Endogenous Money or Sticky Prices?

Peter N. Ireland
Boston College and NBER

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1. The Model

1.1. The Economic Environment

A representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms, and a monetary authority populate an economy in which time periods are indexed by $t = 0, 1, 2, ...$. The intermediate goods-producing firms are indexed by $i \in [0, 1]$; each produces a distinct, perishable intermediate good. Hence, intermediate goods may also be indexed by $i \in [0, 1]$, where firm i produces good i. The model imposes enough symmetry on tastes and technologies, however, to permit the analysis to focus on the behavior of a representative intermediate goods-producing firm, identified by the generic index i.

The representative household has preferences defined over consumption of the finished good, leisure, and real cash balances, giving rise to a conventional specification for money demand as a function of a scale variable, aggregate consumption, and an opportunity cost variable, the nominal interest rate. The household purchases output from the representative finished goods-producing firm and supplies capital and labor to the intermediate goods-producing firms in competitive markets. The representative finished goods-producing firm purchases the intermediate goods as inputs, with which it produces the finished good.

The representative intermediate goods-producing firm produces intermediate good i with labor and capital supplied by the representative household. Since intermediate goods substitute imperfectly for one another in the representative finished goods-producing firm's production function, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market.

Hence, firm i acts as a price-setter in the market for good i. In addition, each intermediate goods-producing firm faces a cost of adjusting its price; this price adjustment cost is parameterized so that the model nests, as the special case in which the cost is zero, a flexible-price specification in which the observed relationships between nominal and real variables primarily reflect the monetary authority's deliberate responses to changes in the state of the real economy. When the price adjustment cost is positive, however, it gives the monetary authority some leverage over real variables in the short run, so that causality may run both ways.

1.2. The Representative Household

The representative household carries M_{t-1} units of money, B_{t-1} bonds, and K_t units of capital into period t. At the beginning of the period, the household receives a lump-sum nominal transfer T_t from the monetary authority. Next, the household's bonds mature, providing B_{t-1} additional units of money. The household uses some of this money to purchase B_t new bonds at the cost of $1/r_t$ units of money per bond, where r_t denotes the gross nominal interest rate between periods t and $t+1$.

During period t, the household supplies $h_t(i)$ units of labor at the nominal wage rate W_t and $K_t(i)$ units of capital at the nominal rental rate Q_t to each intermediate goods-producing firm $i \in [0, 1]$. The household's choices of $h_t(i)$ and $K_t(i)$ must satisfy

$$h_t = \int_0^1 h_t(i)di,$$

where h_t denotes total hours worked, and

$$K_t = \int_0^1 K_t(i)di$$

for all $t = 0, 1, 2, \ldots.$ The household, therefore, receives total nominal factor payments $W_t h_t + Q_t K_t$ during period t. In addition, the household receives total nominal profits $D_t(i)$ from each firm $i \in [0, 1]$, for a total of

$$D_t = \int_0^1 D_t(i)di$$

during period t.

The household uses its funds to purchase output at the nominal price P_t from the representative finished goods-producing firm, which it divides between consumption C_t and investment I_t. In order to transform invested units of the finished good into new units of productive capital, the household must pay an adjustment cost, measured in terms of the finished good and given by

$$\frac{\phi_K}{2}\left(\frac{K_{t+1}}{gK_t}-1\right)^2 K_t,$$

where g is the steady-state growth rate of the capital stock and where

$$K_{t+1} = (1-\delta)K_t + x_t I_t \tag{1}$$

for all $t = 0, 1, 2, ...$ The adjustment cost specification is similar to Abel and Blanchard's (1983), but implies that the cost is zero in the model's steady state. In (1), x_t is Greenwood, Hercowitz, and Huffman's (1988) shock to the marginal efficiency of investment; it follows the autoregressive process

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}. \tag{2}$$

The household then carries M_t units of money, B_t bonds, and K_{t+1} units of capital into period $t+1$; these quantities must satisfy the budget constraint

$$\begin{aligned} & \frac{M_{t-1}+T_t+B_{t-1}+W_t h_t+Q_t K_t+D_t}{P_t} \\ \geq\ & C_t + I_t + \frac{\phi_K}{2}\left(\frac{K_{t+1}}{gK_t}-1\right)^2 K_t + \frac{B_t/r_t + M_t}{P_t} \end{aligned} \tag{3}$$

for all $t = 0, 1, 2, ...$

The household's preferences are described by the expected utility function

$$E\sum_{t=0}^{\infty}\beta^t u(C_t, M_t/P_t, h_t),$$

where

$$u(C_t, M_t/P_t, h_t) = a_t[\gamma/(\gamma-1)]\ln[C_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}] + \eta\ln(1-h_t).$$

As shown by McCallum and Nelson (1999), the preference shock a_t resembles, in equilibrium, a disturbance to the IS curve in more traditional, Keynesian analyses; it follows the autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}. \tag{4}$$

As shown below, the preference shock e_t translates into a disturbance to the model's money demand curve; it follows the autoregressive process

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}. \tag{5}$$

The household, therefore, chooses C_t, h_t, M_t, B_t, I_t, and K_{t+1} for all $t = 0, 1, 2, ...$ to maximize its utility function subject to the constraints (1) and (3). The first-order conditions for this problem are

$$a_t = \Lambda_t C_t^{1/\gamma}[C_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}], \tag{6}$$

$$\eta = \Lambda_t(W_t/P_t)(1-h_t), \tag{7}$$

$$a_t e_t^{1/\gamma} = (M_t/P_t)^{1/\gamma}[C_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}][\Lambda_t - \beta E_t(\Lambda_{t+1}P_t/P_{t+1})], \tag{8}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}P_t/P_{t+1}), \tag{9}$$

$$\begin{aligned} &\Lambda_t\left[\frac{1}{x_t} + \left(\frac{\phi_K}{g}\right)\left(\frac{K_{t+1}}{gK_t} - 1\right)\right] \\ = \quad &\beta E_t\left[\Lambda_{t+1}\left(\frac{Q_{t+1}}{P_{t+1}} + \frac{1-\delta}{x_{t+1}}\right)\right] - \left(\frac{\beta\phi_K}{2}\right)E_t\left[\Lambda_{t+1}\left(\frac{K_{t+2}}{gK_{t+1}} - 1\right)^2\right] \\ &+\beta\phi_K E_t\left[\Lambda_{t+1}\left(\frac{K_{t+2}}{gK_{t+1}} - 1\right)\left(\frac{K_{t+2}}{gK_{t+1}}\right)\right], \end{aligned} \tag{10}$$

and with (1) and (3) with equality for all $t = 0, 1, 2, ...$, where Λ_t denotes the multiplier on (3). Note that (6), (8), and (9) can be combined to obtain

$$C_t^{1/\gamma} e_t^{1/\gamma} = \left(\frac{M_t}{P_t}\right)^{1/\gamma}\left(1 - \frac{1}{r_t}\right).$$

Letting $R_t = r_t - 1$ denote the net nominal interest rate and using the approximation

$$\frac{1}{r_t} = \frac{1}{1+R_t} = f(R_t) \approx f(0) + f'(0)R_t = 1 - R_t,$$

this last equation can be rewritten as

$$\ln(M_t/P_t) \approx \ln(C_t) - \gamma\ln(R_t) + \ln(e_t),$$

which confirms that e_t acts as a serially correlated shock to money demand and also shows that γ measures the interest elasticity of money demand.

1.3. The Representative Finished Goods-Producing Firm

The representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0,1]$ to produce Y_t units of the finished good according to the constant returns to scale technology described by

$$\left[\int_0^1 Y_t(i)^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)} \geq Y_t.$$

Intermediate good i sells at the nominal price $P_t(i)$, while the finished good sells at the nominal price P_t; given these prices, the finished goods-producing firm chooses Y_t and $Y_t(i)$ for all $i \in [0,1]$ to maximize its profits,

$$P_t Y_t - \int_0^1 P_t(i) Y_t(i) di,$$

subject to the constraint imposed by the production function.

The first-order conditions for this problem imply that

$$Y_t(i) = [P_t(i)/P_t]^{-\theta} Y_t$$

for all $i \in [0,1]$ and $t = 0,1,2,....$ Competition in the market for the finished good requires that the representative firm earn zero profits in equilibrium. This zero-profit condition determines P_t as

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta} di\right]^{1/(1-\theta)}$$

for all $t = 0,1,2,....$

1.4. The Representative Intermediate Goods-Producing Firm

The representative intermediate goods-producing firm hires $h_t(i)$ units of labor and rents $K_t(i)$ units of capital from the representative household during period t in order to produce $Y_t(i)$ units of intermediate good i according to the constant returns to scale technology described by

$$K_t(i)^\alpha [g^t z_t h_t(i)]^{1-\alpha} \geq Y_t(i),$$

where g denotes the gross rate of labor-augmenting technological progress. The technology shock z_t follows the autoregressive process

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}. \quad (11)$$

Since intermediate goods substitute imperfectly for one another as inputs to producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market; during each period t, it sets a nominal price $P_t(i)$, subject to the requirement that it satisfy the representative finished goods-producing firm's demand, taking P_t and Y_t as given.

In addition, the representative intermediate goods-producing firm faces a cost of adjusting its nominal price, measured in units of the finished good and given by

$$\frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)}-1\right]^2 Y_t,$$

where π is the gross steady-state rate of inflation. In the special case where $\phi_P = 0$, the model collapses to a flexible-price specification.

In general, the cost of price adjustment makes the firm's problem dynamic; it chooses $h_t(i)$, $K_t(i)$, $Y_t(i)$, and $P_t(i)$ to maximize its total market value,

$$E\sum_{t=0}^{\infty}\beta^t\Lambda_t[D_t(i)/P_t],$$

where $\beta^t\Lambda_t/P_t$ measures the marginal utility to the representative household provided by an additional dollar of profits during period t and where

$$\frac{D_t(i)}{P_t}=\left[\frac{P_t(i)}{P_t}\right]Y_t(i)-\frac{W_th_t(i)+Q_tK_t(i)}{P_t}-\frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)}-1\right]^2 Y_t$$

or

$$\frac{D_t(i)}{P_t}=\left[\frac{P_t(i)}{P_t}\right]^{1-\theta}Y_t-\frac{W_th_t(i)+Q_tK_t(i)}{P_t}-\frac{\phi_P}{2}\left[\frac{P_t(i)}{\pi P_{t-1}(i)}-1\right]^2 Y_t \tag{12}$$

subject to the constraint

$$K_t(i)^\alpha[g^tz_th_t(i)]^{1-\alpha}\geq[P_t(i)/P_t]^{-\theta}Y_t \tag{13}$$

for all $t = 0, 1, 2, ...$

The first-order conditions for this problem are

$$\frac{\Lambda_tW_th_t(i)}{P_t}=(1-\alpha)\Xi_tK_t(i)^\alpha[g^tz_th_t(i)]^{1-\alpha}, \tag{14}$$

$$\frac{\Lambda_tQ_tK_t(i)}{P_t}=\alpha\Xi_tK_t(i)^\alpha[g^tz_th_t(i)]^{1-\alpha}, \tag{15}$$

$$
\begin{aligned}
& \phi_P \Lambda_t \left[\frac{P_t(i)}{\pi P_{t-1}(i)} - 1\right] \left[\frac{P_t}{\pi P_{t-1}(i)}\right] \qquad (16) \\
= \; & (1-\theta)\Lambda_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta} + \theta \Xi_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta-1} \\
& + \beta \phi_P E_t \left\{ \Lambda_{t+1} \left[\frac{P_{t+1}(i)}{\pi P_t(i)} - 1\right] \left[\frac{P_{t+1}(i) P_t}{\pi P_t(i)^2}\right] \left(\frac{Y_{t+1}}{Y_t}\right) \right\},
\end{aligned}
$$

and (13) with equality for all $t = 0, 1, 2, \ldots$, where Ξ_t denotes the multiplier on (13).

1.5. The Monetary Authority

The monetary authority conducts monetary policy by adjusting a linear combination of the short-term nominal interest rate r_t and the money growth rate $\mu_t = M_t / M_{t-1}$ in response to deviations of inflation $\pi_t = P_t / P_{t-1}$ and detrended output $y_t = Y_t / g^t$ from their steady-state values. It uses the policy rule

$$\omega_r \ln(r_t / r) = \omega_\mu \ln(\mu_t / \mu) + \omega_\pi \ln(\pi_t / \pi) + \omega_y \ln(y_t / y) + \ln(v_t), \qquad (17)$$

which nests as special cases a money growth rule (when $\omega_r = 0$) and an interest rate rule (when $\omega_\mu = 0$). In (17), r, μ, π, and y are the steady-state values of r_t, μ_t, π_t, and y_t. The policy shock v_t follows the autoregressive process

$$\ln(v_t) = \rho_v \ln(v_{t-1}) + \varepsilon_{vt}. \qquad (18)$$

Since it is useful to allow for the possibility that any of the ω parameters in (17) equals zero, the standard deviation of ε_{vt} must be normalized; a convenient choice is $\sigma_v = 0.01$.

1.6. Symmetric Equilibrium

Let $N_t = Y_t / h_t$ denote labor's average product. In a symmetric equilibrium, $P_t(i) = P_t$, $Y_t(i) = Y_t$, $h_t(i) = h_t$, $K_t(i) = K_t$, and $D_t(i) = D_t$ for all $i \in [0, 1]$ and $t = 0, 1, 2, \ldots$. In addition, $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$ for all $t = 0, 1, 2, \ldots$. Substituting these conditions into (1)-(18) yields

$$K_{t+1} = (1-\delta) K_t + x_t I_t, \qquad (1)$$

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \qquad (2)$$

$$Y_t = C_t + I_t + \frac{\phi_K}{2}\left(\frac{K_{t+1}}{gK_t} - 1\right)^2 K_t + \frac{\phi_P}{2}\left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 Y_t, \tag{3}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{4}$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{5}$$

$$a_t = \Lambda_t C_t^{1/\gamma}[C_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma}(M_t/P_t)^{(\gamma-1)/\gamma}], \tag{6}$$

$$\eta = \Lambda_t (W_t/P_t)(1-h_t), \tag{7}$$

$$e_t C_t = (M_t/P_t)(1-1/r_t)^\gamma, \tag{8}$$

$$\Lambda_t = \beta r_t E_t(\Lambda_{t+1}P_t/P_{t+1}), \tag{9}$$

$$\begin{aligned} &\Lambda_t\left[\frac{1}{x_t} + \left(\frac{\phi_K}{g}\right)\left(\frac{K_{t+1}}{gK_t} - 1\right)\right] \qquad (10) \\ = \;& \beta E_t\left[\Lambda_{t+1}\left(\frac{Q_{t+1}}{P_{t+1}} + \frac{1-\delta}{x_{t+1}}\right)\right] - \left(\frac{\beta\phi_K}{2}\right)E_t\left[\Lambda_{t+1}\left(\frac{K_{t+2}}{gK_{t+1}} - 1\right)^2\right] \\ &+\beta\phi_K E_t\left[\Lambda_{t+1}\left(\frac{K_{t+2}}{gK_{t+1}} - 1\right)\left(\frac{K_{t+2}}{gK_{t+1}}\right)\right], \end{aligned}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{11}$$

$$\frac{D_t}{P_t} = Y_t - \frac{W_t h_t + Q_t K_t}{P_t} - \frac{\phi_P}{2}\left(\frac{P_t}{\pi P_{t-1}} - 1\right)^2 Y_t, \tag{12}$$

$$Y_t = K_t^\alpha (g^t z_t h_t)^{1-\alpha}, \tag{13}$$

$$\Lambda_t (W_t/P_t) h_t = (1-\alpha)\Xi_t Y_t, \tag{14}$$

$$\Lambda_t (Q_t/P_t) K_t = \alpha \Xi_t Y_t, \tag{15}$$

$$\begin{aligned} &\phi_P \Lambda_t \left(\frac{P_t}{\pi P_{t-1}} - 1\right)\left(\frac{P_t}{\pi P_{t-1}}\right) \qquad (16) \\ = \;& (1-\theta)\Lambda_t + \theta\Xi_t + \beta\phi_P E_t\left[\Lambda_{t+1}\left(\frac{P_{t+1}}{\pi P_t} - 1\right)\left(\frac{P_{t+1}}{\pi P_t}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\right], \end{aligned}$$

$$\omega_r \ln(r_t/r) = \omega_\mu \ln(\mu_t/\mu) + \omega_\pi \ln(\pi_t/\pi) + \omega_y \ln(y_t/y) + \ln(v_t), \tag{17}$$

$$\ln(v_t) = \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \tag{18}$$

and

$$N_t = Y_t/h_t. \tag{19}$$

These 19 equations determine equilibrium values for the 19 variables Y_t, C_t, I_t, h_t, N_t, M_t, W_t, Q_t, D_t, P_t, r_t, K_t, Λ_t, Ξ_t, a_t, e_t, z_t, x_t, and v_t.

2. Representing the Equilibrium

2.1. The Transformed System

In addition to $\mu_t = M_t/M_{t-1}$, $y_t = Y_t/g^t$, and $\pi_t = P_t/P_{t-1}$ defined above, let $c_t = C_t/g^t$, $i_t = I_t/g^t$, $n_t = N_t/g^t$, $m_t = (M_t/P_t)/g^t$, $w_t = (W_t/P_t)/g^t$, $q_t = Q_t/P_t$, $d_t = (D_t/P_t)/g^t$, $k_t = K_t/g^t$, $\lambda_t = g^t\Lambda_t$, and $\xi_t = g^t\Xi_t$. In terms of these transformed variables, (1)-(19) become

$$gk_{t+1} = (1-\delta)k_t + x_t i_t, \tag{1}$$

$$\ln(x_t) = \rho_x \ln(x_{t-1}) + \varepsilon_{xt}, \tag{2}$$

$$y_t = c_t + i_t + (\phi_K/2)(k_{t+1}/k_t - 1)^2 k_t + (\phi_P/2)(\pi_t/\pi - 1)^2 y_t, \tag{3}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \varepsilon_{at}, \tag{4}$$

$$\ln(e_t) = (1-\rho_e)\ln(e) + \rho_e \ln(e_{t-1}) + \varepsilon_{et}, \tag{5}$$

$$a_t = \lambda_t c_t^{1/\gamma}[c_t^{(\gamma-1)/\gamma} + e_t^{1/\gamma} m_t^{(\gamma-1)/\gamma}], \tag{6}$$

$$\eta = \lambda_t w_t (1-h_t), \tag{7}$$

$$e_t c_t = m_t (1 - 1/r_t)^{\gamma}, \tag{8}$$

$$g\lambda_t = \beta r_t E_t(\lambda_{t+1}/\pi_{t+1}), \tag{9}$$

$$\begin{aligned} & g(\lambda_t/x_t) + \phi_K \lambda_t (k_{t+1}/k_t - 1) \\ = \; & \beta E_t\{\lambda_{t+1}[q_{t+1} + (1-\delta)/x_{t+1}]\} \\ & -\beta(\phi_K/2)E_t[\lambda_{t+1}(k_{t+2}/k_{t+1} - 1)^2] \\ & +\beta\phi_K E_t[\lambda_{t+1}(k_{t+2}/k_{t+1} - 1)(k_{t+2}/k_{t+1})], \end{aligned} \tag{10}$$

$$\ln(z_t) = (1-\rho_z)\ln(z) + \rho_z \ln(z_{t-1}) + \varepsilon_{zt}, \tag{11}$$

$$d_t = y_t - w_t h_t - q_t k_t - (\phi_P/2)(\pi_t/\pi - 1)^2 y_t, \tag{12}$$

$$y_t = k_t^{\alpha}(z_t h_t)^{1-\alpha}, \tag{13}$$

$$\lambda_t w_t h_t = (1-\alpha)\xi_t y_t, \tag{14}$$

$$\lambda_t q_t k_t = \alpha \xi_t y_t, \tag{15}$$

$$\begin{aligned} & \phi_P \lambda_t (\pi_t/\pi - 1)(\pi_t/\pi) \\ = \; & (1-\theta)\lambda_t + \theta\xi_t + \beta\phi_P E_t[\lambda_{t+1}(\pi_{t+1}/\pi - 1)(\pi_{t+1}/\pi)(y_{t+1}/y_t)], \end{aligned} \tag{16}$$

$$\omega_r \ln(r_t/r) = \omega_\mu \ln(\mu_t/\mu) + \omega_\pi \ln(\pi_t/\pi) + \omega_y \ln(y_t/y) + \ln(v_t), \tag{17}$$

$$\ln(v_t) = \rho_v \ln(v_{t-1}) + \varepsilon_{vt}, \tag{18}$$

and

$$n_t = y_t/h_t, \tag{19}$$

where, in addition,

$$\mu_t = g(m_t/m_{t-1})\pi_t. \tag{20}$$

These 20 equations determine equilibrium values for the 20 stationary variables y_t, c_t, i_t, h_t, n_t, m_t, μ_t, w_t, q_t, d_t, π_t, r_t, k_t, λ_t, ξ_t, a_t, e_t, z_t, x_t, and v_t.

2.2. The Steady State

In the absence of shocks, the economy converges to a steady state, in which all stationary variables are constant. Use (2), (4), (5), (11), and (18) to solve for

$$x_t = x = 1,$$

$$a_t = a = 1,$$

$$e_t = e,$$

$$z_t = z,$$

and

$$v_t = v = 1.$$

Next, let the steady-state money growth rate μ be determined by policy, and use (9) and (20) to solve for

$$\pi_t = \pi = \mu/g$$

and

$$r_t = r = \mu/\beta.$$

Use (10) and (16) to solve for

$$q_t = q = g/\beta - 1 + \delta$$

and

$$\xi_t = \xi = [(\theta - 1)/\theta]\lambda.$$

Use (6) and (8) to solve for

$$c_t = c = \left[1 + e\left(\frac{r}{r-1}\right)^{\gamma-1}\right]^{-1}\left(\frac{1}{\lambda}\right)$$

and

$$m_t = m = e\left(\frac{r}{r-1}\right)^{\gamma} c.$$

Use (1), (3), (12)-(15), and (19) to solve for

$$y_t = y = \left[1 - (g-1+\delta)\left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha}{q}\right)\right]^{-1} c,$$

$$k_t = k = \left(\frac{\theta-1}{\theta}\right)\left(\frac{\alpha y}{q}\right),$$

$$i_t = i = (g-1+\delta)k,$$

$$h_t = h = \frac{1}{z}\left(\frac{y}{k^{\alpha}}\right)^{1/(1-\alpha)},$$

$$w_t = w = (1-\alpha)\left(\frac{\theta-1}{\theta}\right)\left(\frac{y}{h}\right),$$

$$d_t = d = y - wh - qk,$$

and

$$n_t = n = y/h.$$

Finally, use (7) to solve for

$$\lambda_t = \lambda = \frac{\eta + (1-\alpha)\left[1 + e\left(\frac{r}{r-1}\right)^{\gamma-1}\right]^{-1}\left[\left(\frac{\theta}{\theta-1}\right) - (g-1+\delta)\left(\frac{\alpha}{q}\right)\right]^{-1}}{(1-\alpha)z\left(\frac{\theta-1}{\theta}\right)^{1/(1-\alpha)}\left(\frac{\alpha}{q}\right)^{\alpha/(1-\alpha)}}.$$

2.3. The Linearized System

Equations (1)-(20) can be log-linearized to describe the behavior of the 20 stationary variables as they fluctuate about their steady-state values in response to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\imath}_t = \ln(i_t/i)$, $\hat{h}_t = \ln(h_t/h)$, $\hat{n}_t = \ln(n_t/n)$, $\hat{m}_t = \ln(m_t/m)$, $\hat{\mu}_t = \ln(\mu_t/\mu)$, $\hat{w}_t = \ln(w_t/w)$, $\hat{q}_t = \ln(q_t/q)$, $\hat{d}_t = \ln(d_t/d)$, $\hat{\pi}_t = \ln(\pi_t/\pi)$, $\hat{r}_t = \ln(r_t/r)$, $\hat{k}_t = \ln(k_t/k)$, $\hat{\lambda}_t = \ln(\lambda_t/\lambda)$, $\hat{\xi}_t = \ln(\xi_t/\xi)$,

$\hat{a}_t = \ln(a_t/a)$, $\hat{e}_t = \ln(e_t/e)$, $\hat{z}_t = \ln(z_t/z)$, $\hat{x}_t = \ln(x_t/x)$, and $\hat{v}_t = \ln(v_t/v)$. Then first-order Taylor approximations to (1)-(20) yield

$$gk\hat{k}_{t+1} = (1-\delta)k\hat{k}_t + i\hat{x}_t + i\hat{\imath}_t, \tag{1}$$

$$\hat{x}_t = \rho_x \hat{x}_{t-1} + \varepsilon_{xt}, \tag{2}$$

$$y\hat{y}_t = c\hat{c}_t + i\hat{\imath}_t, \tag{3}$$

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \varepsilon_{at}, \tag{4}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \varepsilon_{et}, \tag{5}$$

$$\gamma r \hat{a}_t = \gamma r \hat{\lambda}_t + r[1 + (\gamma - 1)\lambda c]\hat{c}_t + (r-1)\lambda m \hat{e}_t + (\gamma - 1)(r-1)\lambda m \hat{m}_t, \tag{6}$$

$$0 = \eta \hat{\lambda}_t + \eta \hat{w}_t - \lambda w h \hat{h}_t, \tag{7}$$

$$(r-1)\hat{e}_t + (r-1)\hat{c}_t = (r-1)\hat{m}_t + \gamma \hat{r}_t, \tag{8}$$

$$\hat{\lambda}_t = \hat{r}_t + E_t \hat{\lambda}_{t+1} - E_t \hat{\pi}_{t+1}, \tag{9}$$

$$\begin{aligned} & g\hat{\lambda}_t - g\hat{x}_t - \phi_K \hat{k}_t \\ = \;& gE_t\hat{\lambda}_{t+1} + \beta q E_t \hat{q}_{t+1} - \beta(1-\delta)E_t\hat{x}_{t+1} + \beta\phi_K E_t \hat{k}_{t+2} - (1+\beta)\phi_K \hat{k}_{t+1}, \end{aligned} \tag{10}$$

$$\hat{z}_t = \rho_z \hat{z}_{t-1} + \varepsilon_{zt}, \tag{11}$$

$$d\hat{d}_t = y\hat{y}_t - wh\hat{w}_t - wh\hat{h}_t - qk\hat{q}_t - qk\hat{k}_t, \tag{12}$$

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{z}_t + (1-\alpha)\hat{h}_t, \tag{13}$$

$$\hat{\lambda}_t + \hat{w}_t + \hat{h}_t = \hat{\xi}_t + \hat{y}_t, \tag{14}$$

$$\hat{\lambda}_t + \hat{q}_t + \hat{k}_t = \hat{\xi}_t + \hat{y}_t, \tag{15}$$

$$\phi_P \hat{\pi}_t = (1-\theta)\hat{\lambda}_t + (\theta - 1)\hat{\xi}_t + \beta\phi_P E_t \hat{\pi}_{t+1}, \tag{16}$$

$$\omega_r \hat{r}_t = \omega_\mu \hat{\mu}_t + \omega_\pi \hat{\pi}_t + \omega_y \hat{y}_t + \hat{v}_t, \tag{17}$$

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \varepsilon_{vt}, \tag{18}$$

$$\hat{n}_t = \hat{y}_t - \hat{h}_t, \tag{19}$$

and

$$\hat{\mu}_t = \hat{m}_t - \hat{m}_{t-1} + \hat{\pi}_t. \tag{20}$$

In solving the model, it is convenient to use (20) to rewrite (6) and (8) as

$$\begin{aligned}\gamma r\hat{a}_t &= \gamma r\hat{\lambda}_t + r[1+(\gamma-1)\lambda c]\hat{c}_t + (r-1)\lambda m\hat{e}_t \\ &\quad +(\gamma-1)(r-1)\lambda m\hat{\mu}_t + (\gamma-1)(r-1)\lambda m\hat{m}_{t-1} - (\gamma-1)(r-1)\lambda m\hat{\pi}_t\end{aligned} \tag{6}$$

and

$$(r-1)\hat{e}_t + (r-1)\hat{c}_t = (r-1)\hat{\mu}_t + (r-1)\hat{m}_{t-1} - (r-1)\hat{\pi}_t + \gamma\hat{r}_t. \tag{8}$$

It is also convenient to use (1) and (2) to rewrite (10) as

$$\begin{aligned}&g\hat{\lambda}_t - (g+\beta\{\phi_K[1-(1-\delta)/g]-(1-\delta)\}\rho_x)\hat{x}_t - \phi_K\hat{k}_t \\ =\ &gE_t\hat{\lambda}_{t+1} + \beta q E_t\hat{q}_{t+1} \\ &+\phi_K[\beta(1-\delta)/g-(1+\beta)]\hat{k}_{t+1} + \beta\phi_K[1-(1-\delta)/g]E_t\hat{\imath}_{t+1}.\end{aligned} \tag{10}$$

These 20 equations describe the behavior of the 20 variables $\hat{y}_t$, $\hat{c}_t$, $\hat{\imath}_t$, $\hat{h}_t$, $\hat{n}_t$, $\hat{m}_t$, $\hat{\mu}_t$, $\hat{w}_t$, $\hat{q}_t$, $\hat{d}_t$, $\hat{\pi}_t$, $\hat{r}_t$, $\hat{k}_t$, $\hat{\lambda}_t$, $\hat{\xi}_t$, $\hat{a}_t$, $\hat{e}_t$, $\hat{z}_t$, $\hat{x}_t$, and $\hat{v}_t$.

3. Solving the Model with Sticky Prices

In the case where $\phi_P > 0$, let

$$f_t^0 = \left[\begin{array}{cccccccccc} \hat{y}_t & \hat{c}_t & \hat{\imath}_t & \hat{h}_t & \hat{n}_t & \hat{\mu}_t & \hat{w}_t & \hat{q}_t & \hat{d}_t & \hat{r}_t \end{array}\right]',$$

$$s_t^0 = \left[\begin{array}{ccccc} \hat{k}_t & \hat{m}_{t-1} & \hat{\pi}_t & \hat{\lambda}_t & \hat{\xi}_t \end{array}\right]',$$

and

$$u_t = \left[\begin{array}{ccccc} \hat{a}_t & \hat{e}_t & \hat{x}_t & \hat{z}_t & \hat{v}_t \end{array}\right]'.$$

The (3), (6)-(8), (12)-(15), (17), and (19) can be written as

$$Af_t^0 = Bs_t^0 + Cu_t, \tag{21}$$

where A is 10×10, B is 10×5, and C is 10×5.

Equation (3) implies

$a_{11} = y$

$a_{12} = -c$

$a_{13} = -i$

Equation (6) implies

$a_{22} = r[1 + (\gamma - 1)\lambda c]$

$a_{26} = (\gamma - 1)(r - 1)\lambda m$

$b_{22} = -(\gamma - 1)(r - 1)\lambda m$

$b_{23} = (\gamma - 1)(r - 1)\lambda m$

$b_{24} = -\gamma r$

$c_{21} = \gamma r$

$c_{22} = -(r - 1)\lambda m$

Equation (7) implies

$a_{34} = \lambda wh$

$a_{37} = -\eta$

$b_{34} = \eta$

Equation (8) implies

$a_{42} = r - 1$

$a_{46} = -(r - 1)$

$a_{410} = -\gamma$

$b_{42} = r - 1$

$b_{43} = -(r - 1)$

$c_{42} = -(r - 1)$

Equation (12) implies

$a_{51} = y$

$a_{54} = -wh$

$a_{57} = -wh$

$a_{58} = -qk$

$a_{59} = -d$

$b_{51} = qk$

Equation (13) implies

$a_{61} = 1$

$a_{64} = -(1 - \alpha)$

$b_{61} = \alpha$

$c_{64} = 1 - \alpha$

Equation (14) implies

$a_{71} = 1$

$a_{74} = -1$

$a_{77} = -1$

$b_{74} = 1$

$b_{75} = -1$

Equation (15) implies

$a_{81} = 1$

$a_{88} = -1$

$b_{81} = 1$

$b_{84} = 1$

$b_{85} = -1$

Equation (17) implies

$a_{91} = \omega_y$

$a_{96} = \omega_\mu$

$a_{910} = -\omega_r$

$b_{93} = -\omega_\pi$

$c_{95} = -1$

Equation (19) implies

$a_{101} = 1$

$a_{104} = -1$

$a_{105} = -1$

Equations (1), (9), (10), (16), and (20) can be written as

$$DE_t s^0_{t+1} + FE_t f^0_{t+1} = Gs^0_t + Hf^0_t + Ju_t, \tag{22}$$

where D and G are 5×5, F and H are 5×10, and J is 5×5.

Equation (1) implies

$d_{11} = gk$

$g_{11} = (1 - \delta)k$

$h_{13} = i$

$j_{13} = i$

Equation (9) implies

$d_{23} = 1$

$d_{24} = -1$

$g_{24} = -1$

$h_{210} = 1$

Equation (10) implies

$d_{31} = \phi_K[\beta(1-\delta)/g - (1+\beta)]$

$d_{34} = g$

$f_{33} = \beta\phi_K[1-(1-\delta)/g]$

$f_{38} = \beta q$

$g_{31} = -\phi_K$

$g_{34} = g$

$j_{33} = -g - \beta\{\phi_K[1-(1-\delta)/g] - (1-\delta)\}\rho_x$

Equation (16) implies

$d_{43} = \beta\phi_P$

$g_{43} = \phi_P$

$g_{44} = \theta - 1$

$g_{45} = -(\theta - 1)$

Equation (20) implies

$d_{52} = 1$

$g_{52} = 1$

$g_{53} = -1$

$h_{56} = 1$

Finally, (2), (4), (5), (11), and (18) can be written as

$$u_t = Pu_{t-1} + \varepsilon_t, \tag{23}$$

where

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & \rho_v \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{xt} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}'.$$

Rewrite (21) as

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}Cu_t,$$

and substitute it into (22) to obtain

$$E_t s_{t+1}^0 = Ks_t^0 + Lu_t, \tag{24}$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(J + HA^{-1}C - FA^{-1}CP).$$

If the 5×5 matrix K has two eigenvalues inside the unit circle and three eigenvalues outside the unit circle, then the system has a unique solution. If K has more than three eigenvalues outside the unit circle, then the system has no solution. If K has less than three eigenvalues outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980).

Assuming from now on that there are exactly three eigenvalues outside the unit circle, write K as

$$K = M^{-1}NM,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of N are the eigenvalues of K, with those in the 2×2 matrix N_1 inside the unit circle and those in the 3×3 matrix N_2 outside the unit circle. The columns of M^{-1} are the eigenvectors of K; M_{11} is 2×2, M_{12} is 2×3, M_{21} is 3×2, and M_{22} is 3×3. In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where L_1 is 2×5 and L_2 is 3×5.

Now (24) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s^0_{t+1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^0_t + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u_t$$

or

$$E_t s^1_{1t+1} = N_1 s^1_{1t} + Q_1 u_t \tag{25}$$

and

$$E_t s^1_{2t+1} = N_2 s^1_{2t} + Q_2 u_t, \tag{26}$$

where

$$s_{1t} = M_{11} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{12} \begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix}, \tag{27}$$

$$s_{2t} = M_{21} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{22} \begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix}, \tag{28}$$

$$Q_1 = M_{11} L_1 + M_{12} L_2,$$

and

$$Q_2 = M_{21} L_1 + M_{22} L_2.$$

Since the eigenvalues in N_2 lie outside the unit circle, (26) can be solved forward to obtain

$$s_{2t} = -N_2^{-1} R u_t,$$

where the 3×5 matrix R is given by

$$\begin{aligned} vec(R) &= vec \sum_{j=0}^{\infty} N_2^{-j} Q_2 P^j = \sum_{j=0}^{\infty} vec(N_2^{-j} Q_2 P^j) \\ &= \sum_{j=0}^{\infty} [P^j \otimes (N_2^{-1})^j] vec(Q_2) = \sum_{j=0}^{\infty} (P \otimes N_2^{-1})^j vec(Q_2) \\ &= [I_{(15\times 15)} - P \otimes N_2^{-1}]^{-1} vec(Q_2). \end{aligned}$$

Use this result, along with (28), to solve for

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \\ \hat{\xi}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_2 u_t, \tag{29}$$

where

$$S_1 = -M_{22}^{-1} M_{21}$$

and

$$S_2 = -M_{22}^{-1} N_2^{-1} R.$$

Equation (27) now provides a solution for s_{1t}^1:

$$s_{1t} = (M_{11} + M_{12} S_1) \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{12} S_2 u_t.$$

Substitute this result into (25) to obtain

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{m}_t \end{bmatrix} = S_3 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_4 u_t, \tag{30}$$

where

$$S_3 = (M_{11} + M_{12} S_1)^{-1} N_1 (M_{11} + M_{12} S_1)$$

and

$$S_4 = (M_{11} + M_{12} S_1)^{-1} (Q_1 + N_1 M_{12} S_2 - M_{12} S_2 P).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1} B s_t^0 + A^{-1} C u_t \\ &= A^{-1} B \begin{bmatrix} I_{(2\times2)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + A^{-1} B \begin{bmatrix} 0_{(2\times5)} \\ S_2 \end{bmatrix} u_t + A^{-1} C u_t, \end{aligned}$$

which can be written more simply as

$$f_t^0 = S_5 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_6 u_t, \tag{31}$$

where

$$S_5 = A^{-1} B \begin{bmatrix} I_{(2\times2)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}B \begin{bmatrix} 0_{(2\times5)} \\ S_2 \end{bmatrix} + A^{-1}C.$$

Equations (23) and (29)-(31) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{32}$$

and

$$f_t = Us_t, \tag{33}$$

where

$$s_t = \begin{bmatrix} \hat{k}_t & \hat{m}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{x}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{y}_t & \hat{c}_t & \hat{\imath}_t & \hat{h}_t & \hat{n}_t & \hat{\mu}_t & \hat{w}_t & \hat{q}_t & \hat{d}_t & \hat{r}_t & \hat{\pi}_t & \hat{\lambda}_t & \hat{\xi}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{xt} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} S_3 & S_4 \\ 0_{(5\times2)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(2\times5)} \\ I_{(5\times5)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

4. Solving the Model with Flexible Prices

In the case where $\phi_P = 0$, (16) reduces to

$$\hat{\xi}_t = \hat{\lambda}_t. \tag{16}$$

Hence, $\hat{\xi}_t$ can be eliminated from the system by dropping (16) and by rewriting (14) and (15) as

$$\hat{w}_t + \hat{h}_t = \hat{y}_t, \tag{14}$$

$$\hat{q}_t + \hat{k}_t = \hat{y}_t, \tag{15}$$

In this case, let

$$f_t^0 = \begin{bmatrix} \hat{y}_t & \hat{c}_t & \hat{\imath}_t & \hat{h}_t & \hat{n}_t & \hat{\mu}_t & \hat{w}_t & \hat{q}_t & \hat{d}_t & \hat{r}_t \end{bmatrix}',$$

$$s_t^0 = \begin{bmatrix} \hat{k}_t & \hat{m}_{t-1} & \hat{\pi}_t & \hat{\lambda}_t \end{bmatrix}',$$

and

$$u_t = \begin{bmatrix} \hat{a}_t & \hat{e}_t & \hat{x}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}'.$$

The (3), (6)-(8), (12)-(15), (17), and (19) can be written as

$$Af_t^0 = Bs_t^0 + Cu_t, \tag{34}$$

where A is 10×10, B is 10×4, and C is 10×5.

Equation (3) implies

$$a_{11} = y$$

$$a_{12} = -c$$

$$a_{13} = -i$$

Equation (6) implies

$$a_{22} = r[1 + (\gamma - 1)\lambda c]$$

$$a_{26} = (\gamma - 1)(r - 1)\lambda m$$

$$b_{22} = -(\gamma - 1)(r - 1)\lambda m$$

$$b_{23} = (\gamma - 1)(r - 1)\lambda m$$

$$b_{24} = -\gamma r$$

$$c_{21} = \gamma r$$

$$c_{22} = -(r - 1)\lambda m$$

Equation (7) implies

$$a_{34} = \lambda w h$$

$$a_{37} = -\eta$$

$$b_{34} = \eta$$

Equation (8) implies

$a_{42} = r - 1$

$a_{46} = -(r - 1)$

$a_{410} = -\gamma$

$b_{42} = r - 1$

$b_{43} = -(r - 1)$

$c_{42} = -(r - 1)$

Equation (12) implies

$a_{51} = y$

$a_{54} = -wh$

$a_{57} = -wh$

$a_{58} = -qk$

$a_{59} = -d$

$b_{51} = qk$

Equation (13) implies

$a_{61} = 1$

$a_{64} = -(1 - \alpha)$

$b_{61} = \alpha$

$c_{64} = 1 - \alpha$

Equation (14) implies

$a_{71} = 1$

$a_{74} = -1$

$a_{77} = -1$

Equation (15) implies

$a_{81} = 1$

$a_{88} = -1$

$b_{81} = 1$

Equation (17) implies

$a_{91} = \omega_y$

$a_{96} = \omega_\mu$

$a_{910} = -\omega_r$

$b_{93} = -\omega_\pi$

$c_{95} = -1$

Equation (19) implies

$a_{101} = 1$

$a_{104} = -1$

$a_{105} = -1$

Equations (1), (9), (10), and (20) can be written as

$$DE_t s^0_{t+1} + FE_t f^0_{t+1} = Gs^0_t + Hf^0_t + Ju_t, \tag{35}$$

where D and G are 4×4, F and H are 4×10, and J is 4×5.

Equation (1) implies

$d_{11} = gk$

$g_{11} = (1 - \delta)k$

$h_{13} = i$

$j_{13} = i$

Equation (9) implies

$d_{23} = 1$

$d_{24} = -1$

$g_{24} = -1$

$h_{210} = 1$

Equation (10) implies

$d_{31} = \phi_K[\beta(1-\delta)/g - (1+\beta)]$

$d_{34} = g$

$f_{33} = \beta\phi_K[1 - (1-\delta)/g]$

$f_{38} = \beta q$

$g_{31} = -\phi_K$

$g_{34} = g$

$j_{33} = -g - \beta\{\phi_K[1 - (1-\delta)/g] - (1-\delta)\}\rho_x$

Equation (20) implies

$d_{42} = 1$

$g_{42} = 1$

$g_{43} = -1$

$h_{46} = 1$

Finally, (2), (4), (5), (11), and (18) can be written as

$$u_t = Pu_{t-1} + \varepsilon_t, \tag{36}$$

where

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 & 0 \\ 0 & 0 & \rho_x & 0 & 0 \\ 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & \rho_v \end{bmatrix}$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{xt} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}'.$$

Rewrite (34) as

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}Cu_t,$$

and substitute it into (35) to obtain

$$E_t s_{t+1}^0 = Ks_t^0 + Lu_t, \tag{37}$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(J + HA^{-1}C - FA^{-1}CP).$$

If the 4×4 matrix K has two eigenvalues inside the unit circle and two eigenvalues outside the unit circle, then the system has a unique solution. If K has more than two eigenvalues outside the unit circle, then the system has no solution. If K has less than two eigenvalues outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980).

Assuming from now on that there are exactly two eigenvalues outside the unit circle, write K as

$$K = M^{-1}NM,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of N are the eigenvalues of K, with those in the 2×2 matrix N_1 inside the unit circle and those in the 2×2 matrix N_2 outside the unit circle. The columns of M^{-1} are the eigenvectors of K; M_{11}, M_{12}, M_{21}, and M_{22} are all 2×2. In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where L_1 and L_2 are 2×5.

Now (37) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s^0_{t+1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^0_t + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} u_t$$

or

$$E_t s^1_{1t+1} = N_1 s^1_{1t} + Q_1 u_t \tag{38}$$

and

$$E_t s^1_{2t+1} = N_2 s^1_{2t} + Q_2 u_t, \tag{39}$$

where

$$s_{1t} = M_{11} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{12} \begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \end{bmatrix}, \tag{40}$$

$$s_{2t} = M_{21} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{22} \begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \end{bmatrix}, \tag{41}$$

$$Q_1 = M_{11} L_1 + M_{12} L_2,$$

and

$$Q_2 = M_{21} L_1 + M_{22} L_2.$$

Since the eigenvalues in N_2 lie outside the unit circle, (39) can be solved forward to obtain

$$s_{2t} = -N_2^{-1} R u_t,$$

where the 2×5 matrix R is given by

$$\begin{aligned} vec(R) &= vec \sum_{j=0}^{\infty} N_2^{-j} Q_2 P^j = \sum_{j=0}^{\infty} vec(N_2^{-j} Q_2 P^j) \\ &= \sum_{j=0}^{\infty} [P^j \otimes (N_2^{-1})^j] vec(Q_2) = \sum_{j=0}^{\infty} (P \otimes N_2^{-1})^j vec(Q_2) \\ &= [I_{(10\times10)} - P \otimes N_2^{-1}]^{-1} vec(Q_2). \end{aligned}$$

Use this result, along with (41), to solve for

$$\begin{bmatrix} \hat{\pi}_t \\ \hat{\lambda}_t \end{bmatrix} = S_1 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_2 u_t, \tag{42}$$

where

$$S_1 = -M_{22}^{-1} M_{21}$$

and

$$S_2 = -M_{22}^{-1} N_2^{-1} R.$$

Equation (40) now provides a solution for s_{1t}^1:

$$s_{1t} = (M_{11} + M_{12} S_1) \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + M_{12} S_2 u_t.$$

Substitute this result into (38) to obtain

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{m}_t \end{bmatrix} = S_3 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_4 u_t, \tag{43}$$

where

$$S_3 = (M_{11} + M_{12} S_1)^{-1} N_1 (M_{11} + M_{12} S_1)$$

and

$$S_4 = (M_{11} + M_{12} S_1)^{-1} (Q_1 + N_1 M_{12} S_2 - M_{12} S_2 P).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1} B s_t^0 + A^{-1} C u_t \\ &= A^{-1} B \begin{bmatrix} I_{(2\times2)} \\ S_1 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + A^{-1} B \begin{bmatrix} 0_{(2\times5)} \\ S_2 \end{bmatrix} u_t + A^{-1} C u_t, \end{aligned}$$

which can be written more simply as

$$f_t^0 = S_5 \begin{bmatrix} \hat{k}_t \\ \hat{m}_{t-1} \end{bmatrix} + S_6 u_t, \tag{44}$$

where

$$S_5 = A^{-1} B \begin{bmatrix} I_{(2\times2)} \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}B \begin{bmatrix} 0_{(2\times 5)} \\ S_2 \end{bmatrix} + A^{-1}C.$$

Equations (36) and (42)-(44) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{45}$$

and

$$f_t = U s_t, \tag{46}$$

where

$$s_t = \begin{bmatrix} \hat{k}_t & \hat{m}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{x}_t & \hat{z}_t & \hat{v}_t \end{bmatrix}',$$

$$f_t = \begin{bmatrix} \hat{y}_t & \hat{c}_t & \hat{\imath}_t & \hat{h}_t & \hat{n}_t & \hat{\mu}_t & \hat{w}_t & \hat{q}_t & \hat{d}_t & \hat{r}_t & \hat{\pi}_t & \hat{\lambda}_t \end{bmatrix}',$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{xt} & \varepsilon_{zt} & \varepsilon_{vt} \end{bmatrix}',$$

$$\Pi = \begin{bmatrix} S_3 & S_4 \\ 0_{(5\times 2)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(2\times 5)} \\ I_{(5\times 5)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} S_5 & S_6 \\ S_1 & S_2 \end{bmatrix}.$$

5. Estimating the Model

Suppose that data are available on consumption C_t, investment I_t, money M_t, prices P_t, and interest rates r_t. These data can be used to construct a series $\{d_t\}_{t=1}^{T}$, where

$$d_t = \begin{bmatrix} \hat{c}_t \\ \hat{\imath}_t \\ \hat{m}_t \\ \hat{\pi}_t \\ \hat{r}_t \end{bmatrix} = \begin{bmatrix} \ln(C_t) - t\ln(g) - \ln(c) \\ \ln(I_t) - t\ln(g) - \ln(i) \\ \ln(M_t) - \ln(P_t) - t\ln(g) - \ln(m) \\ \ln(P_t) - \ln(P_{t-1}) - \ln(\pi) \\ \ln(r_t) - \ln(r) \end{bmatrix}.$$

Equations (32) and (33) or (45) and (46) then given rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{47}$$

and

$$d_t = Cs_t, \tag{48}$$

where $A = \Pi$, $B = W$, C is formed from the rows of Π and U as

$$C = \begin{bmatrix} U_2 \\ U_3 \\ \Pi_2 \\ U_{11} \\ U_{10} \end{bmatrix},$$

and the vector of serially uncorrelated innovations

$$\varepsilon_{t+1} = \begin{bmatrix} \varepsilon_{at+1} & \varepsilon_{et+1} & \varepsilon_{xt+1} & \varepsilon_{zt+1} & \varepsilon_{vt+1} \end{bmatrix}'$$

is assumed to be normally distributed with zero mean and diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} \sigma_a^2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_e^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_x^2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_z^2 & 0 \\ 0 & 0 & 0 & 0 & 0.01^2 \end{bmatrix}.$$

The model defined by (47) and (48) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (47) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(7\times1)} \tag{49}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1 s_1') = [I_{(49\times 49)} - A \otimes A]^{-1} vec(BVB'). \qquad (50)$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (48) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned}
\hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1} u_t \\
&= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} u_t.
\end{aligned}$$

Hence, from (47),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} u_t.$$

Using this last result, along with (47) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1} C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t | d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t | d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1} C\Sigma_t A',$$

and initial conditions $\hat{s}_1$ and Σ_1 are provided by (49) and (50).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\frac{5T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T} u_t'\Omega_t^{-1}u_t.$$

6. Evaluating the Model

6.1. Testing for Sticky Prices

One way of testing the null hypothesis that $\phi_P = 0$ against the alternative that $\phi_P > 0$ is to compare the estimate of ϕ_P from the model with sticky prices to its standard error σ_ϕ:

$$W = \phi_P/\sigma_\phi.$$

Under the null hypothesis that $\phi_P = 0$, W is asymptotically distributed as a standard normal random variable, while W^2 is asymptotically distributed as a chi-square random variable with one degree of freedom.

Alternatively, let $\ln L^s$ be the maximized value of the log likelihood function for the unconstrained model with sticky prices ($\phi_P \geq 0$), and let $\ln L^f$ be the maximized value of the log likelihood function for the constrained model with flexible prices ($\phi_p = 0$). Then under the null hypothesis that $\phi_P = 0$, the likelihood ratio statistic

$$LR = 2(\ln L^s - \ln L^f)$$

is asymptotically distributed as a chi-square random variable with one degree of freedom.

6.2. Testing for Parameter Stability

The procedures described by Andrews and Fair (1988) can be used to test for the stability of the model's estimated parameters. Let Θ^1 and Θ^2 denote the estimated parameters from two disjoint subsamples, and let H^1 and H^2 denote the associated covariance matrices, so that asymptotically,

$$\Theta^1 \sim N(\Theta^{10}, H^1)$$

and

$$\Theta^2 \sim N(\Theta^{20}, H^2).$$

One way of testing for the stability of all of the estimated parameters is with the likelihood ratio statistic

$$LR = 2[\ln L(\Theta^1) + \ln L(\Theta^2) - \ln L(\Theta)],$$

where $\ln L(\Theta^1)$, $\ln L(\Theta^2)$, and $\ln L(\Theta)$ are the maximized log likelihood functions for the first subsample, the second subsample, and the entire sample. According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with p degrees of freedom under the null hypothesis of parameter stability, where p is the number of estimated parameters.

Alternatively, the stability of some or all of the parameters can be tested with the Wald statistic

$$W = g(\Theta^1, \Theta^2)'(G\hat{H}G')^{-1}g(\Theta^1, \Theta^2),$$

where the stability restrictions are written as

$$g(\Theta^1, \Theta^2) = 0$$

and where

$$G = \frac{\partial g(\Theta^1, \Theta^2)}{\partial(\Theta^1, \Theta^2)'}$$

and

$$\hat{H} = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}.$$

If Θ^1_q and Θ^2_q denote the subsets of Θ^1 and Θ^2 of interest, and if H^1_q and H^2_q denote the covariance matrices of Θ^1_q and Θ^2_q, then this Wald statistic can be written more simply as

$$W = (\Theta^1_q - \Theta^2_q)'(H^1_q + H^2_q)^{-1}(\Theta^1_q - \Theta^2_q).$$

According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of parameter stability, where q is the number of parameters being tested for stability.

6.3. Variance Decompositions

Begin by considering (47), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t$$

or

$$(1 - AL)s_t = B\varepsilon_t$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2 BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (47) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s = Es_t s_t'$$

is given by

$$vec(\Sigma^s) = [I_{(49\times 49)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (33), (46), and (48), which imply that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U',$$

$$\Sigma^f = \lim \Sigma_k^f = E f_t f_t' = U\Sigma^s U',$$

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = C\Sigma_k^s C',$$

and

$$\Sigma^d = \lim \Sigma_k^d = E d_t d_t' = C\Sigma^s C'.$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

The elements of Σ_k^s, Σ^s, Σ_k^f, Σ^f, Σ_k^d, and Σ^d can all be expressed as nonlinear functions of Θ, with

$$\Sigma = g(\Theta).$$

Thus, asymptotic standard errors for these elements can be found by calculating

$$GHG',$$

where

$$G = \frac{\partial g(\Theta)}{\partial \Theta'}.$$

In practice, G can be evaluated numerically, as suggested by Runkle (1987).

6.4. Vector Autocorrelations

Return again to (47), which implies

$$s_t = A^k s_{t-k} + \sum_{j=0}^{k-1} A^j B \varepsilon_{t-j}.$$

Hence,

$$E s_t s_{t-k}' = A^k \Sigma^s,$$

where, as above,

$$vec(\Sigma^s) = [I_{(49\times49)} - A \otimes A]^{-1} vec(BVB').$$

Let y_t be a vector of variables selected from s_t, f_t, or d_t. Then

$$y_t = Fs_t,$$

where F is formed from the rows of U or C. Then

$$\Gamma_k = Ey_t y'_{t-k} = FA^k \Sigma^s F',$$

so that the autocorrelations can be computed as

$$\frac{\Gamma_k(i,j)}{[\Gamma_0(i,i)]^{1/2}[\Gamma_0(j,j)]^{1/2}}.$$

Now consider using data on y_t to estimate the autoregression

$$y_t = Gy_{t-1} + H\varepsilon_t,$$

where more than one lag of y_t can be accommodated by writing the system in companion form. Then

$$y_t = G^k y_{t-k} + \sum_{j=0}^{k-1} G^j H\varepsilon_{t-j}.$$

Hence,

$$\Gamma_k = Ey_t y'_{t-k} = G^k \Sigma^y,$$

where

$$vec(\Sigma^y) = [I_{(n^2p^2 \times n^2p^2)} - G \otimes G]^{-1} vec(HVH'),$$

where n is the number of variables in the vector y_t, p is the number of lags in the autoregression, and

$$V = E\varepsilon_t \varepsilon'_t.$$

Once again, the autocorrelations can be computed as

$$\frac{\Gamma_k(i,j)}{[\Gamma_0(i,i)]^{1/2}[\Gamma_0(j,j)]^{1/2}}.$$