Chapter 20

Small Open Economy OG Model

In this chapter, we take the S-period-lived agent model from Chapter 7 and open up international markets by treating the country as a small open economy. As a small open economy, the interest rate will be taken as exogenous, being determined by capital markets in the rest of the world. Capital will flow into and out of this small open economy to equilibrate the interest rate in this economy to this world interest rate.

20.1 Households

The basic structure of the household's problem from Chapter 7 remains the same. We must only change our notation regarding the interest rate that enters the household problem to denote that it is exogenous. Let us denote the world interest rate in the period t as r_t^* and the steady-state world interest rate as $\bar{r}^*$. With this, we can write the household's budget constraint as:

$$\begin{aligned} c_{s,t} + b_{s+1,t+1} = (1+r_t^*)b_{s,t} + w_t n_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{aligned} \tag{20.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^{S},\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^{S}\beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{7.6}$$

$$\text{s.t.} \quad c_{s,t} + b_{s+1,t+1} \tag{20.2}$$

$$= (1+r_t^*)b_{s,t} + w_t n_{s,t} \tag{20.3}$$

$$\text{where} \quad u\big(c_{s,t},n_{s,t}\big) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} + \chi_s^n b\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \tag{7.7}$$

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (20.4), the following $S-1$ dynamic savings Euler equations (20.5), and a budget constraint that binds in all S periods (20.3),

$$\begin{aligned} &w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ &\Rightarrow \quad w_t\left(c_{s,t}\right)^{-\sigma} = \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \end{aligned} \tag{20.4}$$

$$\begin{aligned} &u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = \beta(1+r_{t+1}^*)u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\} \\ &\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta(1+r_{t+1}^*)\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{20.5}$$

$$c_{s,t} + b_{s+1,t+1} = (1+r_t^*)b_{s,t} + w_t n_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{20.3}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$.

These $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u, r_u\}_{u=t}^{t+S-1}$.

20.2 Firms

Firms are characterized similarly to Section 5.2, with the firm's aggregate capital decision K_t governed by first order condition (5.20) and its aggregate labor decision L_t governed by first order condition (5.21). In a small open economy, the interest rate is exogenous. So the firm's first order condition for the choice of capital will now be used to find its capital demand as this exogenous interest rate.

The firm seeks to maximize profits and thus solves,

$$\max_{K_t,L_t} Y_t - w_t L_t - (r_t^* + \delta)K_t \tag{20.6}$$

The two first order conditions that characterize firm optimization are the following.

$$r_t^* = \left(\alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta\right) \tag{20.7}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{5.21}$$

20.3 Market Clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$\begin{aligned}
&L_t^d = L_t^s = \sum_{s=1}^{S} n_{s,t} \quad \forall t \qquad\qquad \text{where} \quad I_t \\
&K_t^d = K_t^s + K_t^f = \sum_{s=2}^{S} b_{s,t} + \text{net capital inflows} \quad \forall t \\
&Y_t = C_t + I_t \quad \forall t \\
&\qquad \text{where} \quad I_t \equiv K_{t+1}^d - (1-\delta)K_t^d
\end{aligned} \tag{20.10}$$

The goods market clearing equation (20.10) is redundant by Walras' Law.

20.4 Equilibrium

An equilibrium is found when:

i. Households optimize according to equations (20.4) and (20.5).

ii. Firms optimize according to (20.7) and (**??**).

iii. Markets clear according to (20.8) and (20.9).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

Given the exogenous interest rate r^*, we can use (20.7) to solve for the capital-labor ratio in each period. Then, given the capital-labor ratio in each period, we can use (5.21) to find the wage rate in each period. Thus, given the exogenous interest rate, we can also pin down the wage rate in each period. So all factor prices are exogenous and determined by the world interest rate.

We can then use the firm's first order conditoins together with the market clearing conditons to show that the equilibrium interest rate and wage rates are functions of the distributions of savings and labor supply.

The system of S nonlinear static equations (**??**) and $S-1$ nonlinear dynamic equations (**??**) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^{S}$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is perfectly identified in this case. This is because the factor prices, r_t and w_t are both functions of the exogenous interest rate.

$\mathrm{K}_t^{(1-\alpha)}\ \forall\ t$

$w_t = (1-\alpha)A\,($

$\mathrm{L}_t^{\alpha} = (1-\alpha)A\,($

$\mathrm{r}^* + \delta^{\frac{\alpha}{1-\alpha}}\ \forall\ t$

We first define the steady-state equilibrium. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 20.1 (**Steady-state equilibrium**). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and r^* such that:

i. households optimize according to (20.4) and (20.5),

ii. firms optimize according to (5.20) and (5.21),

iii. markets clear according to (20.8) and (15.17).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s(r^*)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s(r^*)$. The function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 20.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\big\{n_{s,t} = \phi_s\big(r^*\big)\big\}_{s=1}^{S}$, $\big\{b_{s+1,t+1} = \psi_s\big(r^*\big)\big\}_{s=1}^{S-1}$ and and stationary price functions $w(r^*)$ and r^* such that:

i. households optimize according to (20.4) and (20.5),

ii. firms optimize according to (5.20) and (5.21),

iii. markets clear according to (20.8) and (15.17).

20.5 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 20.1 and the transition path equilibrium from Definition **??**.

20.5.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 20.1. The parameters needed for the steady-state solution of this model are

$\Big\{S,\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S,A,\alpha,\delta,r^*\Big\}$, where S is the number of periods in an individual's life, $\Big\{\beta,\sigma,\tilde{l},b,\upsilon,\{\chi^n_s\}_{s=1}^S\Big\}$ are household utility function parameters, $\{A,\alpha,\delta\}$ are firm production function parameters, and r^* is the exogenous world interest rate. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{s,t} = \bar{c}_s, \quad n_{s,t} = \bar{n}_s, \quad b_{s,t} = \bar{b}_s \quad \forall s,t \tag{20.11}$$

From the market clearing conditions (20.8) and (6.16) and the firms' first order equations (5.20) and (5.21), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K^d_t = \bar{K}^d_t, \quad K^s_t = \bar{K}^d, \quad L_t = \bar{L} \quad \forall t \tag{20.12}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}\Big([1+r^*]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{20.13}$$

$$\Big([1+r^*]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \beta(1+r^*)\Big([1+r^*]\bar{b}_{s+1} + \bar{w}\bar{n}_{s+1} - \bar{b}_{s+2}\Big)^{-\sigma} \quad \text{for} \quad s=\{1,2,...S-1\} \tag{20.14}$$

where $\bar{w}$ is a function of r$^*asinequation(20.4).Giventhatbothfactorpricesaredeterminedthroughtheexogen$
$1equationsandunknownsintoaseriesofmanyunivariaterootfinderproblemsandonebivariaterootfinderpr$

Given r^* and $\bar{w}$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

i. Given r* and $\bar{w}$, guess an initial steady-state consumption $\bar{c}_1^m$, where m is the index of the inner-loop (household problem given r^*, $\bar{w}$) iteration.

ii. Given $\bar{r}^i$, $\bar{w}^i$, $\bar{x}^i$, and $\bar{c}_1^m$, use the sequence of $S-1$ dynamic savings Euler equations (20.5) to solve for the implied series of steady-state consumptions $\{\bar{c}_s^m\}_{s=1}^S$. This sequence has an analytical solution.

$$\bar{c}_{s+1}^m = \bar{c}_s^m \left[\beta(1+r^*)\right]^{\frac{1}{\sigma}} \quad \text{for} \quad s = \{1,2,...S-1\} \tag{20.15}$$

iii. Given r^*, $\bar{w}$, and $\{\bar{c}_s^m\}_{s=1}^S$, solve for the series of steady-state labor supplies $\{\bar{n}_s^m\}_{s=1}^S$ using the S static labor supply Euler equations (20.4). This will require a series of S separate univariate root finders or one multivariate root finder.

$$\bar{w}\left(\bar{c}_s^m\right)^{-\sigma} = \chi_s^n \left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s = \{1,2,...S\} \tag{20.16}$$

It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

iv. Given r^*, $\bar{w}$, and implied steady-state consumption $\{\bar{c}_s^m\}_{s=1}^S$ and labor supply $\{\bar{n}_s^m\}_{s=1}^S$, solve for implied time path of savings $\{\bar{b}_{s+1}^m\}_{s=1}^S$ across all ages of the representative lifetime using the household budget constraint (5.1).

$$\bar{b}_{s+1}^m = (1+r^*)\bar{b}_s^m + \bar{w}\bar{n}_s^m - \bar{c}_s^m \quad \text{for} \quad s = \{1,2,...S\} \tag{20.17}$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}_1^m$ as in step (a).

v. Update the initial guess for $\bar{c}_1^m$ to $\bar{c}_1^{m+1}$ until the implied savings in the last period equals zero $\bar{b}_{S+1}^{m+1} = 0$.

Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ we can sollve for the aggregate capital supplied $\bar{K}^s$ and aggregate labor $\bar{L}$ implied by the household solutions and market clearing conditions.

With $\bar{L}$ in hand, we can use the firms' FOC for choice of labor to solve for capital demand, $\bar{K}^d$.

Finally, the net capital inflows, $\bar{K}^f$ and be found as the difference between the capital demand from firms and the capital supply by households: $\bar{K}^f = \bar{K}^d - \bar{K}^s$.

Figure 20.1: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$

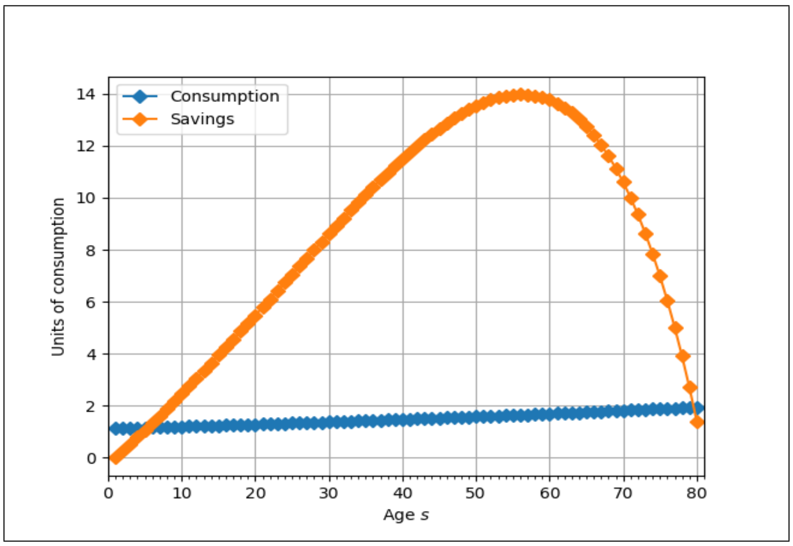
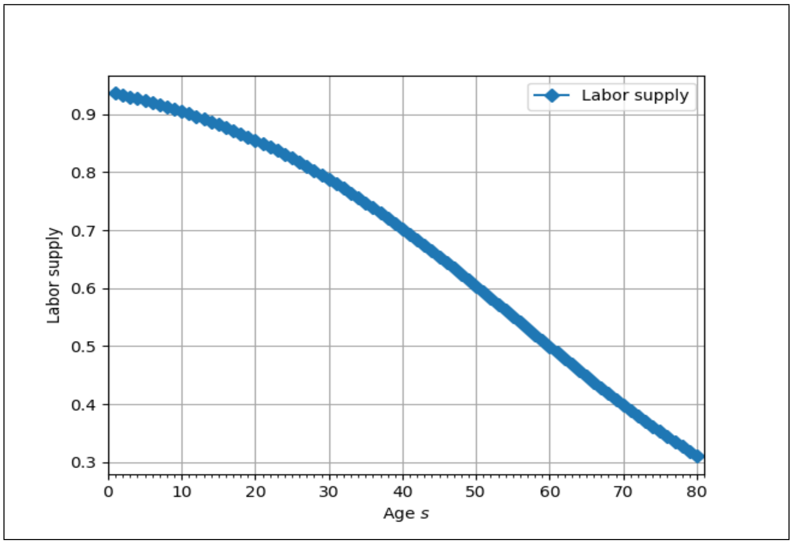


Table 20.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.082	Max. absolute savings Euler error	7.44e-11
$\bar{w}$	1.037	Max. absolute labor supply Euler error	1.47e-11
$\bar{K}$	252.648	Absolute final period savings $\bar{b}_{S+1}$	-1.16e-13
$\bar{L}$	66.423	Resource constraint error	4.20e-08
$\bar{Y}$	106.019		
$\bar{C}$	79.293		

Figure 20.1 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 20.3 in Section 20.6. Figure 20.2 shows the steady-state distribution of individual labor supply by age. The left side of Table 20.1 gives the resulting steady-state values for the prices and aggregate variables.

Figure 20.2: Steady-state distribution of labor supply $\bar{n}_s$



As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 20.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.5e-16, the final period individual savings is less than 6.1e-13, and the resource constraint error is less than 1.8e-06, we can be confident that we have successfully solved for the steady-state.

20.5.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor and a small open economy can also be simplified from method described in Section 7.6.2.

To solve for the transition path (non-steady-state) equilibrium from Definition **??**, we must know the parameters from the steady-state problem $\Big\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta, r^*, \Big\}$, the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\mathbf{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 20.3 and 20.1 show a particular calibration of the model and a steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

i. Given calibration for initial distribution of savings (wealth) $\mathbf{\Gamma}_1$, which implies an initial capital stock K^s_1.

ii. Given the exogenous interest rate, r^*, we can solve for the entire time path of the wage rate, using the firms' first order consitions.

iii. Given the time paths for the interest rate $\boldsymbol{r}^i$, wage $\boldsymbol{w}^i$, transfers $\boldsymbol{x}^i$ and the period-1 distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, solve for the lifetime decisions $c_{s,t}$, $n_{s,t}$, and $b_{s,t}$ of each household alive during periods 1 and $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 20.5.1.

iv. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions we can compute the paths of aggregate capital supply and aggregate labor supply, K_t^s and L_t^s.

v. With the time path of aggregate labor supply and the labor market clearing condition, we can use the firms' first order condition for capital to solve for capital demand, K_t^d.

vi. The time paths for aggregate capital supply and demand imply the path for net capital inflows: $K^f = K_t^s - K_t^d$.

Table 20.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	4.87e-13
Maximum absolute savings Euler error	8.07e-16
Maximum absolute final period savings $\bar{b}_{S+1,t}$	0.00
Maximum absolute resource constraint error	3.20e-08

The 6 panels of Figure 20.3 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The three panels of Figure 20.4 show the transition paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$. Table 20.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

20.6 Calibration

20.7 Exercises

TBD...