Hence, from Equation (5.8),

$$F_s(x) = 1 - \frac{s+a}{x+s+a} = \frac{x}{x+s+a}$$

To invert this, suppose that $x = F_s^{-1}(u)$, and so

$$u = F_s(x) = \frac{x}{x+s+a}$$

or, equivalently,

$$x = \frac{u(s+a)}{1-u}$$

That is,

$$F_s^{-1}(u) = (s+a)\frac{u}{1-u}$$

We can therefore generate the successive event times $S_1, S_2, \ldots$ by generating random numbers $U_1, U_2, \ldots$ and then recursively setting

$$S_1 = \frac{aU_1}{1-U_1}$$

$$S_2 = S_1 + (S_1 + a)\frac{U_2}{1-U_2} = \frac{S_1 + aU_2}{1-U_2}$$

and, in general,

$$S_j = S_{j-1} + (S_{j-1} + a)\frac{U_j}{1-U_j} = \frac{S_{j-1} + aU_j}{1-U_j}, \quad j \geq 2 \qquad \square$$

Exercises

1. Give a method for generating a random variable having density function

$$f(x) = e^x/(e-1), \quad 0 \leq x \leq 1$$

2. Give a method to generate a random variable having density function

$$f(x) = \begin{cases} \frac{x-2}{2} & \text{if } 2 \leq x \leq 3 \\ \frac{2-x/3}{2} & \text{if } 3 \leq x \leq 6 \end{cases}$$

3. Use the inverse transform method to generate a random variable having distribution function

$$F(x) = \frac{x^2 + x}{2}, \quad 0 \leq x \leq 1$$

4. Give a method for generating a random variable having distribution function

$$F(x) = 1 - \exp(-\alpha x^\beta), \quad 0 < x < \infty$$

A random variable having such a distribution is said to be a Weibull random variable.

5. Give a method for generating a random variable having density function

$$f(x) = \begin{cases} e^{2x}, & -\infty < x < 0 \\ e^{-2x}, & 0 < x < \infty \end{cases}$$

6. Let X be an exponential random variable with mean 1. Give an efficient algorithm for simulating a random variable whose distribution is the conditional distribution of X given that $X < 0.05$. That is, its density function is

$$f(x) = \frac{e^{-x}}{1 - e^{-0.05}}, \quad 0 < x < 0.05$$

Generate 1000 such variables and use them to estimate of $E[X \mid X < 0.05]$. Then determine the exact value of $E[X \mid X < 0.05]$.

7. (The Composition Method) Suppose it is relatively easy to generate random variables from any of the distributions F_i, $i = 1, \ldots, n$. How could we generate a random variable having the distribution function

$$F(x) = \sum_{i=1}^{n} p_i F_i(x)$$

where p_i, $i = 1, \ldots, n$, are nonnegative numbers whose sum is 1?

8. Using the result of Exercise 7, give algorithms for generating random variables from the following distributions.

(a) $F(x) = \frac{x+x^3+x^5}{3}, 0 \leq x \leq 1$

(b) $F(x) = \begin{cases} \frac{1-e^{-2x}+2x}{3} & \text{if } 0 < x < 1 \\ \frac{3-e^{-2x}}{3} & \text{if } 1 < x < \infty \end{cases}$

(c) $F(x) = \sum_{i=1}^{n} \alpha_i x^i, 0 \leq x \leq 1, \quad$ where $\alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1$

9. Give a method to generate a random variable having distribution function

$$F(x) = \int_0^\infty x^y e^{-y}\,dy, \quad 0 \le x \le 1$$

[Hint: Think in terms of the composition method of Exercise 7. In particular, let F denote the distribution function of X, and suppose that the conditional distribution of X given that $Y = y$ is

$$P\{X \le x \mid Y = y\} = x^y, \quad 0 \le x \le 1]$$

10. A casualty insurance company has 1000 policyholders, each of whom will independently present a claim in the next month with probability .05. Assuming that the amounts of the claims made are independent exponential random variables with mean \$800, use simulation to estimate the probability that the sum of these claims exceeds \$50,000.

11. Write an algorithm that can be used to generate exponential random variables in sets of 3. Compare the computational requirements of this method with the one presented after Example 5c which generates them in pairs.

12. Suppose it is easy to generate random variable from any of the distribution F_i, $i = 1, \ldots, n$. How can we generate from the following distributions?

(a) $F(x) = \prod_{i=1}^n F_i(x)$

(b) $F(x) = 1 - \prod_{i=1}^n [1 - F_i(x)]$

[Hint: If X_i, $i = 1, \ldots, n$, are independent random variables, with X_i having distribution F_i, what random variable has distribution function F?]

13. Using the rejection method and the results of Exercise 12, give two other methods, aside from the inverse transform method, that can be used to generate a random variable having distribution function

$$F(x) = x^n, \quad 0 \le x \le 1$$

Discuss the efficiency of the three approaches to generating from F.

14. Let G be a distribution function with density g and suppose, for constants $a < b$, we want to generate a random variable from the distribution function

$$F(x) = \frac{G(x) - G(a)}{G(b) - G(a)}, \quad a \le x \le b$$

(a) If X has distribution G, then F is the conditional distribution of X given what information?

(b) Show that the rejection method reduces in this case to generating a random variable X having distribution G and then accepting it if it lies between a and b.

15. Give two methods for generating a random variable having density function

$$f(x) = xe^{-x}, \quad 0 \leq x < \infty$$

and compare their efficiency.

16. Suppose that we want to generate a random variable X whose density function is

$$f(x) = \frac{1}{2}x^2 e^{-x} \quad x > 0$$

by using the rejection method with an exponential density having rate λ. Find the value of λ that minimizes the expected number of iterations of the algorithm used to generate X.

17. Give an algorithm that generates a random variable having density

$$f(x) = 30(x^2 - 2x^3 + x^4), \quad 0 \leq x \leq 1$$

Discuss the efficiency of this approach.

18. Give an efficient method to generate a random variable X having density

$$f(x) = \frac{1}{.000336}x(1-x)^3, \quad .8 < x < 1$$

19. In Example 5f we simulated a normal random variable by using the rejection technique with an exponential distribution with rate 1. Show that among all exponential density functions $g(x) = \lambda e^{-\lambda x}$ the number of iterations needed is minimized when $\lambda = 1$.

20. Write a program that generates normal random variables by the method of Example 5f.

21. Let (X, Y) be uniformly distributed in a circle of radius 1. Show that if R is the distance from the center of the circle to (X, Y) then R^2 is uniform on $(0, 1)$.

22. Write a program that generates the first T time units of a Poisson process having rate λ.

23. To complete a job a worker must go through k stages in sequence. The time to complete stage i is an exponential random variable with rate λ_i, $i = 1, \ldots, k$. However, after completing stage i the worker will only go to the next stage with probability α_i, $i = 1, \ldots, k-1$. That is, after completing stage i the worker will stop working with probability $1 - \alpha_i$. If we let X denote the amount of time that the worker spends on the job, then X is called a *Coxian* random variable. Write an algorithm for generating such a random variable.

24. Buses arrive at a sporting event according to a Poisson process with rate 5 per hour. Each bus is equally likely to contain either $20, 21, \ldots, 40$ fans, with the numbers in the different buses being independent. Write an algorithm to simulate the arrival of fans to the event by time $t = 1$.

25. (a) Write a program that uses the thinning algorithm to generate the first 10 time units of a nonhomogeneous Poisson process with intensity function

$$\lambda(t) = 3 + \frac{4}{t+1}$$

(b) Give a way to improve upon the thinning algorithm for this example.

26. Give an efficient algorithm to generate the first 10 times units of a nonhomogeneous Poisson process having intensity function

$$\lambda(t) = \begin{cases} \frac{t}{5}, & 0 < t < 5 \\ 1 + 5(t-5), & 5 < t < 10 \end{cases}$$

References

Dagpunar, T., *Principles of Random Variate Generation.* Clarendon Press, Oxford, 1988.

Devroye, L., *Nonuniform Random Variate Generation.* Springer-Verlag, New York, 1986.

Fishman, G. S., *Principles of Discrete Event Simulation.* Wiley, New York, 1978.

Knuth, D., *The Art of Computer Programming*, Vol. 2, 2nd ed., *Seminumerical Algorithms.* Addison-Wesley, Reading, MA, 2000.

Law, A. M., and W. D. Kelton, *Simulation Modelling and Analysis*, 3rd ed. McGraw-Hill, New York, 1997.

Lewis, P. A. W., and G. S. Shedler, "Simulation of Nonhomogeneous Poisson Processes by Thinning," *Nav. Res. Log. Quart.*, **26**, 403–413, 1979.

Marsaglia, G., "Generating Discrete Random Variables in a Computer," *Commun. Assoc. Comput. Mach.*, **6**, 37–38, 1963.

Morgan, B. J. T., *Elements of Simulation.* Chapman and Hall, London, 1983.

Ripley, B. D., "Computer Generation of Random Variables: A Tutorial," *Inst. Statist. Rev.*, **51**, 301–319, 1983.

Ripley, B. D., *Stochastic Simulation.* Wiley, New York, 1986.

Rubenstein, R. Y., *Simulation and the Monte Carlo Method.* Wiley, New York, 1981.

Schmeiser, B. W., "Random Variate Generation, a Survey," *Proc. 1980 Winter Simulation Conf.,* Orlando, FL; pp. 79–104, 1980.