Productivity and US Macroeconomic Performance: Interpreting the Past and Predicting the Future with a Two-Sector Real Business Cycle Model

Peter N. Ireland
Boston College and NBER

Scott Schuh
Federal Reserve Bank of Boston

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1 The Model with Habit Persistence, Adjustment Costs for Labor, and Constant Depreciation and Capital Utilization Rates

The representative household has preferences described by the expected utility function

$$E\sum_{t=0}^{\infty}\beta^t[\ln(C_t-\gamma C_{t-1})-(H_{ct}+H_{it})/A_t].$$

The two-sector production structure is described by production function and adjustment cost specifications for consumption and investment goods,

$$\left[1-\frac{\phi_{hc}}{2}\left(\frac{H_{ct}}{H_{ct-1}}-\eta_c\right)^2\right]\left[1-\frac{\phi_{kc}}{2}\left(\frac{I_{ct}}{K_{ct}}-\kappa_c\right)^2\right]K_{ct}^{\theta_c}(Z_{ct}H_{ct})^{1-\theta_c}\geq C_t$$

and

$$\left[1-\frac{\phi_{hi}}{2}\left(\frac{H_{it}}{H_{it-1}}-\eta_i\right)^2\right]\left[1-\frac{\phi_{ki}}{2}\left(\frac{I_{it}}{K_{it}}-\kappa_i\right)^2\right]K_{it}^{\theta_i}(Z_{it}H_{it})^{1-\theta_i}\geq I_{ct}+I_{it},$$

as well as the capital accumulation constraints

$$(1-\delta_c)K_{ct}+I_{ct}\geq K_{ct+1}$$

and

$$(1-\delta_i)K_{it} + I_{it} \geq K_{it+1}.$$

2 Optimal Resource Allocations

Since the two welfare theorems apply, equilibrium resource allocations can be characterized by solving the social planner's problem: choose C_t, H_{ct}, H_{it}, I_{ct}, I_{it}, K_{ct+1}, and K_{it+1} for all $t = 0, 1, 2, ...$ to maximize the representative household's utility subject to the production possibility and capital accumulation constraints. Letting Λ_{ct} and Λ_{it} denote nonnegative Largrange multipliers on the production possibility constraints and Ξ_{ct} and Ξ_{it} denote nonnegative multipliers on the capital accumulation constraints, the first-order conditions for the planner's problem can be written as

$$\frac{1}{C_t - \gamma C_{t-1}} - \beta\gamma E_t \left(\frac{1}{C_{t+1} - \gamma C_t}\right) = \Lambda_{ct}, \tag{1}$$

$$\begin{aligned}\frac{1}{A_t} \quad = \quad & \frac{(1-\theta_c)\Lambda_{ct} C_t}{H_{ct}} - \phi_{hc}\Lambda_{ct}\left(\frac{H_{ct}}{H_{ct-1}} - \eta_c\right)\left(\frac{1}{H_{ct-1}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{I_{ct}}{K_{ct}} - \kappa_c\right)^2\right] K_{ct}^{\theta_c}(Z_{ct}H_{ct})^{1-\theta_c} \\ & +\beta\phi_{hc}E_t\left\{\Lambda_{ct+1}\left(\frac{H_{ct+1}}{H_{ct}} - \eta_c\right)\left(\frac{H_{ct+1}}{H_{ct}}\right)\left(\frac{1}{H_{ct}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{I_{ct+1}}{K_{ct+1}} - \kappa_c\right)^2\right] K_{ct+1}^{\theta_c}(Z_{ct+1}H_{ct+1})^{1-\theta_c}\right\},\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{1}{A_t} = \quad & \frac{(1-\theta_i)\Lambda_{it} I_t}{H_{it}} - \phi_{hi}\Lambda_{it}\left(\frac{H_{it}}{H_{it-1}} - \eta_i\right)\left(\frac{1}{H_{it-1}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{I_{it}}{K_{it}} - \kappa_i\right)^2\right] K_{it}^{\theta_i}(Z_{it}H_{it})^{1-\theta_i} \\ & +\beta\phi_{hi}E_t\left\{\Lambda_{it+1}\left(\frac{H_{it+1}}{H_{it}} - \eta_i\right)\left(\frac{H_{it+1}}{H_{it}}\right)\left(\frac{1}{H_{it}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{I_{it+1}}{K_{it+1}} - \kappa_i\right)^2\right] K_{it+1}^{\theta_i}(Z_{it+1}H_{it+1})^{1-\theta_i}\right\},\end{aligned} \tag{3}$$

$$\Xi_{ct} = \Lambda_{it} + \phi_{kc}\Lambda_{ct}\left[1 - \frac{\phi_{hc}}{2}\left(\frac{H_{ct}}{H_{ct-1}} - \eta_c\right)^2\right]\left(\frac{I_{ct}}{K_{ct}} - \kappa_c\right)\left(\frac{1}{K_{ct}}\right) K_{ct}^{\theta_c}(Z_{ct}H_{ct})^{1-\theta_c}, \tag{4}$$

$$\Xi_{it} = \Lambda_{it}\left\{1 + \phi_{ki}\left[1 - \frac{\phi_{hi}}{2}\left(\frac{H_{it}}{H_{it-1}} - \eta_i\right)^2\right]\left(\frac{I_{it}}{K_{it}} - \kappa_i\right)\left(\frac{1}{K_{it}}\right) K_{it}^{\theta_i}(Z_{it}H_{it})^{1-\theta_i}\right\}, \tag{5}$$

$$
\begin{aligned}
\Xi_{ct} &= \beta E_t[(1-\delta_c)\Xi_{ct+1}] + \beta\theta_c E_t\left(\frac{\Lambda_{ct+1}C_{t+1}}{K_{ct+1}}\right) \\
&+\beta\phi_{kc}E_t\left\{\Lambda_{ct+1}\left[1-\frac{\phi_{hc}}{2}\left(\frac{H_{ct+1}}{H_{ct}}-\eta_c\right)^2\right]\left(\frac{I_{ct+1}}{K_{ct+1}}-\kappa_c\right)\left(\frac{I_{ct+1}}{K_{ct+1}}\right)\left(\frac{1}{K_{ct+1}}\right)K_{ct+1}^{\theta_c}(Z_{ct+1}H_{ct+1})^{1-\theta_c}\right\},
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\Xi_{it} &= \beta E_t[(1-\delta_i)\Xi_{it+1}] + \beta\theta_i E_t\left(\frac{\Lambda_{it+1}I_{t+1}}{K_{it+1}}\right) \\
&+\beta\phi_{ki}E_t\left\{\Lambda_{it+1}\left[1-\frac{\phi_{hi}}{2}\left(\frac{H_{it+1}}{H_{it}}-\eta_i\right)^2\right]\left(\frac{I_{it+1}}{K_{it+1}}-\kappa_i\right)\left(\frac{I_{it+1}}{K_{it+1}}\right)\left(\frac{1}{K_{it+1}}\right)K_{it+1}^{\theta_i}(Z_{it+1}H_{it+1})^{1-\theta_i}\right\},
\end{aligned}
\tag{7}
$$

$$
C_t = \left[1-\frac{\phi_{hc}}{2}\left(\frac{H_{ct}}{H_{ct-1}}-\eta_c\right)^2\right]\left[1-\frac{\phi_{kc}}{2}\left(\frac{I_{ct}}{K_{ct}}-\kappa_c\right)^2\right]K_{ct}^{\theta_c}(Z_{ct}H_{ct})^{1-\theta_c}, \tag{8}
$$

$$
I_t = \left[1-\frac{\phi_{hi}}{2}\left(\frac{H_{it}}{H_{it-1}}-\eta_i\right)^2\right]\left[1-\frac{\phi_{ki}}{2}\left(\frac{I_{it}}{K_{it}}-\kappa_i\right)^2\right]K_{it}^{\theta_i}(Z_{it}H_{it})^{1-\theta_i}, \tag{9}
$$

$$
(1-\delta_c)K_{ct} + I_{ct} = K_{ct+1}, \tag{10}
$$

and

$$
(1-\delta_i)K_{it} + I_{it} = K_{it+1} \tag{11}
$$

for all $t = 0, 1, 2, ...$, where aggregate investment has been defined as

$$
I_t = I_{ct} + I_{it}, \tag{12}
$$

and aggregate hours worked can be defined similarly as

$$
H_t = H_{ct} + H_{it}. \tag{13}
$$

3 Driving Processes

The system gets closed by making assumptions about the stochastic behavior of the three shocks: A_t, Z_{ct}, and Z_{it}. Suppose, in particular, that each shock contains separate autoregressive components governing its level and its growth rate:

$$\ln(A_t) = \ln(a_t^l) + \ln(A_t^g), \tag{14}$$

$$\ln(a_t^l) = \rho_a^l \ln(a_{t-1}^l) + \varepsilon_{at}^l, \tag{15}$$

$$\ln(A_t^g/A_{t-1}^g) = (1-\rho_a^g)\ln(a^g) + \rho_a^g \ln(A_{t-1}^g/A_{t-2}^g) + \varepsilon_{at}^g, \tag{16}$$

$$\ln(Z_{ct}) = \ln(z_{ct}^l) + \ln(Z_{ct}^g), \tag{17}$$

$$\ln(z_{ct}^l) = \rho_c^l \ln(z_{ct-1}^l) + \varepsilon_{ct}^l, \tag{18}$$

$$\ln(Z_{ct}^g/Z_{ct-1}^g) = (1-\rho_c^g)\ln(z_c^g) + \rho_c^g \ln(Z_{ct-1}^g/Z_{ct-2}^g) + \varepsilon_{ct}^g, \tag{19}$$

$$\ln(Z_{it}) = \ln(z_{it}^l) + \ln(Z_{it}^g), \tag{20}$$

$$\ln(z_{it}^l) = \rho_i^l \ln(z_{it-1}^l) + \varepsilon_{it}^l, \tag{21}$$

and

$$\ln(Z_{it}^g/Z_{it-1}^g) = (1-\rho_i^g)\ln(z_i^g) + \rho_i^g \ln(Z_{it-1}^g/Z_{it-2}^g) + \varepsilon_{it}^g. \tag{22}$$

Suppose, in addition, that the innovations ε_{at}^l, ε_{at}^g, ε_{ct}^l, ε_{ct}^g, ε_{it}^l, and ε_{it}^g are serially and mutually uncorrelated and normally distributed with zero means and standard deviations σ_a^l, σ_a^g, σ_c^l, σ_c^g, σ_i^l, and σ_i^g.

4 The Stationary System

Equations (1)-(22) now describe the behavior of the 22 variables C_t, H_t, H_{ct}, H_{it}, I_t, I_{ct}, I_{it}, K_{ct}, K_{it}, Λ_{ct}, Λ_{it}, Ξ_{ct}, Ξ_{it}, A_t, a_t^l, A_t^g, Z_{ct}, z_{ct}^l, Z_{ct}^g, Z_{it}, z_{it}^l, and Z_{it}^g. At the optimum, many of these variables inherit unit roots from the nonstationary components of the shocks. However, the variables $c_t = C_t/[A_{t-1}^g(Z_{it-1}^g)^{\theta_c}(Z_{ct-1}^g)^{1-\theta_c}]$, $h_t = H_t/A_{t-1}^g$, $h_{ct} = H_{ct}/A_{t-1}^g$, $h_{it} = H_{it}/A_{t-1}^g$, $i_t = I_t/(A_{t-1}^g Z_{it-1}^g)$, $i_{ct} = I_{ct}/(A_{t-1}^g Z_{it-1}^g)$, $i_{it} = I_{it}/(A_{t-1}^g Z_{it-1}^g)$, $k_{ct} = K_{ct}/(A_{t-1}^g Z_{it-1}^g)$, $k_{it} = K_{it}/(A_{t-1}^g Z_{it-1}^g)$, $\lambda_{ct} = [A_{t-1}^g(Z_{it-1}^g)^{\theta_c}(Z_{ct-1}^g)^{1-\theta_c}]\Lambda_{ct}$, $\lambda_{it} = A_{t-1}^g Z_{it-1}^g \Lambda_{it}$, $\xi_{ct} = A_{t-1}^g Z_{it-1}^g \Xi_{ct}$, $\xi_{it} = A_{t-1}^g Z_{it-1}^g \Xi_{it}$, $a_t = A_t/A_{t-1}^g$, a_t^l, $a_t^g = A_t^g/A_{t-1}^g$, $z_{ct} = Z_{ct}/Z_{ct-1}^g$, z_{ct}^l, $z_{ct}^g = Z_{ct}^g/Z_{ct-1}^g$, $z_{it} = Z_{it}/Z_{it-1}^g$, z_{it}^l, and $z_{it}^g = Z_{it}^g/Z_{it-1}^g$ remain stationary, and the system can be rewritten in terms of these new variables as

$$\frac{1}{c_t - \gamma\{c_{t-1}/[a_{t-1}^g(z_{it-1}^g)^{\theta_c}(z_{ct-1}^g)^{1-\theta_c}]\}} - \beta\gamma E_t\left\{\frac{1}{[a_t^g(z_{it}^g)^{\theta_c}(z_{ct}^g)^{1-\theta_c}]c_{t+1} - \gamma c_t}\right\} = \lambda_{ct}, \tag{1}$$

$$\frac{1}{a_t} = \frac{(1-\theta_c)\lambda_{ct}c_t}{h_{ct}} - \phi_{hc}\lambda_{ct}\left(\frac{a^g_{t-1}h_{ct}}{h_{ct-1}} - \eta_c\right)\left(\frac{a^g_{t-1}}{h_{ct-1}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{i_{ct}}{k_{ct}} - \kappa_c\right)^2\right]k_{ct}^{\theta_c}(z_{ct}h_{ct})^{1-\theta_c}$$
$$+\beta\phi_{hc}E_t\left\{\lambda_{ct+1}\left(\frac{a^g_t h_{ct+1}}{h_{ct}} - \eta_c\right)\left(\frac{a^g_t h_{ct+1}}{h_{ct}}\right)\left(\frac{1}{h_{ct}}\right)\left[1 - \frac{\phi_{kc}}{2}\left(\frac{i_{ct+1}}{k_{ct+1}} - \kappa_c\right)^2\right]k_{ct+1}^{\theta_c}(z_{ct+1}h_{ct+1})^{1-\theta_c}\right\}, \tag{2}$$

$$\frac{1}{a_t} = \frac{(1-\theta_i)\lambda_{it}i_t}{h_{it}} - \phi_{hi}\lambda_{it}\left(\frac{a^g_{t-1}h_{it}}{h_{it-1}} - \eta_i\right)\left(\frac{a^g_{t-1}}{h_{it-1}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{i_{it}}{k_{it}} - \kappa_i\right)^2\right]k_{it}^{\theta_i}(z_{it}h_{it})^{1-\theta_i}$$
$$+\beta\phi_{hi}E_t\left\{\lambda_{it+1}\left(\frac{a^g_t h_{it+1}}{h_{it}} - \eta_i\right)\left(\frac{a^g_t h_{it+1}}{h_{it}}\right)\left(\frac{1}{h_{it}}\right)\left[1 - \frac{\phi_{ki}}{2}\left(\frac{i_{it+1}}{k_{it+1}} - \kappa_i\right)^2\right]k_{it+1}^{\theta_i}(z_{it+1}h_{it+1})^{1-\theta_i}\right\}, \tag{3}$$

$$\xi_{ct} = \lambda_{it} + \phi_{kc}\lambda_{ct}\left[1 - \frac{\phi_{hc}}{2}\left(\frac{a^g_{t-1}h_{ct}}{h_{ct-1}} - \eta_c\right)^2\right]\left(\frac{i_{ct}}{k_{ct}} - \kappa_c\right)k_{ct}^{\theta_c-1}(z_{ct}h_{ct})^{1-\theta_c}, \tag{4}$$

$$\xi_{it} = \lambda_{it}\left\{1 + \phi_{ki}\left[1 - \frac{\phi_{hi}}{2}\left(\frac{a^g_{t-1}h_{it}}{h_{it-1}} - \eta_i\right)^2\right]\left(\frac{i_{it}}{k_{it}} - \kappa_i\right)k_{it}^{\theta_i-1}(z_{it}h_{it})^{1-\theta_i}\right\}, \tag{5}$$

$$a^g_t z^g_{it}\xi_{ct} = \beta(1-\delta_c)E_t\xi_{ct+1} + \beta\theta_c E_t\left(\frac{\lambda_{ct+1}c_{t+1}}{k_{ct+1}}\right)$$
$$+\beta\phi_{kc}E_t\left\{\lambda_{ct+1}\left[1 - \frac{\phi_{hc}}{2}\left(\frac{a^g_t h_{ct+1}}{h_{ct}} - \eta_c\right)^2\right]\left(\frac{i_{ct+1}}{k_{ct+1}} - \kappa_c\right)\left(\frac{i_{ct+1}}{k_{ct+1}}\right)k_{ct+1}^{\theta_c-1}(z_{ct+1}h_{ct+1})^{1-\theta_c}\right\}, \tag{6}$$

$$a^g_t z^g_{it}\xi_{it} = \beta(1-\delta_i)E_t\xi_{it+1} + \beta\theta_i E_t\left(\frac{\lambda_{it+1}i_{t+1}}{k_{it+1}}\right)$$
$$+\beta\phi_{ki}E_t\left\{\lambda_{it+1}\left[1 - \frac{\phi_{hi}}{2}\left(\frac{a^g_t h_{it+1}}{h_{it}} - \eta_i\right)^2\right]\left(\frac{i_{it+1}}{k_{it+1}} - \kappa_i\right)\left(\frac{i_{it+1}}{k_{it+1}}\right)k_{it}^{\theta_i-1}(z_{it+1}h_{it+1})^{1-\theta_i}\right\}, \tag{7}$$

$$c_t = \left[1 - \frac{\phi_{hc}}{2}\left(\frac{a^g_{t-1}h_{ct}}{h_{ct-1}} - \eta_c\right)^2\right]\left[1 - \frac{\phi_{kc}}{2}\left(\frac{i_{ct}}{k_{ct}} - \kappa_c\right)^2\right]k_{ct}^{\theta_c}(z_{ct}h_{ct})^{1-\theta_c}, \tag{8}$$

$$i_t = \left[1 - \frac{\phi_{hi}}{2}\left(\frac{a_{t-1}^g h_{it}}{h_{it-1}} - \eta_i\right)^2\right]\left[1 - \frac{\phi_{ki}}{2}\left(\frac{i_{it}}{k_{it}} - \kappa_i\right)^2\right] k_{it}^{\theta_i}(z_{it}h_{it})^{1-\theta_i}, \tag{9}$$

$$(1-\delta_c)k_{ct} + i_{ct} = a_t^g z_{it}^g k_{ct+1}, \tag{10}$$

$$(1-\delta_i)k_{it} + i_{it} = a_t^g z_{it}^g k_{it+1}, \tag{11}$$

$$i_t = i_{ct} + i_{it}, \tag{12}$$

$$h_t = h_{ct} + h_{it}, \tag{13}$$

$$\ln(a_t) = \ln(a_t^l) + \ln(a_t^g), \tag{14}$$

$$\ln(a_t^l) = \rho_a^l \ln(a_{t-1}^l) + \varepsilon_{at}^l, \tag{15}$$

$$\ln(a_t^g) = (1-\rho_a^g)\ln(a^g) + \rho_a^g \ln(a_{t-1}^g) + \varepsilon_{at}^g, \tag{16}$$

$$\ln(z_{ct}) = \ln(z_{ct}^l) + \ln(z_{ct}^g), \tag{17}$$

$$\ln(z_{ct}^l) = \rho_c^l \ln(z_{ct-1}^l) + \varepsilon_{ct}^l, \tag{18}$$

$$\ln(z_{ct}^g) = (1-\rho_c^g)\ln(z_c^g) + \rho_c^g \ln(z_{ct-1}^g) + \varepsilon_{ct}^g, \tag{19}$$

$$\ln(z_{it}) = \ln(z_{it}^l) + \ln(z_{it}^g), \tag{20}$$

$$\ln(z_{it}^l) = \rho_i^l \ln(z_{it-1}^l) + \varepsilon_{it}^l, \tag{21}$$

and

$$\ln(z_{it}^g) = (1-\rho_i^g)\ln(z_i^g) + \rho_i^g \ln(z_{it-1}^g) + \varepsilon_{it}^g. \tag{22}$$

for all $t = 0, 1, 2, ...$

In order to keep track of the model's observable variables, define the growth rates of consumption, investment, and hours worked as

$$g_t^c = C_t/C_{t-1} = a_{t-1}^g (z_{it-1}^g)^{\theta_c} (z_{ct-1}^g)^{1-\theta_c} (c_t/c_{t-1}), \tag{23}$$

$$g_t^i = I_i/I_{t-1} = a_{t-1}^g z_{it-1}^g (i_t/i_{t-1}), \tag{24}$$

and

$$g_t^h = H_t/H_{t-1} = a_{t-1}^g (h_t/h_{t-1}). \tag{25}$$

5 The Steady State

Equations (1)-(25) imply that in the absence of shocks, the economy converges to a steady-state growth path along which all of the stationary variables are constant, with $c_t = c$, $h_t = h$, $h_{ct} = h_c$, $h_{it} = h_i$, $i_t = i$, $i_{ct} = i_c$, $i_{it} = i_i$, $u_{ct} = u_c$, $u_{it} = u_i$, $k_{ct} = k_c$, $k_{it} = k_i$, $\lambda_{ct} = \lambda_c$, $\lambda_{it} = \lambda_i$, $\xi_{ct} = \xi_c$, $\xi_{it} = \xi_i$, $a_t = a$, $a_t^l = a^l$, $a_t^g = a^g$, $z_{ct} = z_c$, $z_{ct}^l = z_c^l$, $z_{ct}^g = z_c^g$, $z_{it} = z_i$, $z_{it}^l = z_i^l$, $z_{it}^g = z_i^g$, $g_t^c = g^c$, $g_t^i = g^i$, and $g_t^h = g^h$ for all $t = 0, 1, 2, ...$

Equations (16), (19), and (22) provide the values for a^g, z_c^g, and z_i^g. Equations (14), (15), (17), (18), (20), (21), and (23)-(24) then imply

$$a^l = 1,$$

$$a = a^g,$$

$$z_c^l = 1,$$

$$z_c = z_c^g,$$

$$z_i^l = 1,$$

$$z_i = z_i^g,$$

$$g^c = a^g (z_i^g)^{\theta_c} (z_c^g)^{1-\theta_c},$$

$$g^i = a^g z_i^g,$$

and

$$g^h = a^g.$$

In light of (1), (2) determines

$$h_c = (1-\theta_c)\left(\frac{g^c - \beta\gamma}{g^c - \gamma}\right) a^g.$$

Now suppose that solutions for k_c and k_i are in hand, and use (9)-(13) to solve for

$$i_c = (a^g z_i^g - 1 + \delta_c) k_c,$$

$$i_i = (a^g z_i^g - 1 + \delta_i) k_i,$$

$$i = i_c + i_i,$$

$$h_i = \frac{1}{z_i^g}\left(\frac{i}{k_i^{\theta_i}}\right)^{1/(1-\theta_i)},$$

and

$$h = h_c + h_i.$$

Then use (1), (3)-(5), and (8) to solve for

$$c = k_c^{\theta_c}(z_c^g h_c)^{1-\theta_c},$$

$$\lambda_c = \left(\frac{g^c - \beta\gamma}{g^c - \gamma}\right)\frac{1}{c},$$

and

$$\lambda_i = \xi_c = \xi_i = \frac{h_i}{(1-\theta_i)a^g i}.$$

Finally, use (6) and (7) to solve for k_c and k_i:

$$k_c = (1-\theta_i)a^g z_i^g \left(\frac{g^c - \beta\gamma}{g^c - \gamma}\right)\left\{\frac{\theta_c[a^g z_i^g - \beta(1-\delta_i)]}{\theta_i[a^g z_i^g - \beta(1-\delta_c)}\right\}\left[\frac{\beta\theta_i}{a^g z_i^g - \beta(1-\delta_i)}\right]^{1/(1-\theta_i)}$$

and

$$k_i = \left\{\frac{\beta\theta_i(a^g z_i^g - 1 + \delta_c)}{a^g z_i^g - \beta(1-\delta_i) - \beta\theta_i(a^g z_i^g - 1 + \delta_i)}\right\} k_c.$$

Hence, to compute all of these steady-state values, start by calculating h_c, then calculate k_c and k_i, then go back and calculate i_c, i_i, i, h_i, h, c, λ_c, λ_i, ξ_c, and ξ_i. Note that the solutions for i_c and i_i reveal that to have zero steady-state capital adjustment costs, the parameters κ_c and κ_i must be given by

$$\kappa_c = a^g z_i^g - 1 + \delta_c$$

and

$$\kappa_i = a^g z_i^g - 1 + \delta_i.$$

Likewise, to have zero steady-state labor adjustment costs, the parameters η_c and η_i must satisfy

$$\eta_c = \eta_i = a^g.$$

6 The Linearized System

Equations (1)-(25) can be log-linearized around the steady state to describe how the model's variables respond to shocks. Let $\hat{c}_t = \ln(c_t/c)$, $\hat{h}_t = \ln(h_t/h)$, $\hat{h}_{ct} = \ln(h_{ct}/h_c)$, $\hat{h}_{it} = \ln(h_{it}/h_i)$, $\hat{\imath}_t = \ln(i_t/i)$, $\hat{\imath}_{ct} = \ln(i_{ct}/i_c)$, $i_{it} = \ln(i_{it}/i_i)$, $\hat{k}_{ct} = \ln(k_{ct}/k_c)$, $\hat{k}_{it} = \ln(k_{it}/k_t)$, $\hat{\lambda}_{ct} = \ln(\lambda_{ct}/\lambda_c)$, $\hat{\lambda}_{it} = \ln(\lambda_{it}/\lambda_i)$, $\hat{\xi}_{ct} = \ln(\xi_{ct}/\xi_c)$, $\hat{\xi}_{it} = \ln(\xi_{it}/\xi_i)$, $\hat{a}_t = \ln(a_t/a)$, $\hat{a}^l_t = \ln(a^l_t/a^l)$, $\hat{a}^g_t = \ln(a^g_t/a^g)$, $\hat{z}_{ct} = \ln(z_{ct}/z_c)$, $\hat{z}^l_{ct} = \ln(z^l_{ct}/z^l_c)$, $\hat{z}^g_{ct} = \ln(z^g_{ct}/z^g_c)$, $\hat{z}_{it} = \ln(z_{it}/z_i)$, $\hat{z}^l_{it} = \ln(z^l_{it}/z^l_i)$, $\hat{z}^g_{it} = \ln(z^g_{it}/z^g_i)$, $\hat{g}^c_t = \ln(g^c_t/g^c)$, $\hat{g}^i_t = \ln(g^i_t/g^i)$, and $\hat{g}^h_t = \ln(g^h_t/g^h)$. Then first-order Taylor approximations to (1)-(25) imply

$$\begin{aligned} & [(g^c)^2 + \beta\gamma^2]\hat{c}_t + (g^c - \gamma)(g^c - \beta\gamma)\hat{\lambda}_{ct} \\ = \;& \gamma g^c \hat{c}_{t-1} - \gamma g^c \hat{a}^g_{t-1} - \theta_c \gamma g^c \hat{z}^g_{it-1} - (1-\theta_c)\gamma g^c \hat{z}^g_{ct-1} + \beta\gamma g^c \hat{a}^g_t + \theta_c \beta\gamma g^c \hat{z}^g_{it} + (1-\theta_c)\beta\gamma g^c \hat{z}^g_{ct} + \beta\gamma g^c E_t \hat{c}_{t+1}, \end{aligned} \tag{1}$$

$$\begin{aligned} & [(1-\theta_c)(1/a^g) + (1+\beta)\phi_{hc} a^g]\hat{h}_{ct} - (1-\theta_c)(1/a^g)\hat{a}_t \\ = \;& (1-\theta_c)(1/a^g)\hat{\lambda}_{ct} + (1-\theta_c)(1/a^g)\hat{c}_t - \phi_{hc} a^g \hat{a}^g_{t-1} + \phi_{hc} a^g \hat{h}_{ct-1} + \beta\phi_{hc} a^g \hat{a}^g_t + \beta\phi_{hc} a^g E_t \hat{h}_{ct+1} \end{aligned} \tag{2}$$

$$\begin{aligned} & [(1-\theta_i)(1/a^g) + (1+\beta)\phi_{hi} a^g]\hat{h}_{it} - (1-\theta_i)(1/a^g)\hat{a}_t \\ = \;& (1-\theta_i)(1/a^g)\hat{\lambda}_{it} + (1-\theta_i)(1/a^g)\hat{\imath}_t - \phi_{hi} a^g \hat{a}^g_{t-1} + \phi_{hi} a^g \hat{h}_{it-1} + \beta\phi_{hi} a^g \hat{a}^g_t + \beta\phi_{hi} a^g E_t \hat{h}_{it+1} \end{aligned} \tag{3}$$

$$\xi_c \hat{\xi}_{ct} = \lambda_i \hat{\lambda}_{it} + \phi_{kc}\lambda_c (i_c/k_c)(c/k_c)\hat{\imath}_{ct} - \phi_{kc}\lambda_c (i_c/k_c)(c/k_c)\hat{k}_{ct}, \tag{4}$$

$$\hat{\xi}_{it} = \hat{\lambda}_{it} + \phi_{ki}(i_i/k_i)(i/k_i)\hat{\imath}_{it} - \phi_{ki}(i_i/k_i)(i/k_i)\hat{k}_{it}, \tag{5}$$

$$\begin{aligned} & a^g z^g_i \xi_c \hat{a}^g_t + a^g z^g_i \xi_c \hat{z}^g_{it} + a^g z^g_i \xi_c \hat{\xi}_{ct} \\ = \;& \beta(1-\delta_c)\xi_c E_t \hat{\xi}_{ct+1} + \beta\theta_c \lambda_c c(1/k_c) E_t \hat{\lambda}_{ct+1} \\ & + \beta\theta_c \lambda_c c(1/k_c) E_t \hat{c}_{t+1} - \beta\lambda_c c(1/k_c)[\theta_c + \phi_{kc}(i_c/k_c)^2]\hat{k}_{ct+1} + \beta\phi_{kc}\lambda_c c(i_c/k_c)^2(1/k_c) E_t \hat{\imath}_{ct+1}, \end{aligned} \tag{6}$$

$$\begin{aligned} & a^g z^g_i \hat{a}^g_t + a^g z^g_i \hat{z}^g_{it} + a^g z^g_i \hat{\xi}_{it} \\ = \;& \beta(1-\delta_i) E_t \hat{\xi}_{it+1} + \beta\theta_i (i/k_i) E_t \hat{\lambda}_{it+1} + \beta\theta_i (i/k_i) E_t \hat{\imath}_{t+1} - \beta(i/k_i)[\theta_i + \phi_{ki}(i_i/k_i)^2]\hat{k}_{it+1} + \beta\phi_{ki}(i_i/k_i)^2 (i/k_i) E_t \hat{\imath}_{it+1}, \end{aligned} \tag{7}$$

$$\hat{c}_t = \theta_c \hat{k}_{ct} + (1-\theta_c)\hat{z}_{ct} + (1-\theta_c)\hat{h}_{ct}, \tag{8}$$

$$\hat{\imath}_t = \theta_i \hat{k}_{it} + (1-\theta_i)\hat{z}_{it} + (1-\theta_i)\hat{h}_{it}, \tag{9}$$

$$(1-\delta_c)k_c\hat{k}_{ct} + i_c\hat{\imath}_{ct} = a^g z_i^g k_c \hat{a}_t^g + a^g z_i^g k_c \hat{z}_{it}^g + a^g z_i^g k_c \hat{k}_{ct+1}, \tag{10}$$

$$(1-\delta_i)k_i\hat{k}_{it} + i_i\hat{\imath}_{it} = a^g z_i^g k_i \hat{a}_t^g + a^g z_i^g k_i \hat{z}_{it}^g + a^g z_i^g k_i \hat{k}_{it+1}, \tag{11}$$

$$i\hat{\imath}_t = i_c\hat{\imath}_{ct} + i_i\hat{\imath}_{it}, \tag{12}$$

$$h\hat{h}_t = h_c\hat{h}_{ct} + h_i\hat{h}_{it}, \tag{13}$$

$$\hat{a}_t = \hat{a}_t^l + \hat{a}_t^g, \tag{14}$$

$$\hat{a}_t^l = \rho_a^l \hat{a}_{t-1}^l + \varepsilon_{at}^l, \tag{15}$$

$$\hat{a}_t^g = \rho_a^g \hat{a}_{t-1}^g + \varepsilon_{at}^g, \tag{16}$$

$$\hat{z}_{ct} = \hat{z}_{ct}^l + \hat{z}_{ct}^g, \tag{17}$$

$$\hat{z}_{ct}^l = \rho_c^l \hat{z}_{ct-1}^l + \varepsilon_{ct}^l, \tag{18}$$

$$\hat{z}_{ct}^g = \rho_c^g \hat{z}_{ct-1}^g + \varepsilon_{ct}^g, \tag{19}$$

$$\hat{z}_{it} = \hat{z}_{it}^l + \hat{z}_{it}^g, \tag{20}$$

$$\hat{z}_{it}^l = \rho_i^l \hat{z}_{it-1}^l + \varepsilon_{it}^l, \tag{21}$$

$$\hat{z}_{it}^g = \rho_i^g \hat{z}_{it-1}^g + \varepsilon_{it}^g, \tag{22}$$

$$\hat{g}_t^c = \hat{a}_{t-1}^g + \theta_c\hat{z}_{it-1}^g + (1-\theta_c)\hat{z}_{ct-1}^g + \hat{c}_t - \hat{c}_{t-1}, \tag{23}$$

$$\hat{g}_t^i = \hat{a}_{t-1}^g + \hat{z}_{it-1}^g + \hat{\imath}_t - \hat{\imath}_{t-1}, \tag{24}$$

and

$$\hat{g}_t^h = \hat{a}_{t-1}^g + \hat{h}_t - (h_c/h)\hat{h}_{ct-1} - (h_i/h)\hat{h}_{it-1}. \tag{25}$$

for all $t = 0, 1, 2, ...$

7 The System in Matrix Form

Let

$$f_t^0 = \left[\begin{array}{ccccccccccc} \hat{h}_t & \hat{\imath}_t & \hat{\imath}_{ct} & \hat{\imath}_{it} & \hat{\lambda}_{it} & \hat{a}_t & \hat{z}_{ct} & \hat{z}_{it} & \hat{g}_t^c & \hat{g}_t^i & \hat{g}_t^h \end{array} \right]',$$

$$s_t^0 = \left[\begin{array}{ccccccccccccccc} \hat{k}_{ct} & \hat{k}_{it} & \hat{c}_{t-1} & \hat{\imath}_{t-1} & \hat{h}_{ct-1} & \hat{h}_{it-1} & \hat{a}_{t-1}^g & \hat{z}_{ct-1}^g & \hat{z}_{it-1}^g & \hat{c}_t & \hat{h}_{ct} & \hat{h}_{it} & \hat{\lambda}_{ct} & \hat{\xi}_{ct} & \hat{\xi}_{it} \end{array} \right]',$$

and

$$v_t = \left[\begin{array}{cccccc} \hat{a}_t^l & \hat{a}_t^g & \hat{z}_{ct}^l & \hat{z}_{ct}^g & \hat{z}_{it}^l & \hat{z}_{it}^g \end{array} \right]'.$$

Then (4), (5), (9), (12)-(14), (17), (20), and (23)-(25) can be written as

$$A f_t^0 = B s_t^0 + C v_t, \tag{26}$$

where A is 11×11, B is 11×15, and C is 11×6.

Equation (4) implies

$a_{13} = \phi_{kc} \lambda_c (i_c/k_c)(c/k_c)$

$a_{15} = \lambda_i$

$b_{11} = \phi_{kc} \lambda_c (i_c/k_c)(c/k_c)$

$b_{114} = \xi_c$

Equation (5) implies

$a_{24} = \phi_{ki}(i_i/k_i)(i/k_i)$

$a_{25} = 1$

$b_{22} = \phi_{ki}(i_i/k_i)(i/k_i)$

$b_{215} = 1$

Equation (9) implies

$a_{32} = 1$

$a_{38} = \theta_i - 1$

$b_{32} = \theta_i$

$b_{312} = 1 - \theta_i$

Equation (12) implies

$a_{42} = i$

$a_{43} = -i_c$

$a_{44} = -i_i$

Equation (13) implies

$a_{51} = h$

$b_{511} = h_c$

$b_{512} = h_i$

Equation (14) implies

$a_{66} = 1$

$c_{61} = 1$

$c_{62} = 1$

Equation (17) implies

$a_{77} = 1$

$c_{73} = 1$

$c_{74} = 1$

Equation (20) implies

$a_{88} = 1$

$c_{85} = 1$

$c_{86} = 1$

Equation (23) implies

$a_{99} = 1$

$b_{93} = -1$

$b_{97} = 1$

$b_{98} = 1 - \theta_c$

$b_{99} = \theta_c$

$b_{910} = 1$

Equation (24) implies

$a_{102} = -1$

$a_{1010} = 1$

$b_{104} = -1$

$b_{107} = 1$

$b_{109} = 1$

Equation (25) implies

$a_{111} = -1$

$a_{1111} = 1$

$b_{115} = -h_c/h$

$b_{116} = -h_i/h$

$b_{117} = 1$

Equations (1)-(3), (6), (7), (8), (10), and (11) can be written as

$$DE_t s^0_{t+1} + FE_t f^0_{t+1} = Gs^0_t + Hf^0_t + Jv_t, \tag{27}$$

where D and G are 15×15, F and H are 15×11, and J is 15×6.

Equation (1) implies

$d_{17} = \beta\gamma g^c$

$d_{18} = (1 - \theta_c)\beta\gamma g^c$

$d_{19} = \theta_c\beta\gamma g^c$

$d_{110} = \beta\gamma g^c$

$g_{13} = -\gamma g^c$

$g_{17} = \gamma g^c$

$g_{18} = (1 - \theta_c)\gamma g^c$

$g_{19} = \theta_c\gamma g^c$

$g_{110} = (g^c)^2 + \beta\gamma^2$

$g_{113} = (g^c - \gamma)(g^c - \beta\gamma)$

Equation (2) implies

$d_{27} = \beta\phi_{hc}a^g$

$d_{211} = \beta\phi_{hc}a^g$

$g_{25} = -\phi_{hc}a^g$

$g_{27} = \phi_{hc}a^g$

$g_{210} = (\theta_c - 1)(1/a^g)$

$g_{211} = (1 - \theta_c)(1/a^g) + (1 + \beta)\phi_{hc}a^g$

$g_{213} = (\theta_c - 1)(1/a^g)$

$h_{26} = (\theta_c - 1)(1/a^g)$

Equation (3) implies

$d_{37} = \beta\phi_{hi}a^g$

$d_{312} = \beta\phi_{hi}a^g$

$g_{36} = -\phi_{hi}a^g$

$g_{37} = \phi_{hi}a^g$

$g_{312} = (1 - \theta_i)(1/a^g) + (1 + \beta)\phi_{hi}a^g$

$h_{32} = (\theta_i - 1)(1/a^g)$

$h_{35} = (\theta_i - 1)(1/a^g)$

$h_{36} = (\theta_i - 1)(1/a^g)$

Equation (6) implies

$d_{41} = -\beta\lambda_c c(1/k_c)[\theta_c + \phi_{kc}(i_c/k_c)^2]$

$d_{410} = \beta\theta_c\lambda_c c(1/k_c)$

$d_{413} = \beta\theta_c\lambda_c c(1/k_c)$

$d_{414} = \beta(1-\delta_c)\xi_c$

$f_{43} = \beta\phi_{kc}\lambda_c c(i_c/k_c)^2(1/k_c)$

$g_{414} = a^g z_i^g \xi_c$

$j_{42} = a^g z_i^g \xi_c$

$j_{46} = a^g z_i^g \xi_c$

Equation (7) implies

$d_{52} = -\beta(i/k_i)[\theta_i + \phi_{ki}(i_i/k_i)^2]$

$d_{515} = \beta(1-\delta_i)$

$f_{52} = \beta\theta_i(i/k_i)$

$f_{54} = \beta\phi_{ki}(i_i/k_i)^2(i/k_i)$

$f_{55} = \beta\theta_i(i/k_i)$

$g_{515} = a^g z_i^g$

$j_{52} = a^g z_i^g$

$j_{56} = a^g z_i^g$

Equation (8) implies

$d_{63} = 1$

$d_{65} = \theta_c - 1$

$g_{61} = \theta_c$

$h_{67} = 1 - \theta_c$

Equation (10) implies

$d_{71} = a^g z_i^g k_c$

$g_{71} = (1 - \delta_c) k_c$

$h_{73} = i_c$

$j_{72} = -a^g z_i^g k_c$

$j_{76} = -a^g z_i^g k_c$

Equation (11) implies

$d_{82} = a^g z_i^g k_i$

$g_{82} = (1 - \delta_i) k_i$

$h_{84} = i_i$

$j_{82} = -a^g z_i^g k_i$

$j_{86} = -a^g z_i^g k_i$

The presence of lagged values in s_t^0 implies

$d_{93} = 1$

$g_{910} = 1$

$d_{104} = 1$

$h_{102} = 1$

$d_{115} = 1$

$g_{1111} = 1$

$d_{126} = 1$

$g_{1212} = 1$

$d_{137} = 1$

$j_{132} = 1$

$d_{148} = 1$

$j_{144} = 1$

$d_{159} = 1$

$j_{156} = 1$

Finally, (15), (16), (18), (19), and (21)-(22) can be written as

$$v_t = Pv_{t-1} + \varepsilon_t, \tag{28}$$

where

$$P = \begin{bmatrix} \rho_a^l & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho_a^g & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_c^l & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_c^g & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_i^l & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_i^g \end{bmatrix}$$

and

$$\varepsilon_t = \left[\begin{array}{cccccc} \varepsilon_{at}^l & \varepsilon_{at}^g & \varepsilon_{ct}^l & \varepsilon_{ct}^g & \varepsilon_{it}^l & \varepsilon_{it}^g \end{array} \right]'.$$

Note that (28) implies

$$E_t v_{t+j} = P^j v_t$$

for all $j = 0, 1, 2, ...$

8 Solving the Linearized Model

Start by using (26) to solve out for f_t^0:

$$f_t^0 = A^{-1}Bs_t^0 + A^{-1}Cv_t. \tag{29}$$

The substitute this result into (27) to obtain

$$KE_t s_{t+1}^0 = Ls_t^0 + Mv_t, \tag{30}$$

where

$$K = D + FA^{-1}B,$$

$$L = G + HA^{-1}B,$$

and

$$M = J + HA^{-1}C - FA^{-1}CP.$$

Equation (30) takes the form of a system of linear expectational difference equations, driven by the exogenous shocks in (28). This system can be solved by uncoupling the unstable and stable components and then solving the unstable component forward. There are a number of algorithms for working through this process; the approach taken here uses the methods outlined by Klein (2000).

Klein's method relies on the complex generalized Schur decomposition, which identifies unitary matrices Q and Z such that

$$QKZ = S$$

and

$$QLZ = T$$

are both upper triangular, where the generalized eigenvalues of L and K can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(L, K) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 15\},$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues appear in ascending order in absolute value. Note that there are nine predetermined variables in the vector s_t^0. Thus, if nine of the generalized eigenvalues in $\lambda(L, K)$ lie inside the unit circle and zix of the generalized eigenvalues lie outside the unit circle, then the system has a unique solution. If more than six of the generalized eigenvalues in $\lambda(L, K)$ lie outside the unit circle, then the system has no solution. If less than six of the generalized eigenvalues in $\lambda(L, K)$ lie outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assume from now on that there are exactly two generalized eigenvalues that lie outside the unit circle, and partition the matrices Q, Z, S, and T conformably, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 is 9×15 and Q_2 is 6×15, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{(6\times 9)} & S_{22} \end{bmatrix},$$

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{(6\times 9)} & T_{22} \end{bmatrix},$$

where Z_{11}, S_{11}, and T_{11} are 9×9, Z_{12}, S_{12}, and T_{12} are 9×6, Z_{21} is 6×9 and Z_{22}, S_{22}, and T_{22} are 6×6.

Next, define the vector s_t^1 of auxiliary variables as

$$s_t^1 = Z' s_t^0$$

so that, in particular,

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z_{11}' \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}_{t-1}^g \\ \hat{z}_{ct-1}^g \\ \hat{z}_{it-1}^g \end{bmatrix} + Z_{21}' \begin{bmatrix} \hat{c}_t \\ \hat{h}_{ct} \\ \hat{h}_{it} \\ \hat{\lambda}_{ct} \\ \hat{\xi}_{ct} \\ \hat{\xi}_{it} \end{bmatrix} \tag{31}$$

is 9×1 and

$$s^1_{2t} = Z'_{12} \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \end{bmatrix} + Z'_{22} \begin{bmatrix} \hat{c}_t \\ \hat{h}_{ct} \\ \hat{h}_{it} \\ \hat{\lambda}_{ct} \\ \hat{\xi}_{ct} \\ \hat{\xi}_{it} \end{bmatrix} \tag{32}$$

is 6×1.

Since Z is unitary, $Z'Z = I$ or $Z' = Z^{-1}$ and hence $s^0_t = Zs^1_t$. Use this fact to rewrite (30) as

$$KZE_t s^1_{t+1} = LZs^1_t + Mv_t.$$

Premultiply this version of (30) by Q to obtain

$$SE_t s^1_{t+1} = Ts^1_t + QMv_t$$

or, in terms of the matrix partitions,

$$S_{11}E_t s^1_{1t+1} + S_{12}E_t s^1_{2t+1} = T_{11}s^1_{1t} + T_{12}s^1_{2t} + Q_1 Mv_t \tag{33}$$

and

$$S_{22}E_t s^1_{2t+1} = T_{22}s^1_{2t} + Q_2 Mv_t. \tag{34}$$

Since the generalized eigenvalues corresponding to the elements of S_{22} and T_{22} all lie outside the unit circle, (34) can be solved forward to obtain

$$s^1_{2t} = -T^{-1}_{22} Rv_t,$$

where the 6×6 matrix R is given by

$$\begin{aligned}
vec(R) &= vec\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2 M P^j = \sum_{j=0}^{\infty} vec[(S_{22}T_{22}^{-1})^j Q_2 M P^j] \\
&= \sum_{j=0}^{\infty}[P^j \otimes (S_{22}T_{22}^{-1})^j] vec(Q_2 M) = \sum_{j=0}^{\infty}[P \otimes (S_{22}T_{22}^{-1})]^j vec(Q_2 M) \\
&= [I_{(36\times 36)} - P \otimes (S_{22}T_{22}^{-1})]^{-1} vec(Q_2 M).
\end{aligned}$$

Use this result, along with (32), to solve for

$$\begin{bmatrix} \hat{c}_t \\ \hat{h}_{ct} \\ \hat{h}_{it} \\ \hat{\lambda}_{ct} \\ \hat{\xi}_{ct} \\ \hat{\xi}_{it} \end{bmatrix} = -(Z'_{22})Z'_{12} \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \end{bmatrix} - (Z'_{22})T_{22}^{-1} R v_t.$$

Since Z is unitary, $Z'Z = I$ or

$$\begin{bmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} I_{(9\times 9)} & 0_{(9\times 6)} \\ 0_{(6\times 9)} & I_{(6\times 6)} \end{bmatrix}.$$

Hence, in particular,

$$Z'_{12}Z_{11} + Z'_{22}Z_{21} = 0$$

or

$$-(Z'_{22})^{-1}Z'_{12} = Z_{21}Z_{11}^{-1}$$

and

$$Z'_{12}Z_{12} + Z'_{22}Z_{22} = I$$

or

$$(Z'_{22})^{-1} = Z_{22} + (Z'_{22})^{-1}Z'_{12}Z_{12} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12},$$

allowing this solution to be written more conveniently as

$$\begin{bmatrix} \hat{c}_t \\ \hat{h}_{ct} \\ \hat{h}_{it} \\ \hat{\lambda}_{ct} \\ \hat{\xi}_{ct} \\ \hat{\xi}_{it} \end{bmatrix} = N_1 \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \end{bmatrix} + N_2 v_t \tag{35}$$

where

$$N_1 = Z_{21} Z_{11}^{-1}$$

and

$$N_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Equation (31) now provides the solution for s^1_{1t}:

$$s^1_{1t} = (Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \end{bmatrix} - Z'_{21}[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R v_t.$$

Using

$$Z'_{11} Z_{11} + Z'_{21} Z_{21} = I$$

or

$$Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1} = Z_{11}^{-1}$$

and

$$Z_{21}'[Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}] = Z_{21}'Z_{22} - Z_{21}'Z_{21}Z_{11}^{-1}Z_{12} = -Z_{11}^{-1}Z_{12},$$

this last result can be written more conveniently as

$$s_{1t}^1 = Z_{11}^{-1}\begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}_{t-1}^g \\ \hat{z}_{ct-1}^g \\ \hat{z}_{it-1}^g \end{bmatrix} + Z_{11}^{-1}Z_{12}T_{22}^{-1}Rv_t.$$

Substitute these results into (33) to obtain to the solution

$$\begin{bmatrix} \hat{k}_{ct+1} \\ \hat{k}_{it+1} \\ \hat{c}_t \\ \hat{\imath}_t \\ \hat{h}_{ct} \\ \hat{h}_{it} \\ \hat{a}_t^g \\ \hat{z}_{ct}^g \\ \hat{z}_{it}^g \end{bmatrix} = N_3\begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}_{t-1}^g \\ \hat{z}_{ct-1}^g \\ \hat{z}_{it-1}^g \end{bmatrix} + N_4v_t, \tag{36}$$

where

$$N_3 = Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}$$

and

$$N_4 = Z_{11}S_{11}^{-1}(T_{11}Z_{11}^{-1}Z_{12}T_{22}^{-1}R + Q_1M + S_{12}T_{22}^{-1}RP - T_{12}T_{22}^{-1}R) - Z_{12}T_{22}^{-1}RP.$$

Finally, return to (29) to solve for f_t^0:

$$f_t^0 = A^{-1}B \begin{bmatrix} I_{(9\times 9)} \\ N_1 \end{bmatrix} \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \end{bmatrix} + A^{-1}B \begin{bmatrix} 0_{(9\times 6)} \\ N_2 \end{bmatrix} v_t + A^{-1}Cv_t,$$

which can be written more simply as

$$f_t^0 = N_5 \begin{bmatrix} \hat{k}_{ct} \\ \hat{k}_{it} \\ \hat{c}_{t-1} \\ \hat{\imath}_{t-1} \\ \hat{h}_{ct-1} \\ \hat{h}_{it-1} \\ \hat{a}^g_{t-1} \\ \hat{z}^g_{ct-1} \\ \hat{z}^g_{it-1} \end{bmatrix} + N_6 v_t, \tag{37}$$

where

$$N_5 = A^{-1}B \begin{bmatrix} I_{(9\times 9)} \\ N_1 \end{bmatrix}$$

and

$$N_6 = A^{-1}C + A^{-1}B \begin{bmatrix} 0_{(9\times 6)} \\ N_2 \end{bmatrix}.$$

Equations (28) and (35)-(37) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{38}$$

and

$$f_t = Us_t, \tag{39}$$

where

$$s_t = \left[\ \hat{k}_{ct} \quad \hat{k}_{it} \quad \hat{c}_{t-1} \quad \hat{\imath}_{t-1} \quad \hat{h}_{ct-1} \quad \hat{h}_{it-1} \quad \hat{a}^g_{t-1} \quad \hat{z}^g_{ct-1} \quad \hat{z}^g_{it-1} \quad \hat{a}^l_t \quad \hat{a}^g_t \quad \hat{z}^l_{ct} \quad \hat{z}^g_{ct} \quad \hat{z}^l_{it} \quad \hat{z}^g_{it}\ \right]',$$

$$f_t = \left[\ \hat{h}_t \quad \hat{\imath}_t \quad \hat{\imath}_{ct} \quad \hat{\imath}_{it} \quad \hat{\lambda}_{it} \quad \hat{a}_t \quad \hat{z}_{ct} \quad \hat{z}_{it} \quad \hat{g}^c_t \quad \hat{g}^i_t \quad \hat{g}^h_t \quad \hat{c}_t \quad \hat{h}_{ct} \quad \hat{h}_{it} \quad \hat{\lambda}_{ct} \quad \hat{\xi}_{ct} \quad \hat{\xi}_{it}\ \right]',$$

$$\varepsilon_t = \left[\ \varepsilon^l_{at} \quad \varepsilon^g_{at} \quad \varepsilon^l_{ct} \quad \varepsilon^g_{ct} \quad \varepsilon^l_{it} \quad \varepsilon^g_{it}\ \right]',$$

$$\Pi = \begin{bmatrix} N_3 & N_4 \\ 0_{(6\times 9)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(9\times 6)} \\ I_{(6\times 6)} \end{bmatrix},$$

and

$$U = \begin{bmatrix} N_5 & N_6 \\ N_1 & N_2 \end{bmatrix}.$$

9 Estimating the Model

The model has implications for the behavior of three observable variables: the growth rates of consumption, investment, and hours worked. The empirical model has 25 parameters: β, γ, θ_c, θ_i, ϕ_{kc}, ϕ_{ki}, ϕ_{hc}, ϕ_{hi}, δ_c, δ_i, a^g, z^g_c, z^g_i, ρ^l_a, ρ^g_a, ρ^l_c, ρ^g_c, ρ^l_i, ρ^g_i, σ^l_a, σ^g_a, σ^l_c, σ^g_c, σ^l_i, and σ^g_i. To estimate these parameters via maximum likelihood, let $\{d_t\}_{t=1}^T$ denote the series for the logarithmic deviations of the growth rates of consumption, investment, and hours worked from their average, or steady-state, values:

$$d_t = \begin{bmatrix} \hat{g}^c_t \\ \hat{g}^i_t \\ \hat{g}^h_t \end{bmatrix} = \begin{bmatrix} \ln(C_t) - \ln(C_{t-1}) - \ln[a^g (z^g_i)^{\theta_c} (z^g_c)^{1-\theta_c}] \\ \ln(I_t) - \ln(I_{t-1}) - \ln(a^g z^g_i) \\ \ln(H_t) - \ln(H_{t-1}) - \ln(a^g) \end{bmatrix},$$

where C_t, I_i, and H_t are the levels of real, per capita consumption, investment, and hours worked.

The empirical model then takes form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{40}$$

and

$$d_t = Cs_t, \tag{41}$$

where $A = \Pi$, $B = W$, C is formed from the rows of U as

$$C = \begin{bmatrix} U_9 \\ U_{10} \\ U_{11} \end{bmatrix},$$

and the vector of zero-mean, serially uncorrelated innovations ε_{t+1} is normally distributed with diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon'_{t+1} = \begin{bmatrix} (\sigma_a^l)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\sigma_a^g)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\sigma_c^l)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\sigma_c^g)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\sigma_i^l)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\sigma_i^g)^2 \end{bmatrix}.$$

The model defined by (40) and (41) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (40) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(15\times1)} \tag{42}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1s_1') = [I_{(225\times225)} - A \otimes A]^{-1}vec(BVB'). \tag{43}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (41) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_tu_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned}
\hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\
&= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.
\end{aligned}$$

Hence, from (40),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (40) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_tu_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_tu_t' = C\Sigma_tC' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_tC'(C\Sigma_tC')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1}C\Sigma_t A',$$

and initial conditions $\hat{s}_1$ and Σ_1 are provided by (42) and (43).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\sum_{t=1}^T \ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^T u_t'\Omega_t^{-1}u_t.$$

10 Evaluating the Model

10.1 Variance Decompositions

Begin by considering (40), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\ &= BVB' + ABVB'A' + A^2BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.\end{aligned}$$

In addition, (40) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(225\times 225)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (39), which implies that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U'$$

and

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U'.$$

Finally, note that

$$\begin{bmatrix} \ln(C_t) - \ln(C_{t-1}) - \ln(g^c) \\ \ln(I_t) - \ln(I_{t-1}) - \ln(g^i) \\ \ln(H_t) - \ln(H_{t-1}) - \ln(g^h) \end{bmatrix} = \begin{bmatrix} U_9 \\ U_{10} \\ U_{11} \end{bmatrix} s_t.$$

Hence,

$$\ln(C_{t+k}) = \ln(C_t) + k\ln(g^c) + U_9 \sum_{j=1}^{k} s_{t+j},$$

$$\ln(I_{t+k}) = \ln(I_t) + k\ln(g^i) + U_{10} \sum_{j=1}^{k} s_{t+j},$$

and

$$\ln(H_{t+k}) = \ln(H_t) + k\ln(g^h) + U_{11} \sum_{j=1}^{k} s_{t+j}.$$

Consequently,

$$\ln(C_{t+k}) - E_t \ln(C_{t+k}) = U_9 \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}),$$

$$\ln(I_{t+k}) - E_t \ln(I_{t+k}) = U_{10} \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}),$$

and

$$\ln(H_{t+k}) - E_t \ln(H_{t+k}) = U_{11} \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}).$$

And hence

$$\Sigma_k^C = E[\ln(C_{t+k}) - E_t \ln(C_{t+k})][\ln(C_{t+k}) - E_t \ln(C_{t+k})]' = U_9 \Sigma_k^S U_9',$$
$$\Sigma_k^I = E[\ln(I_{t+k}) - E_t \ln(I_{t+k})][\ln(I_{t+k}) - E_t \ln(I_{t+k})]' = U_{10} \Sigma_k^S U_{10}',$$

and

$$\Sigma_k^H = E[\ln(H_{t+k}) - E_t \ln(H_{t+k})][\ln(H_{t+k}) - E_t \ln(H_{t+k})]' = U_{11} \Sigma_k^S U_{11}',$$

where

$$\begin{aligned}
\Sigma_k^S &= \left[\sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j})\right] \left[\sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j})\right]' \\
&= \left[\sum_{j=1}^{k} \sum_{l=0}^{j-1} A^l B \varepsilon_{t+j-l}\right] \left[\sum_{j=1}^{k} \sum_{l=0}^{j-1} A^l B \varepsilon_{t+j-l}\right]' \\
&= [B\varepsilon_{t+k} + (I+A)B\varepsilon_{t+k-1} + (I+A+A^2)B\varepsilon_{t+k-2} + ... + (I+A+...+A^{k-1})B\varepsilon_{t+1}] \\
&\quad \times [B\varepsilon_{t+k} + (I+A)B\varepsilon_{t+k-1} + (I+A+A^2)B\varepsilon_{t+k-2} + ... + (I+A+...+A^{k-1})B\varepsilon_{t+1}]' \\
&= BVB' + (I+A)BVB'(I+A)' + (I+A+A^2)BVB'(I+A+A^2)' \\
&\quad +... + (I+A+...+A^{k-1})BVB'(I+A+...+A^{k-1})'
\end{aligned}$$

10.2 Producing Smoothed Estimates of the Shocks

Hamilton (Ch.13, Sec.6, pp.394-397) shows how to generate a sequence of smoothed estimates $\{\hat{s}_{t|T}\}_{t=1}^{T}$ of the unobservable state vector, where

$$\hat{s}_{t|T} = E(s_t|d_T, d_{T-1}, ..., d_1).$$

As before, for $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Also as before, let

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

so that again,

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Then

$$\hat{s}_{1|0} = Es_1 = 0_{(15\times 1)}$$

and

$$vec(\Sigma_{1|0}) = vec(Ex_1x_1') = [I_{(225\times 225)} - (A\otimes A)]^{-1}vec(BVB').$$

From these starting values, the sequences $\{\hat{s}_{t|t}\}_{t=1}^{T}$, $\{\hat{s}_{t|t-1}\}_{t=1}^{T}$, $\{\Sigma_{t|t}\}_{t=1}^{T}$, and $\{\Sigma_{t|t-1}\}_{t=1}^{T}$ can be generated recursively using

$$u_t = d_t - C\hat{s}_{t|t-1},$$

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t,$$

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t},$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1},$$

and

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t}A'$$

for $t = 1, 2, ..., T$.

Now, to begin, construct a sequence $\{J_t\}_{t=1}^{T-1}$ using Hamilton's equation (13.6.11):

$$J_t = \Sigma_{t|t} A' \Sigma_{t+1|t}^{-1}.$$

Then note that $\hat{s}_{T|T}$ is just the last element of $\{\hat{s}_{t|t}\}_{t=1}^{T}$. From this terminal condition, the rest of the sequence can be generated recursively using Hamilton's equation (13.6.16):

$$\hat{s}_{T-j|T} = \hat{s}_{T-j|T-j} + J_{T-j}(\hat{s}_{T-j+1|T} - \hat{s}_{T-j+1|T-j})$$

for $j = 1, 2, ..., T-1$. Kohn and Ansley (1983) show that in cases where $\Sigma_{t+1|t}$ turns out to be singular, its inverse can be replaced by its Moore-Penrose pseudoinverse in the expression for J_t.