APPENDIX D OF THE SUPPLEMENTAL MATERIAL: DERIVATIONS AND ESTIMATION OF THE BENCHMARK DSGE MODEL (*NOT INTENDED FOR PUBLICATION*)

ZHENG LIU, PENGFEI WANG, AND TAO ZHA

I. The Model

The economy is populated by two types of agents—households and entrepreneurs—with a continuum and unit measure of each type. There are four types of commodities: labor, goods, land, and loanable bonds. Goods production requires labor, capital, and land as inputs. The output can be used for consumption (by both types of agents) and for capital investment (by the entrepreneurs). The representative household's utility depends on consumption goods, land services (housing), and leisure; the representative entrepreneur's utility depends on consumption goods only.

I.1. **The representative household.** Similar to Iacoviello (2005), the household has the utility function

$$\mathrm{E}\sum_{t=0}^{\infty}\beta^t A_t\left\{\log(C_{ht}-\gamma_h C_{h,t-1})+\varphi_t\log L_{ht}-\psi_t N_{ht}\right\}, \tag{1}$$

where C_{ht} denotes consumption, L_{ht} denotes land holdings, and N_{ht} denotes labor hours. The parameter $\beta\in(0,1)$ is a subjective discount factor, the parameter γ_h measures the degree of habit persistence, and the term E is a mathematical expectation operator. The terms A_t, φ_t, and ψ_t are preference shocks. We assume that the intertemporal preference shock A_t follows the stochastic process

$$A_t = A_{t-1}(1+\lambda_{at}),\quad \ln\lambda_{at}=(1-\rho_a)\ln\bar{\lambda}_a+\rho_a\ln\lambda_{a,t-1}+\varepsilon_{at}, \tag{2}$$

where $\bar{\lambda}_a>0$ is a constant, $\rho_a\in(-1,1)$ is the persistence parameter, and ε_{at} is an identically and independently distributed (i.i.d.) white noise process with mean zero

Date: April 7, 2013.

Supplemental materials for the paper "Land-price dynamics and macroeconomic fluctuations" by Zheng Liu, Pengfei Wang, and Tao Zha. We are grateful to Pat Higgins, who provides invaluable research assistance. The views expressed herein are those of the authors and do not necessarily reflect the views of the Federal Reserve Banks of Atlanta and San Francisco or the Federal Reserve System.

and variance σ_a^2. The housing preference shock φ_t follows the stationary process

$$\ln \varphi_t = (1 - \rho_\varphi) \ln \bar{\varphi} + \rho_\varphi \ln \varphi_{t-1} + \varepsilon_{\varphi t}, \tag{3}$$

where $\bar{\varphi} > 0$ is a constant, $\rho_\varphi \in (-1, 1)$ measures the persistence of the shock, and $\varepsilon_{\varphi t}$ is a white noise process with mean zero and variance σ_φ^2. The labor supply shock ψ_t follows the stationary process

$$\ln \psi_t = (1 - \rho_\psi) \ln \bar{\psi} + \rho_\psi \ln \psi_{t-1} + \varepsilon_{\psi t}, \tag{4}$$

where $\bar{\varphi} > 0$ is a constant, $\rho_\psi \in (-1, 1)$ measures the persistence, and $\varepsilon_{\psi t})$ is a white noise process with mean zero and variance σ_ψ^2.

Denote by q_{lt} the relative price of housing (in consumption units), R_t the gross real loan rate, and w_t the real wage; denote by S_t the household's purchase in period t of the loanable bond that pays off one unit of consumption good in all states of nature in period $t+1$. In period 0, the household begins with $L_{h,-1} > 0$ units of housing and $S_{-1} > 0$ units of the loanable bond. The flow of funds constraint for the household is given by

$$C_{ht} + q_{lt}(L_{ht} - L_{h,t-1}) + \frac{S_t}{R_t} \leq w_t N_{ht} + S_{t-1}. \tag{5}$$

The household chooses C_{ht}, $L_{h,t}$, N_{ht}, and S_t to maximize (1) subject to (2)-(5) and the borrowing constraint $S_t \geq -\bar{S}$ for some large number $\bar{S}$.

I.2. **The representative entrepreneur.** The entrepreneur has the utility function

$$\mathrm{E} \sum_{t=0}^{\infty} \beta^t \left[\log(C_{et} - \gamma_e C_{e,t-1}) \right], \tag{6}$$

where C_{et} denotes the entrepreneur's consumption and γ_e is the habit persistence parameter.

The entrepreneur produces goods using capital, labor, and land as inputs. The production function is given by

$$Y_t = Z_t [L_{e,t-1}^{\phi} K_{t-1}^{1-\phi}]^{\alpha} N_{et}^{1-\alpha}, \tag{7}$$

where Y_t denotes output, K_{t-1}, N_{et}, and $L_{e,t-1}$ denote the inputs capital, labor, and land, respectively, and the parameters $\alpha \in (0, 1)$ and $\phi \in (0, 1)$ measure the output elasticities of these production factors. We assume that the total factor productivity Z_t is composed of a permanent component Z_t^p and a transitory component ν_t such that $Z_t = Z_t^p \nu_{zt}$, where the permanent component Z_t^p follows the stochastic process

$$Z_t^p = Z_{t-1}^p \lambda_{zt}, \quad \ln \lambda_{zt} = (1 - \rho_z) \ln \bar{\lambda}_z + \rho_z \ln \lambda_{z,t-1} + \varepsilon_{zt}, \tag{8}$$

and the transitory component follows the stochastic process

$$\ln \nu_{zt} = \rho_{\nu_z} \ln \nu_{z,t-1} + \varepsilon_{\nu_z t}. \tag{9}$$

The parameter $\bar{\lambda}_z$ is the steady-state growth rate of Z_t^p; the parameters ρ_z and ρ_{ν_z} measure the degree of persistence. The innovations ε_{zt} and $\varepsilon_{\nu_z t}$ are i.i.d. white noise processes that are mutually independent with mean zero and variances given by σ_z^2 and $\sigma_{\nu_z}^2$, respectively.

The entrepreneur is endowed with K_{-1} units of initial capital stock and $L_{-1,e}$ units of initial land. Capital accumulation follows the law of motion

$$K_t = (1-\delta)K_{t-1} + \left[1 - \frac{\Omega}{2}\left(\frac{I_t}{I_{t-1}} - \bar{\lambda}_I\right)^2\right] I_t, \tag{10}$$

where I_t denotes investment, $\bar{\lambda}_I$ denotes the steady-state growth rate of investment, and $\Omega > 0$ is the adjustment cost parameter.

The entrepreneur faces the flow of funds constraint

$$C_{et} + q_{lt}(L_{et} - L_{e,t-1}) + B_{t-1} = Z_t[L_{e,t-1}^{\phi} K_{t-1}^{1-\phi}]^{\alpha} N_{et}^{1-\alpha} - \frac{I_t}{Q_t} - w_t N_{et} + \frac{B_t}{R_t}, \tag{11}$$

where B_{t-1} is the amount of matured debt and B_t/R_t is the value of new debt. Following Greenwood, Hercowitz, and Krusell (1997), we interpret Q_t as the investment-specific technological change. Specifically, we assume that $Q_t = Q_t^p \nu_{qt}$, where the permanent component Q_t^p follows the stochastic process

$$Q_t^p = Q_{t-1}^p \lambda_{qt}, \quad \ln \lambda_{qt} = (1-\rho_q)\ln \bar{\lambda}_q + \rho_q \ln \lambda_{q,t-1} + \varepsilon_{qt}, \tag{12}$$

and the transitory component follows the stochastic process

$$\ln \nu_{qt} = \rho_{\nu_q} \ln \nu_{q,t-1} + \varepsilon_{\nu_q t}. \tag{13}$$

The parameter $\bar{\lambda}_q$ is the steady-state growth rate of Q_t^p; the parameters ρ_q and ρ_{ν_q} measure the degree of persistence. The innovations ε_{qt} and $\varepsilon_{\nu_q t}$ are i.i.d. white noise processes that are mutually independent with mean zero and variances given by σ_q^2 and $\sigma_{\nu_q}^2$, respectively.

The entrepreneur faces the credit constraint

$$B_t \leq \theta_t E_t[q_{l,t+1} L_{et} + q_{k,t+1} K_t], \tag{14}$$

where $q_{k,t+1}$ is the shadow price of capital in consumption units.[1] Under this credit constraint, the amount that the entrepreneur can borrow is limited by a fraction of

[1]Since the price of new capital is $1/Q_t$, Tobin's q in this model is given by $q_{kt}Q_t$, which is the ratio of the value of installed capital to the price of new capital.

the value of the collateral assets—land and capital. Following Kiyotaki and Moore (1997), we interpret this type of credit constraints as reflecting the problem of costly contract enforcement: if the entrepreneur fails to pay the debt, the creditor can seize the land and the accumulated capital; since it is costly to liquidate the seized land and capital stock, the creditor can recoup up to a fraction θ_t of the total value of the collateral assets. We interpret θ_t as a "collateral shock" that reflects the uncertainty in the tightness of the credit market. We assume that θ_t follows the stochastic process

$$\ln \theta_t = (1 - \rho_\theta) \ln \bar{\theta} + \rho_\theta \ln \theta_{t-1} + \varepsilon_{\theta t}, \tag{15}$$

where $\bar{\theta}$ is the steady-state value of θ_t, $\rho_\theta \in (0, 1)$ is the persistence parameter, and $\varepsilon_{\theta t}$ is an i.i.d. white noise process with mean zero and variance σ_θ^2.

The entrepreneur chooses C_{et}, N_{et}, I_t, $L_{e,t}$, K_t, and B_t to maximize (6) subject to (7) through (15).

I.3. **Market clearing conditions and equilibrium.** In a competitive equilibrium, the markets for goods, labor, land, and loanable bonds all clear. The goods market clearing condition implies that

$$C_t + \frac{I_t}{Q_t} = Y_t, \tag{16}$$

where $C_t = C_{ht} + C_{et}$ denotes aggregate consumption. The labor market clearing condition implies that labor demand equals labor supply:

$$N_{et} = N_{ht} \equiv N_t. \tag{17}$$

The land market clearing condition implies that

$$L_{ht} + L_{et} = \bar{L}, \tag{18}$$

where $\bar{L}$ is the fixed aggregate land endowment. Finally, the bond market clearing condition implies that

$$S_t = B_t. \tag{19}$$

A competitive equilibrium consists of sequences of prices $\{w_t, q_{lt}, R_t\}_{t=0}^{\infty}$ and allocations $\{C_{ht}, C_{et}, I_t, N_{ht}, N_{et}, L_{ht}, L_{et}, S_t, B_t, K_t, Y_t\}_{t=0}^{\infty}$ such that (i) taking the prices as given, the allocations solve the optimizing problems for the household and the entrepreneur and (ii) all markets clear.

II. Derivations of excess returns and equilibrium conditions

II.1. **The excess returns.** In this section, we provide an intuitive derivation of the first-order excess returns in the presence of binding credit constraints.

The representative entrepreneur has two types of assets: land and capital. Each asset can be intuitively thought of as a Lucas tree bearing fruits and growing at a gross rate of g_γ. The entrepreneur can trade a portion of the tree in the market, and the return on this tree depends on the price of a unit of the tree as well as the marginal product (fruit) of the remaining tree. In steady state, it should be g_γ/β. To see if this intuition works in the model when the entrepreneur faces the borrowing constraint, we first derive the expected return on each of these assets. We begin with the return on land.

Suppose the entrepreneur purchases one unit of land at the price q_{lt} in period t. Since she can pledge a fraction θ_t of the present value of the land as a collateral, the net out-of-pocket payment (i.e., the down payment) to purchase the land is given by

$$u_t \equiv q_{lt} - \theta_t \mathrm{E}_t \frac{q_{l,t+1}}{R_t}, \tag{20}$$

where R_t is the loan rate. The land is used for period $t+1$ production and yields $\phi\alpha Y_{t+1}/L_{et}$ units of extra output. In addition, the entrepreneur can keep the remaining value of the land in period $t+1$ after repaying the debt so that the total payoff from the land is $\phi\alpha Y_{t+1}/L_{et} + q_{l,t+1} - \theta_t \mathrm{E}_t q_{l,t+1}$. The return on the land from period t to $t+1$ is thus given by

$$R_{l,t+1} = \frac{\phi\alpha Y_{t+1}/L_{et} + q_{l,t+1} - \theta_t \mathrm{E}_t q_{l,t+1}}{q_{lt} - \theta_t \mathrm{E}_t \frac{q_{l,t+1}}{R_t}}. \tag{21}$$

We can similarly derive the return on capital, which is given by

$$R_{k,t+1} = \frac{\phi\alpha Y_{t+1}/K_t + q_{k,t+1}(1-\delta) - \theta_t \mathrm{E}_t q_{k,t+1}}{q_{kt} - \theta_t \mathrm{E}_t \frac{q_{k,t+1}}{R_t}}. \tag{22}$$

To see how these returns relate to the entrepreneur's optimal decisions, we denote by μ_{et} the Lagrangian multiplier for the flow of funds constraint (11), μ_{kt} the multiplier for the capital accumulation equation (10), and μ_{bt} the multiplier for the credit constraint (14). With these notations, the shadow price of capital in consumption units is given by

$$q_{kt} = \frac{\mu_{kt}}{\mu_{et}},$$

and the marginal utility of income, μ_{et}, is equal to the marginal utility of consumption:

$$\mu_{et} = \frac{1}{C_{et} - \gamma_e C_{e,t-1}} - \mathrm{E}_t \frac{\beta\gamma_e}{C_{e,t+1} - \gamma_e C_{et}}.$$

The optimal decision on the entrepreneur's borrowing can be described by

$$\frac{1}{R_t} = \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} + \frac{\mu_{bt}}{\mu_{et}}. \tag{23}$$

The above Euler equation implies that the credit constraint is binding (i.e., $\mu_{bt} > 0$) if and only if the interest rate is lower than the entrepreneur's intertemporal marginal rate of substitution. The entrepreneur's optimal decisions on land and capital can be described by the following two Euler equations:

$$q_{lt} = \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} \left[\alpha\phi \frac{Y_{t+1}}{L_{et}} + q_{l,t+1} \right] + \frac{\mu_{bt}}{\mu_{et}} \theta_t \mathrm{E}_t q_{l,t+1}, \tag{24}$$

$$q_{kt} = \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} \left[\alpha(1-\phi) \frac{Y_{t+1}}{K_t} + q_{k,t+1}(1-\delta) \right] + \frac{\mu_{bt}}{\mu_{et}} \theta_t \mathrm{E}_t q_{k,t+1}. \tag{25}$$

Using (23), we can rewrite Equations (24) and (25) as

$$1 = \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} R_{j,t+1}, \quad j \in \{l, k\}. \tag{26}$$

Since consumption grows at the rate g_λ in equilibrium and the utility function is of logarithmic form, (26) implies that $R_j = g_\lambda / \beta$.

On the other hand, the loan rate R_t is determined by the household's intertemporal Euler equation:

$$\frac{1}{R_t} = \beta \mathrm{E}_t \frac{\mu_{h,t+1}}{\mu_{ht}}, \tag{27}$$

where μ_{ht} is the Lagrangian multiplier for the flow of funds constraint (5). It represents the marginal utility of income and is equal to the marginal utility of consumption:

$$\mu_{ht} = A_t \left[\frac{1}{C_{ht} - \gamma_h C_{h,t-1}} - \mathrm{E}_t \frac{\beta\gamma_h}{C_{h,t+1} - \gamma_h C_{ht}} (1 + \lambda_{a,t+1}) \right].$$

It follows from (27) that in steady state, $R = \frac{g_\gamma}{\beta(1+\bar{\lambda}_a)}$, where $\bar{\lambda}_a > 0$ measures the extent to which the household is more patient than the entrepreneur. The steady state excess return is then given by

$$R_j^e \equiv R_j - R = \frac{g_\gamma}{\beta} \frac{\bar{\lambda}_a}{1 + \bar{\lambda}_a}, \quad j \in \{l, k\}. \tag{28}$$

Clearly, the steady-state excess return is positive if and only if the patience factor, $\bar{\lambda}_a$, is positive.

To see how a positive first-order excess return is related to the entrepreneur's credit constraint, one can derive from (23) the following steady state relationship:

$$\frac{\beta\bar{\lambda}_a}{g_\gamma} = \frac{\tilde{\mu}_b}{\tilde{\mu}_e}.$$

Thus, the credit constraint is binding (i.e., $\tilde{\mu}_b > 0$) if and only if the household is more patient than the entrepreneur (i.e., $\bar{\lambda}_a > 0$).

This result carries over to the dynamics of excess returns. Denote by $R^e_{j,t+1} \equiv R_{j,t+1} - R_t$ the excess return for asset $j \in \{l, k\}$. By combining the bond Euler equation (23) and the asset-pricing equation (26), we obtain

$$\beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} R^e_{j,t+1} = \frac{\mu_{bt}}{\mu_{et}} R_t, \quad j \in \{l, k\}. \tag{29}$$

As in the standard asset-pricing model, the mean excess return depends on the asset's riskiness measured by the covariance between the return and the marginal utility of consumption. Unlike the standard model, however, the excess return in our model contains a first-order term that is positive if and only if the borrowing constraint is binding (i.e., $\mu_{bt} > 0$).

II.2. **Euler equations.** Denote by μ_{ht} the Lagrangian multiplier for the flow of funds constraint (5). The first-order conditions for the household's optimizing problem are given by

$$\mu_{ht} = A_t \left[\frac{1}{C_{ht} - \gamma_h C_{h,t-1}} - \mathrm{E}_t \frac{\beta\gamma_h}{C_{h,t+1} - \gamma_h C_{ht}} (1 + \lambda_{a,t+1}) \right], \tag{30}$$

$$w_t = \frac{A_t}{\mu_{ht}} \psi_t, \tag{31}$$

$$q_{lt} = \beta \mathrm{E}_t \frac{\mu_{h,t+1}}{\mu_{ht}} q_{l,t+1} + \frac{A_t \varphi_t}{\mu_{ht} L_{ht}}, \tag{32}$$

$$\frac{1}{R_t} = \beta \mathrm{E}_t \frac{\mu_{h,t+1}}{\mu_{ht}}. \tag{33}$$

Equation (30) equates the marginal utility of income and of consumption; equation (31) equates the real wage and the marginal rate of substitution (MRS) between leisure and income; equation (32) equates the current relative price of land to the marginal benefit of purchasing an extra unit of land, which consists of the current utility benefits (i.e., the MRS between housing and consumption) and the land's discounted future resale value; and equation (33) is the standard Euler equation for the loanable bond.

Denote by μ_{et} the Lagrangian multiplier for the flow of funds constraint (11), μ_{kt} the multiplier for the capital accumulation equation (10), and μ_{bt} the multiplier for the borrowing constraint (14). With these notations, the shadow price of capital in consumption units is given by

$$q_{kt} = \frac{\mu_{kt}}{\mu_{et}}. \tag{34}$$

The first-order conditions for the entrepreneur's optimizing problem are given by

$$\begin{aligned}
\mu_{et} &= \frac{1}{C_{et} - \gamma_e C_{e,t-1}} - \mathrm{E}_t \frac{\beta \gamma_e}{C_{e,t+1} - \gamma_e C_{et}}, && (35)\\
w_t &= (1-\alpha) Y_t / N_{et}, && (36)\\
\frac{1}{Q_t} &= q_{kt} \left[1 - \frac{\Omega}{2} \left(\frac{I_t}{I_{t-1}} - \bar{\lambda}_I \right)^2 - \Omega \left(\frac{I_t}{I_{t-1}} - \bar{\lambda}_I \right) \frac{I_t}{I_{t-1}} \right] \\
&\quad + \beta \Omega \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} q_{k,t+1} \left(\frac{I_{t+1}}{I_t} - \bar{\lambda}_I \right) \left(\frac{I_{t+1}}{I_t} \right)^2, && (37)\\
q_{kt} &= \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} \left[\alpha (1-\phi) \frac{Y_{t+1}}{K_t} + q_{k,t+1}(1-\delta) \right] + \frac{\mu_{bt}}{\mu_{et}} \theta_t \mathrm{E}_t q_{k,t+1}, && (38)\\
q_{lt} &= \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} \left[\alpha \phi \frac{Y_{t+1}}{L_{et}} + q_{l,t+1} \right] + \frac{\mu_{bt}}{\mu_{et}} \theta_t \mathrm{E}_t q_{l,t+1}, && (39)\\
\frac{1}{R_t} &= \beta \mathrm{E}_t \frac{\mu_{e,t+1}}{\mu_{et}} + \frac{\mu_{bt}}{\mu_{et}}. && (40)
\end{aligned}$$

Equation (35) equates the marginal utility of income to the marginal utility of consumption since consumption is the numéraire; equation (36) is the labor demand equation which equates the real wage to the marginal product of labor; equation (37) is the investment Euler equation, which equates the cost of purchasing an additional unit of investment good and the benefit of having an extra unit of new capital, where the benefit includes the shadow value of the installed capital net of adjustment costs and the present value of the saved future adjustment costs; equation (38) is the capital Euler equation, which equates the shadow price of capital to the present value of future marginal product of capital and the resale value of the un-depreciated capital, plus the value of capital as a collateral asset for borrowing; equation (39) is the land Euler equation, which equates the price of the land to the present value of the future marginal product of land and the resale value, plus the value of land as a collateral asset for borrowing; equation (40) is the bond Euler equation for the entrepreneur, which reveals that the borrowing constraint is binding (i.e., $\mu_{bt} > 0$) if and only if the interest rate is lower than the entrepreneur's intertemporal marginal rate of substitution.

II.3. **Stationary equilibrium.** We are interested in studying the fluctuations around the balanced growth path. For this purpose, we focus on a stationary equilibrium by appropriately transforming the growing variables. Specifically, we make the following transformations of the variables

$$\tilde{Y}_t \equiv \frac{Y_t}{\Gamma_t}, \quad \tilde{C}_{ht} \equiv \frac{C_{ht}}{\Gamma_t}, \quad \tilde{C}_{et} \equiv \frac{C_{et}}{\Gamma_t}, \quad \tilde{I}_t \equiv \frac{I_t}{Q_t\Gamma_t}, \quad \tilde{K}_t \equiv \frac{K_t}{Q_t\Gamma_t}, \quad \tilde{B}_t \equiv \frac{B_t}{\Gamma_t},$$
$$\tilde{w}_t \equiv \frac{w_t}{\Gamma_t}, \quad \tilde{\mu}_{ht} \equiv \frac{\mu_{ht}\Gamma_t}{A_t}, \quad \tilde{\mu}_{et} \equiv \mu_{et}\Gamma_t, \quad \tilde{\mu}_{bt} \equiv \mu_{bt}\Gamma_t, \quad \tilde{q}_{lt} \equiv \frac{q_{lt}}{\Gamma_t}, \quad \tilde{q}_{kt} \equiv q_{kt}Q_t, \quad (41)$$

where $\Gamma_t \equiv [Z_t Q_t^{(1-\phi)\alpha}]^{\frac{1}{1-(1-\phi)\alpha}}$. In Appendix II, we describe the stationary equilibrium and derive the log-linearized equilibrium conditions around the steady state for solving the model. To solve the log-linearized equilibrium system requires the input of several key steady-state values. These include the shadow value of the loanable funds $\frac{\tilde{\mu}_b}{\tilde{\mu}_e}$, the ratio of commercial real estate to aggregate output $\frac{\tilde{q}_l L_e}{\tilde{Y}}$, the ratio of residential land to commercial real estate $\frac{L_h}{L_e}$, the ratio of loanable funds to output $\frac{\tilde{B}}{\tilde{Y}}$, the capital-output ratio $\frac{\tilde{K}}{\tilde{Y}}$, and the "big ratios" $\frac{\tilde{C}_h}{\tilde{Y}}$, $\frac{\tilde{C}_e}{\tilde{Y}}$, and $\frac{\tilde{I}}{\tilde{Y}}$. The model implies a set of restrictions between these steady-state ratios and the parameters, and we will use these restrictions along with the first moments of selected time series in the data to sharpen our priors and to help identify a subset of the parameters in our estimation.

Denote by $g_{\gamma t} \equiv \frac{\Gamma_t}{\Gamma_{t-1}}$ and $g_{qt} \equiv \frac{Q_t}{Q_{t-1}}$ the growth rates for the exogenous variables Γ_t and Q_t. Denote by g_γ the steady-state value of $g_{\gamma t}$ and $\lambda_k \equiv g_\gamma \bar{\lambda}_q$ the steady-state growth rate of capital stock. On the balanced growth path, investment grows at the same rate as does capital, so we have $\bar{\lambda}_I = \lambda_k$.

The stationary equilibrium is the solution to the following system of equations:

$$\tilde{\mu}_{ht} = \frac{1}{\tilde{C}_{ht} - \gamma_h \tilde{C}_{h,t-1}\Gamma_{t-1}/\Gamma_t} - \mathrm{E}_t \frac{\beta\gamma_h}{\tilde{C}_{h,t+1}\Gamma_{t+1}/\Gamma_t - \gamma_h \tilde{C}_{ht}}(1+\lambda_{a,t+1}), \quad (42)$$

$$\tilde{w}_t = \frac{\psi_t}{\tilde{\mu}_{ht}}, \quad (43)$$

$$\tilde{q}_{lt} = \beta \mathrm{E}_t \frac{\tilde{\mu}_{h,t+1}}{\tilde{\mu}_{ht}}(1+\lambda_{a,t+1})\tilde{q}_{l,t+1} + \frac{\varphi_t}{\tilde{\mu}_{ht}L_{ht}}, \quad (44)$$

$$\frac{1}{R_t} = \beta \mathrm{E}_t \frac{\tilde{\mu}_{h,t+1}}{\tilde{\mu}_{ht}} \frac{\Gamma_t}{\Gamma_{t+1}}(1+\lambda_{a,t+1}). \quad (45)$$

$$\tilde{\mu}_{et} = \frac{1}{\tilde{C}_{et} - \gamma_e \tilde{C}_{e,t-1}\Gamma_{t-1}/\Gamma_t} - \mathrm{E}_t \frac{\beta\gamma_e}{\tilde{C}_{e,t+1}\Gamma_{t+1}/\Gamma_t - \gamma_e \tilde{C}_{et}}, \quad (46)$$

$$\tilde{w}_t = (1-\alpha)\tilde{Y}_t/N_t, \quad (47)$$

$$1 = \tilde{q}_{kt}\left[1 - \frac{\Omega}{2}\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}} - \bar{\lambda}_I\right)^2 - \Omega\left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}} - \bar{\lambda}_I\right)\frac{\tilde{I}_t}{\tilde{I}_{t-1}}\frac{Q_t\Gamma_t}{Q_{t-1}\Gamma_{t-1}}\right]$$

$$+\beta\Omega \mathrm{E}_t \frac{\tilde{\mu}_{e,t+1}}{\tilde{\mu}_{et}} \frac{Q_t \Gamma_t}{Q_{t+1}\Gamma_{t+1}} \tilde{q}_{k,t+1} \left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t} \frac{Q_{t+1}\Gamma_{t+1}}{Q_t \Gamma_t} - \bar{\lambda}_I \right) \left(\frac{\tilde{I}_{t+1}}{\tilde{I}_t} \frac{Q_{t+1}\Gamma_{t+1}}{Q_t \Gamma_t} \right)^2, \quad (48)$$

$$\tilde{q}_{kt} = \beta \mathrm{E}_t \frac{\tilde{\mu}_{e,t+1}}{\tilde{\mu}_{et}} \left[\alpha(1-\phi) \frac{\tilde{Y}_{t+1}}{\tilde{K}_t} + \tilde{q}_{k,t+1} \frac{Q_t \Gamma_t}{Q_{t+1}\Gamma_{t+1}} (1-\delta) \right] + \frac{\tilde{\mu}_{bt}}{\tilde{\mu}_{et}} \theta_t \mathrm{E}_t \tilde{q}_{k,t+1} \frac{Q_t}{Q_{t+1}}, \quad (49)$$

$$\tilde{q}_{lt} = \beta \mathrm{E}_t \frac{\tilde{\mu}_{e,t+1}}{\tilde{\mu}_{et}} \left[\alpha\phi \frac{\tilde{Y}_{t+1}}{L_{et}} + \tilde{q}_{l,t+1} \right] + \frac{\tilde{\mu}_{bt}}{\tilde{\mu}_{et}} \theta_t \mathrm{E}_t \tilde{q}_{l,t+1} \frac{\Gamma_{t+1}}{\Gamma_t}, \quad (50)$$

$$\frac{1}{R_t} = \beta \mathrm{E}_t \frac{\tilde{\mu}_{e,t+1}}{\tilde{\mu}_{et}} \frac{\Gamma_t}{\Gamma_{t+1}} + \frac{\tilde{\mu}_{bt}}{\tilde{\mu}_{et}}. \quad (51)$$

$$\tilde{Y}_t = \left(\frac{Z_t Q_t}{Z_{t-1} Q_{t-1}} \right)^{-\frac{(1-\phi)\alpha}{1-(1-\phi)\alpha}} [L_{e,t-1}^{\phi} \tilde{K}_{t-1}^{1-\phi}]^{\alpha} N_t^{1-\alpha}, \quad (52)$$

$$\tilde{K}_t = (1-\delta)\tilde{K}_{t-1} \frac{Q_{t-1}\Gamma_{t-1}}{Q_t \Gamma_t} + \left[1 - \frac{\Omega}{2} \left(\frac{\tilde{I}_t}{\tilde{I}_{t-1}} \frac{Q_t \Gamma_t}{Q_{t-1}\Gamma_{t-1}} - \bar{\lambda}_I \right)^2 \right] \tilde{I}_t, \quad (53)$$

$$\tilde{Y}_t = \tilde{C}_{ht} + \tilde{C}_{et} + \tilde{I}_t, \quad (54)$$

$$\bar{L} = L_{ht} + L_{et}, \quad (55)$$

$$\alpha \tilde{Y}_t = \tilde{C}_{et} + \tilde{I}_t + \tilde{q}_{lt}(L_{et} - L_{e,t-1}) + \tilde{B}_{t-1} \frac{\Gamma_{t-1}}{\Gamma_t} - \frac{\tilde{B}_t}{R_t}, \quad (56)$$

$$\tilde{B}_t = \theta_t \mathrm{E}_t \left[\tilde{q}_{l,t+1} \frac{\Gamma_{t+1}}{\Gamma_t} L_{et} + \tilde{q}_{k,t+1} \tilde{K}_t \frac{Q_t}{Q_{t+1}} \right]. \quad (57)$$

We solve these 16 equations for 16 variables summarized in the vector

$$[\tilde{\mu}_{ht}, \tilde{w}_t, \tilde{q}_{lt}, R_t, \tilde{\mu}_{et}, N_t, \tilde{I}_t, \tilde{Y}_t, \tilde{C}_{ht}, \tilde{C}_{et}, \tilde{q}_{kt}, L_{et}, L_{ht}, \tilde{K}_t, \tilde{B}_t, \tilde{\mu}_{bt}]'.$$

II.4. **Steady state.** To get the steady-state value for $\frac{\tilde{\mu}_b}{\tilde{\mu}_e}$, we use the stationary bond Euler equations (45) for the household and (51) (described in the Appendix) to obtain

$$\frac{1}{R} = \frac{\beta(1+\bar{\lambda}_a)}{g_\gamma}, \quad \frac{\tilde{\mu}_b}{\tilde{\mu}_e} = \frac{\beta\bar{\lambda}_a}{g_\gamma}. \quad (58)$$

Since $\bar{\lambda}_a > 0$, we have $\tilde{\mu}_b > 0$ and the borrowing constraint is binding in the steady-state equilibrium.

To get the ratio of commercial real estate to output, we use the land Euler equation (50) for the entrepreneur, the definition of $\tilde{\mu}_e$ in (46), and the solution for $\frac{\tilde{\mu}_b}{\tilde{\mu}_e}$ in (58). In particular, we have

$$\frac{\tilde{q}_l L_e}{\tilde{Y}} = \frac{\beta\alpha\phi}{1-\beta-\beta\bar{\lambda}_a\bar{\theta}}. \quad (59)$$

To get the investment-output ratio, we first solve for the investment-capital ratio by using the law of motion for capital stock in (53) and then solve for the capital-output ratio using the capital Euler equation (49). Specifically, we have

$$\frac{\tilde{I}}{\tilde{K}} = 1 - \frac{1-\delta}{\lambda_k}, \tag{60}$$

$$\frac{\tilde{K}}{\tilde{Y}} = \left[1 - \frac{\beta}{\lambda_k}(\bar{\lambda}_a \bar{\theta} + 1 - \delta)\right]^{-1} \beta\alpha(1-\phi), \tag{61}$$

where we have used the steady-state condition that $\tilde{q}_k = 1$, as implied by the investment Euler equation (48). The investment-output ratio is then given by

$$\frac{\tilde{I}}{\tilde{Y}} = \frac{\tilde{I}}{\tilde{K}}\frac{\tilde{K}}{\tilde{Y}} = \frac{\beta\alpha(1-\phi)[\lambda_k - (1-\delta)]}{\lambda_k - \beta(\bar{\lambda}_a\bar{\theta} + 1 - \delta)}. \tag{62}$$

Given the solution for the ratios $\frac{\tilde{q}_l L_e}{\tilde{Y}}$ and $\frac{\tilde{K}}{\tilde{Y}}$ in (59) and (61), the binding borrowing constraint (57) implies that

$$\frac{\tilde{B}}{\tilde{Y}} = \bar{\theta} g_\gamma \frac{\tilde{q}_l L_e}{\tilde{Y}} + \frac{\bar{\theta}}{\bar{\lambda}_q}\frac{\tilde{K}}{\tilde{Y}}. \tag{63}$$

The entrepreneur's flow of funds constraint (56) implies that

$$\frac{\tilde{C}_e}{\tilde{Y}} = \alpha - \frac{\tilde{I}}{\tilde{Y}} - \frac{1-\beta(1+\bar{\lambda}_a)}{g_\gamma}\frac{\tilde{B}}{\tilde{Y}}. \tag{64}$$

The aggregate resource constraint (54) then implies that

$$\frac{\tilde{C}_h}{\tilde{Y}} = 1 - \frac{\tilde{C}_e}{\tilde{Y}} - \frac{\tilde{I}}{\tilde{Y}}. \tag{65}$$

To solve for $\frac{L_h}{L_e}$, we first use the household's land Euler equation (i.e., the housing demand equation) (44) and the definition for the marginal utility (42) to obtain

$$\frac{\tilde{q}_l L_h}{\tilde{C}_h} = \frac{\bar{\varphi}(g_\gamma - \gamma_h)}{g_\gamma(1 - g_\gamma/R)(1-\gamma_h/R)}, \tag{66}$$

where the steady-state loan rate is given by (58).

Taking the ratio between (66) and (59) results in the solution

$$\frac{L_h}{L_e} = \frac{\bar{\varphi}(g_\gamma - \gamma_h)(1-\beta-\beta\bar{\lambda}_a\bar{\theta})}{\beta\alpha\phi g_\gamma(1-g_\gamma/R)(1-\gamma_h/R)}\frac{\tilde{C}_h}{\tilde{Y}}. \tag{67}$$

Finally, we can solve for the steady-state hours by combining the labor supply equation (43) and the labor demand equation (47) to get

$$N = \frac{(1-\alpha)g_\gamma(1-\gamma_h/R)}{\bar{\psi}(g_\gamma - \gamma_h)}\frac{\tilde{Y}}{\tilde{C}_h}. \tag{68}$$

II.5. **Log-linearized equilibrium system.** Upon obtaining the steady-state equilibrium, we log-linearize the equilibrium conditions (42) through (57) around the steady state. We define the constants $\Omega_h \equiv (g_\gamma - \beta(1+\bar{\lambda}_a)\gamma_h)(g_\gamma - \gamma_h)$ and $\Omega_e \equiv (g_\gamma - \beta\gamma_e)(g_\gamma - \gamma_e)$. The log-linearized equilibrium conditions are given by

$$\begin{aligned}
\Omega_h \hat{\mu}_{ht} &= -[g_\gamma^2 + \gamma_h^2 \beta(1+\bar{\lambda}_a)]\hat{C}_{ht} + g_\gamma \gamma_h (\hat{C}_{h,t-1} - \hat{g}_{\gamma t}) \\
&\quad -\beta \bar{\lambda}_a \gamma_h (g_\gamma - \gamma_h) \mathrm{E}_t \hat{\lambda}_{a,t+1} + \beta(1+\bar{\lambda}_a) g_\gamma \gamma_h \mathrm{E}_t (\hat{C}_{h,t+1} + \hat{g}_{\gamma,t+1}), && (69) \\
\hat{w}_t + \hat{\mu}_{ht} &= \hat{\psi}_t, && (70) \\
\hat{q}_{lt} + \hat{\mu}_{ht} &= \beta(1+\bar{\lambda}_a)\mathrm{E}_t\left[\hat{\mu}_{h,t+1} + \hat{q}_{l,t+1}\right] \\
&\quad +[1-\beta(1+\bar{\lambda}_a)](\hat{\varphi}_t - \hat{L}_{ht}) + \beta\bar{\lambda}_a \mathrm{E}_t \hat{\lambda}_{a,t+1}, && (71) \\
\hat{\mu}_{ht} - \hat{R}_t &= \mathrm{E}_t\left[\hat{\mu}_{h,t+1} + \frac{\bar{\lambda}_a}{1+\bar{\lambda}_a}\hat{\lambda}_{a,t+1} - \hat{g}_{\gamma,t+1}\right], && (72) \\
\Omega_e \hat{\mu}_{et} &= -(g_\gamma^2 + \beta\gamma_e^2)\hat{C}_{e,t} + g_\gamma\gamma_e(\hat{C}_{e,t-1} - \hat{g}_{\gamma t}) + \beta g_\gamma \gamma_e \mathrm{E}_t(\hat{C}_{e,t+1} + \hat{g}_{\gamma,t+1}), && (73) \\
\hat{w}_t &= \hat{Y}_t - \hat{N}_t, && (74) \\
\hat{q}_{kt} &= (1+\beta)\Omega\lambda_k^2 \hat{I}_t - \Omega\lambda_k^2 \hat{I}_{t-1} + \Omega\lambda_k^2(\hat{g}_{\gamma t} + \hat{g}_{qt}) \\
&\quad -\beta\Omega\lambda_k^2 \mathrm{E}_t[\hat{I}_{t+1} + \hat{g}_{\gamma,t+1} + \hat{g}_{q,t+1}], && (75) \\
\hat{q}_{kt} + \hat{\mu}_{et} &= \frac{\tilde{\mu}_b}{\tilde{\mu}_e}\frac{\bar{\theta}}{\bar{\lambda}_q}(\hat{\mu}_{bt} + \hat{\theta}_t) + \frac{\beta(1-\delta)}{\lambda_k}\mathrm{E}_t(\hat{q}_{k,t+1} - \hat{g}_{q,t+1} - \hat{g}_{\gamma,t+1}) + \left(1 - \frac{\tilde{\mu}_b}{\tilde{\mu}_e}\frac{\bar{\theta}}{\bar{\lambda}_q}\right)\mathrm{E}_t\hat{\mu}_{e,t+1} \\
&\quad + \frac{\tilde{\mu}_b}{\tilde{\mu}_e}\frac{\bar{\theta}}{\bar{\lambda}_q}\mathrm{E}_t(\hat{q}_{k,t+1} - \hat{g}_{q,t+1}) + \beta\alpha(1-\phi)\frac{\tilde{Y}}{\tilde{K}}\mathrm{E}_t(\hat{Y}_{t+1} - \hat{K}_t), && (76) \\
\hat{q}_{lt} + \hat{\mu}_{et} &= \frac{\tilde{\mu}_b}{\tilde{\mu}_e} g_\gamma \bar{\theta}(\hat{\theta}_t + \hat{\mu}_{bt}) + \left(1 - \frac{\tilde{\mu}_b}{\tilde{\mu}_e} g_\gamma \theta\right)\mathrm{E}_t\hat{\mu}_{e,t+1} + \frac{\tilde{\mu}_b}{\tilde{\mu}_e} g_\gamma \bar{\theta}\mathrm{E}_t(\hat{q}_{l,t+1} + \hat{g}_{\gamma,t+1}) \\
&\quad +\beta\mathrm{E}_t\hat{q}_{l,t+1} + (1-\beta-\beta\bar{\lambda}_a\bar{\theta})\mathrm{E}_t[\hat{Y}_{t+1} - \hat{L}_{et}], && (77) \\
\hat{\mu}_{et} - \hat{R}_t &= \frac{1}{1+\bar{\lambda}_a}\left[\mathrm{E}_t(\hat{\mu}_{e,t+1} - \hat{g}_{\gamma,t+1}) + \bar{\lambda}_a\hat{\mu}_{bt}\right], && (78) \\
\hat{Y}_t &= \alpha\phi\hat{L}_{e,t-1} + \alpha(1-\phi)\hat{K}_{t-1} + (1-\alpha)\hat{N}_t - \frac{(1-\phi)\alpha}{1-(1-\phi)\alpha}[\hat{g}_{zt} + \hat{g}_{qt}], && (79) \\
\hat{K}_t &= \frac{1-\delta}{\lambda_k}[\hat{K}_{t-1} - \hat{g}_{\gamma t} - \hat{g}_{qt}] + \left(1 - \frac{1-\delta}{\lambda_k}\right)\hat{I}_t, && (80) \\
\hat{Y}_t &= \frac{\tilde{C}_h}{\tilde{Y}}\hat{C}_{ht} + \frac{C_e}{\tilde{Y}}\hat{C}_{e,t} + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t, && (81) \\
0 &= \frac{L_h}{\bar{L}}\hat{L}_{ht} + \frac{L_e}{\bar{L}}\hat{L}_{et}, && (82) \\
\alpha\hat{Y}_t &= \frac{\tilde{C}_e}{\tilde{Y}}\hat{C}_{e,t} + \frac{\tilde{I}}{\tilde{Y}}\hat{I}_t + \frac{\tilde{q}_l L_e}{\tilde{Y}}(\hat{L}_{et} - \hat{L}_{e,t-1})
\end{aligned}$$

$$
+\frac{1}{g_\gamma}\frac{\tilde{B}}{\tilde{Y}}(\hat{B}_{t-1}-\hat{g}_{\gamma t})-\frac{1}{R}\frac{\tilde{B}}{\tilde{Y}}(\hat{B}_t-\hat{R}_t), \tag{83}
$$

$$
\begin{aligned}
\hat{B}_t &= \hat{\theta}_t+g_\gamma\bar{\theta}\frac{\tilde{q}_l L_e}{\tilde{B}}\mathrm{E}_t(\hat{q}_{l,t+1}+\hat{L}_{et}+\hat{g}_{\gamma,t+1}) \\
&\quad +\left(1-g_\gamma\bar{\theta}\frac{\tilde{q}_l L_e}{\tilde{B}}\right)\mathrm{E}_t(\hat{q}_{k,t+1}+\hat{K}_t-\hat{g}_{q,t+1}).
\end{aligned} \tag{84}
$$

The terms $\hat{g}_{zt}$, $\hat{g}_{qt}$, and $\hat{g}_{\gamma t}$ are given by

$$
\hat{g}_{zt} = \hat{\lambda}_{zt}+\hat{\nu}_{zt}-\hat{v}_{z,t-1}, \tag{85}
$$

$$
\hat{g}_{qt} = \hat{\lambda}_{qt}+\hat{\nu}_{qt}-\hat{v}_{q,t-1}, \tag{86}
$$

$$
\hat{g}_{\gamma t} = \frac{1}{(1-(1-\phi)\alpha)}\hat{g}_{zt}+\frac{(1-\phi)\alpha}{(1-(1-\phi)\alpha)}\hat{g}_{qt}. \tag{87}
$$

The technology shocks follow the processes

$$
\hat{\lambda}_{zt} = \rho_z\hat{\lambda}_{z,t-1}+\hat{\varepsilon}_{zt}, \tag{88}
$$

$$
\hat{\nu}_{zt} = \rho_{\nu_z}\hat{\nu}_{z,t-1}+\hat{\varepsilon}_{\nu_z t}, \tag{89}
$$

$$
\hat{\lambda}_{qt} = \rho_q\hat{\lambda}_{q,t-1}+\hat{\varepsilon}_{qt}, \tag{90}
$$

$$
\hat{\nu}_{qt} = \rho_{\nu_q}\hat{\nu}_{q,t-1}+\hat{\varepsilon}_{\nu_q t}. \tag{91}
$$

$$
\tag{92}
$$

There preference shocks follow the processes

$$
\hat{\lambda}_{at} = \rho_a\hat{\lambda}_{a,t-1}+\hat{\varepsilon}_{at}, \tag{93}
$$

$$
\hat{\varphi}_t = \rho_\varphi\hat{\varphi}_{t-1}+\hat{\varepsilon}_{\varphi t}, \tag{94}
$$

$$
\hat{\psi}_t = \rho_\psi\hat{\psi}_{t-1}+\hat{\varepsilon}_{\psi t}. \tag{95}
$$

The liquidity shock follows the process

$$
\hat{\theta}_t = \rho_\theta\hat{\theta}_{t-1}+\hat{\varepsilon}_{\theta t}. \tag{96}
$$

We use Sims (2002)'s algorithm to solve the 19 rational expectations equations, (69) through (87), for the 19 unknowns summarized in the column vector

$$
x_t = [\hat{\mu}_{ht},\hat{w}_t,\hat{q}_{lt},\hat{R}_t,\hat{\mu}_{et},\hat{\mu}_{bt},\hat{N}_t,\hat{I}_t,\hat{Y}_t,\hat{C}_{ht},\hat{C}_{et},\hat{q}_{kt},\hat{L}_{ht},\hat{L}_{et},\hat{K}_t,\hat{B}_t,\hat{g}_{\gamma t},\hat{g}_{zt},\hat{g}_{qt}]',
$$

where x_t is referred to as a vector of state variables. The system of solved-out equations forms a system of state equations.

III. Estimation

We log-linearized the model around the steady state in which the credit constraint is binding. We use the Bayesian method to fit the linearized model to 6 quarterly U.S. time series: the relative price of land (q_{lt}^{Data}), the inverse of the relative price of investment (Q_t^{Data}), real per capita consumption (C_t^{Data}), real per capita investment in consumption units (I_t^{Data}), real per capita nonfinancial business debt (B_t^{Data}), and per capita hours (L_t^{Data}). All these series are constructed to be consistent with the corresponding series in Greenwood, Hercowitz, and Krusell (1997), Cummins and Violante (2002), and Davis and Heathcote (2007). The sample period covers the first quarter of 1975 through the fourth quarter of 2010.

A system of measurement equations links the observable variables to the state variables. A standard Kalman-filter algorithm can then be applied to the system of measurement and state equations in form the likelihood function. Multiplying the likelihood by the prior distribution leads to a posterior kernel (proportional to the posterior density function).

In our model with credit constraints, we find that the posterior kernel is full of thin and winding ridges as well as local peaks. Finding the mode of the posterior distribution has proven a difficult task. Indeed, the popular dynare software fails to find the posterior mode with its various built-in optimizing methods.

To see how such difficulty arises, we first use dynare 4.2 to estimate our model. We choose many sets of reasonably calibrated parameters as different starting points and the dynare program has difficulty to converge. For quasi-newton based optimization methods (e.g., options mode_compute=1 to 5 in dynare), we encounter the message "POSTERIOR KERNEL OPTIMIZATION PROBLEM! (minus) the hessian matrix at the 'mode' is not positive definite!," meaning that the results are unreliable. One method (with the option mode_compute=6 in dynare), which triggers a Monte-Carlo based optimization routine, is very inefficient and seems to be able to converge to a local peak only.

In the examples given at `http://www.tzha.net/articles#CREDITCONSTRAINTS`[2], we summarize all the output produced by different methods of dynare:

[2]The complete set of materials–source code, figures, and tables—is stored in the zip file "DynareCode4LWZpaper.zip". In the zip file, the estimated results under different methods can be found under the subdirectory "/Output."

- When the method “options mode_compute=1” is used, the program converges with ill-behaved hessian matrix. According to these estimated results, a housing demand shock plays almost no role in macroeconomic fluctuations. Instead, at the fourth year horizon, a permanent investment-specific technology shock contributes to 67.64% of investment fluctuations and a labor supply shock contributes to 61.68% of consumption fluctuations.
- The method “options mode_compute=2” (Lester Ingber’s Adaptive Simulated Annealing) is no longer available for dynare 4.2.
- The method “options mode_compute=2” cannot converge and the solver stops prematurely.
- When the method “options mode_compute=4” is used, the program converges with ill-behaved hessian matrix. According to these estimated results, a housing demand shock contributes to a majority of fluctuations in the land price (for example, 76.44% at the fourth year horizon) but little in other macroeconomic variables. Instead, at the fourth year horizon, a permanent investment-specific technology shock contributes to a majority of fluctuations in investment (81.12%) and consumption (78.9%).
- When the method “options mode_compute=5” is used, the program converges with ill-behaved hessian matrix. According to these estimated results, a housing demand shock has a numerically zero impact on any variable. At the fourth year horizon, contributions to investment fluctuations are 42.17% from a preference shock, 15.21% from a labor supply shock, 18.15% from a permanent investment-specific technology shock, and 16.26% from a collateral shock.
- When the method “options mode_compute=6” is used, the program converges but the converged results turn out to be at a local posterior peak. A housing demand shock plays almost no role in affecting any macroeconomic variables. A preference shock affects most of fluctuations in the land price. A permanent investment-specific technology shock explains a majority of fluctuations in macroeconomic variables (78.90% for consumption and 81.12% for investment at the fourth year horizon).

As we have discussed before, we have experimented with different sets of reasonably guessed parameter values as starting points and none of the options in the optimization routine in dynare can achieve decent convergence.

Our own optimization routine, based on Sims, Waggoner, and Zha (2008) and coded in C/C++, has proven to be both efficient and able to find the posterior mode.[3] Given an initial guess of the values of the parameters, our program uses a combination of a constrained optimization algorithm and a hill-climbing quasi-Newton optimization routine, with the Broyden–Fletcher–Goldfarb–Shanno (BFGS) updates of the inverse of the Hessian, to find a local peak. We use this initial local peak to run Monte Carlo Markov Chain (MCMC) simulations and then use simulated draws as different starting points for our optimization routine to find a potentially higher peak. We iterate this process until it converges. The computation typically takes three and a half days on a single processor but less time if one avails oneself to a multi-processor computer (a cluster of nodes, for example).

Once we complete the posterior mode estimation using our own program, we use the estimated results as a starting point for the dynare optimization routine. The dynare program converges instantly.[4] We are currently working with Dynare to use their preprocessor and compile part of our C/C++ code into Dynare so that the general user will have be able to use our estimation procedure.

IV. Convergence

In this paper we use the Bayesian criterion to compare several models. Specifically, we compute the marginal data density (MDD) for each model and compare the MDDs. There are two related issues. One is to use the MDD to select a model. Potential problems of taking this approach blindly are addressed in Sims (2003), Geweke and Amisano (2011), and Waggoner and Zha (forthcoming). The other pertains to the accuracy of estimating the MDD.

In this section, we focus on discussing the second issue. We adopt two techniques. First, we use an extremely long sequence, ten millions, of MCMC draws.[5] We divide

[3]The source code (with the main function file “dsgelinv1_estmcmc.c”) can be downloaded from `http://www.tzha.net/articles#CREDITCONSTRAINTS`. The user must be familiar with C/C++ and needs a C/C++ compiler to link the C code (in the zip file “C_Cpp_Library4LWZpaper.zip”) to the C library (in the zip file “C_Cpp_Library4LWZpaper.zip”). After linking and compiling all the C functions, the user needs to generate an executable file for obtaining the estimation.

[4]See the complete set of results stored under the subdirectories /Output/Method4FromLWZmode and /Output/Method5FromLWZmode.

[5]On a standard desktop computer with dual cores, the computation would have taken more than two months. We utilize a cluster of computers to reduce an exceedingly large amount of computing time.

this sequence into ten subsequences of one million draws and then compute the MDD from the entire sequence and from each of the subsequences. The variation among the subsequences is very small (under one in log value for all models studied in the paper).

Second, we use draws from the prior as starting points for multiple MCMC chains, each of which has a length of one million draws. Selecting an *appropriate* starting point is crucial for reliable MCMC draws. If the initial value is in an extremely low probability region, an unreasonably long burn-in period would be required to obtain convergence of the MCMC chain. Most parameter values drawn from the prior have extremely low likelihood values. Thus, we draw from the prior until it reaches a reasonable likelihood value. We use ten such randomly selected starting points and record the minimum and maximum values of the MDDs calculated from these chains. The difference is under four in log value for all the models.

V. Linear vs. nonlinear models

In addition to the benchmark model, we estimate two variants of the benchmark model in which we fix the value of $\bar{\lambda}_a$ at a relatively high value (0.012) or a low value (0.0015). We find that the parameter estimates with $\bar{\lambda}_a$ fixed a priori do not change our main results obtained from the benchmark model (where $\bar{\lambda}_a$ is estimated). Figure 1 displays the estimated sample paths of the Lagrangian multiplier for the credit constraint for the benchmark model and the two variants. As one can see, the multipliers are above zero.

In general, the estimated parameter values for the benchmark model are almost indistinguishable from those for the model with $\bar{\lambda}_a$ fixed at 0.012. Figure 2 shows the impulse responses to both a TFP shock and a housing demand shock for these two models. It is clear that for the most part the responses are hard to distinguish by eyes.

As discussed in the main text of the paper, the results reported in Figure 1 by no means imply that the original nonlinear model has binding constraints always. It is possible and even probable that the original nonlinear model has occasionally binding constraints. In that case, one must estimate the original nonlinear model with occasionally binding constraints. Such a task is infeasible and beyond the scope of the paper.

From Figure 3 to Figure 6, however, we compare the impulse responses in the benchmark log-linearized model with those from two alternative nonlinear models for several key macroeconomic variables. We display the impulse responses to a positive housing demand shock and a positive collateral shock with one standard deviation as well as with three standard deviations. We solve two different nonlinear models, one in which we impose the credit constraint is always binding (so that the multiplier for the credit constraint may be negative) and the other in which we allow the credit constraint to be occasionally binding (so that the multiplier is greater than or equal to zero). For both nonlinear models, we use a shooting algorithm to compute impulse responses, and we use the parameter estimates obtained from our log-linearized model.

Figures 3 and 5 show that, when the shock is moderate, the difference between all these models is negligible. Figures 4 and 6 show that, when the shock is large, the difference remains small. Even when the constraint is occasionally binding as shown in the initial responses of the multiplier in Figures 4 and 6 in response to a large shock, much of the difference is driven by other parts of nonlinearity in the model rather than the occasionally binding constraint: although the responses of the Lagrangian multiplier following large shocks to housing demand (or credit limit) are very different between the linearized model and the nonlinear model, the responses of the land price and macroeconomic variables are very similar. For the impulse responses to other structural shocks, we obtain similar results.

While the preceding exercise is reassuring, it would be misleading to infer that one can simply calibrate the original nonlinear model with the estimates obtained from the log-linearized version. As shown in Section III, the estimation of the log-linearized version has already posed a challenging task as many estimation procedures have been inadequate and consequently, misleading conclusions may be drawn if the model parameters are not properly estimated. This lesson is particularly true for the nonlinear model with occasionally binding constraints. We are in the process of developing a robust empirical method that can tackle the estimation of such a model.

VI. Estimation issues

In this appendix, we discuss several estimation challenges we have faced during this project.

We use our own algorithm to estimate the log-linearized model and the corresponding model with regime-switching volatilities. One natural question is why we do not use Dynare to estimate these models. Dynare does not yet have capability to estimate the DSGE model with Markov-switching features. For the benchmark model, one could use Dynare. But because the posterior distribution is full of thin winding ridges as well as local peaks, finding its mode has proven to be a difficult task. To see exactly how such difficulty arises, we first use Dynare 4.2 to estimate our model. We choose many sets of reasonably calibrated parameters as different starting points, but the Dynare program has difficulty converging. Most options in Dynare lead to an ill-behaved Hessian matrix due to thin winding ridges in the posterior distribution. One option, similar to a simulated annealing algorithm, converges but to a local posterior peak (see details in Section III of Online Supplemental Appendix I).

Our own optimization routine, based on Sims, Waggoner, and Zha (2008) and coded in C/C++, has proven to be both efficient and able to find the posterior mode. The routine relies in part on the Broyden–Fletcher–Goldfarb–Shanno (BFGS) updates of the inverse of the Hessian matrix. When the inverse Hessian matrix is close to being numerically ill-conditioned, our program resets it to a diagonal matrix. Given an initial guess of the values of the parameters, our program uses a combination of a constrained optimization algorithm and an unconstrained BFGS optimization routine to find a local peak. We then use the local peak to generate a long sequence of Monte Carlo Markov Chain (MCMC) posterior draws. These simulated draws are randomly selected as different starting points for our optimization routine to find a potentially higher peak. We iterate this process until the highest peak is found. The computation typically takes four and a half days on a cluster of five dual-core processors. We are in the process of collaborating with the Dynare team to incorporate our estimation software into the Dynare package.

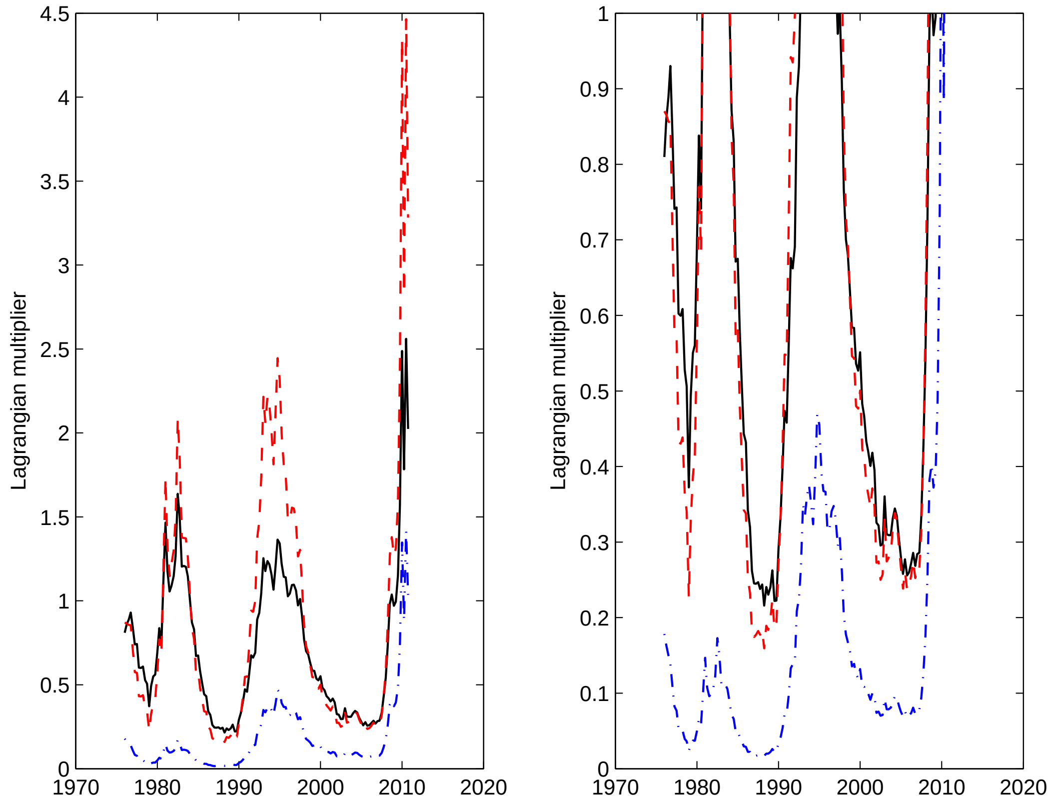

FIGURE 1. Lagrangian multipliers for the benchmark model (solid lines), the model with $\bar{\lambda}_a = 0.012$ (dashed lines), and the model with $\bar{\lambda}_a = 0.0015$ (dotted-dashed lines). Note that the right column is the same plot as the left column except the vertical axis is restricted to between 0 and 1 so that one can easily see how far the Lagrangian multipliers are away from zero.

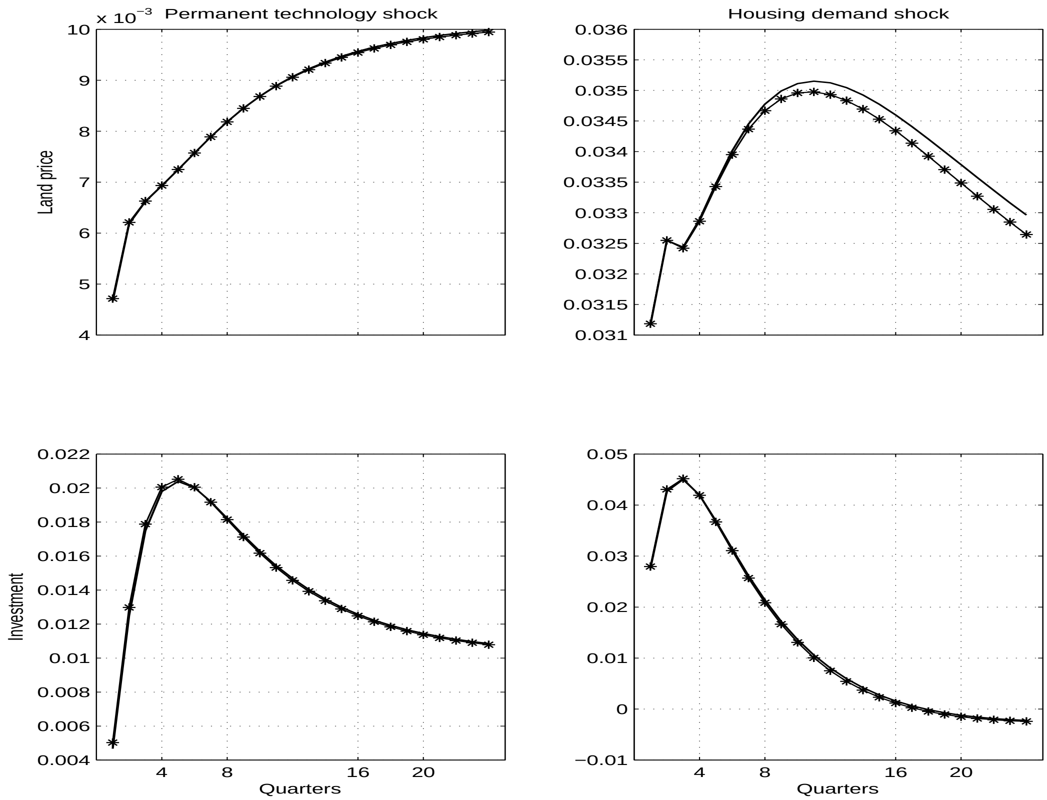

FIGURE 2. Impulse responses to a positive shock to neutral technology growth (left column) and to a positive shock to housing demand (right column). Lines marked by asterisks represent the responses for the benchmark model; thin solid lines represent the model with $\bar{\lambda}_a = 0.012$. Note that the results are so close that some lines are on top of one another.

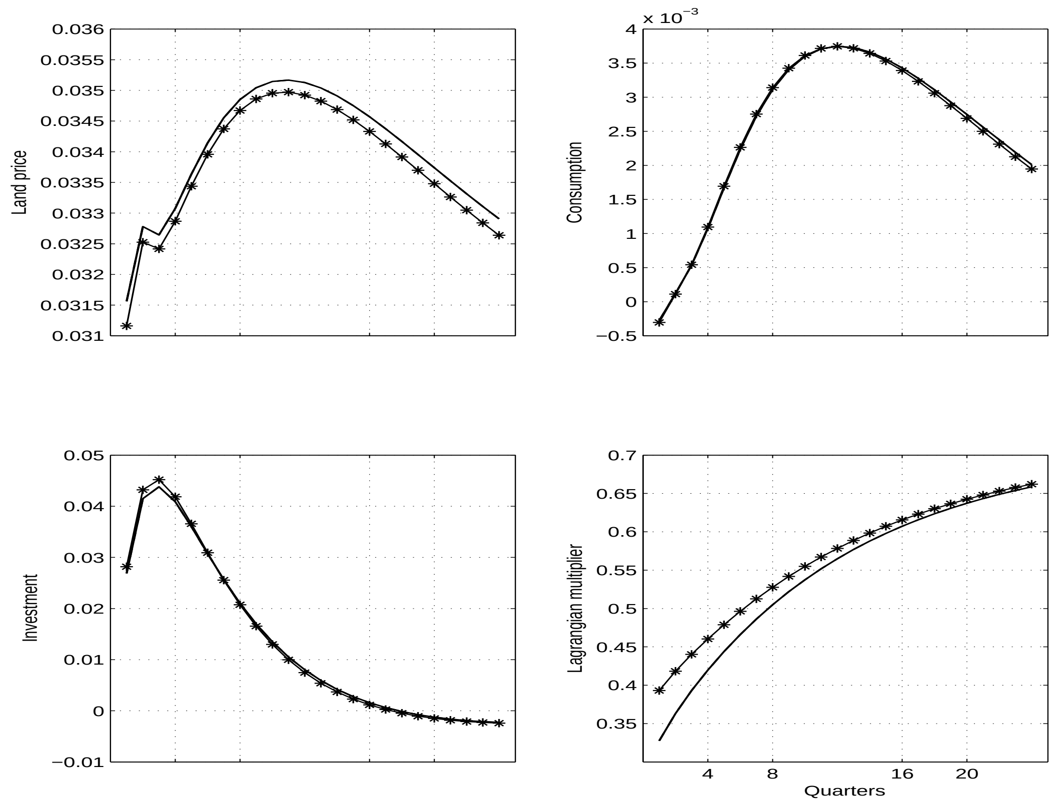

FIGURE 3. Impulse responses to a moderate positive housing demand shock (one standard deviation) in the benchmark model. Lines marked by asterisks represent the impulse responses for the log-linearized model; solid lines represent the original nonlinear model but with the credit constraint imposed to be always binding; dashed lines represent the original nonlinear model with the credit constraint allowed to be occasionally binding. Note that solid and dashed lines are on top of each other so that one cannot distinguish by eyes.

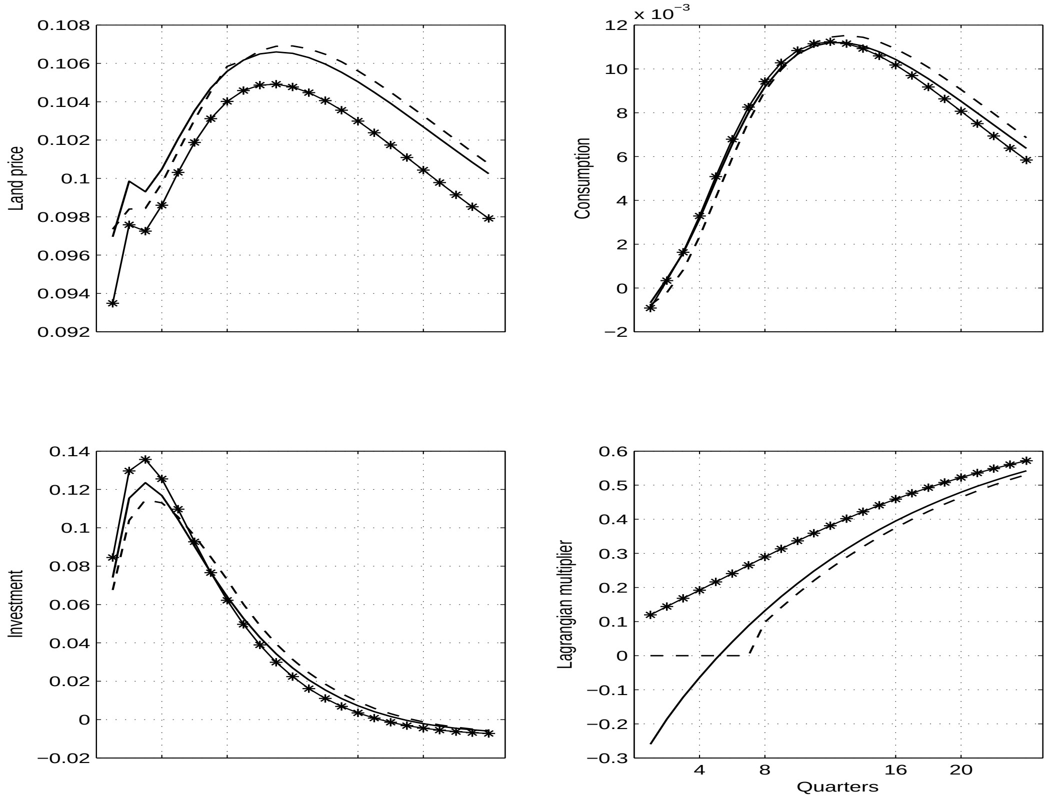

FIGURE 4. Impulse responses to a large positive housing demand shock (three standard deviations) in the benchmark model. Lines marked by asterisks represent the impulse responses for the log-linearized model; solid lines represent the original nonlinear model but with the credit constraint imposed to be always binding; dashed lines represent the original nonlinear model with the credit constraint allowed to be occasionally binding.

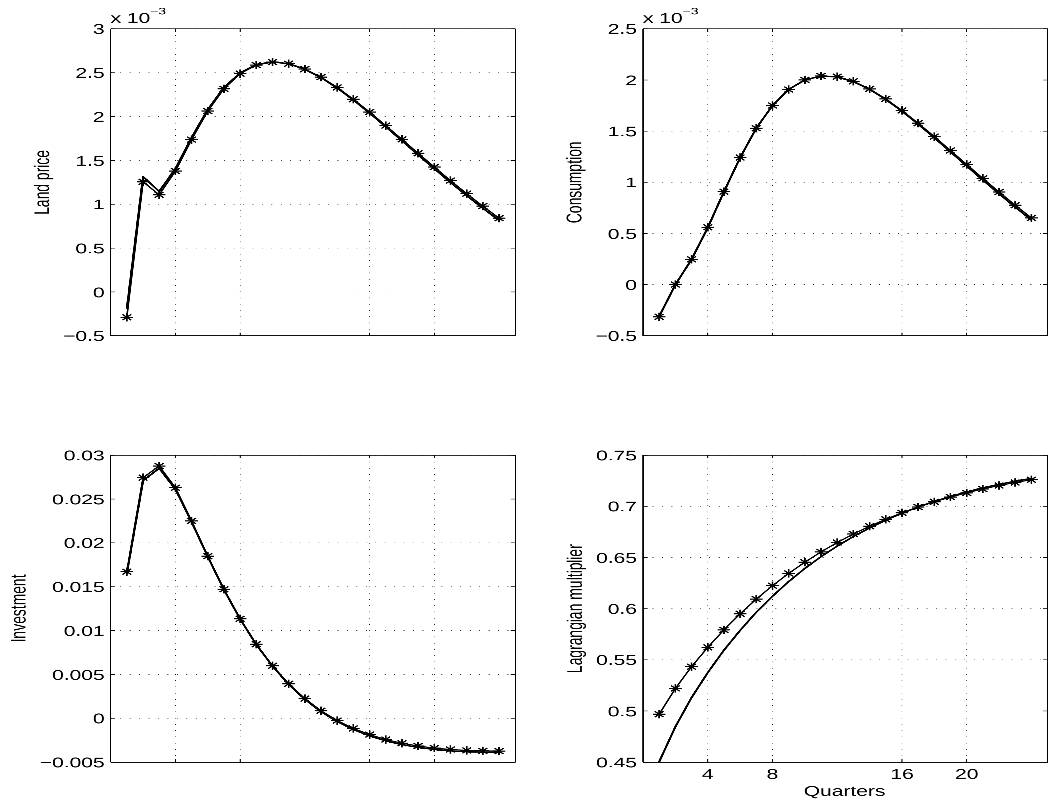

FIGURE 5. Impulse responses to a moderate positive collateral shock (one standard deviation) in the benchmark model. Lines marked by asterisks represent the impulse responses for the log-linearized model; solid lines represent the original nonlinear model but with the credit constraint imposed to be always binding; dashed lines represent the original nonlinear model with the credit constraint allowed to be occasionally binding. Note that solid and dashed lines are on top of each other so that one cannot distinguish by eyes.

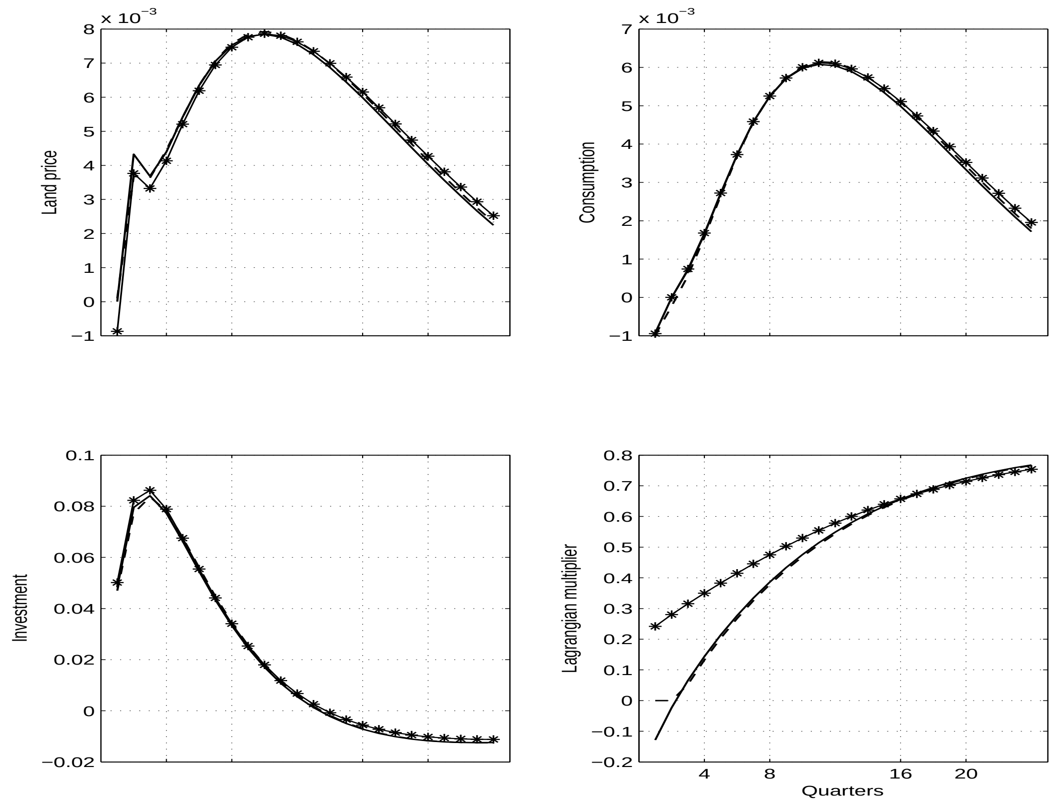

FIGURE 6. Impulse responses to a large positive collateral shock (three standard deviations) in the benchmark model. Lines marked by asterisks represent the impulse responses for the log-linearized model; solid lines represent the original nonlinear model but with the credit constraint imposed to be always binding; dashed lines represent the original nonlinear model with the credit constraint allowed to be occasionally binding.

References

Cummins, J. G., and G. L. Violante (2002): "Investment-Specific Technical Change in the United States (1947-2000): Measurement and Macroeconomic Consequences," *Review of Economic Dynamics*, 5, 243–284.

Davis, M. A., and J. Heathcote (2007): "The Price and Quantity of Residential Land in the United States," *Journal of Monetary Economics*, 54, 2595–2620.

Geweke, J., and G. Amisano (2011): "Optimal Prediction Pools," *Journal of Econometrics*, 164, 130–141.

Greenwood, J., Z. Hercowitz, and P. Krusell (1997): "Long-Run Implications of Investment-Specific Technological Change," *American Economic Review*, 87, 342–362.

Iacoviello, M. (2005): "House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle," *American Economic Review*, 95(3), 739–764.

Kiyotaki, N., and J. Moore (1997): "Credit Cycles," *Journal of Political Economy*, 105(2), 211–248.

Sims, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, 20(1), 1–20.

——— (2003): "Probability Models for Monetary Policy Decisions," Manuscript, Princeton University.

Sims, C. A., D. F. Waggoner, and T. Zha (2008): "Methods for Inference in Large Multiple-Equation Markov-Switching Models," *Journal of Econometrics*, 146(2), 255–274.

Waggoner, D. F., and T. Zha (forthcoming): "Confronting Model Misspecification in Macroeconomics," *Journal of Econometrics*.

Federal Reserve Bank of San Francisco, Hong Kong University of Science and Technology, Federal Reserve Bank of Atlanta, Emory University, and NBER