Monetary Policy Switching and Indeterminacy

Jean Barthélemy   Magali Marx*

March 31, 2015

Abstract

This paper determines conditions for the existence of a unique rational expectations equilibrium -determinacy- in a monetary policy switching economy. We depart from the existing literature by considering all bounded equilibria. We prove that two monetary policy regimes with very different responses to inflation may trigger indeterminacy even if both regimes satisfy the Taylor principle. Indeterminacy arises because the policymaker with a stronger response to inflation reacts aggressively to inflationary expectations due to expected weaker policies in the future. We thus identify situations in which policy switching itself, rather than one or the other policy, is responsible for indeterminacy.

Keywords: Markov-switching, indeterminacy, monetary policy.

JEL: E31, E43, E52

*Barthélemy: Department of Economics, Sciences Po, 28 rue des Saints Pères, 75007 PARIS. Email: jean.barthelemy@sciencespo.fr. Marx: Banque de France, 39 rue Croix des Petits Champs, 75001 Paris. Email: magali.marx@banque-france.fr. We thank Eric Leeper, Dan Waggoner and Tao Zha for stimulating discussions. We are also grateful to Klaus Adam, Pamfili Antipa, Robert Barsky, Jess Benhabib, Francesco Bianchi, Vincent Bignon, Florin Bilbiie, Regis Breton, Braz Camargo, Carlos Carvalho, Seonghoon Cho, Lawrence Christiano, Nicolas Coeurdacier, Filippo Ferroni, Gaetano Gaballo, Jordi Gali, Christian Hellwig, Michel Juillard, Raphaël Jungers, Bruce McGough, Eric Mengus, Leonardo Melosi, Christopher Sims, François Velde, Toni Yates, and Michael Woodford for their very helpful comments on this paper. We are also grateful to seminar participants at the 2012 and 2013 Conference on Computing in Economics and Finance, the 2013 T2M conference, the 2013 EEA-ESEM conference, the 5th Joint French Macro Workshop, the CREI and the Bristol University. The views expressed in this paper do not necessarily reflect the opinion of the Banque de France. All remaining errors are ours.

1 Introduction

Good monetary policy should prevent indeterminacy, i.e. the existence of multiple stable equilibria. Indeed, in the absence of a policy tool to coordinate expectations on a particular equilibrium, an economy experiencing indeterminacy may respond to non-fundamental -sunspot- disturbances, and hence, may be affected by extrinsic volatility. Since limiting inflation volatility is a well-accepted objective for monetary policy, this extrinsic volatility that is not under policymaker's control is undesirable from a policy perspective. Consequently, determinacy is a desirable feature of monetary policy.

Since Taylor (1993), economists have been used to modeling the nominal interest rate as a state-contingent rule with constant parameters. In this setup, monetary authorities should increase the nominal interest rate by more than one-for-one in response to inflation to ensure determinacy. By doing so, monetary authorities rule out equilibria driven by non-fundamental beliefs and hence prevent the economy from extrinsic volatility. This condition is known as the Taylor principle.

Monetary policy, however, does not necessarily follow a constant-parameter rule.[1] Many empirical works (for instance Clarida et al., 2000; Lubik and Schorfheide, 2004; Bianchi, 2013) document the existence of monetary policy switching in the post-world war II US economy. Economic agents should, thus, internalize the possibility of future policy switches when forming their expectations. Since determinacy depends on economic agents' expectations, regime switching affects stability conditions calling for an update of the Taylor principle.

In a regime switching environment, economic agents can expect different outcomes according to the exact regimes trajectory in the future. These beliefs can trigger equilibria that depend on past policy regimes. Indeterminacy may hence result from equilibria with complex history dependence. Previous literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2013) however restrict admissible equilibria by disregarding equilibria that depend on past

[1] There are at least three reasons to believe that the parameters of the monetary policy rule may vary over time. First, if monetary policy is optimal, any structural change in the economy should result in a change in monetary policy. Second, the monetary policy rule stems from multiple beliefs regarding the structure of the economy, the role of monetary policy and monetary policy transmission mechanisms. All these beliefs may change over time according to empirical as well as theoretical advances in macroeconomics. Third, the rule captures economic preferences, which have little reason to be stable over time. Governors of major central banks are chosen by the government according to the latter's own preferences, and hence depend on political cycles.

regimes or that are non-Markovian.[2] Omitting certain equilibria leads to underestimate policy configurations associated with indeterminacy. Monetary authorities aiming at minimizing inflation volatility and agnostic about the exact structure of economic beliefs should instead guarantee determinacy given the broadest class of equilibria.

In this paper, we characterize stable equilibria when the economy faces regime switching without restrictive assumptions on the class of equilibria, and, in particular, without excluding equilibria depending on past regimes. We especially study determinacy in the context of a new-Keynesian economy experiencing switching between multiple monetary policy regimes, described as periods for which the interest rate follows a constant-parameters Taylor rule. Our findings are threefold.

First, we show that a two-regime economy may be indeterminate even if the two regimes meet the Taylor principle. Why does the Taylor principle fail? In a purely forward-looking model, the existence of converging non-zero expectations is a necessary and sufficient condition of indeterminacy. In an economy without regime switching, the Taylor principle ensures that any non-zero expectations diverge and are hence disregarded by economic agents. However, expectations diverge only asymptotically and this divergence may take time. In a regime switching environment, if the regimes are too short-lasting to ensure such explosiveness, indeterminacy can emerge even if the two regimes satisfy the Taylor principle.

We illustrate this result in an economy that deterministically switches between a flexible-price regime and a regime characterized by strict inflation targeting and sticky prices. The question is under which conditions, non-zero expectations that converge exist. First, as these two regimes alternate with certainty, expectations oscillate between two polar situations, namely non-zero output-gap and zero inflation and vice-versa. Second, this oscillation is damped -converging to the steady state eventually- if the monetary policy response to inflation in the flexible-price regime is sufficiently small or and if prices are sufficiently sticky. Indeed, in the strict inflation targeting regime, inflation expectations trigger a fall in the output-gap and the stickier the prices, the larger this fall. In the flexible-price regime, the smaller the response of monetary policy to inflation the higher inflation if positive output-gap is anticipated. In both cases, insufficient response to inflation or sufficient price stickiness

[2]Some of these papers consider stability concepts that differ from the one considered in this paper - i.e. boundedness. Different stability concepts generally lead to different indeterminacy conditions. In this paper, we however stress that previous papers always miss some classes of equilibria. These missing rather than different stability concepts explain disagreements found in the literature.

dampen the expectations, eventually leading to indeterminacy. Finally, for a sufficient high degree of price stickiness, indeterminacy arises even if the response to inflation in the flexible regime is larger than one.

We then confirm this result in a new-Keynesian economy, in which monetary policy's reaction to inflation follows a Markov process while private sector behaviors remain unchanged. Indeterminacy arises when the policy response to inflation changes dramatically across regime. We show that regimes trajectories responsible for indeterminacy are those constituted by short-lasting regime with a strong reaction to inflation followed by a long-lasting regime with a weak reaction to inflation. Indeed, along these regimes trajectories, the dynamics of expectations resembles the one described above. Let suppose economic agents expect a positive output-gap after such a succession of regimes. On the one hand, such expectations generate protracted inflationary pressure as long as monetary policy responds moderately to inflation. On the other hand, an aggressive monetary policy that reacts strongly to such inflationary pressures generates a large fall in the output-gap. The overall effect is a large drop in the output-gap if the latter policy regime follows the former. Eventually, if such successions of regimes occur with significantly large probabilities, indeterminacy arises.

Second, we provide the necessary and sufficient determinacy condition for forward-looking rational expectations models with parameters following a Markov process. We also extend our results when the model includes pre-determined variables. The model is determinate if a certain limit, involving the computation of all possible future regimes trajectories, is smaller than one. This result extends Blanchard and Kahn (1980) conditions to regime-switching economy. In general, this limit cannot be computed using a finite number of regimes trajectories. Indeed, when indeterminate, multiple equilibria depending on (possibly distant) past regimes exist.

Since necessary and sufficient conditions are computationally hard to verify, we also provide for a sufficient condition of indeterminacy and a sufficient condition of determinacy that are easier to check. But due to the intrinsic nature of the problem, there will always be a region of parameters for which the economists will not be able to conclude.[3] As a follow up of this paper, recent contributions by Ogura and Jungers (2014) and Ogura et al. (2015) suggest that the region of parameters for which we cannot conclude could be substantially -but never completely- reduced in the near future.

[3]In other words, in general, settling determinacy is an 'undecidable' problem using mathematical terminology.

Third, we, by-product, settle a controversy in the literature on determinacy conditions (Davig and Leeper, 2007; Farmer et al., 2010a; Davig and Leeper, 2010). Using a new-Keynesian model with monetary policy switching, we show that restricting to equilibria only depending on a limited number of past regimes or to Markovian equilibria generally leads to underestimate the indeterminacy region. All the existing literature implicitly or explicitly restricts the class of equilibria, see Davig and Leeper (2007), Farmer et al. (2009b), Cho (2013) or Foerster et al. (2013) for instance. When a unique stable equilibrium among all possible equilibria exists, the unique stable equilibrium only depends on the current shocks and current regime. Therefore, the unique stable equilibrium always belongs to the classes of equilibria considered in these papers. However, for certain configurations of policy parameters, such restrictions lead to conclude to determinacy while multiple stable equilibria exist. Thus, the existing literature only provides sufficient conditions for indeterminacy but no information about determinacy in the whole class of equilibria. To our knowledge, this paper is the first to provide sufficient determinacy conditions for the whole class of equilibria.

The remainder of the paper is organized as follows. In section 2, we expose some insights on indeterminacy in the absence of regime switching that are useful to understand indeterminacy in a regime switching environment. In section 3, we develop a simple regime switching model in which the economy oscillates between a flexible-price regime and an inflation targeting regime. We then turn to a general class of models in section 4. We provide for a necessary and sufficient determinacy condition. We also derive practical sufficient conditions for determinacy and for indeterminacy. Finally, we show how one can use our determinacy conditions to settle determinacy for models with pre-determined variables. In section 5, we extend the Taylor principle when monetary policy switches between different regimes. In section 6, we conclude.

2 Some insights on indeterminacy in the absence of regime switching

In this section, we recall the economics of indeterminacy when monetary policy follows a constant-parameters rule and when private behaviors are consistent with a log-linearized new-Keynesian model following Clarida et al. (2000) and Woodford (2003). We say that this economy is determinate if it admits a unique bounded equilibrium, indeterminate otherwise.[4]

[4] Section A gives a formal definition of indeterminacy.

This section allows us to introduce notations in a well-known model as well as to review the key forces underlying indeterminacy.

We assume that the economy is described by this canonical 3-equation new-Keynesian model:

$$y_t = \mathbb{E}_t y_{t+1} - \sigma(r_t - \mathbb{E}_t \pi_{t+1} - \epsilon_t^d), \tag{1}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t + \epsilon_t^s, \tag{2}$$

$$r_t = \alpha \pi_t + \gamma y_t + \epsilon_t^r, \tag{3}$$

where variables y_t, π_t and r_t are respectively the output-gap, inflation (in log) and the nominal interest rate (in deviation around a certain steady state). The operator $\mathbb{E}_t$ denotes expectations at time t. Equation (1) is an IS curve that links the output-gap to all the future *ex-ante* real interest rates and future and current shocks, ϵ_t^d. Parameter σ measures the risk aversion. Equation (2) is a New-Keynesian Phillips Curve linking inflation to all the future marginal costs summarized by the output-gap. Parameter κ measures the degree of nominal rigidities while β stands for the discount factor. Shock ϵ_t^s denotes a cost-push shock translating the Phillips curve. Equation (3) is a simplified Taylor rule. The parameters α and γ measure the sensitivity of the interest rate to inflation and to the output-gap. Finally, shock ϵ_t^r stands for the unsystematic part of monetary policy. We assume that shocks are bounded[5] and, without loss of generality, are i.i.d. and zero mean.

By plugging the monetary policy rule, equation (3), into the IS curve, equation (1), we end up with a system of two forward-looking equations determining simultaneously inflation and the output gap:

$$\Gamma z_t = \mathbb{E}_t z_{t+1} + C\epsilon_t, \tag{4}$$

where the column vector z_t denotes endogenous variables, $[\pi_t \quad y_t]'$ and the column vector ϵ_t denotes the shocks, $[\epsilon_t^S \quad \epsilon_t^R \quad \epsilon_t^D]'$. The matrices Γ and C gather the parameters of the model.[6] We can check that inflation, π^F, and the output-gap, y^F defined in equation 5 is a bounded equilibrium satisfying the model equations.

$$\pi_t^F = \frac{(1+\sigma\gamma)\varepsilon_t^S + \sigma\kappa\varepsilon_t^D - \varepsilon_t^R}{1+\sigma\gamma+\sigma\alpha\kappa} \quad \text{and} \quad y_t^F = \frac{-\sigma\alpha\varepsilon_t^S + \sigma\kappa\varepsilon_t^D - \varepsilon_t^R}{1+\sigma\gamma+\sigma\alpha\kappa} \tag{5}$$

[5] Boundedness of shocks is required since we consider bounded equilibria.

[6] See Appendix section B for the formula giving these matrices

The question is whether this equilibrium is the unique bounded solution to the model or not.

Expectations and indeterminacy This particular equilibrium (sometimes called the Minimum State Variables or the fundamental equilibrium) is the only one consistent with zero expectations. It is hence natural to investigate whether expectations can be or not different from zero. Following Sims (2002), we thus analyze expectations of inflation and the output-gap, that we denote by the column vector $z^e_t = E_t z_{t+1}$. We also introduce the associated forecast error, $\xi_{t+1} = z_{t+1} - z^e_t$. Finally, it is convenient to rewrite equation (4) as follows:

$$z^e_{t+1} = \Gamma z^e_t - \Gamma \xi_{t+1} - C\epsilon_{t+1}. \tag{6}$$

The equilibrium, z^F_t, is the unique bounded equilibrium if it is impossible to find non-zero stable expectations satisfying equation (6). Let us suppose that at period 0, expectations are different from zero. Then, depending on the eigenvalues of Γ, expectations will explode or implode. If all eigenvalues of Γ are larger than one then expectations diverge. As we rule out unbounded equilibrium, it means that having a non zero expectations in the first place is inconsistent with a stable equilibrium. This proves that z^F_t is the only stable solution of the model. Thus, determinacy ultimately depends on the lowest eigenvalue of this matrix.

Determinacy condition This condition on eigenvalues translates into a condition on the sensitivity of the nominal interest rate to inflation and the output gap. Indeed, if this sensitivity is too low - i.e. if $\alpha + \frac{1-\beta}{\kappa}\gamma < 1$ -[7] then the matrix Γ has only one eigenvalue larger than one, the other being strictly lower than one. The unstable eigenvalue gives only one restriction to the expectations failing to ruling out multiple bounded expectations. In this case, forecast errors as well as expectations are not uniquely pinned down by structural shocks but rather may depend on sunspots, that we denote by a scalar, u_t. We denote by z_0 a vector belonging to the contracting direction of Γ. Expectations and forecast errors satisfying equation (7) are bounded by definition of z_0.

$$z^e_{t+1} = \Gamma z^e_t - \Gamma z_0 u_t, \quad \text{and} \quad \xi_{t+1} = \Gamma^{-1} C\epsilon_{t+1} + z_0 u_t, \tag{7}$$

where the sunspot, u_t, is an arbitrary bounded Martingale ($E_{t-1}u_t = 0$).

[7] The proof can be found in Woodford (2003).

The existence of multiple mapping between shocks and expectations triggers indeterminacy. Indeed, any stable equilibrium can be written as the sum of the particular equilibrium given by equation (5) and a self-fulfilling part determined by expectations:

$$z_t = z_t^F + \Gamma^{-1} z_t^e. \tag{8}$$

Determinacy case | Indeterminacy case

$\alpha = 1.5$ and $\gamma = 0$ | $\alpha = 0.9$ and $\gamma = 0$

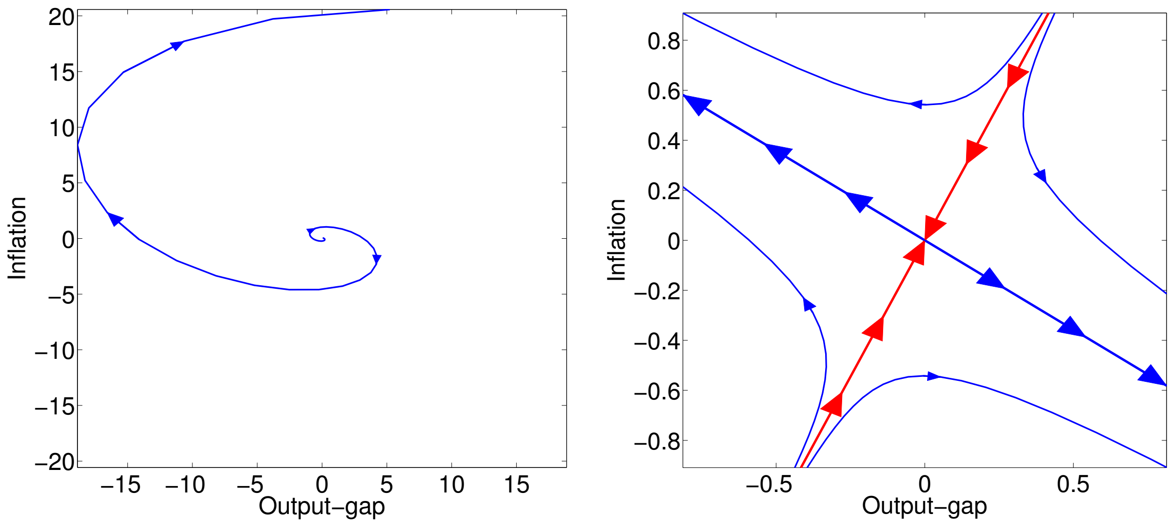

Table 1: Dynamics of expectations in new-Keynesian model

Note: On the left panel, monetary policy reacts sufficiently strongly to inflation to prevent from indeterminacy. If economic agents expect an initial positive inflation and output-gap in first period: $\mathbb{E}_0 y_1 = \mathbb{E}_0 \pi_1 = 1$ *then inflation and output-gap expectations,* $\mathbb{E}_0 z_t^e$*, diverge following a diverging spiral. On the right panel, we depict the dynamics of expectations if monetary policy does not sufficiently respond to inflation. The red (blue) line with contracting (diverging) arrows corresponds to the eigenvector associated with the stable (unstable, resp.) eigenvalue of matrix* Γ*. Any expectations,* $E_0 z_1$*, that are not on the stable eigenvector will lead to diverging expectations path, i.e.* $|\mathbb{E}_0 z_t| \to \infty$ *when* t *tends to* ∞*. Blue thin curves report such diverging trajectories.*

Calibration of parameters: $\beta = 0.99$*,* $\kappa = 0.17$ *and* $\sigma = 1$*.*

The role of monetary policy The right panel of Table 2 plots possible stable expectations paths when monetary policy fails to guarantee determinacy, $\alpha = 0.9$ and $\gamma = 0$. As emphasized by Cochrane (2011), the key mechanism is the following. The larger the reaction to inflation, the stronger the autocorrelation of expectations.[8] When this response is sufficiently large, expectations diverge whatever the initial expectations, z_0^e. Such an example is given in the left panel of Table 2. In this case, the only possible bounded equilibrium is the one associated with zero expectations, z_t^F.

In the absence of regime switching, monetary policy can rule out indeterminacy by increasing the nominal interest rate by more than one-for-one in response to inflation. Through such a policy, monetary authorities guarantee that non-zero expectations diverge asymptotically. However, expectations of some endogenous variables can take time to diverge and can even converge transitory. If, in the meanwhile, the economy switches to another regime in which the variables that converged in previous regime now diverge and those which diverged converge, then indeterminacy can arise. We provide such a policy combination in subsection 5.3.

3 A simple regime switching economy

We now depart from the assumption of time-invariant monetary policy. We start with a very stylized example in which monetary policy parameters and the slope of the Phillips curve oscillates between two different set of values. In the first regime, corresponding to odd periods, monetary policy follows the Taylor rule presented in equation (3) with no response to the output-gap, $\gamma = 0$. In the second regime, at even periods, monetary authorities choose nominal interest rate such as achieving zero inflation. This second regime can be viewed as a strict inflation targeting regime.[9] In addition, we assume that prices are completely flexible in the first regime, i.e. the slope of the Phillips curve, captured by the parameter κ, is zero. The simultaneous change of monetary policy and price rigidities allows to solve the model analytically and to provide for simple intuitions.

First regime In the first regime, as prices are flexible, the output-gap is equal to zero. Thus, this regime is characterized by the following equations:

[8] This mechanism is straightforward when prices are flexible, in this case the real interest rate is exogenous to monetary policy and the monetary policy response to inflation exactly corresponds to the expectations autocorrelation.

[9] This regime also corresponds to the extreme case of a monetary policy following a degenerated Taylor rule with an infinite weight on inflation.

$$0 = \mathbb{E}_t y_{t+1} - \sigma(\alpha\pi_t + \epsilon_t^r - \mathbb{E}_t\pi_{t+1} - \epsilon_t^d) \quad \text{and} \quad y_t = 0 \qquad \text{(odd periods)}.$$

If this regime holds forever, this regime would exactly correspond to a flexible price model and we end up with a standard Fisherian equation of inflation determination. In this case, the Taylor principle, recommending a sufficiently large response to inflation (i.e. $\alpha > 1$), coincides with determinacy conditions. However, if this regime is succeeded by an inflation targeting regime, expectations of inflation are zero and inflation determination depends on expectations of future output-gap.

Second regime In the second regime, monetary policy follows a strict inflation targeting and hence inflation is zero. The output-gap may however move if economic agents expect future inflation. This regime can be summarized as follows:

$$\pi_t = 0 \quad \text{and} \quad 0 = \beta\mathbb{E}_t\pi_{t+1} + \kappa y_t + \varepsilon_t^s \qquad \text{(even periods)}.$$

If this regime holds forever, inflation is zero so as inflation expectations. The output-gap is hence determined by the current cost-push shocks, ϵ_t^s. This economy is thus always determinate. However, if this regime is followed by the flexible price regime described above, then the output-gap determination will depend on expected inflation, that in turn will depend on expected output-gap in two periods and so on and so forth.

Switching model To find determinacy conditions we proceed as in the no regime switching case and study the evolution of expectations in the two regimes.

$$\pi_{2t}^e = -\kappa/\beta y_{2t-1}^e + \kappa/\beta\xi_{2t} - \epsilon_{2t}^d, \tag{9}$$

$$y_{2t+1}^e = \sigma\alpha\pi_{2t}^e - \sigma\alpha\xi_{2t+1} + \sigma\alpha\epsilon_{2t+1}^d + \sigma\epsilon_{2t+1}^s. \tag{10}$$

By combining these two relations, we find a time-invariant relationship between expectations at time $2t+1$ and $2t-1$:

$$y_{2t+1}^e = -\kappa\alpha\sigma/\beta y_{2t-1}^e + \kappa\alpha\sigma/\beta\xi_{2t} - \sigma\alpha\epsilon_{2t}^d - \sigma\alpha\xi_{2t+1} + \sigma\alpha\epsilon_{2t+1}^d + \sigma\epsilon_{2t+1}^s$$

This relationship proves that the uniqueness of stable expectations is given by the condition $|\kappa\alpha\sigma/\beta| < 1$ and we hence obtain the following result:

Result 1. *The regime switching model is determinate if and only if* $|\kappa\alpha\sigma/\beta| > 1$.

The intuition of this result is the following. As in the non-switching case, indeterminacy requires the existence of bounded expectations different from zero. At time $t = 0$, economic agents expect zero inflation as monetary policy completely stabilizes inflation in the next period. Thus, the question is whether expecting a positive output-gap is consistent with a stable expectations path or not.

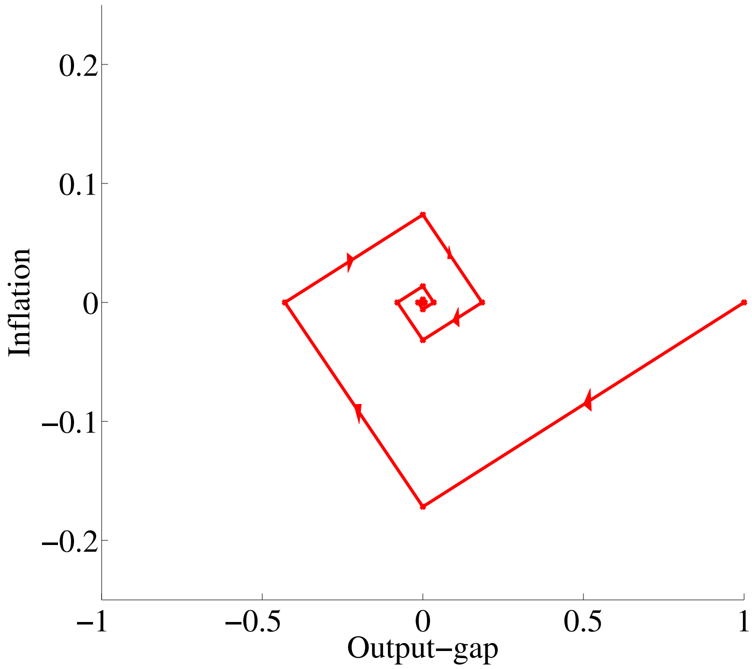

Figure 1: Expectations path in a simple regime switching economy

Note: The red lines with arrows plot the trajectory of expectations in a regime switching economy described by equations (9) *and* (10). *The initial regime (at time* $t = 1$*) is the flexible-price regime. We assume a positive output-gap expectation in first period,* $y_1^e = 1$. *The expectations converge to the steady state even if the Taylor principle is satisfied in both regimes. Calibration of parameters:* $\kappa = 0.17$, $\beta = 0.99$, $\sigma = 1$ *and* $\alpha = 2.5$.

Suppose that economic agents expect a strictly positive level of output-gap in period 1, for instance $y_1^e = 1$. In period 1, as prices do not move because of the strict inflation targeting, a positive output-gap should be counterbalanced by negative expectations of inflation, $\pi_2^e = -\kappa/\beta$ (otherwise, inflation would not be zero, due to the new-Keynesian Philips curve). This expectation of negative inflation in period 2 means that in period 2 the expected real interest rate has to be negative. In addition, the output-gap has to be zero because prices are flexible in the flexible-price regime. Thus, the only situation consistent with negative real rate and zero output-gap is when economic agents expect negative output-gap in period 3. In the end, expectations of period-3 output-gap have to be equal to $-\kappa\alpha\sigma/\beta$. This expectation is smaller than initial expectations only if this quantity is strictly smaller than one in absolute value. If it is the case, then initial expectations are consistent with an expectation path converging to the steady state, otherwise expectations diverge and the model is determinate. Figure 1 plots a non-zero expectations path converging to the steady state when the model is indeterminate.

In the mechanics leading to indeterminacy, we see two key determinants. First, a large reaction of monetary policy to inflation in the first regime reinforces the impact of output-gap expectations on period 2 inflation expectations. As long as prices are not rigid in the second regime, there always exists a sufficiently large response of inflation leading to determinacy, $\alpha < \beta/(\sigma\kappa)$. However, this condition differs from the Taylor principle and the economy can be indeterminate even if this principle is satisfied (see Figure 1). Second, the more flexible the prices, the stronger the relationship between inflation expectations and the output-gap in the inflation targeting regime. If prices are too sticky in regime 2 (i.e. κ lower than $\beta/(\alpha\sigma)$), indeterminacy arises. This suggests that indeterminacy can emerge in contexts where regimes taken in isolation are determinate. This happens especially if the two regimes are very different. In Section 5, we confirm these results when regime switching follows a Markovian process and without assuming changes in the degree of price stickiness. Before providing these results we derive general determinacy conditions for Markov switching rational expectations models in Section 4.

4 Determinacy conditions for Markov-Switching Rational Expectations Models

4.1 The class of models

Most micro-founded macroeconomic models may be summarized by a system of non-linear equations involving structural parameters governing economic agents' preferences, technology, market structures and economic policies. Allowing these parameters to switch over time results in non-linear regime-switching models. When shocks are small enough, the stability of this class of models can be checked by studying the determinacy of linear regime-switching models as proved by Barthélemy and Marx (2011). In this paper, we focus on linear models of the following form:

$$\Gamma_{s_t} z_t = \mathbb{E}_t z_{t+1} + C_{s_t} \epsilon_t, \tag{11}$$

where vector z_t is a $(n \times 1)$ real vector of endogenous variables, vector ε_t is a $(p \times 1)$ real vector of exogenous shocks and index s_t indicates the current regime, in $\{1, \cdots, N\}$. For any index $i \in \{1, \cdots, N\}$, matrices Γ_i and C_i are respectively $(n \times n)$ and $(n \times p)$ real matrices. Without loss of generality, we assume that matrices Γ_i are invertible. We assume that the vector of shocks, ϵ_t, is bounded[10] and independent of current and past regimes. Finally, we assume that regimes, s_t, follow a Markov-chain with constant transition probabilities:[11]

$$p_{ij} = Pr(s_t = j | s_{t-1} = i). \tag{12}$$

This class of models contains two main limitations. First, it precludes pre-determined variables like capital and debt. Under some restrictions, we can extend our results to such a context (see subsection 4.4). Second, we suppose that transition probabilities are constant over time and over space. In a companion paper, Barthélemy and Marx (2011), we propose a perturbation approach to deal with endogenous regime switching. In any cases, finding determinacy conditions for this class of models appears as a pre-requisite for solving more sophisticated models.

We consider that an equilibrium is stable if it is bounded.

Definition 1. *A stable equilibrium is a bounded continuous process satisfying Equation* (11)*.*

This definition of stability is similar to the one used in non-linear rational expectations models (e.g. Woodford, 1986; Jin and Judd, 2002), we detail this definition in appendix A.

Alternative stability concept Farmer et al. (2009b) claim that the complexity of determinacy conditions in Markov-switching models may be circumvented by choosing an alternative stability concept. By contrast, they put forward Mean Square Stability following the influential book by Costa et al. (2005). This latter concept consists in assuming that the first

[10] To the best of our knowledge, assuming bounded shocks is a prerequisite for linearizing a non-linear rational expectations model. Otherwise, the application of a perturbation approach is questionable. See Woodford (1986) for more details on the perturbation approach.

[11] The simple example presented in subsection 3 corresponds to a degenerated situation in which $p_{12} = p_{21} = 1$.

two conditional moments converge. Mean square stability appears more convenient when restricting the class of equilibria to some specific equilibria (Farmer et al., 2009b, 2010b). By adopting this latter concept and restricting the class of equilibria, they find determinacy conditions for purely forward-looking models. Cho (2013) refines and complements their findings for models with pre-determined variables.

However, the Mean Square Stability concept presents no decisive advantage over boundedness when considering the broadest class of equilibria. The findings of Farmer et al. (2009b); Cho (2013) result from restrictions on the solutions space rather than from the choice of a different stability concept. Besides, mean square stability allows unbounded support for economic variables, which is, in general, inconsistent with a local approach (see Woodford, 1986; Barthélemy and Marx, 2011). *De facto*, boundedness is the only stability concept whose consistency with a perturbation approach has been proved. That is why we stick to the standard stability concept, i.e. boundedness.

4.2 Determinacy condition

In this section, we establish the necessary and sufficient conditions for the existence and uniqueness of a stable equilibrium. Our strategy generalizes the approach developed in Section 2. Equation (11) can be rewritten in terms of expectations as follows:

$$z_t^e = \Gamma_{s_t} z_{t-1}^e - \Gamma_{s_t} \xi_t - C_{s_t} \varepsilon_t, \tag{13}$$

where the vector z_t^e is the expectation at time t of endogenous variables at time $t+1$, i.e. $\mathbb{E}_t z_{t+1}$ and the column vector, ξ_t is the associated forecast errors, $z_t - \mathbb{E}_{t-1} z_t$. Determinacy conditions of the original model are equivalently given by the existence and uniqueness of a couple (z^e, ξ) of bounded process and Martingale satisfying equation (13). By pre-multiplying this equation by $\Gamma_{s_t}^{-1}$ and taking the expectations, we have the following relation:

$$\mathbb{E}_t \Gamma_{s_{t+1}}^{-1} z_{t+1}^e = z_t^e, \tag{14}$$

which leads to

$$z_t^e = \mathbb{E}_t (\Gamma_{s_{t+1}}^{-1} \cdots \Gamma_{s_{t+N}}^{-1}) z_{t+N}^e. \tag{15}$$

Applying the same ideas as before, we see that determinacy conditions correspond to conditions ensuring that the only stable solution of (15) is zero. These conditions rely on a sufficient decrease of $\mathbb{E}_t(||\Gamma_{s_t}^{-1}\Gamma_{s_{t+1}}^{-1}\cdots\Gamma_{s_{t+N}}^{-1}||)$. We denote by the scalar u_k, a sequence that measures the rate of decrease of the expected product of Γ_i:

$$u_k = \left(\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}k}\|\Gamma_{i_1}^{-1}\Gamma_{i_2}^{-1}\cdots\Gamma_{i_k}^{-1}\|\right)^{1/k}. \tag{16}$$

The determinacy condition is then given by:

Proposition 1. *There exists a unique stable equilibrium if and only if the limit of u_k when k tends to infinity is smaller than* 1.

We prove Proposition 1 in the Appendix under section C. We first prove that the sequence u_k converges and admits a limit independent of the chosen norm. If the limit is smaller than 1, then the sequence $\mathbb{E}_t\Gamma_{s_{t+1}}^{-1}\cdots\Gamma_{s_{t+k}}^{-1}X_{t+k}$ tends to 0 whatever the bounded stochastic process X and hence the model admits a unique stable equilibrium. We prove the reciprocal by showing that if the limit is larger than unity, by applying the Gelfand theorem and by analytic continuation, we can construct multiple bounded equilibria. Finally, when unique, the unique equilibrium only depends on current shocks and regime (see appendix F.1).

Proposition 1 generalizes Blanchard and Kahn (1980) conditions to Markov-switching rational expectations models. When there is no regime switching ($\Gamma_{s_t} = \Gamma$ for any regime), the existence of a unique stable equilibrium depends on the asymptotic behavior of $u_k \sim ||\Gamma^{-k}||^{1/k}$. This sequence behaves as a decreasing exponential if and only if all the eigenvalues of Γ^{-1} are less than one. This coincides with the well-known Blanchard and Kahn conditions.

This Proposition also extends the Farmer et al. (2009a) results to multivariate models. When the model is univariate, the matrices Γ_{s_t} are commutative as they are scalars. The limit of u_k hence depends on a simple combination between probabilities and these scalars.

However, in general, the computation of the limit is challenging as the number of terms to compute grows exponentially. This complexity comes from the non-commutativity of the product of the matrices appearing in the definition of the sequence, u_k. Consequently, the sequence is not necessarily monotonous and the speed of convergence is unknown. The limit of the sequence u_k shares similar properties with mathematical objects such as the joint spectral radius (e.g. Theys, 2005) and the p-radius.[12] The complexity of these concepts is well-known

[12]For a two-regime model, if the transition probabilities are symmetric ($p_{11} = p_{22} = 1/2$), the limit of u_k when k tends to infinity exactly corresponds to the 1-radius of $\{\Gamma_1^{-1}, \Gamma_2^{-1}\}$. We refer to Jungers and Protasov (2011) for a detailed presentation of this quantity.

in control theory. The complexity of finding determinacy conditions is also discussed by Costa et al. (2005); Farmer et al. (2009b).

The insight behind this complexity is the following. In some circumstances, economic agents' decisions may depend on the exact order of future regimes and the attached expectations conditional on the considered path. Consequently, far past shocks and regimes may impact current decisions in case of indeterminacy. The existing literature partially rules out such a possibility (see Section 5.2 for an extended discussion).

4.3 Sufficient conditions of determinacy and indeterminacy

Since computing the exact limit of the sequence u_k is impossible for any model and parameters, we derive sufficient conditions ensuring determinacy (Proposition 2) and indeterminacy (Proposition 3).

Proposition 2. *If there exists k such that $u_k < 1$, then there exists a unique equilibrium.*

The proof of Proposition 2 is shown in the Appendix under section E. If it can be proved that there exists k such that u_k is smaller than 1, then its limit is smaller than 1. This condition converges to the determinacy frontier established in Proposition 1 as k tends to infinity.

The intuition is the following. We consider the case of u_k lower than one. The question is whether non-zero expectations, z_t^e, are consistent with a bounded equilibrium satisfying equation (15). Suppose that these expectations are non-zero. Because $u_k < 1$, we see that expectations increases every k periods, i.e. $||z_t^e||/(u_k^p) < ||z_{t+kp}||$. Expectations hence diverge. This proves that $z_t^e = 0$ is the only possible equilibrium consistent with bounded expectations. This reasoning is similar to the standard Blanchard and Kahn (1980) forward iteration but instead of studying the relationship between the current and the immediate future periods we link the current economic outcome with economic agents' expectations k-period ahead.

Finally, this result provides sufficient determinacy conditions. Such conditions are absent from the literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2013). Determinacy conditions are usually given for a subclass of equilibria, and hence, *de facto* correspond to sufficient indeterminacy conditions. We complement these existing conditions by providing new sufficient indeterminacy conditions in a larger class of stable equilibria.[13]

[13]This class of equilibria is the largest for which we can establish formal and general sufficient conditions.

Proposition 3. *If for a certain integer q, there exist N^{q+1} real numbers in the (interior of the) unit disk, $\alpha(i_0, \cdots, i_q)$, such that the highest eigenvalue of the matrix*

$$\left[\sum_{(i_1,\cdots,i_{q-1})\in\{1,N\}^{q-1}} p_{ii_1}p_{i_1i_2}\cdots p_{i_{q-1}j}\Gamma_i^{-1}\cdots\Gamma_{i_{q-1}}^{-1}\alpha(i,i_1,\cdots,i_{q-1},j) \right]_{(i,j)} \tag{17}$$

is larger than 1, then there exist multiple bounded solutions.

We prove this Proposition by constructing a continuum of stable equilibria in Appendix F.2. For a given length, q, the equilibria we consider depend on the q past regimes, shocks and endogenous variables. Every q periods, they belong to and are contracting on a fixed span. Nevertheless, the equilibrium may be locally explosive. Finally, when q increases, the size of the state variables needed to describe the equilibrium increases. When q is infinitely large, the dimension of the state space is infinite. In this case the equilibrium is fully history-dependent and non-recursive.

To give intuitions of this Proposition, let us first consider the special case when a specific q-regime trajectory, $(k_0, \cdots, k_{q-1}, k_0)$, is sufficient to generate a larger-than-one eigenvalue of the matrix defined by equation (17).[14] In this case, there exists an vector u such that $p_{k_0k_1}p_{k_1k_2}\cdots p_{k_{q-1}k_0}\Gamma_{k_0}^{-1}\cdots\Gamma_{k_{q-1}}^{-1}u = \lambda u$, with $\lambda > 1$. Suppose that the economy is initially in regime k_0 and that expectations z_t^e belongs to the unstable eigenvector, u. In addition, suppose that all the future expectations are zero except along the regimes trajectory $(k_0, \cdots, k_{q-1}, k_0, \cdots, k_{q-1}, k_0)$. Then, according to equation (15), the expectations decrease exponentially every q periods. Hence, economic agents can form non-zero expectations consistent with converging expectations. Thus, the economy is indeterminate. The example given in section 3 corresponds to this case, for $p_{12} = p_{21} = 1$ and $q = 2$. More generally, Proposition 3 proves that indeterminacy may arise even if such a regimes trajectory does not exist. This happens when a combination of regimes trajectories instead of only one regimes trajectory leads to a larger-than-one eigenvalue. In this case, non-zero expectations are consistent with converging expectations in many different regimes trajectories.

A generalization of this proposition has been recently proposed by Ogura and Jungers (2014). Basically, these authors refine our results by replacing the weights, $\alpha(i, i_1, ..., i_{q-1}, j)$, in formula (17), by particular matrices. This generalization improves the accuracy of the

However, we lack equilibria that are not the limit of recursive equilibria. In general, we know that such equilibria exist when the model is indeterminate, see Blondel et al. (2001)

[14]This corresponds to $\alpha(i_0, \cdots, i_q) = \delta_{k_0i_0}\cdots\delta_{k_0i_q}$ in equation (17), where $\delta_{ij} = 1$ when $i = j$.

sufficient indeterminacy conditions reducing the parameters region where the economist is not able to settle determinacy for a given computing time constraint. This generalization applied to economics should help applied economists to implement fast algorithm checking determinacy.

The classes of equilibria that we consider in Proposition 3 encompass those considered in the literature (Davig and Leeper, 2007; Farmer et al., 2009b; Cho, 2013; Foerster et al., 2013). The equilibria put forward after Farmer et al. (2009b) correspond to the specific case of $q = 1$. By focusing on smaller classes of equilibria, the literature underestimates the size of the indeterminacy region (section 5.2). In turn, taking into consideration the whole class of stable equilibria modifies our understanding of monetary policy in a context of monetary policy switching (sections 5.3).

4.4 Extension to models with pre-determined variables

We now extend our analysis to models with pre-determined variables. The extended class of models we consider is as follows:

$$\Gamma_{s_t} z_t = \mathbb{E}_t z_{t+1} + \Delta_{s_t} z_{t-1} + C_{s_t} \epsilon_t, \tag{18}$$

where Δ_{s_t} is a regime-dependent $(n \times n)$ real matrix.

As noted by Cho (2013), the existence of a unique stable equilibrium of such a model depends on (i) the existence of a stable equilibrium (ii) determinacy of an associated purely forward-looking model as the one described by equation (11). The first step goes beyond the scope of this paper and we do not tackle it in this paper. Let suppose that there exists a bounded equilibrium verifying:

$$z_t^F = -R_{s_t}^{-1} \Delta_{s_t} z_{t-1} - R_{s_t}^{-1} C_{s_t} \epsilon_t$$

where the $(n \times n)$ matrix R_{s_t} is a solution of the following matricial equation:[15]

$$R_{s_t} = \Gamma_{s_t} - E_t R_{s_{t+1}}^{-1} \Delta_{s_{t+1}}, \tag{19}$$

such that the joint spectral radius[16] of $(R_1^{-1}\Delta_1, ..., R_N^{-1}\Delta_N)$ is smaller than one. The joint

[15] Cho (2013); Foerster et al. (2013) provide algorithms solving this matricial equation.

[16] The joint spectral radius is the maximum growth rate of the product of matrices. For a set of matrices $(A_1, ..., A_N)$, the joint spectral radius is defined as $\rho(A_1, ..., A_N) = \limsup_k \{||A_{s_1} ... A_{s_k}||^{1/k}\}$. See Jungers

spectral radius ensures the boundedness of the equilibrium defined above.

We can observe that any equilibrium satisfies:

$$z_t = -R_{s_t}^{-1}\Delta_{s_t} z_{t-1} + w_t,$$

where w_t solves the following forward-looking model:

$$E_t w_{t+1} + R_{s_t} w_t + C_{s_t}\epsilon_t = 0. \tag{20}$$

We then have the following proposition:

Proposition 4. *The model 18 is determinate if and only if the forward-looking model 20 admits a unique stable equilibrium.*

This last proposition can be checked by applying propositions described above and especially Proposition 1 and links the propositions established for purely forward-looking models with those including pre-determined variables.

5 Monetary policy switching

In this section, we generalize the Taylor principle in a regime-switching monetary policy context. This section differs from the simple example of Section 3 by considering that the economy switches from one regime to another stochastically. We hence complement Davig and Leeper (2007) who prove determinacy conditions in such an environment by restricting the class of equilibria to Markovian equilibria. We first expose the model, we then explicit the relationship between determinacy conditions and restrictions to the class of equilibria. Finally, we provide for results on determinacy when monetary policy switches without restraining the class of equilibria.

5.1 The model

We consider the log-linearized New-Keynesian model described by equations (1) and (2). But instead of assuming a constant policy rule, we suppose that monetary policy follows a Taylor rule with recurring shifts in parameters.

$$r_t = \alpha_{s_t}\pi_t + \gamma_{s_t} y_t + \epsilon_t^r, \tag{21}$$

(2009) for more details about this quantity.

where the index $s_t \in \{1, 2\}$ stands for the current observable regime of monetary policy and the shock ϵ_t^r is a disturbance measuring the unsystematic part of monetary policy.

As discussed in Section 2, when there is no regime switching, i.e. $\alpha_1 = \alpha_2 = \alpha$, and no reaction to the output-gap, i.e. $\gamma_1 = \gamma_2 = 0$, the model admits a unique stable equilibrium if and only if the Taylor principle is verfied, meaning if the real interest rate increases with inflation, $\alpha > 1$. Following Leeper (1991), we say that the regime is active when it satisfies the Taylor principle. Otherwise, the regime is passive. When monetary policy reacts to the output gap, the determinacy condition becomes: $\alpha + \frac{1-\beta}{\kappa}\gamma > 1$.

The question we want to address here is how these conditions evolve in a context of regime-switching monetary policy. This question has been at the core of the controversy opposing Farmer et al. (2010a) and Davig and Leeper (2007, 2010) but remains unsolved.

To establish our main results, we calibrate the model in accordance with Davig and Leeper (2007). The discount factor, β, is set to 0.99, reflecting an annualized steady-state real interest rate equal to 4%. We assume log utility ($\sigma = 1$). The price stickiness is such that the slope of the new-Keynesian Philips Curve, κ, is 0.17. We assume that the persistence of the first regime, p_{11} is relatively low, 0.8, while the persistence of the second regime is high, $p_{22} = 0.95$. These probabilities correspond to average durations of 4 and 19 quarters for regimes 1 and 2 respectively.

5.2 Classes of equilibria and indeterminacy

While the considered class of equilibria does not affect determinacy conditions in the standard linear rational expectations model, determinacy conditions change with the class of equilibria in a regime-switching environment. Nonetheless, in both cases (with and without regime switching), there exist history-dependent equilibria when the economy is indeterminate. In the absence of regime switching, restriction on the stochastic properties of the equilibria, such as their correlation with past fundamental shocks, do not affect stability. By contrast, in a regime-switching economy, the stability of equilibria depends on their co-movement with past regimes. By ruling out some classes of equilibria, the existing literature thus underestimates the size of the indeterminacy region.

Result 2. *Excluding equilibria whose autocorrelation depends on past regimes overestimates the determinacy region, i.e. the region of parameters for which determinacy is ensured.*

We apply Proposition 3 to analyze the relationship between the determinacy conditions and

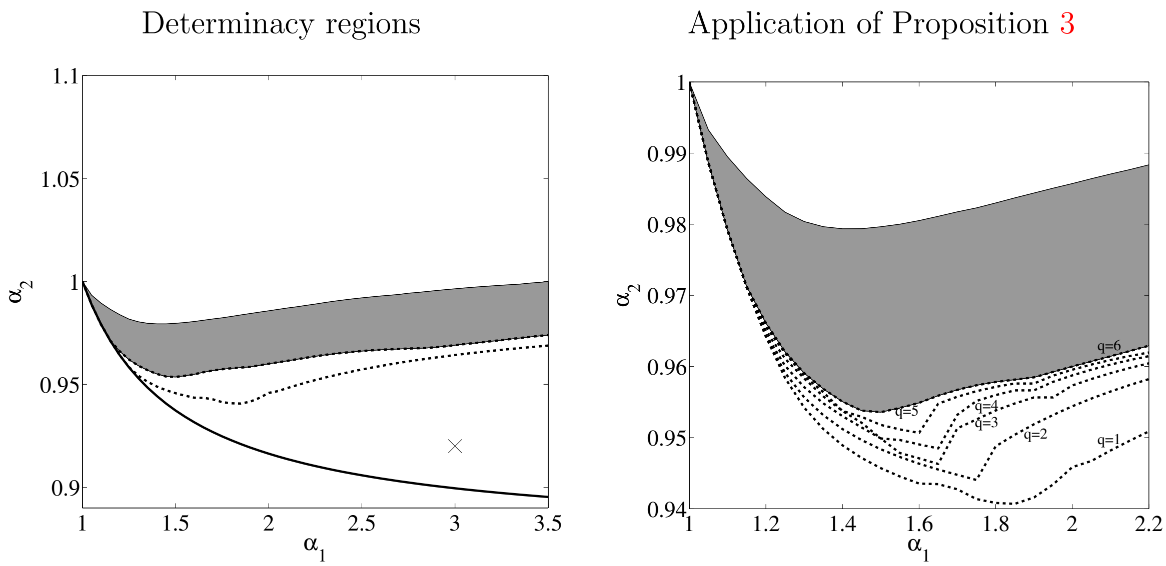

Table 2: Indeterminacy regions and classes of equilibria: new-Keynesian model with Markov-switching monetary policy

Note: In both figures, we represent the regions corresponding to the different classes of equilibria. The thick line is the limit of indeterminacy for Markovian solutions, the dashed line is the limit of the determinacy region for the solutions of Farmer et al. (2009b), the gray area is the region in which we cannot conclude. Below the gray area, equilibria whose autocorrelation depends on the last six regimes at the most exist. The cross corresponds to the counterexample of Farmer et al. (2010a). The right figure is a zoom of the left one close to the gray area, depecting the different determinacy frontiers depending on the number of regimes that can matter at the equilibrium (see Proposition 3. Probabilities are set to $p_{11} = 0.8$ *and* $p_{22} = 0.95$.

the restriction on the class of equilibria. The left figure of Table 2 plots different determinacy regions according to different restrictions. The area above the dashed line corresponds to the determinacy region for equilibria that only depend on a finite number of past regimes -i.e. so-called Markovian equilibria. In this class of equilibria, we show in Proposition 2 (appendix F.1) that the autocorrelation does not depend on past regimes. This class of equilibria is consistent with Davig and Leeper (2007). The dashed line stands for the determinacy conditions among equilibria that are first-order recursive and for which the autocorrelation depends on current and past regimes only. This class of equilibria corresponds to those detected by Proposition 3 when q is equal to 1. This also corresponds to the class of equilibria put forward by Farmer et al. (2009b). Finally, the lower bound of the gray area corresponds to determinacy conditions for the class of equilibria whose autocorrelation depends on the past six regimes at the most. The right figure of Table 2 plots the different determinacy conditions for class of equilibria considered in Proposition 3 for q from 1 to 6. An extended description of the equilibria is given in Appendix F.1. In both figures, the gray area corresponds to a region for which we cannot conclude in a reasonable computing time. Obviously, there are two ways to reduce this area: an improvement in the power and time of calculus, and a refinement of Proposition 3 in the spirit of Ogura and Jungers (2014). Above this area, Proposition 2 ensures the existence of a unique equilibrium *whatever the class of equilibria*. To compute the upper bound of the gray area we check whether the values of u_k for k from 0 to 16 are smaller than one.

Between the dashed and the bold lines, there exists a unique Markovian equilibrium and multiple equilibria with more complex autocorrelation structure. Farmer et al. (2010a) made this point by constructing a counterexample to the determinacy conditions derived by Davig and Leeper (2007). The cross in Figure 2 corresponds to the policy parameters allowing them to build this counterexample. Between the dashed line and the lower bound of the gray area, a unique equilibrium among equilibria whose autocorrelation depends on the current and past regimes exists but multiple equilibria of higher orders also coexist. This proves that determinacy conditions depend on restriction on the structure of the equilibria autocorrelation.

An increasing literature (Cho, 2013; Foerster et al., 2013) adopts the restriction firstly proposed by Farmer et al. (2009b) consisting of first-order recursive equilibria, sometimes referred to as Minimum State Variables solutions. This literature also considers a different concept of stability as noticed in section 4.1. However, Result 2 is independent of and robust to alternative stability concepts. Restricting the considered class of equilibria results in underestimating the size of the indeterminacy region. In the next subsection, we provide for results that are valid for the broadest solutions space, and are hence the more relevant for policymakers.

5.3 Determinacy conditions

We now turn to general determinacy conditions when considering the whole class of equilibria. We first put forward a case in which allowing higher order equilibria generate new policy conclusion. We then revisit one of the main contributions of Davig and Leeper (2007) stating that determinacy can be guaranteed even if one regime fails to satisfy the Taylor principle.

Result 3. *An economy may suffer from indeterminacy even if the two monetary policy regimes satisfy the Taylor principle.*

We identify such a configuration when the two monetary policies share the same rule - nominal interest rate reacts proportionally to inflation in both regimes- but with different intensities. The two regimes satisfy the Taylor principle. In the first regime, the less active one, the central bank reacts moderately to inflation ($\alpha_1 = 1.01$). In the second, the more active regime, the central bank reacts (extremely) strongly to inflation ($\alpha_2 = 6$). When monetary policy switches between prolonged periods of the less active regime ($p_{11} = 0.95$) and short-lasting periods of the more active regime ($p_{22} = 0.5$), Proposition 3 proves that the economy is indeterminate. To understand this result we first isolate the regimes trajectory that explain the most indeterminacy (figure 2), then, we describe the dynamics of expectations along these trajectories.

Figure 2 plots the ten largest contributions to sequence u_k, which asymptotically measures model stability. The larger the contribution, the more converging the expectations along the regimes trajectory. Unsurprisingly, a prolonged less active monetary policy regime significantly contributes to increase this measure of stability. Indeed, in this regime, monetary policy makes expectations diverging but only slowly as the response to inflation is weak in this regime (see Section 2). This regimes trajectory is however insufficient to explain indeterminacy by itself as the less active regime satisfies the Taylor principle, and hence, induces determinacy when taken in isolation. The other largest contributions correspond to the alternation between a short period of the more active regime and a protracted period of the less active regime. We give intuitions in Figure 3 why these regimes trajectory can explain indeterminacy.

Indeterminacy arises when expectations can be non-zero without generating diverging expectations (see section 2). To understand why the identified regimes trajectories trigger indeterminacy we thus analyze the dynamics of expectations along the regimes trajectories identified above. Figure 3 reports the responses of the output-gap and inflation to a 17-quarter-ahead expectation conditional on the future regimes path identified as the largest contributors to a measure of model stability, u_{16}, in Figure 2. We plot the responses to an expected rise in the output-gap ($\mathbb{E}_t y_{t+17} = 1$ while $\mathbb{E}_t \pi_{t+17} = 0$) in 17 quarters. If the model is determinate such expectations should necessarily lead to small expectations in first period

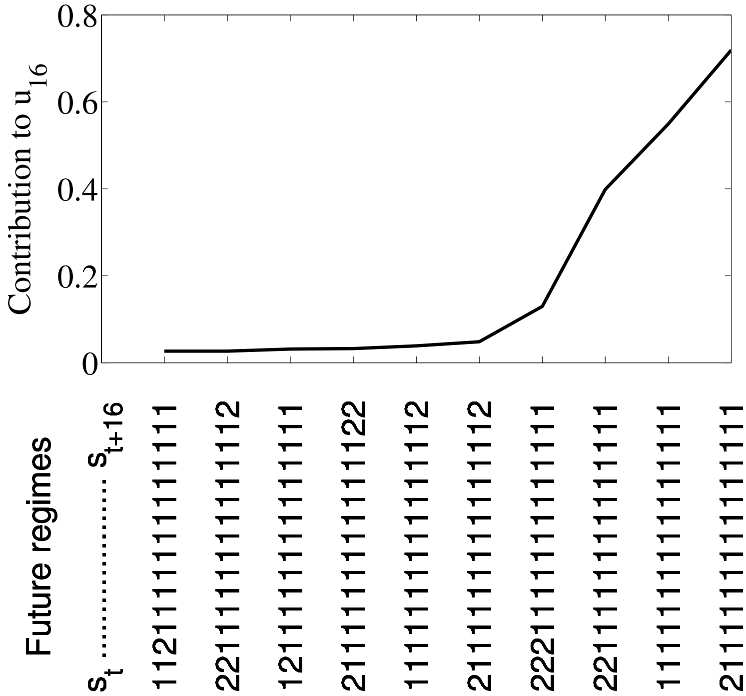

Figure 2: Ten largest contributions to a measure of expectations explosiveness

Note: we report the ten largest contributions -in terms of the future regime trajectory and amongst 2^{36} *possible trajectories- to a measure of model stability,* u_{16} *(see Proposition 1). The x-axis represents a particular regimes trajectory. The y-axis stands for the associated contribution to* u_{16}*. A higher contribution suggests that the expectations are less diverging along this regimes trajectory. When all the contributions add up to less than one, this proves that the economy is determinate. Otherwise, this suggests (without formally proving) indeterminacy. Finally, if one contribution is larger than one, the economy is indeterminate and we can build converging non-zero expectations along this regimes trajectory. Policy parameters and probabilities are set to* $\alpha_1 = 1.01$, $\gamma_1 = 0$ *and* $p_{11} = 0.95$ *and* $\alpha_2 = 6$, $\gamma_2 = 0$ *and* $p_{22} = 0.5$.

(when weighted by regimes trajectory's probability) as non-zero expectations diverge in a determinate economy.

Along the black line without markers, we plot the expectations dynamics if the economy remains in the less active regime. In this regime, a positive expectation of the output-gap in remote future leads to inflationary pressure through the new-Keynesian Phillips curve explaining the downward dynamics of inflation expectations. On the other hand, these inflation expectations lead to a moderate increase in the real rate that contributes to slightly moderate the output-gap through the IS-curve. This explains that output-gap expectations are

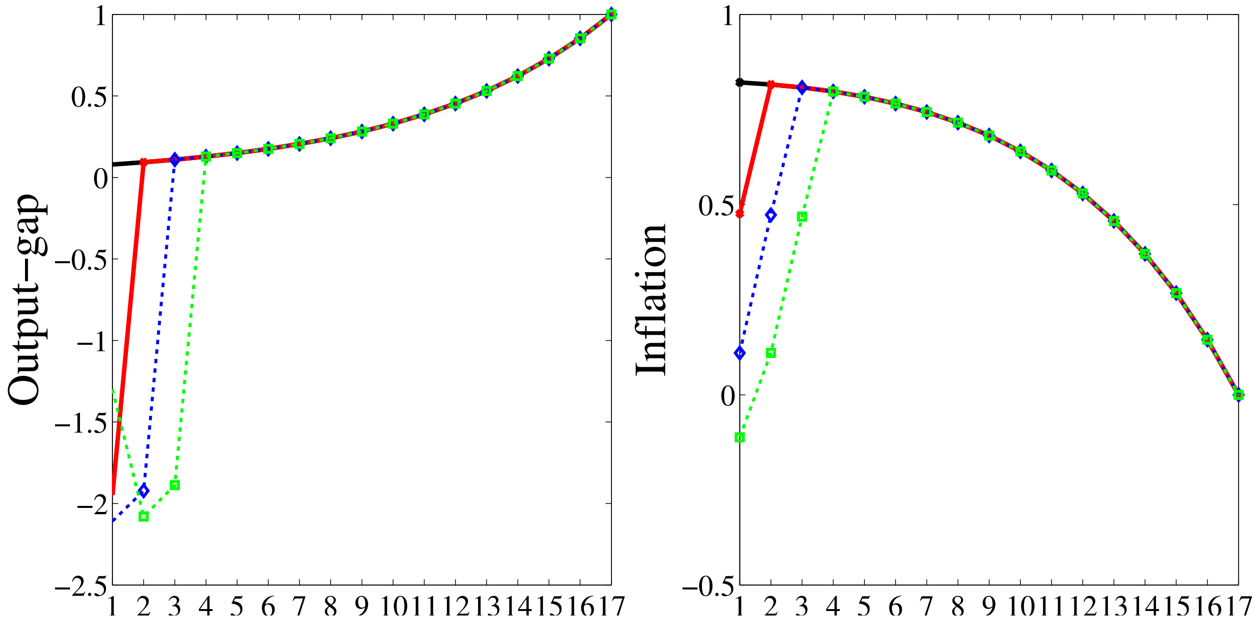

Figure 3: Impact of a positive output-gap expectation in a new-Keynesian model with Markov-switching monetary policy

Note: the two figures report the dynamics of the output-gap (left) and inflation (right) expectations to an expectation of a rise in the output-gap in 17 periods. We relate the four largest contributors to u_{16} *(see Figure 2). The bold red line with crosses displays the trajectory if the future regimes are the more active (regime* 2*) in first period and the less active afterwards (regime* 1*). The bold black line represents the trajectory conditional on staying in the less active regime (regime* 1*). In dashed blue line with diamonds, we plot the response when the economy is in the more active regime during two periods before being in the other regime and in dashed green line with squares, three times in the more active regime and the less active regime afterwards. Policy parameters and probabilities are set to* $\alpha_1 = 1.01$*,* $\gamma_1 = 0$ *and* $p_{11} = 0.95$ *and* $\alpha_2 = 6$*,* $\gamma_2 = 0$ *and* $p_{22} = 0.5$*.*

increasing up to the final period. If the period of less active monetary policy lasts longer then, inflation expectations are finally completely stabilized through positive real rates as the Taylor principle is satisfied in this regime. Hence, inflation expectations are positive in first period only because the duration of the sample is too short.

Along the red line with crosses, dashed blue line with diamonds and dashed green line with squares, we plot the expectations dynamics when the more active regime lasts one, two, and three periods respectively, and is followed by a long-lasting period of the less active regime. The long lasting period of the less active regime coincides with the case described above (black lines without markers). Hence, in the more active monetary policy, the monetary authorities face positive inflation expectations but small output-gap expectations. These inflation expectations due to the less active regime lead to positive expectations of inflation because of the new-Keynesian Phillips curve, that, in turn, contribute to a dramatic increase in the real rate that triggers a fall in the output-gap, economic agents preferring to postpone their consumption to benefit from high real rates. Eventually, inflation expectations in first period can be close to zero because of the aggressiveness of monetary policy at the cost of a large fall in the output-gap. In the end, we see that period-1 expectations of the output-gap are larger than period-17 output-gap expectations while inflation expectations are close to zero in both cases. It means that conditional on these regimes trajectory, non-zero expectations in first period are consistent with converging expectations. If such successions of regimes occur with sufficiently large probabilities, multiple equilibria arise eventually.

The dynamics of expectations described above is similar to the one described in the Section 3 simple example. The first period of the more active monetary policy regime corresponds to the inflation targeting regime in the simple example. In this regime, expectations of inflation are close to zero (zero in the simple example) because monetary policy is very aggressive against any deviation of inflation. However, inflation expectations due to next less active policy regime cause negative output-gap through the new-Keynesian Phillips curve. In the beginning of the less active monetary policy regime, the output-gap is close to zero while inflation is positive due to expectations of positive output-gap in the future. This echoes the situation in the flexible price regime where the output-gap is zero but inflation can be positive due to expectations of positive output-gap in next period. Finally, last period with positive output-gap could coincide with a return to a more active monetary policy as in the simple example. Finally, besides this apparent similarity, what is key to explain indeterminacy, in both cases, is switching between very different policies that both make expectations diverging but according to inconsistent patterns: inflation expectations explode first in the more active regime, output gap expectations then explode in the less active regime.

This configuration is more likely when the more active monetary policy regime is infrequent and short-lasting. Conversely, indeterminacy occurs when the less active monetary policy regime is very persistent. Figure 4 displays determinacy regions with respect to the probabilities in each regime. Thus, contrary to what one could expect, expectations of infrequent highly active monetary policy regimes may introduce converging non-zero expectations

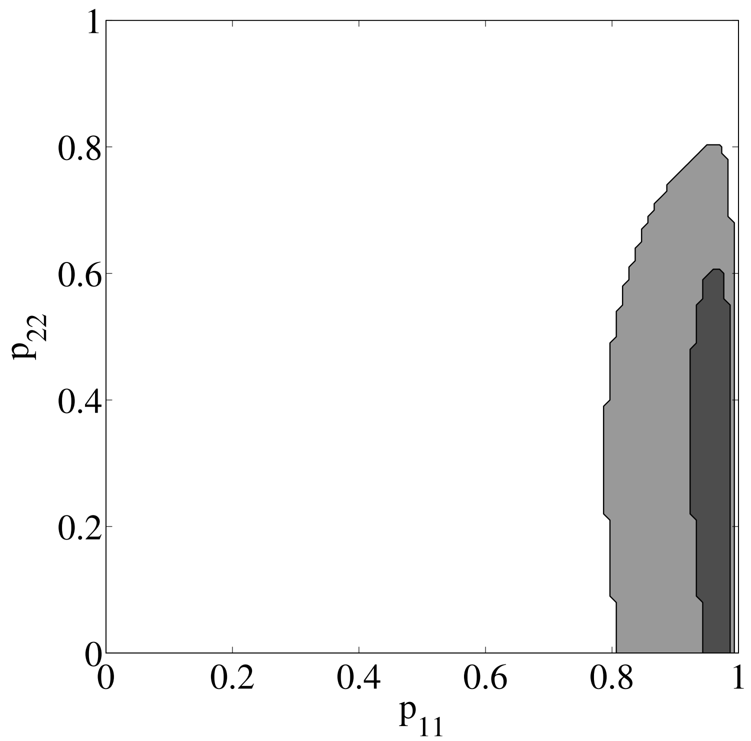

Figure 4: Determinacy regions and persistence of regimes: new-Keynesian model with Markov-switching monetary policy

Note: the white area represents the determinacy region; the light-shaded area represents a region in which we cannot decide whether or not there is a unique bounded equilibrium in a reasonable amount of time; the dark-shaded area represents a region in which there exist multiple bounded equilibria. Policy parameters are calibrated such that the two regimes satisfy the Taylor principle. Response to inflation in the first (second) regime is $\alpha_1 = 1.01$ *(*$\alpha_2 = 6$*, respectively).*

rather than making expectations more diverging, leading to indeterminacy eventually.

For two active monetary policy regimes to trigger indeterminacy, the economy should suffer from price stickiness. In a flexible-price economy ($\kappa \rightarrow \infty$), inflation is only determined by inflation expectations since the feedback force between output and inflation observed in Figure 3 does not hold anymore. If monetary policy satisfies the Taylor principle in the two regimes, current and future monetary policies (whatever the regime is) makes any non-zero inflation expectations exploding.[17] Hence, non-zero inflation expectations can be discarded

[17]In this environment, determinacy conditions depend on a simple combination of transition probabilities and inflation reaction in both regimes (see Farmer et al., 2009a).

and inflation is uniquely defined. This is the reason why Result 3 does not hold in a Fisherian model of inflation determination.

Result 4. *A regime-switching economy may be determinate even if one of the two regimes does not satisfy the Taylor principle.*

Monetary policy may deviate from the Taylor principle moderately ($\alpha_1 = 0.98$) but relatively persistently ($p_{11} = 0.95$) without implying indeterminacy if monetary policy reacts sufficiently to inflation in the other regime (for instance if $\alpha_2 = 1.5$) as shown in Figure 2. While the first regime is not active enough to ensure determinacy by its own, expectations of a switch to a more active regime suffice to anchor expectations, and hence to rule out indeterminacy. Thus, expectations of a more aggressive monetary policy may be effective in guaranteeing macroeconomic stability.

This result is in line with the contribution of Davig and Leeper (2007). In this paper, the authors argue that 'a unique bounded equilibrium does not require the Taylor principle to hold in every period'. They find this result by restricting the solutions space (see Section F.1 for further details). We prove that this result is unchanged when considering the whole class of equilibria, and hence, is not an artifact due to *ad hoc* restrictions.

One of the consequences of the two previous results is to potentially impose some restrictions on the admissible monetary policy in a regime switching economy. Consider a central banker that would like to prevent the economy from indeterminacy as this can be an uncontrollable source of volatility. If this central banker knows that he will be replaced by another central banker occasionally, determinacy conditions will generate upper and lower bound on his degree of reaction to inflation. We suppose that monetary policy reacts moderately to inflation in the most frequent regime and long-lasting regime ($p_{11} = 0.95$ and $\alpha_1 = 0.99$) while the other regime is short-lasting ($p_{22} = 0.5$). Restrictions on admissible policies are summed up below.

Result 5. *In a long-lasting passive monetary policy regime, the anticipation of a short-lived monetary policy switch stabilizes the economy if the response to inflation in the latter regime is neither too weak nor too strong.*

Too weak or too strong a response to inflation in the infrequent regime leads to indeterminacy, while an intermediate response to inflation stabilizes the economy. Figure 5 shows determinacy regions with respect to the inflation response in the infrequent regime. The darkest area represents the indeterminacy region, the lightest gray area, the parameters region for

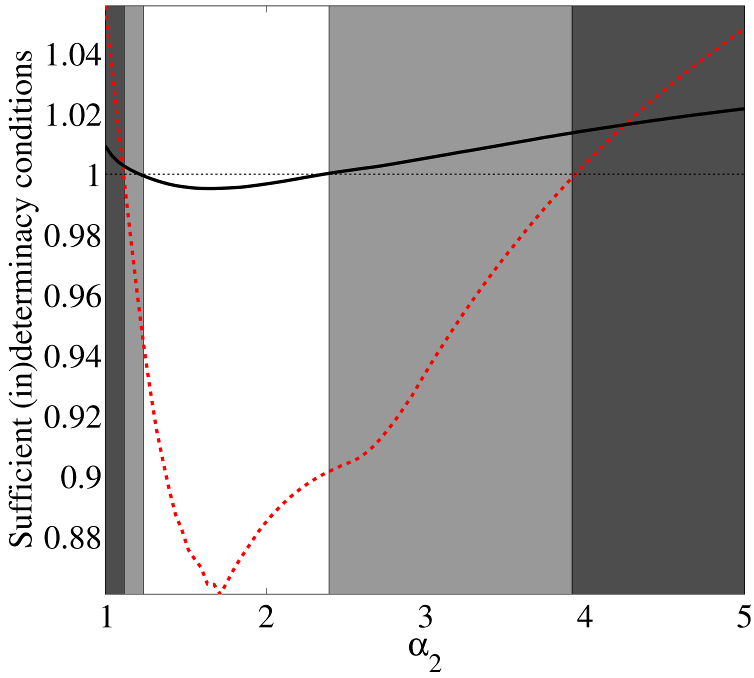

Figure 5: Optimal degrees of activism: new-Keynesian model with Markov-switching monetary policy

Note: the white area represents the determinacy region; the light-shaded area represents a region in which we cannot decide whether or not there is a unique bounded equilibrium in a reasonable amount of time; the dark-shaded area represents a region in which there exists multiple bounded equilibria. Policy parameters are calibrated as follows: $\alpha_1 = 0.99$, $p_{11} = 0.95$ *and* $p_{22} = 0.5$. *The dark line shows the index of stability,* u_{16}. *When this line is below one, the economy is determinate. The dashed red line shows the largest eigenvalue of Proposition 3 for* $q = 6$ *(and for any scalars* α_{ijkl}*). When this largest eigenvalue is larger than one, we can construct multiple equilibria that depend on the past six regimes.*

which we cannot conclude and the white area, the determinacy region. Determinacy arises for responses to inflation, α_2, between at least 1.2 and 2.4 (white area). Indeterminacy arises when the response is at least smaller than 1.1 or larger than 3.9.

This result stems from the combination of two effects. First, a large enough reaction to inflation helps to rule out indeterminacy that arises due to the passive monetary policy in the long-lasting regime. By raising the nominal interest rate when facing inflationary pressure, the central banker helps reduce the interplay between inflation and inflation expectations (result 4). Second, an excessively large response to inflation in one monetary policy regime increases the sensitivity of the output-gap to expectations and may eventually lead to indeterminacy (result 3).

Thus, when choosing their policies, central bankers should internalize the possibility of a switch to a passive monetary policy and may be forced to moderate their reaction compared to what it would otherwise have been.

The role of price stickiness In the simple example of Section 3, one of the two regimes is characterized by flexible price suggesting that the degree of price stickiness matters for determinacy. We thus consider in this paragraph a simultaneous switch of monetary policy and the degree of price stickiness. In a state-dependent price setting, one can think that the probability of resetting the prices depend on monetary policy as it influences the volatility of inflation. While the standard new-Keynesian model abstracts from such a mechanism, a parsimonious way to analysis it is to assume that the monetary policy regims are associated with different degree of price stickiness. The following result proves that indeed price stickiness matters for indeterminacy in a regime switching economy.

Result 6. *The stickier the prices are in one regime, the larger the reaction to inflation in the other regime should be to prevent from indeterminacy.*

To prove this result we assume that, in regime 2, the degree of price stickiness is equal to the benchmark case, $\kappa_2 = 0.17$, and monetary policy reaction to inflation is moderate and denoted by α_2. Concerning the price rigidity in regime 1, we compare two scenarii: either it is similar to regime 1, $\kappa_1^* = 0.17$ or prices are more rigid, $\kappa_1 = 0.06$.[18] Determinacy regions are represented in Figure 6, when probabilities of remaining in each regime are equal to 0.9.

For a moderate response to inflation in regime 2 ($\alpha_2 = 0.95$) but a more aggressive one in regime 1 ($\alpha_1 = 2.5$), the economy is determinate for the benchmark price stickiness, but indeterminate for a higher degree of price rigidity. This result echoes Result 1 in the simple example in which we prove that the stickier the price the larger the reaction to inflation should be in the other regime.

To understand the complementarity between price stickiness and monetary policy, we analyze the impact of more rigid prices in regime 1 on the dynamics of expectations as described in figure 3. The regimes trajectories that explain indeterminacy when monetary policy regimes are very different are those of short-lasting period of more active monetary

[18]For standard Calvo-type of price setting, this parametrization corresponds to a probability of not resetting prices of 0.75 and 0.85 per quarter, respectively.

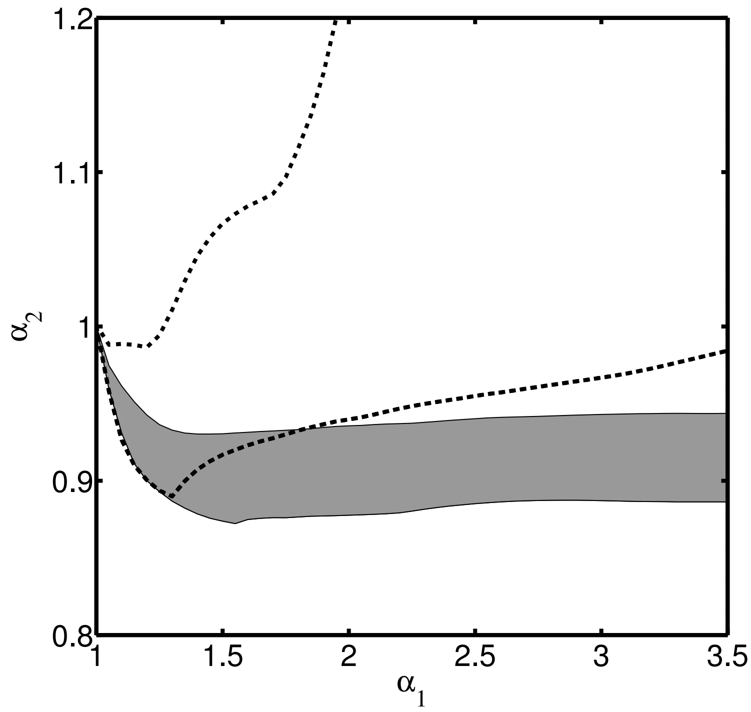

Figure 6: Price rigidities and determinacy in a new-Keynesian model with Markov-switching monetary policy

Note: the light-shaded area represents a region in which we cannot decide whether or not there is a unique bounded equilibrium in a reasonable amount of time for the benchmark case ($\kappa = 0.17$*). Below the model is indeterminate, above it is determinate. The region between the two dashed lines describes the region of indecision for stickier prices (*$\kappa_1 = 0.06$*). Above this area, the economy is determinate and indeterminate below. Transition probabilities are set to* $p_{11} = p_{22} = 0.9$.

policy followed by a long-lasting period of less active monetary policy. The main mechanism as described above is the fall of the output-gap in first period that results from inflation expectations due to less active monetary policy in the future (if we assume positive output-gap expectations in distant future). The fall of the output-gap increases with the degree of price stickiness in the more active regime. Indeed, the stickier the prices, the less sensitive inflation is to the output-gap because of the new-Keynesian Phillips curve. Thus, to stabilize inflation the monetary authority needs to raise the real interest rate to lower the output-gap and counterbalances the effect of inflation expectations. Consequently, the stickier the price the more the output-gap falls and hence the more likely indeterminacy arises.

6 Conclusion

From a theoretical standpoint, this paper establishes a necessary and sufficient condition of determinacy for purely forward-looking rational expectations models with Markov-switching. This condition depends on the asymptotic behavior of all matrix products. The complexity raised by regime-switching models reflects the path-dependency of economic agents' expectations. Hence, determinacy conditions in the presence of regime switching intimately depend on the restrictions on the solutions space. To overcome this difficulty, we derive verifiable sufficient conditions of determinacy and indeterminacy. Furthermore, ongoing progress in applied mathematics (Ogura and Jungers, 2014) should result in rapid improvements in the accuracy and speed of determinacy check.

We then generalize the Taylor principle to a canonical monetary model in which monetary policy switches between different Taylor rules and in which the degree of price stickiness can change. We establish a non-trivial relationship between monetary policy regimes and determinacy. On the one hand, an active monetary policy may help anchor inflation expectations if the other regime fails to satisfy the Taylor principle. On the other hand, an over-active monetary policy may lead to indeterminacy. This aggressive monetary policy destabilizes the output-gap by over-reacting to inflation expectations due to the other regime. This may happen even if the two regimes satisfy the Taylor principle.

This last result emphasizes that, even if the different regimes ensures determinacy when taken in isolation, regime switching may itself destabilize the economy, when these regimes are very different. We thus identify situations in which policy switching itself, rather than one or the other policy, is responsible for indeterminacy.

APPENDIX

Appendices A to F.3 are for Online Publication only.

A Definition of a solution

We assume that F is a bounded set such that $\varepsilon_t \in F$, and we denote by $\varepsilon^t = \{\varepsilon_t, \cdots, \varepsilon_{-\infty}\}$ and $s^t = \{s_t, \cdots, s_{-\infty}\}$ the history of shocks and regimes. We define a stable equilibrium as follows:

Definition 2. *A stable equilibrium of* (11) *is function z on $\{1, \cdots, N\}^\infty \times F^\infty$, satisfying equation* (11) *and such that*

$$\|z\|_\infty = \sup_{z^t, \varepsilon^t} \|z(s^t, \varepsilon^t)\| < \infty \tag{22}$$

We denote by $\mathcal{B}$ the set of all the bounded functions on $\{1, \cdots, N\}^\infty \times F^\infty$. The set $\mathcal{B}$, with the norm $\|\cdot\|_\infty$ defined in equation (22) is a Banach space.

B Elements in the absence of switching

We transform equations 1, 2 and 3 by replacing r_t in equations 1 and 2.

$$\begin{aligned} \sigma\alpha\pi_t + (1+\sigma\gamma)y_t - \sigma\epsilon_t^d + \sigma\epsilon_t^r &= \mathbb{E}_t y_{t+1} + \sigma\mathbb{E}_t\pi_{t+1} \\ \pi_t - \kappa y_t - \epsilon_t^s &= \beta\mathbb{E}_t\pi_{t+1} \end{aligned}$$

This leads to

$$\begin{bmatrix} 1/\beta & -\kappa/\beta \\ \sigma(\alpha - 1/\beta) & 1+\sigma\gamma+\kappa\sigma/\beta \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \mathbb{E}_t \begin{bmatrix} \pi_{t+1} \\ y_{t+1} \end{bmatrix} + \begin{bmatrix} 1/\beta & 0 & 0 \\ -\sigma/\beta & -\sigma & \sigma \end{bmatrix} \begin{bmatrix} \epsilon_t^s \\ \epsilon_t^r \\ \epsilon_t^d \end{bmatrix}$$

and finally to equation 4 with

$$\Gamma = \begin{bmatrix} 1/\beta & -\kappa/\beta \\ \sigma(\alpha - 1/\beta) & 1+\sigma\gamma+\kappa\sigma/\beta \end{bmatrix}, \qquad C = \begin{bmatrix} 1/\beta & 0 & 0 \\ -\sigma/\beta & -\sigma & \sigma \end{bmatrix}$$

An equilibrium is given by

$$\begin{bmatrix} \pi_t^F \\ y_t^F \end{bmatrix} = \Gamma^{-1} C \varepsilon_t$$

C Proof of Proposition 1

In this section, we prove Proposition 1. Assuming that Γ_i is invertible for any $i \in \{1, \cdots, N\}$, we rewrite (11) as:

$$z_t + \Gamma_{s_t}^{-1}\mathbb{E}_t z_{t+1} = -\Gamma_{s_t}^{-1}C_{s_t}\varepsilon_t \tag{23}$$

Then, considering $z_t = z(s^t, \varepsilon^t)$ as a function of all the past shocks $\{\varepsilon_t, \cdots, \varepsilon_{-\infty}\}$ and regimes $\{s_t, \cdots, s_{-\infty}\}$, introducing ψ_0 such that $\psi_0(s^t, \varepsilon^t) = -\Gamma_{s_t}^{-1}C_{s_t}\varepsilon_t$ and defining the operator $\mathcal{R}$ as

$$\mathcal{R} : z \mapsto ((s^t, \varepsilon^t) \mapsto -\Gamma_{s_t}^{-1}\mathbb{E}_t z(s^{t+1}, \varepsilon^{t+1})) \tag{24}$$

Equation (23) is equivalent to the functional equation:

$$(\mathbb{1} - \mathcal{R})z = \psi_0 \tag{25}$$

This equation admits a unique solution if the operator $\mathbb{1} - \mathcal{R}$ is invertible, and thus, if $1 \notin \sigma(\mathcal{R})$. Consequently, conditions of existence and uniqueness of a solution of (11) rely on the spectrum of $\mathcal{R}$, this spectrum depending on the space of solutions we consider.

Before characterizing this spectrum, we first show that the sequence (u_k) in equation (16) is convergent.

C.1 Behavior of the sequence u_p

In this section, we prove the following result

Lemma 1. *The sequence (u_k) in equation (16) has the following properties.*

- *The sequence $(u_k)^k$ is sub-multiplicative ($(u_{m+n})^{m+n} \leq u_m^m u_n^n$), and thus, convergent.*
- *The limit, ν, does not depend on the chosen norm.*

We first show that (u_k^k) is sub-multiplicative. Using the sub-multiplicativity of a matricial norm, u_{m+n}^{m+n} satisfies:

$$\begin{aligned} & \sum_{(i_1,\cdots,i_m,i_{m+1},\cdots,i_{m+n})\in\{1,\cdots,N\}^{m+n}} p_{i_1 i_2}\cdots p_{i_{m-1}i_m}\times \\ & p_{i_m i_{m+1}}\cdots p_{i_{m+n-1}i_{m+n}}\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\| \\ \leq & \sum_{(i_1,\cdots,i_m,i_{m+1})\in\{1,\cdots,N\}^{m+1}} p_{i_1 i_2}\cdots p_{i_{m-1}i_m}p_{i_m i_{m+1}}\|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\| \end{aligned}$$

$$\times \left(\sum_{(i_{m+2},\cdots,i_{m+n})\in\{1,\cdots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}} \|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\| \right)$$

We find an upper bound for the second term by adding up i_{m+1} and as all the terms are positive:

$$\sum_{(i_{m+2},\cdots,i_{m+n})\in\{1,\cdots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}} \|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\|$$

$$\leq \sum_{(i_{m+1},i_{m+2},\cdots,i_{m+n})\in\{1,\cdots,N\}^{n-1}} p_{i_{m+1}i_{m+2}}\cdots p_{i_{m+n-1}i_{m+n}} \|\Gamma_{i_{m+1}}^{-1}\cdots\Gamma_{i_{m+n}}^{-1}\| = (u_n)^n$$

Thus,

$$(u_{n+m})^{n+m} \leq u_n^n \sum_{(i_1,\cdots,i_m,i_{m+1})\in\{1,\cdots,N\}^{m+1}} p_{i_1i_2}\cdots p_{i_{m-1}i_m}p_{i_mi_{m+1}} \|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_m}^{-1}\| = u_n^n \times u_m^m$$

since $\sum_{i_{m+1}\in\{1,\cdots,N\}} p_{i_mi_{m+1}} = 1$.

This shows that (u_k^k) is sub-multiplicative.

Besides, if a sequence of non-negative real numbers (v_k) is sub-multiplicative, then $v_k^{1/k}$ is converging and $\lim_{k\to+\infty} v_k^{1/k} = \inf_k v_k^{1/k}$, see for instance Lemma 21 p.8 in Müller (2003). Thus, (u_k) is convergent.

Finally, because of the equivalence of the norms in $\mathcal{M}_n(\mathbb{R})$, it is immediate that ν does not depend on the chosen norm.

C.2 Characterization of the spectral radius of $\mathcal{R}$

We will prove the following lemma, describing the spectrum of $\mathcal{R}$ in $\mathcal{B}$.

Lemma 2. *The operator $\mathcal{R}$ is bounded in $\mathcal{B}$ and its spectrum is given by:*

$$\sigma(\mathcal{R}) = [-\nu, \nu]$$

First, $\mathcal{R}$ is bounded as the expectation operator is a bounded operator. The rest of the proof is based on two main arguments:

- The spectrum of $\mathcal{R}$ is symmetric convex.
- $$\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} = \nu$$

The second point ensures that $\rho(\mathcal{R}) = \nu$ by applying the Gelfand characterization of the spectral radius for an operator, see for instance Theorem 22 p.8 in Müller (2003), while the first point leads to the equality $\sigma(\mathcal{R}) = [-\nu, \nu]$.

First, we introduce operators $\mathcal{F}_i$, for $i \in \{1, \cdots N\}$, $\mathcal{F}$ and $\mathcal{L}$ on $\mathcal{B}$ defined by:

$$\mathcal{F}_i : \phi \mapsto ((s^t, \varepsilon^t) \mapsto \int_V \phi(is^t, \varepsilon\varepsilon^t) d\varepsilon$$

$$\mathcal{L} : \phi \mapsto ((s^t, \varepsilon^t) \mapsto \phi(s^{t-1}, \varepsilon^{t-1})$$

$$\mathcal{F}(\phi)(s^t, \varepsilon^t) = (p_{s_t 1}\mathcal{F}_1 + p_{s_t 2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t)$$

Operators $\mathcal{F}_i$ and $\mathcal{L}$ have the following straightforward properties.

1. $\mathcal{F}_i\mathcal{L} = \mathbb{1}$, and $\mathcal{F}\mathcal{L} = \mathbb{1}$
2. $|||\mathcal{F}_i||| = 1$ and $|||\mathcal{L}||| = 1$

where $|||\cdot|||$ is the triple norm associated with the infinite norm $\|\cdot\|_\infty$ on $\mathcal{B}$. Then $\mathcal{R}$ can be rewritten as:

$$\mathcal{R}(\phi)(s^t, \varepsilon^t) = \Gamma_{s_t}^{-1}(p_{s_t 1}\mathcal{F}_1 + p_{s_t 2}\mathcal{F}_2)(\phi)(s^t, \varepsilon^t)$$

We define $\tilde{\mathcal{R}}$ by

$$\tilde{\mathcal{R}} : \phi \mapsto \Gamma_{s_{t-1}}\mathcal{L}(\phi)(s^t, \varepsilon^t)$$

We have that:

$$\tilde{\mathcal{R}}\mathcal{R} = \mathcal{L}\mathcal{F}, \qquad \mathcal{R}\tilde{\mathcal{R}} = \mathbb{1}$$

We copy the techniques used to study the spectrum of isometries in Banach spaces such as that of Conway (1990). We refer to this publication and to Müller (2003) for the different types of spectrum. We know that the spectrum of $\mathcal{R}$ is a closed subset of $[-\|\mathcal{R}\|, \|\mathcal{R}\|]$, and that the boundary $\partial\sigma(\mathcal{R})$ of $\sigma(\mathcal{R})$ is included in the point approximate spectrum, i.e. the set of values λ such that $\mathcal{R} - \lambda\mathbb{1}$ is neither injective nor bounded below. We assume that $\sigma(\mathcal{R})$ is not convex, and that there exists $\lambda_0 \in (0, \nu)$ such that $\lambda \in \partial\sigma(\mathcal{R})$. We then prove that λ_0 is an eigenvalue. Actually, $\mathcal{R} - \lambda\mathbb{1}$ is bounded below for any $\lambda < \|\mathcal{R}\|$. $\mathcal{R}$ is the composition of an invertible operator and an isometry, and is thus bounded below. Moreover, we notice that:

$$\|\mathcal{R}\| = \sup_{v \in \mathrm{Im}(\tilde{\mathcal{R}})} \frac{\|\mathcal{R}v\|}{\|v\|} = \left(\inf_u \frac{\|\tilde{\mathcal{R}}u\|}{\|u\|}\right)^{-1}$$

which implies that:

$$\|u - \lambda\tilde{\mathcal{R}}u\| \leq (1 - \frac{\lambda}{\|\mathcal{R}\|})\|u\|$$

We show now that for any α such that $|\alpha| < 1$, then $\lambda\alpha$ belongs to $\sigma(\mathcal{R})$. We know that λ is an eigenvalue of $\mathcal{R}$, let $\phi_0 \in \mathcal{B}$ an eigenvector of $\mathcal{R}$ associated with λ,

$$\mathcal{R}\phi_0 = \lambda\phi_0$$

We define f by:

$$f = \phi_0 - \lambda\tilde{\mathcal{R}}\phi_0$$

We notice that $\mathcal{R}(f) = o$, and that $\|(\lambda\tilde{\mathcal{R}})^k(f)\| \leq \|\phi_0\|$. Fix α such that $|\alpha| < 1$. We define $\tilde{\phi}_0$ by :

$$\tilde{\phi}_0 = \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f)$$

We compute:

$$\mathcal{R}(\tilde{\phi}_0) = \sum_{k=0}^{\infty} \alpha^k \mathcal{R}(\lambda\tilde{\mathcal{R}})^k(f)$$

$$\mathcal{R}(\tilde{\phi}_0) = \alpha\lambda \sum_{k=0}^{\infty} \alpha^k (\lambda\tilde{\mathcal{R}})^k(f) = \alpha\lambda\tilde{\phi}_0$$

Thus, $\alpha\lambda$ is an eigenvalue of $\mathcal{R}$, which contradicts $\lambda \in \partial\sigma(\mathcal{R})$, and $\partial\sigma(\mathcal{R}) = \nu$.

As regards the second point, we first prove that $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \leq \nu$. We then construct, for any k, a function ϕ_k, such that:

$$\|\mathcal{R}^k(\phi_k)\|^{1/k} \geq \rho(S_k)^{1/k}$$

This construction is a generalization to the multivariate cases of Farmer et al. (2009a) and Farmer et al. (2010a).

We compute

$$\mathcal{R}^k(\phi)(s^t) = \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} A_{i_1} \cdots \Gamma_{i_{k-1}}^{-1} \mathcal{F}_{i_1} \cdots \mathcal{F}_{i_k}(\phi)(s^t)$$

We will find an upper bound and a lower bound for $\|\mathcal{R}^k\|$, in terms of a sequence (u_k) associated with well-chosen norms on $\mathcal{M}_n(\mathbb{R})$. First, we consider the triple norm associated with the infinite norm on $\mathcal{M}_n(\mathbb{R})$ and the associated sequence u_k. For any ϕ such that $\|\phi\|_\infty = 1$, we obtain by sub-additivity of the norm,

$$\|\mathcal{R}^k(\phi)\|_\infty \leq \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |||\Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1}||| = u_k^k$$

which leads to $\lim_{k\to+\infty} \|\mathcal{R}^k\|^{1/k} \leq \nu$.

Reciprocally, we consider on $\mathcal{M}_{r,s}(\mathbb{R})$ the norm $|\cdot|$ defined by:

$$|M| = \sum_{i,j} |m_{i,j}|, \quad \text{where} \quad M = [m_{i,j}]_{(i,j)\in\{1,\cdots,r\}\times\{1,\cdots,s\}}$$

This norm satisfies:

- $|M| \leq r|||M|||_\infty$
- If we write $M = [M_1, M_2, \cdots, M_l]$ by blocks, we notice the following useful property:

$$|M| = \sum_{i=1}^{l} |M_i|$$

Fix $s_t \in \{1, \cdots, N\}$ and let us denote by $\{w_{i_1\cdots i_{k+1}}, \forall(i_1 \cdots i_{k+1} \in \{1, \cdots, N\})\}$ a family of $n \times 1$ vectors and rewrite the following sum as a product of matrices by blocks:

$$\sum_{(i_1,\cdots,i_k)\in\{1,\cdots,N\}^k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} A_{s_t} A_{i_1} \cdots A_{i_{k-1}} w_{s_t i_1 \cdots i_{k-1}}$$

$$= \left[\begin{array}{ccc} p_{s_t 1} \cdots p_{11} [\Gamma_{s_t}^{-1} \cdots \Gamma_1^{-1}] & \cdots & | p_{s_t N} \cdots p_{NN} [\Gamma_{s_t}^{-1} \cdots \Gamma_N^{-1}] \end{array} \right] \times \left[\begin{array}{c} w_{s_t 1\cdots 1} \\ \vdots \\ w_{s_t N \cdots N} \end{array} \right]$$

Thus,

$$\sup_{\|w_{i_1\cdots i_p}\|_\infty \leq 1} \| \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} A_{s_t} A_{i_1} \cdots A_{i_{k-1}} w_{i_1 \cdots i_p} \|_\infty$$

$$= \sup_{\|w_{i_1\cdots i_p}\|_\infty \leq 1} \| \left[\begin{array}{ccc} p_{s_t 1} \cdots p_{11} [\Gamma_{s_t}^{-1} \cdots \Gamma_1^{-1}] & \cdots & | p_{s_t N} \cdots p_{NN} [\Gamma_{s_t}^{-1} \cdots \Gamma_N^{-1}] \end{array} \right] \times \left[\begin{array}{c} w_{s_t 1\cdots 1} \\ \vdots \\ w_{s_t N \cdots N} \end{array} \right] \|_\infty$$

$$= ||| \left[\begin{array}{ccc} p_{s_t 1} \cdots p_{11} [\Gamma_{s_t}^{-1} \cdots \Gamma_1^{-1}] & \cdots & | p_{s_t N} \cdots p_{NN} [\Gamma_{s_t}^{-1} \cdots \Gamma_N^{-1}] \end{array} \right] |||_\infty$$

$$\geq \frac{1}{Nn} | \left[\begin{array}{ccc} p_{s_t 1} \cdots p_{11} [\Gamma_{s_t}^{-1} \cdots \Gamma_1^{-1}] & \cdots & | p_{s_t N} \cdots p_{NN} [\Gamma_{s_t}^{-1} \cdots \Gamma_N^{-1}] \end{array} \right] |$$

$$\geq \frac{1}{Nn} \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |A_{s_t} A_{i_1} \cdots A_{i_{k-1}}|$$

Furthermore, as the considered space is a bounded subset of finite-dimensional vectorial space, the supremum is reached and there exist N^k vectors $(w_{s_t i_1 \cdots i_{k-1}})$ for $(i_1, \cdots, i_{k-1}) \in \{1, \cdots, N\}^{k-1}$ such that:

$$\|\sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} \Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1} w_{i_1 \cdots i_p}\|$$
$$\geq \frac{1}{Nn} \sum_{i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |\Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1}|$$

We define the function ϕ_0 by:$\phi_0(s^t) = w_{s_t s_{t-1} s_{t-2} \cdots s_{t-k}}$. This function is bounded and of norm 1. Moreover, ϕ_0 satisfies:

$$\sum_{s_t} \|\mathcal{R}^k(\phi_0)(s^t)\| \geq \frac{1}{Nn} \sum_{s_t, i_1,\cdots,i_k} p_{s_t i_1} p_{i_1 i_2} \cdots p_{i_{k-1} i_k} |\Gamma_{s_t}^{-1} \Gamma_{i_1}^{-1} \cdots \Gamma_{i_{k-1}}^{-1}| = \frac{1}{Nn} (\tilde{u}_k)^k$$

which leads to:

$$\|\mathcal{R}^k(\phi_0)\|_\infty \geq \frac{1}{N^2 n} (\tilde{u}_k)^k$$

Finally, this implies that:

$$\|\mathcal{R}^k\|^{1/k} \geq (N^2 n)^{-1/k} \tilde{u}_k$$

Taking the limit, we obtain $\lim_{k \to +\infty} \|\mathcal{R}^k\|^{1/k} \geq \nu$. This ends the proof of Lemma 2.

C.3 Proof of Proposition 1

A consequence of Lemma 2 is that $1 \in \sigma(\mathcal{R})$ if and only if $\nu \geq 1$, and thus, $(\mathbb{1} - \mathcal{R})$ is invertible if and only if $\nu < 1$, which proves Proposition 1.

D Asymptotic behavior of u_p

For computational reasons, it is often quicker to compute eigenvalues rather than sums with an increasing number of terms.

Let us start with the univariate case. In this case, the sequence u_p^p is the sum of the terms of a matrix S^p where S is defined as follows:

$$S = \left(p_{ij} \|\Gamma_i^{-1}\| \right)_{ij}.$$

Actually, matrices Γ_i^{-1} reduce to scalars in this case, and hence, are commutative. Thus, for univariate model, u_p^p behaves as $\|S^p\|$ and limit ν is equal to the spectral radius of S, $\rho(S)$. Farmer et al. (2009a) find a comparable result in the specific context of the Fisherian model of inflation determination.

In the general case, we introduce matrix S_k.

$$S_k = \left(\sum_{(i_1,\cdots,i_{k-1})\in\{1,\cdots,N\}^{k-1}} p_{ii_1}\cdots p_{i_{k-1}j} \|\Gamma_i^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{k-1}}^{-1}\| \right)_{ij}.$$

For any k, an (i,j) element of matrix S_k corresponds to an upper bound of the expected impact (expressed as a norm) of the endogenous variables along trajectories from regime i to regime j in k steps weighted by the probability of each trajectory.

The following result links the behavior of u_p to that of the spectral radius $\rho(S_p)$.

Lemma 3. *Sequence* $\left(\rho(S_p)^{1/p}\right)$ *is equivalent to* (u_p) *when* p *tends to* ∞.

We now consider norm $|\cdot|_\infty$ on $\mathcal{M}_2(\mathbb{R})$ defined by $|M|_\infty = \sum_{i,j} |m_{ij}|$. One may observe that:

$$|S_p|_\infty = \sum_{i,i_1,\cdots,i_{p-1},j} p_{ii_1}\cdots p_{i_{p-1}j} \|\Gamma_i^{-1}\Gamma_{i_1}^{-1}\cdots\Gamma_{i_{p-1}}^{-1}\| = u_{p-1}^{p-1} \tag{26}$$

As the spectral radius is the infimum of matricial norms, Equation 26 leads to:

$$\rho(S_p) \le u_{p-1}^{p-1} \tag{27}$$

Furthermore,

$$\left(S_p^q\right)_{ij} = \sum_{i_1,\cdots,i_{p-1},i_p,i_{p+1},\cdots,i_{2p},\cdots,i_{p(q-1)+1},\cdots,i_{pq-1}} p_{ii_1}\cdots p_{i_{p-1}i_p} p_{i_{(q-1)p}i_{(q-1)p+1}}\cdots p_{i_{qp-1}j}\times$$
$$\|\Gamma_i^{-1}\cdots\Gamma_{i_{p-1}}^{-1}\|\cdots\|\Gamma_{i_{(q-1)p}}^{-1}\cdots\Gamma_{i_{qp-1}}^{-1}\|$$

And using the sub-multiplicativity of matricial norms:

$$\left(S_p^q\right)_{ij} \ge \sum_{i_1,\cdots,i_{p-1},i_p,i_{p+1},\cdots,i_{2p},\cdots,i_{p(q-1)+1},\cdots,i_{pq-1}} p_{ii_1}\cdots p_{i_{p-1}i_p} p_{i_{(q-1)p}i_{(q-1)p+1}}\cdots p_{i_{qp-1}j}\times$$
$$\|\Gamma_i^{-1}\cdots\Gamma_{i_{p-1}}^{-1}\cdots\Gamma_{i_{(q-1)p}}^{-1}\cdots\Gamma_{i_{qp-1}}^{-1}\|$$

and hence,

$$|S_p^q|_\infty \ge u_{pq-1}^{pq-1} \tag{28}$$

Equation 27 can be rewritten as follows:

$$|S_p^q|_\infty^{(1/q)} \ge (u_{pq-1})^{p-1/q}$$

For any norm, Gelfand's Theorem shows that $\lim_{q\to\infty} ||X^q||_\infty^{(1/q)} = \rho(X)$. Thus, when q tends to infinity, 27 leads to:

$$\lim_{k\to\infty} u_k^p \leq \rho(S_p)$$

Thus, as $p > 1$,

$$\lim_{k\to\infty} u_k \leq \rho(S_p)^{1/p} \tag{29}$$

Combining Equations (27) and (29), we find the following upper and lower bounds:

$$\lim_{k\to\infty} u_k \leq \rho(S_p)^{1/p} \leq u_{p-1}^{1-1/p}$$

and thus, $\left(\rho(S_p)^{1/p}\right)$ is convergent and has the same limit as (u_p).

E Proof of Proposition 2

Proposition 2 follows directly from Equation (29).

To prove Proposition 2, we notice that

$$u_k^k = \sum_{(i_1,\cdots,i_k,i_{k+1})\in\{1,\cdots,N\}^k} p_{i_1 i_2}\cdots p_{i_{k-1}i_k} p_{i_k i_{k+1}} \|\Gamma_{i_1}^{-1}\cdots\Gamma_{i_k}^{-1}\|$$

Then by considering the multiples of p ($k = np$) and by only keeping the diverging trajectory (the hypothesis of the Lemma), we can rewrite the above equation as follows:

$$u_{np}^{np} \geq [p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\|]^n$$

and hence,

$$u_{np} \geq [p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\|]^{1/p}$$

Besides,

$$[p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \|\Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1}\|]^{1/p} \geq \rho(p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1})$$

Thus,

$$\lim_{n\to\infty} u_{np} \geq \rho(p_{i_0 i_1} p_{i_1 i_2}\cdots p_{i_{p_0} i_0} \Gamma_{i_0}^{-1}\cdots\Gamma_{i_{p_0}}^{-1})$$

The right-hand-side of the inequality is larger than one by hypothesis which implies that $\nu > 1$.

F Different classes of equilibria

F.1 Markovian equilibria

In this section, we define the assumptions under which Proposition 1 of Davig and Leeper (2007) is valid, and thus explain the debate between Davig and Leeper (2007) and Farmer et al. (2010a). The authors restrict the solution space to bounded Markovian solutions.

Definition 3. *A solution, z_t, is said to be Markovian of order p, if it depends on the p past regimes, $\{s_t, s_{t-1}, \cdots, s_{t-p}\}$, and all past shocks, ε^t, i.e. there exists a measurable function, ϕ, mapping $\{1, \cdots, N\}^{p+1} \times (\mathbb{R}^p)^\infty$ into $\mathbb{R}^n$ such that $z_t = \phi(\{s_t, s_{t-1}, \cdots, s_{t-p}\}, \varepsilon^t)$.*

This definition is a generalization of what Branch et al. (2007) call a "regime dependent equilibrium". In this paper, the authors define a regime dependent equilibrium as a solution depending on the current regime only. In our terminology, such an equilibrium is a 0-order Markovian solution. Davig and Leeper (2007) consider 0-order Markovian solution, whereas the counter-example provided by Farmer et al. (2010a) does not belong to this solution space.

We introduce a matrix, $\mathbf{M}$, as a combination between transition probabilities matrix, P, and a $(n.N \times n.N)$ real matrix, diagonal by blocks, $\text{diag}(\Gamma_1, \cdots, \Gamma_N)$: $\mathbf{M} = (P \otimes \mathbb{1}_n) \times \text{diag}(\Gamma_1, \cdots, \Gamma_N)$. The mathematical symbol $\otimes$ denotes the standard Kronecker product. Proposition 2 refines and complements the main Proposition by Davig and Leeper (2007).

Proposition 5. 1. *Model (11) admits a unique Markovian bounded solution, z_t^0, if and only if the spectral radius of $\mathbf{M}$, i.e. the largest eigenvalue in absolute value, $\rho(\mathbf{M})$, is strictly less than one.*

2. *Otherwise, all the bounded Markovian equilibria can be put into the following form:*

$$z_t = z_t^0 + V_{s_t} w_t, \quad \text{where,} \quad w_t = J_w w_{t-1} + \xi_t,$$

with, ξ_t being any bounded martingale ($\mathbb{E}_t \xi_{t+1} = 0$) independent of current and past regimes. The martingale can be either a sunspot shock as defined in Cass and Shell (1983) or a fundamental disturbance. Matrices J_w and V_{s_t} are defined in equations (32) and (33).

Proving the first point is done in two steps:

- If $\phi \in \mathcal{M}$ is solution of Equation (11), then $\phi \in \mathcal{M}_0$

- Furthermore if $\phi \in \mathcal{M}$, then defining $\mathbf{\Phi}$ by:

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1s^{t-1}, \varepsilon^t) \\ \vdots \\ \phi(Ns^{t-1}, \varepsilon^t) \end{bmatrix}$$

  $\mathbf{\Phi}$ is the solution of a linear rational expectations model with regime-independent parameters. We can thus apply Blanchard and Kahn (1980) technique.

Assuming that there exists a p-order Markovian solution of (11), ϕ, we define $\mathcal{P}(q)$ as the statement that the solution only depends on the past q regimes:

$$\mathcal{P}(q): \qquad \phi(is_1 \cdots s_q w, \varepsilon^t) = \phi(is_1 \cdots s_q w', \varepsilon^t)$$

$$\forall (s_1, \cdots, s_q) \in \{1, \cdots, N\}^q, \qquad \forall w \in \{1, \cdots, N\}^\infty, \qquad \forall w' \in \{1, \cdots, N\}^\infty, \qquad \forall \varepsilon^t \in V^\infty$$

$\mathcal{P}(p)$ is satisfied by assumption. Let us assume that $\mathcal{P}(q)$ is met for $q \in \{1, \cdots, p\}$. Since ϕ is a solution of (11), for any w, we compute:

$$\phi(s_t s_1 \cdots s_{q-1} w, \varepsilon^t) = -\Gamma_{s_t}^{-1}(\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w, \varepsilon \varepsilon^t) d\varepsilon - \Gamma_{s_t}^{-1} C_{s_t} \varepsilon_t$$

Due to $\mathcal{P}(q)$, we know that:

$$-\Gamma_{s_t}^{-1}(\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w, \varepsilon \varepsilon^t) d\varepsilon = -\Gamma_{s_t}^{-1}(\sum_i p_{s_t i} \int \phi(is_t s_1 \cdots s_{q-1} w', \varepsilon \varepsilon^t) d\varepsilon$$

for any w', and hence, ϕ does not depend on w. $\mathcal{P}(q-1)$ is thus satisfied. By decreasing induction we eventually show that ϕ is Markovian of order 0.

More generally if the solution is Markovian, its order is the same as ψ_0. Here, ψ_0 is Markovian of order 0, thus ϕ is also Markovian of order 0.

If $\phi \in \mathcal{M}_0$ is a solution of 11, ϕ is a solution of:

$$\forall i \in \{1, \cdots, N\} \qquad \phi(i, \varepsilon^t) + \Gamma_i^{-1} \left(p_{i1} \int \phi(1, \varepsilon \varepsilon^t) d\varepsilon + p_{i2} \int \phi(2, \varepsilon \varepsilon^t) d\varepsilon \right) = -\Gamma_i^{-1} C_i \varepsilon_t$$

We consider

$$\mathbf{\Phi}(\varepsilon^t) = \begin{bmatrix} \phi(1, \varepsilon^t) \\ \phi(2, \varepsilon^t) \\ \vdots \\ \phi(N, \varepsilon^t) \end{bmatrix} \tag{30}$$

Thus, by introducing $\mathbf{D} = -\begin{bmatrix} \Gamma_1^{-1} C_1 \\ \vdots \\ \Gamma_N^{-1} C_N \end{bmatrix}$, the system is rewritten as :

$$\mathbf{\Phi}(\varepsilon^t) + \mathbf{M} \int \mathbf{\Phi}(\varepsilon \varepsilon^t) d\varepsilon = D\varepsilon_t \tag{31}$$

where $\mathbf{\Phi}$ is defined in Equation (30). Model (31) is a standard linear rational expectations model with constant parameters. We hence easily prove Proposition 2 by applying the Blanchard and Kahn (1980) technique.

We denote by $\mathcal{B}_0$ the set of bounded functions on V^∞, and by $\mathcal{F}$ the bounded operator acting in $\mathcal{B}_0$:

$$\mathcal{F} : \mathbf{\Phi} \mapsto \left((\varepsilon^t) \mapsto \int \mathbf{\Phi}(\varepsilon \varepsilon^t) d\varepsilon \right)$$

We rewrite equation (31) as: $[\mathbb{1} + \mathbf{M}\mathcal{F}]\mathbf{\Phi} = \mathbf{\Psi_0}$, where $\mathbf{\Psi_0}(\varepsilon_t) = \mathbf{D}\varepsilon_t$. The solution $\mathbf{\Phi}$ is then: $\mathbf{\Phi} = \sum_{k=0}^{\infty} [-\mathbf{M}\mathcal{F}]^k \mathbf{\Psi_0}$. Knowing that:

$$(\mathcal{F}^k \mathbf{\Psi_0})(\varepsilon_t) = \mathbf{D}\mathbb{E}_t \varepsilon_{t+k}$$

The solution is then given by:

$$\mathbf{\Phi}(\varepsilon^t) = -\sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D} \mathbb{E}_t \varepsilon_{t+k}$$

thus,

$$\phi(s_t, \varepsilon^t) = U_{s_t} \sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D} \mathbb{E}_t \varepsilon_{t+k}$$

where

$$\begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_N \end{bmatrix} = -\mathbb{1}$$

Defining $z_t^0 = U_{s_t} \sum_{k=0}^{\infty} \mathbf{M}^k \mathbf{D} \mathbb{E}_t \varepsilon_{t+k}$, we notice that the solution z_t^0 only depends on s_t and ε_t. In existing literature, z_t^0 is called the fundamental or the Minimum State Variable solution.

In the case where $\rho(\mathbf{M}) > 1$, we apply the strategy of solving linear rational expectations models (see Blanchard and Kahn (1980) and Lubik and Schorfheide (2004)). There exists an invertible matrix Q such that

$$\mathbf{M} = \mathbf{Q} \times \begin{bmatrix} \Delta_u & R_u \\ 0 & \Delta_s \end{bmatrix} \mathbf{Q}^{-1}$$

with $\rho(\Delta_u) > 1$ and $\rho(\Delta_s) < 1$. Writing $Z_t = Q \begin{bmatrix} Z_t^u \\ Z_t^s \end{bmatrix}$, Z^s and Z^u are such that:

$$Z_t^s = \sum_{k=0}^{\infty} (-\Delta_s)^k [\ 0 \quad \mathbb{1}\ ] QC\mathbb{E}_t \varepsilon_{t+k}$$

$$\Delta_u \mathbb{E}_t Z_{t+1}^u + R_u \mathbb{E}_t Z_{t+1}^s + Z_t^u = [\ \mathbb{1} \quad 0\ ] QC\varepsilon_t$$

A general solution of the previous equation is then

$$Z_t^u = Z_t^{u,0} + \sum_{k=0} (-\Delta_u)^{-k} \xi_{t-k}$$

where ξ_t is any martingale and solution $Z_t^{u,0}$ is such that:

$$Z_t^{u,0} = \sum_{k=0}^{\infty} (-\Delta_u)^{-k} [\ \mathbb{1} \quad 0\ ] QC\varepsilon_{t-k-1} - \sum_{k=0}^{\infty} (-\Delta_u)^{-k} Z_{t-k}^s$$

Then, the solutions are given by

$$Z_t = Z_t^0 - Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} (-\Delta_u)^{-k} \xi_{t-k}$$

And finally,

$$z_t = z_t^0 + V_{s_t} w_t$$

with

$$V_1 = -[\ \mathbb{1} \quad 0\ ] Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \text{ and } V_2 = -[\ 0 \quad \mathbb{1}\ ] Q \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \tag{32}$$

w satisfies:

$$w_t = (-\Delta_u)^{-1} w_{t-1} + \xi_t$$

Defining J_w by

$$J_w = (-\Delta_u)^{-1} \tag{33}$$

leads to the result.

F.2 Non-Markovian equilibria : a proof of Proposition 3

For $\alpha = [\alpha(i_0, i_1, \cdots, i_{q-1}, i_q)]_{(i_0, i_1, \cdots, i_{q-1}, i_q) \in \{1, \cdots, N\}^{q+1}}$, we denote by $K(\alpha)$ the matrix introduced in Proposition 3 as follows:

$$K(\alpha) = \left[\sum_{(i_1, \cdots, i_{q-1}) \in \{1, N\}^{q-1}} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{q-1} j} \Gamma_i^{-1} \cdots \Gamma_{i_{q-1}}^{-1} \alpha(i, i_1, \cdots, i_{q-1}, j) \right]_{(i,j)}. \tag{34}$$

The assumptions of Proposition 3 imply that, for a certain integer q, there exist N^{q+1} real numbers in the (interior of the) unit disk, $\alpha(i_0, \cdots, i_q)$, and a $(nN \times 1)$ column vector, U, satisfying:

$$K(\alpha)U = U, \tag{35}$$

Then, we introduce, for $k \in \{0, \cdots, q\}$ the $(k+1)^N$ vectors $V(s_{t-k}, \cdots, s_t)$. For $k = q$, $V(s_{t-q}, \cdots, s_t)$ satisfies:

$$V(s_{t-q}, \cdots, s_t) = \alpha(s_{t-q}, \cdots, s_t)U_{s_t} \tag{36}$$

and, for k from $q-1$ to 0, $V(s_{t-k}, \cdots, s_t)$ is defined by backward induction by

$$\Gamma_{s_t} V(s_{t-k}, \cdots, s_t) = p_{s_t 1} V(s_{t-k}, \cdots, s_t, 1) + p_{s_t 2} V(s_{t-k}, \cdots, s_t, 2) \tag{37}$$

By construction, we see that

$$V(s_t) = U_{s_t}$$

Now, we construct some specific solutions.

Lemma 4. *Under the assumptions of Proposition 3, a unique Markovian stable equilibrium co-exists with multiple stable cyclical equilibria. An example of such bounded equilibria is, for any given t_0.*

$$\begin{cases} z_t = z_t^0, \text{ for } t < t_0 \\ z_t = z_t^0 + w_t, \text{ for } t \geq t_0 \\ w_t = \frac{V'(s_{t-1-(t-1-t_0)[q]}, \cdots, s_{t-1})w_{t-1}}{V'(s_{t-1-(t-1-t_0)[q]}, \cdots, s_{t-1})V(s_{t-1-(t-1-t_0)[q]}, \cdots, s_{t-1})} V(s_{t-(t-t_0)[q]}, \cdots, s_t) + V(s_{t-(t-t_0)[q]}, \cdots, s_t)\xi_t, \end{cases}$$

where $(t-t_0)[q]$ represents the rest of the division of $(t-t_0)$ by q. Vectors V are defined in equations (36), ξ_t is any bounded real-valued martingale independent of s^t.

Lemma 4 gives the explicit form of sunspots and thus implies Proposition 3. Moreover, we notice that for any $\lambda \in]1, \frac{1}{\max(\alpha)}[$, matrix $K(\lambda\alpha)$ has an eigenvalue larger than 1, thus according to (36), there exists a continuum of solutions w.

To prove Lemma 4, we have to check that w is a solution.

We first notice that by construction, w_t is collinear with $V(s_{t-(t-t_0)[q]}, \cdots, s_t)$, for any $t \geq t_0$. Moreover, we notice that $(t-t_0)[q]$ belongs to $\{0, \cdots, q-1\}$, thus:

$$\mathbb{E}_t(w_{t+1}) = \left(\frac{V'(s_{t-(t-t_0)[q]}, \cdots, s_t)w_t}{V'(s_{t-(t-t_0)[q]}, \cdots, s_t)V(s_{t-(t-t_0)[q]}, \cdots, s_t)} + \mathbb{E}_t\xi_{t+1}\right)\mathbb{E}_t V(s_{t+1-(t+1-t_0)[q]}, \cdots, s_{t+1})$$

We know that $\mathbb{E}_t(\xi_{t+1}) = 0$, and according to equation (37), that

$$\mathbb{E}_t V(s_{t+1-(t+1-t_0)[q]}, \cdots, s_{t+1}) = \Gamma_{s_t} V(s_{t+1-(t+1-t_0)[q]}, \cdots, s_t)$$

And finally:

$$\mathbb{E}_t(w_{t+1}) = \frac{V'(s_{t-(t-t_0)[q]}, \cdots, s_t) w_t}{V'(s_{t-(t-t_0)[q]}, \cdots, s_t) V(s_{t-(t-t_0)[q]}, \cdots, s_t)} \Gamma_{s_t} V(s_{t+1-(t+1-t_0)[q]}, \cdots, s_t) = \Gamma_{s_t} w_t$$

We represent in Figure 2, the different determinacy regions, for $q = 1$ and $q = 6$ for the calibration chosen in Davig and Leeper (2007).

F.3 The solutions of Farmer et al. (2009b)

Farmer et al. (2009b) focus on equilibria that can be put under the following form:

$$z_t = z_t^0 + w_t,$$

$$w_t = \Lambda_{s_{t-1}s_t} w_{t-1} + V_{s_t} V'_{s_t} \xi_t$$

This class of equilibria is a subclass of our solutions in Lemma 4 and Proposition 3, for $q = 1$.

The result of Farmer et al. (2009b) can be extended in the following Proposition.

Proposition 6. *If $\rho(\mathbf{M}) \neq 1$, any solution of model (11) can then be written as:*

$$z_t = z_t^0 + V_{s_t} V'_{s_t} w_t,$$

$$w_t = V_{s_t} \Phi_{s_t, \cdots, s_{t-p}} V'_{s_{t-p}} w_{t-p} + \eta_t,$$

where η_t is a martingale when $p = 1$ and satisfies the following property otherwise:

$$\mathbb{E}_{t-p} \prod_{i=t-p}^{t-1} \Gamma_{s_i}^{-1} \eta_t = 0.$$

$\Phi_{s_t, \cdots, s_{t-p}}$ and V_{s_t} satisfy:

$$\Gamma_{s_t} V_{s_t} = \sum_{s_{t+1} \ldots s_{t+p}} p_{s_t s_{t+1}} \ldots p_{s_{t+p-1} s_{t+p}} \Gamma_{s_{t+1}}^{-1} \ldots \Gamma_{s_{t+p}}^{-1} V_{s_{t+p}} \Phi_{s_t \ldots s_{t+p}} \tag{38}$$

The stability of processes described in Proposition 6 is given by the maximum eigenvalue of all the possible products involving $V_{s_t} \Phi_{s_t, \cdots, s_{t-p}} V'_{s_{t-p}}$ as already mentioned in Farmer et al. (2009b). This maximum is known in mathematics as the joint spectral radius. The joint spectral radius of a set of matrices $\mathcal{M} = \{M_1, \cdots M_N\}$ is defined by:

$$\overline{\rho}(\mathcal{M}) = \lim_{t \to +\infty} \max\{\|M_{i_1} \cdots M_{i_t}\|, \quad , (M_{i_1}, \cdots, M_{i_t}) \in \mathcal{M}^t\} \tag{39}$$

We thus have the following corollary of Proposition 6:

Corollary 1. *If there exists $p > 0$ such that matrices $\Phi_{\{.\}}$ and $V_{\{.\}}$ satisfy equations 38 and*

$$\overline{\rho}\left(\{V_{i_1}\Phi_{i_1,\cdots,i_{p+1}}V'_{i_{p+1}},(i_1,\cdots,i_{p+1})\in\{1,\cdots,N\}^{p+1}\}\right)<1 \tag{40}$$

Then, there exists at least one non-Markovian bounded solution to model (11).

In Farmer et al. (2009b), the authors give the determinacy conditions for first-order Markov-switching Vectorial Autoregressive processes, i.e. when for $p = 1$ the solution is stable (mean square stable in their paper, but that is not key) and η_t only depends on ε^t but is independent of current and past regimes.

However, there exists a region of parameters for which there is no such a stable solution but indeterminacy. That may be the case when for $p = 1$, η_t depends on past regimes and is unbounded as well as the autoregressive process. Indeed, the sum of two unbounded processes can be bounded. This proves that determinacy region provided by the application of Farmer et al. (2009b) is too large.

Proposition 6 combined with Corollary 1 theoretically provides a way of refining the region of indeterminacy. However, it requires finding all the possible matrices $\Phi_{\{.\}}$ and $V_{\{.\}}$ that satisfy equations 38 which is computationally expensive.

The proof of Proposition 6 consists in extending the strategy of Farmer et al. (2009b). Let us consider a solution to model 11, z_t. We define w_t as the difference between this solution and the solution z_t^0. w_t is thus a solution of the homogenous model:

$$\Gamma_{s_t}w_t = \mathbb{E}_t w_{t+1}, \tag{41}$$

We denote by V_{s_t} a $(n \times k_{s_t})$ matrix whose column vectors are a base of the linear space spanned by w_t in regime s_t.

Let us consider an element, w_k of V_i ($k \leq k_{s_t}$) solution of the homogenous model (41).

$$\begin{aligned}
\Gamma_i w_k &= \mathbb{E}(\Gamma_{s_t} w_t | s_t = i, w_t = w_k) && (42)\\
&= \mathbb{E}(\mathbb{E}_t(w_{t+1}) | s_t = i, w_t = w_k) && (43)\\
&= \mathbb{E}(\sum_{s_{t+1}} p_{s_t s_{t+1}} \mathbb{E}_t(w_{t+1} | s_{t+1}) | s_t = i, w_t = w_k) && (44)\\
&= \sum_{s_{t+1}} p_{s_t s_{t+1}} \mathbb{E}(w_{t+1} | s_t = i, w_t = w_k, s_{t+1}) && (45)\\
&= \sum_{s_{t+1}} p_{s_t s_{t+1}} E(\Gamma_{s_{t+1}}^{-1} \mathbb{E}_{t+1} w_{t+2} | s_t = i, w_t = w_k, s_{t+1}) && (46)\\
&= \sum_{s_{t+1}} p_{s_t s_{t+1}} \Gamma_{s_{t+1}}^{-1} \mathbb{E}(\mathbb{E}_{t+1} w_{t+2} | s_t = i, w_t = w_k, s_{t+1}) && (47)\\
&= \sum_{s_{t+1}, s_{t+2}} p_{s_t s_{t+1}} p_{s_{t+1} s_{t+2}} \Gamma_{s_{t+1}}^{-1} \mathbb{E}(w_{t+2} | s_t = i, w_t = w_k, s_{t+1}, s_{t+2}) && (48)\\
& && (49)
\end{aligned}$$

And by recurrence,

$$\Gamma_i w_k = \sum_{s_{t+1}, \ldots s_{t+p}} p_{s_t s_{t+1}} \ldots p_{s_{t+p-1} s_{t+p}} \Gamma_{s_{t+1}}^{-1} \ldots \Gamma_{s_{t+p}}^{-1} \mathbb{E}(w_{t+p} | s_t = i, w_t = w_k, s_{t+1}, s_{t+2}, s_{t+p})$$

As $(w_{t+p} | s_t = i, w_t = w_k, s_{t+1}, s_{t+2}, s_{t+p})$ is in $V_{s_{t+p}}$, there exists a matrix such that:

$$\mathbb{E}(w_{t+p} | s_t = k, w_t = w_i, s_{t+1}, s_{t+2}, s_{t+p}) = V_{s_{t+p}} \phi_{s_t \ldots s_{t+p}}^{\{k\}}$$

By doing so for any column vector of V_{s_t} we obtain:

$$\Gamma_{s_t} V_{s_t} = \sum_{s_{t+1}, \ldots s_{t+p}} p_{s_t s_{t+1}} \ldots p_{s_{t+p-1} s_{t+p}} \Gamma_{s_{t+1}}^{-1} \ldots \Gamma_{s_{t+p}}^{-1} V_{s_{t+p}} \Phi_{s_t \ldots s_{t+p}}$$

where $\Phi_{s_t \ldots s_{t+p}} = [\phi_{s_t \ldots s_{t+p}}^{\{1\}} \ldots \phi_{s_t \ldots s_{t+p}}^{\{k_{s_t}\}}]$.
Now, we define η_t, the difference between w_t and the autoregressive term: $V_{s_t} \Phi_{s_{t-p} \ldots s_t} V'_{s_{t-p}} w_{t-p}$.
We notice that this difference satisfies:

$$\mathbb{E}_{t-p} \prod_{i=t-p}^{t-1} \Gamma_{s_i}^{-1} \eta_t = 0.$$

References

Barthélemy, J. and M. Marx (2011): "State-Dependent Probability Distributions in Non Linear Rational Expectations Models," *Working Papers 347, Banque de France.*

Bianchi, F. (2013): "Regime Switches, Agents' Beliefs, and Post-World War II U.S. Macroeconomic Dynamics," *Review of Economic Studies*, 80, 463–490.

Blanchard, O. and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48.

Blondel, V., J. Theys, and A. Vladimirov (2001): "An Elementary Counterexample to the Finiteness Conjecture," *SIAM Journal of Matrix Analysis.*

Branch, W., T. Davig, and B. McGough (2007): "Expectational Stability in Regime-Switching Rational Expectations Models," *Federal Reserve Bank of Kansas City Research Working Paper.*

Cass, D. and K. Shell (1983): "Do Sunspots Matter?" *Journal of Political Economy*, 91, 193–227.

Cho, S. (2013): "Characterizing Markov-Switching Rational Expectations Models," *mimeo, School of Economics, Yonsei University.*

Clarida, R., J. Galí, and M. Gertler (2000): "Monetary Policy Rules And Macroeconomic Stability: Evidence And Some Theory," *The Quarterly Journal of Economics*, 115, 147–180.

Cochrane, J. (2011): "Determinacy and Identification with Taylor Rules," *Journal of Political Economy*, 119, 565 – 615.

Conway, J. (1990): *A Course in Functional Analysis*, Springer.

Costa, O., M. Fragoso, and R. Marques (2005): *Discrete-Time Markov Jump Linear Systems*, Springer.

Davig, T. and E. M. Leeper (2007): "Generalizing the Taylor Principle," *American Economic Review*, 97, 607–635.

——— (2010): "Generalizing the Taylor Principle: Reply," *American Economic Review*, 100, 618–624.

Farmer, R. E. A., D. F. Waggoner, and T. Zha (2009a): "Indeterminacy in a Forward-Looking Regime Switching Model," *International Journal of Economic Theory*, 5, 69–84.

——— (2009b): "Understanding Markov-switching rational expectations models," *Journal of Economic Theory*, 144, 1849–1867.

——— (2010a): "Generalizing the Taylor Principle: a Comment," *American Economic Review.*

——— (2010b): "Minimal State Variable Solutions to Markov-Switching Rational Expectations Models," *to appear in Journal of Economic Dynamics and Control.*

Foerster, A., J. Rubio-Ramirez, D. Waggoner, and T. Zha (2013): "Perturbation Methods for Markov-switching DSGE model," *Research Working Paper, Federal Reserve Bank of Kansas City.*

Jin, H. and K. Judd (2002): "Perturbation Methods for General Dynamic Stochastic Models," *Working Paper, Stanford University.*

Jungers, R. (2009): "The Joint Spectral Radius: Theory and Applications," *Lecture Notes in Control and Information Sciences, Springer-Verlag. Berlin Heidelberg.*, 385.

Jungers, R. and V. Protasov (2011): "Fast Algorithm for the p-radius Computation," *SIAM Journal on Scientific Computing*, 33(3), 1246–1266.

Leeper, E. M. (1991): "Equilibria under 'Active' and 'Passive' Monetary and Fiscal Policies," *Journal of Monetary Economics*, 27, 129–147.

Lubik, T. A. and F. Schorfheide (2004): "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review*, 94, 190–217.

Müller, V. (2003): "Spectral Theory of Linear Operators: and Spectral Systems in Banach Algebras," *Operator Theory, Advances and Applications, vol. 139. Birkhaüser.*

Ogura, M. and R. Jungers (2014): "Efficiently Computable Lower Bounds for the p-radius of Switching Linear Systems," *53rd IEEE Conference on Decision and Control (Los Angeles, USA, 2014), Proceedings of CDC 2014.*

Ogura, M., V. M. Preciado, and R. Jungers (2015): "Effcient method for computing lower bounds on the p-radius of switched linear systems," *preprint- arXiv:1503.03034*.

Sims, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20, 1–20.

Taylor, J. B. (1993): "Discretion versus Policy Rules in Practice," *Carnegie-Rochester Conference Series on Public Policy*, 39.

Theys, J. (2005): "Joint Spectral Radius: Theory and Approximations," *PhD Thesis, Center for Systems Engineering and Applied Mechanics, Université Catholique de Louvain.*

Woodford, M. (1986): "Stationary Sunspot Equilibria: The Case of Small Fluctuations around a Deterministic Steady State," *mimeo.*

——— (2003): "Interest and prices: Foundations of a Theory of Monetary Policy," *Princeton University Press.*