(Strictly speaking we need to justify exchanging the limit and expectation. Although we won't do this here, it can be done provided f is bounded and continuous.) Thus if $U_1, \ldots, U_n$ are an iid sample of $U(a,b)$ random variables, then our estimate of I is

$$\hat{I} = \frac{1}{n}\sum_{i=1}^{n} f(U_i)(b-a).$$

The following function performs Monte-Carlo integration of the function `ftn` over the interval $[a,b]$.

```
mc.integral <- function(ftn, a, b, n) {
  # Monte-Carlo integral of ftn over [a, b] using a sample of size n
  u <- runif(n, a, b)
  x <- sapply(u, ftn)
  return(mean(x)*(b-a))
}
```

19.2.2 Accuracy in higher dimensions

The big-O notation is used to describe how fast a function grows. We say $f(x)$ is $O(x^{-\alpha})$ if $\limsup_{x\to\infty} f(x)/x^{-\alpha} = \limsup_{x\to\infty} f(x)x^{\alpha} < \infty$.

Let d be the dimension of our integral and n the number of function calls used, then the accuracy of the different numerical integration techniques we have seen is as follows:

Method	Error
Trapezoid	$O(n^{-2/d})$
Simpson's rule	$O(n^{-4/d})$
Hit-and-miss Monte-Carlo	$O(n^{-1/2})$
Improved Monte-Carlo	$O(n^{-1/2})$

We see that the size of the error for the Monte-Carlo methods does not depend on d and that, asymptotically, they are preferable when $d > 8$.

19.3 Exercises

1. Suppose that X and Y are iid $U(0,1)$ random variables.

   (a). What is $\mathbb{P}((X,Y) \in [a,b] \times [c,d])$ for $0 \le a \le b \le 1$ and $0 \le c \le d \le 1$?
   Based on your previous answer, what do you think you should get for $\mathbb{P}((X,Y) \in A)$, where A is an arbitrary subset of $[0,1] \times [0,1]$?

   (b). Let $A = \{(x,y) \in [0,1] \times [0,1] : x^2 + y^2 \le 1\}$. What is the area of A?

(c). Define the rv Z by

$$Z = \begin{cases} 1 & \text{if } X^2 + Y^2 \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\mathbb{E}Z$?

(d). By simulating Z, write a program to estimate π.

2. Which is more accurate, the hit-and-miss method or the improved Monte-Carlo method? Suppose that $f : [0,1] \to [0,1]$ and we wish to estimate $I = \int_0^1 f(x)\,dx$.

   Using the hit-and-miss method, we obtain the estimate

   $$\hat{I}_{HM} = \frac{1}{n}\sum_{i=1}^{n} X_i,$$

   where $X_1, \ldots, X_n$ are an iid sample and $X_i \sim \text{binom}(1, I)$ (make sure you understand why this is the case).

   Using the improved Monte-Carlo method, we obtain the estimate

   $$\hat{I}_{MC} = \frac{1}{n}\sum_{i=1}^{n} f(U_i),$$

   where $U_1, \ldots, U_n$ are an iid sample of $U(0,1)$ random variables.

   The accuracy of the hit-and-miss method can be measured by the standard deviation of $\hat{I}_{HM}$, which is just $1/\sqrt{n}$ times the standard deviation of X_1. Similarly the accuracy of the basic Monte-Carlo method can be measured by the standard deviation of $\hat{I}_{MC}$, which is just $1/\sqrt{n}$ times the standard deviation of $f(U_1)$.

   Show that

   $$\operatorname{Var} X_1 = \int_0^1 f(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2,$$

   and that

   $$\operatorname{Var} f(U_1) = \int_0^1 f^2(x)\,dx - \left(\int_0^1 f(x)\,dx\right)^2.$$

   Explain why (in this case at least) the improved Monte-Carlo method is more accurate than the hit-and-miss method.

3. The previous exercise gave a theoretical comparison of the hit-and-miss and improved Monte-Carlo method. Can you verify this experimentally?

   Repeat the example of Section 19.1 using the improved Monte-Carlo method. How many function calls are required to get 2 decimal places accuracy?

4. The trapezoidal rule for approximating the integral $I = \int_0^1 f(x)\,dx$ can be broken into two steps

   Step 1: $I = \sum_{i=0}^{n-1}$ (Area under the curve from i/n to $(i+1)/n$);