

Chapter 8

S-period Lives, Endogenous Labor, and Heterogeneous Abilities

In this chapter, we augment the model from Section 7 with individuals of each age s having different abilities. The abilities will enter the labor income part of the budget constraint in a way that an individual will earn the average wage plus a premium or a discount based on their different abilities. Age and ability are two of the most important dimensions of individual heterogeneity to have in an economic model.

8.1 Heterogeneous lifetime ability paths

Differences among workers' productivity in terms of ability is one of the key dimensions of heterogeneity to model in a micro-founded macroeconomy. In this chapter, we model this heterogeneity as deterministic lifetime productivity paths to which new cohorts of agents in the model are randomly assigned.¹ In our model, agents' labor income comes from the equilibrium wage and the agent's endogenous quantity of labor supply. In this section, we augment the labor income expression with an individual productivity $e_{j,s}$, where j is the index of the ability type or path of the individual and s is the age of the individual with

¹Stochastic and persistent idiosyncratic ability is another common way to model income heterogeneity. We treat this in Chapter 9.

that ability path.

$$\text{labor income: } w_t e_{j,s} n_{j,s,t} \quad \forall j, s, t \quad (8.1)$$

In this specification, w_t is an equilibrium wage representing a portion of labor income that is common to all workers. Individual quantity of labor supply is $n_{j,s,t}$, and $e_{j,s}$ represents a labor productivity factor that augments or diminishes the productivity of a worker's labor supply relative to average productivity.

We calibrate deterministic ability paths such that each lifetime income group has a different life-cycle profile of earnings. The distribution on income and wealth are often focal components of macroeconomic models. As such, we use a calibration of deterministic life-time ability paths from DeBacker et al. (2017b) that can represent U.S. earners in the top 1% of the distribution of lifetime income. Piketty and Saez (2003) show that income and wealth attributable to these households has shown the greatest growth in recent decades. The data come from the U.S. Internal Revenue Services's (IRS) Statistics of Income program (SOI) Continuous Work History Sample (CWHS). DeBacker et al. (2017b) match the SOI data with Social Security Administration (SSA) data on age and Current Population Survey (CPS) data on hours in order to generate a non-top-coded measure of hourly wage.

Figure 8.1: Exogenous life cycle income ability paths $\log(e_{j,s})$ with $S = 80$ and $J = 7$

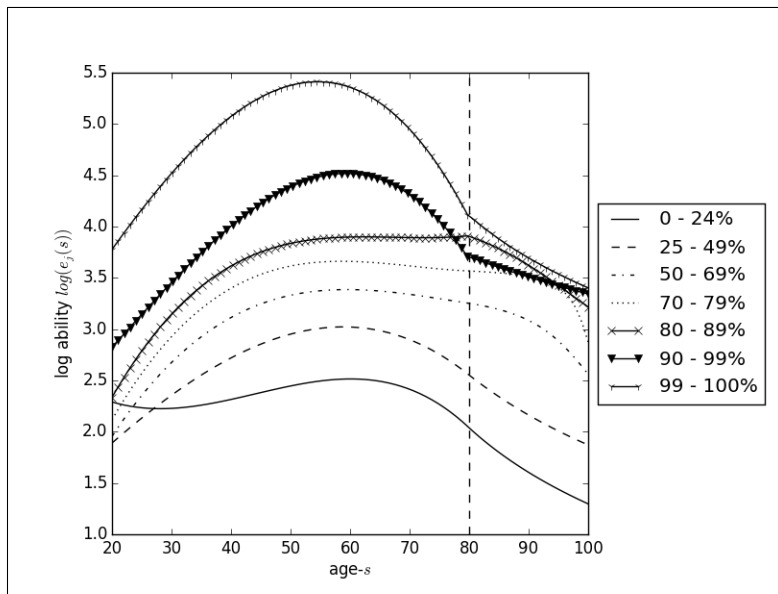


Figure 8.1 shows a calibration for $J = 7$ deterministic lifetime ability paths $e_{j,s}$ corresponding to labor income percentiles $\lambda = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Because there are few individuals above age 80 in the data, DeBacker et al. (2017b) extrapolate these estimates for model ages 80-100 using an arctan function.

8.2 Households

Individuals are born at age $s = 1$ and live to age $S \in [3, 80]$. We choose 80 as the upper bound of periods to live so that the minimum amount of time represented by a model period is one year. But this restriction is not important.

At birth, each individual age $s = 1$ is randomly assigned one of J ability groups, indexed by j . Let λ_j represent the fraction of individuals in each ability group, such that $\sum_j \lambda_j = 1$. Note that this implies that the distribution across ability types in each age is given by $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_J]$. Once an individual is born and assigned to an ability type, he remains that ability type for his entire lifetime. This is deterministic ability heterogeneity. Let $e_{j,s} > 0$ be a matrix of ability-levels such that an individual of ability type j will have lifetime abilities of $[e_{j,1}, e_{j,2}, \dots, e_{j,S}]$. The household budget constraint is now the following,

$$\begin{aligned} c_{j,s,t} + b_{j,s+1,t+1} &= (1 + r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \quad \forall j, s, t \\ \text{where } b_{j,1,t}, b_{j,S+1,t} &= 0 \quad \forall j, t \end{aligned} \tag{8.2}$$

where many of the variables now have j subscripts. The variables with three subscripts (j, s, t) tell you to which ability type j and age s individual the variable belongs and in which period t .

The labor-leisure constraint and the period utility function are similar to (7.1) and (7.7), respectively, except the endogenous variables have an additional j subscript.

$$n_{j,s,t} + l_{j,s,t} = \tilde{l} \tag{8.3}$$

$$u(c_{j,s,t}, n_{j,s,t}) = \frac{c_{j,s,t}^{1-\sigma} - 1}{1-\sigma} + \chi_s^n b \left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}} \right)^v \right]^{\frac{1}{v}} \quad (8.4)$$

We describe the properties and derivation of the elliptical disutility of labor function in (8.4) in Section 7.1. It is an approximation of the standard constant Frisch elasticity disutility of labor supply function, but (8.4) has Inada conditions at both the upper and lower bounds of labor supply. This provides a significant increase in computational tractability with very little cost.

Households choose lifetime consumption $\{c_{j,s,t+s-1}\}_{s=1}^S$, labor supply $\{n_{j,s,t+s-1}\}_{s=1}^S$, and savings $\{b_{j,s+1,t+s}\}_{s=1}^{S-1}$ to maximize discounted lifetime utility,

$$\max_{\{c_{j,s,t+s-1}, n_{j,s,t+s-1}\}_{s=1}^S, \{b_{j,s+1,t+s}\}_{s=1}^{S-1}} \sum_{u=0}^{E+S-s} \beta^u u(c_{j,s+u,t+u}, n_{j,s+u,t+u}) \quad \forall j, t \quad (8.5)$$

$$\text{s.t.} \quad c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_t e_{j,s} n_{j,s,t} \quad (8.2)$$

$$\text{where} \quad u(c_{j,s,t}, n_{j,s,t}) = \frac{c_{j,s,t}^{1-\sigma} - 1}{1-\sigma} + \chi_s^n b \left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}} \right)^v \right]^{\frac{1}{v}} \quad (8.4)$$

where χ_s^n is a scale parameter that can potentially vary by age s influencing the relative disutility of labor to the utility of consumption. The household's lifetime problem (8.5) can be reduced to choosing S labor supplies $\{n_{j,s,t+s-1}\}_{s=1}^S$ and $S-1$ savings $\{b_{j,s+1,t+s}\}_{s=1}^{S-1}$ by substituting the budget constraints (8.2) in for $c_{j,s,t}$ in each period utility function (8.4) of the lifetime utility function.

The set of optimal lifetime choices for an agent of type j born in period t are characterized by the following S static labor supply Euler equations (8.6), the following $S-1$ dynamic savings Euler equations (8.7), and a budget constraint that binds in all S periods (8.2),

$$\begin{aligned} w_t e_{j,s} u_1(c_{j,s,t+s-1}, n_{j,s,t+s-1}) &= -u_2(c_{j,s+1,t+s}, n_{j,s+1,t+s}) \quad \forall j, t \quad \text{and} \quad s \in \{1, 2, \dots, S\} \\ \Rightarrow \quad w_t e_{j,s} (c_{j,s,t})^{-\sigma} &= \chi_s^n \left(\frac{b}{\tilde{l}} \right) \left(\frac{n_{j,s,t}}{\tilde{l}} \right)^{v-1} \left[1 - \left(\frac{n_{j,s,t}}{\tilde{l}} \right)^v \right]^{\frac{1-v}{v}} \end{aligned} \quad (8.6)$$

$$\begin{aligned}
u_1(c_{j,s,t+s-1}, n_{j,s,t+s-1}) &= \beta(1+r_{t+1})u_1(c_{j,s+1,t+s}, n_{j,s+1,t+s}) \quad \forall j, t \quad \text{and} \quad s \in \{1, 2, \dots, S-1\} \\
\Rightarrow (c_{j,s,t})^{-\sigma} &= \beta(1+r_{t+1})(c_{j,s+1,t+1})^{-\sigma}
\end{aligned} \tag{8.7}$$

$$c_{j,s,t} + b_{j,s+1,t+1} = (1+r_t)b_{j,s,t} + w_te_{j,s}n_{j,s,t} \quad \forall j, t \quad \text{and} \quad s \in \{1, 2, \dots, S\}, \quad b_{1,t}, b_{S+1,t+S} = 0 \tag{8.2}$$

where u_1 is the partial derivative of the period utility function with respect to its first argument $c_{j,s,t}$, and u_2 is the partial derivative of the period utility function with respect to its second argument $n_{j,s,t}$. As was demonstrated in detail in Section 6.1, the dynamic Euler equations (8.7) do not include marginal utilities of all future periods because of the principle of optimality and the envelope condition.

Note that these $2S-1$ household decisions are perfectly identified if the household knows what prices will be over its lifetime $\{w_u, r_u\}_{u=t}^{t+S-1}$. As in section 6.1, let the distribution of capital and household beliefs about the evolution of the distribution of capital be characterized by (8.8) and (5.17).

$$\mathbf{\Gamma}_t \equiv \{b_{j,s,t}\}_{s=2}^S \quad \forall j, t \tag{8.8}$$

$$\mathbf{\Gamma}_{t+u}^e = \Omega^u(\mathbf{\Gamma}_t) \quad \forall t, \quad u \geq 1 \tag{5.17}$$

8.3 Firms

Firms are characterized exactly as in Section 5.2, with the firm's aggregate capital decision K_t governed by first order condition (5.20) and its aggregate labor decision L_t governed by first order condition (5.21).

$$r_t = \alpha A \left(\frac{L_t}{K_t} \right)^{1-\alpha} - \delta \tag{5.20}$$

$$w_t = (1-\alpha)A \left(\frac{K_t}{L_t} \right)^\alpha \tag{5.21}$$

The per-period depreciation rate of capital is $\delta \in [0, 1]$, the capital share of income is $\alpha \in (0, 1)$, and total factor productivity is $A > 0$.

The only change to note, which is described more carefully in Section 8.4 in the labor

market clearing condition (8.9), is that aggregate labor L_t in the production function is now in efficiency units. This means that the aggregate labor L_t used in production is made up of both labor hours $n_{j,s,t}$ and ability $e_{j,s}$.

8.4 Market clearing

Three markets must clear in this model: the labor market, the capital market, and the goods market. Each of these equations amounts to a statement of supply equals demand. The market clearing conditions for this version of the model slightly different from those in the previous sections because we must sum not only over ages s but also over ability types j .

$$L_t = \sum_{s=1}^S \sum_{j=1}^J \lambda_j e_{j,s} n_{j,s,t} \quad \forall t \quad (8.9)$$

$$K_t = \sum_{s=2}^S \sum_{j=1}^J \lambda_j b_{j,s,t} \quad \forall t \quad (8.10)$$

$$Y_t = C_t + K_{t+1} - (1 - \delta)K_t \quad \forall t \quad (8.11)$$

where $C_t \equiv \sum_{s=1}^S \sum_{j=1}^J \lambda_j c_{j,s,t}$

The goods market clearing equation (8.11) is redundant by Walras' Law. As noted in Section 8.3, the aggregate labor supply in (8.9) is in efficiency units (i.e., L_t sums over $e_{j,s}n_{j,s,t}$).

It is important to note that the distributional assumptions here are that each age- s cohort has a population of one, which means the total population is S and the population age- s and ability- j is λ_j . Each aggregate variable L_t , K_t , and C_t (and Y_t indirectly) sums over individual decisions multiplied by the individual population measure λ_j .

8.5 Equilibrium

As in previous sections, we give a rough sketch of the equilibrium before providing exact definitions of the functional equilibrium concepts so you can see what the functions look like

and understand the exact equilibrium definition more clearly. A rough description of the equilibrium solution to the problem above is the following three points.

- i. Households optimize according to equations (8.6) and (8.7).
- ii. Firms optimize according to (5.20) and (5.21).
- iii. Markets clear according to (8.9) and (8.10).

These equations characterize the equilibrium and constitute a system of nonlinear difference equations.

The easiest way to understand the equilibrium solution is to substitute the market clearing conditions (8.9) and (8.10) into the firm's optimal conditions (5.20) and (5.21) solve for the equilibrium wage and interest rate as functions of the distribution of capital.

$$w_t(\mathbf{\Gamma}_t) : \quad w_t = (1 - \alpha)A \left(\frac{\sum_{s=2}^S \sum_{j=1}^J \lambda_j b_{j,s,t}}{\sum_{s=1}^S \sum_{j=1}^J \lambda_j e_{j,s} n_{j,s,t}} \right)^\alpha \quad \forall t \quad (8.12)$$

$$r_t(\mathbf{\Gamma}_t) : \quad r_t = \alpha A \left(\frac{\sum_{s=1}^S \sum_{j=1}^J \lambda_j e_{j,s} n_{j,s,t}}{\sum_{s=2}^S \sum_{j=1}^J \lambda_j b_{j,s,t}} \right)^{1-\alpha} - \delta \quad \forall t \quad (8.13)$$

It is worth noting here that the equilibrium wage (8.12) and interest rate (8.13) are written as functions of the period- t distribution of savings (wealth) $\mathbf{\Gamma}_t$ from (8.8) and are not functions of the period- t distribution of labor supply, which labor distribution shows up in (8.12) and (8.13). This is because, similar to next period savings $b_{j,s+1,t+1}$, current period labor supply $n_{j,s,t}$ must be chosen in period t and is therefore a function of the current state $\mathbf{\Gamma}_t$ distribution of savings.

Now (8.12), (8.13), and the budget constraint (8.2) can be substituted into household Euler equations (8.6) and (8.7) to get the following $(2S - 1)$ -equation system of lifetime characterizing equations for an individual of ability type j . Extended across all time periods, this system completely characterizes the equilibrium.

$$w_t(\mathbf{\Gamma}_t) e_{j,s} \left(\left[1 + r_t(\mathbf{\Gamma}_t) \right] b_{j,s,t} + w_t(\mathbf{\Gamma}_t) e_{j,s} n_{j,s,t} - b_{j,s+1,t+1} \right)^{-\sigma} = \dots \quad (8.14)$$

$$\chi_s^n \left(\frac{b}{\bar{l}} \right) \left(\frac{n_{j,s,t}}{\bar{l}} \right)^{v-1} \left[1 - \left(\frac{n_{j,s,t}}{\bar{l}} \right)^v \right]^{\frac{1-v}{v}} \quad \text{for } s \in \{1, 2, \dots, S\} \quad \text{and } \forall j, t$$

$$\begin{aligned}
& \left(\left[1 + r_t(\mathbf{\Gamma}_t) \right] b_{j,s,t} + w_t(\mathbf{\Gamma}_t) e_{j,s} n_{j,s,t} - b_{j,s+1,t+1} \right)^{-\sigma} = \\
& \beta \left[1 + r_{t+1}(\mathbf{\Gamma}_{t+1}) \right] \left(\left[1 + r_{t+1}(\mathbf{\Gamma}_{t+1}) \right] b_{j,s+1,t+1} + w_{t+1}(\mathbf{\Gamma}_{t+1}) e_{j,s} n_{j,s+1,t+1} - b_{j,s+2,t+2} \right)^{-\sigma} \\
& \text{for } s \in \{1, 2, \dots, S-1\} \quad \text{and} \quad \forall j, t
\end{aligned} \tag{8.15}$$

The system of S nonlinear static equations (8.14) and $S-1$ nonlinear dynamic equations (8.15) characterizing the the lifetime labor supply and savings decisions for each household $\{n_{j,s,t+s-1}\}_{j,s=1}^{J,S}$ and $\{b_{j,s+1,t+s}\}_{j,s=1}^{J,S-1}$ is not identified. Each individual knows the current distribution of capital $\mathbf{\Gamma}_t$. However, we need to solve for policy functions for the entire distribution of capital in the next period $\mathbf{\Gamma}_{t+1} = \{b_{j,s+1,t+1}\}_{j,s=1}^{J,S-1}$ and a number of subsequent periods for all agents alive in those subsequent periods. We also need to solve for a policy function for the individual $b_{j,s+2,t+2}$ from these $S-1$ equations. Even if we pile together all the sets of individual lifetime Euler equations, it looks like this system is unidentified. This is because it is a series of second order difference equations. But the solution is a fixed point of stationary functions.

We first define the steady-state equilibrium, which is exactly identified. Let the steady state of endogenous variable x_t be characterized by $x_{t+1} = x_t = \bar{x}$ in which the endogenous variables are constant over time. Then we can define the steady-state equilibrium as follows.

Definition 8.1 (Steady-state equilibrium). A non-autarkic steady-state equilibrium in the perfect foresight overlapping generations model with S -period lived agents, endogenous labor supply, and deterministic heterogeneous lifetime ability paths is defined as constant allocations of consumption $\{\bar{c}_{j,s}\}_{j,s=1}^{J,S}$, labor supply $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$, and savings $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$, and prices \bar{w} and \bar{r} such that:

- i. households optimize according to (8.6) and (8.7),
 - ii. firms optimize according to (5.20) and (5.21),
 - iii. markets clear according to (8.9) and (8.10).
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The relevant examples of stationary functions in this model are the policy functions for labor and savings. Let the equilibrium policy functions for labor supply be represented by $n_{j,s,t} = \phi_{j,s}(\mathbf{\Gamma}_t)$, and let the equilibrium policy functions for savings be represented by $b_{j,s+1,t+1} = \psi_{j,s}(\mathbf{\Gamma}_t)$. The arguments of the functions (the state) may change overtime

causing the labor and savings levels to change over time, but the function of the arguments is constant (stationary) across time.

With the concept of the state of a dynamical system and a stationary function, we are ready to define a functional non-steady-state (transition path) equilibrium of the model.

Definition 8.2 (Non-steady-state functional equilibrium). A non-steady-state functional equilibrium in the perfect foresight overlapping generations model with S -period lived agents, endogenous labor supply, and deterministic heterogeneous lifetime ability paths is defined as stationary allocation functions of the state $\{n_{j,s,t} = \phi_{j,s}(\mathbf{\Gamma}_t)\}_{j,s=1}^{J,S}$, $\{b_{j,s+1,t+1} = \psi_{j,s}(\mathbf{\Gamma}_t)\}_{j,s=1}^{J,S-1}$ and stationary price functions $w(\mathbf{\Gamma}_t)$ and $r(\mathbf{\Gamma}_t)$ such that:

- i. households have symmetric beliefs $\Omega(\cdot)$ about the evolution of the distribution of savings as characterized in (5.17), and those beliefs about the future distribution of savings equal the realized outcome (rational expectations),

$$\mathbf{\Gamma}_{t+u} = \mathbf{\Gamma}_{t+u}^e = \Omega^u(\mathbf{\Gamma}_t) \quad \forall t, \quad u \geq 1$$

- ii. households optimize according to (8.6) and (8.7),
 - iii. firms optimize according to (5.20) and (5.21),
 - iv. markets clear according to (8.9) and (8.10).
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8.6 Solution Method

In this section we characterize computational approaches to solving for the the steady-state equilibrium from Definition 8.1 and the transition path equilibrium from Definition 8.2.

8.6.1 Steady-state equilibrium

This section outlines the steps for computing the solution to the steady-state equilibrium in Definition 8.1. The parameters needed for the steady-state solution of this model are $\{J, S, \{\lambda_j\}_{j=1}^J, \beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, v, \{\chi_s^n\}_{s=1}^S, A, \alpha, \delta\}$, where S is the number of periods in an individual's life, J is the number of ability groups, $\{\lambda_j\}_{j=1}^J$ is the income percentiles of ability groups, $\{\beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, v, \{\chi_s^n\}_{s=1}^S\}$ are household utility function parameters, and $\{A, \alpha, \delta\}$ are firm production function parameters. These parameters are chosen, calibrated, or estimated outside of the model and are inputs to the solution method.

The steady-state is defined as the solution to the model in which the distributions of individual consumption, labor supply, and savings have settled down and are no longer changing over time. As such, it can be thought of as a long-run solution to the model in which the effects of any shocks or changes from the past no longer have an effect.

$$c_{j,s,t} = \bar{c}_{j,s}, \quad n_{j,s,t} = \bar{n}_{j,s}, \quad b_{j,s,t} = \bar{b}_{j,s} \quad \forall j, s, t \quad (8.16)$$

From the market clearing conditions (8.9) and (8.10) and the firms' first order equations (5.20) and (5.21), the household steady-state conditions imply the following steady-state conditions for prices and aggregate variables.

$$r_t = \bar{r}, \quad w_t = \bar{w}, \quad K_t = \bar{K} \quad L_t = \bar{L} \quad \forall t \quad (7.16)$$

The steady-state is characterized by the steady-state versions of the set of $2S - 1$ Euler equations over the lifetime of an individual (after substituting in the budget constraint) and the $2S - 1$ unknowns $\{\bar{n}_s\}_{s=1}^S$ and $\{\bar{b}_{s+1}\}_{s=1}^{S-1}$,

$$\bar{w}e_{j,s} \left([1 + \bar{r}] \bar{b}_{j,s} + \bar{w}e_{j,s} \bar{n}_{j,s} - \bar{b}_{j,s+1} \right)^{-\sigma} = \chi_s^n \left(\frac{\bar{b}}{\bar{l}} \right) \left(\frac{\bar{n}_{j,s}}{\bar{l}} \right)^{v-1} \left[1 - \left(\frac{\bar{n}_{j,s}}{\bar{l}} \right)^v \right]^{\frac{1-v}{v}} \quad (8.17)$$

$$\forall j \quad \text{and} \quad s = \{1, 2, \dots, S\}$$

$$\left([1 + \bar{r}] \bar{b}_{j,s} + \bar{w}e_{j,s} \bar{n}_{j,s} - \bar{b}_{j,s+1} \right)^{-\sigma} = \beta(1 + \bar{r}) \left([1 + \bar{r}] \bar{b}_{j,s+1} + \bar{w}e_{j,s+1} \bar{n}_{j,s+1} - \bar{b}_{j,s+2} \right)^{-\sigma}$$

$$\forall j \quad \text{and} \quad s = \{1, 2, \dots, S-1\}$$

$$(8.18)$$

where both \bar{w} and \bar{r} are functions of the distributions of labor supply and savings as shown in (8.12) and (8.13). This represents a system of $J(2S - 1)$ nonlinear dynamic equations and $J(2S - 1)$ unknowns.

One approach to solving this system would be to use a multivariate root finder that chooses the $J(2S - 1)$ steady-state variables $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ simultaneously to solve the zeros of the $J(2S - 1)$ Euler equations (8.17) and (8.18). However, the trade off between labor supply and savings in each period create a series of saddle paths in the ob-

jective function that render the simultaneous equations root finder unreliable and somewhat intractable. We have found that this approach only works when your initial guess for the steady-state $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$ is close to the solution. It breaks down when the underlying parameters are changed.

Another method would be to perform an outer loop root finder on a guess for \bar{r} and \bar{w} (or equivalently \bar{K} and \bar{L}), such that the household's problem is solved for the values of r and w in each iteration and the outer loop root finder uses the firm's first order condition as the error equations. This method also works well if one's initial guess is good. However, it can fail to find the equilibrium for a large range of initial guesses.

The most robust solution method that we have found takes significantly more time than the two methods described in the preceding paragraphs, but it successfully finds the steady-state equilibrium for most feasible initial guesses that we have constructed. This method is a bisection method in the outer loop guesses for steady-state equilibrium \bar{K} and \bar{L} . In the inner loop of the household j 's problem given r and w implied by K and L , we solve the problem by breaking the multivariate root finder problem with $2S - 1$ equations and unknowns into a series of many univariate root finder problems and one bivariate root finder problem. The algorithm is the following.

- i. Make a guess for the steady-state aggregate capital stock \bar{K}^i and aggregate labor \bar{L}^i .
 - (a) Values for \bar{K}^i and \bar{L}^i will imply values for the interest rate \bar{r}^i and wage \bar{w}^i from (5.20) and (5.21)
 - (b) For the bisection method, we must use guesses for K and L because those uniquely determine r and w , whereas the converse is not true. From the firms' first order conditions (5.20) and (5.21), we see that r and w are functions of the same capital-labor ratio K/L . Infinitely many combinations of K and L determine a given r and w .
- ii. Given \bar{r}^i and \bar{w}^i , solve for the steady-state household's lifetime decisions $\{\bar{n}_{j,s}\}_{j,s=1}^{J,S}$ and $\{\bar{b}_{j,s+1}\}_{j,s=1}^{J,S-1}$.
 - (a) Given \bar{r}^i and \bar{w}^i , guess an initial steady-state consumption $\bar{c}_{j,1}^m$, where m is the

index of the inner-loop (household problem given \bar{r}^i, \bar{w}^i) iteration.

- (b) Given \bar{r}^i, \bar{w}^i , and $\bar{c}_{j,1}^m$, use the sequence of $S - 1$ dynamic savings Euler equations (8.7) to solve for the implied series of steady-state consumptions $\{\bar{c}_{j,s}^m\}_{j,s=1}^{J,S}$. This sequence has an analytical solution.

$$\bar{c}_{j,s+1}^m = \bar{c}_{j,s}^m [\beta(1 + \bar{r}^i)]^{\frac{1}{\sigma}} \quad \forall j \quad \text{and} \quad s = \{1, 2, \dots, S-1\} \quad (8.19)$$

- (c) Given \bar{r}^i, \bar{w}^i , and $\{\bar{c}_{j,s}^m\}_{j,s=1}^{J,S}$, solve for the series of steady-state labor supplies $\{\bar{n}_{j,s}^m\}_{j,s=1}^{J,S}$ using the S static labor supply Euler equations (8.6). This will require a series of S separate univariate root finders or one multivariate root finder.

$$\bar{w}^i e_{j,s} (\bar{c}_{j,s}^m)^{-\sigma} = \chi_s^n \left(\frac{b}{\bar{l}} \right) \left(\frac{\bar{n}_{j,s}^m}{\bar{l}} \right)^{v-1} \left[1 - \left(\frac{\bar{n}_{j,s}^m}{\bar{l}} \right)^v \right]^{\frac{1-v}{v}} \quad \forall j \quad \text{and} \quad s = \{1, 2, \dots, S\} \quad (8.20)$$

It is this separation of the labor supply decisions from the consumption-savings decisions that gets rid of the saddle paths in the objective function that are so difficult for global optimization.

- (d) Given \bar{r}^i, \bar{w}^i , and implied steady-state consumption $\{\bar{c}_{j,s}^m\}_{j,s=1}^{J,S}$ and labor supply $\{\bar{n}_{j,s}^m\}_{j,s=1}^{J,S}$, solve for implied time path of savings $\{\bar{b}_{j,s+1}^m\}_{j,s=1}^{J,S}$ across all ages of the representative lifetime using the household budget constraint (8.2).

$$\bar{b}_{j,s+1}^m = (1 + \bar{r}^i) \bar{b}_{j,s}^m + \bar{w}^i e_{j,s} \bar{n}_{j,s}^m - \bar{c}_{j,s}^m \quad \forall j \quad \text{and} \quad s = \{1, 2, \dots, S\} \quad (8.21)$$

Note that this sequence of savings includes savings in the last period of life for the next period $\bar{b}_{j,S+1}$. This savings amount is zero in equilibrium, but is not zero for an arbitrary guess for $\bar{c}_{j,1}^m$ as in step (a).

- (e) Update the initial guess for $\bar{c}_{j,1}^m$ to $\bar{c}_{j,1}^{m+1}$ until the implied savings in the last period equals zero $\bar{b}_{j,S+1}^{m+1} = 0$.

- iii. Given solution for optimal household decisions $\{\bar{c}_{j,s}^m\}_{j,s=1}^{J,S}$, $\{\bar{n}_{j,s}^m\}_{j,s=1}^{J,S}$, and $\{\bar{b}_{j,s+1}^m\}_{j,s=1}^{J,S-1}$ based on the guesses for aggregate capital \bar{K}^i and aggregate labor \bar{L}^i , solve for the

aggregate capital $\bar{K}^{i'}$ and aggregate labor $\bar{L}^{i'}$ implied by the household solutions and market clearing conditions.

$$\bar{K}^{i'} = \sum_{s=2}^S \sum_{j=1}^J \lambda_j \bar{b}_{j,s}^m \quad (8.22)$$

$$\bar{L}^{i'} = \sum_{s=1}^S \sum_{j=1}^J \lambda_j e_{j,s} \bar{n}_{j,s}^m \quad (8.23)$$

Update guesses for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ until the the aggregates implied by household optimization equal the initial guess for the aggregates $(\bar{K}^{i'+1}, \bar{L}^{i'+1}) = (\bar{K}^{i+1}, \bar{L}^{i+1})$.

- (a) The bisection method characterizes the updated guess for the aggregate capital stock and aggregate labor $(\bar{K}^{i+1}, \bar{L}^{i+1})$ as a convex combination of the initial guess (\bar{K}^i, \bar{L}^i) and the values implied by household and firm optimization $(\bar{K}^{i'}, \bar{L}^{i'})$, where the weight put on the new values $(\bar{K}^{i'}, \bar{L}^{i'})$ is given by $\xi \in (0, 1]$. The value for ξ must sometimes be small—between 0.05 and 0.2—for certain parameterizations of the model to solve.

$$(\bar{K}^{i+1}, \bar{L}^{i+1}) = \xi(\bar{K}^{i'}, \bar{L}^{i'}) + (1 - \xi)(\bar{K}^i, \bar{L}^i) \quad \text{for } \xi \in (0, 1] \quad (7.24)$$

- (b) Let $\|\cdot\|$ be a norm on the space of feasible aggregate capital and aggregate labor values (K, L) . We often use a sum of squared errors or a maximum absolute error. Check the distance between the initial guess and the implied values as in (7.25). If the distance is less than some tolerance `toler` > 0 , then the problem has converged. Otherwise continue updating the values of aggregate capital and labor using (7.24).

$$\text{dist} \equiv \left\| (\bar{K}^{i'}, \bar{L}^{i'}) - (\bar{K}^i, \bar{L}^i) \right\| \quad (7.25)$$

Define the updating of aggregate variable values (\bar{K}^i, \bar{L}^i) in step (iii) indexed by i as the “outer loop” of the fixed point solution. Although computationally intensive, the bisection method described above is often the most robust solution method.

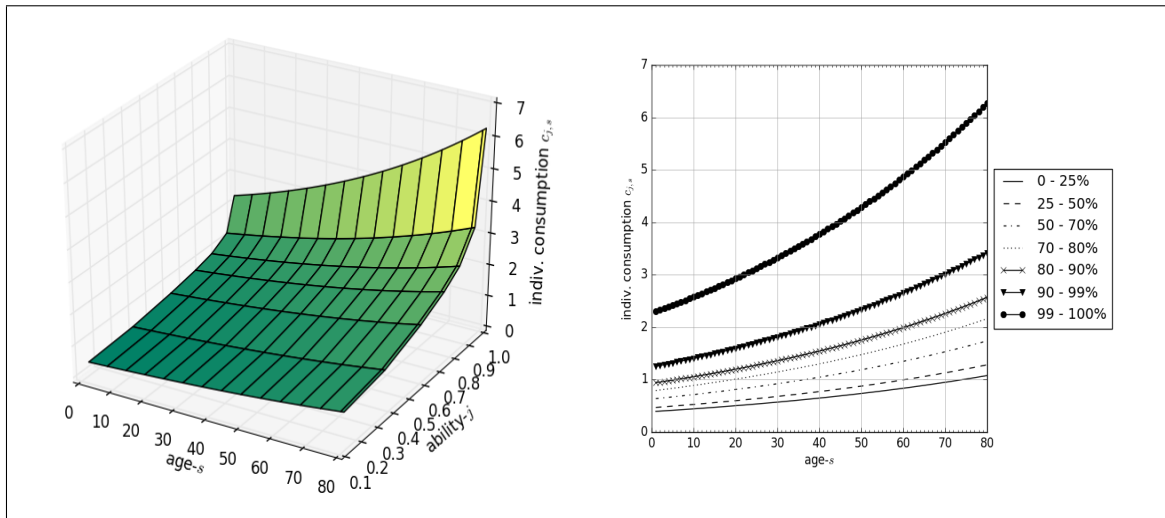
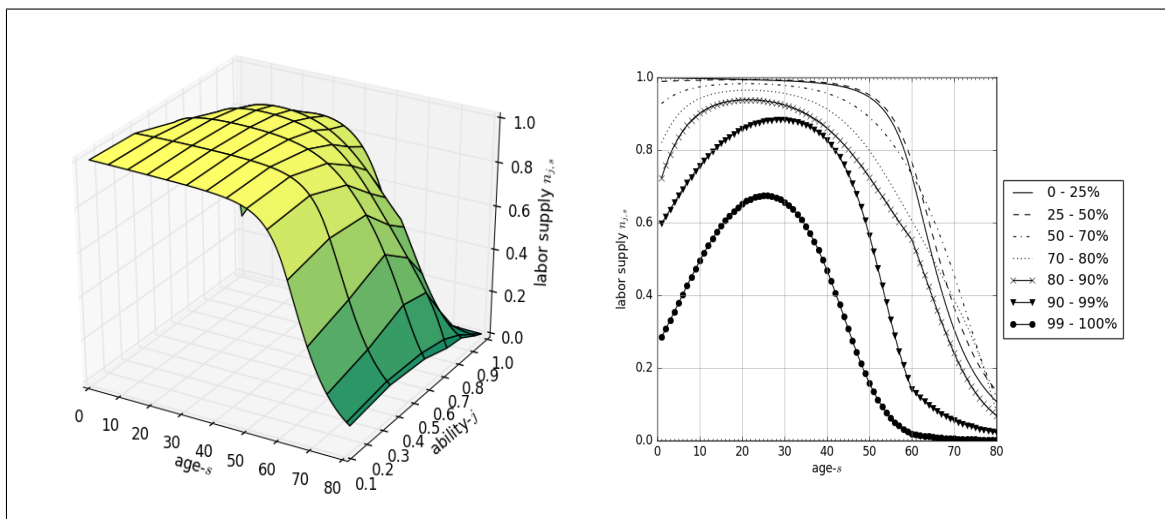
Figure 8.2: Steady-state distribution of consumption $\bar{c}_{j,s}$ Figure 8.3: Steady-state distribution of labor supply $\bar{n}_{j,s}$ 

Figure 8.4: Steady-state distribution of savings/wealth $\bar{b}_{j,s}$

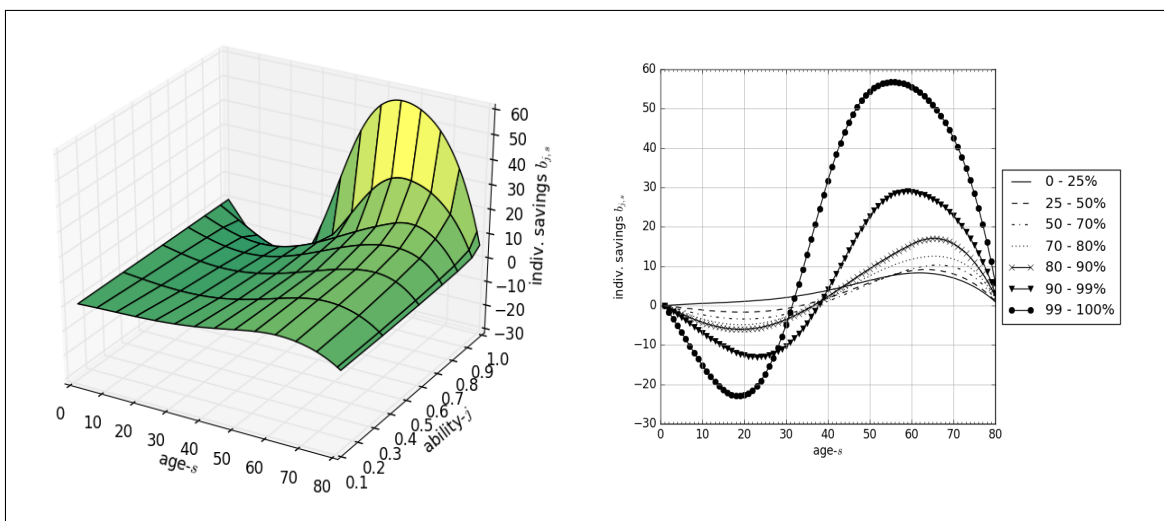


Table 8.1: Steady-state prices, aggregate variables, and maximum errors

Variable	Value	Equilibrium error	Value
\bar{r}	0.075	Max. absolute savings Euler error	1.78e-15
\bar{w}	1.130	Max. absolute labor supply Euler error	7.02e-14
\bar{K}	290.632	Max. absolute final period savings $\bar{b}_{j,S+1}$	8.89e-12
\bar{L}	59.815	Resource constraint error	-0.576
\bar{Y}	104.015		
\bar{C}	90.060	Computation time	8 min. 4 sec.

Figures 8.2, 8.3, and 8.4 show the steady-state distributions of individual consumption, labor supply, and savings, respectively, by age s and ability j in an 80-period-lived agent model with parameter values listed above the line in Table 8.3 in Section 8.7. The left side of Table 8.1 gives the resulting steady-state values for the prices and aggregate variables.

As a final note, it is important to make sure that all of the characterizing equations are satisfied in order to verify that the steady-state has been found. In this model, we must check the $J(2S - 1)$ Euler errors from the labor supply and savings decisions, the final period savings decision (should be zero), the two firm first order conditions, and the three market clearing conditions (including the goods market clearing condition, which is redundant by Walras law). The right side of Table 8.1 shows the maximum errors in all these characterizing conditions. Because all Euler errors are smaller than 7.0e-14, the final period individual savings is less than 8.9e-12, and the resource constraint error is smaller than -0.58, we can be confident that we have successfully solved for the steady-state.

8.6.2 Transition path equilibrium

The TPI solution method for the non-steady-state equilibrium transition path for the S -period-lived agent model with endogenous labor and deterministic heterogeneous lifetime ability paths is similar to the method described in Section 7.6.2. The key assumption is that the economy will reach the steady-state equilibrium $\bar{\Gamma}$ described in Definition 8.1 in a finite number of periods $T < \infty$ regardless of the initial state Γ_1 .

To solve for the transition path (non-steady-state) equilibrium from Definition 8.2, we

must know the parameters from the steady-state problem,

$$\left\{ J, S, \{\lambda_j\}_{j=1}^J, \beta, \sigma, \tilde{l}, \{e_{j,s}\}_{j,s=1}^{J,S}, b, v, \{\chi_s^n\}_{s=1}^S, A, \alpha, \delta \right\}$$

the steady-state solution values $\{\bar{K}, \bar{L}\}$, initial distribution of savings $\mathbf{\Gamma}_1$, and TPI parameters $\{T1, T2, \xi\}$. Tables 8.1 and 8.3 show a particular calibration of the model and a steady-state solution. The algorithm for solving for the transition path equilibrium by time path iteration (TPI) is the following.

- i. Choose a period $T1$ in which the initial guess for the time paths of aggregate capital and aggregate labor will arrive at the steady state and stay there. Choose a period $T2$ upon which and thereafter the economy is assumed to be in the steady state. You must have the guessed time path hit the steady state before individual optimal decisions will hit their steady state.
- ii. Given calibration for initial distribution of savings (wealth) $\mathbf{\Gamma}_1$, which implies an initial capital stock K_1 , guess initial time paths for the aggregate capital stock $\mathbf{K}^i = \{K_1^i, K_2^i, \dots, K_{T1}^i\}$ and aggregate labor $\mathbf{L}^i = \{L_1^i, L_2^i, \dots, L_{T1}^i\}$. Both of these time paths will have to be extended with their respective steady-state values so that they are $T2 + S - 1$ elements long. This is the time-path length that will allow you to solve the lifetime of every individual alive in period $T2$.
- iii. Given time paths \mathbf{K}^i and \mathbf{L}^i , solve for the lifetime consumption $c_{j,s,t}$, labor supply $n_{j,s,t}$, and savings $b_{j,s+1,t+1}$ decisions of all households alive in periods $t = 1$ to $t = T2$.
 - (a) The initial paths for aggregate capital \mathbf{K}^i and aggregate labor \mathbf{L}^i imply time paths for the interest rate $\mathbf{r}^i = \{r_1^i, r_2^i, \dots, r_{T2+S-1}^i\}$ and wage $\mathbf{w}^i = \{w_1^i, w_2^i, \dots, w_{T2+S-1}^i\}$ using the firms' first order equations (5.20) and (5.21).
 - (b) Given the time paths for the interest rate \mathbf{r}^i and wage \mathbf{w}^i and the period-1 distribution of savings (wealth) $\mathbf{\Gamma}_1$, solve for the lifetime decisions $c_{j,s,t}$, $n_{j,s,t}$, and $b_{j,s,t}$ of each household alive during periods 1 through $T2$. This is done using the method outlined in steps (ii)(a) through (ii)(e) of the steady-state computational algorithm outlined in Section 8.6.1.

- iv. Use time path of the distribution of labor supply $n_{j,s,t}$ and savings $b_{j,s,t}$ from households optimal decisions given \mathbf{K}^i and \mathbf{L}^i to compute new paths for aggregate capital and aggregate labor $\mathbf{K}^{i'}$ and $\mathbf{L}^{i'}$ implied by capital and labor market clearing conditions (8.9) and (8.10).
- v. Compare the distance of the time paths of the new implied paths for the aggregate capital and labor $(\mathbf{K}^{i'}, \mathbf{L}^{i'})$ versus the initial aggregate capital and labor $(\mathbf{K}^i, \mathbf{L}^i)$.

$$\text{dist} = \left\| (\mathbf{K}^{i'}, \mathbf{L}^{i'}) - (\mathbf{K}^i, \mathbf{L}^i) \right\| \geq 0 \quad (7.26)$$

Let $\|\cdot\|$ be a norm on the space of time paths for the aggregate capital stock and aggregate labor $(\mathbf{K}^i, \mathbf{L}^i)$. Common norms to use are the L^2 and the L^∞ norms.

- (a) If the distance is less than or equal to some tolerance level $\text{dist} \leq \text{TPI_toler} > 0$, then the fixed point, and therefore the equilibrium transition path, has been found.
- (b) If the distance is greater than some tolerance level, then update the guess for a new set of initial time paths to be a convex combination current initial time paths and the implied time paths.

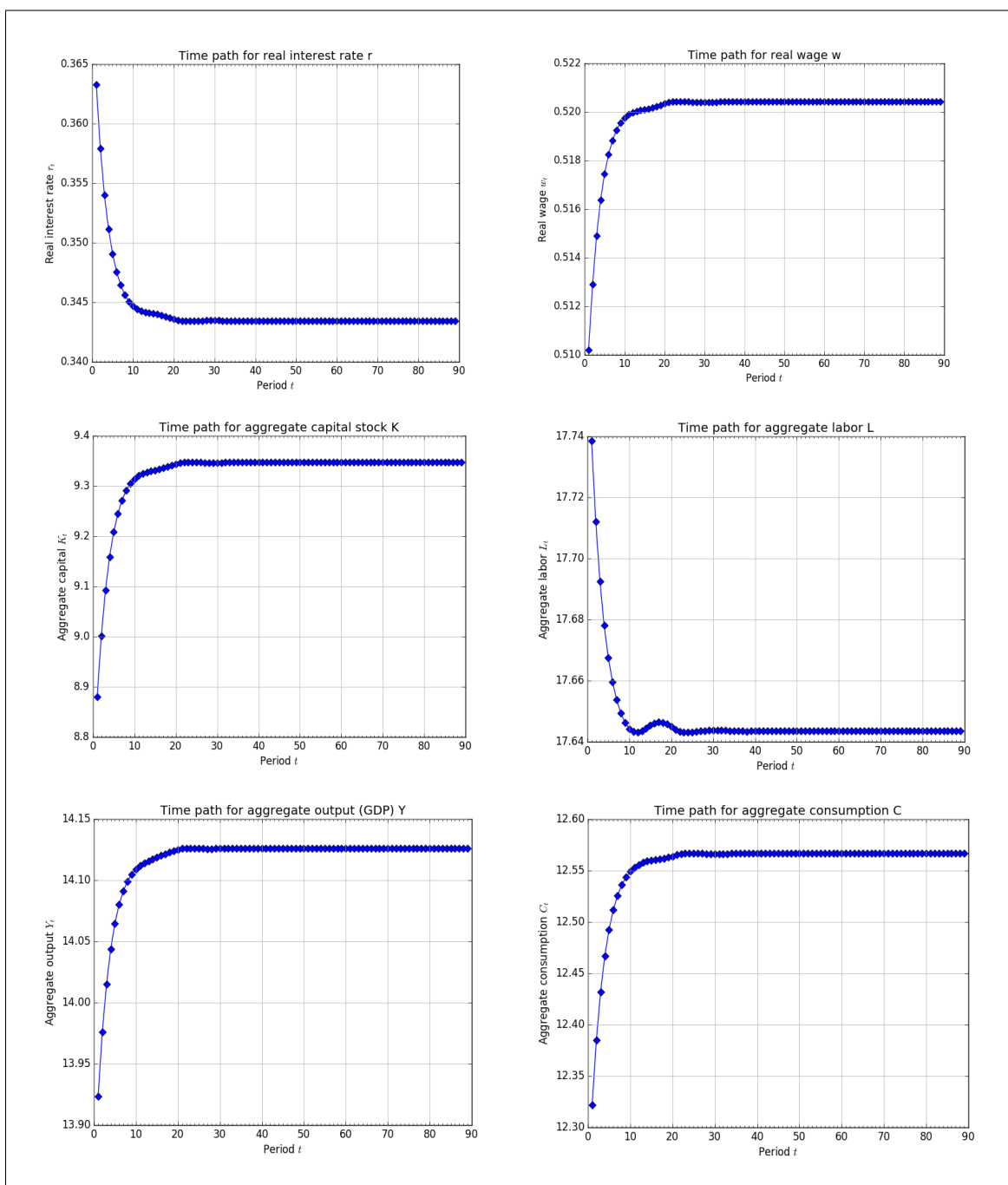
$$(\mathbf{K}^{i+1}, \mathbf{L}^{i+1}) = \xi(\mathbf{K}^{i'}, \mathbf{L}^{i'}) + (1 - \xi)(\mathbf{K}^i, \mathbf{L}^i) \quad \text{for } \xi \in (0, 1] \quad (7.27)$$

Table 8.2: Maximum absolute errors in characterizing equations across transition path

Description	Value
Maximum absolute labor supply Euler error	1.90e-12
Maximum absolute savings Euler error	2.13e-14
Maximum absolute final period savings $\bar{b}_{j,S+1,t}$	1.88e-13
Maximum absolute resource constraint error	1.74e-01

The 6 panels of Figure 8.5 show the equilibrium time paths of the interest rate r_t , wage w_t , and aggregate variables K_t , L_t , Y_t , and C_t . The calibration is the same as that of Section 8.6.1, except that we reduced the number of ages to $S = 20$ and the number of abilities to

Figure 8.5: Equilibrium transition paths of prices and aggregate variables



$J = 3$ with $\lambda = [0.40, 0.35, 0.25]$. We do not show the time paths of individual consumption, labor supply, and savings because they are higher dimensional distributions that would be difficult to show over time. Table 8.2 shows the maximum absolute Euler errors, end-of-life savings, and resource constraint errors across the transition path. All of these should be zero in equilibrium. The fact that none of them is greater than $2.0\text{e-}12$ in absolute value is evidence that we have successfully solved for the non-steady-state equilibrium transition path of the model.

8.7 Calibration

For the steady-state solution in Section 8.6.1 we used the calibration displayed in Table 8.3. We let agents live for $S = 80$ periods, which implies that each model period of life is one year. The annual discount factor is estimated to be 0.96, so the discount factor per period in the S -period model should be $\beta = 0.96^{80/S} = 0.96$. Let the annual depreciation rate of capital be 0.05. Then the depreciation rate in the S -period model should be $\delta = 1 - (1 - 0.05)^{80/S} = 0.05$. Let the coefficient of relative risk aversion be $\sigma = 2.5$, let the productivity scale parameter of firms be $A = 1$, and let the capital share of income be $\alpha = 0.35$. For the disutility of labor, let $\chi_s^n = 1$ for all s , and let $[b, v] = [0.501, 1.554]$. We assume $J = 7$ ability types with percentages $\{\lambda_j\}_{j=1}^7 = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$, with a lifetime ability profiles $e_{j,s}$ given in Figure 8.1.

Because the computation of the non-steady-state equilibrium time path of the calibration used for the steady-state would take more than 24 hours, we use a simpler calibration for the solution in Section 8.6.2. We assume that $S = 20$, which implies that each model period equals 4 years. We also assume only three ability types $J = 3$ with $\{\lambda_j\}_{j=1}^3 = [0.40, 0.35, 0.25]$. This changes the value of the discount factor to $\beta = 0.96^{80/S} = 0.849$ and $\delta = 1 - (1 - 0.05)^{80/S} = 0.185$. We also use the method described in Exercise 8.1 to linearly interpolate a 20×3 ability matrix $e_{j,s}$ from the original 80×7 matrix shown in Figure 8.1. All the other parameters are the same as shown in Table 8.3.

Table 8.3: Calibrated parameter values for simple endogenous labor model

Parameter	Description	Value
J	Number of lifetime income (ability) groups	7
S	Number of periods in individual life	80
$\{\lambda_j\}_{j=1}^J$	Population distribution among ability types	(see Sec. 8.1)
$\{e_{j,s}\}_{j,s=1}^{J,S}$	Individual ability profile factor	(see Fig. 8.1)
β	Per-period discount factor	0.96
σ	Coefficient of relative risk aversion	2.5
\tilde{l}	Time endowment per period	1.0
b	Elliptical disutility of labor scale parameter	0.501 ^a
v	Elliptical disutility of labor shape parameter	1.554 ^a
$\{\chi_s^n\}_{s=1}^S$	Disutility of labor relative scale factor by age	1.0
A	Total factor productivity	1.0
α	Capital share of income	0.35
δ	Per-period depreciation rate of capital	0.05
Γ_1	Initial distribution of savings (wealth)	(see Fig. 7.7)
$T1$	Time period in which initial path guess hits steady state	160
$T2$	Time period in which the model is assumed to hit the steady state	200
ξ	TPI path updating parameter	0.2

^a The calibration of b and v is based on matching the marginal disutility of labor supply of a constant Frisch elasticity of labor supply functional form with a Frisch elasticity of 0.8. See [Evans and Phillips \(2017\)](#).

8.8 Exercises

Exercise 8.1. Import the comma delimited data file `emat.txt` as a NumPy array `emat`. The file `emat.txt` is data for the 80×7 matrix of ability levels $e_{j,s}$. Each row represents $s = \{1, 2, \dots, 80\}$, and each column represents ability levels $j = \{1, 2, \dots, 7\}$ associated with income percentiles $\lambda = [0.25, 0.25, 0.20, 0.10, 0.10, 0.09, 0.01]$. Assume that the population distribution by age is uniform $1/S$. The data in `emat` are scaled such that the average ability is 1 so each value $e_{j,s}$ can be interpreted as the percent premium or discount of the average wage w_t .

$$\frac{1}{S} \sum_{s=1}^S \sum_{j=1}^J \lambda_j e_{j,s} = 1$$

- Make a 2D plot of the ability matrix `emat` made up of seven lines showing ability $e_{j,s}$ over the life cycle s for each ability type j . That is, each line on the graph should be a column of the $e_{j,s}$ matrix. Include a legend on the graph that shows “0-24%” for the line for $j = 1$, “25-49%” for the line for $j = 2$, “50-69%” for the line for $j = 3$, “70-79%” for the line for $j = 4$, “80-89%” for the line for $j = 5$, “90-98%” for the line for $j = 6$, and “99-100%” for the line for $j = 7$.
- Now assume that $S = 10$ (each model year is 8 years) and $J = 3$ with $\{\lambda_j\}_{j=1}^3 = [0.4, 0.4, 0.3]$. Create a function that uses the original 80×7 `emat` matrix that you imported in the previous part of this exercise as the baseline, and creates a new matrix that is $S_{new} \times J_{new}$, which is 10×3 in this case. Solve for $e_{j,s}$ in the new matrix as a linear interpolation between points on the original matrix.

Exercise 8.2. Using a small version calibration similar to the one that was used for the non-steady-state equilibrium solution in Section 8.6.2, and the steady-state equilibrium Definition 8.1, solve for the steady-state equilibrium values of $\{\{\bar{c}_{j,s}\}_{s=1}^S\}_{j=1}^J$, $\{\{\bar{n}_{j,s}\}_{s=1}^S\}_{j=1}^J$, $\{\{\bar{b}_{j,s}\}_{s=2}^S\}_{j=1}^J$, \bar{w} , \bar{r} , \bar{K} , and \bar{L} numerically. More specifically, let $S = 20$, $J = 2$, $\lambda = [0.6, 0.4]$, $\beta = 0.849$, and $\delta = 0.185$. Use the 20×2 $e_{j,s}$ matrix `emat20x2.txt`. And let all other values equal their values from Table 8.3.

Exercise 8.3. Use the calibration from Exercise 8.2 and time path iteration (TPI) to solve for the non-steady state equilibrium transition path of the economy from $\Gamma_1 = 0.95\bar{\Gamma}$ to the

steady-state $\bar{\Gamma}$. You must guess a value for T and a time path updating parameter $\xi \in (0, 1)$, but it is likely that $T < 3 * S$ is sufficient. Use an L^2 norm for your distance measure (sum of squared percent deviations), and use a convergence parameter of $\varepsilon = 10^{-9}$. Use a linear or quadratic initial guess for the time path of the aggregate capital stock from the initial state K_1^1 to the steady state K_T^1 at time T .

Exercise 8.4. Plot the equilibrium time paths of the aggregate capital stock $\{K_t\}_{t=1}^{T+10}$ and aggregate labor $\{L_t\}_{t=1}^{T+10}$. How many periods did it take for the economy to get within 0.0001 of the steady-state aggregate capital stock \bar{K} ? That is, what is the true T ?

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