Chapter 7

S-period-lived Agents with Endogenous Labor

In this chapter, we take the S-period-lived agent model from Chapter 6, and add an endogenous labor decision in every period for every household. That is, now the household must choose in every period how much to work $n_{s,t}$ and how much to save $b_{s+1,t+1}$.

In adding the labor decision to the household optimization problem, we must add a utility of leisure or a disutility of labor to the period utility function. In this chapter, we will introduce a new functional form for the disutility of labor following Evans and Phillips (2017). This approach fits an ellipse to the standard constant Frisch elasticity disutility of labor specification. The elliptical disutility of labor functional form provides Inada conditions at both the upper and lower bounds of labor supply, which greatly simplifies the computation.

7.1 Disutility of labor

In previous chapters, the labor decision was exogenously imposed and was inelastic to changes in underlying parameters or other variables of the model. With endogenous labor supply $n_{s,t}$, we must specify how labor enters an agent's utility function and what are the constraints. Assume that each household is endowed with a measure of time $\tilde{l}$ each period that it can choose to spend as either labor $n_{s,t} \in [0,\tilde{l}]$ or leisure $l_{s,t} \in [0,\tilde{l}]$.

$$n_{s,t} + l_{s,t} = \tilde{l} \quad \forall s,t \tag{7.1}$$

In contrast to the CRRA period utility function (5.6) from the previous section, the endogenous labor of this version of the model requires that we add either a utility of leisure term to the period utility function or a disutility of labor term. This is usually done in one of three ways. One can add a multiplicative constant elasticity of substitution term for leisure in the utility function in which households can substitute between consumption and leisure. The other two options are additively separable terms. One can model the utility of leisure with constant relative risk aversion (CRRA), similar to our period utility function from (5.6). Or one can model the disutility of labor using a constant Frisch elasticity (CFE) functional form.

Our preferred specification in this chapter will be an approximation to the constant Frisch elasticity (CFE) functional form. The following equation is the period utility function with a CRRA utility of consumption as in (5.6) and an additively separable CFE disutility of labor,

$$u\big(c_{s,t},n_{s,t}\big) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} - \chi^n_s\frac{\big(n_{s,t}\big)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}} \tag{7.2}$$

where $\sigma \geq 1$ is the coefficient of relative risk aversion on consumption and $\theta > 0$ is the Frisch elasticity of labor supply. The constant $\chi^n_s > 0$ for all s is a scale parameter influencing the relative disutility of labor to the utility of consumption that can potentially vary by age s.

The first term in the period utility function (7.2) represents the utility from consumption $c_{s,t}$. Consumption has a lower bound in that it cannot be negative $c_{s,t} \geq 0$. Notice that $c_{s,t}$ is included. You can imagine someone consuming nothing in some period. However, the CRRA functional form in (7.2) puts an extra restriction on consumption in that $(c_{s,t}^{1-\sigma}-1)/(1-\sigma)$ is not defined for $c_{s,t}=0$ as well as $c_{s,t}<0$. Furthermore, the marginal utility of consumption goes to ∞ as consumption gets close to 0.

$$\lim_{c\rightarrow 0^+} c^{-\sigma} = \infty \tag{7.3}$$

This infinite marginal utility as consumption declines from above toward zero is often called an Inada condition.[1] It is a natural condition in the theory that bounds solutions away from the corners and forces interior solutions. Consumption has no natural upper bound, so the one Inada condition on consumption is sufficient to avoid a needing an occasionally binding Lagrange multiplier on the lower bound of consumption $c \geq 0$. Occasionally binding constraints are a notoriously difficult problem in optimization science as noted by Guerrieri and Iacoviello (2015), Brumm and Grill (2014), Judd et al. (2003), and Christiano and Fisher (2000).

All three utility of leisure or disutility of labor specifications mentioned in this section have at most one Inada condition at either the upper or lower bound of household labor supply. CRRA utility of leisure has an Inada condition that bounds solutions away from nonpositive leisure $l_{s,t} \leq 0$, and therefore bounds solutions away from labor supply at or above its upper bound $n_{s,t} \geq \tilde{l}$. But CRRA utility of leisure has no Inada condition on the upper bound of leisure or the lower bound of labor. The CFE disutility of labor specification in (7.2) has an Inada condition for the lower bound of labor supply, but has no Inada condition for the upper bound of labor supply. For this reason, one must take care in computing solutions to make answers respect both the upper and lower bounds of labor supply.

Evans and Phillips (2017) propose a useful approximation to both the CRRA utility of leisure and the CFE disutility of labor specifications, which has some nice properties as an independent specification rather than just an approximation. Evans and Phillips propose using the upper-right quadrant of an ellipse as an approximation to the CFE functional form for the disutility of labor.[2] This elliptical disutility of labor provides Inada conditions at both the upper and lower bounds of labor supply.

The functional form for a general ellipse in x and y space is the following, where the centroid of the ellipse is at coordinates $(x_0, y_0) = (h, k)$, the horizontal radius is $a > 0$, the vertical radius is $b > 0$, and the curvature is controlled by $\mu > 1$.

$$\left(\frac{x-h}{a}\right)^{\upsilon} + \left(\frac{y-k}{b}\right)^{\upsilon} = 1, \quad a, b > 0 \quad \text{and} \quad \upsilon > 1 \tag{7.4}$$

[1] See Inada (1963).

[2] Evans and Phillips (2017) provide approximations for both CFE disutility of labor as well as CRRA utility of leisure.

Figure 7.1: Ellipse with $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$

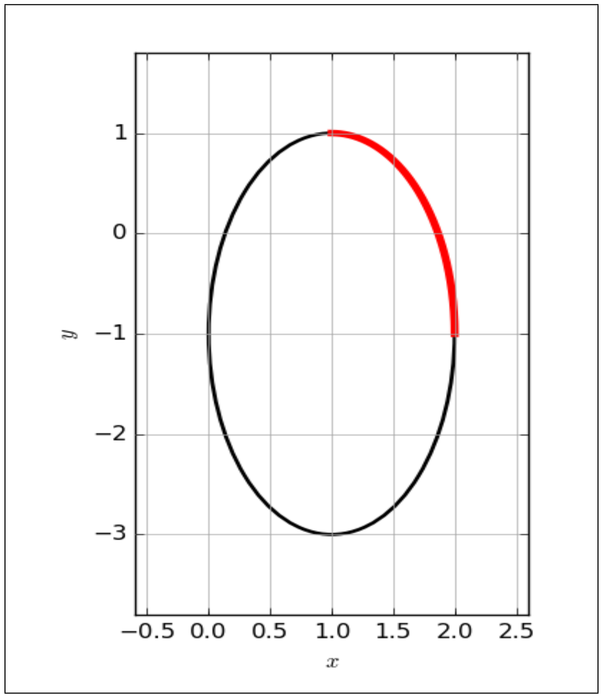


Figure 7.1 shows an ellipse with the parameterization $[h, k, a, b, \upsilon] = [1, -1, 1, 2, 2]$. The upper-right quadrant of the ellipse is highlighted because we focus on this portion of the function.

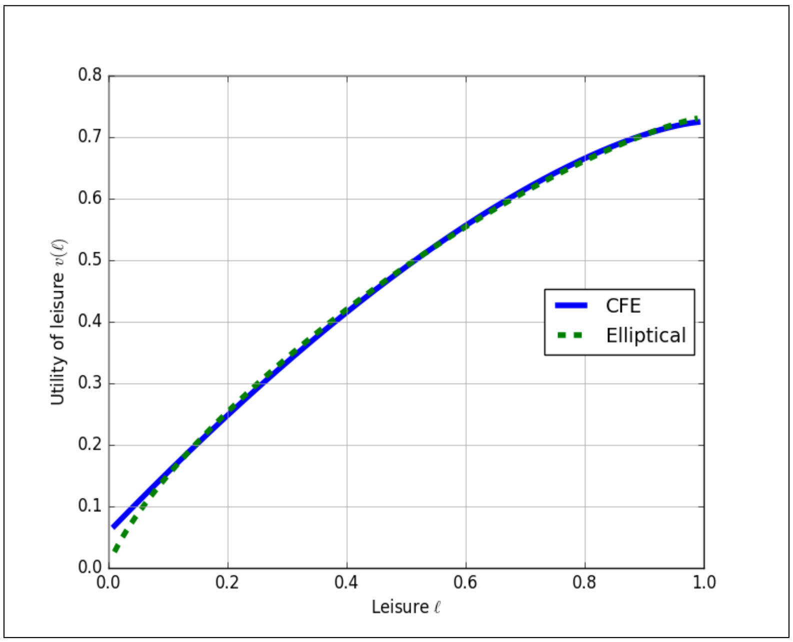
We want to rename the variables in (7.4) such that x is labor supply $n_{s,t}$ or $1 - l_{s,t}$ and y is the utility of labor $-g\big(n_{s,t}\big)$.[3] We want the x-coordinate of the centroid to be at zero $h = 0$ and the horizontal radius to be the labor endowment $a = \tilde{l}$ so that the ellipse is defined for $n_{s,t} \in [0, \tilde{l}]$ and has marginal benefit of zero at $n_{s,t} = 0$ and marginal benefit of $-\infty$ at $n_{s,t} = \tilde{l}$. We can normalize the centroid in the y dimension or $g(n)$ dimension to zero $k = 0$ because it will drop out of any marginal utility calculation. The vertical radius b and the curvature υ are free parameters that we can use to match another functional form. Using this specification and solving for $g(n)$, we get the following functional form for the elliptical

[3]The upper-right-quadrant of the ellipse represents the "utility" of labor because it is decreasing in n. With this interpretation, the functional forms we are matching are the negative of the CFE disutility of labor function $-g(n) = -\frac{(n)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$ to the upper-right-quadrant of the ellipse formula $b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}}$. Another interpretation of this is to match the lower-right-quadrant of the ellipse (disutility of labor) to the CFE "disutility" of labor function.

disutility of labor representing the upper-right quadrant of the ellipse in Figure 7.1.

$$g\big(n_{s,t}\big) = -b\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \quad \forall s,t \tag{7.5}$$

Figure 7.2: Comparison of CFE utility of leisure $\theta = 2$ to fitted elliptical utility



Peterman (2014) shows that in a macro-model that has no extensive labor margin but represents individuals that are making both intensive and extensive margin labor supply decisions in reality, a Frisch elasticity of around 2.0 is probably appropriate.[4] In Exercise 7.1, you will estimate the parameters b and υ from (7.5) to match a CFE disutility of labor function (7.2) with a Frisch elasticity of $\theta = 2.0$. Figure 7.2 shows the CFE utility of leisure function plotted against an elliptical utility of leisure (disutility of labor) function that was estimated to closely approximate the CFE function.

[4]Peterman (2014) tests the implied macro elasticity when the assumed micro elasticities are small on the intensive margin but only macro aggregates—which include both extensive and intensive margin agents—are observed.

7.2 Households

A unit measure of identical individuals are born each period and live for S periods. The endogenous labor decision does not change the budget constraint from Section 5.

$$\begin{split} c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \forall s,t \\ \text{with} \quad b_{1,t}, b_{S+1,t} = 0 \end{split} \tag{5.1}$$

Households choose lifetime consumption $\{c_{s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{s,t+s-1}\}_{s=1}^S$, and savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ to maximize lifetime utility, subject to the budget constraints and non negativity constraints,

$$\max_{\{c_{s,t+s-1},n_{s,t+s-1}\}_{s=1}^S,\{b_{s+1,t+s}\}_{s=1}^{S-1}} \sum_{s=1}^S \beta^{s-1}u(c_{s,t+s-1},n_{s,t+s-1}) \tag{7.6}$$

$$\text{s.t.} \quad c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \tag{5.1}$$

$$\text{where} \quad u\bigl(c_{s,t},n_{s,t}\bigr) = \frac{c_{s,t}^{1-\sigma}-1}{1-\sigma} + \chi^n_s b\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1}{\upsilon}} \tag{7.7}$$

where $u(c_{s,t},n_{s,t})$ is the period utility function with elliptical disutility of labor (7.7), and χ^n_s is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (7.6) can be reduced to choosing S labor supplies $\{n_{s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (5.1) in for $c_{s,t}$ in each period utility function (7.7) of the lifetime utility function.

The set of optimal lifetime choices for an agent born in period t are characterized by the following S static labor supply Euler equations (7.8), the following $S-1$ dynamic savings Euler equations (7.9), and a budget constraint that binds in all S periods (5.1),

$$\begin{split} w_t u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = -u_2\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S\} \\ \Rightarrow \quad w_t\left(c_{s,t}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}} \end{split} \tag{7.8}$$

$$\begin{aligned} &u_1\left(c_{s,t+s-1},n_{s,t+s-1}\right) = \beta(1+r_{t+1})u_1\left(c_{s+1,t+s},n_{s+1,t+s}\right) \quad \text{for} \quad s\in\{1,2,...S-1\} \\ &\Rightarrow \quad \left(c_{s,t}\right)^{-\sigma} = \beta(1+r_{t+1})\left(c_{s+1,t+1}\right)^{-\sigma} \end{aligned} \tag{7.9}$$

$$c_{s,t} + b_{s+1,t+1} = (1+r_t)b_{s,t} + w_t n_{s,t} \quad \text{for} \quad s\in\{1,2,...S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{5.1}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{s,t}$. As was demonstrated in detail in Section 6.1, the dynamic Euler equations (7.9) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u, r_u\}_{u=t}^{t+S-1}$. As in section 6.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (6.14) and (5.17).

$$\mathbf{\Gamma}_t \equiv \left\{b_{s,t}\right\}_{s=2}^{S} \quad \forall t \tag{6.14}$$

$$\mathbf{\Gamma}^e_{t+u} = \Omega^u\left(\mathbf{\Gamma}_t\right) \quad \forall t, \quad u\geq 1 \tag{5.17}$$

7.3 Firms

Firms are characterized exactly as in Section 5.2, with the firm's aggregate capital decision K_t governed by first order condition (5.20) and its aggregate labor decision L_t governed by first order condition (5.21).

$$r_t = \alpha A\left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{5.20}$$

$$w_t = (1-\alpha)A\left(\frac{K_t}{L_t}\right)^{\alpha} \tag{5.21}$$

The per-period depreciation rate of capital is $\delta\in[0,1]$, the capital share of income is $\alpha\in(0,1)$, and total factor productivity is $A>0$.

7.4 Market Clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand.

$$L_t = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{7.10}$$

$$K_t = \sum_{i=2}^{S} b_{s,t} \quad \forall t \tag{6.16}$$

$$\begin{split} &Y_t = C_t + I_t \quad \forall t \\ &\quad \text{where} \quad I_t \equiv K_{t+1} - (1-\delta)K_t \end{split} \tag{5.24}$$

The goods market clearing equation (5.24) is redundant by Walras' Law.

The market clearing conditions for this version of the model are nearly equivalent to the three conditions described in Section 6.3. The exception is the labor market clearing condition (7.10) in which individual labor supply levels $n_{s,t}$ can vary endogenously with age s and time t.

7.5 Equilibrium

Before providing exact definitions of the functional equilibrium concepts, we give a rough sketch of the equilibrium, so you can see what the functions look like and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points.

i. Households optimize according to equations (7.8) and (7.9).

ii. Firms optimize according to (5.20) and (5.21).

iii. Markets clear according to (7.10) and (6.16).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (7.10) and (6.16) into the firm's optimal conditions (5.20) and (5.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t\big(\mathbf{\Gamma}_t\big): \quad w_t = (1-\alpha)A\left(\frac{\sum_{s=2}^S b_{s,t}}{\sum_{s=1}^S n_{s,t}}\right)^\alpha \quad \forall t \tag{7.11}$$

$$r_t\big(\mathbf{\Gamma}_t\big): \quad r_t = \alpha A\left(\frac{\sum_{s=1}^S n_{s,t}}{\sum_{s=2}^S b_{s,t}}\right)^{1-\alpha} - \delta \quad \forall t \tag{7.12}$$

It is worth noting here that the equilibrium wage (7.11) and interest rate (7.12) are written as functions of the period-t distribution of savings (wealth) $\mathbf{\Gamma}_t$ from (6.14) and are not functions of the period-t distribution of labor supply, which labor distribution shows up in (7.11) and (7.12). This is because, similar to next period savings $b_{s+1,t+1}$, current period labor supply $n_{s,t}$ must be chosen in period t and is therefore a function of the current state $\mathbf{\Gamma}_t$ distribution of savings.

Now (7.11), (7.12), and the budget constraint (5.1) can be substituted into household Euler equations (7.8) and (7.9) to get the following $(2S-1)$-equation system. Extended across all time periods, this system completely characterizes the equilibrium.

$$w_t\big(\mathbf{\Gamma}_t\big)\left(w_t\big(\mathbf{\Gamma}_t\big)n_{s,t} + \Big[1 + r_t\big(\mathbf{\Gamma}_t\big)\Big]b_{s,t} - b_{s+1,t+1}\right)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{n_{s,t}}{\tilde{l}}\right)^{\upsilon-1}\left[1 - \left(\frac{n_{s,t}}{\tilde{l}}\right)^\upsilon\right]^{\frac{1-\upsilon}{\upsilon}}$$
$$\text{for} \quad s \in \{1,2,...S\} \quad \text{and} \quad \forall t \tag{7.13}$$

$$\left(w_t\big(\mathbf{\Gamma}_t\big)n_{s,t} + \Big[1 + r_t\big(\mathbf{\Gamma}_t\big)\Big]b_{s,t} - b_{s+1,t+1}\right)^{-\sigma} =$$
$$\beta\Big[1 + r_{t+1}\big(\mathbf{\Gamma}_{t+1}\big)\Big]\left(w_{t+1}\big(\mathbf{\Gamma}_{t+1}\big)n_{s+1,t+1} + \Big[1 + r_{t+1}\big(\mathbf{\Gamma}_{t+1}\big)\Big]b_{s+1,t+1} - b_{s+2,t+2}\right)^{-\sigma} \tag{7.14}$$
$$\text{for} \quad s \in \{1,2,...S-1\} \quad \text{and} \quad \forall t$$

The system of S nonlinear static equations (7.13) and $S-1$ nonlinear dynamic equations (7.14) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{s,t+s-1}\}_{s=1}^S$ and $\{b_{s+1,t+s}\}_{s=1}^{S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{\{b_{s+1,t+1}\}_{s=1}^{S-1}\}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 7.1 (**Steady-state equilibrium**). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as constant allocations of consumption $\{\bar{c}_s\}_{s=1}^S$, labor supply $\{\bar{n}_s\}_{s=1}^S$, and savings $\{\bar{b}_s\}_{s=2}^S$, and prices $\bar{w}$ and $\bar{r}$ such that:

i. households optimize according to (7.8) and (7.9),

ii. firms optimize according to (5.20) and (5.21),

iii. markets clear according to (7.10) and (6.16).

The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{s,t} = \phi_s(\mathbf{\Gamma}_t)$, and let the equilibrium policy functions for savings be represented by $b_{s+1,t+1} = \psi_s(\mathbf{\Gamma}_t)$. The arguments of the functions (the state) may change overtime causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 7.2 (**Non-steady-state functional equilibrium**). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S-period lived agents and endogenous labor supply is defined as stationary allocation functions of the state $\big\{n_{s,t} = \phi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^S$, $\big\{b_{s+1,t+1} = \psi_s\big(\mathbf{\Gamma}_t\big)\big\}_{s=1}^{S-1}$ and and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (5.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\boldsymbol{\Gamma}_{t+u} = \boldsymbol{\Gamma}^e_{t+u} = \Omega^u\left(\boldsymbol{\Gamma}_t\right) \quad \forall t, \quad u \geq 1$$

ii. households optimize according to (7.8) and (7.9),

iii. firms optimize according to (5.20) and (5.21),

iv. markets clear according to (7.10) and (6.16).

7.6 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 7.1 and the transition path equilibrium from Definition 7.2.

7.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 7.1. The parameters needed for the steady-state solution of this model are $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\right\}$, where S is the number of periods in an individual's life, $\left\{\beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S\right\}$ are household utility function parameters, and $\{A, \alpha, \delta\}$ are firm production function parameters. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{s,t} = \bar{c}_s, \quad n_{s,t} = \bar{n}_s, \quad b_{s,t} = \bar{b}_s \quad \forall s, t \tag{7.15}$$

From the market clearing conditions (7.10) and (6.16) and the firms' first order equations (5.20) and (5.21), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K_t = \bar{K} \quad L_t = \bar{L} \quad \forall t \tag{7.16}$$

The steady-state is characterized by the steady-state versions of the set of $2S-1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S-1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}\Big([1+\bar{r}]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \chi^n_s\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s = \{1,2,...S\} \tag{7.17}$$

$$\Big([1+\bar{r}]\bar{b}_s + \bar{w}\bar{n}_s - \bar{b}_{s+1}\Big)^{-\sigma} = \beta(1+\bar{r})\Big([1+\bar{r}]\bar{b}_{s+1} + \bar{w}\bar{n}_{s+1} - \bar{b}_{s+2}\Big)^{-\sigma} \quad \text{for} \quad s = \{1,2,...S-1\} \tag{7.18}$$

where both $\bar{w}$ and $\bar{r}$ are functions of the distribution of labor supply and savings as shown in (7.11) and (7.12).

One approach to solving this system would be to use a multivariate root finder that chooses the $2S-1$ steady-state variables $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ simultaneously to solve the zeros of the $2S-1$ Euler equations (7.17) and (7.17). However, the tradeoffs between labor supply and savings in each period create a series of saddle paths in the objective function that render the simultaneous equations root finder unreliable and somewhat intractable. We have found that this approach only works when your initial guess for the steady-state $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$ is close to the solution. It breaks down when the underlying parameters are changed.

Another method would be to perform an outer loop root finder on a guess for $\bar{r}$ and $\bar{w}$ (or equivalently $\bar{K}$ and $\bar{L}$), such that the household's problem is solved for the values of r and w in each iteration and the outer loop root finder uses the firm's first order condition as the error equations. This method also works well if one's initial guess is good. However, it can fail to find the equilibrium for a large range of initial guesses.

The most robust solution method that we have found takes significantly more time than the two methods described in the preceding paragraphs, but it successfully finds the steady-state equilibrium for most feasible initial guesses that we have constructed. This method is a bisection method in the outer loop guesses for steady-state equilibrium $\bar{K}$ and $\bar{L}$. In the inner loop of the household's problem given r and w implied by K and L, we solve the problem by breaking the multivariate root finder problem with $2S-1$ equations and unknowns into a series of many univariate root finder problems and one bivariate root finder problem. The algorithm is the following.

i. Make a guess for the steady-state aggregate capital stock $\bar{K}^i$ and aggregate labor $\bar{L}^i$.

   (a) Values for $\bar{K}^i$ and $\bar{L}^i$ will imply values for the interest rate $\bar{r}^i$ and wage $\bar{w}^i$ from (5.20) and (5.21)

   (b) For the bisection method, we must use guesses for K and L because those uniquely determine r and w, whereas the converse is not true. From the firms' first order conditions (5.20) and (5.21), we see that r and w are functions of the same capital-labor ratio K/L. Infinitely many combinations of K and L determine a given r and w.

ii. Given $\bar{r}^i$ and $\bar{w}^i$, solve for the steady-state household's lifetime decisions $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$.

   (a) Given $\bar{r}^i$ and $\bar{w}^i$, guess an initial steady-state consumption $\bar{c}_1^m$, where m is the index of the inner-loop (household problem given $\bar{r}^i$, $\bar{w}^i$) iteration.

   (b) Given $\bar{r}^i$, $\bar{w}^i$, and $\bar{c}_1^m$, use the sequence of $S-1$ dynamic savings Euler equations (7.9) to solve for the implied series of steady-state consumptions $\{\bar{c}_s^m\}_{s=1}^S$. This sequence has an analytical solution.

$$\bar{c}_{s+1}^m = \bar{c}_s^m \left[\beta(1+\bar{r}^i)\right]^{\frac{1}{\sigma}} \quad \text{for} \quad s = \{1, 2, ...S-1\} \tag{7.19}$$

   (c) Given $\bar{r}^i$, $\bar{w}^i$, and $\{\bar{c}_s^m\}_{s=1}^S$, solve for the series of steady-state labor supplies $\{\bar{n}_s^m\}_{s=1}^S$ using the S static labor supply Euler equations (7.8). This will require a series of S separate univariate root finders or one multivariate root finder.

$$\bar{w}^i\left(\bar{c}_s^m\right)^{-\sigma} = \chi_s^n\left(\frac{b}{\tilde{l}}\right)\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{\bar{n}_s^m}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad s=\{1,2,...S\} \tag{7.20}$$

It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

(d) Given $\bar{r}^i$, $\bar{w}^i$, and implied steady-state consumption $\{\bar{c}_s^m\}_{s=1}^S$ and labor supply $\{\bar{n}_s^m\}_{s=1}^S$, solve for implied time path of savings $\{\bar{b}_{s+1}^m\}_{s=1}^S$ across all ages of the representative lifetime using the household budget constraint (5.1).

$$\bar{b}_{s+1}^m = (1+\bar{r}^i)\bar{b}_s^m + \bar{w}^i\bar{n}_s^m - \bar{c}_s^m \quad \text{for} \quad s=\{1,2,...S\} \tag{7.21}$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}_1^m$ as in step (a).

(e) Update the initial guess for $\bar{c}_1^m$ to $\bar{c}_1^{m+1}$ until the implied savings in the last period equals zero $\bar{b}_{S+1}^{m+1}=0$.

iii. Given solution for optimal household decisions $\{\bar{c}_s^m\}_{s=1}^S$, $\{\bar{n}_s^m\}_{s=1}^S$, and $\{\bar{b}_s^m\}_{s=2}^S$ based on the guesses for aggregate capital $\bar{K}^i$ and aggregate labor $\bar{L}^i$, solve for the aggregate capital $\bar{K}^{i'}$ and aggregate labor $\bar{L}^{i'}$ implied by the household solutions and market clearing conditions.

$$\bar{K}^{i'} = \sum_{s=2}^S \bar{b}_s^m \tag{7.22}$$

$$\bar{L}^{i'} = \sum_{s=1}^S \bar{n}_s^m \tag{7.23}$$

Update guesses for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1},\bar{L}^{i+1})$ until the the aggregates implied by household optimization equal the initial guess for the aggregates $(\bar{K}^{i'+1},\bar{L}^{i'+1})=(\bar{K}^{i+1},\bar{L}^{i+1})$.

(a) The bisection method characterizes the updated guess for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ as a convex combination of the initial guess $(\bar{K}^{i}, \bar{L}^{i})$ and the values implied by household and firm optimization $(\bar{K}^{i'}, \bar{L}^{i'})$, where the weight put on the new values $(\bar{K}^{i'}, \bar{L}^{i'})$ is given by $\xi \in (0,1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$(\bar{K}^{i+1}, \bar{L}^{i+1}) = \xi(\bar{K}^{i'}, \bar{L}^{i'}) + (1-\xi)(\bar{K}^{i}, \bar{L}^{i}) \quad \text{for} \quad \xi \in (0,1] \tag{7.24}$$

(b) Let $\|\cdot\|$ be a norm on the space of feasible aggregate capital and aggregate labor values (K, L). We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (15.37). If the distance is less than some tolerance `toler` > 0, then the problem has converged. Otherwise continue updating the values of aggregate capital and labor using (15.36).

$$\texttt{dist} \equiv \left\|(\bar{K}^{i'}, \bar{L}^{i'}) - (\bar{K}^{i}, \bar{L}^{i})\right\| \tag{7.25}$$

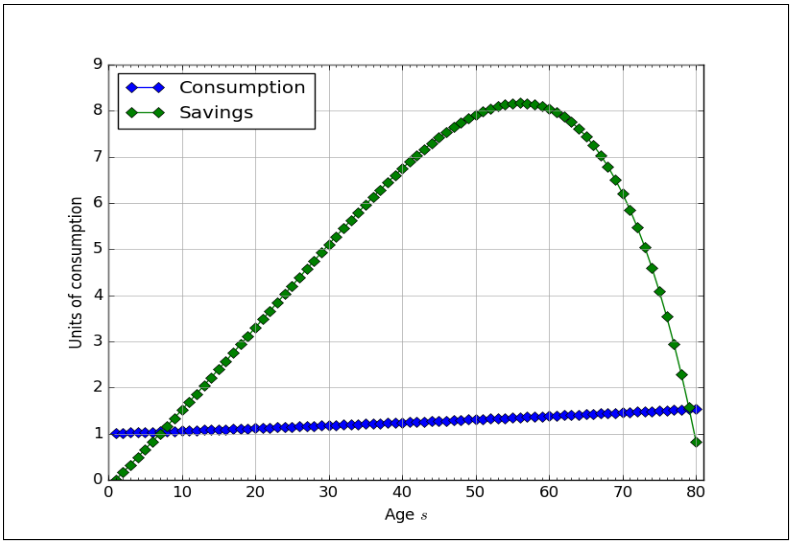
Define the updating of aggregate variable values $(\bar{K}^{i}, \bar{L}^{i})$ in step (iii) indexed by i as the "outer loop" of the fixed point solution. Although computationally intensive, the bisection method described above is the most robust solution method we have found.

Table 7.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
$\bar{r}$	0.055	Max. absolute savings Euler error	2.22e-16
$\bar{w}$	1.240	Max. absolute labor supply Euler error	4.44e-16
$\bar{K}$	399.875	Absolute final period savings $\bar{b}_{S+1}$	6.14e-13
$\bar{L}$	63.186	Resource constraint error	1.84e-06
$\bar{Y}$	120.525		
$\bar{C}$	100.531		

Figure 7.3 shows the steady-state distribution of individual consumption and savings in an 80-period-lived agent model with parameter values listed above the line in Table 7.3 in Section 7.7. Figure 7.4 shows the steady-state distribution of individual labor supply by age.

Figure 7.3: Steady-state distribution of consumption $\bar{c}_s$ and savings $\bar{b}_s$



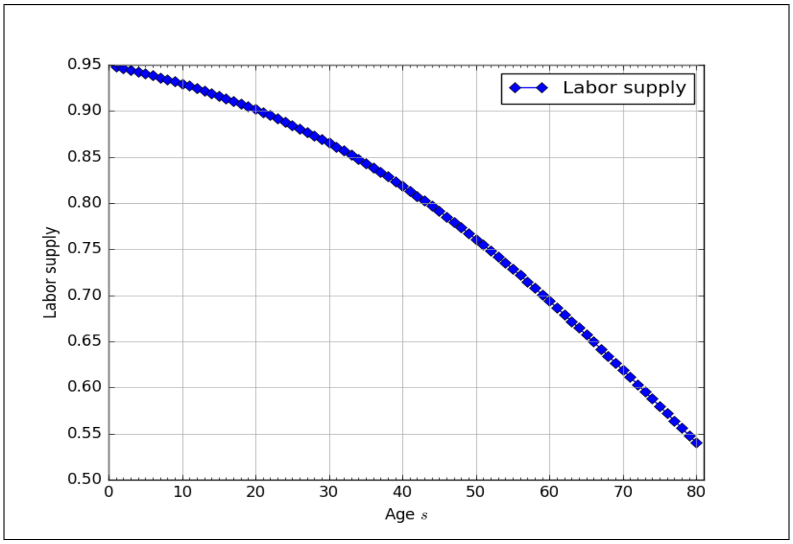
The left side of Table 7.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $2S-1$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 7.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 4.5e-16, the final period individual savings is less than 6.1e-13, and the resource constraint error is less than 1.8e-06, we can be confident that we have successfully solved for the steady-state.

7.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S-period-lived agent model with endogenous labor is similar to the method described in Section 6.5. However, there is also a significant change. To see this change, we refer back to the household budget constraint (5.1), firms' first order conditions (5.20) and (5.21), and market clearing conditions (7.10) and (6.16).

Figure 7.4: Steady-state distribution of labor supply $\bar{n}_s$



In the S-period-lived agent model with exogenous labor from Chapter 6, we only had to guess the time path of the aggregate capital stock from the initial period to the period in which the economy reached its steady-state $\boldsymbol{K}^i = \{K_1^i, K_2^i, ...K_T^i\}$. Because aggregate labor L_t was exogenously fixed, knowing K_t allowed us to determine both the wage w_t and the interest rate r_t in every period as evidenced from the firm first order conditions.

$$r_t = \alpha A \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta \tag{5.20}$$

$$w_t = (1-\alpha) A \left(\frac{K_t}{L_t}\right)^{\alpha} \tag{5.21}$$

Knowing the respective time paths of wages w_t and interest rates r_t gives us everything we need to know to solve every household's lifetime decisions independently from every other household. With exogenous labor supply from Chapter 6, prices were determined solely by the aggregate capital stock.

With endogenous labor supply from this chapter, the aggregate labor becomes a function of the sum of all individual labor supply which can vary by age and by time period.

$$L_t = \sum_{s=1}^{S} n_{s,t} \quad \forall t \tag{7.10}$$

The key assumption is that the economy will reach the steady-state equilibrium $\bar{\boldsymbol{\Gamma}}$ described in Definition 7.1 in a finite number of periods $T < \infty$ regardless of the initial state $\boldsymbol{\Gamma}_1$.

To solve for the transition path (non-steady-state) equilibrium from Definition 7.2, we must know the parameters from the steady-state problem $\left\{S, \beta, \sigma, \tilde{l}, b, \upsilon, \{\chi^n_s\}_{s=1}^S, A, \alpha, \delta\right\}$, the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\boldsymbol{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 7.3 and 7.1 show a particular calibration of the model and a steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

i. Choose a period $T1$ in which the initial guess for the time paths of aggregate capital and aggregate labor will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.

ii. Given calibration for initial distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, which implies an initial capital stock K_1, guess initial time paths for the aggregate capital stock $\boldsymbol{K}^i = \{K^i_1, K^i_2, ...K^i_{T1}\}$ and aggregate labor $\boldsymbol{L}^i = \{L^i_1, L^i_2, ...L^i_{T1}\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T2 + S - 1$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period $T2$.

iii. Given time paths $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$, solve for the lifetime consumption $c_{s,t}$, labor supply $n_{s,t}$, and savings $b_{s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.

   (a) The initial paths for aggregate capital $\boldsymbol{K}^i$ and aggregate labor $\boldsymbol{L}^i$ imply time paths for the interest rate $\boldsymbol{r}^i = \left\{r^i_1, r^i_2, ...r^i_{T2+S-1}\right\}$ and wage $\boldsymbol{w}^i = \left\{w^i_1, w^i_2, ...w^i_{T2+S-1}\right\}$ using the firms' first order equations (5.20) and (5.21).

   (b) Given the time paths for the interest rate $\boldsymbol{r}^i$ and wage $\boldsymbol{w}^i$ and the period-1 distribution of savings (wealth) $\boldsymbol{\Gamma}_1$, solve for the lifetime decisions $c_{s,t}$, $n_{s,t}$, and $b_{s,t}$ of each household alive during periods 1 and $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 7.6.1.

iv. Use time path of the distribution of labor supply $n_{s,t}$ and savings $b_{s,t}$ from households optimal decisions given $\boldsymbol{K}^i$ and $\boldsymbol{L}^i$ to compute new paths for aggregate capital and aggregate labor $\boldsymbol{K}^{i'}$ and $\boldsymbol{L}^{i'}$ implied by capital and labor market clearing conditions (7.10) and (6.16).

v. Compare the distance of the time paths of the new implied paths for the aggregate capital and labor $\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right)$ versus the initial aggregate capital and labor $\left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right)$.

$$\texttt{dist} = \left\| \left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) - \left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right) \right\| \geq 0 \tag{7.26}$$

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and aggregate labor $(\boldsymbol{K}^i, \boldsymbol{L}^i)$. Common norms to use are the L^2 and the L^∞ norms.

(a) If the distance is less than or equal to some tolerance level $\texttt{dist} \leq \texttt{TPI\_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.

(b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

$$\left(\boldsymbol{K}^{i+1}, \boldsymbol{L}^{i+1}\right) = \xi\left(\boldsymbol{K}^{i'}, \boldsymbol{L}^{i'}\right) + (1-\xi)\left(\boldsymbol{K}^{i}, \boldsymbol{L}^{i}\right) \quad \text{for} \quad \xi \in (0,1] \tag{7.27}$$

Table 7.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	8.88e-16
Maximum absolute savings Euler error	5.55e-16
Maximum absolute final period savings $\bar{b}_{S+1,t}$	1.78e-12
Maximum absolute resource constraint error	1.36e-12

The 6 panels of Figure 7.5 show the equilibrium time paths of the interest rate r_t, wage w_t, and aggregate variables K_t, L_t, Y_t, and C_t. The three panels of Figure 7.6 show the transition

Figure 7.5: Equilibrium transition paths of prices and aggregate variables

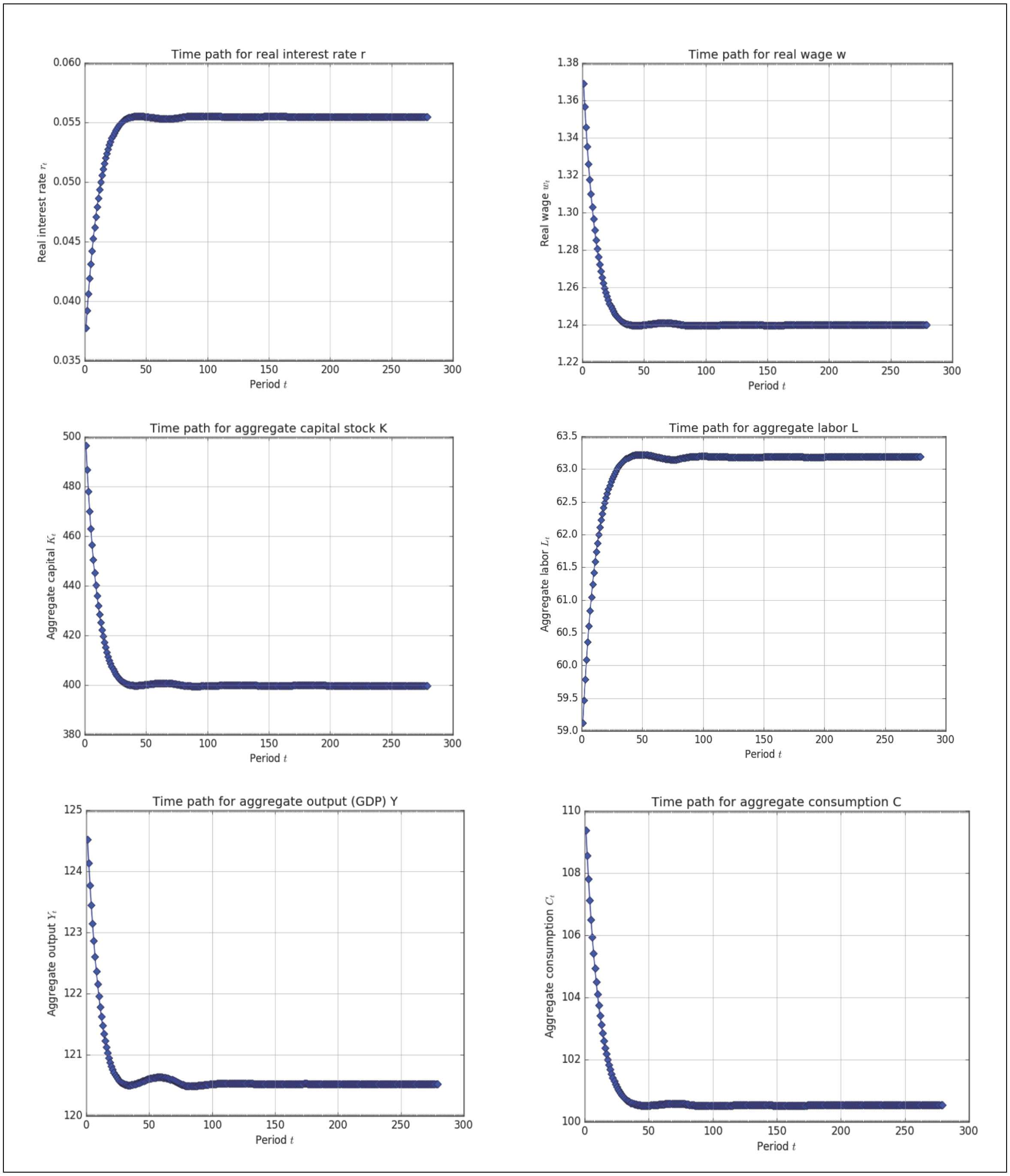
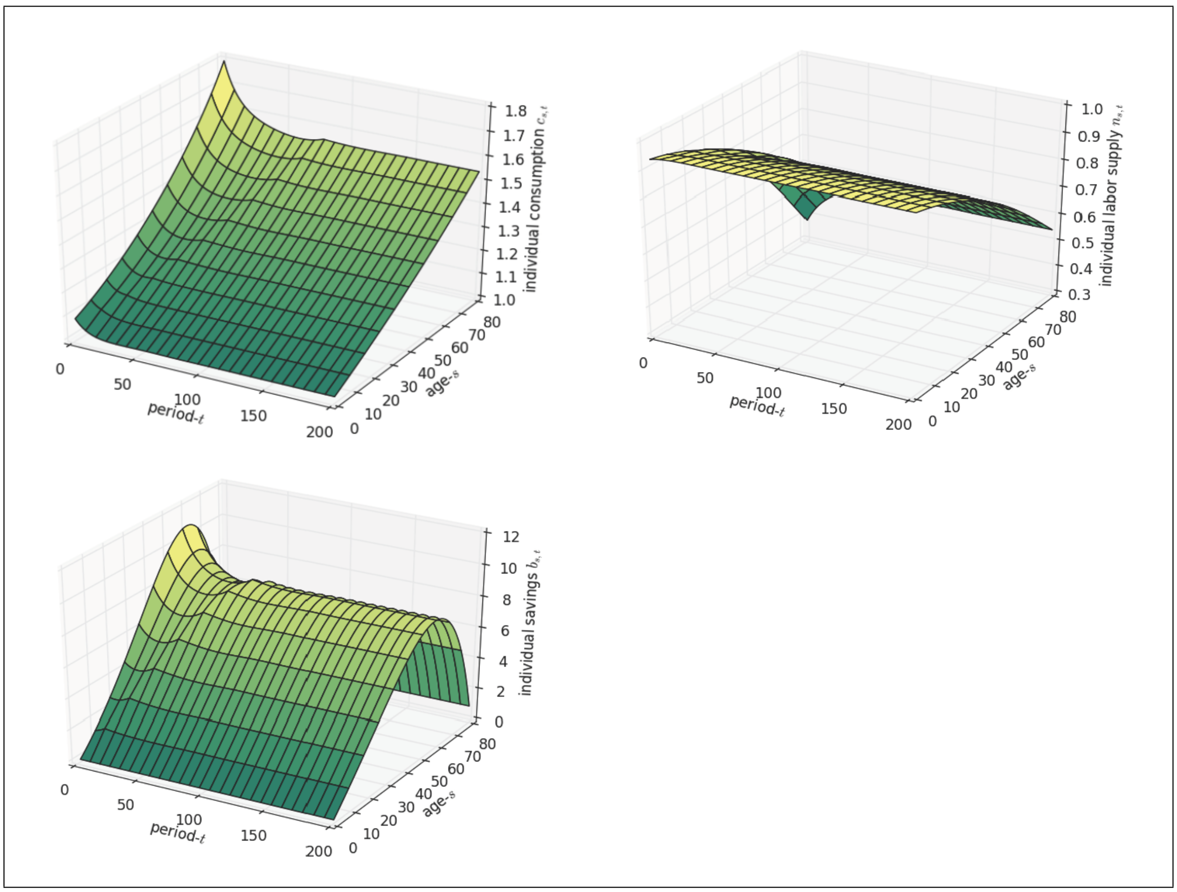


Figure 7.6: Equilibrium transition paths of distributions of consumption, labor supply, and savings



paths of the distributions of consumption $c_{s,t}$, labor supply $n_{s,t}$ and savings $b_{s,t}$. Table 7.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than 2.0e-12 in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

7.7 Calibration

Many of the parameters of the model can be calibrated by simply taking values from other studies or by setting them to intuitive values. Assume that agents are born at age 21 and die at age 100 (80 years of life). Your time dependent parameters can be written as functions of S, because each period of the model is $80/S$ years. If the annual discount factor is estimated to be 0.96, then the model period discount factor is $\beta = 0.96^{80/S}$. Assume initially that $S = 80$. Let the annual depreciation rate of capital be 0.05. Then the model period depreciation rate is $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 3$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$. Assume that each individual's time endowment in each period is $\tilde{l} = 1$.

Figure 7.7: Initial vs. steady-state distributions of wealth (savings) $b_{s,t}$

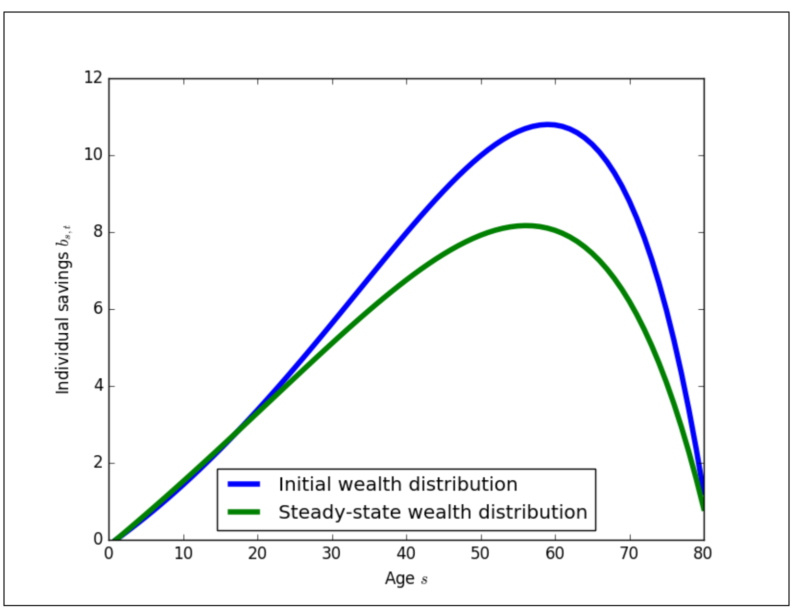


Table 7.3: Calibrated parameter values for simple endogenous labor model

Parameter	Description	Value
S	Number of periods in individual life	80
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
$\tilde{l}$	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501[a]
υ	Elliptical disutility of labor shape parameter	1.554[a]
$\{\chi^n_s\}_{s=1}^S$	Disutility of labor relative scale factor by age	(See Sec. 7.7.1)
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
$\mathbf{\Gamma}_1$	Initial distribution of savings (wealth)	(see Fig. 7.7)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2

[a] The calibration of b and υ is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See Evans and Phillips (2017).

7.7.1 Calibrating χ^n_s and elliptical utility parameters

We might want to be more careful about our calibration of the parameters $\{\chi^n_s\}_{s=1}^S$ that can vary by age s and scale the disutility of labor on the right-hand-side of the household's Euler equation for labor supply (7.8). One approach is to choose values of $\{\chi^n_s\}_{s=1}^S$ that make the steady-state values of labor supply as a percent of total time endowment as close as possible to their data analogues. This is a generalized method of moments (GMM) calibration in that we are choosing $\{\chi^n_s\}_{s=1}^S$ to match model moments to data moments.

Let the vector $\boldsymbol{\theta}$ represent all the parameters of the model, including $\{\chi^n_s\}_{s=1}^S$. Let $\tilde{\boldsymbol{x}}$ represent the endogenous variables of the model, and let $\boldsymbol{x}$ represent the data variables that are the real world data analogues of the model variables. The model moments $\boldsymbol{m}\left(\tilde{\boldsymbol{x}}|\boldsymbol{\theta}\right)$ are functions of model data $\tilde{\boldsymbol{x}}$ given parameters $\boldsymbol{\theta}$. Define the model moments as the steady-state labor supply by age as a percent of total time endowment.

$$\text{model moments:}\quad \boldsymbol{m}\left(\tilde{\boldsymbol{x}}|\boldsymbol{\theta}\right) = \left\{\frac{\bar{n}_s}{\tilde{l}}\right\}_{s=1}^S \tag{7.28}$$

The data moments ($\boldsymbol{x}$) that correspond to the model moments are the average hours of work as a percent of the total time endowment.

$$\text{data moments:} \quad \boldsymbol{m}\left(\boldsymbol{x}\right) = \left\{\frac{\text{avg. hours}_s}{\text{total hours available}}\right\}_{s=1}^{S} \tag{7.29}$$

To calculate average hours by age from the data for our data moments (7.29) we use the U.S. Current Population Survey (CPS) Basic Monthly Data.[5] We use three variables from the survey to calculate average weekly hours by age.[6] We then use the maximum hours worked in the data plus a small amount as our total hours available—the analogue of $\tilde{l}$.

We can now specify the GMM estimation we use to calibrate $\{\chi^n_s\}_{s=1}^S$ to minimize the distance between the model moments and the data moments,

$$\begin{aligned} &\min_{\{\chi^n_s\}_{s=1}^S} \; \boldsymbol{e}\big(\tilde{\boldsymbol{x}},\boldsymbol{x}|\boldsymbol{\theta}\big)^{\mathrm{T}}\,\boldsymbol{W}\,\boldsymbol{e}\big(\tilde{\boldsymbol{x}},\boldsymbol{x}|\boldsymbol{\theta}\big) \\ &\quad\text{where}\quad \boldsymbol{e}\big(\tilde{\boldsymbol{x}},\boldsymbol{x}|\boldsymbol{\theta}\big) \equiv \left(\frac{\boldsymbol{m}\left(\tilde{\boldsymbol{x}}|\boldsymbol{\theta}\right) - \boldsymbol{m}\left(\boldsymbol{x}\right)}{\boldsymbol{m}\left(\boldsymbol{x}\right)}\right) \end{aligned} \tag{7.30}$$

where T is the transpose operator and $\boldsymbol{W}$ is a weighting matrix.[7] This approach to estimating $\{\chi^n_s\}_{s=1}^S$ is exactly identified in that you are choosing S moments to estimate S parameters. This estimation is also nice inasmuch as each parameter value χ^n_s is most closely associated with one of the moments although each moment has some dependence on all the other parameters.

One problem with this approach to calibrating the parameters $\{\chi^n_s\}_{s=1}^S$ is that we are matching steady-state moments from the model with recent-period moments from the data. It is often unlikely that the recent periods of the economy are close to a steady-state. In other words, it is likely that the economy is usually in transition due to shocks it has experienced. For this reason, one might want to calibrate $\{\chi^n_s\}_{s=1}^S$ to some initial period model moments. Two problems that arise with this different approach are that initial-period endogenous

[5]The CPS Basic Monthly Data are available through the National Bureau of Economic Research data portal at http://nber.org/data/cps_basic.html.

[6]See `README.md` at https://github.com/OpenSourceMacro/CPS_hrs_age for a description of how we calculated average weekly hours from the CPS data.

[7]See this GMM Jupyter notebook and Davidson and MacKinnon (2004, chap. 9) for a discussion of estimators of the optimal weighting matrix $\hat{\boldsymbol{W}}$. But using the identity matrix is an unbiased yet inefficient estimator of the optimal weighting matrix.

variables require solving for the entire non-steady-state equilibrium and many initial period moments require scaling in order to match real-world data with model data. Given these two difficulties, we think the steady-state approach described above is acceptable.

You will solve for the elliptical disutility of labor supply parameters b and υ in Exercise 7.1 below. We do this by estimating b and υ such that the marginal disutilities of labor supply along the support of $n_{s,t}$ match the disutilities of labor supply implied by a constant Frisch elasticity (CFE) functional form for the disutility of labor supply with a Frisch elasticity of θ.

7.8 Exercises

Exercise 7.1. Assume that an individual's time endowment each period is one $\tilde{l} = 1$. Let the period utility of a household be an additively separable function of consumption and labor,

$$U(c,n) = u(c) - g(n)$$

where the disutility of labor function $g(n)$ is the constant Frisch elasticity (CFE) disutility of labor functional form.

$$g_{cfe}(n) = \frac{(n)^{1+\frac{1}{\theta}}}{1+\frac{1}{\theta}}$$

Assume that an approximation to this disutility of labor function is the following elliptical disutility of labor function.

$$g_{elp}(n) = -b\left[1 - \left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1}{\upsilon}} \quad \text{for} \quad \tilde{l}, b > 0 \quad \text{and} \quad \upsilon \geq 1$$

i. The marginal disutility of labor $\frac{\partial g(n)}{\partial n}$ governs the household decision of how much to work. Give the expression for the marginal disutility of labor for the CFE specification $g_{cfe}(n)$ and for the elliptical specification $g_{elp}(n)$.

ii. Write a function `fit_ellip()` that takes as inputs the Frisch elasticity of labor supply θ and the time endowment per period $\tilde{l}$ and returns the estimated values of the elliptical utility parameters b and υ.

```
b_ellip, upsilon = fit_ellip(elast_Frisch, l_tilde)
```

Assume that the Frisch elasticity of labor supply in v_{cfe} is $\theta = 2.0$. Using 1,000 evenly spaced points from the support of leisure between 0.15 and 0.95, estimate the elliptical disutility of labor parameters b and υ that minimize the sum of squared deviations between the two marginal utility of leisure functions $g'_{cfe}(n)$ and $g'_{elp}(n)$ from part (i). Plot the two marginal disutility of labor functions.

Exercise 7.2. Optimizers (root finders and minimizers) can sometimes choose values that are outside the feasible set of solutions even if Inada conditions are present to theoretically constrain the values of a given iteration to the feasible set. This is often due to a step-size issue in the optimizer. One solution is to smoothly "stitch" to the original constrained function $f(x)$ a new function $h(x)$ that is defined over the entire direction of the real line. This stitched function $h(x)$ has the same value and slope at the stitching point x_0, and it should preserve the monotonicity of the function as x moves in that direction.

i. The marginal utility of consumption for the period utility function in this model is $c^{-\sigma}$, which is only defined for $c \geq 0$ and has an Inada condition at $c = 0$ because $\lim_{c\rightarrow 0} c^{-\sigma} = \infty$. Write a function `MU_c_stitch()` that takes as arguments a scalar or vector of values for c and a value for $\sigma \geq 1$ and returns a scalar or vector of marginal utilities $c^{-\sigma}$ associated with those values.

```
MU_c_vec = MU_c_stitch(cvec, sigma)
```

Create the function such that values of $c < 0.0001$ are a linear function that has the same slope and value as the marginal utility at $c = 0.0001$.

$$u'(c) = \begin{cases} c^{-\sigma} \quad \text{for} \quad c \geq 0.0001 \\ m_1 c + m_2 \quad \text{for} \quad c < 0.0001 \end{cases}$$
$$\text{s.t.} \quad m_1(0.0001) + m_2 = (0.0001)^{-\sigma} \quad \text{and} \quad m_1 = -\sigma(0.0001)^{-\sigma-1}$$

Report your marginal utility values $u'(c)$ for `cvec=np.array([-0.01,-0.004, 0.5, 2.6])` given $\sigma = 2.2$.

ii. This model uses an elliptical disutility of labor function which has the following marginal disutility of labor function.

$$\texttt{MDU\_n} = g'(n) = \left(\frac{b}{\tilde{l}}\right)\left(\frac{n}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad b,\tilde{l}>0 \quad \text{and} \quad \upsilon \geq 1$$

This function has Inada conditions at $n=0$ and $n=\tilde{l}$ because $\lim_{n\to 0} g'(n) = 0$ and $\lim_{n\to\tilde{l}} g'(n) = \infty$. Write a function `MU_n_stitch()` that takes as arguments a scalar or vector of values for n and values for $\tilde{l}, b > 0$ and $\upsilon \geq 1$ and returns a scalar or vector of marginal disutilities of labor $g'(n)$ associated with those values.

```
MU_n_vec = MU_n_stitch(nvec, ltilde, b_ellip, upsilon)
```

Create the function such that values of $n < 0.000001$ are a linear function that has the same slope and value as the marginal disutility at $n = 0.000001$ and such that values of $n > \tilde{l} - 0.000001$ are a linear function that has the same slope and value as the marginal disutility at $n = \tilde{l} - 0.000001$.

$$g'(n) = \begin{cases} m_1 n + m_2 \quad \text{for} \quad n < 0.000001 \\ \left(\frac{b}{\tilde{l}}\right)\left(\frac{n}{\tilde{l}}\right)^{\upsilon-1}\left[1-\left(\frac{n}{\tilde{l}}\right)^{\upsilon}\right]^{\frac{1-\upsilon}{\upsilon}} \quad \text{for} \quad 0.000001 \leq n \leq \tilde{l} - 0.000001 \\ q_1 n + q_2 \quad \text{for} \quad n > \tilde{l} - 0.000001 \end{cases}$$

$$\text{s.t.} \quad m_1(0.000001) + m_2 = g'(0.000001) \quad \text{and} \quad m_1 = g''(0.000001)$$
$$\text{and} \quad q_1(\tilde{l} - 0.000001) + q_2 = g'(\tilde{l} - 0.000001) \quad \text{and} \quad q_1 = g''(\tilde{l} - 0.000001)$$

Report your resulting marginal disutility values $g'(n)$ for `nvec=np.array([-0.013, -0.002, 0.42, 1.007, 1.011])` given $\tilde{l} = 1.0$, $b_{ellip} = 0.5$, and $\upsilon = 1.5$.

Exercise 7.3. Using the calibration from Section 7.7 and the steady-state equilibrium Definition 7.1, solve for the steady-state equilibrium values of $\{\bar{c}_s, \bar{n}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$, $\{\bar{r}, \bar{w}, \bar{K}, \bar{L}, \bar{Y}, \bar{C}\}$.

i. Plot the steady-state distributions of consumption, labor supply, and savings $\{\bar{c}_s, \bar{n}_s\}_{s=1}^S$, $\{\bar{b}_s\}_{s=2}^S$.

ii. Show in a table the steady-state values for prices and aggregate variables $\{\bar{r}, \bar{w}, \bar{K}, \bar{L}, \bar{Y}, \bar{C}\}$.

Display in the same table the maximum absolute errors in the savings Euler errors, labor supply Euler errors, final period savings $\bar{b}_{S+1}$, and resource constraint errors.

Exercise 7.4. Use time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from $\mathbf{\Gamma}_1 = 1.08(\bar{\mathbf{\Gamma}})$ from the steady-state solution and calibration in Exercise 7.3. You'll have to choose a guess for $T1$ and $T2$ and a time path updating parameter $\xi \in (0,1]$, but I can assure you that $T1 < 250$ and $T2 < 300$. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of `TPI_tol` $= 10^{-12}$. Use a linear or quadratic initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_{T1}^1 at time $T1$.

i. Make 3D surface plots of the equilibrium time path of the distribution of consumption $c_{s,t}$, labor supply $n_{s,t}$, and savings $b_{s,t}$

ii. Make line plots of the equilibrium time paths for the prices and aggregate variables $\{r_t, w_t, K_t, L_t, Y_t, C_t\}$

iii. Show in a table the maximum absolute errors across the equilibrium time path of the savings Euler errors, labor supply Euler errors, final period savings $\bar{b}_{S+1,t}$, and resource constraint errors.

iv. How many periods did it take for the economy to get within 0.0001 of the steady-state aggregate capital stock $\bar{K}$? That is, what is T?

Exercise 7.5. [TODO: Need to update this calibration for the 80-period model.]Assume that $S = 20$. If we divide heads of household in the United States in to 20 age bins $\{21-24, 25-28, 29-32, ...93-96, 97-100\}$, the average hours for each of those age categories (as a percent of the maximum annual hours in the survey) is the following.

```
AnnHrs_US = np.array([0.5, 0.6, 0.7, 0.75, 0.77, 0.78, 0.785, 0.79, 0.8,
    0.81, 0.82, 0.82, 0.8, 0.77, 0.74, 0.7, 0.6, 0.45, 0.3, 0.2])
```

Calibrate χ_s^n for $s = 1, 2, ...20$ so that the steady-state labor supply produced by the model $\{\bar{n}_s\}_{s=1}^{20}$ is close to the vector of empirical hours above.