Since the density function of (X, Y) is constant in the square, it thus follows (by definition) that (X, Y) is uniformly distributed in the square. Now if U is uniform on (0, 1) then $2U$ is uniform on (0, 2), and so $2U - 1$ is uniform on $(-1, 1)$. Therefore, if we generate random numbers U_1 and U_2, set $X = 2U_1 - 1$ and $Y = 2U_2 - 1$, and define

$$I = \begin{cases} 1 & \text{if } X^2 + Y^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$E[I] = P\{X^2 + Y^2 \leq 1\} = \frac{\pi}{4}$$

Hence we can estimate $\pi/4$ by generating a large number of pairs of random numbers u_1, u_2 and estimating $\pi/4$ by the fraction of pairs for which $(2u_1 - 1)^2 + (2u_2 - 1)^2 \leq 1$. □

Thus, random number generators can be used to generate the values of uniform (0, 1) random variables. Starting with these random numbers we show in Chapters 4 and 5 how we can generate the values of random variables from arbitrary distributions. With this ability to generate arbitrary random variables we will be able to simulate a probability system—that is, we will be able to generate, according to the specified probability laws of the system, all the random quantities of this system as it evolves over time.

Exercises

1. If $x_0 = 5$ and

$$x_n = 3x_{n-1} \bmod 150$$

find $x_1, \ldots, x_{10}$.

2. If $x_0 = 3$ and

$$x_n = (5x_{n-1} + 7) \bmod 200$$

find $x_1, \ldots, x_{10}$.

In Exercises 3–9 use simulation to approximate the following integrals. Compare your estimate with the exact answer if known.

3. $\int_0^1 \exp\{e^x\}\, dx$

4. $\int_0^1 (1-x^2)^{3/2}\,dx$

5. $\int_{-2}^{2} e^{x+x^2}\,dx$

6. $\int_0^\infty x(1+x^2)^{-2}\,dx$

7. $\int_{-\infty}^{\infty} e^{-x^2}\,dx$

8. $\int_0^1\int_0^1 e^{(x+y)^2}\,dy\,dx$

9. $\int_0^\infty\int_0^x e^{-(x+y)}\,dy\,dx$

[*Hint:* Let $I_y(x) = \begin{cases} 1 & \text{if } y < x \\ 0 & \text{if } y \geq x \end{cases}$ and use this function and use this function to equate the integral to one in which both terms go from 0 to ∞.]

10. Use simulation to approximate $\text{Cov}(U, e^U)$, where U is uniform on (0, 1). Compare your approximation with the exact answer.

11. Let U be uniform on (0, 1). Use simulation to approximate the following:

(a) $\text{Corr}(U, \sqrt{1-U^2})$.
(b) $\text{Corr}(U^2, \sqrt{1-U^2})$.

12. For uniform (0, 1) random variables $U_1, U_2, \ldots$ define

$$N = \text{Minimum}\left\{n : \sum_{i=1}^{n} U_i > 1\right\}$$

That is, N is equal to the number of random numbers that must be summed to exceed 1.

(a) Estimate $E[N]$ by generating 100 values of N.
(b) Estimate $E[N]$ by generating 1000 values of N.
(c) Estimate $E[N]$ by generating 10,000 values of N.
(d) What do you think is the value of $E[N]$?

13. Let $U_i, i \geq 1$, be random numbers. Define N by

$$N = \text{Maximum}\left\{n : \prod_{i=1}^{n} U_i \geq e^{-3}\right\}$$

where $\prod_{i=1}^{0} U_i \equiv 1$.

(a) Find $E[N]$ by simulation.
(b) Find $P\{N = i\}$, for $i = 0, 1, 2, 3, 4, 5, 6,$ by simulation.