

# Numerical Methods for New Keynesian ZLB Models: Part I

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# What is covered

- Equilibrium with a simple Taylor rule
- Tauchen's (1986) method for approximating AR(1) process
- Equilibrium under the optimal discretionary policy (Adam and Billi, 2007)

# Two-state shock process

- Exogenous shocks take only  $N_s = 2$  values,  $s_t \in \{s_H, s_L\}$ . The stochastic process follows a Markov chain with the transition matrix:

$$\begin{bmatrix} 1 - p_H & p_H \\ 1 - p_L & p_L \end{bmatrix}.$$

- $p_H$  is the frequency of crisis and  $p_L$  is the duration of crisis.

# Equilibrium with Taylor rule

- Equilibrium conditions are

$$y_t = E_t y_{t+1} - (i_t - E_t \pi_{t+1} - s_t),$$

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1},$$

$$i_t^* = r^* + \phi_\pi E_t \pi_{t+1},$$

and the zero lower bound

$$i_t = \max \{0, i_t^*\}.$$

# Two routes

- The solution has a form of (we omit time subscripts for the policy function)

$$y = y(s), \quad \pi = \pi(s), \quad i = i(s).$$

- We are solving the same model with two different methods.
  - Analytical solution.
  - Numerical solution, using the policy function iteration (time iteration).

- We know that the functions have only two values, i.e.,

$$y = \begin{cases} y_H, \\ y_L, \end{cases} \quad \pi = \begin{cases} \pi_H, \\ \pi_L, \end{cases} \quad i = \begin{cases} i_H, \\ i_L. \end{cases}$$

- We assume that  $i_H > 0$  and  $i_L = 0$ . Then we have

$$\begin{aligned}y_H &= (1 - p_H)y_H + p_H y_L - (i_H - [(1 - p_H)\pi_H + p_H \pi_L] - s_H), \\ \pi_H &= \kappa y_H + \beta [(1 - p_H)\pi_H + p_H \pi_L], \\ i_H &= r^* + \phi_\pi [(1 - p_H)\pi_H + p_H \pi_L], \\ y_L &= (1 - p_L)y_H + p_L y_L - (0 - [(1 - p_L)\pi_H + p_L \pi_L] - s_L), \\ \pi_L &= \kappa y_L + \beta [(1 - p_L)\pi_H + p_L \pi_L], \\ i_L &= 0.\end{aligned}$$

- There are 6 equations and 6 unknowns, so we can solve for the unknowns.

# Policy function iteration

- We can solve the same model with the policy function iteration method.
- The method takes four steps:
  - 1 Make an initial guess for the policy function
  - 2 Solve for the endogenous variables at each grid
  - 3 Update the policy function
  - 4 Repeat 2-3 until convergence



# Policy function iteration: Initial guess

- A guess of the policy functions

$$y = y^{(0)}(s), \quad \pi = \pi^{(0)}(s), \quad i = i^{(0)}(s).$$

- Consider the general case of  $N_s \geq 2$ . We know the values of the functions only at each *grid point*, i.e.,

$$y^{(0)}(s) = [y_1, y_2, \dots, y_{N_s}]',$$

$$\pi^{(0)}(s) = [\pi_1, \pi_2, \dots, \pi_{N_s}]',$$

$$i^{(0)}(s) = [i_1, i_2, \dots, i_{N_s}]'.$$

# Policy function iteration: Solving

- At each grid point  $k = 1, \dots, N_s$ , we solve

$$\begin{aligned}y_k &= y^e - (i_k - \pi^e - s_k), \\ \pi_k &= \kappa y_k + \beta \pi^e, \\ i_k &= \max \{0, r^* + \phi_\pi \pi^e\},\end{aligned}$$

for  $(y_k, \pi_k, i_k)$ , where

$$\begin{aligned}y^e &= \sum_{l=1}^{N_s} p(k, l) y^{(0)}(s_l), \\ \pi^e &= \sum_{l=1}^{N_s} p(k, l) \pi^{(0)}(s_l).\end{aligned}$$

and  $p(k, l)$  is the  $(k, l)$  element of the transition matrix.

# Policy function iteration: Updating

- Once this is done for all the grid points, we update

$$y^{(1)}(s) = [y_1, y_2, \dots, y_{N_s}]',$$

$$\pi^{(1)}(s) = [\pi_1, \pi_2, \dots, \pi_{N_s}]',$$

$$i^{(1)}(s) = [i_1, i_2, \dots, i_{N_s}]'.$$

- We repeat the procedure until the policy functions converge, i.e.,  
 $\|x^{(j)}(s) - x^{(j-1)}(s)\| < \epsilon$  for  $x \in \{y, \pi, i\}$ .

# Tauchen's method

- Tauchen (1986) developed a method for approximating AR(1) stochastic process by using Markov chain.
- We have the following AR(1) stochastic process

$$x' = c + \rho x + \varepsilon', \varepsilon' \sim N(0, \sigma_\varepsilon^2).$$

- We want to approximate the stochastic process by a Markov chain  $x_k \in \{x_1, x_2, \dots, x_N\}$ .

# Tauchen's method: Grid points

- We set the grid points for  $x$ :

$$x_k \in \mathcal{I} = \{x_1, x_2, \dots, x_N\},$$

where  $k$  is an index for the set of grid points  $\mathcal{I}$ .

- For example, we set  $x_1 = \frac{-m\sigma_\varepsilon}{\sqrt{1-\rho^2}}$ ,  $x_N = \frac{m\sigma_\varepsilon}{\sqrt{1-\rho^2}}$  and  $x_k = x_{k-1} + w$  for  $k = 2, \dots, N-1$ , where  $w = \frac{x_N - x_1}{N-1}$ .

# Tauchen's method: Transition matrix

- Given the grid points for  $x$ . What is the probability of moving from one point  $x_k$  to another  $x_l$ ?
- We know

$$\varepsilon' = x' - c - \rho x_k \sim N(0, \sigma_\varepsilon^2).$$

# Tauchen's method: Transition matrix, cont'd

- Then, the probability of  $x' \in [x_l - \frac{w}{2}, x_l + \frac{w}{2}]$  can be used as approximation. That is,

$$p_{kl} = \Phi\left(x_l + \frac{w}{2} - c - \rho x_k\right) - \Phi\left(x_l - \frac{w}{2} - c - \rho x_k\right),$$

where  $\Phi(\cdot)$  is the cdf of  $N(0, \sigma_\varepsilon^2)$ . Be careful at the boundary points.

- Once this is done for all  $k, l$ , we have the transition matrix

$$P = \begin{bmatrix} p_{11} & \cdots & \cdots & p_{1N} \\ p_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{N1} & \cdots & \cdots & p_{NN} \end{bmatrix}.$$

# Optimal Discretionary Policy

- The policymaker chooses  $\{\pi_t, y_t, i_t\}$  so as to maximize

$$V_0 \equiv -E_0 \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \lambda y_t^2)$$

subject to

$$\begin{aligned} y_t &= E_t y_{t+1} - (i_t - E_t \pi_{t+1}) + g_t, \\ \pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t, \\ i_t &\geq 0, \end{aligned}$$

taking  $E_t y_{t+1}$  and  $E_t \pi_{t+1}$  as given.



- Exogenous shocks are given by

$$g_t = (1 - \rho_g)g + \rho_g g_{t-1} + \varepsilon_{g,t},$$

$$u_t = \rho_u u_{t-1} + \varepsilon_{u,t},$$

where  $\varepsilon_{g,t} \sim N(0, \sigma_g^2)$  and  $\varepsilon_{u,t} \sim N(0, \sigma_u^2)$ .

- Note that  $g = r^*$ .

- We know that Markov-perfect equilibrium has only natural state variables.
- Lagrangean is

$$\begin{aligned}\mathcal{L} \equiv E_0 \sum \beta^t & (\pi_t^2 + \lambda y_t^2) + 2\phi_{PC,t}(-\pi_t + \kappa y_t + \beta E_t \pi_{t+1} + u_t) \\ & + 2\phi_{EE,t}(-y_t - i_t + E_t y_{t+1} + E_t \pi_{t+1} + g_t) + 2\phi_{ZLB,t} i_t.\end{aligned}$$

- First-order necessary conditions are

$$\partial \pi_t : \pi_t - \phi_{PC,t} = 0,$$

$$\partial y_t : \lambda y_t + \kappa \phi_{PC,t} - \phi_{EE,t} = 0,$$

$$\partial i_t : -\phi_{EE,t} + \phi_{ZLB,t} = 0.$$

# Complementary slackness

- Complementary slackness condition:

$$\phi_{ZLB,t} > 0 \perp i_t > 0.$$

- When  $i_t > 0$ ,  $\phi_{ZLB,t} = 0$ . Equilibrium conditions are

$$i_t = -y_t + E_t y_{t+1} + E_t \pi_{t+1} + g_t,$$

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t,$$

$$0 = \lambda y_t + \kappa \pi_t.$$

- When  $i_t = 0$ ,  $\phi_{ZLB,t} > 0$ . Equilibrium conditions are

$$0 = -y_t + E_t y_{t+1} + E_t \pi_{t+1} + g_t,$$

$$\pi_t = \kappa y_t + \beta E_t \pi_{t+1} + u_t,$$

$$\phi_{ZLB,t} = \lambda y_t + \kappa \pi_t.$$

# Solving the model

- The solution has a form of

$$y = y(g, u), \quad \pi = \pi(g, u), \quad i = i(g, u).$$

- Now consider the case in which there are only two-state  $g$  shocks. We know that the functions have only two values, i.e.,

$$y = \begin{cases} y_H, \\ y_L, \end{cases} \quad \pi = \begin{cases} \pi_H, \\ \pi_L, \end{cases} \quad i = \begin{cases} i_H, \\ i_L. \end{cases}$$

- Again, we will look at analytical solution and numerical solution.

- We assume that  $i_H > 0$  and  $i_L = 0$ . Then we have

$$y_H = (1 - p_H)y_H + p_H y_L - (i_H - [(1 - p_H)\pi_H + p_H \pi_L]) + g_H,$$

$$\pi_H = \kappa y_H + \beta [(1 - p_H)\pi_H + p_H \pi_L],$$

$$0 = \lambda y_H + \kappa \pi_H,$$

$$y_L = (1 - p_L)y_H + p_L y_L - (0 - [(1 - p_L)\pi_H + p_L \pi_L]) + g_L,$$

$$\pi_L = \kappa y_L + \beta [(1 - p_L)\pi_H + p_L \pi_L],$$

$$\phi_L = \lambda y_L + \kappa \pi_L.$$

- There are 6 equations and 6 unknowns, so we can solve for the unknowns.

# Joint shock process

- Let's get back to the general case. The shock processes are approximated by Markov chains. That is,

$$g_m \in \{g_1, g_2, \dots, g_{N_g}\},$$

$$u_n \in \{u_1, u_2, \dots, u_{N_u}\},$$

and

$$P^g = \begin{bmatrix} p_{11}^g & \cdots & \cdots & p_{1N_g}^g \\ p_{21}^g & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{N_g1}^g & \cdots & \cdots & p_{N_gN_g}^g \end{bmatrix}, \quad P^u = \begin{bmatrix} p_{11}^u & \cdots & \cdots & p_{1N_g}^u \\ p_{21}^u & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{N_g1}^u & \cdots & \cdots & p_{N_gN_g}^u \end{bmatrix}.$$

# Joint shock process, cont'd

- 2D grid coordinate with  $g$  and  $u$  represents the joint shock process.
  - For example, when  $N_g = N_u = 2$

$$s_1 = (g_1, u_1),$$

$$s_2 = (g_1, u_2),$$

$$s_3 = (g_2, u_1),$$

$$s_4 = (g_2, u_2).$$

- Note that each index points to a pair of shocks,  $s_k = (g_{m(k)}, u_{n(k)})$ .
- A kronecker product of the transition matrices  $P^s = P^g \otimes P^u$  is the transition matrix of the joint shock process.

# Policy function iteration: Initial guess

- A guess of the policy functions

$$y = y^{(0)}(s), \quad \pi = \pi^{(0)}(s), \quad i = i^{(0)}(s).$$

- We know the values of the functions only at each *grid point*, e.g.,

$$y^{(0)}(s) = [y_1, y_2, \dots, y_{N_s}]',$$

$$\pi^{(0)}(s) = [\pi_1, \pi_2, \dots, \pi_{N_s}]',$$

$$i^{(0)}(s) = [i_1, i_2, \dots, i_{N_s}]'.$$



# Policy function iteration: Solving

- At each grid point  $k = 1, \dots, N$ , we solve

$$i_k = -y_k + y^e + \pi^e + g_{m(k)},$$

$$\pi_k = \kappa y_k + \beta \pi^e + u_{n(k)},$$

$$0 = \lambda y_k + \kappa \pi_k,$$

for  $(y_k, \pi_k, i_k)$ , where

$$y^e = \sum_{l=1}^{N_s} P^s(k, l) y^{(0)}(s_l),$$

$$\pi^e = \sum_{l=1}^{N_s} P^s(k, l) \pi^{(0)}(s_l).$$

# Policy function iteration: Solving, cont'd

- Check  $i_k \geq 0$ . If not, we solve instead

$$0 = -y_k + y^e + \pi^e + g_{m(k)},$$

$$\pi_k = \kappa y_k + \beta \pi^e + u_{n(k)},$$

for  $(y_k, \pi_k)$ , and set  $i_k = 0$ .

# Policy function iteration: Updating

- Once this is done for all the grid points, we update

$$y^{(1)}(s) = [y_1, y_2, \dots, y_{N_s}]',$$

$$\pi^{(1)}(s) = [\pi_1, \pi_2, \dots, \pi_{N_s}]',$$

$$i^{(1)}(s) = [i_1, i_2, \dots, i_{N_s}]'.$$

- We repeat the procedure until the policy functions converge, i.e.,  
 $\|x^{(j)}(s) - x^{(j-1)}(s)\| < \epsilon$  for  $x \in \{y, \pi, i\}$ .

Figure 4 in Adam and Billi (2007)

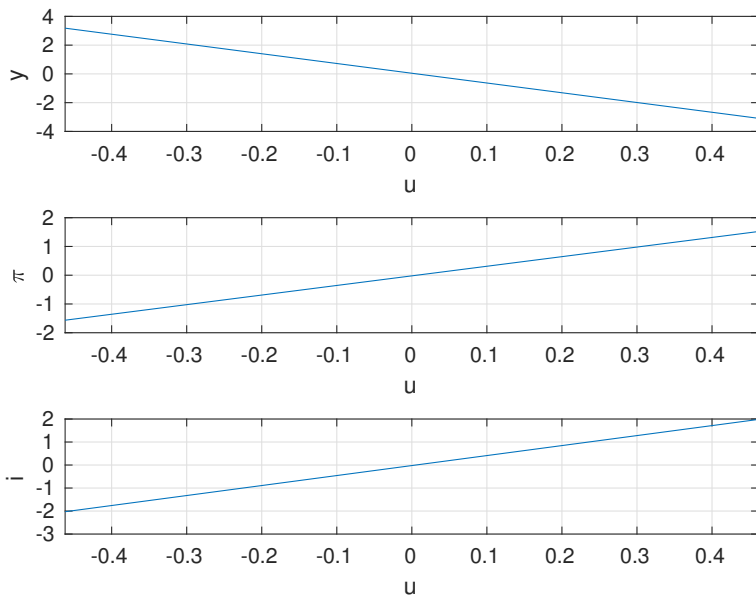


Figure 5 in Adam and Billi (2007)

