Tutorial Session

Solution and Estimation Methods for Nonlinear DSGE Models

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Introduction

- Dynamic stochastic general equilibrium (DSGE) models have been one of the primary tools in macroeconomic analysis.
  - Microfoundation
  - Immune to the Lucas Critique
  - Suitable for welfare analysis
  - Rational expectations
- Following the development of Bayesian estimation techniques, many economists have estimated DSGE models.
  - Cross-equation restrictions
  - Avoid weak-instrument problems that would arise in GMM estimation.
  - Priors for structural parameters
    - Deal with identification issues.
    - Estimate parameters with bounded domains.
    - Make the likelihood function well-shaped.

Introduction

- While DSGE models are inherently nonlinear, the nonlinearities are often small.
- Equilibrium conditions are linearly approximated.
  - Rational expectations solution (policy function) can be obtained using a standard solution method.
    - Blanchard and Kahn (1980); Sims (2002)
  - Likelihood can be evaluated using the Kalman filter.
    - Additional assumption: Normality of shocks

Steps for Estimating Linearized DSGE Models

- DSGE model:

$$\Gamma_0(\theta)s_t = \Gamma_1(\theta)s_{t-1} + \Psi_0(\theta)\varepsilon_t + \Pi_0(\theta)\eta_t$$

- Rational expectations solution (state transition equations):

$$s_t = \Phi_1(\theta)s_{t-1} + \Phi_\varepsilon(\theta)\varepsilon_t \qquad \varepsilon_t \sim N(0, \Sigma_\varepsilon)$$

- Relation between model variables and data (observation equations):

$$y_t = \Psi_0(\theta) + \Psi_1(\theta)s_t + u_t \qquad u_t \sim N(0, \Sigma_u)$$

- State transition equations & observation equations $\Rightarrow$ Kalman filter $\Rightarrow$ Likelihood function: $L(\theta|Y)$
- Prior distribution: $p(\theta)$
- Bayes' Theorem $\Rightarrow$ Posterior distribution

$$p\left(\theta|Y\right) \propto L(\theta|Y)p\left(\theta\right)$$

Needs for Nonlinear DSGE models

- However, linearized DSGE models cannot deal with features that generate pronounced nonlinearities:
  - Occasionally binding constraints
  - Stochastic volatilities
  - Markov switching coefficients
  - Asymmetric adjustment costs
- Nonlinear solution and estimation methods are required.

Session Plan

1. Sunakawa: How to solve nonlinear DSGE models
2. Hirose: How to estimate nonlinear DSGE models

- A survey paper will be published in *Japanese Economic Review*.

1. How to solve nonlinear DSGE models

Plan for the first part

- What is the time iteration method?
  - Our explanation tends to be intuitive and self-contained to some extent, but not necessarily be strict.
- Applications to New Keynesian models
- MATLAB codes available at `https://github.com/tkksnk/NKZLB`

What is the time iteration method?

- Neoclassical growth model
  - Interpolation
  - Nonlinear optimization (or how to avoid it)
- Stochastic neoclassical growth model
  - Two-dimensional interpolation
  - Nonlinear optimization (or how to avoid it)
  - Numerical integration (or how to avoid it)

Neoclassical growth model

- We consider an example of neoclassical growth model. An individual maximizes life-time utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} \leq f(k_t).$$

where $u(\cdot)$ and $f(\cdot)$ satisfy standard conditions.

- The first-order necessary condition is given by

$$u_c(c_t) = \beta u_c(c_{t+1}) f_k(k_{t+1})$$

where $u_c(\cdot)$ denotes the derivative of u wrt c and $f_k(\cdot)$ denotes the derivative of f wrt k.

Collman operator

- There is a mapping $\sigma = K\sigma$ that solves

$$u_c(c) = \beta u_c\left(\sigma(f(k) - c)\right) f_k(f(k) - c)$$

for $c = \sigma(k)$. σ is called policy function.

  - Note that $k' = f(k) - c$ and $c' = \sigma(k') = \sigma(f(k) - c)$.
  - Bellman operator

$$V(k) = \max_{c \in (0, f(k)]} \left\{u(c) + \beta V\left(f(k) - c\right)\right\}.$$

    is to solve the Bellman equation, whereas the Collman operator is helpful to solve the Euler equation.

Collman operator, cont'd

- Collman (1990) proves the existence of the fixed point of K in a stochastic neoclassical growth model with distortionary tax.
    - Greenwood and Huffman (1995) extend it to several cases. Also see Richter, Throckmorton and Walker (2014) and Sargent and Stachurski (2018).

Time iteration: Algorithm

- Time iteration is a method to solve the Collman operator.
- The time iteration method takes the following steps:
  1. Make an initial guess for the policy function $\sigma^{(0)}$.
  2. Given the policy function previously obtained $\sigma^{(i-1)}$ (i is an index for the number of iteration), solve

$$u_c(c) = \beta u_c\left(\sigma^{(i-1)}(f(k)-c)\right) f_k(f(k)-c)$$

     for c.
  3. Update the policy function by setting $c = \sigma^{(i)}(k)$.
  4. Repeat 2-3 until $\left\|\sigma^{(i)} - \sigma^{(i-1)}\right\|$ is small enough.

Optimization and interpolation

- We discretize the state space of k by grid points:

$$k_j \in \{k_1, k_2, \cdots, k_N\},$$

where j is an index for grid points.

- Then, given $\sigma^{(i-1)}$, we solve

$$\tilde{R}(c; k_j, \sigma^{(i-1)}) \equiv -u_c(c) + \beta u_c\left(\sigma^{(i-1)}(f(k_j) - c)\right) f_k(f(k_j) - c) = 0$$

for c at each grid point k_j. That is, the residual function is equal to zero at each grid point. This is called collocation.

- We need to know the value of $\sigma^{(i-1)}(f(k_j) - c)$, which may be off the grid points.

Optimization and interpolation, cont'd

- More generally, we want to:
- solve $f(x) = 0$ for x (optimization),
- when we know only the values of $f(x_j)$ at $x_j \in \{x_1, ..., x_N\}$ (interpolation).

Polynomial function

- The policy functions are approximated by a higher-order polynomial function

$$\hat{\sigma}(x;\boldsymbol{\theta}) = \theta_0 + \theta_1 x + \theta_2 x^2 + \cdots + \theta_{N-1} x^{N-1}.$$

- We need to know the values of $\sigma(x)$ (at least) at N grid points to fit the polynomial.

- Polynomial function is a global approximation.

Chebyshev polynomial

- We define the basis functions $T(x) : [-1, 1] \to [-1, 1]$, and have an univariate polynomial

$$\hat{\sigma}(x; \boldsymbol{\theta}) = \theta_0 + \theta_1 T_1(x) + \theta_2 T_2(x) + \cdots + \theta_{N-1} T_{N-1}(x).$$

- An example of $T(x)$:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ &\vdots \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x). \end{aligned}$$

  where $x \in [-1, 1]$. These are called Chebyshev polynomials, or Chebyshev basis functions.

Chebyshev collocation points

- The polynomial is evaluated at the collocation points
- Chebyshev zeros:

$$x_j = \cos\left(\frac{(2j+1)\pi}{2(N-1)}\right) \text{ for } j = 0, 1, ..., N-1.$$

- Chebyshev extrema:

$$x_j = \cos\left(\frac{j\pi}{N-1}\right) \text{ for } j = 0, 1, ..., N-1.$$

Transforming grid points

- For a general function which takes a value $k_j \in [k_1, k_N]$ at each grid point, we have to transform k_j to $x_j \in [-1, 1]$ to (or x_j to k_j) by applying

$$x_j = \varphi(k_j) = \frac{2(k_j - k_1)}{k_N - k_1} - 1,$$

or

$$k_j = \varphi^{-1}(x_j) = k_1 + 0.5(1 + x_j)(k_N - k_1).$$

Fitting polynomial

- Once we have the collocation points $\{x_j\}$ and the function values $\{\sigma(x_j)\}$ evaluated at x_j for $j = 1, 2, ..., N$, we can fit $\hat{\sigma}(x; \theta)$ to the data to obtain θ.

$$\begin{bmatrix} \sigma(x_1) \\ \sigma(x_2) \\ \\ \sigma(x_N) \end{bmatrix} = \begin{bmatrix} 1 & T_1(x_1) & T_2(x_1) & T_{N-1}(x_1) \\ 1 & T_1(x_2) & T_2(x_2) & T_{N-1}(x_2) \\ & & & \\ 1 & T_1(x_N) & T_2(x_1) & T_{N-1}(x_N) \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \\ \theta_{N-1} \end{bmatrix},$$

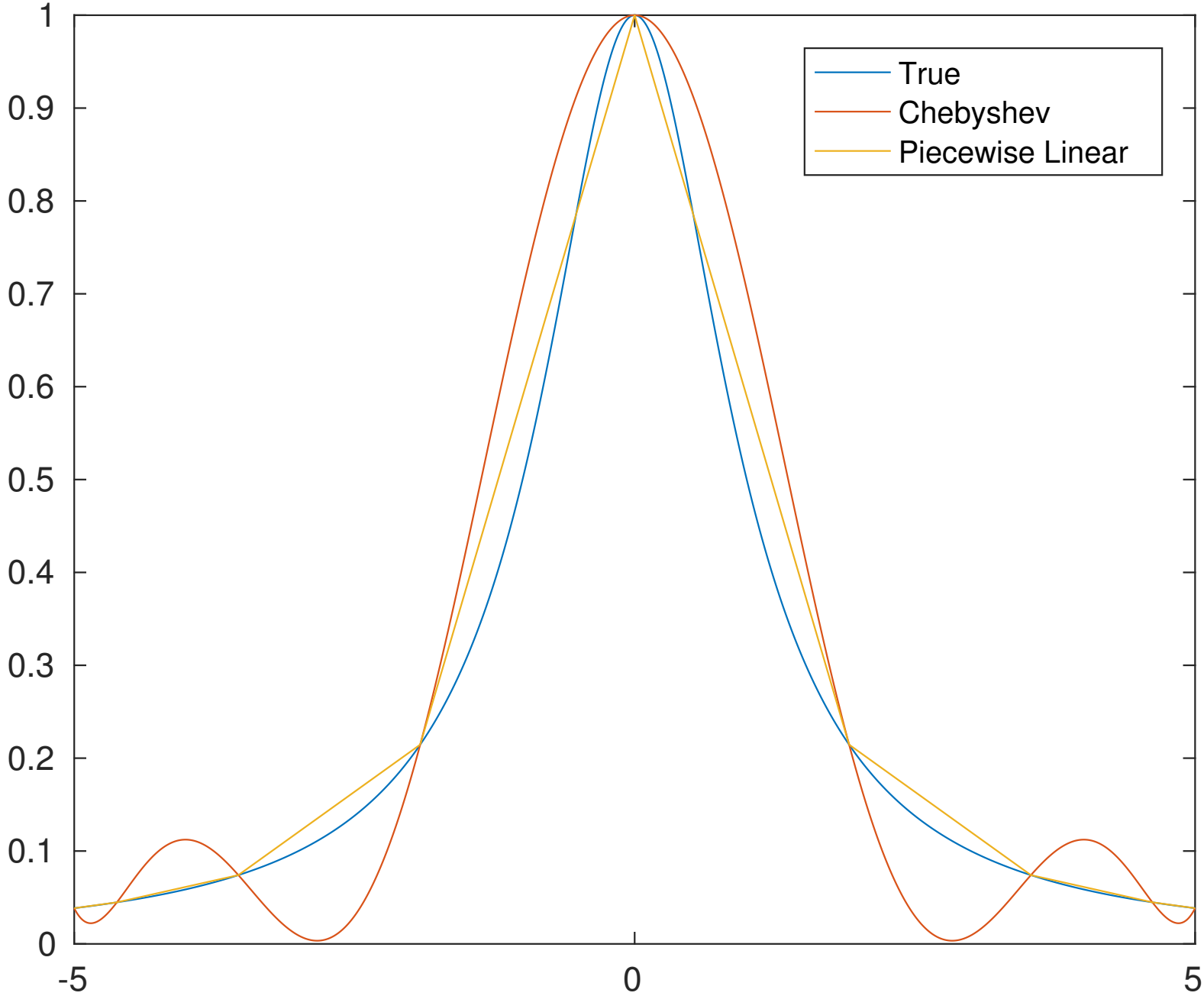
  or

$$\sigma(\boldsymbol{x}) = T(\boldsymbol{x})\boldsymbol{\theta}.$$

  Then we have $\boldsymbol{\theta} = T(\boldsymbol{x})^{-1}\sigma(\boldsymbol{x})$. $T(\boldsymbol{x})$ needs to be nonsingular.

- The basis functions with Chebyshev zeros/extrema satisfy orthogonality property. That is, each column of $T(\boldsymbol{x})$ is uncorrelated to each other.

Example: $1/(1+x^2)$ with $N = 9$



Nonlinear optimization

- Given $\hat{\sigma}^{(i-1)}(k;\boldsymbol{\theta}) = T(\varphi(k))\boldsymbol{\theta}$, we solve a nonlinear equation

  $$\tilde{R}(c;k_j,\hat{\sigma}^{(i-1)}) \approx -u_c(c)+\beta u_c\left(\hat{\sigma}^{(i-1)}(f(k_j)-c;\boldsymbol{\theta})\right) f_k(f(k_j)-c) = 0$$

  for c at each grid point k_j.

- We use Newton's method to solve the nonlinear equation. For example, Matlab's command `fsolve or` Chris Sims' `csolve` does such a job.

- But, such a nonlinear optimization can be costly when the number of grid points and/or the number of nonlinear equations is large.

Parameterized expectation

- By applying a version of the parameterized expectation algorithm (PEA), we can avoid solving nonlinear equations (Maliar and Maliar, 2015; Gust et al., 2017, Hirose and Sunakawa, 2015; 2017).
  - Marcet (1988) uses a stochastic approach based on Monte Carlo simulations.
  - Christiano and Fisher (2000) propose a non-stochastic approach called PEA collocation.
  - Endogenous grid-point method (EGM) is another popular method to avoid nonlinear optimization.

PEA collocation

- There are two ways for applying PEA collocation (Christiano and Fisher, 2000):
  - One is to fit polynomials to future variables (Marcet, 1988).
  - The other is to fit polynomials to current variables (Williams and Wright, 1982a; 1982b; 1984).

Fitting future variables

- We define
$$e(k) \equiv \beta u_c\left(c'\right) f_k(k').$$
- Then, given the values of $e^{(i-1)}(k_j)$ at each grid point, we have
$$c = u_c^{-1}(e^{(i-1)}(k_j)),$$
$$k' = f(k_j) - c,$$

  and an intermediate policy function $c = \sigma^{(i)}(k_j)$. Note that we don't have to solve the nonlinear equation here.

Fitting future variables, cont'd

- We also update

$$e^{(i)}(k_j) = \beta u_c \left(\sigma^{(i)}(k')\right) f_k(k')$$

where k' is obtained in the previous step.

  - Note that $c' = \sigma^{(i)}(k')$, or equivalently $e^{(i-1)}(k')$, needs to be interpolated here. That is,

$$\begin{aligned} c' =& \sigma^{(i)}(k') \\ \approx& u_c^{-1}(\hat{e}^{(i-1)}(k'; \boldsymbol{\theta})) \end{aligned}$$

where $\hat{e}^{(i-1)}(k; \boldsymbol{\theta}) = \theta_0 + \theta_1 T_1(k) + \theta_2 T_2(k) + \cdots + \theta_{N-1} T_{N-1}(k)$. $\boldsymbol{\theta}$ is obtained by fitting the polynomial to the data of future variables $e^{(i-1)}(k_j)$ at each grid point k_j.

What is the time iteration method?

- Neoclassical growth model
  - Univariate Interpolation
  - Nonlinear optimization (or how to avoid it)
- Stochastic neoclassical growth model
  - Multivariate interpolation
  - Nonlinear optimization (or how to avoid it)
  - Numerical integration (or how to avoid it)

Stochastic neoclassical growth model

- Now we extend the earlier example with stochastic technology. An individual maximizes expected life-time utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$c_t + k_{t+1} \leq f(k_t, z_t).$$

$\mathbb{E}_0$ is expectation operator at time 0.

- z_t follows an AR(1) process

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \sigma_\epsilon^2)$$

Collman operator

- The first-order necessary condition is given by

$$u_c(c_t) = \beta \mathbb{E}_t \left\{ u_c(c_{t+1}) f_k(k_{t+1}, z_{t+1}) \right\}.$$

- There is a mapping $\sigma = K\sigma$ that solves

$$u_c(c) = \beta \int u_c \left(\sigma(f(k,z) - c, z') \right) f_k(f(k,z) - c, z') p(z'|z) dz'$$

for $c = \sigma(k, z)$.

Time iteration

- The time iteration method takes the following steps:
  1. Make an initial guess for the policy function $\sigma^{(0)}$.
  2. Given the policy function previously obtained $\sigma^{(i-1)}$, solve

     $$u_c(c) = \beta \int u_c\left(\sigma^{(i-1)}(f(k,z)-c,z')\right) f_k(f(k,z)-c,z')p(z'|z)dz'$$

     for c.
  3. Update the policy function by setting $c = \sigma^{(i)}(k,z)$.
  4. Repeat 2-3 until $\left\|\sigma^{(i)} - \sigma^{(i-1)}\right\|$ is small enough.

Optimization, interpolation and integration

- We discretize the state space of (k, z) by grid points:

$$k_j \in \{k_1, k_2, \cdots, k_N\}, \quad z_m \in \{z_1, z_2, \cdots, z_N\},$$

where (j, m) is an index for the set of grid points.

- Then, given $\sigma^{(i-1)}$, we solve

$$\begin{aligned}
&\tilde{R}(c; k_j, z_m, \sigma^{(i-1)}) \\
\approx& -u_c(c) \\
&+\beta \int \left[u_c\left(\sigma^{(i-1)}(f(k_j, z_m) - c, z')\right) f_k(f(k_j, z_m) - c, z') p(z'|z_m) \right] dz' \\
=& 0
\end{aligned}$$

for c at each grid point (k_j, z_m) (optimization).

Optimization, interpolation and integration, cont'd

- We need to know the value of $\sigma^{(i-1)}(f(k_j, z_m) - c, z')$, which may be off the grid points (interpolation).
  - $\sigma(k, z)$ is a two-dimensional object, which can be approximated by two-dimensional Chebyshev polynomial.
- Also, we compute an integral with regard to z' for the next period's expectation (integration).

2-D Chebyshev polynomial

- The policy functions are approximated by basis functions. For example, if $N_x = N_y = 3$,

$$\begin{aligned}\hat{\sigma}(x, y; \boldsymbol{\theta}) = \theta_{0,0} &+ \theta_{1,0}T_1(x) + \theta_{2,0}T_2(x) + \theta_{0,1}T_1(y) + \theta_{0,2}T_2(y) \\ &+ \theta_{1,1}T_1(x)T_1(y) + \theta_{1,2}T_1(x)T_2(y) \\ &+ \theta_{2,1}T_2(x)T_1(y) + \theta_{2,2}T_2(x)T_2(y)\end{aligned}$$

  There are 9 coefficients, so we need at least $N = N_x N_y = 9$ collocation points.

- Chebyshev extrema is used as collocation points

$$\begin{aligned}(x, y) \in \{&(0,0), (-1,0), (1,0), (0,-1), (0,1), \\ &(-1,-1), (1,-1), (-1,1), (1,1)\}.\end{aligned}$$

Fitting 2-D Chebyshev polynomial

- Once we have the collocation points $\{x_i, y_j\}$ and the function values $\{\sigma(x_i, y_j)\}$ evaluated at (x_i, y_j) for $i = 1, 2, ..., N_x$ and $j = 1, 2, ..., N_y$, we can fit $\hat{\sigma}(x, y; \boldsymbol{\theta})$ to the data to obtain $\boldsymbol{\theta}$.

- For example, if $N_x = N_y = 2$,

$$\begin{bmatrix} \sigma(x_1, y_1) \\ \sigma(x_2, y_1) \\ \sigma(x_1, y_2) \\ \sigma(x_2, y_2) \end{bmatrix} = \begin{bmatrix} 1 & T_1(x_1) & T_1(y_1) & T_1(x_1)T_1(y_1) \\ 1 & T_1(x_2) & T_1(y_1) & T_1(x_2)T_1(y_1) \\ 1 & T_1(x_1) & T_1(y_2) & T_1(x_1)T_1(y_2) \\ 1 & T_1(x_2) & T_1(y_2) & T_1(x_2)T_1(y_2) \end{bmatrix} \begin{bmatrix} \theta_{0,0} \\ \theta_{1,0} \\ \theta_{1,0} \\ \theta_{1,1} \end{bmatrix}$$

  or $\sigma(\boldsymbol{x}, \boldsymbol{y}) = T(\boldsymbol{x}, \boldsymbol{y})\boldsymbol{\theta}$. Then we have $\boldsymbol{\theta} = T(\boldsymbol{x}, \boldsymbol{y})^{-1}\sigma(\boldsymbol{x}, \boldsymbol{y})$.

- More generally, we have a tensor product for $T(\boldsymbol{x}, \boldsymbol{y}) = T(\boldsymbol{x}) \otimes T(\boldsymbol{y})$. The total number of grid points is exponentially increasing in the number of varialbles.

Approximating stochastic process

- How to compute an integral with regard to z'?
  - Tauchen's method (Rouwenhorst's method): AR(1) process is approximated by a Markov chain.
  - Gaussian Quadrature: The quadrature nodes $\{x_i\}_{i=1}^M$ and quadrature weights $\{w_i\}_{i=1}^M$ approximate

$$\int f(x)w(x)dx \approx \sum w_i f(x_i).$$

- The total number of quadrature points is exponentially increasing in the number of exogeneous varialbles.

PEA collocation

- There are two ways for applying PEA collocation (Christiano and Fisher, 2000):
  - One is to fit polynomials to future variables (Marcet, 1988).
  - The other is to fit polynomials to current variables (Williams and Wright, 1982a; 1982b; 1984)
  - In the latter approach, we can also avoid computing numerical integration in the expectation terms by precomputation technique of Judd, Maliar, Maliar and Tsener (2017).

Numerical examples I

- Solve the stochastic neoclassical model by the time iteration method
  - Interpolation: Chebyshev polynomial (either $N_d = 3$ or 5 for each $d \in \{k, z\}$).
  - Optimization:
    - Chris Sims' csolve is used in Time iteration with Newton's method (TI).
    - No optimization is required in PEA collocation.
  - Integration:
    - Gaussian-Hermite quadrature is used in TI and PEA collocation with fitting future variables (future PEA).
    - Precomputation technique in Judd et al. (2017) is used in PEA collocation with fitting current variables (current PEA).

Euler equation errors

- Accuracy and computation speed are compared. Look at the Euler equation errors:

$$\mathcal{E}(k,z) \equiv 1-\beta \int \left\{ \left(\frac{\sigma_c(k',z')}{\sigma_c(k,z)} \right)^{-\tau} \left(1-\delta+\alpha z' k'^{\alpha-1}\right) \right\} p(z'|z)dz'$$

  where $k' = f(k,z) - \sigma_c(k,z)$.

Summary

(N_d, τ)	TI			future PEA			current PEA		
	L_1	L_∞	CPU	L_1	L_∞	CPU	L_1	L_∞	CPU
$(3, 1.0)$	-5.12	-4.60	4.03	-4.23	-3.69	0.04	-3.13	-2.44	0.02
$(5, 1.0)$	-7.08	-6.72	9.76	-5.92	-5.59	0.09	-3.13	-2.44	0.04
$(3, 2.0)$	-4.82	-4.35	0.86	-3.99	-3.53	0.03	-2.95	-2.26	0.01
$(5, 2.0)$	-6.76	-6.45	2.88	-5.63	-5.36	0.11	-2.96	-2.27	0.04
$(3, 5.0)$	-4.48	-3.87	0.62	-3.57	-2.88	0.05	-2.67	-1.99	0.02
$(5, 5.0)$	-6.43	-5.38	1.91	-5.10	-3.90	0.15	-2.69	-2.00	0.05

Notes: L_1 and L_∞ are, respectively, the average and maximum of absolute Euler errors (in log 10 units) on a 10,000 period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds).

Applications to New Keynesian models

- We solve small-scale nonlinear New Keynesian DSGE model with
  - Smolyak's method with sparse grid points
  - Simulation-based method with EDS grid
  - The techniques we mentioned earlier (PEA collocation and precomputing integrals) are also applied.

Time iteration in the New Keynesian literature

- Time iteration is a popular method to solve nonlinear New Keynesian models.
  - We have to look at the decentralized economy, as the second welfare theorem fails to hold.
  - An incomplete list includes: Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez, 2015; Maliar and Maliar, 2015; Gavin, Keen, Richter, and Throckmorton, 2015; Gust, Herbst, López-Salido, and Smith, 2017; Iiboshi, Ueda, and Shintani, 2018; Nakata, 2016a, 2016b; Hills, Nakata, and Schmitt, 2016; Dennis, 2016; Ngo, 2014; Hirose and Sunakawa, 2015, 2017; Hills, Nakata, and Sunakawa, 2018.

A small scale New Keynesian DSGE model

- The example here is taken from An and Schorfheide (2007) and Herbst and Schorfheide (2015).
- The model economy consists of
  - Final-good and intermediate-good producing firms
  - Households
  - Monetary and fiscal authorities
- Prices are sticky due to Rotemberg-type (1982) adjustment cost.

Model overview

- Equilibrium conditions (after detrending):

$$1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{R_t}{\gamma_{t+1} \pi_{t+1}} \right]$$

$$0 = (1 - \nu^{-1}) + \nu^{-1} \chi_H c_t^{\tau} - \phi (\pi_t - \bar{\pi}) \left[\pi_t - \frac{1}{2\nu} (\pi_t - \bar{\pi}) \right]$$

$$+ \beta \phi \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{y_{t+1}}{y_t} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right],$$

$$R_t^* = \left(r \bar{\pi} \left(\frac{\pi}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y_t}{y_t^*} \right)^{\psi_2} \right)^{1-\rho_R} R_{t-1}^{*\rho_R} e^{\epsilon_{R,t}},$$

$$c_t + \frac{\phi}{2} (\pi_t - \pi)^2 y_t = g_t^{-1} y_t,$$

$$R_t = \max \{ R_t^*, 1 \}.$$

There are 5 equations and 5 endogenous variables $\{c_t, \pi_t, R_t^*, R_t, y_t\}$ and 3 exogenous variables $\{\gamma_t, g_t, \epsilon_{R,t}\}$. The natural level of output is given by $y_t^* = (1 - \nu)^{1/\tau} g_t$.

Model overview, cont'd

- A_t has a deterministic trend $\bar{\gamma}$ and a shock to the trend z_t such as $\ln \gamma_t \equiv \ln(A_t/A_{t-1}) = \ln \bar{\gamma} + \ln z_t$. Also, $\{z_t, g_t\}$ follow

$$\ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t},$$
$$\ln g_t = (1 - \rho_g) \ln \bar{g} + \rho_g \ln g_{t-1} + \epsilon_{g,t}.$$

Collman operator

- The mapping $\sigma = K\sigma$ solves

$$0 = -c^{-\tau} + \beta R \int \left[\frac{\sigma_c(R, s')^{-\tau}}{\gamma' \sigma_\pi(R, s')} \right] p(s'|s) ds',$$

$$0 = \left((1 - \nu^{-1}) + \nu^{-1} c^{\tau} - \phi (\pi - \bar{\pi}) \left[\pi - \frac{1}{2\nu} (\pi - \bar{\pi}) \right] \right) c^{-\tau} y$$

$$+ \beta \phi \int \left[\sigma_c(R, s')^{-\tau} \sigma_y(R, s') (\sigma_\pi(R, s') - \bar{\pi}) \sigma_\pi(R, s') \right] p(s'|s) ds',$$

$$R = \left(r \bar{\pi} \left(\frac{\pi}{\bar{\pi}} \right)^{\psi_1} \left(\frac{y}{y^*} \right)^{\psi_2} \right)^{1-\rho_R} R_{-1}^{\rho_R} e^{\epsilon_R},$$

$$c + \frac{\phi}{2} (\pi - \bar{\pi})^2 y = g^{-1} y.$$

$$R = \max \{ R^*, 1 \},$$

for c, π, R^*, R, y.

Time iteration

- The time iteration method takes the following steps:
  1. Make an initial guess for the policy function $\sigma^{(0)}$.
  2. Given the policy function previously obtained $\sigma^{(i-1)}$, solve the relevant equations for (c, π, R^*, y).
  3. Update the policy function by setting $c = \sigma_c^{(i)}(R^*_{-1}, s)$, $\pi = \sigma_\pi^{(i)}(R^*_{-1}, s)$, $R^* = \sigma_{R^*}^{(i)}(R^*_{-1}, s)$, and $y = \sigma_y^{(i)}(R^*_{-1}, s)$.
  4. Repeat 2-3 until $\left\|\sigma^{(i)} - \sigma^{(i-1)}\right\|$ is small enough.

Optimization, interpolation, and integration

- Here, we need to solve the system of nonlinear equations (optimization).
  - PEA collocation can also be used here to avoid costly nonlinear optimization.
- Also, we need to evaluate the function $\sigma(R^*, s')$ off the grid points (interpolation).
  - $\sigma(R^*_{-1}, s)$ is a high dimensional object (our model have 4 state variables, $(R^*_{-1}, s) = (R^*_{-1}, \gamma, g, \epsilon_R)$), which can be dealt with Smolyak's method with sparse grid points (Fernandez-Villaverde et al., 2015) or simulation based method with EDS grid (Maliar and Maliar, 2015).
- We compute an integral wrt s' for the next period's expectation (integration).
  - Precomputing integrals can also be applied here to avoid numerical integration.

Smolyak's method

- We introduce Smolyak's (1963) method with sparse grid points to handle such a high-dimensional object.
  - Krueger and Kubler (2004) first introduced Smolyak sparse grid points to solve heterogeneous OLG models with aggregate uncertainty.
  - Applications of Smolyak's method to New Keynesian models are found in Fernandez-Villaverde et al., (2015), Gust et al. (2017), Hirose and Sunakawa (2015; 2017).
- We look at simple cases with second-order polynomials with $N_d = 3$ for each dimension.
  - Cases with higher-order polynomials (for example $N_d = 5$ or 9) are a bit more complicated but manageable. See Judd, Maliar, Maliar and Valero (2014).

Smolyak's method: Simple cases

- The policy functions are approximated by basis functions. For example, if we use second-order polynomials and $N_x = N_y = 3$,

$$\hat{\sigma}(x, y; \boldsymbol{\theta}) = \theta_{0,0} + \theta_{1,0} T_1(x) + \theta_{2,0} T_2(x) + \theta_{0,1} T_1(y) + \theta_{0,2} T_2(y).$$

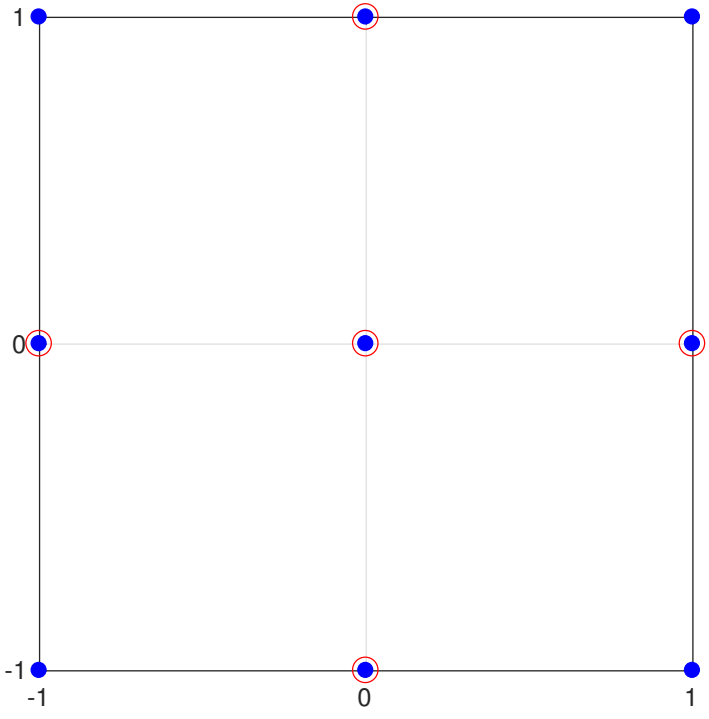
  There are 5 coefficients, so we need 5 collocation points. We have just eliminated all the cross terms!

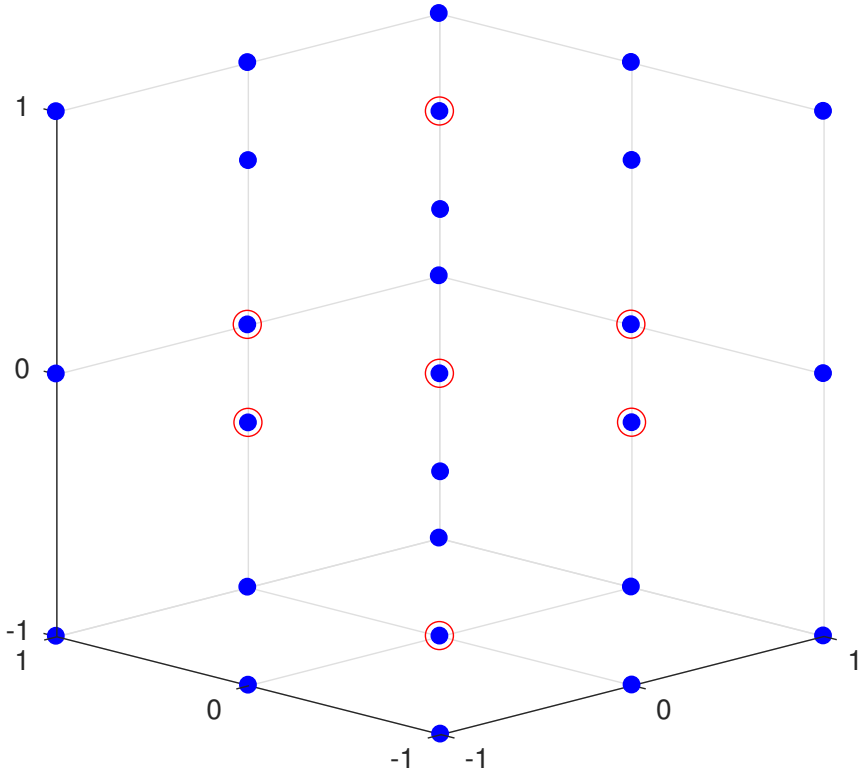
Simple cases with second-order polynomials

- Chebyshev extrema is used as collocation points:

$$N_d = 2: \quad (x, y) \in \{(0,0), (1,0), (-1,0), (0,1), (0,-1)\}$$

$$N_d = 3: \quad (x, y, z) \in \begin{array}{c}\{(0,0,0), (1,0,0), (-1,0,0),\\ (0,1,0), (0,-1,0), (0,0,1), (0,0,-1)\}\end{array}$$





Smolyak's method: Simple cases, cont'd

- The total number of the grid points is $1 + 2N_d$, whereas it is 3^{N_d} with the standard Chebyshev polynomials.

N_d	$1 + 2N_d$	3^{N_d}
2	5	9
3	7	27
4	9	81
5	11	243
⋮	⋮	⋮
10	21	59,049
20	41	3,486,784,401

Simulation-based method

- We solve for the policy functions on simulated grid points based on ergodic distribution of the state variables.
  - Judd, Maliar and Maliar (2011) and Maliar and Maliar (2015, MM hereafter) developed simulation-based method, based on the original work of Marcet's (1988) parameterized expectation algorithm.
  - Applications to New Keynesian models are found in Maliar and Maliar (2015), Lepetuyk, Maliar, and Maliar (2017), Aruoba, Cuba-borda, and Schorfheide (2018), and Hills, Nakata, and Sunakawa (2018).

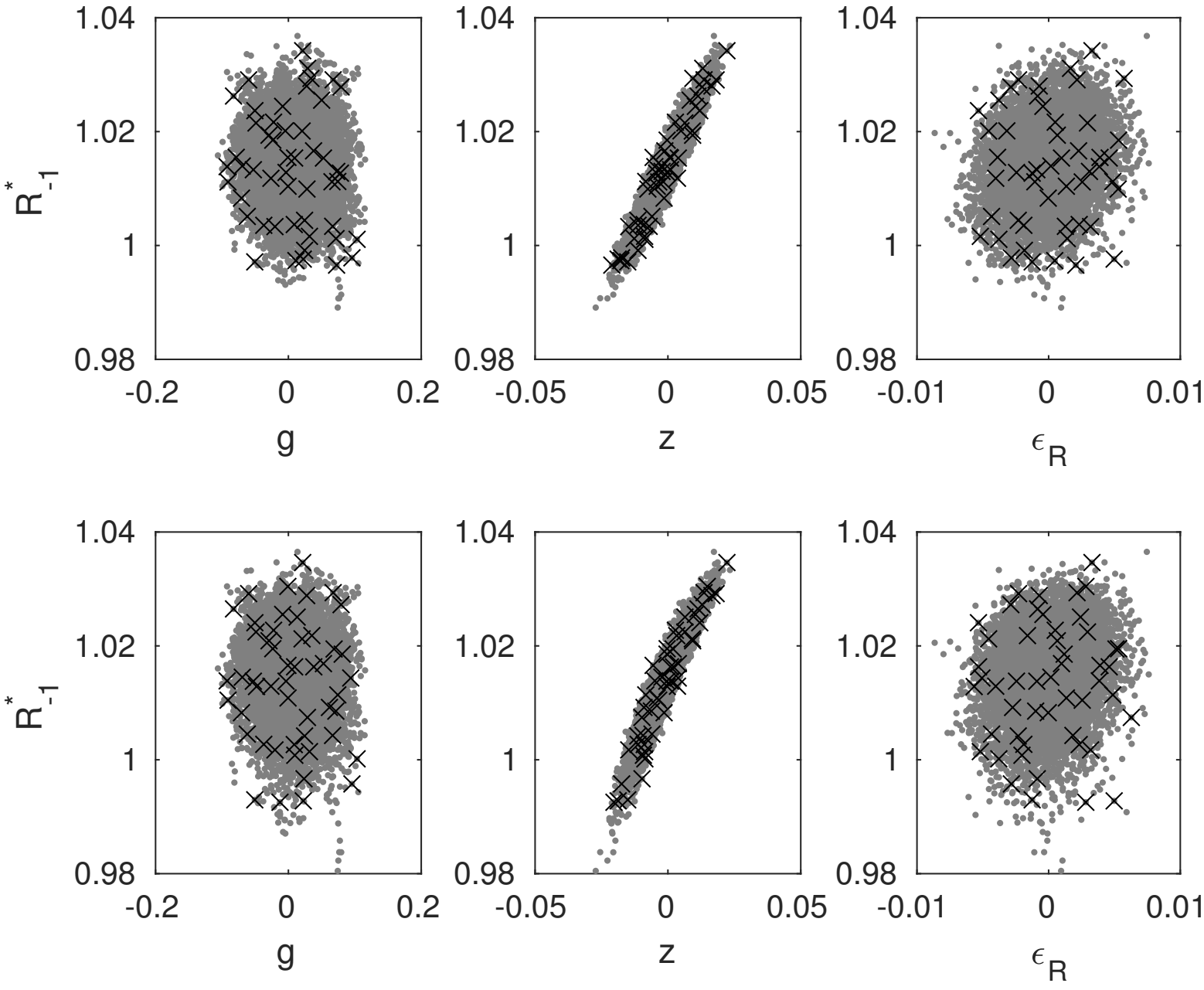
Constructing EDS grid

- In constructing an EDS grid from the ergodic set, we do the following two step procedure (See MM for more details):

1. Selecting points within an essentially ergodic set (called Algorithm $\mathcal{A}^{\eta}$ in MM)

2. Constructing a uniformly spaced set of points that covers the essentially ergodic set (called Algorithm P^{ϵ} in MM)

Constructing EDS grid



Numerical examples II

- Solve the nonlinear NK model by the time iteration method
  - Interpolation:
    - Non-stochastic method: Chebyshev polynomial with Smolyak sparse grid points (in the latter two). $(N_d, N) = (3, 81), (3, 9), (5, 41)$.
    - Simulation-based method: Second-order polynomials with cross terms. The number of grid points $N = 25, 50$ or 100 EDS grid points. As we have 4 state variables, the number of coefficients is 15.
  - Optimization:
    - Chris Sims' csolve is used in Time iteration with Newton's method (TI).
    - No optimization is required in PEA collocation.
  - Integration:
    - Gaussian-Hermite quadrature is used with $M = 3^3 = 27$ in TI and PEA collocation with fitting future variables (future PEA).
    - Precomputation technique in Judd et al. (2017) is used in PEA collocation with fitting current variables (current PEA, only with non-stochastic method).

Non-stochastic method with ZLB

(N_d, N)	TI								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	$\Pr_{\text{ZLB}}$	CPU
$(3, 81)$	-3.73	-2.62	-2.07	-1.42	0.76	2.05	2.53	1.53	1127.3
$(3, 9)$	-3.40	-2.38	-2.06	-1.09	0.76	2.02	2.50	1.79	12.98
$(5, 41)$	-3.97	-3.14	-2.07	-1.73	0.76	2.04	2.50	1.40	270.65

(N_d, N)	future PEA								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	$\Pr_{\text{ZLB}}$	CPU
$(3, 81)$	-3.73	-2.67	-2.08	-1.44	0.76	2.11	2.59	1.71	82.28
$(3, 9)$	-3.26	-2.66	-1.92	-1.48	0.76	2.19	2.68	3.42	0.96
$(5, 41)$	-4.04	-3.54	-2.13	-1.49	0.76	2.03	2.48	1.18	14.66

(N_d, N)	current PEA								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	$\Pr_{\text{ZLB}}$	CPU
$(3, 81)$	-4.05	-2.71	-2.12	-1.53	0.76	2.01	2.45	1.03	4.05
$(3, 9)$	-3.35	-2.42	-1.97	-1.25	0.76	2.01	2.47	1.92	0.19
$(5, 41)$	-4.17	-2.95	-2.12	-1.44	0.76	2.03	2.47	1.07	0.99

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of absolute Euler errors (in log 10 units) on a 10,000 period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_π, and σ_R are the standard deviation of output growth, inflation, and the policy rate.

Simulation-based method with ZLB

N	TI								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	Pr_{ZLB}	CPU
25	-3.39	-2.93	-2.21	-1.66	0.77	2.14	2.63	1.21	44.41
50	-3.00	-2.56	-1.83	-1.48	0.77	2.21	2.71	1.92	65.37
100	-3.28	-2.80	-2.08	-1.69	0.76	2.10	2.57	1.25	106.17

N	future PEA								
	$L_{1,c}$	$L_{1,\pi}$	$L_{\infty,c}$	$L_{\infty,\pi}$	$\sigma_{\Delta y}$	σ_π	σ_R	Pr_{ZLB}	CPU
25	-2.44	-1.24	-1.58	-0.61	0.77	2.18	2.68	1.31	3.37
50	-2.44	-1.24	-1.69	-0.63	0.77	2.15	2.63	1.46	5.08
100	-2.46	-1.26	-1.71	-0.67	0.76	2.11	2.58	1.31	7.93

Notes: $L_{1,c}$, $L_{1,\pi}$, $L_{\infty,c}$, and $L_{\infty,\pi}$ are, respectively, the average and maximum of absolute Euler errors (in log 10 units) on a 10,000 period stochastic simulation. CPU is the elapsed time for computing equilibrium (in seconds). $\sigma_{\Delta y}$, σ_π, and σ_R are the standard deviation of output growth, inflation, and the policy rate.

Parameter values

Parameter		Value
ν	Inverse of demand elasticity	1/6
$\bar{g}$	Steady state government expenditure	1.25
γ	Steady state technology growth	1.0052
β	Discount factor	0.9990
$\bar{\pi}$	Steady state inflation	1.0083
τ	CRRA parameter	2.83
ϕ	Price adjustment cost	17.85
ψ_1	Interest rate elasticity to inflation	1.80
ψ_2	Interest rate elasticity to output gap	0.63
ρ_r	Interest rate smoothing	0.77
ρ_g	Persistence of government shock	0.98
ρ_z	Persistence of technology growth shock	0.88
σ_r	Std. dev. of monetary policy shock	0.0022
σ_g	Std. dev. of government shock	0.0071
σ_z	Std. dev. of technology growth shock	0.0031

Notes: Taken from Schorfheide and Herbst (2015). The observations used in the estimation range from 1983:I to 2002:IV, giving us a total of $T = 80$ observations.

2. How to estimate nonlinear DSGE models

State-Space Representation

- Linear case
  1. State transition equations:

$$s_t = \Phi_1(\theta) s_{t-1} + \Phi_\varepsilon(\theta)\varepsilon_t, \qquad \varepsilon_t \sim N(0, \Sigma_\varepsilon)$$

  2. Observation equations:

$$y_t = \Psi_0(\theta) + \Psi_1(\theta) s_t + u_t, \qquad u_t \sim N(0, \Sigma_u)$$

- Nonlinear case
  1. State transition equations:

$$s_t = \Phi(s_{t-1}, \varepsilon_t; \theta), \qquad \varepsilon_t \sim F_\varepsilon(\cdot; \theta)$$

  2. Observation equations:

$$y_t = \Psi(s_t; \theta) + u_t, \qquad u_t \sim F_u(\cdot; \theta)$$

Kalman Filter

- In a linear case, the *Kalman filter* is available to evaluate likelihood.
- Distributional assumption about the initial state s_0:

$$s_0 \sim N(\bar{s}_{0|0}, P_{0|0})$$

  - It is common to set $\bar{s}_{0|0}$ and $P_{0|0}$ equal to the unconditional first and second moments of implied by the law of motion of s_t.

- Given $\bar{s}_{0|0}$ and $P_{0|0}$, the one-period-ahead forecasts of these moments are

$$\begin{aligned}\bar{s}_{1|0} &= \Phi_1 \bar{s}_{0|0}, \\ P_{1|0} &= \Phi_1 P_{0|0} \Phi_1' + \Phi_\varepsilon \Sigma_\varepsilon \Phi_\varepsilon'.\end{aligned}$$

- Given $\bar{s}_{1|0}$ and $P_{1|0}$, the conditional mean and variances of the observables y_1 are

$$\begin{aligned}\bar{y}_{1|0} &= A + B\bar{s}_{1|0}, \\ F_{1|0} &= BP_{1|0}B'.\end{aligned}$$

- Then, the forecast error of the observables y_1 is

$$\nu_{1|0} = y_1 - \bar{y}_{1|0}.$$

- It has been known that the optimal updating leads to

$$\bar{s}_{1|1} = \bar{s}_{1|0} + P_{1|0}B'F_{1|0}^{-1}\nu_{1|0},$$
$$P_{1|1} = P_{1|0} - P_{1|0}B'F_{1|0}^{-1}BP_{1|0},$$

  where $\bar{s}_{1|1}$ and $P_{1|1}$ are the mean and variances of s_1 conditional on y_1.
- In the same way as above, given $\bar{s}_{1|1}$ and $P_{1|1}$, we can obtain $\bar{s}_{2|1}$, $P_{2|1}$, $\bar{y}_{2|1}$, $F_{2|1}$, and $\nu_{2|1}$.
  - Also, updating gives $\bar{s}_{2|2}$ and $P_{2|2}$, conditional on $\{y_1, y_2\}$.
- Repeating these steps yields the sequences of $\bar{s}_{t|t-1}$, $P_{t|t-1}$, $\bar{y}_{t|t-1}$, $F_{t|t-1}$, and $\nu_{t|t-1}$ conditional on $\{y_1, y_2, ..., y_{t-1}\}$ for $t = 1, 2, ...T$.

- Since $\varepsilon_t \sim N(0, \Sigma_\varepsilon)$,

$$y_t|Y^{t-1} \sim N\left(A + B\hat{s}_{t|t-1}, F_{t|t-1}\right),$$

  where $Y^{t-1} = \{y_1, y_2, ..., y_{t-1}\}$.
- Its probability density function is

$$p\left(y_t|Y^{t-1}\right) = (2\pi)^{-\frac{n}{2}} \left|F_{t|t-1}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\nu'_{t|t-1}F^{-1}_{t|t-1}\nu_{t|t-1}\right).$$

- Therefore, the log-likelihood is given by

$$\begin{aligned}\ln L(\theta|Y) &= \sum_{t=1}^{T} \ln p\left(y_t|Y^{t-1}\right) \\ &= -\frac{nT}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\ln\left|F_{t|t-1}\right| - \frac{1}{2}\sum_{t=1}^{T}\nu'_{t|t-1}F^{-1}_{t|t-1}\nu_{t|t-1}.\end{aligned}$$

Particle Filter

- In a nonlinear case, the *Kalman filter* is NOT available because the distribution of $y_t|Y^{t-1}$ is non-normal.
- A particle filter can approximate the likelihood function.
- While there are many particle filters, we will focus on the *bootstrap particle filter*.
    - Gordon, Salmond, and Smith (1993)

Bootstrap Particle Filter

- Idea: Particles representing s_t are propagated according to state space representation, so that the distribution of $y_t|Y^{t-1}$ can be approximated.
- Draw the initial particles $\{s_0^j\}_{j=1}^M$ from the distribution:

$$s_0^j \sim N(\bar{s}_0, P_0).$$

  - M denotes the number of particles.
  - Set $\bar{s}$ and P equal to the unconditional first and second moments of implied by the law of motion of s_t.
  - Set particle weights $W_0^j = 1$, for $j = 1, ..., M$.

For $t = 1, ..., T$:

1. Draw shocks in period t:

$$\varepsilon_t^j \sim F_\varepsilon(\cdot; \theta),$$

and propagate particles $\{s_{t-1}^j\}$ using the state-transition equation,

$$\tilde{s}_t^j = \Phi(s_{t-1}^j, \varepsilon_t; \theta).$$

2. Define the incremental weights:

$$\tilde{w}_t^j = p(y_t | \tilde{s}_t^j, \theta).$$

- The predictive density $p(y_t | Y_{1:t-1}, \theta)$ can be approximated by

$$\hat{p}(y_t | Y_{1:t-1}, \theta) = \frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j.$$

- If the measurement errors $u_t \sim N(0, \Sigma_u)$, then the incremental weights are evaluated as

$$\tilde{w}_t^j = (2\pi)^{-\frac{n}{2}} |\Sigma_u|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (y_t - \Psi(\tilde{s}_t^j; \theta))' \Sigma_u^{-1} (y_t - \Psi(\tilde{s}_t^j; \theta)) \right\},$$

where n denotes the number of observables.

3. Define the normalized weights:

$$\tilde{W}_t^j = \frac{\tilde{w}_t^j W_{t-1}^j}{\frac{1}{M}\sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j}$$

   - An approximation of $E[s_t|Y_{1:t}, \theta]$ is given by

$$\hat{E}[s_t|Y_{1:t}, \theta] = \frac{1}{M}\sum_{j=1}^{M} \tilde{s}_t^j \tilde{W}_t^j.$$

4. If resampling is needed, M iid draws $\{s_t^j\}_{j=1}^M$ from a multinomial distribution characterized by support points and weights $\{\tilde{s}_t^j, \tilde{W}_t^j\}$ and set $W_t^j = 1$, for $j = 1, ..., M$.
   - If resampling is not needed, let $s_t^j = \tilde{s}_t^j$ and $W_t^j = \tilde{W}_t^j$ for $j = 1, ..., M$.
5. Repeat steps 1–4 for next t.

Resampling

- A resampling step is necessary to avoid the degeneracy of the distribution of particle weights.
  - A situation in which all but a few of the weights are near zero
- Since resampling is done with replacement, a particle with a large weight is likely to be drawn many times and particles with small weights are not likely to be drawn at all.
  - Resampling effectively deals with the degeneracy problem by eliminating the particles with very small weights.
- Resampling is done whenever the effective sample size

$$\widehat{ESS}_t = \frac{M}{\frac{1}{M}\sum_{j=1}^{M}(\tilde{W}_t^j)^2}$$

  falls below a threshold, e.g., $M/2$.

Likelihood Approximation

- Repeating steps 1–4 for $t = 1, ..., T$ gives particle weights $\{\tilde{w}_t^j, W_{t-1}^j\}_{j=1}^M$ for each t.
- The approximation of the log-likelihood function is given by

$$\ln \hat{p}(Y_{1:T}|\theta) = \sum_{t=1}^{T} \ln \left(\frac{1}{M} \sum_{j=1}^{M} \tilde{w}_t^j W_{t-1}^j \right).$$

Central Difference Kalman Filter

- Likelihood approximation using a particle filter causes a huge computational cost.
- If a DSGE model is approximated by a 2nd- or 3rd-order perturbation method, the Central Difference Kalman Filter (CDFK) can approximate the likelihood more efficiently.
  - Andreasen (2013)
- Idea: Approximate the filtering equations that compute and update first and second moments of state variables by 2nd-order multivariate Stirling interpolations.
- Andreasen (2013) show that a quasi maximum likelihood estimator based on the CDFK can be consistent and asymptotically normal for DSGE models solved up to third order.

Bayesian Estimation

- Once we approximate and evaluate the likelihood function $\hat{p}(Y|\theta)$, the Bayesian likelihood approach is applicable.
  - Prior distribution: $p(\theta)$
  - Bayes' Theorem $\Rightarrow$ Posterior distribution

$$\hat{p}\left(\theta|Y\right) \propto \hat{p}(Y|\theta)p\left(\theta\right)$$

- Generate draws from the posterior distribution using Markov Chain Monte Carlo (MCMC) algorithm.
  - Random-Walk Metropolis Hasting (RWMH) algorithm is widely used.

Sequential Monte Carlo Algorithm

- Issues in the RWMH algorithm:
  - The posterior distribution is possibly multimodal.
  - The RWMH algorithm can get stuck near a local mode and fail to find the entire posterior distribution.
  - It is often very difficult to find a model.
- Herbst and Schorfheide (2014, 2015) propose the Sequential Monte Carlo (SMC) algorithm.
  - Particles representing θ are propagated, similar to a particle filter.
  - Overcome the issues by building a particle approximation to the posterior gradually through tempering the likelihood function.
  - Sequence of tempered posteriors:

$$\pi_n(\theta) = \frac{[p(Y|\theta)]^{\phi_n} p(\theta)}{\int [p(Y|\theta)]^{\phi_n} p(\theta) d\theta}, \qquad n = 0, ..., N_\phi.$$

    - Tempering schedule: $\phi_n = (n/N_\phi)^\chi$

Parallelization

- Both the particle filter and the SMC algorithm can be parallelized.
  - Nonlinear solution methods can be also parallelized.
  - Massive speed gains in estimation
- Matlab parallel toolbox is very easy to use, but not so fast.
- Explicit parallelization:
  - OpenMP
  - MPI
  - GPU programming: CUDA and OpenCL
- Hardware:
  - Workstation with multi-core and multi-processors
  - Computer cluster
  - Cloud computing

Faster Programming Languages

- Aruoba and Fernández-Villaverde (2015): "A Comparison of Programming Languages in Macroeconomics," *Journal of Economic Dynamics and Control*, 58, 265–273.
  - Solve the stochastic neoclassical growth model using C++, Fortran, Java, Julia, Python, Matlab, Mathematica, and R.
  - Report the execution time of the codes.
- Aruoba and Fernández-Villaverde (2018) update results for new versions of each language.

Aruoba and Fernández-Villaverde (2018)

Language	Compiler	Time	Rel. Time
C++	GCC	1.60	1.00
	Intel C++	1.67	1.04
	Clang	1.64	1.03
Fortran	GCC	1.61	1.01
	Intel Fortran	1.74	1.09
Java		3.20	2.00
Julia		2.35	1.47
	fast	2.14	1.34
Matlab		4.80	3.00
Python	CPython	145.27	90.79
	CPython	166.75	104.22
R		57.06	35.66
Mathematica	base	1634.94	1021.84

Aruoba and Fernández-Villaverde (2018)

Language	Compiler	Time	Rel. Time
Matlab, Mex		2.01	1.26
Rcpp		6.60	4.13
Python	Numba	2.31	1.44
Cython	Cython	2.13	1.33
Mathematica	idiomatic	4.42	2.76

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