admissible range. This methodology allows us to determine the occurrence of local indeterminacy. When the consumption good is capital intensive ($b < 0$), we obtain the following proposition.

Proposition 2 *Under Assumptions 1–3, there exist* $\underline{b} < \overline{b} < 0$, $\underline{\varepsilon_{rk}} > 0$, *and* $\gamma^{\mathcal{F}} > \gamma^{\mathcal{T}} > 1$ *such that, for* $b \in (\underline{b}, \overline{b})$ *and* $\varepsilon_{rk} > \underline{\varepsilon_{rk}}$, *the following results prevail:*
(i) *the steady state is locally indeterminate when* $\gamma \in (\gamma^{\mathcal{T}}, \gamma^{\mathcal{F}})$;
(ii) *the steady state is locally determinate when* $\gamma \in (1, \gamma^{\mathcal{T}}) \cup (\gamma^{\mathcal{F}}, +\infty)$.

PROOF: See Appendix A3.2. □

This proposition extends the result of Nourry and Venditti (2011) with a decreasing returns to scale technology in the consumption good sector. In Proposition 2, $\gamma^{\mathcal{T}}$ is generically a transcritical bifurcation[8] leading to the existence of a second steady state which is locally unstable (saddle-point stable) in a right (left) neighborhood of $\gamma^{\mathcal{T}}$, whereas $\gamma^{\mathcal{F}}$ is generically a flip bifurcation value giving rise to period-two cycles which are locally indeterminate (unstable) in a right (left) neighborhood of $\gamma^{\mathcal{F}}$ (see Figure 1).

The intuition of Proposition 2 is as follows. Suppose that agents expect that the gross rate of return of investment will rise in period t. Since the elasticity of intertemporal substitution in consumption is higher than 1 (i.e., $\gamma > 1$), this expectation causes agents to reduce their current consumption and save more. In addition, if γ is sufficiently high ($\gamma > \gamma^{\mathcal{T}}$), there is a large increase in the future consumption. At the same time, agents increase their saving and the capital will rise in period $t + 1$. Since the consumption good sector is the most capital-intensive sector, it implies through the Rybczynski effect that there is an increase in the production of the consumption good in period $t + 1$. The intertemporal substitution effect generates a rise in the future consumption demand and thus the relative price of the investment good p_t decreases in order to equilibrate the consumption good market and the investment good market. This implies, through the Stolper–Samuelson effect, that the rate of investment r_t increases so that the expectation is self-fulfilling.

In contrast, if γ is too small ($\gamma < \gamma^{\mathcal{T}}$), the intertemporal substitution effect is small and generates a low future consumption demand. Since the consumption good sector is the most capital-intensive sector, it follows, through the Rybczynski effect, that there is an increase in the production of the consumption good in period $t + 1$. The relative price of the investment good p_t increases in order to equilibrate the consumption good market and the investment good market. This implies, through

[8] When the bifurcation parameter γ crosses a $\gamma^{\mathcal{T}}$, one characteristic root crosses 1. We cannot *a priori* differentiate between the transcritical, the pitchfork or the saddle-node bifurcation from the difference equation (12). Under Assumption 2, the existence of the NSS is always ensured and a saddle-node bifurcation cannot occur. Moreover, the pitchfork bifurcation requires non-generic conditions (see Ruelle 1989). In order to simplify the exposition we focus on the generic case and relate the existence of one characteristic root going through 1 to a transcritical bifurcation.

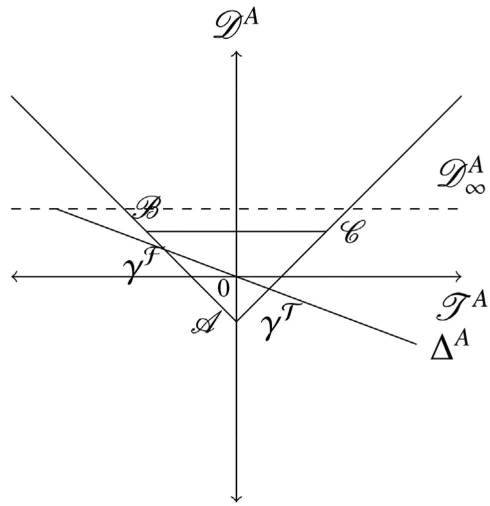

Figure 1 Dynamic efficiency and local indeterminacy in the autarky regime.

the Stolper–Samuelson effect, that the rate of investment r_t decreases so that the expectation is not self-fulfilling.

Finally, if γ is too high ($\gamma > \gamma^{\mathcal{F}}$), agents are willing to save a significant amount with a small decrease in their current consumption and an increase in their future consumption. Meanwhile, the capital stock in the next period will rise to a significant level. Since the consumption good sector is the most capital-intensive one, it implies through the Rybczynski effect that there is a large increase in the production of the consumption good at time $t+1$. This rise in consumption good production may exceed the increase in future consumption demand. As a result, the expectation is not realized when $\gamma > \gamma^{\mathcal{F}}$.

4 The two-country model

In this section we consider a two-country model (the North and the South). First, we present the main assumptions of the model and derive the perfect-foresight equilibrium. Second, we describe the existence of an NSS as well as the efficiency properties. Finally, assuming that the North is the more patient country, we examine the pattern of trade and analyze the local stability.

4.1 Assumption of the two-country model

Consider a world that consists of two countries, the North (N) and the South (S), which differ only with respect to their discount rate $\delta^N \neq \delta^S$. In other words, the two countries have different preferences and thus a different capital accumulation path. Capital and labor are mobile across countries such that $r_t^N = r_t^S$ and $w_t^N = w_t^S$. We assume free trade, thus the relative price of the investment good is the same in the two countries, $p_t^N = p_t^S$. As the investment good is mobile, there are investment flows across countries. Let $K_t^W = K_t^N + K_t^S$ be the world capital stock at time t, $Y_t^W = Y_t^N + Y_t^S$ the world production of investment good at time t, and $L_t^W = L_t^N + L_t^S = 2$ the world labor force at time t. At each period the balance of trade is at equilibrium for both countries. For an economy $i \in \{N, S\}$, the balance of trade is given by $\chi_t^{i0} + p_t\chi_t^{i1} = 0$, where χ_t^{i0} is the net

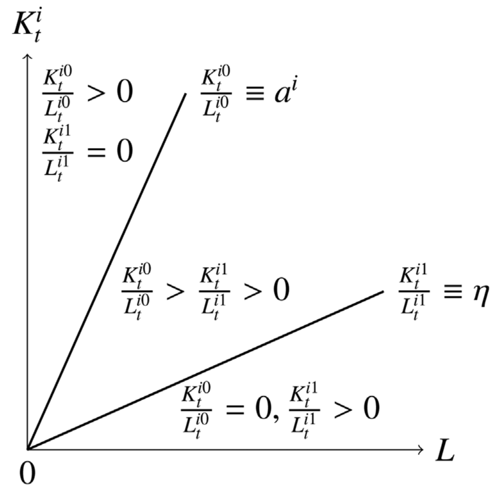

Figure 2 Cone of diversification of country $i \in \{N, S\}$ at time t.

export of the consumption good in country i at time t and χ_t^{i1} is the net export of the investment good in country i at time t. Since the net export of goods of one country is the net import of the other country, we have $\chi_t^{N0} + \chi_t^{S0} = 0, \quad \chi_t^{N1} + \chi_t^{S1} = 0$.

We suppose that the consumption good sector is capital intensive ($b^i < 0$), as in Assumption 3, and there is no factor intensity reversal. In other words, the capital–labor ratio used in the consumption good sector ($a_t^i \equiv K_t^{i0}/L_t^{i0}$) is higher than the capital–labor ratio used in the investment good sector ($\eta \equiv K_t^{i1}/L_t^{i1}$) in each period (see Figure 2). These two capital–labor ratios are crucial levels which describe the specialization pattern of a country $i \in \{N, S\}$ at time t. If $\eta < k_t^i < a_t^i$, country i diversifies in the production of the consumption good and the investment good. The range of capital–labor ratios between η and a_t^i is called the cone of diversification. When the North and South have their capital–labor ratio in the cone of diversification, the free-trade equilibrium is characterized by an interior equilibrium in which the two countries produce both consumption and investment goods. In our model, the cone of diversification evolves over time since the capital–labor ratio changes over time. The two countries produce both goods in a free-trade equilibrium if the two countries stay in the corresponding cone of diversification for each period t. Note that if the two countries are in the cone of diversification at the steady state, the North and South stay in the cone of diversification in the neighborhood of the steady state. This allows us to provide an analysis of the local dynamics of a world in which each country produces both goods.[9]

4.2 World dynamic equilibrium

Trade modifies the restrictions that an economy faces. Indeed, an economy in the autarky regime can increase its own capital stock only by producing more investment good, but an economy in the trade regime may in addition import the investment good from another economy. For an economy

[9] In Section 4.4 we show that each country produces both goods at the steady state. If at least one country specializes, the dynamics of the world economy examined in the next section is different.

$i \in \{N, S\}$ in the trade regime, the capital accumulation is now given by

$$p_t y_t^i = p_t k_{t+1}^i + p_t \chi_t^{i1} = \phi_t^i. \tag{18}$$

For our purposes, we study the dynamic behavior of the two countries at a world level. Such an approach allows us to compare the dynamic behavior of an economy in the autarky regime with the dynamic behavior of the world economy. We will thus characterize the world dynamic equilibrium by solving the following optimal allocation of factors problem:

$$\begin{aligned} \max_{K_t^i, Y_t^i, L_t^i, i \in \{N,S\}} \ & \tau\left(K_t^W, Y_t^W, 2\right) = T\left(K_t^N, Y_t^N, L_t^N\right) + T\left(K_t^S, Y_t^S, L_t^S\right) \\ \text{s.t. } & K_t^N + K_t^S \leq K_t^w, \\ & Y_t^N + Y_t^S \leq Y_t^w, \\ & L_t^N + L_t^S \leq 2. \end{aligned} \tag{19}$$

From the envelope theorem we get

$$\begin{aligned} r\left(K_t^W, Y_t^W, 2\right) &= r\left(K_t^N, Y_t^N, L_t^N\right) = r\left(K_t^S, Y_t^S, L_t^S\right), \\ p\left(K_t^W, Y_t^W, 2\right) &= p\left(K_t^N, Y_t^N, L_t^N\right) = p\left(K_t^S, Y_t^S, L_t^S\right), \\ w\left(K_t^W, Y_t^W, 2\right) &= w\left(K_t^N, Y_t^N, L_t^N\right) = w\left(K_t^S, Y_t^S, L_t^S\right). \end{aligned} \tag{20}$$

We define a world perfect-foresight equilibrium as follows.

Definition 2 *A sequence* $\left\{K_t^W, Y_t^W\right\}_{t=0}^{\infty}$ *with* $K_0^W = K_0^N + K_0^S$ *given, is a world perfect-foresight competitive equilibrium if:*
(i) *Producers and households are at their optimum – the first-order condition (4) and (10)–(11) are satisfied and* $R_{t+1}^W = r_{t+1}/p_t$*;*
(ii) *each period the balance of trade is at the equilibrium for both countries;*
(iii) *the capital accumulation is determined by* $p_t(Y_t^N + Y_t^S) = \phi_t^N + \phi_t^S$ *with* $K_{t+1}^N + K_{t+1}^S = Y_t^N + Y_t^S$*;*
(iv) *the market clearing condition for the consumption good is given by* $A_t^N + C_t^N + D_t^N + A_t^S + C_t^S + D_t^S = \tau(K_t^W, Y_t^W, 2)$*.*

We derive from Definition 2 that the dynamics of the world economy is represented by the evolution of the world capital stock

$$K_{t+1}^W - \frac{w\left(K_t^W, K_{t+1}^W, 2\right)\left\{2 - \alpha^N\left[\frac{r\left(K_{t+1}^W, K_{t+2}^W, 2\right)}{\Gamma^N p\left(K_t^W, K_{t+1}^W, 2\right)}\right] - \alpha^S\left[\frac{r\left(K_{t+1}^W, K_{t+2}^W, 2\right)}{\Gamma^S p\left(K_t^W, K_{t+1}^W, 2\right)}\right]\right\}}{p\left(K_t^W, K_{t+1}^W, 2\right)} = 0. \tag{21}$$

The set of admissible paths is defined as

$$\Omega^W = \left\{(K_t^W, K_{t+1}^W) \in \mathbb{R}_+^2 \mid K_t^W \leq \overline{K}^W,\ K_{t+1}^W \leq \tfrac{1}{2}\left[F^1\left(K_t^N, 1\right) + F^1\left(K_t^S, 1\right)\right]\right\}, \tag{22}$$

where the maximal admissible value of $\overline{K}^W$ is solution of $K^W - \frac{1}{2}[F^1(K^N,1) + F^1(K^S,1)] = 0$. Defining the GDP function as $\tau\left(K_t^W, K_{t+1}^W, 2\right) + p\left(K_t^W, K_{t+1}^W, 2\right) K_{t+1}^W = w\left(K_t^W, K_{t+1}^W, 2\right) + r\left(K_t^W, K_{t+1}^W, 2\right) K_t^W + \pi_{c,t}^N + \pi_{c,t}^S$, we get the share of capital in the world economy:

$$s^W\left(K_t^W, K_{t+1}^W, 2\right) = \frac{r\left(K_t^W, K_{t+1}^W, 2\right) K_t^W}{\tau\left(K_t^W, K_{t+1}^W, 2\right) + p\left(K_t^W, K_{t+1}^W, 2\right) K_{t+1}^W - \pi_c^N - \pi_c^S} \in (0,1). \tag{23}$$

4.3 World steady-state and efficiency properties

A world steady state $K_t^W = K_{t+1}^W = K_{t+2}^W = K^{W*}$ is defined by

$$K^{W*} - \frac{w\left(K^{W*}, K^{W*}, 2\right)\left\{2 - \alpha^N\left[\frac{r(K^{W*},K^{W*},2)}{\Gamma^N p(K^{W*},K^{W*},2)}\right] - \alpha^S\left[\frac{r(K^{W*},K^{W*},2)}{\Gamma^S p(K^{W*},K^{W*},2)}\right]\right\}}{p\left(K^{W*}, K^{W*}, 2\right)} = 0. \tag{24}$$

We follow the same procedure as in Section 2.4. Building on the homogeneity property of the utility function, we exploit the scaling parameter Γ^N in order to give conditions for the existence of an NSS $K^{W*} \in (0, \overline{K}^W)$ in the world economy which remains unaltered as γ is made to vary. Let us express $\xi^N = R/\Gamma^N$. Under gross substitutability, $\alpha^N(\xi^N)$ is a monotone decreasing function with $\lim_{\xi^N \to 0} \alpha^N(\xi^N) = \alpha_{\sup}^N$, $\lim_{\xi^N \to +\infty} \alpha^N(\xi^N) = \alpha_{\inf}^N$ and $(\alpha_{\inf}^N, \alpha_{\sup}^N) \subseteq (0,1)$. Let us define the inverse function of $\alpha^N(\xi^N)$ as

$$\Phi_{K^{W*}} = 2 - \alpha^S - \frac{2K^{W*} p\left(K^{W*}, K^{W*}, 2\right)}{w\left(K^{W*}, K^{W*}, 2\right)}. \tag{25}$$

By adopting a proper value for K^{W*}, we may find a corresponding value for $\Phi_{K^{W*}} \in (\alpha_{\inf}^N, \alpha_{\sup}^N)$. Then the following proposition holds.

Proposition 3 *Under Assumption 1, let $K^{W*} \in (0, \overline{K}^W)$ be such that $\Phi_{K^{W*}} \in (\alpha_{\inf}^N, \alpha_{\sup}^N)$. There exists a unique value $\Gamma^N\left(K^{W*}\right) > 0$ solving (24) such that K^{W*} is a steady state if and only if $\Gamma^N = \Gamma^N\left(K^{W*}\right)$.*

PROOF: : See Appendix A4. □

We introduce the following assumption to guarantee the existence of an NSS K^{W*} in the trade regime.

Assumption 4 $\Gamma^N = \Gamma^N(K^{W*})$.

As in the autarky regime, the golden-rule level of capital $\hat{K}^W$ is characterized by the total stationary consumption $A^N + C^N + D^N + A^S + C^S + D^S = \tau(\hat{K}^W, \hat{K}^W, 2)$. Denoting $R^W(\hat{K}^W, \hat{K}^W, 2) = -\tau_1(\hat{K}^W, \hat{K}^W, 2)/\tau_2(\hat{K}^W, \hat{K}^W, 2)$, $\hat{K}^W$ satisfies $R^W(\hat{K}^W, \hat{K}^W, 2) = 1$. We derive from (24) the propensity to consume of young agent in the world economy α^W and the stationary gross rate of return R^W at the NSS K^{W*}:

$$\alpha^W = \frac{1}{2}\left(\alpha^N + \alpha^S\right), \quad R^W = \frac{2s^W}{(2 - \alpha^N - \alpha^S)(1 - s^W)}. \tag{26}$$

Following the same methodology as in the autarky regime, we derive a condition on the propensity to consume of young agent α^W to obtain an NSS K^{W*} lower than the golden-rule level and to ensure the dynamic efficiency of equilibria.

Lemma 2 *Under Assumptions 1 and 4, let $\underline{\alpha}^W = (1-2s^W)/(1-s^W)$. An intertemporal dynamic equilibrium converging towards an NSS is dynamically efficient if $\alpha^W \in (\underline{\alpha}^W, 1)$, and dynamically inefficient if $\alpha^W \in (0, \underline{\alpha}^W)$.*

Under free trade and Equation (20), we have $p_t = p_t^N = p_t^S$ and $r_t = r_t^N = r_t^S$. This implies that $R = R^N = R^S$. Thus if the NSS is dynamically efficient in the world economy, the NSSs of the North and South are dynamically efficient.

4.4 Pattern of trade

Once we have the world dynamic equilibrium, we need to consider the trade pattern. In our two-sector OLG models, during the dynamic transition the pattern of specialization may change over time since the capital–labor ratio $K_t^i, i \in \{N, S\}$, evolves. As shown by the following proposition, the two countries produce both goods at the NSS K^{W*}.

Proposition 4 *Under Assumptions 1, 3, and 4, there exists $\overline{\eta}$ such that for $\eta \in (0, \overline{\eta})$, each country produces both goods at the steady state.*

PROOF: : See Appendix A5. □

This proposition allows us to provide an analysis of the stability of the world equilibrium in the neighborhood of the NSS in which both countries produce both goods. We introduce the following assumption to guarantee the existence of a world economy in which each country produces both goods at the steady state.

Assumption 5 $\eta \in (0, \overline{\eta})$.

We suppose in the following that the North and South differ only with respect to their discount rate $\delta^N \neq \delta^S$ and that the North has a comparative advantage in the production of the capital intensive good. By using Equations (10), (20), and (A1), we can show that $\phi_t^N > \phi_t^S$. This implies that the North saves more than the South and thus the North is the more patient country. Then the following proposition holds.

Proposition 5 *Under Assumptions 1 and 3–5, consider a world NSS equilibrium in which North and South differ only with respect to their discount rate, $\delta^N > \delta^S$. Then the North is the exporter of the capital-intensive good while the South is the exporter of the labor-intensive good.*

PROOF: : See Appendix A6. □

4.5 Local indeterminacy in the trade regime

We now consider the local stability results in the trade regime (see Appendix). In the autarky regime, we study a capital-intensive consumption good, and then consider a similar configuration and show the following proposition.

Proposition 6 *Under Assumptions 1 and 3–5, there exist* $\underline{b}^S < \overline{b}^S < 0$, $\underline{b}^N < \overline{b}^N < 0$, $\underline{\varepsilon}_{rk}^W > 0$, *and* $\gamma^{W,\mathcal{F}} > \gamma^{W,\mathcal{T}} > 1$ *such that for* $b^S \in (\underline{b}^S, \overline{b}^S)$, $b^N \in (\underline{b}^N, \overline{b}^N)$, $\varepsilon_{rk}^W > \underline{\varepsilon}_{rk}^W$, *the following results prevail:*
(i) *the steady state is locally indeterminate when* $\gamma \in (\gamma^{W,\mathcal{T}}, \gamma^{W,\mathcal{F}})$;
(ii) *the steady state is locally determinate when* $\gamma \in (1, \gamma^{W,\mathcal{T}}) \cup (\gamma^{W,\mathcal{F}}, +\infty)$.

PROOF: : See Appendix A7.2 . □

The intuition of Proposition 6 is similar to that of Proposition 2 in the autarky regime. Proposition 6 provides conditions under which local indeterminacy together with dynamic efficiency arise in the world economy. These conditions on elasticities and shares are based on joint restrictions on the parameters of the North and South. As our results on the occurrence of sunspot fluctuations of an economy in the autarky regime and the world economy differ, international trade may have various effects on the stability properties of each countries. In order to determine the impact of international trade we need to compare the conditions on elasticities and shares of the North and South in the autarky regime (Proposition 2) with respect to the conditions on elasticities and shares in the trade regime (Proposition 6).

5 The effect of international trade

This section investigates the impact of opening up to international trade on the dynamical properties of the North and South. In order to determine the impact of international trade we compare the conditions on the bifurcation parameter γ of the North and South in the autarky regime with the bifurcation parameter γ in the world economy. The steady-state values of all the bifurcation parameters γ at the closed and free-trade levels are not the same. Indeed, each bifurcation value depends on different elasticities, shares and thus depends on all parameters of the model. This makes the analytical comparison between the different steady-state values of all the bifurcation values somewhat complex and thus we rely on a numerical example.

We now present the numerical conditions such that local indeterminacy holds in the autarky regime for the North and South. The numerical example uses the following approach. We choose μ, η, ρ, ν, k^i, and γ in their admissible ranges. We then compute the values of α^i, b^i, s^i, ε_{rk}^i and the critical bounds on b^i, ε_{rk}^i, and γ. In order to illustrate the result of Proposition 2, we consider the set of parameters in Table 1. Then we derive the conditions for which Proposition 2 is satisfied. Dynamic efficiency holds for $\alpha^N \simeq 0.159 \in (\underline{\alpha}^N, 1/2)$ and $\alpha^S \approx 0.1581 \in (\underline{\alpha}^S, 1/2)$ with $\underline{\alpha}^N \approx 0.092$ and $\underline{\alpha}^S \approx 0.082$. Restrictions on technology in Proposition 2 are satisfied in the North and South for any $s^N \approx 0.476 \in (1/3, 1/2)$, $s^S \approx 0.478 \in (1/3, 1/2)$, $b^N \approx -2.6 \in (-5.292, -1.926)$, $b^S \approx -2.598 \in (-5.321, -1.916)$, $\varepsilon_{rk}^N \approx 2.618 > 2.454$, and $\varepsilon_{rk}^S \approx 2.609 > 2.374$. Finally, conditions on γ are ensured in the North for any $\gamma \in (1.586, 1.768)$ and the South for any $\gamma \in (1.597, 1.775)$. The condition on γ are represented in Figure 3. Figure 3 shows that the bifurcation parameters for the North and South are different. In other words, the stability properties of the two countries in the autarky regime are different. For example, when $\gamma \in (\gamma^{N,\mathcal{T}}, \gamma^{N,\mathcal{F}})$, the NSS of the North is locally

Table 1 Parameter values

μ	η	ρ	ν	K^N	K^S	γ	Θ
0.99996	0.1002	9	0.99	0.7501	0.7499	1.6	1

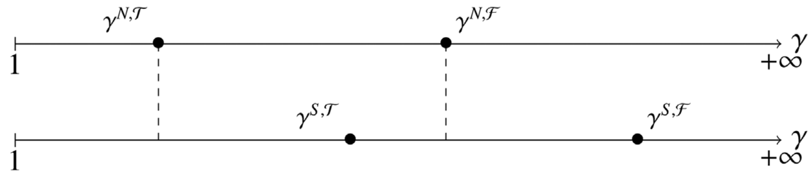

Figure 3 Bifurcation parameters of the North and South.

indeterminate while the NSS of the South can either be locally determinate or indeterminate.

We now follow the same approach as in the autarky regime. We consider the same set of parameters as in Table 1. As in the autarky regime, the existence of local indeterminacy is based on conditions on technology (b^i and ε^i_{rk}) and preference (γ). The conditions on technology in Proposition 6 are satisfied in the world economy for any $b^N \equiv -2.601 \in (-5.297, -2.071)$, $b^S \equiv -2.601 \in (-5.317, -2.071)$, and $\varepsilon^W_{rk} \equiv 2.607 > 2.454$. The conditions on preference in Proposition 6 are satisfied in the world economy for any $\gamma \in (1.591, 1.769)$. Figure 4 gathers the bifurcation parameters of the North and South in the autarky regime and the world economy.

As mentioned above, the stability properties of the two countries are not the same. Moreover, the bifurcation parameters of the world economy are different from those of the two countries. In particular, when $\gamma \in (\gamma^{W,\mathcal{T}}, \gamma^{W,\mathcal{F}})$, the NSS of the world economy is locally indeterminate while the NSS of the North and South can either be locally determinate or indeterminate.

In Figure 5, we represent the different values of γ defined in Figures 3 and 4 in a single line. Then, for a given value of γ, we can deduce the different stability properties of the two countries and thus the effect of international trade. We derive, from Figure 5 and the numerical example, the following proposition which is the main result on the impact of international trade on the dynamical properties of the two countries.

Proposition 7 *Under Assumptions 1 and 3–5, there exists a set of parameters (η, μ, ρ, ν, γ, $\gamma^{N,\mathcal{F}}$, $\gamma^{W,\mathcal{F}}$) and allocations (K^{N*}, K^{S*}) such that the NSS of one country is locally determinate and the NSS of the other country is locally indeterminate in the autarky regime, while local indeterminacy holds for both countries in the trade regime if $\gamma \in (\gamma^{W,\mathcal{T}}, \gamma^{S,\mathcal{T}}) \cup (\gamma^{N,\mathcal{F}}, \gamma^{W,\mathcal{F}})$.*

Opening up to trade creates a contagion of cycles from one country to another. The intuition of Proposition 7 when $\gamma \in (\gamma^{N,\mathcal{F}}, \gamma^{W,\mathcal{F}})$ is as follows. Let us suppose that all agents in the world economy anticipate an increase of the gross rate of return.

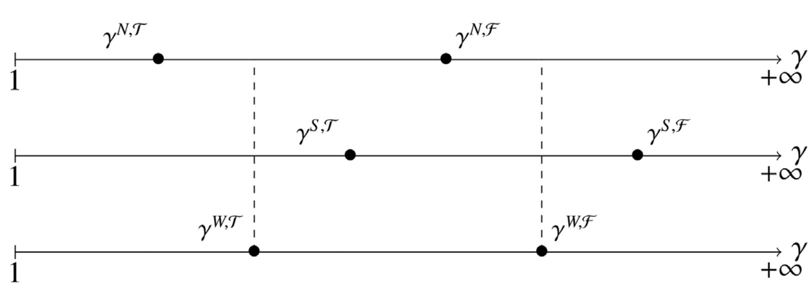

Figure 4 Bifurcation parameters of the North, the South and the world economy.

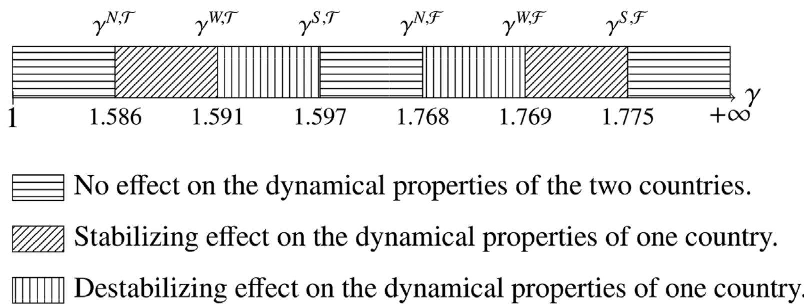

Figure 5 Trade effect on the occurrence of endogenous fluctuations.

Let us first consider the expectation of agents in the autarky equilibrium. Since $\gamma > 1$, this expectation causes agents in both countries to reduce their current consumption and save more. Through the Rybczynski effect, there is an increase in the production of the consumption good at period $t + 1$.

On the one hand, in the North, agents are most patient of the two countries (δ^N relatively high). In view of the propensity to consume (10), agents are willing to save an important amount with a small decrease in their current consumption and an increase in their future consumption. Meanwhile, the capital stock in the next period will rise at a significant level. Since the consumption good sector is the most capital-intensive sector, it implies through the Rybczynski effect that there is a large increase in the production of the consumption good at time $t + 1$. This rise in consumption good production may exceed the increase in future consumption demand. As a result, the expectation is not fulfilled for agents in the North.

On the other hand, in the South, the intertemporal substitution effect generates a rise in the future consumption good demand and thus the relative price of investment decreases in order to equilibrate the consumption good and investment good markets. This implies through the Stolper–Samuelson effect that the rate of investment r_t increases so the expectation is fulfilled for the agents in the South.

Let us now consider the free-trade equilibrium. Since the consumption good is exported from the North to the South, there is less production of the future consumption that remains in the North. Then the consumption good market is equilibrated in the North, and as a result the expectation can be realized as an equilibrium for both countries in the trade regime.

Remark 1 *Based on our numerical example and Figure 5, international trade may have other effects. On the one hand, opening up to international trade may rule out sunspot fluctuations that exist in the autarky regime. On the other hand, the two countries in the trade regime replicate either local determinacy or local indeterminacy that exists in the autarky regime.*

6 Concluding remarks

This paper has studied the effect of international trade on the stability properties of trading countries in a two-country, two-sector overlapping generations model. The two countries are characterized by CES life cycle utility, CES technology in the consumption good sector and Leontief technology in the investment good sector. Our main result shows that international trade can be a source of macroeconomic volatility. Indeed, by considering a free-trade equilibrium in which one country is a net exporter of the capital-intensive consumption good and the other country is a net exporter of the labor-intensive investment good, we have proved that sunspot fluctuations can occur in the trade regime although one country is characterized by saddle-point stability in the autarky regime.

Appendix

A1 The partial derivatives of $T(K_t, Y_t, L)$ in the autarky regime

We provide preliminary results necessary for the proof. We start by recalling that $T_g = T_g(K_t, Y_t, L)$ and $T_{gh} = T_{gh}(K_t, Y_t, L)$, with $g \in \{1, 2, 3\}$ and $h \in \{1, 2\}$. The first partial derivatives of $T(K_t, Y_t, L)$ directly follow from computations of (3) and are given by the following lemma.

Lemma A1 *The first partial derivatives of $T(K_t, Y_t, L)$ satisfy*

$$T_1 = \Theta\mu\nu(K_t - \eta Y_t)^{-(1+\rho)}\left[\mu(K_t - \eta Y_t)^{-\rho} + (1-\mu)(L - Y_t)^{-\rho}\right]^{-\frac{\nu+\rho}{\rho}},$$

$$T_2 = -T_1\left[\eta + \left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t - \eta Y_t}{L - Y_t}\right)^{1+\rho}\right],$$

$$T_3 = \left(\frac{1-\mu}{\mu}\right)\left(\frac{K_t - \eta Y_t}{L - Y_t}\right)^{1+\rho} T_1.$$

□

Using a_t and b_t defined in (5) and computations of Lemma A1, we derive the second partial derivatives of $T(K_t, Y_t, L)$.

Lemma A2 *The second partial derivatives of $T(K_t, Y_t, L)$ satisfy the following:*

$$T_{11} = \frac{T_1\left[(\nu-1) - (1+\rho)\frac{T_3}{T_1 a_t}\right]}{(K_t - \eta Y_t)\left(1 + \frac{T_3}{T_1 a_t}\right)},$$

$$T_{21} = \frac{T_{11}\left[(\nu-1)\frac{T_2}{T_1} + (1+\rho)\frac{T_3 b_t}{T_1 a_t}\right]}{\nu - 1 - (1+\rho)\frac{T_3}{T_1 a_t}}, \quad T_{22} = -\frac{T_{11}\left[-(\nu-1) + (1+\rho)\frac{T_3 b_t^2}{T_1 a_t}\right]}{\nu - 1 - (1+\rho)\frac{T_3}{T_1 a_t}}$$

$$T_{31} = \frac{T_{11}(\nu+\rho)\frac{T_3}{T_1}}{\nu - 1 - (1+\rho)\frac{T_3}{T_1 a_t}}, \quad T_{32} = -\frac{T_{11}\frac{T_3}{T_1}\left[-(\nu-1)\frac{T_2}{T_1} + (1+\rho)b_t\right]}{\nu - 1 - (1+\rho)\frac{T_3}{T_1 a_t}}.$$

□

Using the fact that $T_3/T_1 a = (1-s)/(1-b)s$, we can rewrite the second partial derivatives of the social production function at the NSS.

Lemma A3 *At the NSS, the second partial derivatives of $T(K^*, K^*, 1)$ satisfy*

$$T_{21} = -\frac{T_{11}(1-s)[(1-\nu)(1-\alpha)(1-b)+(1+\rho)b]}{(1-\nu)(1-b)s+(1+\rho)(1-s)}, \quad T_{31} = -\frac{T_{11}(\nu+\rho)(1-s)a}{(1-\nu)(1-b)s+(1+\rho)(1-s)},$$

$$T_{22} = \frac{T_{11}\left[(1-\nu)(1-b)s+(1-s)(1+\rho)b^2\right]}{(1-\nu)(1-b)s+(1+\rho)(1-s)}, \quad T_{32} = \frac{\frac{T_{11}a(1-s)}{s}[(\nu-1)(1-\alpha)(1-s)+(1+\rho)sb]}{(1-\nu)(1-b)s+(1+\rho)(1-s)}.$$

□

A2 Proof of Proposition 1

From the set of admissible paths defined by (13), we have $K^* \in \left(0, \overline{K}\right)$. K^* is a solution of (14) if

$$\frac{1}{1 + \delta^{\gamma}\left(\frac{r(K^*,K^*,1)}{\Gamma p(K^*,K^*,1)}\right)^{\gamma-1}} = 1 - \frac{K^* p(K^*, K^*, 1)}{w(K^*, K^*, 1)} \equiv \Phi_{K^*} \in (0, 1).$$

Let us express $\xi = R/\Gamma$. Under Assumption 1, $\alpha(\xi)$ is a monotone decreasing function with $\lim_{\xi\to 0}\alpha(\xi) = \alpha_{\sup}$, $\lim_{\xi\to+\infty}\alpha(\xi) = \alpha_{\inf}$ and $(\alpha_{\inf}, \alpha_{\sup}) \subseteq (0, 1)$. It follows that $\alpha(\xi)$ admits an inverse function defined over $(\alpha_{\inf}, \alpha_{\sup})$. Let $K^* \in \left(0, \overline{K}\right)$ be such that $\Phi_{K^*} \in (\alpha_{\inf}, \alpha_{\sup})$. We then derive that K^* is a steady state if and only if $\Gamma = \Gamma(K^*)$. Using Lemma A1, the scaling parameter Γ is a unique solution of (14) and is defined by

$$\Gamma\left(K^*\right) = R^A \left\{ \frac{\delta^\gamma \left[(1-s)a(1-K^*) - s\eta(1-b)K^*\right]}{K^* \left[s\eta(1-b) + (1-s)a\right]} \right\}^{\frac{1}{\gamma-1}}. \tag{A1}$$

□

A3 Local stability in the autarky regime

A3.1 Characteristic polynomial in autarky regime

From (10), the derivative of $\alpha(R^A/\Gamma)$ is given by

$$\frac{d\alpha(R^A/\Gamma)}{d(R^A/\Gamma)} = -\frac{\alpha\,(\gamma-1)\,(1-\alpha)\,\Gamma}{R^A}. \tag{A2}$$

Consider Equations (3), (5), (17) and (A2) evaluated at the NSS. Totally differentiating equation (12) evaluated at the NSS gives the characteristic polynomial $\mathscr{P}(\lambda) = \lambda^2 - \lambda\mathscr{T}^A(\gamma) + \mathscr{D}^A(\gamma)$ with trace $\mathscr{T}^A(\gamma)$ and determinant $\mathscr{D}^A(\gamma)$ given by

$$\mathscr{T}^A(\gamma) = \frac{1 + \varepsilon_{rk}\left\{ R^A\left[\frac{T_{32}(1-\alpha)}{T_{11}K^*} + \frac{T_{22}}{T_{11}}\right] + \alpha\,(\gamma-1)\left[1 + \frac{T_{22}R^A}{T_{11}}\right]\right\}}{\alpha\,(\gamma-1)\left(-\frac{T_{21}\varepsilon_{rk}}{T_{11}}\right)},$$

$$\mathscr{D}^A(\gamma) = \frac{R^A\left[1 + \alpha\,(\gamma-1) + (1-\alpha)\,\frac{T_{31}}{T_{21}K^*}\right]}{\alpha\,(\gamma-1)}, \tag{A3}$$

with the first and second partial derivatives of $T\,(K_t, Y_t, L)$ given in Lemma A1 and A2.

□

A3.2 Proof of Proposition 2

Under Assumption 2, when $\Gamma = \Gamma(K^*)$, the NSS K^*, α, s and ε_{rk} remain constant as γ varies within $(1, +\infty)$. We can analyze the variation of the trace $\mathscr{T}^A(\gamma)$ and the determinant $\mathscr{D}^A(\gamma)$ in the $(\mathscr{T}^A(\gamma), \mathscr{D}^A(\gamma))$ plane. The relationship between the trace and the determinant is given by a half-line $\Delta^A\left(\mathscr{T}^A\right)$ which is characterized from the consideration of its extremities. The starting point is the pair $(\lim_{\gamma\to+\infty}\mathscr{T}^A \equiv \mathscr{T}^A_\infty, \lim_{\gamma\to+\infty}\mathscr{D}^A \equiv \mathscr{D}^A_\infty)$, while the end point is the pair $(\lim_{\gamma\to 1}\mathscr{T}^A \equiv \mathscr{T}^A_1, \lim_{\gamma\to 1}\mathscr{D}^A \equiv \mathscr{D}^A_1)$. Solving $\mathscr{T}^A$ and $\mathscr{D}^A$ with respect to $\alpha\,(\gamma-1)$ yields the linear relationship

$$\mathscr{D}^A = \Delta^A\left(\mathscr{T}^A\right) = \mathscr{S}^A\mathscr{T}^A + \mathscr{D}^A_\infty - \mathscr{S}^A\mathscr{T}_\infty,$$

where the slope $\mathscr{S}^A$, $\mathscr{D}^A_\infty$ and $\mathscr{T}^A_\infty$ are given by

$$\mathscr{S}^A = -\frac{R^A\,\frac{T_{21}K^*}{T_{11}}\,\varepsilon_{rk}\left[1 + (1-\alpha)\,\frac{T_{31}}{T_{21}K^*}\right]}{1 + \varepsilon_{rk}R^A\left[\frac{T_{32}(1-\alpha)}{T_{11}K^*} + \frac{T_{22}}{T_{11}}\right]}, \quad \mathscr{D}^A_\infty = R^A,$$

$$\mathscr{T}^A_\infty = \frac{(1-\nu)(1-b)\left(\frac{1+R^A}{R^A}\right)s + (1+\rho)(1-s)(1+b^2R^A)}{(1-s)\left[(1-\nu)(1-\alpha) + b(\rho+\alpha+\nu(1-\alpha))\right]}. \tag{A4}$$

Assume that $b < 0$, $\nu \leq 1$, $\alpha \in (\underline{\alpha}, 1/2)$, and $s \in (1/3, 1/2)$. Let us first consider the starting point $(\mathscr{T}^A_\infty, \mathscr{D}^A_\infty)$. Under dynamic efficiency ($\alpha > \underline{\alpha}$), we get from (A4) that $\mathscr{D}^A_\infty > 1$. To establish the precise location of the starting point $(\mathscr{T}^A_\infty, \mathscr{D}^A_\infty)$ we need to determine the sign of $\mathscr{T}^A_\infty$, $\mathscr{P}^A_\infty(1) = 1 - \mathscr{T}^A_\infty + \mathscr{D}^A_\infty$, and $\mathscr{P}^A_\infty(-1) = 1 + \mathscr{T}^A_\infty + \mathscr{D}^A_\infty$. When $b < \overline{b}$, with $\overline{b} =$

$-(1-\alpha)(1-\nu)/(\nu+\rho+(1-\nu)\alpha)$, we get $\mathscr{T}^A_\infty < 0$. Since $\mathscr{T}^A_\infty < 0$ when $b < \overline{b}$, it follows that $\mathscr{P}^A_\infty(1) > 0$. Using Equation (A4), we derive that $\mathscr{P}^A_\infty(-1)$ is given by

$$\mathscr{P}^A_\infty(-1) = \frac{2(1-\nu)(1-b)\left(\frac{1+R^A}{R^A}\right)s + (1+\rho)(1-s)(1+bR^A)(1+b)}{(1-s)\left[(1-\nu)(1-\alpha)+b(\rho+\alpha+\nu(1-\alpha))\right]}.$$

Under $b < \overline{b}$, we derive that $\mathscr{P}^A_\infty(-1) < 0$ since $\overline{b} < -1/R^A < -1$. As a result, the starting point is in the left area outside the triangle $\mathscr{ABC}$ (see Figure 1).

Let us now consider the end point $(\mathscr{T}^A_1, \mathscr{D}^A_1)$. To determine its precise location, it is sufficient to determine whether $\Delta^A(\mathscr{T}^A)$ is pointing upward or downward. We thus study the sign of $\mathscr{D}^{A'}(\gamma)$,

$$\mathscr{D}^{A'}(\gamma) = -\frac{R^A(1+\rho)\left[1-\alpha+b\alpha\right]}{\left[(1-\nu)(1-\alpha)+b(\rho+\alpha+\nu(1-\alpha))\right]\alpha(\gamma-1)^2}.$$

Under $b \in (\underline{b}, \overline{b})$, with $\underline{b} = -(1-\alpha)/\alpha$, we get $\mathscr{D}^{A'}(\gamma) > 0$ and thus $\Delta^A(\mathscr{T}^A)$ is pointing downward. Therefore when $b \in (\underline{b}, \overline{b})$, the only possibility for obtaining local indeterminacy is that $\Delta^A(\mathscr{T}^A)$ crosses the interior of the segment $\mathscr{AC}$, as depicted in Figure 1. This requires that $\mathscr{T}^A > 0$ when $\mathscr{D}^A = -1$. From (A3), solving $\mathscr{D}^A = -1$ and using Lemma A3 yields

$$\alpha(\gamma-1) = -\frac{R^A(1+\rho)(\alpha b+1-\alpha)}{\left(1+R^A\right)\left[(1-\nu)(1-\alpha)(1-b)+(1+\rho)b\right]},$$

which holds if $b \in (\underline{b}, \overline{b})$. Substituting $\mathscr{D}^A = -1$ in $\mathscr{T}^A$, defined in (A3), gives

$$\mathscr{T}^A = -\frac{(1-\alpha)[(1-\nu)(1-b)s+(1+\rho)(1-s)]+(1+\rho)(\alpha b+1-\alpha)\varepsilon_{rk}\left\{1-\frac{R^A\left[(1-\alpha)(1-b)(1-\nu)s\left(\frac{1+R^A}{R^A}\right)+(1+\rho)(1-s)\left(1+R^Ab^2\right)\right]}{\left(1+R^A\right)[(1-\nu)(1-\alpha)(1-b)+(1+\rho)b]}\right\}}{\frac{R^A\varepsilon_{rk}(1+\rho)(\alpha b+1-\alpha)(1-\alpha)(1-s)[(1-\nu)(1-\alpha)(1-b)+(1+\rho)b]}{\left(1+R^A\right)[(1-\nu)(1-\alpha)(1-b)+(1+\rho)b]}}.$$

We get that $\mathscr{T}^A > 0$ when $\mathscr{D}^A = -1$ if $\alpha \in (\underline{\alpha}, 1/2)$, $s \in (1/3, 1/2)$, $b \in (\underline{b}, \overline{b})$, and $\varepsilon_{rk} > \underline{\varepsilon}_{rk}$. The bound $\underline{\varepsilon}_{rk}$ is defined as

$$\underline{\varepsilon}_{rk} = \frac{\left(1+R^A\right)\left[(1-\alpha)(1-\nu)(1-b)+(1+\rho)b\right]\left[(1-\nu)(1-b)s+(1+\rho)(1-s)\right]}{R^A(1+\rho)(1-b)^2(\alpha b+1-\alpha)\left[(\nu-1)\left[(1-s)(1-\alpha)-s\right]+(1+\rho)(1+b)s\right]}. \tag{A5}$$

The bifurcation values $\gamma^{A,\mathcal{T}}$ and $\gamma^{A,\mathcal{F}}$ are, respectively, defined as the solutions of $\mathscr{P}^A(1) = 1-\mathscr{T}^A+\mathscr{D}^A = 0$, and $\mathscr{P}^A(-1) = 1+\mathscr{T}^A+\mathscr{D}^A = 0$, and given by

$$\gamma^{A,\mathcal{T}} = 1 - \frac{1+\frac{\varepsilon_{rk}R^A}{\nu-1-(1+\rho)\frac{T_3}{T_1a}}\left[(\nu-1)\left(1-\left(\frac{1}{R^A}\right)^2\right)+(1+\rho)(1-b)\left(\frac{1}{R^A}+\frac{T_3b}{T_1a}\right)\right]}{\frac{\varepsilon_{rk}\alpha}{\nu-1-(1+\rho)\frac{T_3}{T_1a}}\left[(\nu-1)\left(R^A-\frac{1}{R^A}\right)-(1+\rho)(1-b)(1-bR^A)\frac{T_3b}{T_1a}\right]}, \tag{A6}$$

$$\gamma^{A,\mathcal{F}} = 1 + \frac{-1+\frac{\varepsilon_{rk}R^A}{\nu-1-(1+\rho)\frac{T_3}{T_1a}}\left[(\nu-1)\left(1-\left(\frac{1}{R^A}\right)^2\right)+(1+\rho)(1-b)\left(\frac{1}{R^A}+\frac{T_3b}{T_1a}\right)\right]}{\frac{\varepsilon_{rk}\alpha}{\nu-1-(1+\rho)\frac{T_3}{T_1a}}\left[(\nu-1)\left(1+R^A\right)\left(1+\frac{1}{R^A}\right)-(1+\rho)(1+b)(1-bR^A)\frac{T_3b}{T_1a}\right]}. \tag{A7}$$

The results follow. □

A4 Proof of Proposition 3

From the set of admissible paths defined by (22), we have $K^{W*} \in (0, \overline{K}^W)$. K^{W*} is a solution of (21) if

$$\frac{1}{1+(\delta^N)^\gamma\left(\frac{r(K^{W*},K^{W*},1)}{\Gamma^N p(K^{W*},K^{W*},1)}\right)^{\gamma-1}} = 2-\alpha^S-\frac{2K^{W*}p\left(K^{W*},K^{W*},1\right)}{w\left(K^{W*},K^{W*},1\right)} \equiv \Phi_{K^{W*}} \in (0,1).$$

Let us express $\xi^N = R^W/\Gamma^N$. Under Assumption 1, $\alpha^N(\xi^N)$ is a monotone decreasing function with $\lim_{\xi\to 0}\alpha^N(\xi^N) = \alpha^N_{\text{sup}}$, $\lim_{\xi\to+\infty}\alpha^N(\xi^N) = \alpha^N_{\text{inf}}$, and $(\alpha^N_{\text{inf}}, \alpha^N_{\text{sup}}) \subseteq (0,1)$. It follows that $\alpha^N(\xi^N)$ admits an inverse function defined over $(\alpha^N_{\text{inf}}, \alpha^N_{\text{sup}})$. Let $K^{W*} \in (0, \overline{K}^W)$ be such that $\Phi_{K^{W*}} \in (\alpha^N_{\text{inf}}, \alpha^N_{\text{sup}})$. We then derive that K^{W*} is a steady state if and only if $\Gamma^N = \Gamma^N(K^{W*})$ with $\Gamma^N(K^{W*})$ defined by

$$\Gamma^N(K^{W*}) = R^W\left\{\frac{(\delta^N)^\gamma\left[(1-s^N)a^N\left(1-K^{W*}-K^{N*}\right)-s^N\eta(1-b^N)(K^{W*}+K^{N*})\right]}{(K^{W*}+K^{N*})\left[s^N\eta(1-b^N)+(1-s^N)a^N\right]}\right\}^{\frac{1}{\gamma-1}}. \tag{A8}$$

□

A5 Proof of Proposition 4

The North and South produce both goods if the capital intensity in the consumption good sector a^i, $i \in \{N, S\}$, and in the investment good sector η are positive. Since we assumed that $\eta > 0$, the only task remaining is to look at a^i. Recall that, when taking into account labor mobility and capital mobility, the North can export/import capital such that K^{N*} corresponds to capital produced in the North and to export/import of capital from the South. Conversely, the North can export/import labor such that L^{N*} corresponds to the labor in the North and to export/import of labor from the South. Similar reasoning can be applied for the South. It follows, by direct inspection of (5), that a^N and a^S are positive if and only if $\eta < \overline{\eta} \equiv \min\left\{\frac{K^{N*}}{Y^{N*}}, \frac{K^{S*}}{Y^{S*}}\right\}$. □

A6 Proof of Proposition 5

Using Equations (10) and (20), $\alpha^S > \alpha^N$ if and only if $(\delta^N)^\gamma(\Gamma^N)^{1-\gamma} > (\delta^S)^\gamma(\Gamma^S)^{1-\gamma}$. By replacing Γ^N and Γ^S, defined in equation (A1), we obtain that $K^{N*} > K^{S*}$ and $\phi^N > \phi^S$, meaning that the North is the more patient country. Let us express K^{S*} and Y^{S*} in terms of K^{N*} and Y^{N*} so that

$$K^{N*} = K^{S*} + dK^S, \quad Y^{N*} = Y^{S*} + dY^S, \quad dK^S, dY^S > 0. \tag{A9}$$

From Definition 2, the world market condition for investment good at the NSS is

$$K^{N*} + K^{S*} = Y^{N*} + Y^{S*}.$$

Using Equation (A9), we get the relationships

$$2K^{N*} - dK^N = 2Y^{N*} - dY^N, \quad 2K^{S*} + dK^S = 2Y^{S*} + dY^S. \tag{A10}$$

From the two full-employment conditions we obtain $dY^{i*}/dK^{i*} = b^i$, where b^i is defined in equation (5). Then equation (A10) can be expressed as

$$K^{N*} - Y^{N*} = -\frac{1-b^N}{2b^N}dK^N, \quad K^{S*} - Y^{S*} = \frac{1-b^S}{2b^S}dK^S.$$

Let $\eta \in (0,1)$. From equation (5) evaluated at the NSS we derive $a^i > 0$ and $b^i < 1$. This implies that $K^{S*} \gtreqless Y^{S*}$ and $K^{N*} \lesseqgtr Y^{N*}$ if and only if $b^i \gtreqless 0$. It follows that, if $b^i \in (0,1)$, the North exports the capital-intensive investment good and the South exports the labor-intensive consumption good, and if $b^i < 0$ the North exports the capital-intensive consumption good and the South exports the labor-intensive investment good. □

A7 Local stability in the trade regime

A7.1 Characteristic polynomial in trade regime

In order to simplify the exposition, we denote

$$\tau\left(K^W, Y^W, 2\right) = T\left(K^N, Y^N, L^N\right) + T\left(K^S, Y^S, L^S\right);$$

$$\tau_{jh} = \tau_{jh}\left(K^W, Y^W\right), \quad j \in \{1,2,3\},\ h \in \{1,2\},$$

where $\tau_{11} = \partial\tau^2/\partial^2 K^W$, $\tau_{12} = \tau_{21} = \partial\tau^2/\partial K^W \partial Y^W$, $\tau_{22} = \partial\tau^2/\partial^2 Y^W$, $\tau_{31} = \partial\tau^2/\partial K^W \partial L^W$ and $\tau_{32} = \partial\tau^2/\partial L^W \partial Y^W$. Furthermore, we denote

$$\mid H^i \mid = \mid H^i\left(K_t^i, Y_t^i\right) \mid, \quad i \in \{N, S\}.$$

In order to obtain a tractable formulation of the characteristic polynomial, we have to define the second partial derivatives of the world social production function $\tau(K_t^W, Y_t^W, 2)$. From (Nishimura and Yano 1993, lemma 7), the second partial derivatives of the social production function are given by the following lemma.

Lemma A4 *In a free-trade equilibrium, the second partial derivatives of* $\tau\left(K_t^W, Y_t^W, 2\right)$ *satisfy*

$$\tau_{11} = \frac{1}{\Xi}\left[T_{11}^N \mid H^S \mid + T_{11}^S \mid H^N \mid\right], \quad \tau_{12} = \frac{1}{\Xi}\left[T_{12}^N \mid H^S \mid + T_{12}^S \mid H^N \mid\right],$$

$$\tau_{22} = \frac{1}{\Xi}\left[T_{22}^N \mid H^S \mid + T_{22}^S \mid H^N \mid\right],$$

$$\tau_{31} = \frac{1}{\Xi}\left[T_{31}^N \mid H^S \mid + T_{31}^S \mid H^N \mid\right], \quad \tau_{32} = \frac{1}{\Xi}\left[T_{32}^N \mid H^S \mid + T_{32}^S \mid H^N \mid\right],$$

where

$$\mid H^i \mid = T_{11}^i T_{22}^i - \left(T_{21}^i\right)^2 > 0,$$

$$\Xi = \left[T_{11}^N + T_{11}^S\right] \times \left[T_{22}^N + T_{22}^S\right] - \left[T_{12}^N + T_{12}^S\right]^2 > 0.$$

Letting

$$\Phi^N = \frac{1-\alpha^N}{2-\alpha^N-\alpha^S}, \quad \Phi^S = \frac{1-\alpha^S}{2-\alpha^N-\alpha^S},$$

consider the derivatives of α^i with respect to (R^W/Γ^i) given in Equation (A2) and $\varepsilon_{rk}^W = -\tau_{11}K^{W*}/\tau_1$. Totally differentiating Equation (21) evaluated at the NSS gives the characteristic $\mathscr{P}^W(\lambda^W) = (\lambda^W)^2 - \lambda^W \mathscr{T}^W(\gamma) + \mathscr{D}^W(\gamma)$ where the trace $\mathscr{T}^W(\gamma)$ and the determinant $\mathscr{D}^W(\gamma)$ are given by

$$\mathscr{T}^W(\gamma) = \frac{1 + \varepsilon_{rk}^W \left\{R^W\left[\frac{\tau_{32}\left(2-\alpha^N-\alpha^S\right)}{\tau_{11}K^{W*}} + \frac{\tau_{22}}{\tau_{11}}\right] + \left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1)\left[1 + \frac{\tau_{22}R^W}{\tau_{11}}\right]\right\}}{\left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1)\left(-\frac{\tau_{21}\varepsilon_{rk}^W}{\tau_{11}}\right)}, \tag{A11}$$

$$\mathscr{D}^W(\gamma) = \frac{R^W\left[1 + \left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1) + \frac{\tau_{31}\left(2-\alpha^N-\alpha^S\right)}{\tau_{21}K^{W*}}\right]}{\left(\alpha^N\Phi^N + \alpha^S\Phi^S\right)(\gamma-1)}, \tag{A12}$$

and where the second partial derivatives of $\tau(K^W, Y^W, 2)$ are defined in Lemma A4.

A7.2 Proof of Proposition 6

The result of this proposition is obtained using the geometric method developed in Grandmont *et al.* (1998) which allows to determine the occurrence of local indeterminacy in terms of a unique parameter. Under Assumption 4, the NSS K^{W*}, α^N, α^S, s^N, s^S, and ε_{rk} remain constant as γ varies continuously within $(1, +\infty)$. We can thus analyze the variation of the trace $\mathscr{T}^W(\gamma)$ and the determinant $\mathscr{D}^W(\gamma)$ in the $(\mathscr{T}^W(\gamma), \mathscr{D}^W(\gamma))$ plane. The relationship between the trace and the determinant is given by a half-line $\Delta^W\left(\mathscr{T}^W\right)$ which is characterized by considering its extremities. The starting point is the pair $(\lim_{\gamma\to+\infty}\mathscr{T}^W \equiv \mathscr{T}_\infty^W, \lim_{\gamma\to+\infty}\mathscr{D}^W \equiv \mathscr{D}_\infty^W)$, while the end point is the pair $(\lim_{\gamma\to1}\mathscr{T}^W \equiv \mathscr{T}_1^W, \lim_{\gamma\to1}\mathscr{D}^W \equiv \mathscr{D}_1^W)$. Solving $\mathscr{T}^W$ and $\mathscr{D}^W$ with respect to $(\alpha^N\Phi^N + \alpha^S\Phi^S)(\gamma-1)$ yields the linear relationship

$$\mathscr{D}^W = \Delta\left(\mathscr{T}^W\right) = \mathscr{S}^W\mathscr{T}^W + \mathscr{D}_\infty^W - \mathscr{S}^W\mathscr{T}_\infty^W,$$

where the slope $\mathscr{S}^W$, $\mathscr{D}_\infty^W$, and $\mathscr{T}_\infty^W$ are given by

$$\mathscr{S}^W = -\frac{R^W\frac{\tau_{21}K^W}{\tau_{11}}\varepsilon_{rk}^W\left[1 + \left(2-\alpha^N-\alpha^S\right)\frac{\tau_{31}}{\tau_{21}K^W}\right]}{1 + \varepsilon_{rk}^W R^W\left[\frac{\tau_{32}\left(2-\alpha^N-\alpha^S\right)}{\tau_{11}K^W} + \frac{\tau_{22}}{\tau_{11}}\right]}, \quad \mathscr{D}_\infty^W = R^W, \quad \mathscr{T}_\infty^W = -\frac{1+\tau_{22}^W R^W/\tau_{11}^W}{\tau_{21}^W/\tau_{11}^W}. \tag{A13}$$

Assume that $b^N < 0$, $b^S < 0$, $\nu \leq 1$, $\alpha^N \in (\underline{\alpha}^N, 1/2)$, $\alpha^S \in (\underline{\alpha}^S, 1/2)$ $s^N \in (1/3, 1/2)$, and $s^S \in (1/3, 1/2)$. Let us first consider the starting point $(\mathscr{T}_\infty^W, \mathscr{D}_\infty^W)$. Under dynamic efficiency we get from (A13) that $\mathscr{D}_\infty^W > 1$. To establish the precise location of the starting point $(\mathscr{T}_\infty^W, \mathscr{D}_\infty^W)$ we need to determine the sign of $\mathscr{T}_\infty^W$, $\mathscr{P}_\infty^W(1) = 1 - \mathscr{T}_\infty^W + \mathscr{D}_\infty^W$, and $\mathscr{P}_\infty^W(-1) = 1 + \mathscr{T}_\infty^W + \mathscr{D}_\infty^W$. Using Lemmas A3 and A4, we get

$$\mathscr{T}_\infty^W = \frac{\left[\frac{(1-\nu)(1-b^N)(1+R^W)s^N+(1+\rho)(1-s^N)(1+R^W(b^N)^2)}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)}\right] T_{11}^N \mid H^S \mid + \left[\frac{(1-\nu)(1-b^S)(1+R^W)s^S+(1+\rho)(1-s^S)(1+R^W(b^S)^2)}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)}\right] T^S \mid H^N \mid}{\left[\frac{(1-s^N)\left[(1-\nu)(1-\alpha^N)(1-b^N)+(1+\rho)b^N\right]}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)}\right] T_{11}^N \mid H^S \mid + \left[\frac{(1-s^S)\left[(1-\nu)(1-\alpha^S)(1-b^S)+(1+\rho)b^S\right]}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)}\right] T_{11}^S \mid H^N \mid}.$$

When $b^N < \overline{b}^N$ and $b^S < \overline{b}^S$, with $\overline{b}^N = -(1-\alpha^N)(1-\nu)/(\nu + \rho + (1-\nu)\alpha^N)$ and $\overline{b}^S = -(1-\alpha^S)(1-\nu)/(\nu + \rho + (1-\nu)\alpha^S)$, we get $\mathscr{T}_\infty^W < 0$. Since $\mathscr{T}_\infty^W < 0$ when $b^N < \overline{b}^N$ and $b^S < \overline{b}^S$, it follows that $\mathscr{P}_\infty^W(1) > 0$. Applying Lemmas A3 and A4 to (A13), we then obtain

$$\mathscr{P}_\infty^W(-1) = \frac{\frac{2(1-\nu)(1-b^N)\left(\frac{1+R^W}{R^W}\right)s^N+(1+\rho)(1-s^N)(1+bR^W)(1+b^N)}{(1-s^N)\left[(1-\nu)(1-\alpha^N)+b^N(\rho+\alpha^N+\nu(1-\alpha^N))\right]} T_{11}^N \mid H^S \mid + \frac{2(1-\nu)(1-b^S)\left(\frac{1+R^W}{R^W}\right)s^S+(1+\rho)(1-s^S)(1+b^S R^W)(1+b^S)}{(1-s^S)\left[(1-\nu)(1-\alpha^S)+b^S(\rho+\alpha^S+\nu(1-\alpha^S))\right]} T_{11}^S \mid H^N \mid}{\frac{(1-s^N)\left[(1-\nu)(1-b^N)(1-\alpha^N)+(1+\rho)b^N\right]}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)} T_{11}^N \mid H^S \mid + \frac{(1-s^S)\left[(1-\nu)(1-b^S)(1-\alpha^S)+(1+\rho)b^S\right]}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)} \tau_{11}^S \mid H^N \mid}.$$

Under $b^N < \overline{b}^N$ and $b^S < \overline{b}^S$, we derive that $\mathscr{P}_\infty^W(-1) < 0$ since $\overline{b}^N < -1/R^W < -1$ and $\overline{b}^S < -1/R^W < -1$. As a result, the starting point is in the left area outside the triangle $\mathscr{ABC}$. Let us now consider the end point $(\mathscr{T}_1^W, \mathscr{D}_1^W)$. To determine the precise location of the end point $(\mathscr{T}_1^W, \mathscr{D}_1^W)$, it is sufficient to determine whether $\Delta^W(\mathscr{T}^W)$ is pointing upward or downward. $\Delta^W(\mathscr{T}^W)$ is pointing upward or downward depending on the sign of $\mathscr{D}^{W'}(\gamma)$. It follows from Lemmas A3 and A4 and (A12) that $\mathscr{D}^{W'}(\gamma)$ is given by

$$\mathscr{D}^{W'}(\gamma) = -\frac{R^W\left[\frac{(1-s^N)(1+\rho)(1-\alpha^N+\alpha^N b^N)}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)} T_{11}^N |H^S| + \frac{(1-s^S)(1+\rho)(1-\alpha^S+\alpha^S b^S)}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)} T_{11}^S |H^N|\right]}{(\alpha^N\Phi^N+\alpha^S\Phi^S)(\gamma-1)^2\left[\frac{(1-s^N)\left[(1-\nu)(1-b^N)(1-\alpha^N)+(1+\rho)b^N\right]}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)} T_{11}^N |H^S| + \frac{(1-s^S)\left[(1-\nu)(1-b^S)(1-\alpha^S)+(1+\rho)b^S\right]}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)} T_{11}^S |H^N|\right]}.$$

Under $b^N \in (\underline{b}^N, \overline{b}^N)$ and $b^S \in (\underline{b}^S, \overline{b}^S)$, with $\underline{b}^N = -(1-\alpha^N)/\alpha^N$ and $\underline{b}^S = -(1-\alpha^S)/\alpha^S$, we get $\mathscr{D}^{W'}(\gamma) > 0$ and thus $\Delta^W(\mathscr{T}^W)$ is pointing downward. Therefore when $b^N \in (\underline{b}^N, \overline{b}^N)$ and $b^S \in (\underline{b}^S, \overline{b}^S)$, the only possibility for obtaining local indeterminacy is that $\Delta^W(\mathscr{T}^W)$ crosses the interior of the segment $\mathscr{AC}$. This requires that $\mathscr{T}^W > 0$ when $\mathscr{D}^W = -1$. From (A12), by solving $\mathscr{D}^W = -1$ and using Lemmas A3 and A4, we have

$$(\alpha^N\Phi^N + \alpha^S\Phi^S)(\gamma - 1) = -\frac{R^W\left[\frac{(1-s^N)(1+\rho)(1-\alpha^N+\alpha^N b^N)}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)} T_{11}^N |H^S| + \frac{(1-s^S)(1+\rho)(1-\alpha^S+\alpha^S b^S)}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)} T_{11}^S |H^N|\right]}{(1+R^W)\left[\frac{(1-s^N)\left[(1-\nu)(1-b^N)(1-\alpha^N)+(1+\rho)b^N\right]}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)} T_{11}^N |H^S| + \frac{(1-s^S)\left[(1-\nu)(1-b^S)(1-\alpha^S)+(1+\rho)b^S\right]}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)} T_{11}^S |H^N|\right]},$$

which holds if $b^N \in (\underline{b}^N, \overline{b}^N)$ and $b^S \in (\underline{b}^S, \overline{b}^S)$. Substituting $\mathscr{D}^W = -1$ in $\mathscr{T}^W$, defined in (A11), we get that $\mathscr{T}^W > 0$ when $\mathscr{D}^W = -1$ if $\alpha^N \in (\underline{\alpha}^N, 1/2)$, $\alpha^S \in (\underline{\alpha}^S, 1/2)$, $s^N \in (1/3, 1/2)$, $s^S \in (1/3, 1/2)$, $b^N \in (\underline{b}^N, \overline{b}^N)$, $b^S \in (\underline{b}^S, \overline{b}^S)$, and $\varepsilon_{rk}^W > \underline{\varepsilon}_{rk}^W$. The bound $\underline{\varepsilon}_{rk}^W$ is defined as

$$\underline{\varepsilon}_{rk}^W = \frac{(1+R^W)\tau_{11}}{\frac{T_{11}^N}{s^N}\mathscr{X}_N \mid H^S \mid + \frac{T_{11}^S}{s^S}\mathscr{X}_S \mid H^N \mid}, \tag{A14}$$

with

$$\mathscr{X}_N = (1+\rho)\left[R^W(1-s^N)(1-b^N) + b^N s^N(1+R^W)\right] + (\nu-1)s^N\left(\frac{1+R^W}{R^W}\right)$$
$$+ \frac{\tau_{31}(2-\alpha^N-\alpha^S)R^W\left[(\nu-1)(1-b^N)(1-R^W)s^N - (1+\rho)(1-s^N)(1-R^W\left(b^N\right))\right]}{\tau_{21}K^{W*}(1-b^N)},$$

$$\mathscr{X}_S = (1+\rho)\left[R^W(1-s^S)(1-b^S) + b^S s^S(1+R^W)\right] + (\nu-1)s^S\left(\frac{1+R^W}{R^W}\right)$$
$$+ \frac{\tau_{31}(2-\alpha^N-\alpha^S)R^W\left[(\nu-1)(1-b^S)(1-R^W)s^S - (1+\rho)(1-s^S)(1-R^W\left(b^S\right))\right]}{\tau_{21}K^{W*}(1-b^S)},$$

$$\frac{\tau_{31}(2-\alpha^N-\alpha^S)}{\tau_{21}K^{W*}} = \frac{\frac{(1-s^N)(1-\alpha^N)(\nu+\rho)}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)}T_{11}^N|H^S|+\frac{(1-s^S)(1-\alpha^S)(\nu+\rho)}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)}T_{11}^S|H^N|}{\frac{(1-s^N)\left[(1-\nu)(1-\alpha^N)(1-b^N)+(1+\rho)b^N\right]}{(1-\nu)(1-b^N)s^N+(1+\rho)(1-s^N)}T_{11}^N|H^S|+\frac{(1-s^S)\left[(1-\nu)(1-\alpha^S)(1-b^S)+(1+\rho)b^S\right]}{(1-\nu)(1-b^S)s^S+(1+\rho)(1-s^S)}T_{11}^S|H^N|}.$$

The bifurcation values $\gamma^{W,\mathcal{T}}$ and $\gamma^{W,\mathcal{F}}$ are respectively defined as the solutions of $\mathscr{P}^W(1) = 1-\mathscr{T}^W+\mathscr{D}^W = 0$ and $\mathscr{P}^W(-1) = 1+\mathscr{T}^W+\mathscr{D}^W = 0$, and given by

$$\gamma^{W,\mathcal{T}} = 1 - \frac{1+\frac{\varepsilon_{rk}^W R^W\left\{\left[(\nu-1)\left(1-\left(\frac{1}{R^W}\right)^2\right)+(1+\rho)(1-b^N)\left(\frac{1}{R^W}+\frac{T_3 b^N}{T_1 a^N}\right)\right]T_{11}^N|H^S|+\left[(\nu-1)\left(1-\left(\frac{1}{R^W}\right)^2\right)+(1+\rho)(1-b^S)\left(\frac{1}{R^W}+\frac{T_3 b^S}{T_1 a^S}\right)\right]T_{11}^S|H^N|\right\}}{\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^S}\right]T_{11}^S|H^N|+\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^N}\right]T_{11}^N|H^S|}}{\frac{(\alpha^N\Phi^N+\alpha^S\Phi^S)\varepsilon_{rk}^W\left\{1+\left[(\nu-1)\left(R^W-1-\frac{1}{R^W}\right)+(1+\rho)(1+R^W-b^N R^W)\frac{T_3 b^N}{T_1 a^N}\right]T_{11}^N|H^S|+\left[(\nu-1)\left(R^W-1-\frac{1}{R^W}\right)+(1+\rho)(1+R^W-b^S R^W)\frac{T_3 b^S}{T_1 a^S}\right]T_{11}^S|H^N|\right\}}{\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^S}\right]T_{11}^S|H^N|+\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^N}\right]T_{11}^N|H^S|}}, \tag{A15}$$

$$\gamma^{W,\mathcal{F}} = 1 + \frac{-1+\frac{\varepsilon_{rk}^W R^W\left\{\left[(\nu-1)\left(1-\left(\frac{1}{R^W}\right)^2\right)+(1+\rho)(1-b^N)\left(\frac{1}{R^W}+\frac{T_3 b^N}{T_1 a^N}\right)\right]T_{11}^N|H^S|+\left[(\nu-1)\left(1-\left(\frac{1}{R^W}\right)^2\right)+(1+\rho)(1-b^S)\left(\frac{1}{R^W}+\frac{T_3 b^S}{T_1 a^S}\right)\right]T_{11}^S|H^N|\right\}}{\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^S}\right]T_{11}^S|H^N|+\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^N}\right]T_{11}^N|H^S|}}{\frac{(\alpha^N\Phi^N+\alpha^S\Phi^S)\varepsilon_{rk}^W\left\{1+\left[(\nu-1)\left(R^W-\frac{1}{R^W}-1\right)+(1+\rho)(1+b^N R^W)\frac{T_3 b^N}{T_1 a^N}\right]T_{11}^N|H^S|+\left[(\nu-1)\left(R^W-\frac{1}{R^W}-1\right)+(1+\rho)(1+b^S R^W)\frac{T_3 b^S}{T_1 a^S}\right]T_{11}^S|H^N|\right\}}{\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^S}\right]T_{11}^S|H^N|+\left[\nu-1-(1+\rho)\frac{T_3}{T_1 a^N}\right]T_{11}^N|H^S|}}. \tag{A16}$$

The results follow. □

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