Solving nonlinear DSGE models within a second: Chebyshev PEA meets precomputation of integrals*

Takeki Sunakawa

May 30, 2018

Abstract

We compare two different but similar methods to solve nonlinear dynamic stochastic general equilibrium models more efficiently. By applying non-stochastic parameterized expectations algorithm (PEA), we can avoid costly nonlinear optimization in the standard time iteration method which can be time-consuming. There are two different ways to apply non-stochastic PEA. One is to fit polynomials to future variables and the other is to fit polynomials to current variables. With the latter approach, we can further avoid costly numerical integration by utilizing the technique of precomputation of integrals. The method can also be applied to models with occasionally binding constraints, such as New Keynesian models with the zero lower bound on nominal interest rates.

1 Introduction

Time iteration method is one of the numerical methods widely used to solve nonlinear dynamic stochastic general equilibrium (DSGE) models.[1] In this paper, we compare two different but similar methods to solve nonlinear DSGE models more efficiently. As is known in the literature, by applying non-stochastic parameterized expectations algorithm (PEA), we can avoid costly nonlinear optimization (i.e., root-finding) in the standard time iteration

*__Very Preliminary. Please do not circulate and/or quote the paper without the author's permission.__

[1]Time iteration method is a variant of projection methods with collocation. Collman (1990) proves the existence of the equilibrium as the fixed point of a functional equation in a stochastic neoclassical growth model with distortionary tax. Greenwood and Huffman (1995) extend it to several cases. Also see Richter, Throckmorton and Walker (2014) and Sargent and Stachurski (2018).

method (Christiano and Fisher, 2000; hereafter CF).[2] There are two different ways to apply non-stochastic PEA. One is to fit polynomials to future variables (Marcet, 1988), and the other is to fit polynomials to current variables (Wright and Williams, 1982a, b, 1984). Compared with the conventional method that requires solving a root-finding problem, the methods considered in this paper are more efficient in terms of computation time.

The contribution of this paper to the literature is to show that, with the latter approach of fitting polynomials to current variables, we can further avoid costly numerical integration by utilizing the technique of *precomputation of integrals* (Judd, Maliar, Maliar and Tsener, 2017) and reduce computational costs. The proposed method is surprisingly faster than the other methods considered in the paper, for example it can solve a medium-scale nonlinear New Keynesian model (Smets and Wouters, 2007) with the zero lower bound on nominal interest rates within a second.

PEA is first developed by Marcet (1988). Marcet uses a stochastic approach based on Monte Carlo simulations to solve for polynomials' coefficients which approximate the next period's expectation functions.[3] On the contrary, CF point out that PEA can be applied to non-stochastic grid points such as Chebyshev zeros or extrema. They use Chebyshev polynomials as the basis function to approximate the expectation functions and solve for the coefficients on the basis function by the projection method (Judd, 1992).[4] They call such a non-stochastic approach Chebyshev PEA. In more recent work, Gust, Herbst, Lopez-Salido, and Smith (2017) apply non-stochastic Chebyshev PEA to solve a nonlinear New Keynesian model with occasionally binding constraints. They fit polynomials which approximate the expectation functions to future variables.[5]

This paper is based on the previous studies, and proposes a new approach which solves DSGE models more efficiently. In the proposed method, we fit polynomials to current variables and precomputes integrals in the polynomials (Judd et al., 2017). Namely, this paper applies the technique of precomputation of integrals by Judd et al. (2017) to non-stochastic Chebyshev PEA by Christiano and Fisher (2000). This is a non-trivial task especially when we deal with occasionally binding constraints by utilizing an index function for binding constraints (such as in Gust et al., 2017; Aruoba, Cuba-Borda and Schorfheide, 2018), because the expectation function is now a piece-wise smooth function based on two smooth functions

[2] Another popular method to avoid nonlinear optimization is endogenous grid point method (Carrol, 2005; Barillas and Fernandez-Villaverde, 2007; Fella, 2014).

[3] Judd, Maliar and Maliar (2011) and Maliar and Maliar (2015) further develop this approach to make the algorithm more robust and efficient.

[4] CF show that the conventional PEA with stochastic simulations can be represented as a variant of weighted residual method. Den Haan (?) also mentions such a non-stochastic PEA in his lecture note.

[5] They also use Smolyak's method for sparse grid points to mitigate the curse of dimensionality.

with which we assume either the constraints always or never bind. When we assume the index function is orthogonal to the difference of the two smooth functions, the expectation function is expressed as a weighted average of the two smooth functions with the probability of binding constraints being the weight.

As we have already mentioned, there are two different approaches to apply PEA. One is to fit polynomials to future variables (Marcet, 1988), and the other is to fit polynomials to current variables (Wright and Williams, 1982a, b, 1984). What are different between these two approaches? To demonstrate the difference, we use a stochastic neoclassical growth model as an example and define an auxiliary function for each approach with regard to the right-hand side of the Euler equation:[6][7]

$$
\begin{aligned}
e(k,z) &\equiv \beta \int u_c\left(c'\right) f_k(k',z')p(z'|z)dz', \\
v(k,z) &\equiv \beta u_c(c) f_k(k,z).
\end{aligned}
$$

We need to interpolate these functions by polynomials to evaluate them off the grid points

$$
\begin{aligned}
\hat{e}(k,z;\boldsymbol{\theta}) &\approx e(k,z), \\
\hat{v}(k,z;\boldsymbol{\theta}) &\approx v(k,z).
\end{aligned}
$$

In the first approach, given the state variables (k,z), we evaluate the integral with the next period's variables $c' = \sigma(k',z')$ and $k' = f(k,z) - \sigma(k,z)$ at hand, where $\sigma(k,z) \approx u_c^{-1}(\hat{e}(k,z;\boldsymbol{\theta}))$ is the policy function for c. That is,

$$
\begin{aligned}
&\beta \int u_c\left(c'\right) f_k(k',z')p(z'|z)dz' \\
\approx &\beta \int \left\{ u_c\left(u_c^{-1}(\hat{e}(k',z';\boldsymbol{\theta}))\right) f_k(k',z') \right\} p(z'|z)dz',
\end{aligned}
$$

where $k' = f(k,z) - u_c^{-1}(\hat{e}(k,z;\boldsymbol{\theta}))$.[8] The function to be integrated is a composite function, and we use numerical integration (e.g., Gaussian quadrature) to evaluate the integral. This can be costly with a higher dimension of the state space, as the quadrature points are exponentially increasing in the dimension of the state space.

[6]Namely, $u_c(c) = \beta \int u_c\left(c'\right) f_k(k',z')p(z'|z)dz'$. See Section 2 for more details.

[7]CF define $v(k',z) = \beta \int u_c\left(c'\right) f_k(k',z')p(z'|z)dz'$.

[8]Note that we can obtain $e(k,z)$ and thus $c = u_c^{-1}(e(k,z))$ without nonlinear optimization, as shown in Maliar and Maliar (2015) and Gust et al. (2017).

On the contrary, in the second approach, given (k, z), we evaluate the integral

$$\begin{aligned}&\beta \int u_c(c') f_k(k', z') p(z'|z) dz' \\ &\approx \int \hat{v}(k', z'; \boldsymbol{\theta}) p(z'|z) dz',\end{aligned}$$

where $k' = f(k, z) - \sigma(k, z)$.[9] The function to be integrated is a polynomial function (e.g., Chebyshev polynomial), and we can apply the precomputation tequninue by Judd et al. (2017). As shown in Judd et al. (2017), the precomputation technique can solve the model as in the deterministic case by computing the integral just once. The precomputation technique reduces computational costs associated with numerical integration and increases the accuracy of solutions.

Note that we also have

$$e(k, z) = \int v(f(k, z) - \sigma(k, z), z') p(z'|z) dz'$$

As shown in Christiano and Fisher (2000), v is likely to be smoother than e.

2 Stochastic neoclassical growth model

2.1 Setup

The social planner maximizes the expected life-time utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to the resource constraint

$$c_t + k_{t+1} \leq f(k_t, z_t) \tag{2}$$

where c_t is consumption, k_t is capital, and z_t is technology. $u(c)$ and $f(k, z)$ are utility and production functions satisfying standard assumptions. β is a discount

[9] Note that, with a successive approximation with regard to $\sigma(k, z)$, we can obtain $\int \hat{v}(k', z'; \boldsymbol{\theta}) p(z'|z) dz'$ and thus $c = u_c^{-1}(\int \hat{v}(k', z'; \boldsymbol{\theta}) p(z'|z) dz')$ without nonlinear optimization. Otherwise, we have to solve the nonlinear equation $c = u_c^{-1}\left(\int v^{(i-1)}(f(k, z) - c, z'; \boldsymbol{\theta}) p(z'|z) dz'\right)$ for c.

factor. $\mathbb{E}_0$ is expectation operator at time 0. z_t follows an AR(1) process

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \sigma_\epsilon^2) \tag{3}$$

where ρ is a parameter for persistence of z_t. ϵ_{t+1} is i.i.d. and follows a normal distribution with the variance of σ_ϵ^2.

The first-order necessary condition is given by

$$u_c(c_t) = \beta \mathbb{E}_t \left\{ u_c(c_{t+1}) f_k(k_{t+1}, z_{t+1}) \right\}.$$

$u_c(c)$ denotes differential of $u(c)$ in terms of c and $f_k(k,z)$ denotes differential of $f(k,z)$ in terms of k. There is a mapping $\sigma = K\sigma$ that solves

$$u_c(c) = \beta \int u_c\left(\sigma(f(k,z) - c, z')\right) f_k(f(k,z) - c, z') p(z'|z) dz'$$

for $c = \sigma(k,z)$ (Collman, 1990). $p(z'|z)$ is the pdf of z' conditional on z, or ϵ' as $z' = \rho z + \epsilon'$ holds. That is, $p(z'|z) = p(\epsilon')$.

2.2 The time iteration method

The time iteration method takes the following steps:

1. Make an initial guess for the policy function $\sigma^{(0)}$.
2. For $i = 1, 2, ...$ (i is an index for the number of iteration), given the policy function previously obtained $\sigma^{(i-1)}$, solve

$$u_c(c) = \beta \int u_c\left(\sigma^{(i-1)}(f(k,z) - c, z')\right) f_k(f(k,z) - c, z') p(z'|z) dz' \tag{4}$$

   for c.
3. Update the policy function by setting $c = \sigma^{(i)}(k,z)$.
4. Repeat 2-3 until $\left\|\sigma^{(i)} - \sigma^{(i-1)}\right\|$ is small enough.

In computer, we cannot solve (4) for c for any (k,z) which are continuous state variables on a compact state space. Therefore, in Step 2, we discretize the state space of (k,z) by grid points:

$$k_j \in \{k_1, k_2, \cdots, k_{N_k}\} \quad z_m \in \{z_1, z_2, \cdots, z_{N_z}\}$$

where (j, m) is an index for grid points. Then we solve a nonlinear equation (i.e., the residual function)

$$\begin{aligned} R(c_{jm}; k_j, z_m, \sigma^{(i-1)}) &= -u_c(c_{jm}) \\ &+ \beta \int \left[u_c \left(\sigma^{(i-1)}(f(k_j, z_m) - c_{jm}, z') \right) f_k(f(k_j, z_m) - c_{jm}, z') p(z'|z_m) \right] dz' \\ &= 0 \end{aligned} \tag{5}$$

for c_{jm} at each grid point (k_j, z_m).[10]

In the standard approach, we solve a root finding problem in (5). Such a nonlinear optimization can be time-consuming when the number of grid points and/or the number of nonlinear equations is large. We interpolate the policy function to compute the value of $\sigma^{(i-1)}(f(k_j, z_m) - c_{jm}, z')$, which is off the grid points. We fit a parameterized function (e.g., a polynomial) $\hat{\sigma}^{(i-1)}(k, z; \boldsymbol{\theta})$ ($\boldsymbol{\theta}$ is a set of the polynomial's coefficients) to the data of $\sigma^{(i-1)}(k_j, z_m)$ at each grid point, to evaluate the policy function off the grid points. Also, we compute integrals in (5) with regard to z'. Numerical integration, such as the method of Gaussian-Hermite quadrature, can also be costly because we calculate the weighted average of the next period's values evaluated at each quadrature point.[11] In the following sections, we will show how to avoid costly nonlinear optimization and numerical integration.

2.3 Non-stochastic PEA

By introducing non-stochastic PEA, we can avoid costly nonlinear optimization. There are two ways to apply non-stochastic PEA. One is to fit polynomials to the future variables, and the other is to fit polynomials to the current variables. With the latter approach, we can also avoid costly numerical integration.

[10]Namely, the residual function is satisfied only on the grid points. This is also called collocation. CF also considered other types of projection methods, such as Galerkin, but in the present paper we focus only on the collocation method.

[11]The quadrature nodes $\{x_i\}_{i=1}^M$ and quadrature weights $\{w_i\}_{i=1}^M$ approximate $\int f(x)w(x)dx \approx \sum w_i f(x_i)$. M is the number of the quadrature nodes and weights for each dimension of the state space. Note that the number of the quadrature nodes and weights is exponentially increasing in the number of the state variables.

2.3.1 Fitting future variables

We will replace Step 2-3 presented in Section 2.2. We define a new auxiliary variable[12]

$$e(k,z) \equiv \beta \int u_c\left(c'\right) f_k(k',z')p(z'|z)dz'.$$

Then, given the value of $e^{(i-1)}(k_j, z_m)$ at each grid point, we solve for

$$c_{jm} = u_c^{-1}\left(e^{(i-1)}(k_j, z_m)\right),$$

and obtain an intermediate policy function $c = \sigma^{(i)}(k,z)$. Note that we don't have to solve the nonlinear equation (5). Instead, there is a one-to-one mapping between $e^{(i-1)}(k,z)$ and $\sigma^{(i)}(k,z)$ when u_c^{-1} is a monotonic transformation.

We also update

$$\begin{aligned} e^{(i)}(k_j, z_m) &= \beta \int u_c\left(\sigma^{(i)}(k',z')\right) f_k(k',z')p(z'|z_m)dz' \\ &\approx \beta \int u_c\left(u_c^{-1}\left(e^{(i-1)}(k',z';\boldsymbol{\theta})\right)\right) f_k(k',z')p(z'|z_m)dz' \end{aligned}$$

where

$$k' = f(k_j, z_m) - c_{jm}$$

is obtained in the previous step. Note that we interpolate $\sigma^{(i)}(k',z')$, or equivalently $e^{(i-1)}(k',z')$ by fitting a polynomial here. That is, $c' = \sigma^{(i)}(k',z') \approx u_c^{-1}\left(e^{(i-1)}(k',z';\boldsymbol{\theta})\right)$ holds, where $\boldsymbol{\theta}$ is obtained by fitting the polynomial to the data of $e^{(i-1)}(k_j, z_m)$ at each grid point. Also, we compute an integral of the composite function with regard to z'. We apply numerical integration with Gaussian-Hermite quadrature.

2.3.2 Fitting current variables

Or, we define

$$v(k,z) \equiv \beta u_c\left(c\right) f_k(k,z).$$

[12] Note that this definition implicitly assumes the relationship between the current and future endogenous state variables (i.e., $c' = \sigma(k',z')$ and $k' = f(k,z) - \sigma(k,z)$).

Then, given the function $v^{(i-1)}(k',z')$ and the value of $\sigma^{(i-1)}(k_j,z_m)$ at each grid point, we have

$$\begin{aligned} c_{jm} &= u_c^{-1}\left(\int v^{(i-1)}(k',z')p(z'|z_m)dz'\right), \\ &\approx u_c^{-1}\left(\int v^{(i-1)}(k',z';\boldsymbol{\theta})p(z'|z_m)dz'\right) \end{aligned}$$

where

$$k' = f(k_j,z_m) - \sigma^{(i-1)}(k_j,z_m).$$

Note that we interpolate $v^{(i-1)}(k',z') \approx v^{(i-1)}(k',z';\boldsymbol{\theta})$ by fitting a polynomial. We compute an integral of the polynomial $v^{(i-1)}(k',z';\boldsymbol{\theta})$ with regard to z'. We can also avoid nonlinear optimization by a successive approximation $k' = f(k_j,z_m) - \sigma^{(i-1)}(k_j,z_m)$.[13]

We also update

$$\begin{aligned} v^{(i)}(k_j,z_m) &= \beta u_c\left(c_{jm}\right) f_k(k_j,z_m), \\ \sigma^{(i)}(k_j,z_m) &= c_{jm}, \end{aligned}$$

where c is obtained in the previous step.

2.3.3 Precomputation of integrals

In each case of the non-stochastic PEAs considered in Sections 2.3.1 and 2.3.2, we compute integrals with regard to z'. Only when we fit polynomials to the current variables, we can utilize the precomputation technique of integrals developed in Judd et al. (2017) to avoid computing numerical integration. For example, we fit a second order polynomial (without cross terms) to the data of $v(k,z) = u_c\left(\sigma(k,z)\right) f_k(k,z)$ on the grid points of the state variables (k,z):[14]

$$\hat{v}(k,z;\boldsymbol{\theta}) = \theta_{0,0} + \theta_{1,0}k + \theta_{2,0}k^2 + \theta_{0,1}z + \theta_{0,2}z^2.$$

[13] This is called fixed point iteration? Alternatively, we can solve the nonlinear equation $c = u_c^{-1}\left(\int v^{(i-1)}(f(k_j,z_m)-c,z';\boldsymbol{\theta})p(z'|z_m)dz'\right)$ for c.

[14] For simplicity, an ordinary polynomial is used. Chebyshev polynomial $\theta_{0,0} + \theta_{1,0}T_1(k) + \theta_{2,0}T_2(k) + \theta_{0,1}T_1(z) + \theta_{0,2}T_2(z)$, where $T(x) : [-1,1] \rightarrow [-1,1]$, can be used instead in a straight forward way.

Then,

$$\begin{aligned}\int \hat{v}(k', z'; \boldsymbol{\theta}) p(z'|z) dz' &= \theta_{0,0} + \theta_{1,0} k' + \theta_{2,0} (k')^2 + \int \left(\theta_{0,1} z' + \theta_{0,2} (z')^2\right) p(z'|z) dz' \\ &= \theta_{0,0} + \theta_{1,0} k' + \theta_{2,0} (k')^2 + \int \left(\theta_{0,1} (\rho z + \epsilon') + \theta_{0,2} (\rho z + \epsilon')^2\right) p(\epsilon') d\epsilon' \\ &= \theta_{0,0} + \theta_{1,0} k' + \theta_{2,0} (k')^2 + \theta_{0,1} \rho z + \theta_{0,2} \rho^2 z^2 + \theta_{0,2} \sigma_\epsilon^2.\end{aligned}$$

As pointed out by Judd et al. (2017), the precomputation technique can be used only when we express the function to be integrated by a simple parameterized form. This is exactly the case when we fit polynomials to current variables. In contrast, when we fit polynomials to future variables, we have to compute integrals of

$$\int u_c \left(u_c^{-1} \left(\hat{e}(k', z'; \boldsymbol{\theta})\right)\right) f_k(k', z') p(z'|z) dz',$$

which is a composite nonlinear function. It is difficult or impossible to calculate integrals of such a function analytically.

2.4 Numerical examples

We use the following functional forms: $u(c) = \frac{c^{1-\tau}-1}{1-\tau}$ and $f(k, z) = zk^\alpha$. We set the parameter values as $\beta = 0.99$, $\alpha = 1/3$, $\delta = 0.025$, $\rho_z = 0.95$, $\sigma_\epsilon = 0.008$ and $\tau = \{1, 2, 5\}$. We use the second- and fourth-order Chebyshev polynomials for k and z for interpolation in each solution algorithm. The number of grid points are $3^2 = 9$ and $5^2 = 25$ for each case of polynomials. The number of Gaussian-Hermite quadrature is set to 3. The simulation is done for 10,500 periods and first 500 periods are discarded. The same sequence of random variables for z is used throughout all the simulations.

The Euler equation error is calculated by

$$\mathcal{E}(k, z) = 1 - \beta \int \left\{ \left(\frac{\sigma_c(k', z')}{\sigma_c(k, z)} \right)^{-\tau} \left(1 - \delta + \alpha z' k'^{\alpha-1}\right) \right\} \mu(z'|z) dz' \tag{6}$$

where $k' = f(k, z) - \sigma_c(k, z)$. [$c = \sigma_c(k, z)$ is fitted to polynomials with the same grid points and basis functions used in each solution algorithm.]

Table 1: Accuracy and speed of TI, future PEA, and current PEA: Stochastic neoclassical growth model

Polynomial, τ	TI			future PEA			current PEA		
	L_1	L_∞	CPU	L_1	L_∞	CPU	L_1	L_∞	CPU
2nd, $\tau = 1.0$	-5.12	-4.60	4.03	-4.23	-3.69	0.04	-3.13	-2.44	0.02
4th, $\tau = 1.0$	-7.08	-6.72	9.76	-5.92	-5.59	0.09	-3.13	-2.44	0.04
2nd, $\tau = 2.0$	-4.82	-4.35	0.86	-3.99	-3.53	0.03	-2.95	-2.26	0.01
4th, $\tau = 2.0$	-6.76	-6.45	2.88	-5.63	-5.36	0.11	-2.96	-2.27	0.04
2nd, $\tau = 5.0$	-4.48	-3.87	0.62	-3.57	-2.88	0.05	-2.67	-1.99	0.02
4th, $\tau = 5.0$	-6.43	-5.38	1.91	-5.10	-3.90	0.15	-2.69	-2.00	0.05

Notes: L_1 and L_∞ are, respectively, the average and maximum of absolute Euler errors (6) (in log 10 units) on a 10,000 period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds).

First of all, the method of PEA collocation with fitting future variables (future PEA) is about 10-100 times faster than the standard time iteration method (TI). Also, PEA collocation with fitting current variables (current PEA) is about 2-4 times faster than future PEA. There is a substantial gain in computational time by using the proposed methods (current or future PEA) and they are potentially useful for structural estimation. However, there is a tradeoff between accuracy and computation speed. Specifically, TI is most accurate in terms of the average and absolute Euler errors and takes more time than current PEA and future PEA. [Future PEA and current PEA are comparable in terms of the Euler errors.]

The higher order polynomial reduces the Euler errors and increases computation time in all the methods. We also find that the larger the risk aversion parameter τ is, the more average and absolute Euler errors are in all the methods. This is not surprising, as the model becomes more nonlinear with an increased τ. Interestingly, a larger τ also reduces computation time only in TI, but not in the other methods.

3 Small-scale New Keynesian model

The example here is taken from Herbst and Schorfheide (2016). The model economy consists of final-good and intermediate-good producing firms, households, and monetary and fiscal authorities. Prices are sticky due to Rotemberg-type (1982) adjustment cost. As in the previous section, we solve the model in three different methods; the standard time iteration, the non-stochastic PEA with fitting current variables, and the non-stochastic PEA with fitting future variables. Details are in the Appendix.

3.1 Setup

Equilibrium conditions (after detrending) are given by:

$$
\begin{aligned}
1 &= \beta R_t \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{1}{\gamma_{t+1}\pi_{t+1}} \right], \\
0 &= (1-\nu^{-1}) + \nu^{-1} c_t^{\tau} - \phi \left(\pi_t - \bar{\pi} \right) \left[\pi_t - \frac{1}{2\nu} \left(\pi_t - \bar{\pi} \right) \right] \\
&+ \beta\phi \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} \left(\pi_{t+1} - \bar{\pi} \right) \pi_{t+1} \right], \\
R_t^* &= \left(\bar{r}\bar{\pi} \left(\frac{\pi}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_t}{y_t^*} \right)^{\psi_2} \right)^{1-\rho_R} R_{t-1}^{*\rho_R} e^{\epsilon_{R,t}}, \\
c_t &+ \frac{\phi}{2} \left(\pi_t - \pi \right)^2 y_t = g_t^{-1} y_t, \\
R_t &= \max \left\{ R_t^*, 1 \right\},
\end{aligned}
$$

where c_t is consumption, π_t is inflation, R_t is the nominal interest rate, y_t is output. g_t is exogenous government expenditure. $(\beta, \tau, \nu, \phi, \bar{r}, \bar{\pi}, \psi_1, \psi_2, \rho_R)$ is a set of the parameters. The natural level of output is given by $y_t^* = (1-\nu)^{1/\tau} g_t$, which is obtained when we set $\phi = 0$ in the equilibrium conditions. The total factor productivity A_t has a deterministic trend $\bar{\gamma}$ and a shock to the trend z_t such as $\ln \gamma_t \equiv \ln(A_t/A_{t-1}) = \ln \bar{\gamma} + \ln z_t$. The exogenous shocks $\{z_t, g_t\}$ follow

$$
\begin{aligned}
\ln z_t &= \rho_z \ln z_{t-1} + \epsilon_{z,t}, \\
\ln g_t &= (1-\rho_g) \ln \bar{g} + \rho_g \ln g_{t-1} + \epsilon_{g,t}.
\end{aligned}
$$

where (ρ_g, ρ_z) are the parameters for persistence of the shocks. The disturbance terms $\{\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{R,t}\}$ are serially uncorrelated and independent to each other. The three disturbances are normally distributed with means zero and standard deviations σ_z, σ_g, and σ_R, respectively. The steady state values are given by

$$
\begin{aligned}
c = (1-\nu)^{1/\tau}, \qquad & \pi = \bar{\pi}, \\
R = \bar{r}\bar{\pi} = \beta^{-1}\gamma\bar{\pi}, \quad & y = gc.
\end{aligned}
$$

3.2 The Collman operator

The solution has a form of

$$
\begin{aligned}
c = \sigma_c(R^*_{-1}, s), \qquad & \pi = \sigma_\pi(R^*_{-1}, s), \\
R^* = \sigma_{R^*}(R^*_{-1}, s), \quad & y = \sigma_y(R^*_{-1}, s),
\end{aligned}
$$

where $s = (z, g, \epsilon_R)$. Note that $R = \max\left\{\sigma_{R^*}(R^*_{-1}, s), 1\right\}$. The mapping $\sigma = K\sigma$ solves

$$
\begin{aligned}
& 0 = -c^{-\tau} + \beta R \int \left[\frac{\sigma_c(R^*, s')^{-\tau}}{\gamma' \sigma_\pi(R^*, s')}\right] p(s'|s) ds', \\
& 0 = \left((1-\nu^{-1}) + \nu^{-1} c^\tau - \phi\left(\pi - \bar{\pi}\right)\left[\pi - \frac{1}{2\nu}\left(\pi - \bar{\pi}\right)\right] \right) c^{-\tau} y \\
& + \beta\phi \int \left[\sigma_c(R^*, s')^{-\tau} \sigma_y(R^*, s') \left(\sigma_\pi(R^*, s') - \bar{\pi}\right) \sigma_\pi(R^*, s')\right] p(s'|s) ds', \\
& R^* = \left(r\bar{\pi} \left(\frac{\pi}{\bar{\pi}}\right)^{\psi_1} \left(\frac{y}{y^*}\right)^{\psi_2} \right)^{1-\rho_R} {R^*_{-1}}^{\rho_R} e^{\epsilon_R}, \\
& c + \frac{\phi}{2}\left(\pi - \bar{\pi}\right)^2 y = g^{-1} y, \\
& R = \max\left\{R^*, 1\right\}
\end{aligned}
$$

for c, π, R^*, R, y, where $\gamma' = \bar{\gamma} z'$, $z' = z^{\rho_z} e^{\epsilon'_z}$, and $g'/\bar{g} = (g/\bar{g})^{\rho_g} e^{\epsilon'_g}$.

The time iteration method takes the following steps:

1. Make an initial guess for the policy function $\sigma^{(0)}$.
2. Given the policy function previously obtained $\sigma^{(i-1)}$ solve the relevant equations for (c, π, R^*, y).
3. Update the policy function by setting $c = \sigma_c^{(i)}(R^*_{-1}, s)$, $\pi = \sigma_\pi^{(i)}(R^*_{-1}, s)$, $R^* = \sigma_\pi^{(i)}(R^*_{-1}, s)$, and $y = \sigma_y^{(i)}(R^*_{-1}, s)$.
4. Repeat 2-3 until $\left\|\sigma^{(i)} - \sigma^{(i-1)}\right\|$ is small enough.

3.3 Non-stochastic PEA with the ZLB

To apply non-stochastic PEA to the New Keynesian model presented in the previous section, we define the expectation functions as follows. In the approach with fitting polynomials to future variables, we define

$$\begin{aligned} e_c(R^*_{-1}, s) &\equiv \frac{\beta}{\bar{\gamma}} R \int \left[\frac{\sigma_c(R^*, s')^{-\tau}}{z' \sigma_\pi(R^*, s')} \right] p(s'|s) ds', \\ e_\pi(R^*_{-1}, s) &\equiv \beta\phi \int \left[\sigma_c(R^*, s')^{-\tau} \frac{\sigma_y(R^*, s')}{y} \left(\sigma_\pi(R^*, s') - \bar{\pi} \right) \sigma_\pi(R^*, s') \right] p(s'|s) ds'. \end{aligned}$$

Or, in the approach with fitting polynomials to current variables, we define

$$\begin{aligned} v_c(R^*_{-1}, s) &\equiv \frac{\beta}{\bar{\gamma}} \left[\frac{c^{-\tau}}{z\pi} \right], \\ v_\pi(R^*_{-1}, s) &\equiv \beta\phi \left[c^{-\tau} y \left(\pi - \bar{\pi} \right) \pi \right]. \end{aligned}$$

To deal with the ZLB, we adapt the index-function approach as in Aruoba et al. (2017), Gust et al. (2017), Nakata (2017), Hirose and Sunakawa (2017). That is, given a pair of regime-specific expectation functions $(v_{\text{NZLB}}, v_{\text{ZLB}})$ (or $(e_{\text{NZLB}}, e_{\text{ZLB}})$) and the policy function for the notional rate $\sigma_{R^*,\text{NZLB}}(R^*_{-1}, s)$, we use an index function to obtain

$$v(R^*_{-1}, s) = \mathbb{I}_{(R^*<1)} v_{\text{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) v_{\text{NZLB}}(R^*_{-1}, s),$$

where the index function depends on the value of the current notional rate R^* and is defined as

$$\mathbb{I}_{(R^*<1)} = \begin{cases} 1 & \text{when } R^* = \sigma_{R^*,\text{NZLB}}(R^*_{-1}, s) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

In the latter approach, we can precompute the integral of $v(R^*, s')$ with regard to s'. Note that $v(R^*, s')$ are composite functions of the ones in NZLB regime and ZLB regime indexed by the value of $\sigma_{R^*,\mathrm{NZLB}}(R^*, s')$. Specifically, we have

$$
\begin{aligned}
&\int v(R^*, s')p(s'|s_m)ds' \\
=&\int \left[\mathbb{I}_{(\sigma_{R^*,\mathrm{NZLB}}(R^*,s')<1)} v_{\mathrm{ZLB}}(R^*, s') + \left(1 - \mathbb{I}_{(\sigma_{R^*,\mathrm{NZLB}}(R^*,s')<1)}\right) v_{\mathrm{NZLB}}(R^*, s')\right] p(s'|s_m)ds' \\
=&\int \left[\mathbb{I}_{(\sigma_{R^*,\mathrm{NZLB}}(R^*,s')<1)} \left(v_{\mathrm{ZLB}}(R^*, s') - v_{\mathrm{NZLB}}(R^*, s')\right) + v_{\mathrm{NZLB}}(R^*, s')\right] p(s'|s_m)ds' \\
=&\Pr(\sigma_{R^*,\mathrm{NZLB}}(R^*, s') < 1) \left(\int v_{\mathrm{ZLB}}(R^*, s')p(s'|s_m)ds' - \int v_{\mathrm{NZLB}}(R^*, s')p(s'|s_m)ds'\right) \\
&+ \int v_{\mathrm{NZLB}}(R^*, s')p(s'|s_m)ds'.
\end{aligned}
$$

We assume that $\mathbb{I}_{(\sigma_{R,\mathrm{NZLB}}(R^*,s')<0)}$ and $\int \{v_{\mathrm{ZLB}}(R^*, s') - v_{\mathrm{NZLB}}(R^*, s')\} p(s'|s_m)ds'$ are orthogonal to each other. Also, to compute $\Pr(\sigma_{R^*,\mathrm{NZLB}}(R^*, s') < 1)$, we approximate $\sigma_{R^*,\mathrm{NZLB}}(R^*, s')$ up to first order by truncating higher-order terms

$$
\begin{aligned}
\hat{\sigma}_{R^*,\mathrm{NZLB}}(R^*, s') =&\theta_0 + \theta_{R^*}R^* + \theta_g g' + \theta_z z' + \theta_r \epsilon_r' \\
=&\theta_0 + \theta_{R^*}R^* + \theta_g \rho g + \theta_z \rho z + \theta_g \epsilon_g' + \theta_z \epsilon_z' + \theta_r \epsilon_r'
\end{aligned}
$$

Then we have

$$
\Pr(\hat{\sigma}_{R^*,\mathrm{NZLB}}(R^*, s') < 1) = \Phi\left(x < 1 - (\theta_0 + \theta_{R^*}R^* + \theta_g \rho g + \theta_z \rho z)\right)
$$

where $\Phi(\bullet)$ is the cumulative distribution function of $x = \theta_g \epsilon_g' + \theta_z \epsilon_z' + \theta_r \epsilon_r'$ which follows a normal distribution[15].

3.4 Numerical examples

Parameter values except for $(\nu, \bar{g})$ are taken from an estimation results on the log-linearized version of the model in Herbst and Schorfheide (2016).[16] They are summarized in Table 2. We use the second- and fourth-order Chebyshev polynomials for R and $s = (z, g, \epsilon_r)$ for interpolation in each solution algorithm. The number of grid points are $3^4 = 81$ and $5^4 = 625$ for each case of polynomials. We also use the Smolyak algorithm for each case

[15] We can possibly use higher order approximation for σ_{R^*} to calculate the probability at the cost of more complicated distribution for x.

[16] The parameter for price adjustment cost ϕ is obtained by setting the values of the elasticity of demand $\nu^{-1} = 6$ and the slope of Phillips curve (which is a composite function of parameters) $\kappa = \frac{\tau(\nu^{-1}-1)}{\bar{\pi}^2\phi} = 0.78$. The steady-state government expenditure shock is given by $\bar{g} = (1-g_y)^{-1}$, where g_y is the ratio of government expenditure to output and set to 0.2.

of polynomials. In these cases, the number of grid points are 9 and 41 (See Judd, Maliar, Maliar and Valero, 2014; Gust et al., 2017). The number of Gaussian-Hermite quadrature is set to $3^3 = 27$. The simulation is done for 10,500 periods and first 500 periods are discarded. The same sequence of random variables for s is used throughout all the simulations.

Table 2: Parameter values of small New Keynesian model

Parameter		Value
ν	Inverse of demand elasticity	1/6
$\bar{g}$	Steady state government expenditure	1.25
γ	Steady state technology growth	1.0052
β	Discount factor	0.9990
$\bar{\pi}$	Steady state inflation	1.0083
τ	CRRA parameter	2.83
ϕ	Price adjustment cost	17.85
ψ_1	Interest rate elasticity to inflation	1.80
ψ_2	Interest rate elasticity to output gap	0.63
ρ_r	Interest rate smoothing	0.77
ρ_g	Persistence of government shock	0.98
ρ_z	Persistence of technology growth shock	0.88
σ_r	Std. dev. of monetary policy shock	0.0022
σ_g	Std. dev. of government shock	0.0071
σ_z	Std. dev. of technology growth shock	0.0031

The Euler equation errors are calculated by

$$\begin{aligned}
\mathcal{E}_c(R_{-1}, s) =& 1 - \frac{\beta}{\bar{\gamma}} R \int \left\{ \left(\frac{\sigma_c(R, s')}{\sigma_c(R_{-1}, s)} \right)^{-\tau} \frac{1}{z'\sigma_\pi(R, s')} \right\} \mu(s'|s) ds' && (7) \\
\mathcal{E}_\pi(R_{-1}, s) =& (1 - \nu^{-1}) + \nu^{-1}\sigma_c(R_{-1}, s)^{-\tau} \\
& - \phi \left(\sigma_\pi(R_{-1}, s) - \bar{\pi} \right) \left[\sigma_\pi(R_{-1}, s) - \frac{1}{2\nu} \left(\sigma_\pi(R_{-1}, s) - \bar{\pi} \right) \right] \\
& + \beta\phi \int \left\{ \left(\frac{\sigma_c(R, s')}{\sigma_c(R_{-1}, s)} \right)^{-\tau} \frac{y'}{y} \left(\sigma_\pi(R, s') - \bar{\pi} \right) \sigma_\pi(R, s') \right\} \mu(s'|s) ds', && (8)
\end{aligned}$$

where

$$y = \left(g^{-1} - \frac{\phi}{2} \left(\sigma_\pi(R_{-1}, s) - \bar{\pi} \right)^2 \right)^{-1} \sigma_c(R_{-1}, s), \tag{9}$$

$$R = \left(\bar{r}\bar{\pi} \left(\frac{\sigma_\pi(R_{-1}, s)}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R_{-1}^{\rho_R} e^{\epsilon_R}. \tag{10}$$

Note that the last two equations of the static equilibrium conditions (9)-(10) hold with equality by taking as given $c = \sigma_c(R_{-1}, s)$ and $\pi = \sigma_\pi(R_{-1}, s)$.[17]

Table 3: Accuracy and speed of TI, future PEA, current PEA: Small New Keynesian model

Polynomial	TI							
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	CPU
2nd	-4.15	-2.45	-3.47	-1.79	0.76	0.48	2.36	318.07
2nd, Smolyak	-3.45	-2.35	-2.34	-1.31	0.76	0.49	2.42	3.71
4th, Smolyak	-5.09	-3.73	-3.72	-2.57	0.76	0.50	2.43	61.94
Polynomial	future PEA							
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	CPU
2nd	-4.86	-2.76	-3.52	-2.04	0.76	0.50	2.46	20.90
2nd, Smolyak	-3.32	-2.66	-2.21	-1.63	0.76	0.53	2.59	0.29
4th, Smolyak	-5.03	-3.69	-3.71	-2.69	0.76	0.50	2.44	4.06
Polynomial	current PEA							
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	CPU
2nd	-4.32	-2.46	-3.30	-1.75	0.76	0.48	2.35	1.39
2nd, Smolyak	-3.36	-2.36	-2.21	-1.34	0.76	0.50	2.43	0.04
4th, Smolyak	-4.93	-3.75	-3.56	-2.53	0.76	0.50	2.43	0.32

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of absolute Euler errors (7)-(8) (in log 10 units) on a 10,000 period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_π, and σ_R are the standard deviation of output growth, inflation, and the policy rate.

Also, we solve the model with the zero lower bound on nominal interest rates.

[17] y' in (8) is calculated by substituting R implied by (10) into (9).

Table 4: Accuracy and speed of TI, future PEA, and current PEA: Small New Keynesian model with the ZLB

Polynomial	TI								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	$\Pr_{(R^*<1)}$	CPU
2nd	-3.73	-2.62	-2.07	-1.42	0.76	0.51	2.53	1.53	1127.3
2nd, Smolyak	-3.40	-2.38	-2.06	-1.09	0.76	0.50	2.50	1.79	12.98
4th, Smolyak	-3.97	-3.14	-2.07	-1.73	0.76	0.51	2.50	1.40	270.65
Polynomial	future PEA								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	$\Pr_{(R^*<1)}$	CPU
2nd	-3.73	-2.67	-2.08	-1.44	0.76	0.53	2.59	1.71	82.28
2nd, Smolyak	-3.26	-2.66	-1.92	-1.48	0.76	0.55	2.68	3.42	0.96
4th, Smolyak	-4.04	-3.54	-2.13	-1.49	0.76	0.51	2.48	1.18	14.66
Polynomial	current PEA								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_{π}	σ_R	$\Pr_{(R^*<1)}$	CPU
2nd	-4.05	-2.71	-2.12	-1.53	0.76	0.50	2.45	1.03	4.05
2nd, Smolyak	-3.35	-2.42	-1.97	-1.25	0.76	0.50	2.47	1.92	0.19
4th, Smolyak	-4.17	-2.95	-2.12	-1.44	0.76	0.51	2.47	1.07	0.99

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of absolute Euler errors (7)-(8) (in log 10 units) on a 10,000 period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_{π}, and σ_R are the standard deviation of output growth, inflation, and the policy rate. $\Pr_{(R^*<1)}$ is the probability of binding the ZLB.

4 Smets-Wouters model

The model in Gust et al. (2017) is to be solved by both current PEA and future PEA.

5 Conclusion

TBW

A Appendix to New Keynesian model

The time iteration method takes the following steps:

1. Make an initial guess for the policy function $\sigma^{(0)}$.

2. For $i = 1, 2, ...,$ given the policy function previously obtained $\sigma^{(i-1)}$, at each grid point

(j, m), having the values of $(R^*_{j,-1}, s_m)$ at hand, we solve the

$$
\begin{aligned}
&c_{jm}^{-\tau} = \frac{\beta}{\bar{\gamma}} R_{jm} \int \left[\frac{\sigma_c^{(i-1)}(R^*_{jm}, s')^{-\tau}}{z' \sigma_\pi^{(i-1)}(R^*_{jm}, s')} \right] p(s'|s) ds', \\
&0 = \left((1 - \nu^{-1}) + \nu^{-1} c_{jm}^{\tau} - \phi \left(\pi_{jm} - \bar{\pi} \right) \left[\pi_{jm} - \frac{1}{2\nu} \left(\pi_{jm} - \bar{\pi} \right) \right] \right) c_{jm}^{-\tau} y_{jm} \\
&+ \beta\phi \int \left[\sigma_c^{(i-1)}(R^*_{jm}, s')^{-\tau} \sigma_y^{(i-1)}(R^*_{jm}, s') \left(\sigma_\pi^{(i-1)}(R^*_{jm}, s') - \bar{\pi} \right) \sigma_\pi^{(i-1)}(R^*_{jm}, s') \right] p(s'|s) ds', \\
&R^*_{jm} = \left(r\bar{\pi} \left(\frac{\pi_{jm}}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_{jm}}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R^{*\rho_R}_{j,-1} e^{\epsilon_{R,m}}, \\
&c_{jm} + \frac{\phi}{2} \left(\pi_{jm} - \bar{\pi} \right)^2 y_{jm} = g_m^{-1} y_{jm}, \\
&R_{jm} = \max \left\{ R^*_{jm}, 1 \right\},
\end{aligned}
$$

for $c_{jm}, \pi_{jm}, R^*_{jm}, R_{jm}, y_{jm}$.

3. Update the policy function by setting $c = \sigma_c^{(i)}(R^*_{-1}, s)$, $\pi = \sigma_\pi^{(i)}(R^*_{-1}, s)$, $R^* = \sigma_\pi^{(i)}(R^*_{-1}, s)$, and $y = \sigma_y^{(i)}(R^*_{-1}, s)$.

4. Repeat 2-3 until $\left\| \sigma^{(i)} - \sigma^{(i-1)} \right\|$ is small enough.

A.1 Non-stochastic PEAs without the ZLB

First let's consider the case without the ZLB, i.e., $R = R^*$. The case with the ZLB will be considered in Section A.2.

A.1.1 Fitting future variables

Following Gust et al. (2012; 2017), we define[18]

$$
\begin{aligned}
e_c(R^*_{-1}, s) &\equiv \frac{\beta}{\bar{\gamma}} R \int \left[\frac{\sigma_c(R^*, s')^{-\tau}}{z' \sigma_\pi(R^*, s')} \right] p(s'|s) ds', \\
e_\pi(R^*_{-1}, s) &\equiv \beta\phi \int \left[\sigma_c(R^*, s')^{-\tau} \frac{\sigma_y(R^*, s')}{y} \left(\sigma_\pi(R^*, s') - \bar{\pi} \right) \sigma_\pi(R^*, s') \right] p(s'|s) ds'.
\end{aligned}
$$

[18] We include R in $e_c(R^*_{-1}, s)$ and y in $e_\pi(R^*_{-1}, s)$ so that we can avoid solving a root-finding problem.

Note that $R^* = \sigma_R(R^*_{-1}, s)$. Then, given the values of $e_c^{(i-1)}(R_{j,-1}, s_m)$ and $e_\pi^{(i-1)}(R^*_{j,-1}, s_m)$ at each grid point, we solve

$$
\begin{aligned}
c_{jm} &= e_c^{(i-1)}(R^*_{j,-1}, s_m)^{-1/\tau}, \\
0 &= (1-\nu^{-1}) - \phi\left(\pi_{jm} - \bar{\pi}\right)\left[\pi_{jm} - \frac{1}{2\nu}\left(\pi_{jm} - \bar{\pi}\right)\right] + e_c^{(i-1)}(R^*_{j,-1}, s_m)^{-1}\left(\nu^{-1} + e_\pi^{(i-1)}(R^*_{j,-1}, s_m)\right), \\
y_{jm} &= \left[g_m^{-1} - \frac{\phi}{2}\left(\pi_{jm} - \bar{\pi}\right)^2\right]^{-1} c_{jm}, \\
R^*_{jm} &= \left(\bar{r}\bar{\pi}\left(\frac{\pi_{jm}}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y_{jm}}{y^*}\right)^{\psi_2}\right)^{1-\rho_R} R^{*\rho_R}_{j,-1} e^{\epsilon_{R,m}},
\end{aligned}
$$

for $c_{jm}, \pi_{jm}, R^*_{jm}, y_{jm}$, which generates intermediate policy functions $\sigma_c^{(i)}(R^*_{-1}, s)$, $\sigma_\pi^{(i)}(R^*_{-1}, s)$, $\sigma_{R^*}^{(i)}(R^*_{-1}, s)$, and $\sigma_y^{(i)}(R^*_{-1}, s)$. In other words, there is a mapping between $e^{(i-1)}$ and $\sigma^{(i)}$. In particular, we solve the following second-order polynomial

$$
\alpha_0 - 2\alpha_1\pi_{jm} + \alpha_2\pi_{jm}^2 = 0
$$

for π_{jm}, where

$$
\begin{aligned}
\alpha_0 =& \frac{\phi\bar{\pi}^2}{2\nu} + (1-\nu^{-1}) + e_c^{(i-1)}(R^*_{j,-1}, s_m)^{-1}\left(\nu^{-1} + e_\pi^{(i-1)}(R^*_{j,-1}, s_m)\right), \\
\alpha_1 =& \phi\bar{\pi}\left(\nu^{-1} - 1\right)/2, \\
\alpha_2 =& \phi\left(\frac{1}{2\nu} - 1\right).
\end{aligned}
$$

We pick up the root $\pi = \alpha_1/\alpha_2 - \sqrt{(\alpha_1/\alpha_2)^2 - \alpha_0}$ of the polynomial and ignore the other root (why minus?).

We also update

$$
\begin{aligned}
e_c^{(i)}(R^*_{j,-1}, s_m) &= \frac{\beta}{\bar{\gamma}} R_{jm} \int \left[\frac{\sigma_c^{(i)}(R^*_{jm}, s')^{-\tau}}{z'\sigma_\pi^{(i)}(R^*_{jm}, s')}\right] p(s'|s_m)ds', \\
e_\pi^{(i)}(R^*_{j,-1}, s_m) &= \beta\phi \int \left[\sigma_c^{(i)}(R^*_{jm}, s')^{-\tau} \frac{\sigma_y^{(i)}(R^*_{jm}, s')}{y_{jm}}\left(\sigma_\pi^{(i)}(R^*_{jm}, s') - \bar{\pi}\right)\sigma_\pi^{(i)}(R^*_{jm}, s')\right] p(s'|s_m)ds'.
\end{aligned}
$$

where R_{jm} and y_{jm} are obtained in the previous step. Note that we interpolate $\sigma^{(i)}(R^*, s')$ (or equivalently $e^{(i-1)}(R^*, s')$ as there is a mapping between $e^{(i-1)}$ and $\sigma^{(i)}$). We need to interpolate $e^{(i-1)}(R^*_{-1}, s)$ by fitting polynomials $e^{(i-1)}(R^*_{-1}, s; \boldsymbol{\theta})$ to the data of $e^{(i-1)}(R^*_{j,-1}, s_m)$ at each grid point. We also need to compute integrals numerically with regard to s'.

A.1.2 Fitting current variables

Or, we define

$$
\begin{aligned}
v_c(R^*_{-1}, s) &\equiv \frac{\beta}{\bar{\gamma}} \left[\frac{c^{-\tau}}{z\pi} \right], \\
v_\pi(R^*_{-1}, s) &\equiv \beta\phi \left[c^{-\tau} y \left(\pi - \bar{\pi} \right) \pi \right].
\end{aligned}
$$

Then, given the function $v^{(i-1)}(R^*, s')$ and the value of $\sigma^{(i-1)}(R^*_{j,-1}, s_m)$ at each grid point, we solve

$$
\begin{aligned}
c_{jm} &= \left\{ \sigma^{(i-1)}_{R^*}(R^*_{j,-1}, s_m) \int v^{(i-1)}_c(\sigma^{(i-1)}_{R^*}(R^*_{j,-1}, s_m), s') p(s'|s_m) ds' \right\}^{-1/\tau}, \\
0 &= \left((1-\nu^{-1}) + \nu^{-1} c^{\tau}_{jm} - \phi \left(\pi_{jm} - \bar{\pi} \right) \left[\pi_{jm} - \frac{1}{2\nu} \left(\pi_{jm} - \bar{\pi} \right) \right] \right) c^{-\tau}_{jm} \sigma^{(i-1)}_y (R^*_{j,-1}, s_m) \\
&+ \beta\phi \left\{ \int v^{(i-1)}_c(\sigma^{(i-1)}_R(R^*_{j,-1}, s_m), s') p(s'|s_m) ds' \right\}, \\
y_{jm} &= \left[g^{-1}_m - \frac{\phi}{2} \left(\pi_{jm} - \bar{\pi} \right)^2 \right]^{-1} c_{jm}, \\
R^*_{jm} &= \left(\bar{r}\bar{\pi} \left(\frac{\pi_{jm}}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_{jm}}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R^{*\rho_R}_{j,-1} e^{\epsilon_{R,m}},
\end{aligned}
$$

for $c_{jm}, \pi_{jm}, R^*_{jm}, y_{jm}$. Note that we interpolate $v^{(i-1)}(R^*_{-1}, s)$ by fitting polynomials to $v^{(i-1)}(R^*_{-1}, s; \boldsymbol{\theta})$ to the data of $v_c(R^*_{j,-1}, s_m)$ at each grid point. We also compute integrals of the polynomials $v^{(i-1)}(R^*, s'; \boldsymbol{\theta})$ with regard to s'. π_{jm} is analytically obtained by solving a second-order polynomial similarly to the previous subsection.

We also update

$$
\begin{aligned}
v^{(i)}_c(R^*_{j,-1}, s_m) &= \frac{\beta}{\bar{\gamma}} \left[\frac{c^{-\tau}_{jm}}{z_m \pi_{jm}} \right], \\
v^{(i)}_\pi(R^*_{j,-1}, s_m) &= \beta\phi \left[c^{-\tau}_{jm} y_{jm} \left(\pi_{jm} - \bar{\pi} \right) \pi_{jm} \right],
\end{aligned}
$$

where c_{jm}, π_{jm}, y_{jm} are obtained in the previous step.

A.2 Non-stochastic PEAs with the ZLB

To deal with the zero lower bound (ZLB) on the nominal interest rate, we adapt the index-function approach as in Aruoba et al. (2017), Gust et al. (2017), Nakata (2017), Hirose and Sunakawa (2017). That is, given regime-specific policy functions $\sigma_{x,\mathrm{NZLB}}(R^*_{-1}, s)$ and

$\sigma_{x,\mathrm{ZLB}}(R^*_{-1}, s)$, we use an index function to obtain

$$\sigma_x(R^*_{-1}, s) = \mathbb{I}_{(R^*<1)}\sigma_{x,\mathrm{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right)\sigma_{x,\mathrm{NZLB}}(R^*_{-1}, s),$$

for $x \in \{c, \pi, y\}$, where the index function depends on the value of the current notional rate R^* and is defined as

$$\mathbb{I}_{(R^*<1)} = \begin{cases} 1 & \text{when } R^* = \sigma_{R^*,\mathrm{NZLB}}(R^*_{-1}, s) < 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\sigma_{x,\mathrm{NZLB}}(R^*_{-1}, s)$ is the policy function assuming that ZLB does not bind and $\sigma_{x,\mathrm{ZLB}}(R^*_{-1}, s)$ is the policy function assuming that ZLB binds. $\mathbb{I}_{\sigma_{R^*,\mathrm{NZLB}}(R^*_{-1},s)\geq 0}$ is the indicator function that takes the value of one when $\sigma_{R^*,\mathrm{NZLB}}(R^*_{-1}, s) \geq 0$, i.e., the assumption made for $\sigma_{\mathrm{NZLB}}(R^*_{-1}, s) \geq 0$ is verified, otherwise takes the value of zero. Then, in Step 2 in Section A.1, the problem becomes finding a pair of policy functions, $\left(\sigma_{\mathrm{NZLB}}(R^*_{-1}, s), \sigma_{\mathrm{ZLB}}(R^*_{-1}, s)\right)$. Then, given the values of $\sigma_x^{(i-1)}(R_{j,-1}, s_m)$ at each grid point and each regime $n \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$, we solve

$$c_{jm\mathrm{NZLB}} = \frac{\beta}{\bar{\gamma}} R^*_{jm\mathrm{NZLB}} \int \left[\frac{\sigma_c^{(i-1)}(R^*_{jm\mathrm{NZLB}}, s')^{-\tau}}{z'\sigma_\pi^{(i-1)}(R^*_{jm\mathrm{NZLB}}, s')}\right] p(s'|s)ds',$$
$$c_{jm\mathrm{ZLB}} = \frac{\beta}{\bar{\gamma}} \int \left[\frac{\sigma_c^{(i-1)}(R^*_{jm\mathrm{ZLB}}, s')^{-\tau}}{z'\sigma_\pi^{(i-1)}(R^*_{jm\mathrm{ZLB}}, s')}\right] p(s'|s)ds',$$

and,

$$0 = \left((1-\nu^{-1}) + \nu^{-1}c_{jmn}^{\tau} - \phi\left(\pi_{jmn} - \bar{\pi}\right)\left[\pi_{jmn} - \frac{1}{2\nu}\left(\pi_{jmn} - \bar{\pi}\right)\right]\right) c_{jmn}^{-\tau} y_{jmn}$$
$$+ \beta\phi \int \left[\sigma_c^{(i-1)}(R^*_{jmn}, s')^{-\tau}\sigma_y^{(i-1)}(R^*_{jmn}, s')\left(\sigma_\pi^{(i-1)}(R^*_{jmn}, s') - \bar{\pi}\right)\sigma_\pi^{(i-1)}(R^*_{jmn}, s')\right] p(s'|s)ds',$$
$$y_{jmn} = \left[g_m^{-1} - \frac{\phi}{2}\left(\pi_{jmn} - \bar{\pi}\right)^2\right]^{-1} c_{jmn},$$
$$R^*_{jmn} = \left(\bar{r}\bar{\pi}\left(\frac{\pi_{jmn}}{\bar{\pi}}\right)^{\psi_1}\left(\frac{y_{jmn}}{y^*}\right)^{\psi_2}\right)^{1-\rho_R} R^{*\rho_R}_{j,-1} e^{\epsilon_{R,m}}.$$

for $(c_{jmn}, \pi_{jmn}, R^*_{jmn}, y_{jmn})$.

A.2.1 Fitting future variables

Given regime-specific expectation functions e_{NZLB}, e_{ZLB} and the policy function $R^* = \sigma_{R^*,\text{NZLB}}(R^*_{-1}, s)$, we use an index function to obtain

$$\begin{aligned}
e_c(R^*_{-1}, s) &= \mathbb{I}_{(R^*<1)} e_{c,\text{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) e_{c,\text{NZLB}}(R^*_{-1}, s), \\
e_\pi(R^*_{-1}, s) &= \mathbb{I}_{(R^*<1)} e_{\pi,\text{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) e_{\pi,\text{NZLB}}(R^*_{-1}, s).
\end{aligned}$$

We define

$$\begin{aligned}
e_{c,\text{NZLB}}(R^*_{-1}, s) &\equiv \frac{\beta}{\bar{\gamma}} R^* \int \left[\frac{\sigma_c(R^*, s')^{-\tau}}{z' \sigma_\pi(R^*, s')} \right] p(s'|s) ds', \\
e_{\pi,\text{NZLB}}(R^*_{-1}, s) &\equiv \beta\phi \int \left[\sigma_c(R^*, s')^{-\tau} \frac{\sigma_y(R^*, s')}{y} \left(\sigma_\pi(R^*, s') - \bar{\pi}\right) \sigma_\pi(R^*, s') \right] p(s'|s) ds', \\
e_{c,\text{ZLB}}(R^*_{-1}, s) &\equiv \beta \int \left[\frac{\sigma_c(R^*, s')^{-\tau}}{\gamma' \sigma_\pi(R^*, s')} \right] p(s'|s) ds', \\
e_{\pi,\text{ZLB}}(R^*_{-1}, s) &\equiv \beta\phi \int \left[\sigma_c(R^*, s')^{-\tau} \frac{\sigma_y(R^*, s')}{y} \left(\sigma_\pi(R^*, s') - \bar{\pi}\right) \sigma_\pi(R^*, s') \right] p(s'|s) ds'.
\end{aligned}$$

Then, given the values of $e_n^{(i-1)}(R_{j,-1}, s_m)$ at each grid point and each regime, we solve

$$\begin{aligned}
& c_{jmn}^{-\tau} = e_{c,n}^{(i-1)}(R^*_{j,-1}, s_m) \\
& 0 = (1 - \nu^{-1}) + \nu^{-1} / e_{c,n}^{(i-1)}(R^*_{j,-1}, s_m) - \phi\left(\pi_{jmn} - \bar{\pi}\right) \left[\pi_{jmn} - \frac{1}{2\nu} \left(\pi_{jmn} - \bar{\pi}\right) \right] \\
& + e_{c,n}^{(i-1)}(R^*_{j,-1}, s_m) e_{\pi,n}^{(i-1)}(R^*_{j,-1}, s_m), \\
& R^*_{jmn} = \left(\bar{r}\bar{\pi} \left(\frac{\pi_{jmn}}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_{jmn}}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R^{*\rho_R}_{j,-1} e^{\epsilon_{R,m}}, \\
& c_{jmn} + \frac{\phi}{2} \left(\pi_{jmn} - \bar{\pi}\right)^2 y_{jmn} = g_m^{-1} y_{jmn}.
\end{aligned}$$

for $(c_{jmn}, \pi_{jmn}, R^*_{jmn}, y_{jmn})$. We also update

$$
\begin{aligned}
e^{(i)}_{c,\mathrm{NZLB}}(R^*_{j,-1}, s_m) &= \frac{\beta}{\bar{\gamma}} R^*_{jm\mathrm{NZLB}} \int \left[\frac{\sigma_c^{(i)}(R^*_{jm\mathrm{NZLB}}, s')^{-\tau}}{z' \sigma_\pi^{(i)}(R^*_{jm\mathrm{NZLB}}, s')} \right] \mu(s'|s_m) ds' \\
e^{(i)}_{\pi,\mathrm{NZLB}}(R^*_{j,-1}, s_m) &= \beta\phi \int \Bigg[\sigma_c^{(i)}(R^*_{jm\mathrm{NZLB}}, s')^{-\tau} \frac{\sigma_y^{(i)}(R^*_{jm\mathrm{NZLB}}, s')}{y_{jm\mathrm{NZLB}}} \\
&\qquad \times \left(\sigma_\pi^{(i)}(R^*_{jm\mathrm{NZLB}}, s') - \bar{\pi} \right) \sigma_\pi^{(i)}(R^*_{jm\mathrm{NZLB}}, s') \Bigg] \mu(s'|s_m) ds', \\
e^{(i)}_{c,\mathrm{ZLB}}(R^*_{j,-1}, s_m) &= \beta \int \left[\frac{\sigma_c^{(i)}(R^*_{jm\mathrm{ZLB}}, s')^{-\tau}}{\bar{\gamma} z' \sigma_\pi^{(i)}(R^*_{jm\mathrm{ZLB}}, s')} \right] \mu(s'|s) ds' \\
e^{(i)}_{\pi,\mathrm{ZLB}}(R^*_{j,-1}, s_m) &= \beta\phi \int \Bigg[\sigma_c^{(i)}(R^*_{jm\mathrm{ZLB}}, s')^{-\tau} \frac{\sigma_y^{(i)}(R^*_{jm\mathrm{ZLB}}, s')}{y_{jm\mathrm{ZLB}}} \\
&\qquad \times \left(\sigma_\pi^{(i)}(R^*_{jm\mathrm{ZLB}}, s') - \bar{\pi} \right) \sigma_\pi^{(i)}(R^*_{jm\mathrm{ZLB}}, s') \Bigg] \mu(s'|s_m) ds',
\end{aligned}
$$

where R^*_{jmn} and y_{jmn} are obtained in the previous step. Note that we interpolate $\sigma^{(i)}(R^*, s')$ (or equivalently $e^{(i-1)}(R^*, s')$). Also, we compute numerical integrals with regard to s'.

A.2.2 Fitting current variables

Given regime-specific expectation functions $v_{\mathrm{NZLB}}, v_{\mathrm{ZLB}}$ and the policy function $R^* = \sigma_{R^*,\mathrm{NZLB}}(R^*_{-1}, s)$, we use an index function to obtain

$$
\begin{aligned}
v_c(R^*_{-1}, s) &= \mathbb{I}_{(R^*<1)} v_{c,\mathrm{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) v_{c,\mathrm{NZLB}}(R^*_{-1}, s), \\
v_\pi(R^*_{-1}, s) &= \mathbb{I}_{(R^*<1)} v_{\pi,\mathrm{ZLB}}(R^*_{-1}, s) + \left(1 - \mathbb{I}_{(R^*<1)}\right) v_{\pi,\mathrm{NZLB}}(R^*_{-1}, s),
\end{aligned}
$$

We define

$$
\begin{aligned}
v_{c,n}(R^*_{-1}, s) &\equiv \frac{\beta}{\bar{\gamma}} \left[\frac{c^{-\tau}}{z\pi} \right], \\
v_{\pi,n}(R^*_{-1}, s) &\equiv \beta\phi \left[c^{-\tau} y \left(\pi - \bar{\pi} \right) \pi \right].
\end{aligned}
$$

for each $n \in \{\mathrm{NZLB}, \mathrm{ZLB}\}$. Then, given the function $v^{(i-1)}(R, s')$ and the value of $\sigma_n^{(i-1)}(R_{j,-1}, s_m)$ at each grid point and each regime, we solve

$$
\begin{aligned}
c_{jm\mathrm{NZLB}} &= \left\{ R^* \int v_c^{(i-1)}(\sigma^{(i-1)}_{R^*,\mathrm{NZLB}}(R^*_{j,-1}, s_m), s') p(s'|s_m) ds' \right\}^{-1/\tau}, \\
c_{jm\mathrm{ZLB}} &= \left\{ \int v_c^{(i-1)}(\sigma^{(i-1)}_{R^*,\mathrm{ZLB}}(R^*_{j,-1}, s_m), s') p(s'|s_m) ds' \right\}^{-1/\tau},
\end{aligned}
$$

and

$$
\begin{aligned}
0 =& \left((1-\nu^{-1}) + \nu^{-1} c_{jmn}^{\tau} - \phi\left(\pi_{jmn} - \bar{\pi}\right) \left[\pi_{jmn} - \frac{1}{2\nu}\left(\pi_{jmn} - \bar{\pi}\right) \right] \right) c_{jmn}^{-\tau} \sigma_{y,n}^{(i-1)}(R_{j,-1}^{*}, s_m) \\
&+ \beta\phi \left\{ \int_{s'|s_m} v_c^{(i-1)}(\sigma_{R^*,n}^{(i-1)}(R_{j,-1}^{*}, s_m), s') p(ds'|s_m) \right\}, \\
y_{jmn} =& \left[g_m^{-1} - \frac{\phi}{2}\left(\pi_{jmn} - \bar{\pi}\right)^2 \right]^{-1} c_{jmn}, \\
R_{jmn}^{*} =& \left(\bar{r}\bar{\pi} \left(\frac{\pi_{jmn}}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_{jmn}}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R_{j,-1}^{*\rho_R} e^{\epsilon_{R,m}}.
\end{aligned}
$$

for $(c_{jmn}, \pi_{jmn}, R_{jmn}^{*}, y_{jmn})$. We use a successive approximation $R^* = \sigma_{R^*,n}^{(i-1)}(R_{j,-1}^{*}, s_m)$ and $y = \sigma_{y,n}^{(i-1)}(R_{j,-1}^{*}, s_m)$.

Note that $v^{(i-1)}(R^*, s')$ are composite functions of the ones in NZLB regime and ZLB regime indexed by the value of $\sigma_{R^*,\mathrm{NZLB}}(R^*, s')$. Specifically, we have

$$
\begin{aligned}
&\int v(R^*, s') p(s'|s_m) ds' \\
=& \int \left[\mathbb{I}_{(\sigma_{R^*,\mathrm{NZLB}}(R^*,s')<1)} v_{\mathrm{ZLB}}(R^*, s') + \left(1 - \mathbb{I}_{(\sigma_{R^*,\mathrm{NZLB}}(R^*,s')<1)}\right) v_{\mathrm{NZLB}}(R^*, s') \right] p(s'|s_m) ds' \\
=& \int \left[\mathbb{I}_{(\sigma_{R^*,\mathrm{NZLB}}(R^*,s')<1)} \left(v_{\mathrm{ZLB}}(R^*, s') - v_{\mathrm{NZLB}}(R^*, s') \right) + v_{\mathrm{NZLB}}(R^*, s') \right] p(s'|s_m) ds' \\
=& \Pr(\sigma_{R^*,\mathrm{NZLB}}(R^*, s') < 1) \left(\int v_{\mathrm{ZLB}}(R^*, s') p(s'|s_m) ds' - \int v_{\mathrm{NZLB}}(R^*, s') p(s'|s_m) ds' \right) \\
&+ \int v_{\mathrm{NZLB}}(R^*, s') p(s'|s_m) ds'.
\end{aligned}
$$

We assume that $\mathbb{I}_{(\sigma_{R,\mathrm{NZLB}}(R^*,s')<1)}$ and $\int \left\{ v_{\mathrm{ZLB}}(R^*, s') - v_{\mathrm{NZLB}}(R^*, s') \right\} p(s'|s_m) ds'$ are orthogonal to each other. Also, to compute $\Pr(\sigma_{R^*,\mathrm{NZLB}}(R^*, s') < 1)$, we approximate $\sigma_{R^*,\mathrm{NZLB}}(R^*, s')$ up to first order by truncating higher-order terms

$$
\begin{aligned}
\hat{\sigma}_{R^*,\mathrm{NZLB}}(R^*, s') =& \theta_0 + \theta_{R^*} R^* + \theta_g g' + \theta_z z' + \theta_r \epsilon_r' \\
=& \theta_0 + \theta_{R^*} R^* + \theta_g \rho g + \theta_z \rho z + \theta_g \epsilon_g' + \theta_z \epsilon_z' + \theta_r \epsilon_r'
\end{aligned}
$$

Then we have

$$
\Pr(\hat{\sigma}_{R^*,\mathrm{NZLB}}(R^*, s') < 1) = \Phi\left(x < 1 - (\theta_0 + \theta_{R^*} R^* + \theta_g \rho g + \theta_z \rho z) \right)
$$

where $\Phi(\bullet)$ is the cumulative distribution function of $x = \theta_g \epsilon_g' + \theta_z \epsilon_z' + \theta_r \epsilon_r'$ which follows a normal distribution[19].

We also update

$$v_{c,n}^{(i)}(R_{j,-1}^{*}, s_m) = \frac{\beta}{\bar{\gamma}} \left[\frac{c_{jmn}^{-\tau}}{z_m \pi_{jmn}} \right],$$
$$v_{\pi,n}^{(i)}(R_{j,-1}^{*}, s_m) = \beta\phi \left[c_{jmn}^{-\tau} y_{jmn} \left(\pi_{jmn} - \bar{\pi}\right) \pi_{jmn} \right],$$

where $c_{jmn}, \pi_{jmn}, y_{jmn}$ are obtained in the previous step.

References

[1] TBA

[19] We can possibly use higher order approximation for σ_{R^*} to calculate the probability at the cost of more complicated distribution for x.