# Chapter 15

# Unbalanced government budget constraint

In this chapter, we build off of the $S$-period-lived agent model from Chapter 7 in order to understand how one may model a government that can run deficits for a finite number of periods. In order to understand such a model of the government's budget constraint, we add linear taxes on labor, capital, and corporate income as the source of the government's revenue. The government uses tax revenues and debt to finance spending on a government consumption good and lump sum transfers to households.

## 15.1 Households

The basic structure of the household's problem from Chapter 7 remains the same, but the addition of taxes and transfers alters the household's budget constraint and thus the necessary conditions describing optimal savings and labor supply.

Letting $\tau^l_t$ and $\tau^k_t$ represent the constant tax rate on labor and capital income, respectively, and letting $x_{s,t}$ represent government transfers to households of age $s$ in period $t$, we can write the household's budget constraint as:

$$\begin{split} c_{s,t} + b_{s+1,t+1} = (1 + (1-\tau^k_t)r_t)b_{s,t} + (1-\tau^l_t)w_t n_{s,t} + x_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{split} \tag{5.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{7.6}$$

$$\text{s.t.} \quad c_{s,t} + b_{s+1,t+1} \tag{15.1}$$

$$= (1+(1-\tau_t^k)r_t)b_{s,t} + (1-\tau_t^l)w_t n_{s,t} + x_t \tag{15.2}$$

$$\text{where} \quad u\big(c_{s,t},n_{s,t}\big) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} + \chi_s^n b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \tag{15.3}$$

The set of optimal lifetime choices for an agent born in period $t$ are characterized by the following $S$ static labor supply Euler equations (15.4), the following $S-1$ dynamic savings Euler equations (15.5), and a budget constraint that binds in all $S$ periods (15.2),

$$\begin{aligned} &(1-\tau_t^l)w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ &\Rightarrow \quad (1-\tau_t^l)w_t\left(c_{s,t}\right)^{-\sigma} = \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{15.4}$$

$$\begin{aligned} &u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = \beta(1+(1-\tau_t^k)r_{t+1})u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\} \\ &\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta(1+(1-\tau_t^k)r_{t+1})\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{15.5}$$

$$c_{s,t} + b_{s+1,t+1} = (1+(1-\tau_t^k r_t)b_{s,t} + (1-\tau_t^l w_t n_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{15.2}$$

where $u_1$ is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and $u_2$ is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$.

These $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u,r_u\}_{u=t}^{t+S-1}$. As in section 6.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by

(6.14) and (5.17).

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S} \quad \forall t \tag{6.14}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1 \tag{5.17}$$

## 15.2 Firms

Firms are characterized similarly to Section 5.2, with the firm's aggregate capital decision $K_t$ governed by first order condition (15.7) and its aggregate labor decision $L_t$ governed by first order condition (15.8). However, the addition of the corporate income tax alters the firm's profit maximization problem slightly.

The firm seeks to maximize after-tax profits and thus solves,

$$\max_{K_t,L_t} \left(1-\tau^c_t\right)\left(Y_t - w_t L_t\right) - (r_t+\delta)K_t + \tau^c_t\delta K_t \tag{15.6}$$

Note that the corporate income tax is levied on accounting profits. Thus wage expenses and depreciation expenses are deductible, but payments to capital are not.

In the presence of the corprate income tax, the two first order conditions that characterize firm optimization are the following.

$$r_t = \left(1-\tau^c_t\right)\left(\alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta\right) \tag{15.7}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{15.8}$$

## 15.3 Government

The government takes in tax revenues and uses those revenues and borrowing to finance government purchases, $G_t$, and transfers, $X_t = \sum_{s=1}^{S} x_{s,t}$. In this model, we'll assume that these transfers are distributed lump-sum to all agents. Thus, $x_{s,t} = \frac{X_t}{S}\ \forall\ s$. We let $D_t$ denote the stock of government debt at time $t$ and $R_t$ denote total government tax revenue. Thus we write the government budget constraint as:

$$D_{t+1} + R_t = (1 + r_t)D_t + G_t + X_t \tag{15.9}$$

Revenues are given by:

$$R_t = \underbrace{\tau^c_t\left(Y_t - w_t L_t\right) - \tau^c_t \delta K_t}_{\text{Corporate income tax revenue}} + \underbrace{\sum_{s=1}^{S} \tau^l_t w_t n_{s,t} + \sum_{s=2}^{S} \tau^k_t r_t b_{s,t}}_{\text{Individual income tax revenue}} \tag{15.10}$$

### 15.3.1 Budget Closure Rule

We've specified how government debt evolves (Equation 15.9) and how revenues are determined. We need to also specify how government purchase and transfers are determined. Before we do that, we point out that that government's budget must balance over the infinite horizon. In this simple model, if government debt grows in the steady-state, then at some point, payments of interest would exceed total economic output, which is not feasible. In a richer model, with economic growth, government debt can grow, but it cannot outpace the growth of total output for the same reason noted in the previous sentence. What we do to impose this fiscal condition is to alter the path of government spending in order to hit a target debt to GDP ratio in the steady-state.

We assume that transfers are a constant fraction of GDP in all periods:

$$X_t = \alpha_X * Y_t \tag{15.11}$$

Given this path for transfers and revenues, $G_t$ adjusts to stabilize the debt to GDP ratio. Let $\alpha_D$ be the target debt to GDP ratio in the steady state. In the initial periods, we assume that the ratio of government spending to GDP remains constaint, $G_t = \alpha_G * Y_t$. At some future period, $t_{G1}$, government spending begins to adjust to move towards this target debt to GDP ratio. At priod $t_{G2}$, government spending is adjusted to hit the target debt to GDP ratio, if it has not already been reached. Letting $\rho_G$ be the paramter that describes how quickly the convergences to the steady state debt to GDP ratio takes place and, we right the law of motion for government spending as:

$$
\begin{aligned}
&\text{For } t < t_{G1}: \\
&\qquad G_t = \alpha_G Y_t \\
&\text{For } t_{G1} \leq t < t_{G2}: \\
&\qquad G_t = [\rho_G \alpha_D Y_t + (1-\rho_G) D_t] - (1+r_t) D_t - X_t + R_t \\
&\text{For } t \geq t_{G2}: \\
&\qquad G_t = \alpha_D Y_t - (1+r_t) D_t - X_t + R_t
\end{aligned}
\tag{15.12}
$$

This law of motion for the fiscal variables, together with the amount of debt in the initial period of the time path, will allow us to solve for the values government spending and debt at each period. We set the intial debt level to match the debt to GDP ratio in the economy in the year we want the model to being. Let $\alpha_{D0}$ represent the debt to GDP ratio in the initial period. Thus $D_1 = \alpha_{D0} * Y_1$.

A few notes on this closure rule are in order. First, we chose to adjust government spending because of the simplicity of doing so. Since government spending does not enter into the household's utility function, it's level does not affect the solution of the household problem. This simplifies the model solution significantly. That said, one could chose to adjust taxes or transfers to close the budget (or a combination of all of the government fiscal policy levers). Second, since government spending is doing all of the lifting to hit the target debt to GDP ratio, it is possible that government spending is forced to be less than zero to make this happen. This would be the case if tax revenues bring in less than is needed to financed transfers and interest payments on the national debt. None of the equations we've specified above preclude that result, but it does raise conceptual difficulties. Namely, what does it mean for government spending to be negative? Is the government selling off pubic assets? We caution those using this budget closure rule to consider carefully how the budget is closed in the long run given their parameterization. We'll also note that such difficulties present themselves across all budget closure rules when analyzing tax or spending proposals that induce strucutral budget deficits.

## 15.4 Market Clearing

Four markets must clear in this model: the labor market, the capital market, the bond market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t^d = L_t^s \quad \forall t \tag{15.13}$$

$$K_t^d = K_t^s \quad \forall t \tag{15.14}$$

$$D_t^d = D_t^s \quad \forall t \tag{15.15}$$

$$\begin{split} &Y_t = C_t + I_t \quad \forall t \\ &\quad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{split} \tag{15.16}$$

The goods market clearing equation (**??**) is redundant by Walras' Law.

Labor supply is straight forward and given by $L_t^s = \sum_{s=1}^{S} n_{s,t}$. In this model households hold two differnt assets; capital and government debt. Both are risk free and thus will yield the same rate of return in equilibrium.[1] Given this, we do not differentiate between the household's holdings of capital and government debt. Rather, $b_{s,t}$ represents total assets held by a household of age $s$ at time $t$ and is (potentially) a mix of capital and government debt. Thus, we have that

$$K_t^s + D_t^s = B_t = \sum_{i=2}^{S} b_{s,t}, \tag{15.17}$$

and thus by the capital and bond market clearing conditions we have:

$$B_t = \sum_{i=2}^{S} b_{s,t} = K_t^d + D_t^d \tag{15.18}$$

## 15.5 Equilibrium

An equilibrium is found when:

[1]Though in priciple this assumption could be relaxed and a wedge placed between the return on government debt and private capital.

i. Households optimize according to equations (15.4) and (15.5).

ii. Firms optimize according to (15.7) and (15.8).

iii. Government debt evolves according to (15.9) and (15.12).

iv. Markets clear according to (15.13) and (15.14).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

We can characterize the equilibrium solution by solving for the amount of government debt, $D_t^d$, as a function of the household's decisions and then substituting the market clearing conditions (15.13), (15.14), and (15.18) into the firm's optimal conditions (15.7) and (15.8) to solve for the equilibrium wage and interest rate as functions of the distribution of capital.

To find debt, we use the government budget constraint as characterized in (15.9) along with the exogenous laws of motion describing government spending nad transfers.

$$
\begin{aligned}
D_{t+1} &= (1+r_t)D_t + G_t + X_t - R_t \\
&= (1+r_t)D_t + \alpha_G Y_t + \alpha_X Y_t - \tau_t^c\left(Y_t - w_t L_t\right) - \tau_t^c \delta K_t^d - \sum_{s=1}^{S}\tau_t^l w_t n_{s,t} - \sum_{s=1}^{S}\tau_t^k r_t b_{s,t} \\
&= (1+r_t)D_t + (\alpha_G + \alpha_X)\left(A_t\left(\sum_{s=1}^{S} b_{s,t}\right)^{\alpha}\left(\sum_{s=1}^{S} n_{s,t}\right)^{1-\alpha}\right) - \tau_t^c\Bigg(\Bigg(A_t\left(\sum_{s=1}^{S} b_{s,t}\right)^{\alpha}\left(\sum_{s=1}^{S} n_{s,t}\right)^{1-\alpha}\Bigg) - \\
&\quad - \tau_t^c\delta\sum_{s=1}^{S} b_{s,t} - \sum_{s=1}^{S}\tau_t^l w_t n_{s,t} - \sum_{s=1}^{S}\tau_t^k r_t b_{s,t}
\end{aligned}
\tag{15.19}
$$

Note that the initial debt to GDP ratio is an exogenous initial condition and the steady-state target debt to GDP ratio is also exogenous. Thus the path for debt is entirely pinned down by the distribution of savings and labor supply, $D_t(\mathbf{\Gamma}_t)$.

We can then use the firm's first order conditoins together with the market clearning conditons to show that the equilibrium interest rate and wage rates are functions of the distributions of savings and labor supply.

$$w_t(\mathbf{\Gamma}_t): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t} - D_t(\Gamma_t)}{\sum_{s=1}^S n_{s,t}}\right)^\alpha \quad \forall t \tag{15.20}$$

$$r_t(\mathbf{\Gamma}_t): \quad r_t = (1-\tau_t^c)\left(\alpha A\left(\frac{\sum_{s=1}^S n_{s,t}}{\sum_{s=2}^S b_{s,t} - D_t(\Gamma_t)}\right)^{1-\alpha} - \delta\right) \quad \forall t \tag{15.21}$$

Now (15.20), (15.21), and the budget constraint (15.2) can be substituted into household Euler equations (15.4) and (15.5) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$\begin{split}(1-\tau_t^l)w_t(\mathbf{\Gamma}_t)\left((1-\tau_t^l)w_t(\mathbf{\Gamma}_t)n_{s,t} + \Big[1+(1-\tau_t^k)r_t(\mathbf{\Gamma}_t)\Big]b_{s,t} - b_{s+1,t+1}\right)^{-\sigma} = \\ \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \\ \text{for} \quad s\in\{1,2,...S\} \quad \text{and} \quad \forall t\end{split} \tag{15.22}$$

$$\begin{split}\left((1-\tau_t^l)w_t(\mathbf{\Gamma}_t)n_{s,t} + \Big[1+(1-\tau_t^k)r_t(\mathbf{\Gamma}_t)\Big]b_{s,t} - b_{s+1,t+1}\right)^{-\sigma} = \\ \beta\Big[1+(1-\tau_t^k)r_{t+1}(\mathbf{\Gamma}_{t+1})\Big]\left((1-\tau_t^l)w_{t+1}(\mathbf{\Gamma}_{t+1})n_{s+1,t+1} + \Big[1+(1-\tau_t^k)r_{t+1}(\mathbf{\Gamma}_{t+1})\Big]b_{s+1,t+1} - b_{s+2,t+2}\right)^{-\sigma} \\ \text{for} \quad s\in\{1,2,...S-1\} \quad \text{and} \quad \forall t\end{split} \tag{15.23}$$

The system of $S$ nonlinear static equations (15.22) and $S-1$ nonlinear dynamic equations (15.23) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^S$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable $x_t$ be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

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**Definition 15.1 (Steady-state equilibrium).** A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with $S$-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (15.4) and (15.5),

ii. firms optimize according to (15.7) and (15.8),

iii. government debt satisfies (15.9)

iv. markets clear according to (15.13) and (15.18).

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The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s(\mathbf{\Gamma}_t)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s(\mathbf{\Gamma}_t)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

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**Definition 15.2 (Non-steady-state functional equilibrium).** A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with $S$-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\big\{n_{s,t} = \phi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^S$, $\big\{b_{s+1,t+1} = \psi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^{S-1}$ and and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (5.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (15.4) and (15.5),

iii. firms optimize according to (15.7) and (15.8),

iv. government debt evolves according to (15.9) and (15.12),

v. markets clear according to (15.13) and (15.18).

## 15.6 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 15.1 and the transition path equilibrium from Definition ??.

### 15.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 15.1. The parameters needed for the steady-state solution of this model are $\Big\{S,\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S,A,\alpha,\delta,\bar{\tau}^l,\bar{\tau}^k,\bar{\tau}^c,\alpha_X,\alpha_D\Big\}$, where $S$ is the number of periods in an individual's life, $\Big\{\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S\Big\}$ are household utility function parameters, $\{A,\alpha,\delta\}$ are firm production function parameters, and $\{\bar{\tau}^l,\bar{\tau}^k,\bar{\tau}^c,\alpha_X,\alpha_D\}$ are parameters that describe steady-state fiscal policies. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{s,t}=\bar{c}_s,\quad n_{s,t}=\bar{n}_s,\quad b_{s,t}=\bar{b}_s\quad\forall s,t \tag{15.24}$$

From the market clearing conditions (15.13) and (6.16) and the firms' first order equations (15.7) and (15.8), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t=\bar{r},\quad w_t=\bar{w},,\quad B_t=\bar{B},\quad D_t=\bar{D},\quad K_t=\bar{K},\quad G_t=\bar{G},\quad L_t=\bar{L}\quad\forall t \tag{15.25}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$(1-\bar{\tau}^l)\bar{w}\Big([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_s + (1-\bar{\tau}^l)\bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{15.26}$$

$$\Big([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_s + (1-\bar{\tau}^l)\bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \beta(1+\bar{r})\Big([1+(1-\bar{\tau}^k)\bar{r}]\bar{b}_{s+1} + (1-\bar{\tau}^l)\bar{w}\bar{n}_{s+1} - \bar{b}_{s+2}\Big)^{-\sigma} \quad \text{for} \quad s=\{1,2,...S-1\} \tag{15.27}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distribution of labor supply and savings as shown in (15.20) and (15.21). We solve this system of equations using the robust solution method proposed in Sectoin 7.6.1. Here, we need to update the method for the new specification of fiscal policy. In particular, we need to know government transfers to households when solving the household problem and we need to know government debt for the asset market clearing condition. This method is a bisection method in the outer loop guesses for steady-state equilibrium $\bar{K}$ and $\bar{L}$. In the inner loop of the household's problem given $r$ and $w$ implied by $K$ and $L$, we solve the problem by breaking the multivariate root finder problem with $2S-1$ equations and unknowns into a series of many univariate root finder problems and one bivariate root finder problem. The algorithm is the following.

i. Make a guess for the steady-state aggregate capital stock $\bar{K}^i$ and aggregate labor $\bar{L}^i$.

   (a) Values for $\bar{K}^i$ and $\bar{L}^i$ will imply values for the interest rate $\bar{r}^i$ and wage $\bar{w}^i$ from (15.7) and (15.8). In addition, $\bar{K}^i$ and $\bar{L}^i$ imply values for $\bar{Y}^i$, which yields $\bar{X}^i = \alpha_X * \bar{Y}^i$. We can thus find $\bar{x}^i = \frac{\bar{X}^i}{S}$. All of the triple, $\{\bar{r}^i, \bar{w}^i, \bar{x}^i\}$ are necessary to solve the households' problems.

   (b) For the bisection method, we must use guesses for $K$ and $L$ because those uniquely determine $r$ and $w$, whereas the converse is not true. From the firms' first order conditions (15.7) and (15.8), we see that $r$ and $w$ are functions of the same capital-labor ratio $K/L$. Infinitely many combinations of $K$ and $L$ determine a given $r$

and $w$.

ii. Given $\bar{r}^i$ and $\bar{w}^i$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

(a) Given $\bar{r}^i$, $\bar{w}^i$ and $\bar{x}^i$, guess an initial steady-state consumption $\bar{c}_1^m$, where $m$ is the index of the inner-loop (household problem given $\bar{r}^i$, $\bar{w}^i$) iteration.

(b) Given $\bar{r}^i$, $\bar{w}^i$, $\bar{x}^i$, and $\bar{c}_1^m$, use the sequence of $S-1$ dynamic savings Euler equations (15.5) to solve for the implied series of steady-state consumptions $\{\bar{c}_s^m\}_{s=1}^S$. This sequence has an analytical solution.

$$\bar{c}_{s+1}^m = \bar{c}_s^m \left[\beta(1 + (1-\bar{\tau}^k)\bar{r}^i)\right]^{\frac{1}{\sigma}} \quad \text{for} \quad s = \{1, 2, ...S-1\} \tag{15.28}$$

(c) Given $\bar{r}^i$, $\bar{w}^i$, and $\{\bar{c}_s^m\}_{s=1}^S$, solve for the series of steady-state labor supplies $\{\bar{n}_s^m\}_{s=1}^S$ using the $S$ static labor supply Euler equations (15.4). This will require a series of $S$ separate univariate root finders or one multivariate root finder.

$$(1-\bar{\tau}^l)\bar{w}^i \left(\bar{c}_s^m\right)^{-\sigma} = \chi_s^n \left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s = \{1, 2, ...S\} \tag{15.29}$$

It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

(d) Given $\bar{r}^i$, $\bar{w}^i$, and implied steady-state consumption $\{\bar{c}_s^m\}_{s=1}^S$ and labor supply $\{\bar{n}_s^m\}_{s=1}^S$, solve for implied time path of savings $\{\bar{b}_{s+1}^m\}_{s=1}^S$ across all ages of the representative lifetime using the household budget constraint (5.1).

$$\bar{b}_{s+1}^m = (1 + (1-\bar{\tau}^k)\bar{r}^i)\bar{b}_s^m + (1-\bar{\tau}^l)\bar{w}^i\bar{n}_s^m + \bar{x}^i - \bar{c}_s^m \quad \text{for} \quad s = \{1, 2, ...S\} \tag{15.30}$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}_1^m$ as in step (a).

(e) Update the initial guess for $\bar{c}_1^m$ to $\bar{c}_1^{m+1}$ until the implied savings in the last period equals zero $\bar{b}_{S+1}^{m+1} = 0$.

iii. Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ based on the guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$, we can solve for aggregate tax revenues using (15.10).

$$\bar{R}^i = \bar{\tau}^c\left(\bar{Y}^i - \bar{w}^i\bar{L}^i\right) - \bar{\tau}^c\delta\bar{K}^i + \sum_{s=1}^S \bar{\tau}^l\bar{w}^i\bar{n}_s + \sum_{s=2}^S \bar{\tau}^k\bar{r}^i\bar{b}_s \tag{15.31}$$

iv. The guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$ imply $\bar{Y}^i$. Our target steady-state debt to GDP ration, $\alpha_D$ then gives us steady-state govenment debt.

$$\bar{D}^i = \alpha_D\bar{Y}^i \tag{15.32}$$

v. With total tax revenues and the steady-state amount of debt determined, we can use the government budget constraint defined in (15.9) to find the steady state amount of government spending.

$$\bar{G}^i = \bar{R}^i - \bar{X}^i - \bar{r}^i\bar{D}^i \tag{15.33}$$

vi. Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ based on the guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$, we can solve for the aggregate capital $\bar{K}^{i'}$ and aggregate labor $\bar{L}^{i'}$ implied by the household solutions and market clearing conditions.

$$\bar{B}^i = \sum_{s=2}^S \bar{b}_s^m \tag{15.34}$$

$$\bar{K}^{i'} = \bar{B}^i - \bar{D}^i \tag{15.35}$$

$$\bar{L}^{i'} = \sum_{s=1}^S \bar{n}_s^m \tag{15.36}$$

Update guesses for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ until the the aggregates implied by household optimization equal the initial guess for the aggregates $(\bar{K}^{i'+1}, \bar{L}^{i'+1}) = (\bar{K}^{i+1}, \bar{L}^{i+1})$.

(a) The bisection method characterizes the updated guess for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ as a convex combination of the initial guess $(\bar{K}^{i}, \bar{L}^{i})$ and the values implied by household and firm optimization $(\bar{K}^{i'}, \bar{L}^{i'})$, where the weight put on the new values $(\bar{K}^{i'}, \bar{L}^{i'})$ is given by $\xi \in (0,1]$. The value for $\xi$ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$(\bar{K}^{i+1}, \bar{L}^{i+1}) = \xi(\bar{K}^{i'}, \bar{L}^{i'}) + (1-\xi)(\bar{K}^{i}, \bar{L}^{i}) \quad \text{for} \quad \xi \in (0,1] \tag{15.37}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible aggregate capital and aggregate labor values $(K, L)$. We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (**??**). If the distance is less than some tolerance `toler` $> 0$, then the problem has converged. Otherwise continue updating the values of aggregate capital and labor using (**??**).

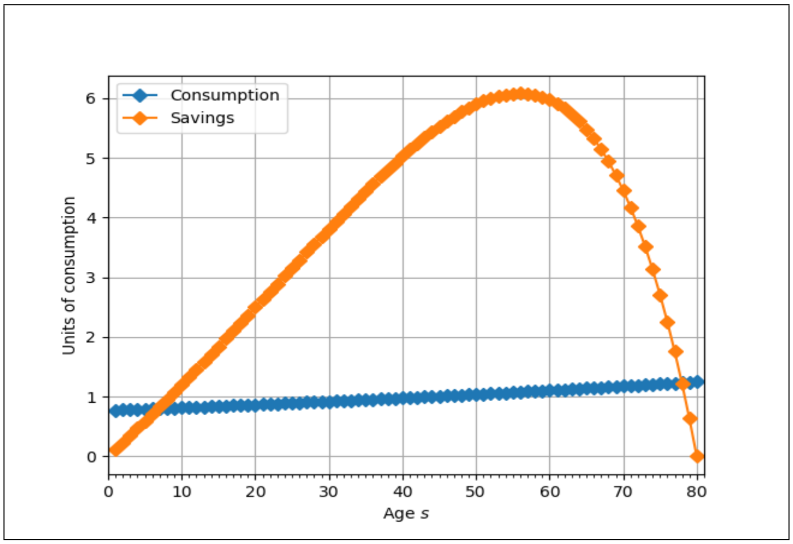
$$\texttt{dist} \equiv \left\|(\bar{K}^{i'}, \bar{L}^{i'}) - (\bar{K}^{i}, \bar{L}^{i})\right\| \tag{15.38}$$

Define the updating of aggregate variable values $(\bar{K}^{i}, \bar{L}^{i})$ in step (iii) indexed by $i$ as the "outer loop" of the fixed point solution. Although computationally intensive, the bisection method described above is the most robust solution method we have found.

Figure 15.1 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 15.3 in Section 15.7. Figure 15.2 shows the steady-state distribution of individual labor supply by age. The left side of Table 15.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing

**Figure 15.1: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$**



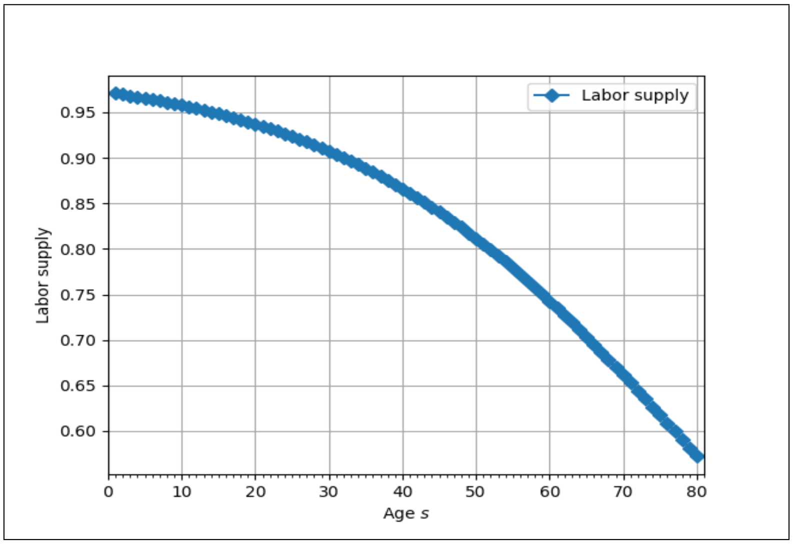

conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 15.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.5e-16, the final period individual savings is less than 6.1e-13, and the resource constraint error is less than 1.8e-06, we can be confident that we have successfully solved for the steady-state.

### 15.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the $S$-period-lived agent model with endogenous labor is similar to the method described in Section 7.6.2.

To solve for the transition path (non-steady-state) equilibrium from Definition **??**, we must know the parameters from the steady-state problem $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, \bar{\tau}^l, \bar{\tau}^k, \bar{\tau}^c, \alpha_X, \alpha_D\right\}$, the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\boldsymbol{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 15.3 and 15.1 show a particular calibration of the model and a steady-state solution. In addition, we need parameters that describe fiscal policy over the time path:$\{\alpha_G, t_{g1}, t_{G2}, \rho_G, \alpha_{D0}, \boldsymbol{\tau}^l, \boldsymbol{\tau}^k, \boldsymbol{\tau}^c\}$ The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

**Figure 15.2: Steady-state distribution of labor supply $\bar{n}_s$**




i. Choose a period $T1$ in which the initial guess for the time paths of aggregate capital and aggregate labor will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Given calibration for initial distribution of savings (wealth) $\mathbf{\Gamma}_1$, which implies an initial capital stock $K_1$, guess initial time paths for the aggregate capital stock $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_{T1}^i\}$ and aggregate labor $\boldsymbol{L}^i = \{L_1^i, L_2^i, ...L_{T1}^i\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T2 + S - 1$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period $T2$.

iii. Given time paths $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$, solve for the lifetime consumption $c_{s,t}$, labor supply $n_{s,t}$, and savings $b_{s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

   (a) The initial paths for aggregate capital $\boldsymbol{K}^i$ and aggregate labor $\boldsymbol{L}^i$ imply time paths for the interest rate $\boldsymbol{r}^i = \left\{r_1^i, r_2^i, ...r_{T2+S-1}^i\right\}$ and wage $\boldsymbol{w}^i = \left\{w_1^i, w_2^i, ...w_{T2+S-1}^i\right\}$ using the firms' first order equations (15.7) and (15.8).

   (b) Aggregate capital and labor aslo imply the time path for aggregate output, $\boldsymbol{Y}^i$.

**Table 15.1: Steady-state prices, aggregate variables, and maximum errors**

| Variable | Value | Equilibrium error | Value |
|---|---|---|---|
| $\bar{r}$ | 0.082 | Max. absolute savings Euler error | 7.44e-11 |
| $\bar{w}$ | 1.037 | Max. absolute labor supply Euler error | 1.47e-11 |
| $\bar{K}$ | 252.648 | Absolute final period savings $\bar{b}_{S+1}$ | -1.16e-13 |
| $\bar{L}$ | 66.423 | Resource constraint error | 4.20e-08 |
| $\bar{Y}$ | 106.019 | | |
| $\bar{C}$ | 79.293 | | |
| $\bar{D}$ | 42.408 | | |
| $\bar{G}$ | 14.094 | | |
| $\bar{X}$ | 10.602 | | |
| $\bar{R}$ | 28.187 | | |

Output, together with the fiscal rule for government transfers from (15.11) yield the path for government trantsfers, $\boldsymbol{X}^i$, which in turn gives the path for transfers per household: $\boldsymbol{x}^i = \frac{\boldsymbol{X}^i}{S}$.

(c) Given the time paths for the interest rate $\boldsymbol{r}^i$, wage $\boldsymbol{w}^i$, transfers $\boldsymbol{x}^i$ and the period-1 distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, solve for the lifetime decisions $c_{s,t}$, $n_{s,t}$, and $b_{s,t}$ of each household alive during periods 1 and $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 15.6.1.

iv. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ to compute total tax revenue given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ using (15.10). Call this $\boldsymbol{R}^i$

v. By putting the time path for aggregate output, $\boldsymbol{Y}^i$, transfers, $\boldsymbol{X}^i$, and revenues, $\boldsymbol{R}^i$, into the budget closure rule from (15.12), we can find the time path for government debt, $\boldsymbol{D}^i_t$, and government spending, $\boldsymbol{G}^i$.

vi. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ to compute new paths for aggregate savings and aggregate labor $\boldsymbol{B}^i$ and $\boldsymbol{L}^{i'}$ implied by the labor market clearing condition (15.13).

vii. Using the asset market clearing condition, (15.18), find the new path for aggregate capital, $\boldsymbol{K}^{i'} = \boldsymbol{B}^i - \boldsymbol{D}^i$.

viii. Compare the distance of the time paths of the new implied paths for the aggregate capital and labor $\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right)$ versus the initial aggregate capital and labor $\left(\boldsymbol{K}^i, \boldsymbol{L}^i\right)$.

$$\texttt{dist} = \left\|\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) - \left(\boldsymbol{K}^i, \boldsymbol{L}^i\right)\right\| \geq 0 \tag{15.39}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and aggregate labor $(\boldsymbol{K}^i, \boldsymbol{L}^i)$. Common norms to use are the $L^2$ and the $L^\infty$ norms.

(a) If the distance is less than or equal to some tolerance level $\texttt{dist} \leq \texttt{TPI\_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

$$\left(\boldsymbol{K}^{i+1}, \boldsymbol{L}^{i+1}\right) = \xi\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) + (1-\xi)\left(\boldsymbol{K}^i, \boldsymbol{L}^i\right) \quad \text{for} \quad \xi \in (0,1] \tag{15.40}$$

**Table 15.2: Maximum absolute errors in characterizing equations across transition path**

| Description | Value |
|---|---|
| Maximum absolute labor supply Euler error | 4.87e-13 |
| Maximum absolute savings Euler error | 8.07e-16 |
| Maximum absolute final period savings $\bar{b}_{S+1,t}$ | 0.00 |
| Maximum absolute resource constraint error | 3.20e-08 |

The 6 panels of Figure 15.3 show the equilibrium time paths of the interest rate $r_t$, wage $w_t$, and aggregate variables $K_t$, $L_t$, $Y_t$, and $C_t$. The three panels of Figure 15.4 show the transition paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$. The time path of fiscal aggregates are shown in Figure 15.5. In all of the time paths, a sharp kink is evident at $t_{G1}$, when the budget closure rule begins. Table 15.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition

**Figure 15.3: Equilibrium transition paths of prices and aggregate variables**

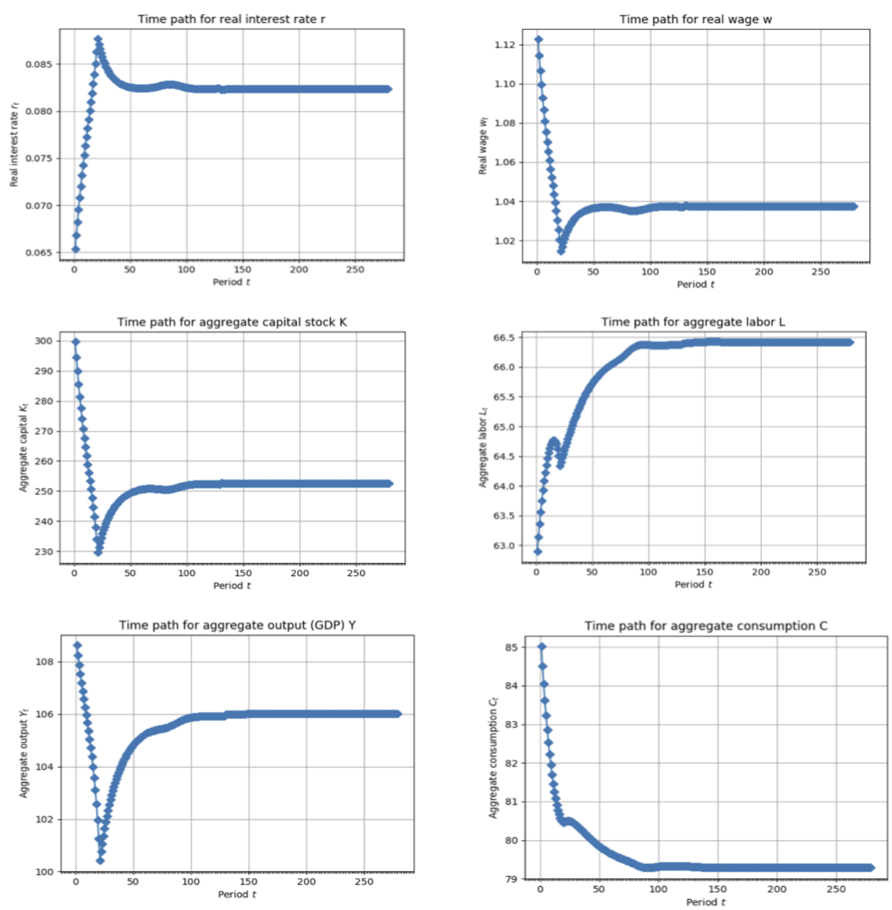


**Figure 15.4: Equilibrium transition paths of distributions of consumption, labor supply, and savings**

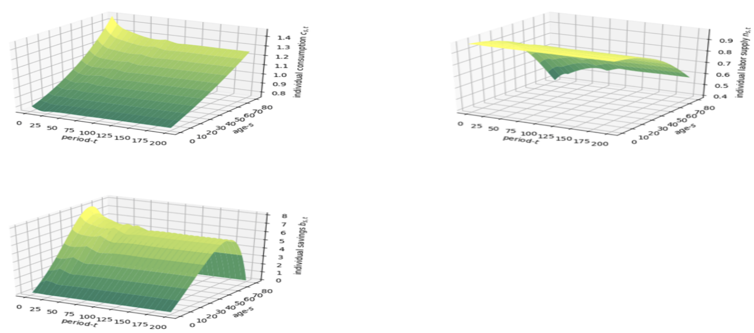
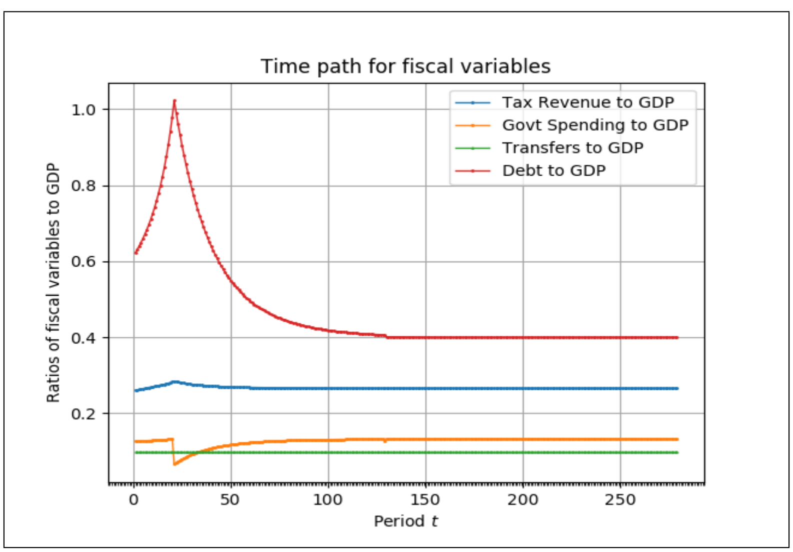


**Figure 15.5: Equilibrium transition paths of government tax revenue, transfers, spending, and debt as percentages of output**

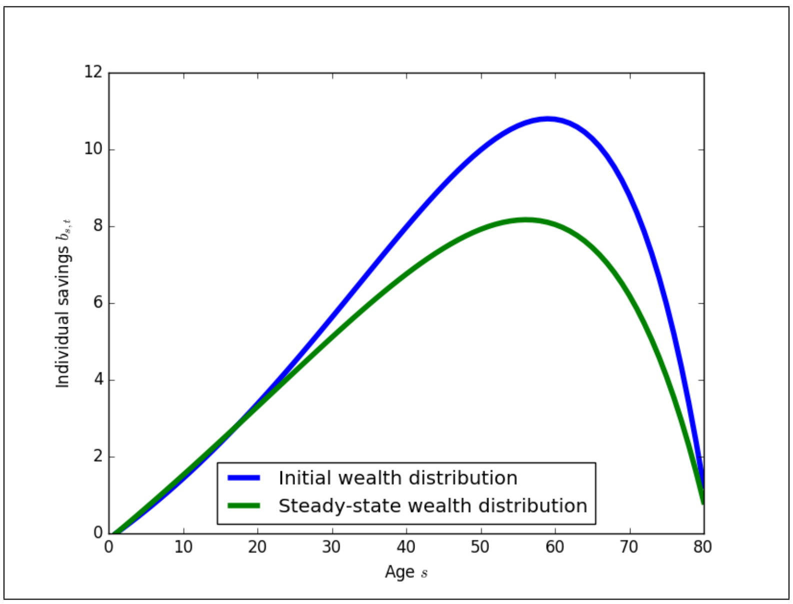


path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

## 15.7 Calibration

Use the following parameterization of the model for the problems below. Assume that agents are born at age 21 and die at age 100 (80 years of life). Your time dependent parameters can be written as functions of $S$, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$.

**Figure 15.6: Initial vs. steady-state distributions of wealth (savings)** $b_{s,t}$




Assume that each individual's time endowment in each period is $\tilde{l} = 1$. Initially, let $\chi^n_s = 1$ for all $s$. However, you could calibrate these values to match the empirical distribution of annual hours by age in the United states (see Exercise 7.5).

**Table 15.3: Calibrated parameter values for simple endogenous labor model with government debt financing**

| Parameter | Description | Value |
|---|---|---|
| $S$ | Number of periods in individual life | 80 |
| $\beta$ | Per-period discount factor | 0.96 |
| $\sigma$ | Coefficient of relative risk aversion | 2.5 |
| $\tilde{l}$ | Time endowment per period | 1.0 |
| $b$ | Elliptical disutility of labor scale parameter | 0.501[a] |
| $\upsilon$ | Elliptical disutility of labor shape parameter | 1.554[a] |
| $\{\chi^n_s\}_{s=1}^S$ | Disutility of labor relative scale factor by age | 1.0 |
| $A$ | Total factor productivity | 1.0 |
| $\alpha$ | Capital share of income | 0.35 |
| $\delta$ | Per-period depreciation rate of capital | 0.05 |
| $\alpha_X$ | Ratio of government transfers to GDP | 0.10 |
| $\alpha_G$ | Ratio of government spending to GDP up to $t_{G1}$ | 0.12 |
| $\alpha_D$ | Steady-state ratio of government debt to GDP | 0.40 |
| $\bar{\tau}^l$ | Steady-state marginal tax rate on labor income | 0.25 |
| $\bar{\tau}^k$ | Steady-state marginal tax rate on capital income | 0.30 |
| $\bar{\tau}^c$ | Steady-state marginal tax rate on corporate income | 0.15 |
| $\boldsymbol{\Gamma}_1$ | Initial distribution of savings (wealth) | (see Fig. 7.7) |
| $T1$ | Time period in which initial path guess hits steady state | 160 |
| $T2$ | Time period in which the model is assumed to hit the steady state | 200 |
| $\xi$ | TPI path updating parameter | 0.2 |
| $\alpha_{D0}$ | Ratio of government spendign to GDP in $t = 1$ | 0.59 |
| $t_{G1}$ | Period when budget closure rule begins | 20 |
| $t_{G2}$ | Period when budget closure rule ends | 128 |
| $\rho_G$ | Rate at which debt to GDP adjusts to steady-state ratio | 0.05 |
| $\boldsymbol{\tau}^l$ | Time path of tax rate on labor income | $0.25\ \forall\ t$ |
| $\boldsymbol{\tau}^k$ | Time path of tax rate on capital income | $0.30\ \forall\ t$ |
| $\boldsymbol{\tau}^c$ | Time path of tax rate on corporate income | $0.15\ \forall\ t$ |

[a] The calibration of $b$ and $\upsilon$ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

## 15.8 Exercises

TBD...

Some potential exercises:

- Write the algorithm to solve for the SS and time path if transfers are adjusted, rather than government spending, in order to hit the target debt to GDP ratio in the SS.
- Solve model with calibration given above. Plot paths of fiscal variables.
- Try to break the model. Can you make fiscal policies that are infeasible?