A Method for Taking Models to the Data

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1 Economic Environment

1.1 Preferences

A representative consumer has preferences defined over consumption C_t and hours worked H_t, as described by the expected utility function

$$E\sum_{t=0}^{\infty}\beta^t[\ln(C_t)-\gamma H_t], \tag{1}$$

where $1>\beta>0$ and $\gamma>0$. The linearity in hours worked can be motived along the lines of Hansen (1985) and Rogerson (1988).

1.2 Technologies

The representative consumer produces output Y_t with capital K_t and labor H_t according to the constant-returns-to-scale technology described by

$$Y_t = A_t K_t^{\theta}(\eta^t H_t)^{1-\theta}, \tag{2}$$

where $\eta>1$ measures the gross rate of labor-augmenting technological progress and where $1>\theta>0$. The technology shock A_t follows the first-order autoregressive process

$$\ln(A_t) = (1-\rho)\ln(A) + \rho\ln(A_{t-1}) + \varepsilon_t, \tag{3}$$

where $A>0$, $1>\rho>-1$, and

$$\varepsilon_t \sim N(0,\sigma^2).$$

During each period, the representative consumer divides output Y_t between consumption C_t and investment I_t, subject to the resource constraint

$$Y_t = C_t + I_t. \tag{4}$$

Investment I_t increases the capital stock according to

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{5}$$

where $1 > \delta > 0$.

1.3 Variables and Parameters

The model has implications for six variables: Y_t, C_t, I_t, H_t, K_t, and A_t. The model has eight parameters:

$$\begin{array}{cc} 1 > \beta > 0 & 1 > \delta > 0 \\ \gamma > 0 & A > 0 \\ 1 > \theta > 0 & 1 > \rho > -1 \\ \eta > 1 & \sigma > 0 \end{array}.$$

2 Characterization of Equilibrium Allocations

2.1 Optimization

Equilibrium allocations can be characterized by solving the representative consumer's problem: choose sequences $\{Y_t, C_t, I_t, H_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize the utility function (1) subject to the constraints (2)-(5) for all $t = 0, 1, 2, \ldots$. By combining (2), (4), and (5), this problem can be simplified to one of choosing $\{C_t, H_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize

$$E \sum_{t=0}^{\infty} \beta^t [\ln(C_t) - \gamma H_t]$$

subject to

$$A_t K_t^{\theta} (\eta^t H_t)^{1-\theta} \geq C_t + K_{t+1} - (1-\delta)K_t$$

for all $t = 0, 1, 2, \ldots$. The first-order conditions for this problem can be written as

$$\gamma C_t H_t = (1-\theta) Y_t \tag{6}$$

and

$$1/C_t = \beta E_t \{(1/C_{t+1})[\theta(Y_{t+1}/K_{t+1}) + 1 - \delta]\} \tag{7}$$

for all $t = 0, 1, 2, \ldots$.

2.2 Equilibrium Conditions

The equilibrium behavior of the six variables Y_t, C_t, I_t, H_t, K_t, and A_t is therefore determined by the six equations

$$Y_t = A_t K_t^{\theta} (\eta^t H_t)^{1-\theta}, \tag{2}$$

$$\ln(A_t) = (1-\rho)\ln(A) + \rho \ln(A_{t-1}) + \varepsilon_t, \tag{3}$$

$$Y_t = C_t + I_t, \tag{4}$$

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{5}$$

$$\gamma C_t H_t = (1-\theta)Y_t, \tag{6}$$

and

$$1/C_t = \beta E_t\{(1/C_{t+1})[\theta(Y_{t+1}/K_{t+1}) + 1 - \delta]\} \tag{7}$$

for all $t = 0, 1, 2, ...$

2.3 Transformed (Stationary) System

Equations (2)-(7) can be rewritten in terms of the six stationary variables $y_t = Y_t/\eta^t$, $c_t = C_t/\eta^t$, $i_t = I_t/\eta^t$, $h_t = H_t$, $k_t = K_t/\eta^t$, and $a_t = A_t$ as

$$y_t = a_t k_t^{\theta} h_t^{1-\theta}, \tag{2}$$

$$\ln(a_t) = (1-\rho)\ln(A) + \rho \ln(a_{t-1}) + \varepsilon_t, \tag{3}$$

$$y_t = c_t + i_t, \tag{4}$$

$$\eta k_{t+1} = (1-\delta)k_t + i_t, \tag{5}$$

$$\gamma c_t h_t = (1-\theta)y_t, \tag{6}$$

and

$$\eta/c_t = \beta E_t\{(1/c_{t+1})[\theta(y_{t+1}/k_{t+1}) + 1 - \delta]\} \tag{7}$$

for all $t = 0, 1, 2, ...$

2.4 Steady State

In the absence of shocks, the economy converges to a steady state, in which each of the six stationary variables is constant, with $y_t = y$, $c_t = c$, $i_t = i$, $h_t = h$, $k_t = k$, and $a_t = a$ for all $t = 0, 1, 2, ...$ Equation (3) immediately provides the solution $a = A$.

Now suppose that the steady-state value y is in hand, and use (7) to solve for

$$k = \left(\frac{\theta}{\eta/\beta - 1 + \delta}\right) y.$$

Use (5) to solve for

$$i = \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta}\right] y,$$

use (4) to solve for

$$c = \left\{1 - \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta}\right]\right\} y,$$

and use (6) to solve for

$$h = \left(\frac{1-\theta}{\gamma}\right) \left\{1 - \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta}\right]\right\}^{-1}.$$

Finally, substitute these results back into (2) to solve for y:

$$y = a^{1/(1-\theta)} \left(\frac{\theta}{\eta/\beta - 1 + \delta}\right)^{\theta/(1-\theta)} \left(\frac{1-\theta}{\gamma}\right) \left\{1 - \left[\frac{\theta(\eta - 1 + \delta)}{\eta/\beta - 1 + \delta}\right]\right\}^{-1}.$$

These equations show how the steady-state values y, c, i, h, k, and a depend on the parameters β, γ, θ, η, δ, and A. By contrast, the parameters ρ and σ have no impact on the model's steady state.

2.5 Linearized System

Equations (2)-(7) can be log-linearized to describe the behavior of the stationary variables as they fluctuate about their steady-state values in response to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\imath}_t = \ln(i_t/i)$, $\hat{h}_t = \ln(h_t/h)$, $\hat{k}_t = \ln(k_t/k)$, and $\hat{a}_t = \ln(a_t/a)$. Then first-order Taylor approximations to (2)-(7) yield

$$\hat{y}_t = \hat{a}_t + \theta\hat{k}_t + (1-\theta)\hat{h}_t, \tag{2}$$

$$\hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t, \tag{3}$$

$$(\eta/\beta - 1 + \delta)\hat{y}_t = [(\eta/\beta - 1 + \delta) - \theta(\eta - 1 + \delta)]\hat{c}_t + \theta(\eta - 1 + \delta)\hat{\imath}_t, \tag{4}$$

$$\eta\hat{k}_{t+1} = (1-\delta)\hat{k}_t + (\eta - 1 + \delta)\hat{\imath}_t, \tag{5}$$

$$\hat{c}_t + \hat{h}_t = \hat{y}_t, \tag{6}$$

and

$$0 = (\eta/\beta)\hat{c}_t - (\eta/\beta)E_t\hat{c}_{t+1} + (\eta/\beta + 1 - \delta)E_t\hat{y}_{t+1} - (\eta/\beta + 1 - \delta)\hat{k}_{t+1} \tag{7}$$

for all $t = 0, 1, 2, ...$

These equations show that the model's dynamics depend on the model's parameters β, θ, η, δ, and ρ. By contrast, the parameters γ and A have no impact on the dynamics; they serve only to determine the steady state. In this linearized model, of course, the standard deviation parameter σ determines the size of the technology shocks, but has no effect on the shapes of the impulse responses.

2.6 The Linear System in Matrix Form

Let

$$\kappa = \eta/\beta - 1 + \delta$$

and

$$\lambda = \eta - 1 + \delta.$$

In addition, let

$$f_t^0 = \begin{bmatrix} \hat{y}_t & \hat{\imath}_t & \hat{h}_t \end{bmatrix}'$$

and

$$s_t^0 = \begin{bmatrix} \hat{k}_t & \hat{c}_t \end{bmatrix}'.$$

Then (2), (4), and (6) can be written as

$$Af_t^0 = Bs_t^0 + C\hat{a}_t, \tag{8}$$

where

$$A = \begin{bmatrix} 1 & 0 & \theta - 1 \\ \kappa & -\theta\lambda & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} \theta & 0 \\ 0 & \kappa - \theta\lambda \\ 0 & 1 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Equations (5) and (7) can be written as

$$DE_t s_{t+1}^0 + FE_t f_{t+1}^0 = Gs_t^0 + Hf_t^0, \tag{9}$$

where

$$D = \begin{bmatrix} \eta & 0 \\ \kappa & \eta/\beta \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & 0 \\ -\kappa & 0 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 - \delta & 0 \\ 0 & \eta/\beta \end{bmatrix},$$

and

$$H = \begin{bmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, note that (3) implies that

$$E_t \hat{a}_{t+j} = \rho^j \hat{a}_t$$

for all $j = 0, 1, 2, ...$

3 Solving the Model

3.1 The Blanchard-Kahn Procedure

Rewrite (8) as

$$f_t^0 = A^{-1} B s_t^0 + A^{-1} C \hat{a}_t.$$

When substituted into (9), this last result yields

$$(D + FA^{-1}B)E_t s_{t+1}^0 = (G + HA^{-1}B)s_t^0 + (HA^{-1}C - FA^{-1}C\rho)\hat{a}_t$$

or, more simply,

$$E_t s_{t+1}^0 = K s_t^0 + L \hat{a}_t, \tag{10}$$

where

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B)$$

and

$$L = (D + FA^{-1}B)^{-1}(HA^{-1}C - FA^{-1}C\rho).$$

Below, it is shown that the 2×2 matrix K has one eigenvalue outside the unit circle and one eigenvalue inside the unit circle, implying that the system has a unique solution. Exploiting this result, write K as

$$K = M^{-1} N M,$$

where

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

and

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}.$$

The diagonal elements of N are the eigenvalues of K, with N_1 inside the unit circle and N_2 outside the unit circle. The columns of M^{-1} are the eigenvectors of K. In addition, let

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}.$$

Now (10) can be rewritten as

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} E_t s^0_{t+1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} s^0_t + \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \hat{a}_t$$

or

$$E_t s^1_{1t+1} = N_1 s^1_{1t} + Q_1 \hat{a}_t \tag{11}$$

and

$$E_t s^1_{2t+1} = N_2 s^1_{2t} + Q_2 \hat{a}_t, \tag{12}$$

where

$$s^1_{1t} = M_{11}\hat{k}_t + M_{12}\hat{c}_t, \tag{13}$$

$$s^1_{2t} = M_{21}\hat{k}_t + M_{22}\hat{c}_t, \tag{14}$$

$$Q_1 = M_{11}L_1 + M_{12}L_2,$$

and

$$Q_2 = M_{21}L_1 + M_{22}L_2.$$

Since N_2 lies outside the unit circle, (12) can be solved forward to obtain

$$\begin{aligned} s^1_{2t} &= (1/N_2)E_t s^1_{2t+1} - (Q_2/N_2)\hat{a}_t \\ &= -(Q_2/N_2)\sum_{j=0}^{\infty}(1/N_2)^j E_t \hat{a}_{t+j} \\ &= -(Q_2/N_2)\sum_{j=0}^{\infty}(\rho/N_2)^j \hat{a}_t \\ &= -\left(\frac{Q_2/N_2}{1-\rho/N_2}\right)\hat{a}_t \\ &= \left(\frac{Q_2}{\rho - N_2}\right)\hat{a}_t. \end{aligned}$$

Use this result, along with (14), to solve for $\hat{c}_t$:

$$\hat{c}_t = -(M_{21}/M_{22})\hat{k}_t + (1/M_{22})\left(\frac{Q_2}{\rho - N_2}\right)\hat{a}_t$$

or, more simply,

$$\hat{c}_t = S_1\hat{k}_t + S_2\hat{a}_t, \tag{15}$$

where

$$S_1 = -M_{21}/M_{22}$$

and

$$S_2 = (1/M_{22}) \left(\frac{Q_2}{\rho - N_2} \right).$$

Equation (13) now provides a solution for s_{1t}^1:

$$s_{1t}^1 = (M_{11} + M_{12}S_1)\hat{k}_t + M_{12}S_2\hat{a}_t.$$

Substitute this result into (11) to obtain

$$(M_{11} + M_{12}S_1)\hat{k}_{t+1} = N_1(M_{11} + M_{12}S_1)\hat{k}_t + (Q_1 + M_{12}S_2 - M_{12}S_2\rho)\hat{a}_t$$

or, more simply,

$$\hat{k}_{t+1} = S_3\hat{k}_t + S_4\hat{a}_t, \tag{16}$$

where

$$S_3 = N_1$$

and

$$S_4 = (Q_1 + N_1M_{12}S_2 - M_{12}S_2\rho)/(M_{11} + M_{12}S_1).$$

Finally, return to

$$\begin{aligned} f_t^0 &= A^{-1}Bs_t^0 + A^{-1}C\hat{a}_t \\ &= A^{-1}B \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + A^{-1}C\hat{a}_t \\ &= A^{-1}B \begin{bmatrix} 1 \\ S_1 \end{bmatrix} \hat{k}_t + \left\{ A^{-1}C + A^{-1}B \begin{bmatrix} 0 \\ S_2 \end{bmatrix} \right\} \hat{a}_t \end{aligned}$$

or, more simply,

$$f_t^0 = S_5\hat{k}_t + S_6\hat{a}_t, \tag{17}$$

where

$$S_5 = A^{-1}B \begin{bmatrix} 1 \\ S_1 \end{bmatrix}$$

and

$$S_6 = A^{-1}C + A^{-1}B \begin{bmatrix} 0 \\ S_2 \end{bmatrix}.$$

Equations (3) and (15)-(17) can now be combined to write the model's solution as

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{18}$$

and

$$f_t = Us_t, \tag{19}$$

where

$$s_t = \left[\begin{array}{cc} \hat{k}_t & \hat{a}_t \end{array}\right]',$$

$$f_t = \left[\begin{array}{cccc} \hat{y}_t & \hat{\imath}_t & \hat{h}_t & \hat{c}_t \end{array}\right],$$

$$\Pi = \left[\begin{array}{cc} S_3 & S_4 \\ 0 & \rho \end{array}\right],$$

$$W = \left[\begin{array}{c} 0 \\ 1 \end{array}\right],$$

and

$$U = \left[\begin{array}{cc} S_5 & S_6 \\ S_1 & S_2 \end{array}\right].$$

3.2 Preliminary Calculations

Using the specific matrices A, B, C, D, F, G, and H as defined above,

$$A^{-1} = \left[\begin{array}{ccc} \frac{1}{\theta} & 0 & \frac{1}{\theta}(\theta-1) \\ \frac{\kappa}{\theta^2\lambda} & -\frac{1}{\theta\lambda} & \frac{\kappa(\theta-1)}{\theta^2\lambda} \\ \frac{1}{\theta} & 0 & -\frac{1}{\theta} \end{array}\right],$$

$$A^{-1}B = \left[\begin{array}{cc} 1 & \frac{1}{\theta}(\theta-1) \\ \frac{\kappa}{\theta\lambda} & \frac{\theta^2\lambda-\kappa}{\theta^2\lambda} \\ 1 & -\frac{1}{\theta} \end{array}\right],$$

$$FA^{-1}B = \left[\begin{array}{cc} 0 & 0 \\ -\kappa & -\frac{\kappa}{\theta}(\theta-1) \end{array}\right],$$

$$HA^{-1}B = \left[\begin{array}{cc} \frac{\kappa}{\theta} & \frac{\theta^2\lambda-\kappa}{\theta^2} \\ 0 & 0 \end{array}\right],$$

$$D + FA^{-1}B = \left[\begin{array}{cc} \eta & 0 \\ 0 & \frac{\eta}{\beta} - \frac{\kappa}{\theta}(\theta-1) \end{array}\right],$$

$$(D + FA^{-1}B)^{-1} = \left[\begin{array}{cc} \frac{1}{\eta} & 0 \\ 0 & \frac{\beta\theta}{\eta\theta-\beta\kappa(\theta-1)} \end{array}\right],$$

$$G + HA^{-1}B = \left[\begin{array}{cc} 1-\delta+\frac{\kappa}{\theta} & \frac{\theta^2\lambda-\kappa}{\theta^2} \\ 0 & \frac{\eta}{\beta} \end{array}\right],$$

and

$$K = (D + FA^{-1}B)^{-1}(G + HA^{-1}B) = \left[\begin{array}{cc} \frac{\kappa+\theta(1-\delta)}{\eta\theta} & \frac{\theta^2\lambda-\kappa}{\eta\theta^2} \\ 0 & \frac{\eta\theta}{\eta\theta-\kappa\beta(\theta-1)} \end{array}\right]$$

or

$$K = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix},$$

where

$$K_{11} = \frac{\eta - \beta(1-\theta)(1-\delta)}{\beta\eta\theta},$$

$$K_{12} = \frac{\beta\eta\theta^2 - \eta + \beta(1-\theta^2)(1-\delta)}{\beta\eta\theta^2},$$

and

$$K_{22} = \frac{\eta\theta}{\eta - \beta(1-\theta)(1-\delta)}.$$

In addition,

$$A^{-1}C = \begin{bmatrix} \frac{1}{\theta} \\ \frac{\kappa}{\theta^2\lambda} \\ \frac{1}{\theta} \end{bmatrix},$$

$$HA^{-1}C = \begin{bmatrix} \frac{\kappa}{\theta^2} \\ 0 \end{bmatrix},$$

$$FA^{-1}C\rho = \begin{bmatrix} 0 \\ -\frac{\kappa\rho}{\theta} \end{bmatrix},$$

and

$$L = (D + FA^{-1}B)^{-1}(HA^{-1}C - FA^{-1}C\rho) = \begin{bmatrix} \frac{\kappa}{\eta\theta^2} \\ \frac{\beta\kappa\rho}{\eta\theta - \beta\kappa(\theta-1)} \end{bmatrix}$$

or

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},$$

where

$$L_1 = \frac{\eta - \beta(1-\delta)}{\beta\eta\theta^2}$$

and

$$L_2 = \frac{\rho[\eta - \beta(1-\delta)]}{\eta - \beta(1-\theta)(1-\delta)}.$$

The eigenvalues and eigenvectors of K are

$$K_{22} \text{ and } \begin{bmatrix} 1 \\ (K_{22} - K_{11})/K_{12} \end{bmatrix}$$

and

$$K_{11} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence

$$N_1 = K_{22} = \frac{\eta\theta}{\eta - \beta(1-\theta)(1-\delta)},$$

$$N_2 = K_{11} = \frac{\eta - \beta(1-\theta)(1-\delta)}{\beta\eta\theta},$$

$$M_{11} = 0,$$

$$M_{12} = K_{12}/(K_{22} - K_{11}),$$

$$M_{21} = 1,$$

and

$$M_{22} = -K_{12}/(K_{22} - K_{11}).$$

In addition,

$$Q_1 = M_{11}L_1 + M_{12}L_2 = K_{12}L_2/(K_{22} - K_{11})$$

and

$$Q_2 = M_{21}L_1 + M_{22}L_2 = L_1 - K_{12}L_2/(K_{22} - K_{11}).$$

Hence,

$$S_1 = -M_{21}/M_{22} = \frac{K_{22} - K_{11}}{K_{12}},$$

$$S_2 = (1/M_{22})\left(\frac{Q_2}{\rho - N_2}\right) = \frac{(K_{22} - K_{11})L_1 - K_{12}L_2}{K_{12}(K_{11} - \rho)},$$

$$S_3 = N_1 = K_{22},$$

$$\begin{aligned} S_4 &= (Q_1 + N_1 M_{12} S_2 - M_{12} S_2 \rho)/(M_{11} + M_{12} S_1) \\ &= \frac{K_{12}L_2}{K_{22} - K_{11}} + \frac{(K_{22} - \rho)[(K_{22} - K_{11})L_1 - K_{12}L_2]}{(K_{22} - K_{11})(K_{11} - \rho)}, \end{aligned}$$

$$S_5 = A^{-1}B\begin{bmatrix} 1 \\ S_1 \end{bmatrix} = \begin{bmatrix} 1 - \left(\frac{1-\theta}{\theta}\right) S_1 \\ S_1 + \left[\frac{\eta/\beta - 1 + \delta}{\theta^2(\eta - 1 + \delta)}\right](\theta - S_1) \\ 1 - \left(\frac{1}{\theta}\right) S_1 \end{bmatrix},$$

and

$$S_6 = A^{-1}C + A^{-1}B\begin{bmatrix} 0 \\ S_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\theta} - \left(\frac{1-\theta}{\theta}\right) S_2 \\ S_2 + \left[\frac{\eta/\beta - 1 + \delta}{\theta^2(\eta - 1 + \delta)}\right](1 - S_2) \\ \frac{1}{\theta} - \left(\frac{1}{\theta}\right) S_2 \end{bmatrix}.$$

3.3 Final Calculations

These results can be summarized as follows. Let

$$K_{11} = \frac{\eta - \beta(1-\theta)(1-\delta)}{\beta\eta\theta},$$

$$K_{12} = \frac{\beta\eta\theta^2 - \eta + \beta(1-\theta^2)(1-\delta)}{\beta\eta\theta^2},$$

$$K_{22} = \frac{\eta\theta}{\eta - \beta(1-\theta)(1-\delta)},$$

$$L_1 = \frac{\eta - \beta(1-\delta)}{\beta\eta\theta^2},$$

and

$$L_2 = \frac{\rho[\eta - \beta(1-\delta)]}{\eta - \beta(1-\theta)(1-\delta)}.$$

Then

$$S_1 = \frac{K_{22} - K_{11}}{K_{12}},$$

$$S_2 = \frac{(K_{22} - K_{11})L_1 - K_{12}L_2}{K_{12}(K_{11} - \rho)},$$

$$S_3 = K_{22},$$

$$S_4 = \frac{K_{12}L_2}{K_{22} - K_{11}} + \frac{(K_{22} - \rho)[(K_{22} - K_{11})L_1 - K_{12}L_2]}{(K_{22} - K_{11})(K_{11} - \rho)},$$

$$S_5 = \left[\frac{1 - \left(\frac{1-\theta}{\theta}\right) S_1}{S_1 + \left[\frac{\eta/\beta - 1 + \delta}{\theta^2(\eta - 1 + \delta)}\right](\theta - S_1)} \Bigg/ 1 - \left(\frac{1}{\theta}\right) S_1 \right],$$

and

$$S_6 = \left[\frac{\frac{1}{\theta} - \left(\frac{1-\theta}{\theta}\right) S_2}{S_2 + \left[\frac{\eta/\beta - 1 + \delta}{\theta^2(\eta - 1 + \delta)}\right](1 - S_2)} \Bigg/ \frac{1}{\theta} - \left(\frac{1}{\theta}\right) S_2 \right].$$

3.4 Existence and Uniqueness

As indicated above, the existence and uniqueness of the model's solution requires that K_{22} lie inside the unit circle and K_{11} lie outside the unit circle. In fact, it can be verified that these conditions always hold, given the restrictions that have been placed directly on the model's parameters.

Note first that

$$K_{22} = \frac{\eta\theta}{\eta - \beta(1-\theta)(1-\delta)} > 0.$$

In addition,

$$\begin{aligned} K_{22} - 1 &= \frac{\eta(\theta - 1) + \beta(1-\theta)(1-\delta)}{\eta - \beta(1-\theta)(1-\delta)} \\ &= \frac{\beta(1-\theta)(1-\delta-\eta/\beta)}{\eta - \beta(1-\theta)(1-\delta)} \\ &< 0. \end{aligned}$$

These two inequalities imply that $1 > K_{22} > 0$ so that, as required, K_{22} lies inside the unit circle.

Next, consider that

$$\begin{aligned} K_{11} - 1 &= \frac{\eta(1-\beta\theta) - \beta(1-\theta)(1-\delta)}{\beta\eta\theta} \\ &= \frac{(1-\beta\theta)(\eta - 1 + \delta) + (1-\beta\theta)(1-\delta) - \beta(1-\theta)(1-\delta)}{\beta\eta\theta} \\ &= \frac{(1-\beta\theta)(\eta - 1 + \delta) + [(1-\beta\theta) - \beta(1-\theta)](1-\delta)}{\beta\eta\theta} \\ &= \frac{(1-\beta\theta)(\eta - 1 + \delta) + (1-\beta)(1-\delta)}{\beta\eta\theta} \\ &> 0. \end{aligned}$$

Hence, $K_{11} > 1$ so that, again as required, K_{11} lies outside the unit circle.

4 Estimating the Model

Suppose that data $\{d_t\}_{t=1}^T$ are available for output, consumption, and hours worked, so that

$$d_t = \left[\begin{array}{ccc} \hat{y}_t & \hat{c}_t & \hat{h}_t \end{array} \right]'$$

for all $t = 0, 1, 2, \ldots$. Then (18) and (19) give rise to an empirical model of the form

$$s_{t+1} = As_t + B\varepsilon_{t+1}, \tag{20}$$

$$d_t = Cs_t + v_t, \tag{21}$$

and

$$v_{t+1} = Dv_t + \xi_{t+1}, \tag{22}$$

where $A = \Pi$, $B = W$, C is formed using the first, fourth, and third rows of U, v_t is a 3×1 vector of serially correlated residuals, and the serially uncorrelated innovations ε_{t+1} satisfy

$$E\varepsilon_{t+1}^2 = V_1 = \sigma^2,$$

$$E\xi_{t+1}\xi_{t+1}' = V_2,$$

and

$$E\varepsilon_{t+1}\xi_{t+1}^{'} = 0_{(1\times 3)}.$$

Altug (1989), McGrattan (1994), Hall (1996), and McGrattan, Rogerson, and Wright (1997) follow Sargent (1989) by assuming that the 3×3 matrices D and V_2 are diagonal, implying that the residuals are uncorrelated across variables. Here, no such restrictions will be imposed: the residuals are allowed to follow an unconstrained, first-order vector autoregression.

Define the augmented state vector

$$x_t = \begin{bmatrix} s_t \\ v_t \end{bmatrix}.$$

Also, define

$$\eta_{t+1} = \begin{bmatrix} B\varepsilon_{t+1} \\ \xi_{t+1} \end{bmatrix}.$$

Then (20)-(22) can be rewritten as

$$x_{t+1} = Fx_t + \eta_{t+1} \tag{23}$$

$$d_t = Gx_t \tag{24}$$

where

$$F = \begin{bmatrix} A & 0_{(2\times 3)} \\ 0_{(3\times 2)} & D \end{bmatrix},$$

$$G = \begin{bmatrix} C & I_{(3\times 3)} \end{bmatrix},$$

and η_{t+1} is serially uncorrelated with

$$\eta_{t+1} \sim N(0, Q)$$

and

$$Q = E\eta_{t+1}\eta_{t+1}^{'} = \begin{bmatrix} BV_1B' & 0_{(2\times 3)} \\ 0_{(3\times 2)} & V_2 \end{bmatrix}. \tag{25}$$

The model defined by (23)-(25) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{x}_{t|t-j} = E(x_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(x_t - \hat{x}_{t|t-j})(x_t - \hat{x}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (23) implies that

$$\hat{x}_{1|0} = Ex_1 = 0 \tag{26}$$

and

$$vec(\Sigma_{1|0}) = vec(Ex_1x_1') = [I_{(25\times 25)} - (F \otimes F)]^{-1}vec(Q). \tag{27}$$

Now suppose $\hat{x}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider calculating $\hat{x}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (24) that

$$\hat{d}_{t|t-1} = G\hat{x}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = G(x_t - \hat{x}_{t|t-1})$$

is such that

$$Eu_tu_t' = G\Sigma_{t|t-1}G'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned} \hat{x}_{t|t} &= \hat{x}_{t|t-1} + [E(x_t - \hat{x}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\ &= \hat{x}_{t|t-1} + \Sigma_{t|t-1}G'(G\Sigma_{t|t-1}G')^{-1}u_t. \end{aligned}$$

Hence, from (23),

$$\hat{x}_{t+1|t} = F\hat{x}_{t|t-1} + F\Sigma_{t|t-1}G'(G\Sigma_{t|t-1}G')^{-1}u_t.$$

Using this last result, along with (23) again,

$$x_{t+1} - \hat{x}_{t+1|t} = F(x_t - \hat{x}_{t|t-1}) + \eta_{t+1} - F\Sigma_{t|t-1}G'(G\Sigma_{t|t-1}G')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = Q + F\Sigma_{t|t-1}F' - F\Sigma_{t|t-1}G'(G\Sigma_{t|t-1}G')^{-1}G\Sigma_{t|t-1}F'.$$

These results can be summarized as follows. Let

$$\hat{x}_t = \hat{x}_{t|t-1}$$

and

$$\Sigma_t = \Sigma_{t|t-1}.$$

Then

$$\hat{x}_{t+1} = F\hat{x}_t + K_t u_t$$

and

$$d_t = G\hat{x}_t + u_t,$$

where

$$\hat{x}_t = E(x_t | d_{t-1}, d_{t-2}, ..., d_1),$$
$$u_t = d_t - E(d_t | d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_t u_t' = G\Sigma_t G' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = F\Sigma_t G'(G\Sigma_t G')^{-1}$$

and

$$\Sigma_{t+1} = Q + F\Sigma_t F' - F\Sigma_t G'(G\Sigma_t G')^{-1} G\Sigma_t F',$$

and initial conditions $\hat{x}_1$ and Σ_1 are provided by (26) and (27).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\frac{3T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T} u_t'\Omega_t^{-1}u_t.$$

5 Evaluating the Model

5.1 Variance Decompositions

Consider (23), which can be rewritten as

$$x_t = Fx_{t-1} + \eta_t$$

or

$$(I - FL)x_t = \eta_t$$
$$x_t = \sum_{j=0}^{\infty} F^j \eta_{t-j}.$$

This implies

$$x_{t+k} = \sum_{j=0}^{\infty} F^j \eta_{t+k-j},$$

$$E_t x_{t+k} = \sum_{j=k}^{\infty} F^j \eta_{t+k-j},$$

$$x_{t+k} - E_t x_{t+k} = \sum_{j=0}^{k-1} F^j \eta_{t+k-k},$$

and hence

$$\begin{aligned} \Sigma_k^x &= E(x_{t+k} - E_t x_{t+k})(x_{t+k} - E_t x_{t+k})' \\ &= Q + FQF' + F^2 Q F^{2\prime} + ... + F^{k-1} Q F^{k-1\prime}. \end{aligned}$$

In addition, (23) implies

$$vec(\Sigma^x) = [I_{(25x25)} - (F \otimes F)]^{-1} vec(Q).$$

Thus, (24) implies that

$$\Sigma_k^d = E(d_{t+k} - E_t d_{t+k})(d_{t+k} - E_t d_{t+k})' = G \Sigma_k^x G'$$

and

$$\Sigma^d = G \Sigma^d G'.$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that elements of Σ_k^d and Σ^d can be expressed as nonlinear functions of Θ,

$$\Sigma_k^d = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\nabla g H \nabla g'.$$

In practice, the gradient ∇g can be evaluated numerically; Runkle (1987) finds that this numerical technique performs about as well as a more elaborate, bootstrapping procedure.

5.2 Testing for Stability

The procedures described by Andrews and Fair (1988) can be used to test for the stability of the model's estimated parameters. Let Θ^1 and Θ^2 denote the estimated parameters from two disjoint subsamples, and let H^1 and H^2 denote the associated covariance matrices, so that asymptotically,

$$\Theta^1 \sim N(\Theta^{10}, H^1)$$

and

$$\Theta^2 \sim N(\Theta^{20}, H^2).$$

One way of testing for the stability of all of the estimated parameters is with the likelihood ratio statistic

$$LR = 2[\ln L(\Theta^1) + \ln L(\Theta^2) - \ln L(\Theta)],$$

where $\ln L(\Theta^1)$, $\ln L(\Theta^2)$, and $\ln L(\Theta)$ are the maximized log likelihood functions for the first subsample, the second subsample, and the entire sample. According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of estimated parameters.

Alternatively, the stability of some or all of the parameters can be tested with the Wald statistic

$$W = g(\Theta^1, \Theta^2)'(G\hat{H}G')^{-1}g(\Theta^1, \Theta^2),$$

when the stability restrictions are written as

$$g(\Theta^1, \Theta^2) = 0$$

and where

$$G = \frac{\partial g(\Theta^1, \Theta^2)}{\partial(\Theta^1, \Theta^2)}$$

and

$$\hat{H} = \begin{bmatrix} H^1 & 0 \\ 0 & H^2 \end{bmatrix}.$$

If Θ_q^1 and Θ_q^2 denote the subsets of Θ^1 and Θ^2 of interest, and if H_q^1 and H_q^2 denote the covariance matrices of Θ_q^1 and Θ_q^2, then this Wald statistic can be written more simply as

$$W = (\Theta_q^1 - \Theta_q^2)'(H_q^1 + H_q^2)^{-1}(\Theta_q^1 - \Theta_q^2).$$

According to Andrews and Fair, this statistic will be asymptotically distributed as a chi-square random variable with q degrees of freedom under the null hypothesis of stability, where q is the number of parameters being tested for stability.

5.3 Forecast Accuracy

Forecasts from the model can be generated as follows. Using (24),

$$E_t d_{t+k} = G E_t x_{t+k}.$$

For $k = 1$,

$$E_t x_{t+1} = \hat{x}_{t+1|t} = \hat{x}_{t+1},$$

and for $k > 1$,

$$E_t x_{t+k} = F^{k-1} E_t x_{t+1} = F^{k-1} \hat{x}_{t+1},$$

so for any $k \geq 1$,

$$E_t d_{t+k} = G F^{k-1} \hat{x}_{t+1}.$$

To evaluate the model's forecast accuracy, let $\{\varepsilon_t\}_{t=1}^T$ denote a sequence of k-step-ahead forecast errors from the model, and let $\{\eta_t\}_{t=1}^T$ denote a sequence of k-step-ahead forecast errors from an alternative model (for example, an unconstrained VAR). Let $g(\varepsilon)$ and $g(\eta)$ denote the corresponding mean squared errors, and let l_t denote the "loss differentials" defined by Diebold and Mariano (1995):

$$g(\varepsilon) = \frac{1}{T} \sum_{t=1}^{T} \varepsilon_t^2,$$

$$g(\eta) = \frac{1}{T} \sum_{t=1}^{T} \eta_t^2,$$

and

$$l_t = \eta_t^2 - \varepsilon_t^2.$$

A comparison of $g(\eta)$ and $g(\varepsilon)$ can provide an informal assessment of the model's forecast accuracy. A formal test, suggested by Diebold and Mariano, is to form the sample mean loss differential,

$$l = \frac{1}{T} \sum_{t=1}^{T} l_t,$$

and to rely on the asymptotic normality of l under the null hypothesis of equal forecast accuracy:

$$\sqrt{T} l \sim N(0, f),$$

where f can be estimated by

$$f = \gamma(0) + 2 \sum_{\tau=1}^{k-1} \gamma(\tau)$$

with

$$\gamma(\tau) = \frac{1}{T} \sum_{t=\tau+1}^{T} l_t l_{t-\tau},$$

using the fact that k-step-ahead forecast errors are at most $(k-1)$-dependent. In particular, the test statistic

$$S = \frac{l}{\sqrt{f/T}}$$

has a standard normal asymptotic distribution.

In principle, this test might be criticized for failing to account for the uncertainty associated with the model's parameters, which are estimated rather than known for sure. But West (1996) shows that in cases like these, where the forecast errors are uncorrelated with the predictors, this parameter uncertainty is asymptotically irrelevant.

5.4 Producing Smoothed Estimates of the Shocks

It may also be useful to use the estimated model to produce estimates of the real business cycle's technology shock and the hybrid model's three residuals. As before, for $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{x}_{t|t-j} = E(x_t | d_{t-j}, d_{t-j-1}, ..., d_1),$$
$$\Sigma_{t|t-j} = E(x_t - \hat{x}_{t|t-j})(x_t - \hat{x}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, ..., d_1).$$

Also, let

$$u_t = d_t - \hat{d}_{t|t-1} = G(x_t - \hat{x}_{t|t-1})$$

so that again as before,

$$E u_t u_t' = G \Sigma_{t|t-1} G'.$$

Then

$$\hat{z}_{1|0} = E z_1 = 0_{(5\times1)}$$

and

$$vec(\Sigma_{1|0}) = vec(E x_1 x_1') = [I_{(25\times25)} - (F \otimes F)]^{-1} vec(Q').$$

From these starting values, the sequences $\{\hat{x}_{t|t}\}_{t=1}^{T}$, $\{\hat{x}_{t|t-1}\}_{t=1}^{T}$, $\{\Sigma_{t|t}\}_{t=1}^{T}$, and $\{\Sigma_{t|t-1}\}_{t=1}^{T}$ can be generated recursively using

$$u_t = d_t - G\hat{x}_{t|t-1},$$
$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \Sigma_{t|t-1} G' (G \Sigma_{t|t-1} G')^{-1} u_t,$$
$$\hat{x}_{t+1|t} = F \hat{x}_{t|t},$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1} G' (G \Sigma_{t|t-1} G')^{-1} G \Sigma_{t|t-1},$$

and

$$\Sigma_{t+1|t} = Q + F \Sigma_{t|t} F'$$

for $t = 1, 2, ..., T$.

Hamilton (Ch.13, Sec.6, pp.394-397) shows how to use these sequences to generate a sequence of smoothed estimates $\{\hat{x}_{t|T}\}_{t=1}^{T}$, where

$$\hat{x}_{t|T} = E(x_t | d_T, d_{T-1}, ..., d_1).$$

To begin, construct a sequence $\{J_t\}_{t=1}^{T}$ using Hamilton's equation (13.6.11):

$$J_t = \Sigma_{t|t} F' \Sigma_{t+1|t}^{-1}.$$

Then note that $\hat{x}_{T|T}$ is just the last element of $\{\hat{x}_{t|t}\}_{t=1}^{T}$. From this terminal condition, the rest of the sequence can be generated recursively using Hamilton's equation (13.6.16):

$$\hat{x}_{T-j|T} = \hat{x}_{T-j|T-j} + J_{T-j} (\hat{x}_{T-j+1|T} - \hat{x}_{T-j+1|T-j})$$

for $j = 1, 2, ..., T-1$.