Numerical Methods for New Keynesian ZLB Models

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What is covered

- Equilibrium with a Taylor rule with interest rate inertia
  - Linear Interpolation
  - Solving nonlinear equations: Golden section search vs. Newton method
- Equilibrium under the optimal commitment policy (Adam and Billi, 2006)

Equilibrium with Taylor rule with interest rate inertia

- Equilibrium conditions are

$$\begin{aligned} y_t &= E_t y_{t+1} - (i_t - E_t \pi_{t+1} - s_t), \\ \pi_t &= \kappa y_t + \beta E_t \pi_{t+1}, \\ i_t^* &= r^* + \rho_r i_{t-1}^* + (1-\rho_r)\phi_\pi E_t \pi_{t+1}, \end{aligned}$$

  and the zero lower bound

$$i_t = \{0, i_t^*\}.$$

- s_t follows a Markov chain.

Solution forms

- The solution has a form of

$$y = y(i^*_{-1}, s_k), \quad \pi = \pi(i^*_{-1}, s_k), \quad i = i(i^*_{-1}, s_k).$$

- Now the solution is a function of infinite object (i.e., $i^*_{-1} \in \mathbb{R}$), indexed by the state of the Markov chain $s_k \in [s_1, s_2, ..., s_{N_s}]$.

- How to deal with it?

Interpolation

- We set grid points for i^*_{-1}:

$$i^*_{m,-1} \in \mathcal{I} = [i^*_1, i^*_2, ..., i^*_N] \subset \mathbb{R},$$

where m is an index for the set of grid points $\mathcal{I}$.

- We may want to evaluate the function $f_k(i^*_{-1}) \equiv f(i^*_{-1}, s_k)$ at $i^*_{-1} \notin \mathcal{I}$.

Numerical solution: Policy function iteration

- A guess of the policy functions

$$y = y^{(0)}(i^*_{-1}, s_k), \quad \pi = \pi^{(0)}(i^*_{-1}, s_k), \quad i = i^{(0)}(i^*_{-1}, s_k).$$

- We know the values of the functions only at each *grid point*, e.g.,

$$\begin{aligned} y^{(0)}(i^*_{m,-1}, s_k) \in y^{(0)}(i^*_{-1}, s_k) &= [y_{1k}, y_{2k}, ..., y_{Nk}]', \\ \pi^{(0)}(i^*_{m,-1}, s_k) \in \pi^{(0)}(i^*_{-1}, s_k) &= [\pi_{1k}, \pi_{2k}, ..., \pi_{Nk}]', \\ i^{(0)}(i^*_{m,-1}, s_k) \in i^{(0)}(i^*_{-1}, s_k) &= [i_{1k}, i_{2k}, ..., i_{Nk}]'. \end{aligned}$$

for $i^*_{m,-1} \in \mathcal{I}$ and $k = 1, ..., N_s$.

Two dimensional grid points

- Now we have two dimensional state space with grid points indexed by (m, k). One is for $i^*_{m,-1}$, the other is for s_k. For example, the set of grid points are represented by $N \times N_s$ matrix:

$$y^{(0)}(i^*_{m,-1}, s_k) \in y^{(0)}(i^*_{-1}, s) = \begin{bmatrix} y_{11} & \cdots & \cdots & y_{1N_s} \\ y_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ y_{N1} & \cdots & \cdots & y_{NN_s} \end{bmatrix}.$$

Policy function iteration

- At each grid point (m,k), we solve

$$\begin{aligned}
y_m &= y_k^e(i_m^*) - \left(i_m - \pi_k^e(i_m^*) - s_k\right), \\
\pi_m &= \kappa y_m + \beta \pi_k^e(i_m^*), \\
i_m^* &= r^* + \rho_r i_{m,-1}^* + (1-\rho_r)\phi_\pi \pi_k^e(i_m^*) \\
i_m &= \max\left\{0, i_m^*\right\},
\end{aligned}$$

for (y_m, π_m, i_m^*, i_m), where

$$\begin{aligned}
y_k^e(i_m^*) &= \sum_{l=1}^{N_s} p(k,l) y^{(0)}(i_m^*, s_l), \\
\pi_k^e(i_m^*) &= \sum_{l=1}^{N_s} p(k,l) \pi^{(0)}(i_m^*, s_l).
\end{aligned}$$

and $p(k,l)$ is the (k,l) element of the transition matrix.

Policy function iteration

- Once this is done for all the grid points, we update

$$
\begin{aligned}
y^{(1)}(i^*_{-1}, s_k) &= [y_{1k}, y_{2k}, ..., y_{Nk}]', \\
\pi^{(1)}(i^*_{-1}, s_k) &= [\pi_{1k}, \pi_{2k}, ..., \pi_{Nk}]', \\
i^{(1)}(i^*_{-1}, s_k) &= [i_{1k}, i_{2k}, ..., i_{Nk}]',
\end{aligned}
$$

  for $k = 1, ..., N_s$.

- We repeat the procedure until the policy functions converge, i.e., $\left\| x^{(j)}(i^*_{-1}, s_k) - x^{(j-1)}(i^*_{-1}, s_k) \right\| < \epsilon$ for $x \in \{y, \pi, i\}$.

Solving nonlinear equation

- We need to solve the following nonlinear equation:

$$i_m^* = r^* + \rho_r i_{m,-1}^* + (1-\rho_r)\phi_\pi \pi_k^e(i_m^*)$$

for i_m^*.

Optimal Commitment Policy

- The policymaker chooses $\{\pi_t, y_t, i_t\}$ so as to maximize

$$V_0 \equiv -E_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \lambda y_t^2\right)$$

subject to

$$\begin{aligned} y_t &= E_t y_{t+1} - (i_t - E_t \pi_{t+1}) + g_t, \\ \pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t, \\ i_t &\geq 0, \end{aligned}$$

Optimal Discretionary Policy

- Exogenous shocks are given by

$$
\begin{aligned}
g_t &= (1-\rho_g)g + \rho_g g_{t-1} + \varepsilon_{g,t}, \\
u_t &= \rho_u u_{t-1} + \varepsilon_{u,t},
\end{aligned}
$$

  where $\varepsilon_{g,t} \sim N(0, \sigma_g^2)$ and $\varepsilon_{u,t} \sim N(0, \sigma_u^2)$.

- Note that $g = r^*$.