Model 1: Nonstationary and Cointegrated Neutral and Investment-Specific Technology Shocks in a Two-Country Model with Capital and Labor Adjustment Costs

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June 2010

1 The Model

1.1 Overview

The basic elements of this two-country model are drawn from Backus, Kehoe, and Kydland's (1994) framework, modified by replacing their assumption of complete international financial markets with Heathcote and Perri's (2002) alternative assumption that only a single, noncontingent bond is traded. In addition, investment-specific technology shocks are introduced as in recent work by Raffo (2009), and both neutral and investment-specific technology shocks are assumed to be nonstationary but cointegrated across countries, borrowing a key element from Rabanal, Rubio-Ramirez, and Tuesta's (2009) specification. Finally, capital and labor adjustment costs are incorporated into the model to help make consumption, investment, and hours worked move together both within and across countries following shocks of various kinds.

Home and foreign variables are denoted with H and F superscripts; time periods are denoted with $t = 0, 1, 2, \ldots$ subscripts. Each economy has a representative consumer, a representative intermediate goods-producing firm, a representative final goods-producing firm, and a government, whose activities will now be described in turn.

1.2 Consumers

The representative home consumer has preferences described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{[(C_t^H)^\mu (1 - L_t^H / M_t^H)^{1-\mu}]^{1-\gamma} - 1}{1-\gamma} \right\},$$

where C_t^H denotes consumption, L_t^H denotes hours worked, the discount factor β lies between zero and one, and the risk aversion parameter γ is strictly positive. The preference shock

M_t^H impacts on the marginal rate of substitution between consumption and leisure and enters into the utility function in a way that associates positive innovations with increases in equilibrium employment. Here, as in Ireland and Schuh (2008), these preference shocks are introduced to compete with the technology shocks as sources of fluctuations in output and hours worked.

For now, all prices in both countries will be expressed in terms of an abstract unit of account. Accordingly, let W_t^H denote the domestic nominal wage, Q_t^H the domestic nominal rental rate for capital, P_t^H the nominal price of the home consumption good, X_t^H the nominal price of the home investment good, and $1/R_t$ the price at time t in units of the home consumption good of a real bond that returns one unit of the home consumption good at time $t+1$. Let K_t^H denote the domestic consumer's holdings of physical capital at the beginning of period t, D_t^H the number of bonds carried by the domestic consumer from period $t-1$ into period t, and T_t^H lump-sum taxes, expressed in nominal terms, paid by the domestic consumer to the domestic government during period t. Then the home consumer's budget constraint can be written in real terms as

$$\frac{W_t^H L_t^H + Q_t^H K_t^H}{P_t^H} + D_t^H \geq C_t^H + \frac{X_t^H I_t^H + T_t^H}{P_t^H} + \frac{D_{t+1}^H}{R_t} + \left(\frac{\phi_d}{2}\right) U_t^H \left(\frac{D_{t+1}^H}{U_t^H}\right)^2 + \left(\frac{\phi_l}{2}\right) U_{t-1}^H \left(\frac{L_t^H}{L_{t-1}^H} - 1\right)^2 L_{t-1}^H. \tag{1}$$

As discussed by Schmitt-Grohe and Uribe (2003) for the case of a small open economy, the second-to-last term introduces arbitrarily small costs of bond (debt) holdings, measured in units of the home consumption good, that guarantee that a suitably-transformed set of conditions describing the model's equilibrium has a stationary solution. The last term introduces adjustment costs for hours worked, also measured in units of the home consumption good. And, as in Rabanal, Rubio-Ramirez, and Tuesta (2009), both adjustment cost terms must be scaled by the country-specific variable

$$U_t^H = (V_t^H)^{\alpha/(1-\alpha)} Z_t^H,$$

defined in terms of the home neutral and investment-specific technology shocks Z_t^H and V_t^H as well as the parameter α measuring capital's share in production, so that the adjustment costs do not vanish asymptotically in this model with stochastic long-run growth. The bond adjustment cost parameter ϕ_d must be strictly positive for the model to have a unique steady state; the labor adjustment cost parameter ϕ_l is nonnegative.

By purchasing I_t^H units of the domestic investment good during period t, the consumer increases the stock of physical capital available in the home country between t and $t+1$ according to

$$(1-\delta)K_t^H + I_t^H - \left(\frac{\phi_k}{2}\right)\left(\frac{I_t^H}{K_t^H} - \eta^H\right)^2 K_t^H \geq K_{t+1}^H, \tag{2}$$

where the parameter δ measuring the depreciation rate lies between zero and one, the capital adjustment cost parameter ϕ_k in nonnegative, and the positive parameter η^H will be set later to equal the steady-state ratio of investment to capital in the home country, so that the capital adjustment costs equal zero in the steady state.

When Λ_t^H and Ξ_t^H are used to denote the Lagrange multipliers on the budget and capital accumulation constraints (1) and (2) for period t, the first-order conditions describing the home consumer's optimizing behavior can be written as

$$\mu[(C_t^H)^\mu(1-L_t^H/M_t^H)^{1-\mu}]^{1-\gamma} = \Lambda_t^H C_t^H, \tag{3}$$

$$\begin{aligned} &\frac{(1-\mu)[(C_t^H)^\mu(1-L_t^H/M_t^H)^{1-\mu}]^{1-\gamma}}{M_t^H(1-L_t^H/M_t^H)} \\ &= \Lambda_t^H\left[\left(\frac{W_t^H}{P_t^H}\right) - \phi_l U_{t-1}^H\left(\frac{L_t^H}{L_{t-1}^H}-1\right)\right] \\ &\quad + \beta\phi_l E_t\left\{\Lambda_{t+1}^H U_t^H\left[\left(\frac{L_{t+1}^H}{L_t^H}-1\right)\left(\frac{L_{t+1}^H}{L_t^H}\right)-\left(\frac{1}{2}\right)\left(\frac{L_{t+1}^H}{L_t^H}-1\right)^2\right]\right\}, \end{aligned} \tag{4}$$

$$\Lambda_t^H\left(\frac{X_t^H}{P_t^H}\right) = \Xi_t^H\left[1-\phi_k\left(\frac{I_t^H}{K_t^H}-\eta^H\right)\right], \tag{5}$$

$$\begin{aligned} \Xi_t^H &= \beta E_t\left[\Lambda_{t+1}^H\left(\frac{Q_{t+1}^H}{P_{t+1}^H}\right)\right] \\ &\quad + \beta E_t\left\{\Xi_{t+1}^H\left[1-\delta+\phi_k\left(\frac{I_{t+1}^H}{K_{t+1}^H}-\eta^H\right)\left(\frac{I_{t+1}^H}{K_{t+1}^H}\right)-\left(\frac{\phi_k}{2}\right)\left(\frac{I_{t+1}^H}{K_{t+1}^H}-\eta^H\right)^2\right]\right\}, \end{aligned} \tag{6}$$

$$\Lambda_t^F\left[\frac{1}{R_t}+\phi_d\left(\frac{D_{t+1}^H}{U_t^H}\right)\right] = \beta E_t\Lambda_{t+1}^H, \tag{7}$$

and (1) and (2) with equality for all $t = 0, 1, 2, \ldots$.

Symmetrically, the representative foreign consumer maximizes the expected utility function

$$E_0\sum_{t=0}^{\infty}\beta^t\left\{\frac{[(C_t^F)^\mu(1-L_t^F/M_t^F)^{1-\mu}]^{1-\gamma}-1}{1-\gamma}\right\},$$

subject to the budget constraint

$$\begin{aligned} \frac{W_t^F L_t^F + Q_t^F K_t^F + P_t^H D_t^F}{P_t^F} &\geq C_t^F + \frac{X_t^F I_t^F + T_t^F}{P_t^F} + \frac{P_t^H D_{t+1}^F}{P_t^F R_t} \\ &\quad + \left(\frac{\phi_d}{2}\right)U_t^F\left(\frac{D_{t+1}^F}{U_t^F}\right)^2 + \left(\frac{\phi_l}{2}\right)U_{t-1}^F\left(\frac{L_t^F}{L_{t-1}^F}-1\right)^2 L_{t-1}^F, \end{aligned} \tag{8}$$

where the costs of bond holdings and the labor adjustment costs are measured in units of the foreign consumption good and scaled by the country-specific factor

$$U_t^F = (V_t^F)^{\alpha/(1-\alpha)}Z_t^F,$$

and the capital accumulation constraint

$$(1-\delta)K_t^F + I_t^F - \left(\frac{\phi_k}{2}\right)\left(\frac{I_t^F}{K_t^F}-\eta^F\right)^2 K_t^F \geq K_{t+1}^F, \tag{9}$$

where the positive parameter η^F will be set later to equal the steady-state ratio of investment to capital in the foreign country, so that once again the capital adjustment costs equal zero in steady state.

Letting Λ_t^F and Ξ_t^F denote the Lagrange multipliers on the budget and capital accumulation constraints (8) and (9) for period t, the first-order conditions describing the foreign consumer's optimizing behavior can be written as

$$\mu[(C_t^F)^\mu(1 - L_t^F/M_t^F)^{1-\mu}]^{1-\gamma} = \Lambda_t^F C_t^F, \tag{10}$$

$$\begin{aligned} &\frac{(1-\mu)[(C_t^F)^\mu(1 - L_t^F/M_t^F)^{1-\mu}]^{1-\gamma}}{M_t^F(1 - L_t^F/M_t^F)} \\ &= \Lambda_t^F \left[\left(\frac{W_t^F}{P_t^F} \right) - \phi_l U_{t-1}^F \left(\frac{L_t^F}{L_{t-1}^F} - 1 \right) \right] \\ &\quad + \beta \phi_l E_t \left\{ \Lambda_{t+1}^F U_t^F \left[\left(\frac{L_{t+1}^F}{L_t^F} - 1 \right) \left(\frac{L_{t+1}^F}{L_t^F} \right) - \left(\frac{1}{2} \right) \left(\frac{L_{t+1}^F}{L_t^F} - 1 \right)^2 \right] \right\}, \end{aligned} \tag{11}$$

$$\Lambda_t^F \left(\frac{X_t^F}{P_t^F} \right) = \Xi_t^F \left[1 - \phi_k \left(\frac{I_t^F}{K_t^F} - \eta^F \right) \right], \tag{12}$$

$$\begin{aligned} \Xi_t^F &= \beta E_t \left[\Lambda_{t+1}^F \left(\frac{Q_{t+1}^F}{P_{t+1}^F} \right) \right] \\ &\quad + \beta E_t \left\{ \Xi_{t+1}^F \left[1 - \delta + \phi_k \left(\frac{I_{t+1}^F}{K_{t+1}^F} - \eta^F \right) \left(\frac{I_{t+1}^F}{K_{t+1}^F} \right) - \left(\frac{\phi_k}{2} \right) \left(\frac{I_{t+1}^F}{K_{t+1}^F} - \eta^F \right)^2 \right] \right\}, \end{aligned} \tag{13}$$

$$\Lambda_t^F \left[\frac{P_t^H}{P_t^F R_t} + \phi_d \left(\frac{D_{t+1}^F}{U_t^F} \right) \right] = \beta E_t \left[\Lambda_{t+1}^F \left(\frac{P_{t+1}^H}{P_{t+1}^F} \right) \right], \tag{14}$$

and (8) and (9) with equality for all $t = 0, 1, 2, ...$

1.3 Intermediate Goods-Producing Firms

The representative home intermediate goods-producing firm hires K_t^H units of capital and L_t^H units of labor to produce Y_t^A units of an internationally-traded intermediate good according to the Cobb-Douglas specification

$$(K_t^H)^\alpha (Z_t^H L_t^H)^{1-\alpha} \geq Y_t^A, \tag{15}$$

where the share parameter α lies between zero and one. The firm sells its output domestically and abroad at the common price P_t^A in order to maximize its profits, which are given by

$$P_t^A Y_t^A - Q_t^H K_t^H - W_t^H L_t^H,$$

subject to (15). The first-order conditions for this problem can be written as

$$\alpha P_t^A Y_t^A = Q_t^H K_t^H, \tag{16}$$

$$(1-\alpha) P_t^A Y_t^A = W_t^H L_t^H, \tag{17}$$

and (15) with equality for all $t = 0, 1, 2, ...$

Symmetrically, the representative foreign intermediate goods-producing firm produces Y_t^B units of a second internationally-traded intermediate good according to

$$(K_t^F)^\alpha (Z_t^F L_t^F)^{1-\alpha} \geq Y_t^B. \tag{18}$$

The firm sells its output in both countries at the common price P_t^B to maximize its profits

$$P_t^B Y_t^B - Q_t^F K_t^F - W_t^F L_t^F,$$

subject to (18). The first-order conditions for this problem can be written as

$$\alpha P_t^B Y_t^B = Q_t^F K_t^F, \tag{19}$$

$$(1-\alpha) P_t^B Y_t^B = W_t^F L_t^F, \tag{20}$$

and (18) with equality for all $t = 0, 1, 2, ...$

1.4 Final Goods-Producing Firms

The representative domestic final goods-producing firm uses A_t^H units of the home intermediate good and B_t^H units of the foreign intermediate good to produce $\tilde{C}_t^H$ units of the home consumption good and $\tilde{I}_t^H$ units of the investment good according to the technology described by

$$[(1-\omega)^{1/\theta}(A_t^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(B_t^H)^{(\theta-1)/\theta}]^{\theta/(\theta-1)} \geq \tilde{C}_t^H + (1/V_t^H)\tilde{I}_t^H, \tag{21}$$

where the term $1/V_t^H$ out in front of investment $\tilde{I}_t^H$ on the right-hand side captures the effects of stochastic, investment-specific technological change of the kind described in a closed economy by Greenwood, Hercowitz, and Huffman (1988) and Greenwood, Hercowitz, and Krusell (1997, 2000) and introduced into small open economy or two-country models by Finn (1999), Boileau (2002), Guerrieri, Henderson, and Kim (2005), Mulraine (2006), Letendre and Luo (2007), Raffo (2009), and Coeurdacier, Kollman, and Martin (2010). The positive parameter θ measures the elasticity of substitution between the domestic and foreign intermediate goods in producing final goods, and the share parameter ω lies between zero and one.

The firm seeks to maximize its profits, given by

$$P_t^H \tilde{C}_t^H + X_t^H \tilde{I}_t^H - P_t^A A_t^H - P_t^B B_t^H.$$

Comparing this equation to the right-hand side of (21) reveals that

$$X_t^H / P_t^H = 1/V_t^H \tag{22}$$

must hold in any equilibrium, confirming that Greenwood, Hercowitz, and Krusell's (1997, 2000) insight that investment-specific technological progress manifests itself, partly, in a falling price of investment relative to consumption carries over to this model as well.

In light of this last observation, the first-order conditions for the firm's problem can be written as

$$P_t^A/P_t^H = [(1-\omega)^{1/\theta}(A_t^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(B_t^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(A_t^H)^{-1/\theta}, \tag{23}$$

$$P_t^B/P_t^H = [(1-\omega)^{1/\theta}(A_t^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(B_t^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(B_t^H)^{-1/\theta}, \tag{24}$$

and (21) with equality for all $t = 0, 1, 2, ...$

Symmetrically, the representative foreign final goods-producing firm uses A_t^F units of the home intermediate good and B_t^F units of the foreign intermediate good to produce $\tilde{C}_t^F$ units of the foreign consumption good and $\tilde{I}_t^F$ units of the foreign investment good according to

$$[\omega^{1/\theta}(A_t^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(B_t^F)^{(\theta-1)/\theta}]^{\theta/(\theta-1)} \geq \tilde{C}_t^F + (1/V_t^F)\tilde{I}_t^F, \tag{25}$$

where $1/V_t^F$ is the foreign investment-specific technology shock.

The firm seeks to maximize its profits, given by

$$P_t^F\tilde{C}_t^F + X_t^F\tilde{I}_t^F - P_t^A A_t^H - P_t^B B_t^H,$$

so that again, symmetrically to the domestic case,

$$X_t^F/P_t^F = 1/V_t^F \tag{26}$$

must hold in any equilibrium. The first-order conditions for the firm's problem can therefore be written as

$$P_t^A/P_t^F = [\omega^{1/\theta}(A_t^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(B_t^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(A_t^F)^{-1/\theta}, \tag{27}$$

$$P_t^B/P_t^F = [\omega^{1/\theta}(A_t^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(B_t^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(B_t^F)^{-1/\theta}, \tag{28}$$

and (25) with equality for all $t = 0, 1, 2, ...$

1.5 Governments

Both governments run balanced budgets, according to which lump-sum taxes raised from consumers are used to finance purchases of the local consumption good; hence

$$T_t^H/P_t^H = G_t^H \tag{29}$$

and

$$T_t^F/P_t^F = G_t^F \tag{30}$$

for all $t = 0, 1, 2, ...$

1.6 Trade Variables and Equilibrium Conditions

Net exports for the home country, expressed in units of the domestic consumption good, are

$$N_t^H = (P_t^A A_t^F - P_t^B B_t^H)/P_t^H. \tag{31}$$

Symmetrically, net exports for the foreign country, expressed in units of the foreign consumption good, are

$$N_t^F = (P_t^B B_t^H - P_t^A A_t^F)/P_t^F. \tag{32}$$

The real exchange rate is given by

$$RER_t = P_t^F / P_t^H \tag{33}$$

and the terms of trade are given by

$$TOT_t = P_t^B / P_t^A. \tag{34}$$

Keeping in mind that the intermediate goods are traded internationally and that bond holding and labor adjustment costs and government purchases are denominated in units of the local consumption goods, the model's market-clearing conditions are

$$Y_t^A = A_t^H + A_t^F, \tag{35}$$

$$Y_t^B = B_t^H + B_t^F, \tag{36}$$

$$\tilde{C}_t^H = C_t^H + \left(\frac{\phi_d}{2}\right) U_t^H \left(\frac{D_{t+1}^H}{U_t^H}\right)^2 + \left(\frac{\phi_l}{2}\right) U_{t-1}^H \left(\frac{L_t^H}{L_{t-1}^H} - 1\right)^2 L_{t-1}^H + G_t^H, \tag{37}$$

$$\tilde{I}_t^H = I_t^H, \tag{38}$$

$$\tilde{C}_t^F = C_t^F + \left(\frac{\phi_d}{2}\right) U_t^F \left(\frac{D_{t+1}^F}{U_t^F}\right)^2 + \left(\frac{\phi_l}{2}\right) U_{t-1}^F \left(\frac{L_t^F}{L_{t-1}^F} - 1\right)^2 L_{t-1}^F + G_t^F, \tag{39}$$

$$\tilde{I}_t^F = I_t^F, \tag{40}$$

and

$$D_t^H + D_t^F = 0 \tag{41}$$

for all $t = 0, 1, 2, ...$

Walras' law implies that one of these equilibrium conditions will be implied by the others. Consider, in particular, multiplying the home budget constraint (1) by P_t^H and the foreign budget constraint (8) by P_t^F, then adding the two results to obtain

$$\begin{aligned}
& W_t^H L_t^H + Q_t^H K_t^H + P_t^H D_t^H + W_t^F L_t^F + Q_t^F K_t^F + P_t^H D_t^F \\
= \; & P_t^H C_t^H + X_t^H I_t^H + T_t^H + P_t^F C_t^F + X_t^F I_t^F + T_t^F \\
& + \frac{P_t^H D_{t+1}^H}{R_t} + P_t^H \left(\frac{\phi_d}{2}\right) U_t^H \left(\frac{D_{t+1}^H}{U_t^H}\right)^2 + P_t^H \left(\frac{\phi_l}{2}\right) U_{t-1}^H \left(\frac{L_t^H}{L_{t-1}^H} - 1\right)^2 L_{t-1}^H \\
& + \frac{P_t^H D_{t+1}^F}{R_t} + P_t^F \left(\frac{\phi_d}{2}\right) U_t^F \left(\frac{D_{t+1}^F}{U_t^F}\right)^2 + P_t^F \left(\frac{\phi_l}{2}\right) U_{t-1}^F \left(\frac{L_t^F}{L_{t-1}^F} - 1\right)^2 L_{t-1}^F
\end{aligned}$$

Now use (16), (17), (19), (20), (29), (30), and (41) to rewrite this last expression as

$$\begin{aligned} P_t^A Y_t^A + P_t^B Y_t^B &= P_t^H C_t^H + X_t^H I_t^H + P_t^H G_t^H + P_t^F C_t^F + X_t^F I_t^F + P_t^F G_t^F \\ &\quad + P_t^H \left(\frac{\phi_d}{2}\right) U_t^H \left(\frac{D_{t+1}^H}{U_t^H}\right)^2 + P_t^H \left(\frac{\phi_l}{2}\right) U_{t-1}^H \left(\frac{L_t^H}{L_{t-1}^H} - 1\right)^2 L_{t-1}^H \\ &\quad + P_t^F \left(\frac{\phi_d}{2}\right) U_t^F \left(\frac{D_{t+1}^F}{U_t^F}\right)^2 + P_t^F \left(\frac{\phi_l}{2}\right) U_{t-1}^F \left(\frac{L_t^F}{L_{t-1}^F} - 1\right)^2 L_{t-1}^F \end{aligned}$$

Now use (22), (26), and (37)-(40) to simplify this last expression to

$$P_t^A Y_t^A + P_t^B Y_t^B = P_t^H [\tilde{C}_t^H + (1/V_t^H)\tilde{I}_t^H] + P_t^F [\tilde{C}_t^F + (1/V_t^F)\tilde{I}_t^F].$$

Finally, use (21), (23)-(25), (27), and (28) to rewrite this last expression as

$$P_t^A Y_t^A + P_t^B Y_t^B = P_t^A A_t^H + P_t^B B_t^H + P_t^A A_t^F + P_t^B B_t^F.$$

Hence, if (35) and all of the other equilibrium conditions hold, then (36) must hold as well.

On the other hand, since all prices have been thus far measured in an abstract unit of account, a choice of numeraire such as

$$P_t^H = 1 \tag{36}$$

can replace (36) as it appeared before in the system.

1.7 Exogenous Shocks

In addition, the model's equilibrium conditions include laws of motion for the eight exogenous shocks: M_t^H, Z_t^H, V_t^H, G_t^H, M_t^F, Z_t^F, V_t^F, and G_t^F.

Assume, in particular, that the preference shocks jointly follow a stationary autoregressive process:

$$\begin{bmatrix} \ln(M_t^H) \\ \ln(M_t^F) \end{bmatrix} = \begin{bmatrix} 1-\rho_m^{HH} & -\rho_m^{HF} \\ -\rho_m^{FH} & 1-\rho_m^{FF} \end{bmatrix} \begin{bmatrix} \ln(m^H) \\ \ln(m^F) \end{bmatrix} + \begin{bmatrix} \rho_m^{HH} & \rho_m^{HF} \\ \rho_m^{FH} & \rho_m^{FF} \end{bmatrix} \begin{bmatrix} \ln(M_{t-1}^H) \\ \ln(M_{t-1}^F) \end{bmatrix} + \begin{bmatrix} \varepsilon_{mt}^H \\ \varepsilon_{mt}^F \end{bmatrix}, \tag{42-43}$$

where the eigenvalues of the matrix of persistence parameters are both less than one in absolute value and the innovations are mutually and serially uncorrelated and normally distributed with zero means and standard deviations σ_m^H and σ_m^F.

Following Rabanal, Rubio-Ramirez, and Tuesta (2009), the neutral and investment-specific technology shocks are assumed to be nonstationary but cointegrated, so that

$$\begin{aligned} \begin{bmatrix} \ln(Z_t^H/Z_{t-1}^H) \\ \ln(Z_t^F/Z_{t-1}^F) \end{bmatrix} &= \begin{bmatrix} 1-\rho_z^{HH} & -\rho_z^{HF} \\ -\rho_z^{FH} & 1-\rho_z^{FF} \end{bmatrix} \begin{bmatrix} \ln(z^H) \\ \ln(z^F) \end{bmatrix} \\ &\quad + \begin{bmatrix} \rho_z^{HH} & \rho_z^{HF} \\ \rho_z^{FH} & \rho_z^{FF} \end{bmatrix} \begin{bmatrix} \ln(Z_{t-1}^H/Z_{t-2}^H) \\ \ln(Z_{t-1}^F/Z_{t-2}^F) \end{bmatrix} \\ &\quad + \begin{bmatrix} \kappa_z^H \\ \kappa_z^F \end{bmatrix} [\ln(Z_{t-1}^H) - \ln(Z_{t-1}^F) - \ln(z^{HF})] + \begin{bmatrix} \varepsilon_{zt}^H \\ \varepsilon_{zt}^F \end{bmatrix}, \end{aligned} \tag{44-45}$$

and

$$\begin{bmatrix}\ln(V_t^H/V_{t-1}^H)\\ \ln(V_t^F/V_{t-1}^F)\end{bmatrix} = \begin{bmatrix}1-\rho_v^{HH} & -\rho_v^{HF}\\ -\rho_v^{FH} & 1-\rho_v^{FF}\end{bmatrix}\begin{bmatrix}\ln(v^H)\\ \ln(v^F)\end{bmatrix} + \begin{bmatrix}\rho_v^{HH} & \rho_v^{HF}\\ \rho_v^{FH} & \rho_v^{FF}\end{bmatrix}\begin{bmatrix}\ln(V_{t-1}^H/V_{t-2}^H)\\ \ln(V_{t-1}^F/V_{t-2}^F)\end{bmatrix} + \begin{bmatrix}\kappa_v^H\\ \kappa_v^F\end{bmatrix}[\ln(V_{t-1}^H)-\ln(V_{t-1}^F)-\ln(v^{HF})] + \begin{bmatrix}\varepsilon_{vt}^H\\ \varepsilon_{vt}^F\end{bmatrix}, \tag{46-47}$$

where the eigenvalues of the matrices of persistence parameters are all less than one in absolute value and where the innovations are mutually and serially uncorrelated and normally distributed with zero means and standard deviations σ_z^H, σ_z^F, σ_v^H, and σ_v^F.

Finally, assume that the government spending variables G_t^H and G_t^F, after they are scaled by the growth U_{t-1}^H and U_{t-1}^F follow a stationary autoregressive process, so that

$$\begin{bmatrix}\ln(G_t^H/U_{t-1}^H)\\ \ln(G_t^F/U_{t-1}^F)\end{bmatrix} = \begin{bmatrix}1-\rho_g^{HH} & -\rho_g^{HF}\\ -\rho_g^{FH} & 1-\rho_g^{FF}\end{bmatrix}\begin{bmatrix}\ln(g^H)\\ \ln(g^F)\end{bmatrix} + \begin{bmatrix}\rho_g^{HH} & \rho_g^{HF}\\ \rho_g^{FH} & \rho_g^{FF}\end{bmatrix}\begin{bmatrix}\ln(G_{t-1}^H/U_{t-2}^H)\\ \ln(G_{t-1}^F/U_{t-2}^F)\end{bmatrix} + \begin{bmatrix}\varepsilon_{gt}^H\\ \varepsilon_{gt}^F\end{bmatrix}, \tag{48-49}$$

where the eigenvalues of the matrix of persistence parameters are both less than one in absolute value and the innovations are mutually and serially uncorrelated and normally distributed with zero means and standard deviations σ_g^H and σ_g^F.

1.8 Equilibrium System

Taken together, therefore, the equilibrium conditions form a system of 49 equations. The corresponding list of variables is:

22 home variables C_t^H, I_t^H, G_t^H, N_t^H, $\tilde{C}_t^H$, $\tilde{I}_t^H$, L_t^H, K_t^H, D_t^H, Λ_t^H, Ξ_t^H, T_t^H, Y_t^A, A_t^H, B_t^H, P_t^H, X_t^H, W_t^H, Q_t^H, M_t^H, Z_t^H, V_t^H.

22 foreign variables C_t^F, I_t^F, G_t^F, N_t^F, $\tilde{C}_t^F$, $\tilde{I}_t^F$, L_t^F, K_t^F, D_t^F, Λ_t^F, Ξ_t^F, T_t^F, Y_t^B, A_t^F, B_t^F, P_t^F, X_t^F, W_t^F, Q_t^F, M_t^F, Z_t^F, V_t^F.

5 international variables R_t, P_t^A, P_t^B, RER_t, TOT_t.

With four shocks for each country, the model can be estimated with data on four variables for each country, the natural choices being consumption, investment, government spending, and hours worked. After the model is estimated, the predicted time paths for international variables like net exports and the real exchange rate can be compared to the series for the corresponding variables in the data.

2 The Stationary System

Although many of the variables listed above inherit unit roots from the nonstationary shocks, appropriately transformed versions of these variables remain stationary, and the equilibrium system can be rewritten in terms of these stationary variables.

Accordingly, define the scaled home variables

$c_t^H = C_t^H / U_{t-1}^H,$

$i_t^H = I_t^H / (V_{t-1}^H U_{t-1}^H),$

$g_t^H = G_t^H / U_{t-1}^H,$

$n_t^H = N_t^H / U_{t-1}^H,$

$\tilde{c}_t^H = \tilde{C}_t^H / U_{t-1}^H,$

$\tilde{i}_t^H = \tilde{I}_t / (V_{t-1}^H U_{t-1}^H),$

$k_t^H = K_t^H / (V_{t-1}^H U_{t-1}^H),$

$d_t^H = D_t^H / U_{t-1}^H,$

$\lambda_t^H = (U_{t-1}^H)^{1-\mu(1-\gamma)} \Lambda_t^H,$

$\xi_t^H = (U_{t-1}^H)^{1-\mu(1-\gamma)} V_{t-1}^H \Xi_t^H,$

$\tau_t^H = T_t^H / (U_{t-1}^H P_t^H),$

$y_t^A = Y_t^A / U_{t-1}^H,$

$a_t^H = A_t^H / U_{t-1}^H,$

$b_t^H = B_t^H / U_{t-1}^H,$

$p_t^H = P_t^H / P_t^H,$

$x_t^H = V_{t-1}^H X_t^H / P_t^H,$

$w_t^H = W_t^H / (U_{t-1}^H P_t^H),$

$q_t^H = V_{t-1}^H Q_t^H / P_t^H,$

$m_t^H = M_t^H,$

$z_t^H = Z_t^H / Z_{t-1}^H$, and

$v_t^H = V_t^H / V_{t-1}^H.$

Symmetrically, define the scaled foreign variables

$c_t^F = C_t^F / U_{t-1}^F,$

$i_t^F = I_t^F / (V_{t-1}^F U_{t-1}^F),$

$g_t^F = G_t^F / U_{t-1}^F,$

$n_t^F = N_t^F / U_{t-1}^F,$

$\tilde{c}_t^F = \tilde{C}_t^F / U_{t-1}^F$,

$\tilde{\imath}_t^F = \tilde{I}_t^F / (V_{t-1}^F U_{t-1}^F)$,

$k_t^F = K_t^F / (V_{t-1}^F U_{t-1}^F)$,

$d_t^F = D_t^F / U_{t-1}^F$,

$\lambda_t^F = (U_{t-1}^F)^{1-\mu(1-\gamma)} \Lambda_t^F$,

$\xi_t^F = (U_{t-1}^F)^{1-\mu(1-\gamma)} V_{t-1}^F \Xi_t^F$,

$\tau_t^F = T_t^F / (U_{t-1}^F P_t^F)$,

$y_t^B = Y_t^B / U_{t-1}^F$,

$a_t^F = A_t^F / U_{t-1}^F$,

$b_t^F = B_t^F / U_{t-1}^F$,

$p_t^F = P_t^F / P_t^H$,

$x_t^F = V_{t-1}^F X_t^F / P_t^F$,

$w_t^F = W_t^F / (U_{t-1}^F P_t^F)$,

$q_t^F = V_{t-1}^F Q_t^F / P_t^F$,

$m_t^F = M_t^F$,

$z_t^F = Z_t^F / Z_{t-1}^F$, and

$v_t^F = V_t^F / V_{t-1}^F$.

Define the scaled international variables

$r_t = R_t$,

$p_t^A = P_t^A / P_t^H$,

$p_t^B = P_t^B / P_t^H$,

$z_t^{HF} = Z_t^H / Z_t^F$, and

$v_t^{HF} = V_t^H / V_t^F$.

Finally, define the scaled observable variables

$g_t^{CH} = C_t^H / C_{t-1}^H$,

$g_t^{IH} = I_t^H / I_{t-1}^H$,

$r_t^{GCH} = G_t^H / C_t^H$,

$l_t^H = L_t^H$,

$r_t^{CFH} = C_t^F/C_t^H$,

$r_t^{IFH} = I_t^F/I_t^H$,

$r_t^{GCF} = G_t^F/C_t^F$, and

$l_t^F = L_t^F$.

In terms of these scaled/stationary variables, the equilibrium conditions can be rewritten as

$$\begin{aligned} &w_t^H l_t^H + q_t^H k_t^H + d_t^H \\ &= c_t^H + x_t^H i_t^H + \tau_t^H + (v_t^H)^{\alpha/(1-\alpha)} z_t^H d_{t+1}^H / r_t \\ &\quad + (\phi_d/2)(v_t^H)^{\alpha/(1-\alpha)} z_t^H (d_{t+1}^H)^2 + (\phi_l/2)(l_t^H/l_{t-1}^H - 1)^2 l_{t-1}^H, \end{aligned} \tag{1}$$

$$(1-\delta)k_t^H + i_t^H - (\phi_k/2)(i_t^H/k_t^H - \eta^H)^2 k_t^H = (v_t^H)^{1/(1-\alpha)} z_t^H k_{t+1}^H, \tag{2}$$

$$\mu[(c_t^H)^\mu (1 - l_t^H/m_t^H)^{1-\mu}]^{1-\gamma} = \lambda_t^H c_t^H, \tag{3}$$

$$\begin{aligned} &\frac{(1-\mu)[(c_t^H)^\mu (1 - l_t^H/m_t^H)^{1-\mu}]^{1-\gamma}}{m_t^H (1 - l_t^H/m_t^H)} \\ &= \lambda_t^H [w_t^H - \phi_l (l_t^H/l_{t-1}^H - 1)] \\ &\quad + \beta\phi_l E_t\{\lambda_{t+1}^H [(v_t^H)^{\alpha/(1-\alpha)} z_t^H]^{\mu(1-\gamma)} [(l_{t+1}^H/l_t^H - 1)(l_{t+1}^H/l_t^H) - (1/2)(l_{t+1}^H/l_t^H - 1)^2]\} \end{aligned} \tag{4}$$

$$\lambda_t^H x_t^H = \xi_t^H [1 - \phi_k (i_t^H/k_t^H - \eta^H)], \tag{5}$$

$$\begin{aligned} &[(v_t^H)^{\alpha/(1-\alpha)} z_t^H]^{1-\mu(1-\gamma)} v_t^H \xi_t^H \\ &= \beta E_t (\lambda_{t+1}^H q_{t+1}^H) \\ &\quad + \beta E_t \{\xi_{t+1}^H [1 - \delta + \phi_k (i_{t+1}^H/k_{t+1}^H - \eta^H)(i_{t+1}^H/k_{t+1}^H) - (\phi_k/2)(i_{t+1}^H/k_{t+1}^H - \eta^H)^2]\}, \end{aligned} \tag{6}$$

$$[(v_t^H)^{\alpha/(1-\alpha)} z_t^H]^{1-\mu(1-\gamma)} \lambda_t^H (1/r_t + \phi_d d_{t+1}^H) = \beta E_t \lambda_{t+1}^H, \tag{7}$$

$$\begin{aligned} &w_t^F l_t^F + q_t^F k_t^F + d_t^F/p_t^F \\ &= c_t^F + x_t^F i_t^F + \tau_t^F + (v_t^F)^{\alpha/(1-\alpha)} z_t^F d_{t+1}^F / (p_t^F r_t) \\ &\quad + (\phi_d/2)(v_t^F)^{\alpha/(1-\alpha)} z_t^F (d_{t+1}^F)^2 + (\phi_l/2)(l_t^F/l_{t-1}^F - 1)^2 l_{t-1}^F, \end{aligned} \tag{8}$$

$$(1-\delta)k_t^F + i_t^F - (\phi_k/2)(i_t^F/k_t^F - \eta^F)^2 k_t^F = (v_t^F)^{1/(1-\alpha)} z_t^F k_{t+1}^F, \tag{9}$$

$$\mu[(c_t^F)^\mu (1 - l_t^F/m_t^F)^{1-\mu}]^{1-\gamma} = \lambda_t^F c_t^F, \tag{10}$$

$$\begin{aligned} &\frac{(1-\mu)[(c_t^F)^\mu (1 - l_t^F/m_t^F)^{1-\mu}]^{1-\gamma}}{m_t^F (1 - l_t^F/m_t^F)} \\ &= \lambda_t^F [w_t^F - \phi_l (l_t^F/l_{t-1}^F - 1)] \\ &\quad + \beta\phi_l E_t\{\lambda_{t+1}^F [(v_t^F)^{\alpha/(1-\alpha)} z_t^F]^{\mu(1-\gamma)} [(l_{t+1}^F/l_t^F - 1)(l_{t+1}^F/l_t^F) - (1/2)(l_{t+1}^F/l_t^F - 1)^2]\} \end{aligned} \tag{11}$$

$$\lambda_t^F x_t^F = \xi_t^F [1 - \phi_k (i_t^F/k_t^F - \eta^F)], \tag{12}$$

$$
\begin{aligned}
&[(v_t^F)^{\alpha/(1-\alpha)} z_t^F]^{1-\mu(1-\gamma)} v_t^F \xi_t^F \\
&= \beta E_t(\lambda_{t+1}^F q_{t+1}^F) \\
&\quad + \beta E_t\{\xi_{t+1}^F[1-\delta+\phi_k(i_{t+1}^F/k_{t+1}^F-\eta^F)(i_{t+1}^F/k_{t+1}^F)-(\phi_k/2)(i_{t+1}^F/k_{t+1}^F-\eta^F)^2]\},
\end{aligned} \tag{13}
$$

$$[(v_t^F)^{\alpha/(1-\alpha)} z_t^F]^{1-\mu(1-\gamma)} \lambda_t^F [1/(p_t^F r_t)+\phi_d d_{t+1}^F] = \beta E_t(\lambda_{t+1}^F/p_{t+1}^F), \tag{14}$$

$$(k_t^H)^\alpha (z_t^H l_t^H)^{1-\alpha} = y_t^A, \tag{15}$$

$$\alpha p_t^A y_t^A = q_t^H k_t^H, \tag{16}$$

$$(1-\alpha) p_t^A y_t^A = w_t^H l_t^H, \tag{17}$$

$$(k_t^F)^\alpha (z_t^F l_t^F)^{1-\alpha} = y_t^B, \tag{18}$$

$$\alpha p_t^B y_t^B = p_t^F q_t^F k_t^F, \tag{19}$$

$$(1-\alpha) p_t^B y_t^B = p_t^F w_t^F l_t^F, \tag{20}$$

$$[(1-\omega)^{1/\theta}(a_t^H)^{(\theta-1)/\theta}+\omega^{1/\theta}(b_t^H)^{(\theta-1)/\theta}]^{\theta/(\theta-1)} = \tilde{c}_t^H + (1/v_t^H)\tilde{\imath}_t^H, \tag{21}$$

$$x_t^H = 1/v_t^H, \tag{22}$$

$$p_t^A = [(1-\omega)^{1/\theta}(a_t^H)^{(\theta-1)/\theta}+\omega^{1/\theta}(b_t^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(a_t^H)^{-1/\theta}, \tag{23}$$

$$p_t^B = [(1-\omega)^{1/\theta}(a_t^H)^{(\theta-1)/\theta}+\omega^{1/\theta}(b_t^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(b_t^H)^{-1/\theta}, \tag{24}$$

$$[\omega^{1/\theta}(a_t^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b_t^F)^{(\theta-1)/\theta}]^{\theta/(\theta-1)} = \tilde{c}_t^F + (1/v_t^F)\tilde{\imath}_t^F, \tag{25}$$

$$x_t^F = 1/v_t^F, \tag{26}$$

$$p_t^A/p_t^F = [\omega^{1/\theta}(a_t^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b_t^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(a_t^F)^{-1/\theta}, \tag{27}$$

$$p_t^B/p_t^F = [\omega^{1/\theta}(a_t^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b_t^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(b_t^F)^{-1/\theta}, \tag{28}$$

$$\tau_t^H = g_t^H, \tag{29}$$

$$\tau_t^F = g_t^F, \tag{30}$$

$$n_t^H = p_t^A a_t^F/[(v_{t-1}^{HF})^{\alpha/(1-\alpha)} z_{t-1}^{HF}] - p_t^B b_t^H, \tag{31}$$

$$n_t^F = [p_t^B b_t^H (v_{t-1}^{HF})^{\alpha/(1-\alpha)} z_{t-1}^{HF} - p_t^A a_t^F]/p_t^F, \tag{32}$$

$$RER_t = p_t^F, \tag{33}$$

$$TOT_t = p_t^B/p_t^A, \tag{34}$$

$$y_t^A = a_t^H + a_t^F/[(v_{t-1}^{HF})^{\alpha/(1-\alpha)} z_{t-1}^{HF}], \tag{35}$$

$$p_t^H = 1, \tag{36}$$

$$\tilde{c}_t^H = c_t^H + (\phi_d/2)(v_t^H)^{\alpha/(1-\alpha)} z_t^H (d_{t+1}^H)^2 + (\phi_l/2)(l_t^H/l_{t-1}^H-1)^2 l_{t-1}^H + g_t^H, \tag{37}$$

$$\tilde{\imath}_t^H = i_t^H, \tag{38}$$

$$\tilde{c}_t^F = c_t^F + (\phi_d/2)(v_t^F)^{\alpha/(1-\alpha)} z_t^F (d_{t+1}^F)^2 + (\phi_l/2)(l_t^F/l_{t-1}^F-1)^2 l_{t-1}^F + g_t^F, \tag{39}$$

$$\tilde{\imath}_t^F = i_t^F, \tag{40}$$

$$(v_{t-1}^{HF})^{\alpha/(1-\alpha)} z_{t-1}^{HF} d_t^H + d_t^F = 0, \tag{41}$$

$$\begin{bmatrix}\ln(m_t^H)\\ \ln(m_t^F)\end{bmatrix} = \begin{bmatrix}1-\rho_m^{HH} & -\rho_m^{HF}\\ -\rho_m^{FH} & 1-\rho_m^{FF}\end{bmatrix}\begin{bmatrix}\ln(m^H)\\ \ln(m^F)\end{bmatrix} + \begin{bmatrix}\rho_m^{HH} & \rho_m^{HF}\\ \rho_m^{FH} & \rho_m^{FF}\end{bmatrix}\begin{bmatrix}\ln(m_{t-1}^H)\\ \ln(m_{t-1}^F)\end{bmatrix} + \begin{bmatrix}\varepsilon_{mt}^H\\ \varepsilon_{mt}^F\end{bmatrix}, \tag{42-43}$$

$$\begin{aligned}\begin{bmatrix}\ln(z_t^H)\\ \ln(z_t^F)\end{bmatrix} &= \begin{bmatrix}1-\rho_z^{HH} & -\rho_z^{HF}\\ -\rho_z^{FH} & 1-\rho_z^{FF}\end{bmatrix}\begin{bmatrix}\ln(z^H)\\ \ln(z^F)\end{bmatrix}\\ &\quad + \begin{bmatrix}\rho_z^{HH} & \rho_z^{HF}\\ \rho_z^{FH} & \rho_z^{FF}\end{bmatrix}\begin{bmatrix}\ln(z_{t-1}^H)\\ \ln(z_{t-1}^F)\end{bmatrix} + \begin{bmatrix}\kappa_z^H\\ \kappa_z^F\end{bmatrix}[\ln(z_{t-1}^{HF}) - \ln(z^{HF})] + \begin{bmatrix}\varepsilon_{zt}^H\\ \varepsilon_{zt}^F\end{bmatrix},\end{aligned} \tag{44-45}$$

$$\begin{aligned}\begin{bmatrix}\ln(v_t^H)\\ \ln(v_t^F)\end{bmatrix} &= \begin{bmatrix}1-\rho_v^{HH} & -\rho_v^{HF}\\ -\rho_v^{FH} & 1-\rho_v^{FF}\end{bmatrix}\begin{bmatrix}\ln(v^H)\\ \ln(v^F)\end{bmatrix}\\ &\quad + \begin{bmatrix}\rho_v^{HH} & \rho_v^{HF}\\ \rho_v^{FH} & \rho_v^{FF}\end{bmatrix}\begin{bmatrix}\ln(v_{t-1}^H)\\ \ln(v_{t-1}^F)\end{bmatrix} + \begin{bmatrix}\kappa_v^H\\ \kappa_v^F\end{bmatrix}[\ln(v_{t-1}^{HF}) - \ln(v^{HF})] + \begin{bmatrix}\varepsilon_{vt}^H\\ \varepsilon_{vt}^F\end{bmatrix},\end{aligned} \tag{46-47}$$

and

$$\begin{bmatrix}\ln(g_t^H)\\ \ln(g_t^F)\end{bmatrix} = \begin{bmatrix}1-\rho_g^{HH} & -\rho_g^{HF}\\ -\rho_g^{FH} & 1-\rho_g^{FF}\end{bmatrix}\begin{bmatrix}\ln(g^H)\\ \ln(g^F)\end{bmatrix} + \begin{bmatrix}\rho_g^{HH} & \rho_g^{HF}\\ \rho_g^{FH} & \rho_g^{FF}\end{bmatrix}\begin{bmatrix}\ln(g_{t-1}^H)\\ \ln(g_{t-1}^F)\end{bmatrix} + \begin{bmatrix}\varepsilon_{gt}^H\\ \varepsilon_{gt}^F\end{bmatrix}, \tag{48-49}$$

where the definitions of the new variables z_t^{HF}, v_t^{HF}, g_t^{CH}, g_t^{IH}, r_t^{GCH}, r_t^{CFH}, r_t^{IFH}, and r_t^{GCF} imply that

$$z_t^{HF} = (z_t^H/z_t^F)z_{t-1}^{HF}, \tag{50}$$

$$v_t^{HF} = (v_t^H/v_t^F)v_{t-1}^{HF}, \tag{51}$$

$$g_t^{CH} = (c_t^H/c_{t-1}^H)(v_{t-1}^H)^{\alpha/(1-\alpha)}z_{t-1}^H, \tag{52}$$

$$g_t^{IH} = (i_t^H/i_{t-1}^H)(v_{t-1}^H)^{1/(1-\alpha)}z_{t-1}^H, \tag{53}$$

$$r_t^{GCH} = g_t^H/c_t^H, \tag{54}$$

$$r_t^{CFH} = (c_t^F/c_t^H)\{[1/(v_{t-1}^{HF})^{\alpha/(1-\alpha)}z_{t-1}^{HF}]\}, \tag{55}$$

$$r_t^{IFH} = (i_t^F/i_t^H)\{1/[(v_{t-1}^{HF})^{1/(1-\alpha)}z_{t-1}^{HF}]\}, \tag{56}$$

and

$$r_t^{GCF} = g_t^F/c_t^F, \tag{57}$$

must hold as well.

This transformed system of 57 equations describes the equilibrium behavior of the 57 stationary variables that can be categorized as follows:

21 home variables c_t^H, i_t^H, g_t^H, n_t^H, $\tilde{c}_t^H$, $\tilde{i}_t^H$, k_t^H, d_t^H, λ_t^H, ξ_t^H, τ_t^H, y_t^A, a_t^H, b_t^H, p_t^H, x_t^H, w_t^H, q_t^H, m_t^H, z_t^H, v_t^H.

21 foreign variables c_t^F, i_t^F, g_t^F, n_t^F, $\tilde{c}_t^F$, $\tilde{i}_t^F$, k_t^F, d_t^F, λ_t^F, ξ_t^F, τ_t^F, y_t^B, a_t^F, d_t^F, p_t^F, x_t^F, w_t^F, q_t^F, m_t^F, z_t^F, v_t^F.

7 international variables r_t, p_t^A, p_t^B, RER_t, TOT_t, z_t^{HF}, v_t^{HF}.

8 observable variables g_t^{CH}, g_t^{IH}, r_t^{GCH}, l_t^H, r_t^{CFH}, r_t^{IFH}, r_t^{GCF}, l_t^F.

3 The Steady State

Equations (1)-(57) imply that in the absence of shocks, and when $z^H = z^F = z$ and $v^H = v^F = v$ so that both neutral and investment-specific technological progress proceed at the same long-run rate across the two countries, the global economy converges to a steady-state growth path along which all of the stationary variables are constant.

Let variables without time subscripts denote steady-state values. Equations (42)-(51) provide the values for m^H, m^F, z, v, g^H, g^F, z^{HF} and v^{HF}. Equations (22), (26), (29), (30), and (36) then imply

$$x^H = 1/v,$$

$$x^F = 1/v,$$

$$\tau^H = g^H,$$

$$\tau^F = g^F,$$

and

$$p^H = 1,$$

and when, as suggested above, η^H and η^F are set equal to the steady-state investment-to-capital ratios in the two countries, (5)-(7), (12)-(14), and (41) imply that

$$\xi^H = \lambda^H/v$$

$$\xi^F = \lambda^F/v$$

$$d^H = 0,$$

$$d^F = 0,$$

$$r = [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}/\beta,$$

$$q^H = [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}/\beta - (1-\delta)/v,$$

and

$$q^F = [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}/\beta - (1-\delta)/v.$$

Next, suppose that steady-state values for p^A, p^B, p^F, y^A and y^B are in hand. Then, use (16) and (19) to compute

$$k^H = (\alpha/q^H)p^A y^A$$

and

$$k^F = (\alpha/q^F)(p^B/p^F)y^B.$$

Use (15) and (18) to compute

$$l^H = (1/z^H)[(q^H/\alpha)(1/p^A)]^{\alpha/(1-\alpha)}y^A$$

and

$$l^F = (1/z^F)[(q^F/\alpha)(p^F/p^B)]^{\alpha/(1-\alpha)}y^B.$$

Use (17) and (20) to compute

$$w^H = (1-\alpha)z^H(\alpha/q^H)^{\alpha/(1-\alpha)}(p^A)^{1/(1-\alpha)}$$

and

$$w^F = (1-\alpha)z^F(\alpha/q^F)^{\alpha/(1-\alpha)}(p^B/p^F)^{1/(1-\alpha)}.$$

Use (2) and (9) to compute

$$i^H = [v^{1/(1-\alpha)}z - 1 + \delta](\alpha/q^H)p^A y^A$$

and

$$i^F = [v^{1/(1-\alpha)}z - 1 + \delta](\alpha/q^F)(p^B/p^F)y^B$$

and to find the appropriate parameter settings

$$\eta^H = v^{1/(1-\alpha)}z - 1 + \delta$$

and

$$\eta^F = v^{1/(1-\alpha)}z - 1 + \delta.$$

And use (4) and (11) to compute

$$c^H = (1-\alpha)\left(\frac{\mu}{1-\mu}\right)\left[z^H m^H \left(\frac{\alpha}{q^H}\right)^{\alpha/(1-\alpha)}(p^A)^{1/(1-\alpha)} - p^A y^A\right]$$

and

$$c^F = (1-\alpha)\left(\frac{\mu}{1-\mu}\right)\left[z^F m^F \left(\frac{\alpha}{q^F}\right)^{\alpha/(1-\alpha)}\left(\frac{p^B}{p^F}\right)^{1/(1-\alpha)} - \left(\frac{p^B}{p^F}\right)y^B\right].$$

Now substitute these solutions into the steady-state versions of (1) and (8) to solve for the steady-state values of y^A and y^B as

$$y^A = \frac{\mu(1-\alpha)z^H m^H[(\alpha/q^H)p^A]^{\alpha/(1-\alpha)} + (1-\mu)g^H/p^A}{1-\alpha\mu-(1/v)(1-\mu)[v^{1/(1-\alpha)}z-1+\delta](\alpha/q^H)}$$

and

$$y^B = \frac{\mu(1-\alpha)z^F m^F[(\alpha/q^F)(p^B/p^F)]^{\alpha/(1-\alpha)} + (1-\mu)p^F g^F/p^B}{1-\alpha\mu-(1/v)(1-\mu)[v^{1/(1-\alpha)}z-1+\delta](\alpha/q^F)}.$$

Next, use (1), (16), (17), (22), (29), (37), and (38) to rewrite the steady-state version of (21) as

$$[(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{\theta/(\theta-1)} = p^A y^A$$

and, similarly, use (8), (19), (20), (26), (30), (39), and (40) to rewrite the steady-state version of (25) as

$$[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{\theta/(\theta-1)} = (p^B/p^F)y^B.$$

These last expressions imply that the steady-state versions of (23), (24), (27), and (28) can be rewritten as

$$a^H = (1-\omega)y^A(p^A)^{1-\theta},$$

$$b^H = \omega y^A p^A (p^B)^{-\theta},$$

$$a^F = \omega y^B(p^B/p^F)(p^A/p^F)^{-\theta},$$

and

$$b^F = (1-\omega)y^B(p^B/p^F)^{1-\theta},$$

which when substituted back into the steady-state versions of (21) and (25), imply that

$$1 = (1-\omega)(p^A)^{1-\theta} + \omega(p^B)^{1-\theta}$$

and

$$(p^F)^{1-\theta} = \omega(p^A)^{1-\theta} + (1-\omega)(p^B)^{1-\theta}$$

or equivalently, that

$$p^A = \left[\frac{1-\omega-\omega(p^F)^{1-\theta}}{1-2\omega}\right]^{1/(1-\theta)}$$

and

$$p^B = \left[\frac{(1-\omega)(p^F)^{1-\theta}-\omega}{1-2\omega}\right]^{1/(1-\theta)}.$$

Now substitute these solutions into the steady-state version of (35),

$$y^A = a^H + a^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}],$$

to obtain an expression that can be solved numerically for the steady-state value of p^F.

To complete the construction of the steady-state, take the values just computed and use (3) and (10) to obtain

$$\lambda^H = \frac{\mu[(c^H)^\mu(1-l^H/m^H)^{1-\mu}]^{1-\gamma}}{c^H}$$

and

$$\lambda^F = \frac{\mu[(c^F)^\mu(1-l^F/m^F)^{1-\mu}]^{1-\gamma}}{c^F}.$$

Use (31)-(34) to compute

$$n^H = \frac{p^Aa^F}{(v^{HF})^{\alpha/(1-\alpha)}z^{HF}} - p^Bb^H$$

$$n^F = \frac{p^Bb^H(v^{HF})^{\alpha/(1-\alpha)}z^{HF} - p^Aa^F}{p^F},$$

$$RER = p^F,$$

and

$$TOT = p^B/p^A.$$

Finally, use (37)-(40) and (52)-(57) to compute

$$\tilde{c}^H = c^H + g^H,$$

$$\tilde{\imath}^H = i^H,$$

$$\tilde{c}^F = c^F + g^F,$$

$$\tilde{\imath}^F = i^F,$$

$$
\begin{aligned}
g^{CH} &= v^{\alpha/(1-\alpha)}z, \\
g^{IH} &= v^{1/(1-\alpha)}z, \\
r^{GCH} &= g^H/c^H, \\
r^{CFH} &= (c^F/c^H)\{[1/(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\}, \\
r^{IFH} &= (i^F/i^H)\{[1/(v^{HF})^{1/(1-\alpha)}z^{HF}]\},
\end{aligned}
$$

and

$$
r^{GCF} = g^F/c^F.
$$

4 The Linearized System

To reduce the size of the equilibrium system, it is helpful at this point to observe that (22), (26), (29), (30), (33), and (36)-(40) can be used to solve out for the variables x_t^H, x_t^F, τ_t^H, τ_t^F, RER_t, p_t^H, $\tilde{c}_t^H$, $\tilde{\imath}_t^H$, $\tilde{c}_t^F$, and $\tilde{\imath}_t^F$. The remaining 47 equations can then be log-linearized around the steady state to describe how the remaining variables respond to shocks.

In particular, let hatted variables keep track of the logarithmic (percentage) deviations of the corresponding unhatted variables from their steady-state values, except for the net export and foreign debt variables, where $\hat{n}_t^H = n_t^H$, $\hat{n}_t^F = n_t^F$, $\hat{d}_t^H = d_t^H$ and $\hat{d}_t^F = d_t^F$ must be used instead since those variables equal zero in steady state. Then first-order Taylor approximations yield

$$
\begin{aligned}
&w^H l^H \hat{w}_t^H + w^H l^H \hat{l}_t^H + q^H k^H \hat{q}_t^H + q^H k^H \hat{k}_t^H + \hat{d}_t^H \\
&= c^H \hat{c}_t^H + (i^H/v)\hat{\imath}_t^H - (i^H/v)\hat{v}_t^H + g^H \hat{g}_t^H + [v^{\alpha/(1-\alpha)}z/r]\hat{d}_{t+1}^H,
\end{aligned} \tag{1}
$$

$$
(1-\delta)k^H \hat{k}_t^H + i^H \hat{\imath}_t^H = [1/(1-\alpha)]v^{1/(1-\alpha)}zk^H \hat{v}_t^H + v^{1/(1-\alpha)}zk^H \hat{z}_t^H + v^{1/(1-\alpha)}zk^H \hat{k}_{t+1}^H, \tag{2}
$$

$$
\begin{aligned}
&(1-\mu)(1-\gamma)l^H \hat{m}_t^H - (1-\mu)(1-\gamma)l^H \hat{l}_t^H \\
&= (m^H - l^H)\hat{\lambda}_t^H + [1-\mu(1-\gamma)](m^H - l^H)\hat{c}_t^H,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&[(1-\mu)c^H(m^H - l^H)^{-1} - \mu w^H]\hat{\lambda}_t^H + (1-\mu)c^H(m^H - l^H)^{-1}\hat{c}_t^H \\
&= \mu w^H \hat{w}_t^H + (1-\mu)c^H m^H (m^H - l^H)^{-2}\hat{m}_t^H + \phi_l \mu \hat{l}_{t-1}^H \\
&\quad - \{\phi_l\mu + \beta\phi_l\mu[v^{\alpha/(1-\alpha)}z]^{\mu(1-\gamma)} + (1-\mu)c^H l^H (m^H - l^H)^{-2}\}\hat{l}_t^H \\
&\quad + \beta\phi_l\mu[v^{\alpha/(1-\alpha)}z]^{\mu(1-\gamma)}E_t\hat{l}_{t+1}^H,
\end{aligned} \tag{4}
$$

$$
\hat{\lambda}_t^H - \hat{v}_t^H = \hat{\xi}_t^H - \phi_k(i^H/k^H)\hat{\imath}_t^H + \phi_k(i^H/k^H)\hat{k}_t^H, \tag{5}
$$

$$
\begin{aligned}
&\{[\alpha/(1-\alpha)][1-\mu(1-\gamma)] + 1\}[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{v}_t^H \\
&\quad + [1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{z}_t^H + [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{\xi}_t^H \\
&= \beta q^H E_t\hat{\lambda}_{t+1}^H + \beta q^H E_t\hat{q}_{t+1}^H + \beta[(1-\delta)/v]E_t\hat{\xi}_{t+1}^H \\
&\quad + \beta(1/v)\phi_k(i^H/k^H)^2 E_t i_{t+1}^H - \beta(1/v)\phi_k(i^H/k^H)^2 k_{t+1}^H,
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&[\alpha/(1-\alpha)][1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{v}_t^H \\
&\quad + [1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{z}_t^H \\
&\quad + [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{\lambda}_t^H - [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{r}_t \\
&= \beta E_t\hat{\lambda}_{t+1}^H - \phi_d[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{d}_{t+1}^H,
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&w^F l^F \hat{w}_t^F + w^F l^F \hat{l}_t^F + q^F k^F \hat{q}_t^F + q^F k^F \hat{k}_t^F + (1/p^F)\hat{d}_t^F \\
&= c^F \hat{c}_t^F + (i^F/v)\hat{i}_t^F - (i^F/v)\hat{v}_t^F + g^F\hat{g}_t^F + [v^{\alpha/(1-\alpha)}z/(p^F r)]\hat{d}_{t+1}^F,
\end{aligned} \tag{8}
$$

$$
(1-\delta)k^F\hat{k}_t^F + i^F\hat{i}_t^F = [1/(1-\alpha)]v^{1/(1-\alpha)}zk^F\hat{v}_t^F + v^{1/(1-\alpha)}zk^F\hat{z}_t^F + v^{1/(1-\alpha)}zk^F\hat{k}_{t+1}^F, \tag{9}
$$

$$
\begin{aligned}
&(1-\mu)(1-\gamma)l^F\hat{m}_t^F - (1-\mu)(1-\gamma)l^F\hat{l}_t^F \\
&= (m^F - l^F)\hat{\lambda}_t^F + [1-\mu(1-\gamma)](m^F - l^F)\hat{c}_t^F,
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
&[(1-\mu)c^F(m^F-l^F)^{-1} - \mu w^F]\hat{\lambda}_t^F + (1-\mu)c^F(m^F-l^F)^{-1}\hat{c}_t^F \\
&= \mu w^F\hat{w}_t^F + (1-\mu)c^F m^F(m^F-l^F)^{-2}\hat{m}_t^F + \phi_l\mu\hat{l}_{t-1}^F \\
&\quad - \{\phi_l\mu + \beta\phi_l\mu[v^{\alpha/(1-\alpha)}z]^{\mu(1-\gamma)} + (1-\mu)c^F l^F(m^F-l^F)^{-2}\}\hat{l}_t^F \\
&\quad + \beta\phi_l\mu[v^{\alpha/(1-\alpha)}z]^{\mu(1-\gamma)}E_t\hat{l}_{t+1}^F,
\end{aligned} \tag{11}
$$

$$
\hat{\lambda}_t^F - \hat{v}_t^F = \hat{\xi}_t^F - \phi_k(i^F/k^F)\hat{i}_t^F + \phi_k(i^F/k^F)\hat{k}_t^F, \tag{12}
$$

$$
\begin{aligned}
&\{[\alpha/(1-\alpha)][1-\mu(1-\gamma)]+1\}[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{v}_t^F \\
&\quad + [1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{z}_t^F + [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}\hat{\xi}_t^F \\
&= \beta q^F E_t\hat{\lambda}_{t+1}^F + \beta q^F E_t\hat{q}_{t+1}^F + \beta[(1-\delta)/v]E_t\hat{\xi}_{t+1}^F \\
&\quad + \beta(1/v)\phi_k(i^F/k^F)^2 E_t i_{t+1}^F - \beta(1/v)\phi_k(i^F/k^F)^2 k_{t+1}^F,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&[\alpha/(1-\alpha)][1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{v}_t^F \\
&\quad + [1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{z}_t^F \\
&\quad + [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{\lambda}_t^F - [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{p}_t^F \\
&\quad - [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}(1/r)\hat{r}_t \\
&= \beta E_t\hat{\lambda}_{t+1}^H - \beta E_t\hat{p}_{t+1}^F - \phi_d[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}p^F\hat{d}_{t+1}^F,
\end{aligned} \tag{14}
$$

$$
\alpha\hat{k}_t^H + (1-\alpha)\hat{z}_t^H + (1-\alpha)\hat{l}_t^H = \hat{y}_t^A, \tag{15}
$$

$$
\hat{p}_t^A + \hat{y}_t^A = \hat{q}_t^H + \hat{k}_t^H, \tag{16}
$$

$$
\hat{p}_t^A + \hat{y}_t^A = \hat{w}_t^H + \hat{l}_t^H, \tag{17}
$$

$$
\alpha\hat{k}_t^F + (1-\alpha)\hat{z}_t^F + (1-\alpha)\hat{l}_t^F = \hat{y}_t^B, \tag{18}
$$

$$
\hat{p}_t^B + \hat{y}_t^B = \hat{p}_t^F + \hat{q}_t^F + \hat{k}_t^F, \tag{19}
$$

$$
\hat{p}_t^B + \hat{y}_t^B = \hat{p}_t^F + \hat{w}_t^F + \hat{l}_t^F, \tag{20}
$$

$$
\begin{aligned}
&[(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}\hat{a}_t^H \\
&\quad + [(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(b^H)^{(\theta-1)/\theta}\hat{b}_t^H \\
&= c^H\hat{c}_t^H + g^H\hat{g}_t^H + (i^H/v)\hat{i}_t^H - (i^H/v)\hat{v}_t^H,
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&[(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}+\omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta p^A\hat{p}_t^A \\
&= -(1-\omega)^{1/\theta}\omega^{1/\theta}(a^H)^{-1/\theta}(b^H)^{(\theta-1)/\theta}\hat{a}_t^H+(1-\omega)^{1/\theta}\omega^{1/\theta}(a^H)^{-1/\theta}(b^H)^{(\theta-1)/\theta}\hat{b}_t^H,
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
&[(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}+\omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta p^B\hat{p}_t^B \\
&=\omega^{1/\theta}(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}(b^H)^{-1/\theta}\hat{a}_t^H-\omega^{1/\theta}(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}(b^H)^{-1/\theta}\hat{b}_t^H,
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}\hat{a}_t^F \\
&\quad+[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}\hat{b}_t^F \\
&=c^F\hat{c}_t^F+g^F\hat{g}_t^F+(i^F/v)\hat{i}_t^F-(i^F/v)\hat{v}_t^F,
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^A/p^F)\hat{p}_t^A \\
&-[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^A/p^F)\hat{p}_t^F \\
&= -\omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{-1/\theta}(b^F)^{(\theta-1)/\theta}\hat{a}_t^F+\omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{-1/\theta}(b^F)^{(\theta-1)/\theta}\hat{b}_t^F,
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^B/p^F)\hat{p}_t^B \\
&-[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}+(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^B/p^F)\hat{p}_t^F \\
&=\omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{(\theta-1)/\theta}(b^F)^{-1/\theta}\hat{a}_t^F-\omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{(\theta-1)/\theta}(b^F)^{-1/\theta}\hat{b}_t^F,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
\hat{n}_t^H=&\,p^Aa^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{p}_t^A+p^Aa^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{a}_t^F \\
&-[\alpha/(1-\alpha)]p^Aa^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{v}_{t-1}^{HF}-p^Aa^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{z}_{t-1}^{HF} \\
&-p^Bb^H\hat{p}_t^B-p^Bb^H\hat{b}_t^H,
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
p^F\hat{n}_t^F=&\,p^Bb^H(v^{HF})^{\alpha/(1-\alpha)}z^{HF}\hat{p}_t^B+p^Bb^H(v^{HF})^{\alpha/(1-\alpha)}z^{HF}\hat{b}_t^H \\
&+[\alpha/(1-\alpha)]p^Bb^H(v^{HF})^{\alpha/(1-\alpha)}z^{HF}\hat{v}_{t-1}^{HF}+p^Bb^H(v^{HF})^{\alpha/(1-\alpha)}z^{HF}\hat{z}_{t-1}^{HF} \\
&-p^Aa^F\hat{p}_t^A-p^Aa^F\hat{a}_t^F,
\end{aligned} \tag{32}
$$

$$
T\hat{O}T_t=\hat{p}_t^B-\hat{p}_t^A, \tag{34}
$$

$$
\begin{aligned}
y^A\hat{y}_t^A=&\,a^H\hat{a}_t^H+a^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{a}_t^F \\
&-[\alpha/(1-\alpha)]a^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{v}_{t-1}^{HF}-a^F/[(v^{HF})^{\alpha/(1-\alpha)}z^{HF}]\hat{z}_{t-1}^{HF},
\end{aligned} \tag{35}
$$

$$
(v^{HF})^{\alpha/(1-\alpha)}z^{HF}\hat{d}_t^H+\hat{d}_t^F=0, \tag{41}
$$

$$
\begin{bmatrix}\hat{m}_t^H\\ \hat{m}_t^F\end{bmatrix}=\begin{bmatrix}\rho_m^{HH} & \rho_m^{HF}\\ \rho_m^{FH} & \rho_m^{FF}\end{bmatrix}\begin{bmatrix}\hat{m}_{t-1}^H\\ \hat{m}_{t-1}^F\end{bmatrix}+\begin{bmatrix}\varepsilon_{mt}^H\\ \varepsilon_{mt}^F\end{bmatrix}, \tag{42-43}
$$

$$
\begin{bmatrix}\hat{z}_t^H\\ \hat{z}_t^F\end{bmatrix}=\begin{bmatrix}\rho_z^{HH} & \rho_z^{HF}\\ \rho_z^{FH} & \rho_z^{FF}\end{bmatrix}\begin{bmatrix}\hat{z}_{t-1}^H\\ \hat{z}_{t-1}^F\end{bmatrix}+\begin{bmatrix}\kappa_z^H\\ \kappa_z^F\end{bmatrix}\hat{z}_{t-1}^{HF}+\begin{bmatrix}\varepsilon_{zt}^H\\ \varepsilon_{zt}^F\end{bmatrix}, \tag{44-45}
$$

$$
\begin{bmatrix}\hat{v}_t^H\\ \hat{v}_t^F\end{bmatrix}=\begin{bmatrix}\rho_v^{HH} & \rho_v^{HF}\\ \rho_v^{FH} & \rho_v^{FF}\end{bmatrix}\begin{bmatrix}\hat{v}_{t-1}^H\\ \hat{v}_{t-1}^F\end{bmatrix}+\begin{bmatrix}\kappa_v^H\\ \kappa_v^F\end{bmatrix}\hat{v}_{t-1}^{HF}+\begin{bmatrix}\varepsilon_{vt}^H\\ \varepsilon_{vt}^F\end{bmatrix}, \tag{46-47}
$$

$$
\begin{bmatrix}\hat{g}_t^H\\ \hat{g}_t^F\end{bmatrix}=\begin{bmatrix}\rho_g^{HH} & \rho_g^{HF}\\ \rho_g^{FH} & \rho_g^{FF}\end{bmatrix}\begin{bmatrix}\hat{g}_{t-1}^H\\ \hat{g}_{t-1}^F\end{bmatrix}+\begin{bmatrix}\varepsilon_{gt}^H\\ \varepsilon_{gt}^F\end{bmatrix}, \tag{48-49}
$$

$$
\hat{z}_t^{HF}=\hat{z}_t^H-\hat{z}_t^F+\hat{z}_{t-1}^{HF}, \tag{50}
$$

$$
\hat{v}_t^{HF}=\hat{v}_t^H-\hat{v}_t^F+\hat{v}_{t-1}^{HF}, \tag{51}
$$

$$\hat{g}_t^{CH} = \hat{c}_t^H - \hat{c}_{t-1}^H + [\alpha/(1-\alpha)]\hat{v}_{t-1}^H + \hat{z}_{t-1}^H, \tag{52}$$

$$\hat{g}_t^{IH} = \hat{\imath}_t^H - \hat{\imath}_{t-1}^H + [1/(1-\alpha)]\hat{v}_{t-1}^H + \hat{z}_{t-1}^H, \tag{53}$$

$$\hat{r}_t^{GCH} = \hat{g}_t^H - \hat{c}_t^H, \tag{54}$$

$$\hat{r}_t^{CFH} = \hat{c}_t^F - \hat{c}_t^H - [\alpha/(1-\alpha)]\hat{v}_{t-1}^{HF} - \hat{z}_{t-1}^{HF}, \tag{55}$$

$$\hat{r}_t^{IFH} = \hat{\imath}_t^F - \hat{\imath}_t^H - [1/(1-\alpha)]\hat{v}_{t-1}^{HF} - \hat{z}_{t-1}^{HF}, \tag{56}$$

and

$$\hat{r}_t^{GCF} = \hat{g}_t^F - \hat{c}_t^F. \tag{57}$$

5 The System in Matrix Form

Let

$$f_t^0 = \begin{bmatrix} \hat{d}_t^F \\ \hat{\lambda}_t^H \\ \hat{\lambda}_t^F \\ \hat{\xi}_t^H \\ \hat{\xi}_t^F \\ \hat{\imath}_t^H \\ \hat{\imath}_t^F \\ \hat{y}_t^A \\ \hat{y}_t^B \\ \hat{a}_t^H \\ \hat{a}_t^F \\ \hat{w}_t^H \\ \hat{w}_t^F \\ \hat{q}_t^H \\ \hat{q}_t^F \\ \hat{n}_t^H \\ \hat{n}_t^F \\ T\hat{O}T_t \\ \hat{p}_t^A \\ \hat{p}_t^B \\ \hat{p}_t^F \\ \hat{g}_t^{CH} \\ \hat{g}_t^{IH} \\ \hat{r}_t^{GCH} \\ \hat{r}_t^{CFH} \\ \hat{r}_t^{IFH} \\ \hat{r}_t^{GCF} \end{bmatrix},$$

$$s_t^0 = \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \\ \hat{c}_t^H \\ \hat{c}_t^F \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{b}_t^H \\ \hat{b}_t^F \\ \hat{r}_t \end{bmatrix},$$

and

$$v_t = \begin{bmatrix} \hat{m}_t^H \\ \hat{m}_t^F \\ \hat{z}_t^H \\ \hat{z}_t^F \\ \hat{z}_t^{HF} \\ \hat{v}_t^H \\ \hat{v}_t^F \\ \hat{v}_t^{HF} \\ \hat{g}_t^H \\ \hat{g}_t^F \end{bmatrix}.$$

Then (3), (5), (10), (12), (15)-(21), (23)-(25), (27), (28), (31), (32), (34), (35), (41), and (52)-(57) can be written as

$$Af_t^0 = Bs_t^0 + Cv_t, \tag{58}$$

where A is 27×27, B is 27×18, and C is 27×10.

Equation (3) implies

$$a_{1,2} = m^H - l^H$$

$$b_{1,12} = -[1 - \mu(1 - \gamma)](m^H - l^H)$$

$$b_{1,14} = -(1 - \mu)(1 - \gamma)l^H$$

$$c_{1,1} = (1 - \mu)(1 - \gamma)l^H$$

Equation (5) implies

$a_{2,2} = 1$

$a_{2,4} = -1$

$a_{2,6} = \phi_k(i^H/k^H)$

$b_{2,1} = \phi_k(i^H/k^H)$

$c_{2,6} = 1$

Equation (10) implies

$a_{3,3} = m^F - l^F$

$b_{3,13} = -[1-\mu(1-\gamma)](m^F - l^F)$

$b_{3,15} = -(1-\mu)(1-\gamma)l^F$

$c_{3,2} = (1-\mu)(1-\gamma)l^F$

Equation (12) implies

$a_{4,3} = 1$

$a_{4,5} = -1$

$a_{4,7} = \phi_k(i^F/k^F)$

$b_{4,2} = \phi_k(i^F/k^F)$

$c_{4,7} = 1$

Equation (15) implies

$a_{5,8} = 1$

$b_{5,1} = \alpha$

$b_{5,14} = 1-\alpha$

$c_{5,3} = 1-\alpha$

Equation (16) implies

$a_{6,8} = 1$

$a_{6,14} = -1$

$a_{6,19} = 1$

$b_{6,1} = 1$

Equation (17) implies

$a_{7,8} = 1$

$a_{7,12} = -1$

$a_{7,19} = 1$

$b_{7,14} = 1$

Equation (18) implies

$a_{8,9} = 1$

$b_{8,2} = \alpha$

$b_{8,15} = 1 - \alpha$

$c_{8,4} = 1 - \alpha$

Equation (19) implies

$a_{9,9} = 1$

$a_{9,15} = -1$

$a_{9,20} = 1$

$a_{9,21} = -1$

$b_{9,2} = 1$

Equation (20) implies

$a_{10,9} = 1$

$a_{10,13} = -1$

$a_{10,20} = 1$

$a_{10,21} = -1$

$b_{10,15} = 1$

Equation (21) implies

$a_{11,6} = i^H/v$

$a_{11,10} = -[(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}$

$b_{11,12} = -c^H$

$b_{11,16} = [(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(b^H)^{(\theta-1)/\theta}$

$c_{11,6} = i^H/v$

$c_{11,9} - -g^H$

Equation (23) implies

$a_{12,10} = (1-\omega)^{1/\theta}\omega^{1/\theta}(a^H)^{-1/\theta}(b^H)^{(\theta-1)/\theta}$

$a_{12,19} = [(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta p^A$

$b_{12,16} = (1-\omega)^{1/\theta}\omega^{1/\theta}(a^H)^{-1/\theta}(b^H)^{(\theta-1)/\theta}$

Equation (24) implies

$a_{13,10} = \omega^{1/\theta}(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}(b^H)^{-1/\theta}$

$a_{13,20} = -[(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta} + \omega^{1/\theta}(b^H)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta p^B$

$b_{13,16} = \omega^{1/\theta}(1-\omega)^{1/\theta}(a^H)^{(\theta-1)/\theta}(b^H)^{-1/\theta}$

Equation (25) implies

$a_{14,7} = i^F/v$

$a_{14,11} = -[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}\omega^{1/\theta}(a^F)^{(\theta-1)/\theta}$

$b_{14,13} = -c^F$

$b_{14,17} = [\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{1/(\theta-1)}(1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}$

$c_{14,7} = i^F/v$

$c_{14,10} = -g^F$

Equation (27) implies

$a_{15,11} = \omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{-1/\theta}(b^F)^{(\theta-1)/\theta}$

$a_{15,19} = [\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^A/p^F)$

$a_{15,21} = -[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^A/p^F)$

$b_{15,17} = \omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{-1/\theta}(b^F)^{(\theta-1)/\theta}$

Equation (28) implies

$a_{16,11} = \omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{(\theta-1)/\theta}(b^F)^{-1/\theta}$

$a_{16,20} = -[\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^B/p^F)$

$a_{16,21} = [\omega^{1/\theta}(a^F)^{(\theta-1)/\theta} + (1-\omega)^{1/\theta}(b^F)^{(\theta-1)/\theta}]^{(\theta-2)/(\theta-1)}\theta(p^B/p^F)$

$b_{16,17} = \omega^{1/\theta}(1-\omega)^{1/\theta}(a^F)^{(\theta-1)/\theta}(b^F)^{-1/\theta}$

Equation (31) implies

$a_{17,11} = p^A a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

$a_{17,16} = -1$

$a_{17,19} = p^A a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

$a_{17,20} = -p^B b^H$

$b_{17,9} = p^A a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

$b_{17,11} = [\alpha/(1-\alpha)]p^A a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

$b_{17,16} = p^B b^H$

Equation (32) implies

$a_{18,11} = p^A a^F$

$a_{18,17} = p^F$

$a_{18,19} = p^A a^F$

$a_{18,20} = -p^B b^H (v^{HF})^{\alpha/(1-\alpha)} z^{HF}$

$b_{18,9} = p^B b^H (v^{HF})^{\alpha/(1-\alpha)} z^{HF}$

$b_{18,11} = [\alpha/(1-\alpha)]p^B b^H (v^{HF})^{\alpha/(1-\alpha)} z^{HF}$

$b_{18,16} = p^B b^H (v^{HF})^{\alpha/(1-\alpha)} z^{HF}$

Equation (34) implies

$a_{19,18} = 1$

$a_{19,19} = 1$

$a_{19,20} = -1$

Equation (35) implies

$a_{20,8} = y^A$

$a_{20,10} = -a^H$

$a_{20,11} = -a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

$b_{20,9} = -a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

$b_{20,11} = -[\alpha/(1-\alpha)]a^F/[(v^{HF})^{\alpha/(1-\alpha)} z^{HF}]$

Equation (41) implies

$a_{21,1} = 1$

$b_{21,3} = -(v^{HF})^{\alpha/(1-\alpha)} z^{HF}$

Equation (52) implies

$a_{22,22} = 1$

$b_{22,4} = -1$

$b_{22,8} = 1$

$b_{22,10} = \alpha/(1-\alpha)$

$b_{22,12} = 1$

Equation (53) implies

$a_{23,6} = -1$

$a_{23,23} = 1$

$b_{23,5} = -1$

$b_{23,8} = 1$

$b_{23,10} = 1/(1-\alpha)$

Equation (54) implies

$a_{24,24} = 1$

$b_{24,12} = -1$

$c_{24,9} = 1$

Equation (55) implies

$a_{25,25} = 1$

$b_{25,9} = -1$

$b_{25,11} = -\alpha/(1-\alpha)$

$b_{25,12} = -1$

$b_{25,13} = 1$

Equation (56) implies

$a_{26,6} = 1$

$a_{26,7} = -1$

$a_{26,26} = 1$

$b_{26,9} = -1$

$b_{26,11} = -1/(1-\alpha)$

Equation (57) implies

$a_{27,27} = 1$

$b_{27,13} = -1$

$c_{27,10} = 1$

Equations (1), (2), (4), (6)-(9), (11), (13), and (14) can be written as

$$DE_t s_{t+1}^0 + FE_t f_{t+1}^0 = Gs_t^0 + Hf_t^0 + Jv_t, \tag{59}$$

where D and G are 18×18, F and H are 18×27, and J is 18×10.

Equation (1) implies

$d_{1,3} = v^{\alpha/(1-\alpha)} z/r$

$g_{1,1} = q^H k^H$

$g_{1,3} = 1$

$g_{1,12} = -c^H$

$g_{1,14} = w^H l^H$

$h_{1,6} = -i^H/v$

$h_{1,12} = w^H l^H$

$h_{1,14} = q^H k^H$

$j_{1,6} = i^H/v$

$j_{1,9} = -g^H$

Equation (2) implies

$d_{2,1} = v^{1/(1-\alpha)} z k^H$

$g_{2,1} = (1-\delta) k^H$

$h_{2,6} = i^H$

$j_{2,3} = -v^{1/(1-\alpha)} z k^H$

$j_{2,6} = -[1/(1-\alpha)] v^{1/(1-\alpha)} z k^H$

Equation (4) implies

$d_{3,14} = \beta\phi_l\mu[v^{\alpha/(1-\alpha)}z]^{\mu(1-\gamma)}$

$g_{3,6} = -\phi_l\mu$

$g_{3,12} = (1-\mu)c^H(m^H - l^H)^{-1}$

$g_{3,14} = \phi_l\mu + \beta\phi_l\mu[v^{\alpha/(1-\alpha)}z]^{\mu(1-\gamma)} + (1-\mu)c^H l^H(m^H - l^H)^{-2}$

$h_{3,2} = (1-\mu)c^H(m^H - l^H)^{-1} - \mu w^H$

$h_{3,12} = -\mu w^H$

$j_{3,1} = -(1-\mu)c^H m^H(m^H - l^H)^{-2}$

Equation (6) implies

$d_{4,1} = -\beta(1/v)\phi_k(i^H/k^H)^2$

$f_{4,2} = \beta q^H$

$f_{4,4} = \beta[(1-\delta)/v]$

$f_{4,6} = \beta(1/v)\phi_k(i^H/k^H)^2$

$f_{4,14} = \beta q^H$

$h_{4,4} = [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{4,3} = [1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{4,6} = \{[\alpha/(1-\alpha)][1-\mu(1-\gamma)] + 1\}[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

Equation (7) implies

$d_{5,3} = -\phi_d[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$f_{5,2} = \beta$

$g_{5,18} = -(1/r)[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$h_{5,2} = (1/r)[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{5,3} = (1/r)[1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{5,6} = (1/r)[\alpha/(1-\alpha)][1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

Equation (8) implies

$f_{6,1} = v^{\alpha/(1-\alpha)}z/(p^F r)$

$g_{6,2} = q^F k^F$

$g_{6,13} = -c^F$

$g_{6,15} = w^F l^F$

$h_{6,1} = 1/p^F$

$h_{6,7} = -i^F/v$

$h_{6,13} = w^F l^F$

$h_{6,15} = q^F k^F$

$j_{6,7} = i^F/v$

$j_{6,10} = -g^F$

Equation (9) implies

$d_{7,2} = v^{1/(1-\alpha)} z k^F$

$g_{7,2} = (1-\delta) k^F$

$h_{7,7} = i^F$

$j_{7,4} = -v^{1/(1-\alpha)} z k^F$

$j_{7,7} = -[1/(1-\alpha)] v^{1/(1-\alpha)} z k^F$

Equation (11) implies

$d_{8,15} = \beta \phi_l \mu [v^{\alpha/(1-\alpha)} z]^{\mu(1-\gamma)}$

$g_{8,7} = -\phi_l \mu$

$g_{8,13} = (1-\mu) c^F (m^F - l^F)^{-1}$

$g_{8,15} = \phi_l \mu + \beta \phi_l \mu [v^{\alpha/(1-\alpha)} z]^{\mu(1-\gamma)} + (1-\mu) c^F l^F (m^F - l^F)^{-2}$

$h_{8,3} = (1-\mu) c^F (m^F - l^F)^{-1} - \mu w^F$

$h_{8,13} = -\mu w^F$

$j_{8,2} = -(1-\mu) c^F m^F (m^F - l^F)^{-2}$

Equation (13) implies

$d_{9,2} = -\beta (1/v) \phi_k (i^F/k^F)^2$

$f_{9,3} = \beta q^F$

$f_{9.5} = \beta[(1-\delta)/v]$

$f_{9,7} = \beta (1/v) \phi_k (i^F/k^F)^2$

$f_{9,15} = \beta q^F$

$h_{9,5} = [v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{9,4} = [1 - \mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{9,7} = \{[\alpha/(1-\alpha)][1-\mu(1-\gamma)] + 1\}[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

Equation (14) impiles

$f_{10,1} = -\phi_d[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}p^F$

$f_{10,3} = \beta$

$f_{10,21} = -\beta$

$g_{10,18} = -(1/r)[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$h_{10,3} = (1/r)[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$h_{10,21} = -(1/r)[v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{10,4} = (1/r)[1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

$j_{10,7} = (1/r)[\alpha/(1-\alpha)][1-\mu(1-\gamma)][v^{\alpha/(1-\alpha)}z]^{1-\mu(1-\gamma)}$

The presence of lagged values in s_t^0 implies

$d_{11,4} = 1$

$g_{11,12} = 1$

$d_{12,5} = 1$

$h_{12,6} = 1$

$d_{13,6} = 1$

$g_{13,14} = 1$

$d_{14,7} = 1$

$g_{14,15} = 1$

$d_{15,8} = 1$

$j_{15,3} = 1$

$d_{16,9} = 1$

$j_{16,5} = 1$

$d_{17,10} = 1$

$j_{17,6} = 1$

$d_{18,11} = 1$

$j_{18,8} = 1$

Finally, (42)-(51) can be written as

$$P_1 v_t = P_2 v_{t-1} + P_3 \varepsilon_t, \tag{60}$$

where

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} \rho_m^{HH} & \rho_m^{HF} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \rho_m^{FH} & \rho_m^{FF} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_z^{HH} & \rho_z^{HF} & \kappa_z^H & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_z^{FH} & \rho_z^{FF} & \kappa_z^F & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_v^{HH} & \rho_v^{HF} & \kappa_v^H & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_v^{FH} & \rho_v^{FF} & \kappa_v^F & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_g^{HH} & \rho_g^{HF} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_g^{FH} & \rho_g^{FF} \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{mt}^H \\ \varepsilon_{mt}^F \\ \varepsilon_{zt}^H \\ \varepsilon_{zt}^F \\ \varepsilon_{vt}^H \\ \varepsilon_{vt}^F \\ \varepsilon_{gt}^H \\ \varepsilon_{gt}^F \end{bmatrix}.$$

Note that (60) can be rewritten as

$$v_t = P v_{t-1} + P_4 \varepsilon_t, \tag{60}$$

where

$$P = P_1^{-1} P_2$$

and

$$P_4 = P_1^{-1} P_3$$

implying that

$$E_t v_{t+j} = P^j v_t$$

for all $j = 0, 1, 2, ...$

6 Solving the Linearized Model

Start by using (58) to solve out for f_t^0:

$$f_t^0 = A^{-1} B s_t^0 + A^{-1} C v_t. \tag{61}$$

Then use this equation together with (60) to rewrite (59) as

$$K E_t s_{t+1}^0 = L s_t^0 + M v_t, \tag{62}$$

where

$$K = D + F A^{-1} B,$$
$$L = G + H A^{-1} B,$$
$$M = J + H A^{-1} C - F A^{-1} C P.$$

Equation (62) takes the form of a system of linear expectational difference equations, driven by the exogenous shocks in (60). This system can be solved by uncoupling the unstable and stable components and then solving the unstable component forward. There are a number of algorithms for working through this process; the approach taken here uses methods outlined by Klein (2000).

Klein's method relies on the complex generalized Schur decomposition, which identifies unitary matrices Q and Z such that

$$QKZ = S$$

and

$$QLZ = T$$

are both upper triangular, where the generalized eigenvalues of L and K can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(L, K) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 18\}.$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues appear in ascending order in absolute value. Note that there are 11 predetermined variables in the vector s_t^0. Thus, if 11 of the generalized eigenvalues in $\lambda(L, K)$ lie inside the unit circle and seven of the generalized eigenvalues lie outside the unit circle, then the system has a unique solution. If more than seven of the eigenvalues in $\lambda(L, K)$ lie outside the unit circle, then the system has no solution. If fewer than seven of the generalized eigenvalues lie outside the unit circle, then the system has multiple solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assume from now on that there are exactly seven generalized eigenvalues that lie outside the unit circle, and partition the matrices Q, Z, S, and T conformably, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 is 11×18 and Q_2 is 7×18, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{(7\times 11)} & S_{22} \end{bmatrix},$$

and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{(7\times 11)} & T_{22} \end{bmatrix},$$

where Z_{11}, S_{11}, and T_{11} are 11×11, Z_{12}, S_{12}, and T_{12} are 11×7, Z_{21} is 7×11, and Z_{22}, S_{22}, and T_{22} are 7×7.

Next, define the vector s_t^1 of auxiliary variables as

$$s_t^1 = Z' s_t^0,$$

so that, in particular,

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{1t}^2 \end{bmatrix},$$

where

$$s_{1t}^1 = Z_{11}' \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + Z_{21}' \begin{bmatrix} \hat{c}_t^H \\ \hat{c}_t^F \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{b}_t^H \\ \hat{b}_t^F \\ \hat{r}_t \end{bmatrix} \tag{63}$$

is 11×1 and

$$s_{2t}^1 = Z_{12}' \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + Z_{22}' \begin{bmatrix} \hat{c}_t^H \\ \hat{c}_t^F \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{b}_t^H \\ \hat{b}_t^F \\ \hat{r}_t \end{bmatrix} \tag{64}$$

is 7×1.

Since Z is unitary, $Z'Z = I$ or $Z' = Z^{-1}$ and hence $s_t^0 = Zs_t^1$. Use this fact to rewrite (62) as

$$KZE_t s_{t+1}^1 = LZs_t^1 + Mv_t.$$

Premultiply this version of (62) by Q to obtain

$$SE_t s_{t+1}^1 = Ts_t^1 + QMv_t$$

or, in terms of the matrix partitions,

$$S_{11}E_t s_{1t+1}^1 + S_{12}E_t s_{2t+1}^1 = T_{11}s_{1t}^1 + T_{12}s_{2t}^1 + Q_1 Mv_t \tag{65}$$

and

$$S_{22}E_t s_{2t+1}^1 = T_{22}s_{2t}^1 + Q_2 Mv_t. \tag{66}$$

Since the generalized eigenvalues corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (66) can be solved forward to obtain

$$s_{2t}^1 = -T_{22}^{-1} Rv_t,$$

where the 7×10 matrix R is given by

$$\begin{aligned}
\mathrm{vec}(R) &= \mathrm{vec}\sum_{j=0}^{\infty}(S_{22}T_{22}^{-1})^j Q_2 M P^j = \sum_{j=0}^{\infty}\mathrm{vec}(S_{22}T_{22}^{-1})^j Q_2 M P^j \\
&= \sum_{j=0}^{\infty}[(P')^j \otimes (S_{22}T_{22}^{-1})^j]\mathrm{vec}(Q_2 M) = \sum_{j=0}^{\infty}[P' \otimes (S_{22}T_{22}^{-1})]^j \mathrm{vec}(Q_2 M) \\
&= [I_{(70\times 70)} - P' \otimes (S_{22}T_{22}^{-1})]^{-1}\mathrm{vec}(Q_2 M).
\end{aligned}$$

Use this result, along with (64), to solve for

$$\begin{bmatrix} \hat{c}_t^H \\ \hat{c}_t^F \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{b}_t^H \\ \hat{b}_t^F \\ \hat{r}_t \end{bmatrix} = -(Z_{22}')^{-1}Z_{12}' \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} - (Z_{22}')^{-1}T_{22}^{-1}Rv_t.$$

Since Z is unitary, $Z'Z = I$ or

$$\begin{bmatrix} Z_{11}' & Z_{21}' \\ Z_{12}' & Z_{22}' \end{bmatrix}\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} I_{(11\times 11)} & 0_{(11\times 7)} \\ 0_{(7\times 11)} & I_{(7\times 7)} \end{bmatrix}.$$

Hence, in particular,

$$Z_{12}'Z_{11} + Z_{22}'Z_{21} = 0_{(7\times 11)}$$

or

$$-(Z_{22}')^{-1}Z_{12}' = Z_{21}Z_{11}^{-1}$$

and

$$Z_{12}'Z_{12} + Z_{22}'Z_{22} = I_{(7\times 7)}$$

or

$$(Z_{22}')^{-1} = Z_{22} + (Z_{22}')^{-1}Z_{12}'Z_{12} = Z_{22} - Z_{21}Z_{11}^{-1}Z_{12},$$

allowing this solution to be written more conveniently as

$$\begin{bmatrix} \hat{c}_t^H \\ \hat{c}_t^F \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{b}_t^H \\ \hat{b}_t^F \\ \hat{r}_t \end{bmatrix} = N_1 \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + N_2 v_t \tag{67}$$

where

$$N_1 = Z_{21} Z_{11}^{-1}$$

and

$$N_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Equation (63) now provides the solution for s_{1t}^1:

$$s_{1t}^1 = (Z_{11}' + Z_{21}' Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} - Z_{21}' [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R v_t.$$

Using

$$Z_{11}' Z_{11} + Z_{21}' Z_{21} = I_{(11 \times 11)}$$

or

$$Z_{11}' + Z_{21}' Z_{21} Z_{11}^{-1} = Z_{11}^{-1}$$

and

$$Z_{21}' [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] = Z_{21}' Z_{22} - Z_{21}' Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12},$$

this last result can be written more conveniently as

$$s_{1t}^1 = Z_{11}^{-1} \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + Z_{11}^{-1} Z_{12} T_{22}^{-1} R v_t.$$

Substitute these results into (65) to obtain the solution

$$\begin{bmatrix} \hat{k}_{t+1}^H \\ \hat{k}_{t+1}^F \\ \hat{d}_{t+1}^H \\ \hat{c}_t^H \\ \hat{\imath}_t^H \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{z}_t^H \\ \hat{z}_t^{HF} \\ \hat{v}_t^H \\ \hat{v}_t^{HF} \end{bmatrix} = N_3 \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + N_4 v_t, \tag{68}$$

where

$$N_3 = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$$

and

$$N_4 = Z_{11} S_{11}^{-1} (T_{11} Z_{11}^{-1} Z_{12} T_{22}^{-1} R + Q_1 M + S_{12} T_{22}^{-1} R P - T_{12} T_{22}^{-1} R) - Z_{12} T_{22}^{-1} R P.$$

Finally, return to (61) to solve for f_t^0:

$$f_t^0 = A^{-1} B \begin{bmatrix} I_{(11\times 11)} \\ N_1 \end{bmatrix} \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + A^{-1} B \begin{bmatrix} 0_{(11\times 10)} \\ N_2 \end{bmatrix} v_t + A^{-1} C v_t,$$

which can be written more simply as

$$f_t^0 = N_5 \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \end{bmatrix} + N_6 v_t, \tag{69}$$

where

$$N_5 = A^{-1}B \begin{bmatrix} I_{(11\times 11)} \\ N_1 \end{bmatrix}$$

and

$$N_6 = A^{-1}C + A^{-1}B \begin{bmatrix} 0_{(11\times 10)} \\ N_2 \end{bmatrix}.$$

Equations (60) and (67)-(69) provide the model's solution:

$$s_{t+1} = \Pi s_t + W\varepsilon_{t+1} \tag{70}$$

and

$$f_t = Us_t, \tag{71}$$

where

$$s_t = \begin{bmatrix} \hat{k}_t^H \\ \hat{k}_t^F \\ \hat{d}_t^H \\ \hat{c}_{t-1}^H \\ \hat{\imath}_{t-1}^H \\ \hat{l}_{t-1}^H \\ \hat{l}_{t-1}^F \\ \hat{z}_{t-1}^H \\ \hat{z}_{t-1}^{HF} \\ \hat{v}_{t-1}^H \\ \hat{v}_{t-1}^{HF} \\ \hat{m}_t^H \\ \hat{m}_t^F \\ \hat{z}_t^H \\ \hat{z}_t^F \\ \hat{z}_t^{HF} \\ \hat{v}_t^H \\ \hat{v}_t^F \\ \hat{v}_t^{HF} \\ \hat{g}_t^H \\ \hat{g}_t^F \end{bmatrix},$$

$$f_t = \begin{bmatrix} \hat{d}_t^F \\ \hat{\lambda}_t^H \\ \hat{\lambda}_t^F \\ \hat{\xi}_t^H \\ \hat{\xi}_t^F \\ \hat{\imath}_t^H \\ \hat{\imath}_t^F \\ \hat{y}_t^A \\ \hat{y}_t^B \\ \hat{a}_t^H \\ \hat{a}_t^F \\ \hat{w}_t^H \\ \hat{w}_t^F \\ \hat{q}_t^H \\ \hat{q}_t^F \\ \hat{n}_t^H \\ \hat{n}_t^F \\ T\hat{O}T_t \\ \hat{p}_t^A \\ \hat{p}_t^B \\ \hat{p}_t^F \\ \hat{g}_t^{CH} \\ \hat{g}_t^{IH} \\ \hat{r}_t^{GCH} \\ \hat{r}_t^{CFH} \\ \hat{r}_t^{IFH} \\ \hat{r}_t^{GCF} \\ \hat{c}_t^H \\ \hat{c}_t^F \\ \hat{l}_t^H \\ \hat{l}_t^F \\ \hat{b}_t^H \\ \hat{b}_t^F \\ \hat{r}_t \end{bmatrix},$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{mt}^H \\ \varepsilon_{mt}^F \\ \varepsilon_{zt}^H \\ \varepsilon_{zt}^F \\ \varepsilon_{vt}^H \\ \varepsilon_{vt}^F \\ \varepsilon_{gt}^H \\ \varepsilon_{gt}^F \end{bmatrix},$$

$$\Pi = \begin{bmatrix} N_3 & N_4 \\ 0_{(10\times 11)} & P \end{bmatrix},$$

$$W = \begin{bmatrix} 0_{(11\times 8)} \\ P_4 \end{bmatrix},$$

and

$$U = \begin{bmatrix} N_5 & N_6 \\ N_1 & N_2 \end{bmatrix}.$$

7 Estimating the Model

The model has implications for the behavior of eight observable variables: the growth rate of home consumption, the growth rate of home investment, the ratio of home government spending to home consumption, home hours worked, the ratio of foreign consumption to home consumption, the ratio of foreign investment to home investment, the ratio of foreign government spending to foreign consumption, and foreign hours worked. Excluding η_H and η_F, which are assigned values to make capital adjustment costs zero in the steady state, the empirical model has 46 parameters: β, μ, γ, ϕ_d, ϕ_l, ϕ_k, δ, α, θ, ω, m^H, m^F, z, z^{HF}, v, v^{HF}, g^H, g^F, ρ_m^{HH}, ρ_m^{HF}, ρ_m^{FH}, ρ_m^{FF}, ρ_z^{HH}, ρ_z^{HF}, ρ_z^{FH}, ρ_z^{FF}, κ_z^H, κ_z^F, ρ_v^{HH}, ρ_v^{HF}, ρ_v^{FH}, ρ_v^{FF}, κ_v^H, κ_v^F, ρ_g^{HH}, ρ_g^{HF}, ρ_g^{FH}, ρ_g^{FF}, σ_m^H, σ_m^F, σ_z^H, σ_z^F, σ_v^H, σ_v^F, σ_g^H, and σ_g^F. To estimate these parameters via maximum likelihood, let $\{d_t\}_{t=1}^T$ denote the series for the observable variables, each expressed as the logarithmic deviation from its steady-state value:

$$d_t = \begin{bmatrix} \hat{g}_t^{CH} \\ \hat{g}_t^{IH} \\ \hat{r}_t^{GCH} \\ \hat{l}_t^H \\ \hat{r}_t^{CFH} \\ \hat{r}_t^{IGH} \\ \hat{r}_t^{GCF} \\ \hat{l}_t^F \end{bmatrix} = \begin{bmatrix} \ln(C_t^H) - \ln(C_{t-1}^H) - [\alpha/(1-\alpha)]\ln(v) - \ln(z) \\ \ln(I_t^H) - \ln(I_{t-1}^H) - [1/(1-\alpha)]\ln(v) - \ln(z) \\ \ln(G_t^H) - \ln(C_t^H) - \ln(g^H) + \ln(c^H) \\ \ln(l_t^H) - \ln(l^H) \\ \ln(C_t^F) - \ln(C_t^H) - \ln(c^F) + \ln(c^H) + [\alpha/(1-\alpha)]\ln(v^{HF}) + \ln(z^{HF}) \\ \ln(I_t^F) - \ln(I_t^H) - \ln(i^F) + \ln(i^H) + [1/(1-\alpha)]\ln(v^{HF}) + \ln(z^{HF}) \\ \ln(G_t^F) - \ln(C_t^F) - \ln(g^F) + \ln(c^F) \\ \ln(l_t^F) - \ln(l^F) \end{bmatrix},$$

where C_t^H, I_t^H, G_t^H, l_t^H, C_t^F, I_t^F, G_t^F, and l_t^F are the original series for real consumption, investment, government spending, and hours worked in the two countries.

The empirical model then takes the form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{72}$$

$$d_t = Cs_t, \tag{73}$$

where $A = \Pi$, $B = W$, C is formed from rows 22 through 27, 30, and 31 of U, and the vector of zero-mean, serially uncorrelated innovations ε_{t+1} is normally distributed with covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon_{t+1}' = \begin{bmatrix} (\sigma_m^H)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\sigma_m^F)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\sigma_z^H)^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\sigma_z^F)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\sigma_v^H)^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\sigma_v^F)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\sigma_G^H)^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & (\sigma_G^F)^2 \end{bmatrix}.$$

The model defined by (72) and (73) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^T$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (72) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(21\times1)} \tag{74}$$

and

$$\text{vec}(\Sigma_{1|0}) = \text{vec}(Es_1s_1') = [I_{(441\times441)} - A \otimes A]^{-1}\text{vec}(BVB'). \tag{75}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (73) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence,

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_tu_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.213) formula for updating a linear projection,

$$\begin{aligned} \hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t. \end{aligned}$$

Hence, from (72),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (72) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1}C\Sigma_t A',$$

and initial conditions for $\hat{s}_1$ and Σ_1 are provided by (74) and (75).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln(L) = -\left(\frac{8T}{2}\right)\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T}u_t'\Omega_t^{-1}u_t.$$

8 Evaluating the Model

8.1 Variance Decompositions

Begin by considering (72), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B\varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B\varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B\varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}
\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\
&= BVB' + ABVBA' + A^2BVB'A^{2\prime} + ... + A^{k-1}BVB'A^{k-1\prime}.
\end{aligned}$$

In addition, (72) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$\text{vec}(\Sigma^s) = [I_{441\times 441} - A \otimes A]^{-1}\text{vec}(BVB').$$

Next, consider (71), which implies that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U'$$

and

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U'.$$

Finally, let

$$D_t = \begin{bmatrix} \ln(C_t^H) \\ \ln(I_t^H) \\ \ln(G_t^H) \\ \ln(l_t^H) \\ \ln(C_t^F) \\ \ln(I_t^F) \\ \ln(G_t^F) \\ \ln(l_t^F) \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$d = \begin{bmatrix} \ln(g^{CH}) \\ \ln(g^{IH}) \\ \ln(r^{GCH}) \\ \ln(l^{H}) \\ \ln(r^{CFH}) \\ \ln(r^{IFH}) \\ \ln(r^{GCF}) \\ \ln(l^{F}) \end{bmatrix}$$

so that (73) can be rewritten as

$$FD_t = GD_{t-1} + d + Cs_t$$

or

$$D_t = HD_{t-1} + F^{-1}d + Js_t$$

where

$$H = F^{-1}G$$

and

$$J = F^{-1}C.$$

Consequently,

$$D_{t+k} = H^k D_t + \left(\sum_{j=1}^{k} H^{k-j}\right) F^{-1}d + \sum_{j=1}^{k} H^{k-j} J s_{t+j},$$

$$E_t D_{t+k} = H^k D_t + \left(\sum_{j=1}^{k} H^{k-j}\right) F^{-1}d + \sum_{j=1}^{k} H^{k-j} J E_t s_{t+j},$$

and

$$D_{t+k} - E_t D_{t+k} = \sum_{j=1}^{k} H^{k-j} J (s_{t+j} - E_t s_{t+j}) = \sum_{j=1}^{k} H^{k-j} J \sum_{l=0}^{j-1} A^l B e_{t+j-l}.$$

Since it can be verified that

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and hence $H^{j-1} = H$ for all $j \geq 2$, this last expression expands out to

$$\begin{aligned} D_{t+k} - E_t D_{t+k} &= H^{k-1} J B \varepsilon_{t+1} \\ &\quad + H^{k-2} J (B\varepsilon_{t+2} + AB\varepsilon_{t+1}) \\ &\quad + H^{k-3} J (B\varepsilon_{t+3} + AB\varepsilon_{t+2} + A^2 B\varepsilon_{t+1}) \\ &\quad + ... + HJ(B\varepsilon_{t+k-1} + AB\varepsilon_{t+k-2} + ... + A^{k-2}B\varepsilon_{t+1}) \\ &\quad + J(B\varepsilon_{t+k} + AB\varepsilon_{t+k-1} + ... + A^{k-1}B\varepsilon_{t+1}) \\ &= (HJ + HJA + HJA^2 + ... + HJA^{k-2} + JA^{k-1})B\varepsilon_{t+1} \\ &\quad + (HJ + HJA + HJA^2 + ... + HJA^{k-3} + JA^{k-2})B\varepsilon_{t+2} \\ &\quad + ... + (HJ + JA)B\varepsilon_{t+k-1} + JB\varepsilon_{t+k}. \end{aligned}$$

Consequently,

$$\begin{aligned} \Sigma_k^D &= E(D_{t+k} - E_t D_{t+k})(D_{t+k} - E_t D_{t+k})' \\ &= JBVB'J' + (HJ + JA)BVB'(HJ + JA)' \\ &\quad + (HJ + HJA + JA^2)BVB'(HJ + HJA + JA^2)' \\ &\quad + (HJ + HJA + HJA^2 + JA^3)BVB'(HJ + HJA + HJA^2 + JA^3)' \\ &\quad + ... + (HJ + ... + HJA^{k-2} + JA^{k-1})BVB'(HJ + ... + HJA^{k-2} + JA^{k-1})'. \end{aligned}$$

8.2 Producing Smoothed Estimates of the Shocks

Hamilton (1994, Ch.13, Sec.6, pp.394-397) shows how to generate a sequence of smoothed estimates $\{\hat{s}_{t|T}\}_{t=1}^T$ of the unobservable state vector, where

$$\hat{s}_{t|T} = E(s_t | d_T, d_{T-1}, ..., d_1).$$

As before, for $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t | d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t | d_{t-j}, d_{t-j-1}, ..., d_1).$$

Also as before, let

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

so that again,

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Then

$$\hat{s}_{1|0} = Es_1 = 0_{(21\times 1)}$$

and

$$\text{vec}(\Sigma_{1|0}) = \text{vec}(Es_1 s_1') = [I_{(441\times 441)} - A \otimes A]^{-1}\text{vec}(BVB').$$

From these starting values, the sequences $\{\hat{s}_{t|t}\}_{t=1}^T$, $\{\hat{s}_{t|t-1}\}_{t=1}^T$, $\{\Sigma_{t|t}\}_{t=1}^T$, and $\{\Sigma_{t|t-1}\}_{t=1}^T$ can be generated recursively using

$$u_t = d_t - C\hat{s}_{t|t-1},$$

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t,$$

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t},$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1},$$

and

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t}A'$$

for $t = 1, 2, ..., T$.

Now, to begin, construct a sequence $\{J_t\}_{t=1}^T$ using Hamilton's equation (13.6.11):

$$J_t = \Sigma_{t|t}A'\Sigma_{t+1|t}^{-1}.$$

Then note that $\hat{s}_{T|T}$, the last element of $\{\hat{s}_{t|t}\}_{t=1}^T$, is also the last element of $\{\hat{s}_{t|T}\}_{t=1}^T$. From this terminal condition, the rest of the sequence can be generated recusively using Hamilton's equation (13.6.16):

$$\hat{s}_{T-j|T} = \hat{s}_{T-j|T-j} + J_{T-j}(\hat{s}_{T-j+1|T} - \hat{s}_{T-j+1|T-j})$$

for $j = 1, 2, ..., T-1$. Kohn and Ansley (1983) show that in cases where $\Sigma_{t+1|t}$ turns out to be singular, its inverse can be replaced by its Moore-Penrose pseudoinverse in the expression for J_t.