Numerical Methods for New Keynesian ZLB Models: Part I

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What is covered

- Equilibrium with a simple Taylor rule
- Tauchen's (1986) method for approximating AR(1) process
- Equilibrium under the optimal discretionary policy (Adam and Billi, 2007)

Two-state shock process

- Exogenous shocks take only $N_s = 2$ values, $s_t \in \{s_H, s_L\}$. The stochastic process follows a Markov chain with the transition matrix:

$$\left[\begin{array}{cc} 1-p_H & p_H \\ 1-p_L & p_L \end{array}\right].$$

- p_H is the frequency of crisis and p_L is the duration of crisis.

Equilibrium with Taylor rule

- Equilibrium conditions are

$$\begin{aligned} y_t &= E_t y_{t+1} - \left(i_t - E_t \pi_{t+1} - s_t\right), \\ \pi_t &= \kappa y_t + \beta E_t \pi_{t+1}, \\ i_t^* &= r^* + \phi_\pi E_t \pi_{t+1}, \end{aligned}$$

and the zero lower bound

$$i_t = \max\left\{0, i_t^*\right\}.$$

Solving the model: Two routes

- The solution has a form of (we omit time subscripts for the policy function)

$$y = y(s), \quad \pi = \pi(s), \quad i = i(s).$$

- We are solving the same model with two different methods.
    - Analytical solution.
    - Numerical solution, using the policy function iteration (time iteration).

Analytical solution

- We know that the functions have only two values, i.e.,

$$y = \begin{cases} y_H, \\ y_L, \end{cases} \qquad \pi = \begin{cases} \pi_H, \\ \pi_L, \end{cases} \qquad i = \begin{cases} i_H, \\ i_L. \end{cases}$$

Analytical solution, cont'd

- We assume that $i_H > 0$ and $i_L = 0$. Then we have

$$
\begin{aligned}
y_H &= (1-p_H)y_H + p_H y_L - \left(i_H - [(1-p_H)\pi_H + p_H\pi_L] - s_H\right), \\
\pi_H &= \kappa y_H + \beta\left[(1-p_H)\pi_H + p_H\pi_L\right], \\
i_H &= r^* + \phi_\pi\left[(1-p_H)\pi_H + p_H\pi_L\right], \\
y_L &= (1-p_L)y_H + p_L y_L - \left(0 - [(1-p_L)\pi_H + p_L\pi_L] - s_L\right), \\
\pi_L &= \kappa y_L + \beta\left[(1-p_L)\pi_H + p_L\pi_L\right], \\
i_L &= 0.
\end{aligned}
$$

- There are 6 equations and 6 unknowns, so we can solve for the unknowns.

Numerical solution: Policy function iteration

- We can solve the same model with the policy function iteration method.
- The method takes four steps:
  1. Initial guess for the policy function
  2. Solving for endogenous variables at each grid
  3. Updating the policy function
  4. Repeat 2-3 until convergence

Policy function iteration: Initial guess

- A guess of the policy functions

$$y = y^{(0)}(s), \quad \pi = \pi^{(0)}(s), \quad i = i^{(0)}(s).$$

- Consider the general case of $N_s \geq 2$. We know the values of the functions only at each *grid point*, i.e.,

$$\begin{aligned} y^{(0)}(s) &= [y_1, y_2, ..., y_{N_s}]', \\ \pi^{(0)}(s) &= [\pi_1, \pi_2, ..., \pi_{N_s}]', \\ i^{(0)}(s) &= [i_1, i_2, ..., i_{N_s}]'. \end{aligned}$$

Policy function iteration: Solving at each grid

- At each grid point $k = 1, ..., N_s$, we solve

$$y_k = y^e - (i_k - \pi^e - s_k),$$
$$\pi_k = \kappa y_k + \beta \pi^e,$$
$$i_k = \max\{0, r^* + \phi_\pi \pi^e\},$$

for (y_k, π_k, i_k), where

$$y^e = \sum_{l=1}^{N_s} p(k,l) y^{(0)}(s_l),$$
$$\pi^e = \sum_{l=1}^{N_s} p(k,l) \pi^{(0)}(s_l).$$

and $p(k,l)$ is the (k,l) element of the transition matrix.

Policy function iteration: Updating and convergence

- Once this is done for all the grid points, we update

$$
\begin{aligned}
y^{(1)}(s) &= [y_1, y_2, ..., y_N]', \\
\pi^{(1)}(s) &= [\pi_1, \pi_2, ..., \pi_N]', \\
i^{(1)}(s) &= [i_1, i_2, ..., i_N]'.
\end{aligned}
$$

- We repeat the procedure until the policy functions converge, i.e., $\left\| x^{(j)}(s) - x^{(j-1)}(s) \right\| < \epsilon$ for $x \in \{y, \pi, i\}$.

Tauchen's method

- Tauchen (1986) developed a method for approximating AR(1) stochastic process by using Markov chain.

- We have the following AR(1) stochastic process

$$x' = c + \rho x + \varepsilon', \varepsilon' \sim N(0, \sigma_\varepsilon^2).$$

- We want to approximate the stochastic process by a Markov chain $x_k \in \{x_1, x_2, ..., x_N\}$.

Tauchen's method: Grid points

- We set the grid points for x:

$$x_k \in \mathcal{I} = \{x_1, x_2, ..., x_N\} \subset \mathbb{R}^N,$$

where k is an index for the set of grid points $\mathcal{I}$.

- For example, we set $x_1 = \frac{-m\sigma_\varepsilon}{\sqrt{1-\rho^2}}$, $x_N = \frac{m\sigma_\varepsilon}{\sqrt{1-\rho^2}}$ and $x_k = x_{k-1} + w$ for $k = 2, ..., N-1$, where $w = \frac{x_N - x_1}{N-1}$.

Tauchen's method: Transition matrix

- Given the grid points for x. What is the probablity of moving from one point x_k to another x_l?
- We know
$$\varepsilon' = x' - c - \rho x_k \sim N(0, \sigma_\varepsilon^2).$$

Tauchen's method: Transition matrix, cont'd

- Then, the probability of $x' \in \left[x_l - \frac{w}{2}, x_l + \frac{w}{2}\right]$ can be used as approximation. That is,

$$p_{kl} = \Phi\left(x_l + \frac{w}{2} - c - \rho x_k\right) - \Phi\left(x_l - \frac{w}{2} - c - \rho x_k\right),$$

  where $\Phi(\cdot)$ is the cdf of $N(0, \sigma_\varepsilon^2)$. Be careful at the boundary points.

- Once this is done for all k, l, we have the transition matrix

$$P = \begin{bmatrix} p_{11} & \cdots & \cdots & p_{1N} \\ p_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p_{N1} & \cdots & \cdots & p_{NN} \end{bmatrix}.$$

Optimal Discretionary Policy

- The policymaker chooses $\{\pi_t, y_t, i_t\}$ so as to maximize

$$V_0 \equiv -E_0 \sum_{t=0}^{\infty} \beta^t \left(\pi_t^2 + \lambda y_t^2\right)$$

subject to

$$\begin{aligned}
y_t &= E_t y_{t+1} - (i_t - E_t \pi_{t+1}) + g_t, \\
\pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t, \\
i_t &\geq 0,
\end{aligned}$$

taking $E_t y_{t+1}$ and $E_t \pi_{t+1}$ as given.

Shock processes

- Exogenous shocks are given by

$$
\begin{aligned}
g_t &= (1-\rho_g)g + \rho_g g_{t-1} + \varepsilon_{g,t}, \\
u_t &= \rho_u u_{t-1} + \varepsilon_{u,t},
\end{aligned}
$$

  where $\varepsilon_{g,t} \sim N(0, \sigma_g^2)$ and $\varepsilon_{u,t} \sim N(0, \sigma_u^2)$.

- Note that $g = r^*$.

Optimal Discretionary Policy: Lagrangean

- We know that Markov-perfect equilibrium has only natural state variables.
- Lagrangean is

$$\mathcal{L} \equiv E_0 \sum \beta^t \left(\pi_t^2 + \lambda y_t^2\right) + 2\phi_{PC,t} \left(-\pi_t + \kappa y_t + \beta E_t \pi_{t+1} + u_t\right)$$
$$+ 2\phi_{EE,t} \left(-y_t - i_t + E_t y_{t+1} + E_t \pi_{t+1} + g_t\right) + 2\phi_{ZLB,t} i_t.$$

- First-order necessary conditions are

$$\begin{aligned}
\partial \pi_t :& \pi_t - \phi_{PC,t} = 0, \\
\partial y_t :& \lambda y_t + \kappa \phi_{PC,t} - \phi_{EE,t} = 0, \\
\partial i_t :& -\phi_{EE,t} + \phi_{ZLB,t} = 0.
\end{aligned}$$

Optimal Discretionary Policy: Complementary slackness

- Complementary slackness condition:

$$\phi_{ZLB,t} > 0 \perp i_t > 0.$$

- When $i_t > 0$, $\phi_{ZLB,t} = 0$. Equilibrium conitions are

$$\begin{aligned}
i_t &= -y_t + E_t y_{t+1} + E_t \pi_{t+1} + g_t, \\
\pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t, \\
0 &= \lambda y_t + \kappa \pi_t.
\end{aligned}$$

- When $i_t = 0$, $\phi_{ZLB,t} > 0$. Equilibrium conitions are

$$\begin{aligned}
0 &= -y_t + E_t y_{t+1} + E_t \pi_{t+1} + g_t, \\
\pi_t &= \kappa y_t + \beta E_t \pi_{t+1} + u_t, \\
\phi_{ZLB,t} &= \lambda y_t + \kappa \pi_t.
\end{aligned}$$

Solving the model

- The solution has a form of

$$y = y(g,u), \quad \pi = \pi(g,u), \quad i = i(g,u).$$

- Now consider the case in which there are only two-state g shocks. We know that the functions have only two values, i.e.,

$$y = \begin{cases} y_H, \\ y_L, \end{cases} \qquad \pi = \begin{cases} \pi_H, \\ \pi_L, \end{cases} \qquad i = \begin{cases} i_H, \\ i_L. \end{cases}$$

- Again, we will look at analytical solution and numerical solution.

Analytical solution

- We assume that $i_H > 0$ and $i_L = 0$. Then we have

$$
\begin{aligned}
y_H &= (1-p_H)y_H + p_H y_L - (i_H - [(1-p_H)\pi_H + p_H \pi_L]) + g_H, \\
\pi_H &= \kappa y_H + \beta\left[(1-p_H)\pi_H + p_H \pi_L\right], \\
0 &= \lambda y_H + \kappa \pi_H, \\
y_L &= (1-p_L)y_H + p_L y_L - (0 - [(1-p_L)\pi_H + p_L \pi_L]) + g_L, \\
\pi_L &= \kappa y_L + \beta\left[(1-p_L)\pi_H + p_L \pi_L\right], \\
\phi_L &= \lambda y_L + \kappa \pi_L.
\end{aligned}
$$

- There are 6 equations and 6 unknowns, so we can solve for the unknowns.

Joint shock process

- Let's get back to the general case. The shock processes are appoximated by Markov chains. That is,

$$
\begin{aligned}
g_m &\in \{g_1, g_2, ..., g_{N_g}\}, \\
u_n &\in \{u_1, u_2, ..., u_{N_u}\},
\end{aligned}
$$

and

$$
P^g = \begin{bmatrix} p^g_{11} & \cdots & \cdots & p^g_{1N_g} \\ p^g_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p^g_{N_g1} & \cdots & \cdots & p^g_{N_gN_g} \end{bmatrix}, \quad P^u = \begin{bmatrix} p^u_{11} & \cdots & \cdots & p^u_{1N_g} \\ p^u_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p^u_{N_g1} & \cdots & \cdots & p^u_{N_gN_g} \end{bmatrix}.
$$

Joint shock process, cont'd

- A kronecker product $s = g \otimes u$ represents the joint shock process.
  - For example, when $N_g = N_u = 2$

$$\begin{aligned} s_1 &= (g_1, u_1), \\ s_2 &= (g_1, u_2), \\ s_3 &= (g_2, u_1), \\ s_4 &= (g_2, u_2). \end{aligned}$$

  - Note that each index points to a pair of shocks, $s_k = (g_{m(k)}, u_{n(k)})$.
- A kronecker product of the transition matrices $P^s = P^g \otimes P^u$ is the transition matrix of the joint shock process.

Policy function iteration: Initial guess

- A guess of the policy functions

$$y = y^{(0)}(s), \quad \pi = \pi^{(0)}(s), \quad i = i^{(0)}(s).$$

- We know the values of the functions only at each *grid point*, e.g.,

$$\begin{aligned} y^{(0)}(s) &= [y_1, y_2, ..., y_N]', \\ \pi^{(0)}(s) &= [\pi_1, \pi_2, ..., \pi_N]', \\ i^{(0)}(s) &= [i_1, i_2, ..., i_N]'. \end{aligned}$$

Policy function iteration: Solving at each grid

- At each grid point $k = 1, ..., N$, we solve

$$
\begin{aligned}
i_k &= -y_k + y^e + \pi^e + g_{m(k)}, \\
\pi_k &= \kappa y_k + \beta \pi^e + u_{n(k)}, \\
0 &= \lambda y_k + \kappa \pi_k,
\end{aligned}
$$

for (y_k, π_k, i_k), where

$$
\begin{aligned}
y^e &= \sum_{l=1}^{N_s} P^s(k,l) y^{(0)}(s_l), \\
\pi^e &= \sum_{l=1}^{N_s} P^s(k,l) \pi^{(0)}(s_l).
\end{aligned}
$$

Policy function iteration: Solving at each grid, cont'd

- Check $i_k \geq 0$. If not, we solve instead

$$0 = -y_k + y^e + \pi^e + g_{m(k)},$$
$$\pi_k = \kappa y_k + \beta \pi^e + u_{n(k)},$$

for (y_k, π_k), and set $i_k = 0$.

Policy function iteration: Updating and convergence

- Once this is done for all the grid points, we update

$$
\begin{aligned}
y^{(1)}(s) &= [y_1, y_2, ..., y_N]', \\
\pi^{(1)}(s) &= [\pi_1, \pi_2, ..., \pi_N]', \\
i^{(1)}(s) &= [i_1, i_2, ..., i_N]'.
\end{aligned}
$$

- We repeat the procedure until the policy functions converge, i.e., $\left\| x^{(j)}(s) - x^{(j-1)}(s) \right\| < \epsilon$ for $x \in \{y, \pi, i\}$.