Changes in the Federal Reserve's Inflation Target: Causes and Consequences

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1 The Model

The model shares its basic features with many recent New Keynesian formulations, but resembles most closely the specification used in Ireland (2004). The economy consists of a representative household, a representative finished goods-producing firm, a continuum of intermediate goods-producing firms indexed by $i \in [0, 1]$, and a central bank. During each period $t = 0, 1, 2, ...$, each intermediate goods-producing firm produces a distinct, perishable, intermediate good. Hence, intermediate goods may also be indexed by $i \in [0, 1]$, where firm i produces good i. The model features enough symmetry, however, to allow the analysis to focus on the behavior of a representative intermediate goods-producing firm, identified by the generic index i. The activities of each agent, and their implications for the evolution of equilibrium prices and quantities, will now be described in turn.

1.1 The Representative Household

The representative household enters each period $t = 0, 1, 2, ...$ with money M_{t-1} and bonds B_{t-1}. At the beginning of period t, the household receives a lump-sum nominal transfer T_t from the central bank. Next, the household's bonds mature, providing B_{t-1} additional units of money. The household uses some of its money to purchase B_t new bonds at the price of $1/R_t$ units of money per bond, where R_t denotes the gross nominal interest rate between t and $t + 1$.

During period t, the household supplies $h_t(i)$ units of labor to each intermediate goods-producing firm $i \in [0, 1]$, for a total of

$$h_t = \int_0^1 h_t(i)di.$$

The household gets paid at the nominal wage rate W_t, earning $W_t h_t$ in total labor income during the period. Also during the period, the household consumes C_t units of the finished good, purchased at the nominal price P_t from the representative finished goods-producing firm.

At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm $i \in [0,1]$, for a total of

$$D_t = \int_0^1 D_t(i)di.$$

The household then carries M_t units of money into period $t+1$, where its budget constraint dictates that

$$\frac{M_{t-1} + T_t + B_{t-1} + W_t h_t + D_t}{P_t} \geq C_t + \frac{M_t + B_t/R_t}{P_t} \tag{1}$$

for all $t = 0, 1, 2, \ldots.$

The household's preferences are described by the expected utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t [\ln(C_t - \gamma C_{t-1}) + \ln(M_t/P_t) - h_t],$$

where both the discount factor and the habit formation parameters lie between zero and one: $1 > \beta > 0$ and $1 > \gamma \geq 0$. The preference shock a_t follows the stationary autoregressive process

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \sigma_a \varepsilon_{at} \tag{2}$$

for all $t = 0, 1, 2, \ldots$, with $1 > \rho_a \geq 0$ and $\sigma_a \geq 0$, where the serially uncorrelated innovation ε_{at} has the standard normal distribution. Utility is additively separable in consumption, real money balances, and hours worked; as shown by Driscoll (2000) and Ireland (2002), this additive separability is needed to derive a conventional specification for the model's IS relationship that, in particular, does not include additional terms involving real balances or employment. Given this additive separability, the logarithmic specification over consumption is needed, as shown by King, Plosser, and Rebelo (1988), for the model to be consistent with balanced growth. Habit persistence is introduced into preferences following Fuhrer (2000), who shows that this feature–and the partially backward-looking consumption behavior it implies–helps New Keynesian models like this one replicate the observed effects of monetary policy shocks.

Thus, the household chooses C_t, h_t, B_t, and M_t for all $t = 0, 1, 2, \ldots$ to maximize its expected utility subject to the budget constraint (1) for all $t = 0, 1, 2, \ldots.$ The first-order conditions for this problem can be written as

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t} \right), \tag{3}$$

$$a_t = \Lambda_t (W_t/P_t), \tag{4}$$

$$\Lambda_t = \beta R_t E_t (\Lambda_{t+1}/\Pi_{t+1}), \tag{5}$$

$$M_t/P_t = (a_t/\Lambda_t)[R_t/(R_t - 1)], \tag{6}$$

and (1) with equality for all $t = 0, 1, 2, \ldots$, where Λ_t denotes the nonnegative Lagrange multiplier on the budget constraint for period t and $\Pi_t = P_t/P_{t-1}$ denotes the gross inflation rate between t and $t+1$.

1.2 The Representative Finished Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative finished goods-producing firm uses $Y_t(i)$ units of each intermediate good $i \in [0, 1]$, purchased at the nominal price $P_t(i)$, to manufacture Y_t units of the finished good according to the constant-returns-to-scale technology described by

$$\left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} \geq Y_t$$

where, as in Smets and Wouters (2003), Steinsson (2003), and Ireland (2004), θ_t translates into a random shock to the intermediate goods-producing firms' desired markup of price over marginal cost and hence acts like a cost-push shock of the kind introduced into the New Keynesian model by Clarida, Gali, and Gertler (1999). Here, this markup shock follows the stationary autoregressive process

$$\ln(\theta_t) = (1 - \rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \sigma_\theta \varepsilon_{\theta t} \tag{7}$$

for all $t = 0, 1, 2, ...$, with $1 > \rho_\theta \geq 0$, $\theta > 1$, and $\sigma_\theta \geq 0$, where the serially uncorrelated innovation $\varepsilon_{\theta t}$ has the standard normal distribution.

Thus, during each period t, the finished goods-producing firm chooses $Y_t(i)$ for all $i \in [0, 1]$ to maximize its profits, which are given by

$$P_t \left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} - \int_0^1 P_t(i)Y_t(i)di.$$

The first-order conditions for this problem are

$$Y_t(i) = [P_t(i)/P_t]^{-\theta_t} Y_t$$

for all $i \in [0, 1]$ and $t = 0, 1, 2, ...$

Competition drives the finished goods-producing firm's profits to zero in equilibrium, determining P_t as

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t} di\right]^{1/(1-\theta_t)}$$

for all $t = 0, 1, 2, ...$

1.3 The Representative Intermediate Goods-Producing Firm

During each period $t = 0, 1, 2, ...$, the representative intermediate goods-producing firm hires $h_t(i)$ units of labor from the representative household to manufacture $Y_t(i)$ units of intermediate good i according to the constant-returns-to-scale technology described by

$$Z_t h_t(i) \geq Y_t(i). \tag{8}$$

The aggregate technology shock Z_t follows a random walk with drift:

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \sigma_z \varepsilon_{zt} \tag{9}$$

for all $t = 0, 1, 2, \ldots$, with $z > 1$ and $\sigma_z \geq 0$, where the serially uncorrelated innovation ε_{zt} has the standard normal distribution.

Since the intermediate goods substitute imperfectly for one another in producing the finished good, the representative intermediate goods-producing firm sells its output in a monopolistically competitive market: during period t, the firm sets its nominal price $P_t(i)$, subject to the requirement that it satisfy the representative finished goods-producing firm's demand at that price. And, following Rotemberg (1982), the intermediate goods-producing firm faces a quadratic cost of adjusting its nominal price between periods, measured in terms of the finished good and given by

$$\frac{\phi}{2}\left[\frac{P_t(i)}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t,$$

where $\phi \geq 0$ governs the magnitude of the price adjustment cost, α is a parameter that lies between zero and one ($1 \geq \alpha \geq 0$), and Π_t^* denotes the central bank's inflation target for period t. According to this specification, the extent to which price setting is forward or backward-looking depends on whether α is closer to zero or one. When, in particular, $\alpha = 0$, then price setting is purely forward-looking, in the sense that it is costless for firms to increase their prices in line with the central bank's inflation target. When, on the other hand, $\alpha = 1$, then price setting is purely backward-looking, in the sense that it is costless for firms to increase their prices in line with the previous period's actual rate of inflation.

In any case, the cost of price adjustment makes the intermediate goods-producing firm's problem dynamic: it chooses $P_t(i)$ for all $t = 0, 1, 2, \ldots$, to maximize its total real market value, given by

$$E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t [D_t(i)/P_t],$$

where $\beta^t \Lambda_t$ measures the marginal utility value to the representative household of an additional unit of real profits received in the form of dividends during period t and where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} \left(\frac{W_t}{P_t}\right)\left(\frac{Y_t}{Z_t}\right) - \frac{\phi}{2}\left[\frac{P_t(i)}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t \quad (10)$$

measures the firm's real profits during the same period t. The first-order conditions for this problem are

$$\begin{aligned} 0 \;=\; & (1-\theta_t)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} + \theta_t \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t - 1}\left(\frac{W_t}{P_t}\right)\left(\frac{1}{Z_t}\right) \qquad (11) \\ & -\phi\left[\frac{P_t(i)}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}P_{t-1}(i)} - 1\right]\left[\frac{P_t}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}P_{t-1}(i)}\right] \\ & +\beta\phi E_t\left\{\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)\left[\frac{P_{t+1}(i)}{\Pi_t^{\alpha}(\Pi_{t+1}^*)^{1-\alpha}P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)}{\Pi_t^{\alpha}(\Pi_{t+1}^*)^{1-\alpha}P_t(i)}\right]\left[\frac{P_t}{P_t(i)}\right]\left(\frac{Y_{t+1}}{Y_t}\right)\right\} \end{aligned}$$

and (8) with equality for all $t = 0, 1, 2, \ldots$.

1.4 Symmetric Equilibrium

In a symmetric equilibrium, all intermediate goods-producing firms make identical decisions, so that $Y_t(i) = Y_t$, $h_t(i) = h_t$, $D_t(i) = D_t$, and $P_t(i) = P_t$ for all $i \in [0,1]$ and $t = 0,1,2,....$ In addition, the market-clearing conditions for money and bonds, $M_t = M_{t-1} + T_t$ and $B_t = B_{t-1} = 0$, must also hold for all $t = 0,1,2,....$ After imposing these equilibrium conditions, and using (4), (6), (8), and (10) to solve out for W_t/P_t, M_t/P_t, h_t, and D_t/P_t, the system consisting for (1)-(11) reduces to

$$Y_t = C_t + \frac{\phi}{2}\left[\frac{\Pi_t}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}} - 1\right]^2 Y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \sigma_a \varepsilon_{at}, \tag{2}$$

$$\Lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right), \tag{3}$$

$$\Lambda_t = \beta R_t E_t(\Lambda_{t+1}/\Pi_{t+1}), \tag{5}$$

$$\ln(\theta_t) = (1-\rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \sigma_\theta \varepsilon_{\theta t}, \tag{7}$$

$$\ln(Z_t) = \ln(z) + \ln(Z_{t-1}) + \sigma_z \varepsilon_{zt}, \tag{9}$$

and

$$\begin{aligned} \theta_t - 1 \quad = \quad & \theta_t\left(\frac{a_t}{\Lambda_t Z_t}\right) - \phi\left[\frac{\Pi_t}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}} - 1\right]\left[\frac{\Pi_t}{\Pi_{t-1}^{\alpha}(\Pi_t^*)^{1-\alpha}}\right] \\ & + \beta\phi E_t\left\{\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)\left[\frac{\Pi_{t+1}}{\Pi_t^{\alpha}(\Pi_{t+1}^*)^{1-\alpha}} - 1\right]\left[\frac{\Pi_{t+1}}{\Pi_t^{\alpha}(\Pi_{t+1}^*)^{1-\alpha}}\right]\left(\frac{Y_{t+1}}{Y_t}\right)\right\} \end{aligned} \tag{11}$$

for all $t = 0,1,2,....$

To help keep track of the model's observable variables, it is useful to define the growth rates of output, inflation, and the nominal interest rate as

$$g_t^y = Y_t/Y_{t-1}, \tag{12}$$

$$g_t^\pi = \Pi_t/\Pi_{t-1}, \tag{13}$$

and

$$g_t^r = R_t/R_{t-1} \tag{14}$$

for all $t = 0,1,2,....$ Likewise, it is helpful to define the ratio of the nominal interest rate to the inflation rate as

$$r_t^{r\pi} = R_t/\Pi_t \tag{15}$$

for all $t = 0,1,2,....$

1.5 The Efficient Level of Output and the Output Gap

A social planner for this economy chooses Q_t and $n_t(i)$ for all $i \in [0,1]$ to maximize the social welfare function

$$E_0 \sum_{t=0}^{\infty} \beta^t a_t \left[\ln(Q_t - \gamma Q_{t-1}) - \int_0^1 n_t(i) di \right]$$

subject to the aggregate feasibility constraint

$$Z_t \left[\int_0^1 n_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\theta_t/(\theta_t - 1)} \geq Q_t$$

for all $t = 0, 1, 2, \ldots$. The first-order conditions for this problem can be written as

$$\Xi_t = \frac{a_t}{Q_t - \gamma Q_{t-1}} - \beta \gamma E_t \left(\frac{a_{t+1}}{Q_{t+1} - \gamma Q_t} \right),$$

$$a_t = \Xi_t Z_t (Q_t/Z_t)^{1/\theta_t} n_t(i)^{-1/\theta_t}$$

for all $i \in [0,1]$, and the aggregate feasibility constraint with equality for all $t = 0, 1, 2, \ldots$, where Ξ_t denotes the nonnegative multiplier on the aggregate feasibility constraint for period t.

The second optimality condition listed above implies that $n_t(i) = n_t$ for all $i \in [0,1]$ and $t = 0, 1, 2, \ldots$, where

$$n_t = (\Xi_t/a_t)^{\theta_t} Z_t^{\theta_t} (Q_t/Z_t).$$

Substituting this last relationship into the aggregate feasibility constraint yields

$$\Xi_t = a_t/Z_t.$$

Hence, the efficient level of output Q_t must satisfy

$$\frac{1}{Z_t} = \frac{1}{Q_t - \gamma Q_{t-1}} - \beta \gamma E_t \left[\left(\frac{a_{t+1}}{a_t} \right) \left(\frac{1}{Q_{t+1} - \gamma Q_t} \right) \right] \tag{16}$$

for all $t = 0, 1, 2, \ldots$. This definition of the efficient level of output implies a corresponding definition of the output gap x_t

$$x_t = Y_t/Q_t. \tag{17}$$

1.6 The Central Bank

The central bank conducts monetary policy according to the modified Taylor (1993) rule

$$\ln(R_t) = \ln(R_{t-1}) + \rho_\pi \ln(\Pi_t/\Pi_t^*) + \rho_x \ln(x_t/x) + \rho_{gy} \ln(g_t^y/g^y) + \ln(v_t) \tag{18}$$

for all $t = 0, 1, 2, \ldots$, where the response coefficients $\rho_\pi > 0$, $\rho_x \geq 0$, and $\rho_{gy} \geq 0$ are chosen by the central bank. Under this policy rule, the central bank raises the short-term nominal interest rate R_t whenever the inflation rate Π_t rises above its target Π_t^*, whenever the output gap x_t rises above its steady-state level x, and whenever the growth rate of output g_t^y rises above its steady-state level g^y,

Two types of shocks enter into (18). First, the inflation target Π_t^* follows a random walk:

$$\ln(\Pi_t^*) = \ln(\Pi_{t-1}^*) + \delta_a \varepsilon_{at} - \delta_\theta \varepsilon_{\theta t} - \delta_z \varepsilon_{zt} + \sigma_\pi \varepsilon_{\pi t} \tag{19}$$

for all $t = 0, 1, 2, ...,$ where the response coefficients $\delta_a \geq 0$, $\delta_e \geq 0$, and $\delta_z \geq 0$ are chosen by the central bank, where $\sigma_\pi \geq 0$, and where the serially uncorrelated innovation $\varepsilon_{\pi t}$ has the standard normal distribution. Meanwhile, the transitory policy shock v_t follows a stationary autoregressive process:

$$\ln(v_t) = \rho_v \ln(v_{t-1}) + \sigma_v \varepsilon_{vt} \tag{20}$$

for all $t = 0, 1, 2, ...,$ with $1 > \rho_v \geq 0$ and $\sigma_v \geq 0$, where the serially uncorrelated innovation ε_{vt} also has the standard normal distribution.

1.7 The Stationary System

Equations (1)-(3), (5), (7), (9), and (11)-(20) now form a system of 16 equations in the 16 variables Y_t, C_t, Π_t, R_t, Q_t, x_t, g_t^y, g_t^π, g_t^r, $r_t^{r\pi}$, Λ_t, a_t, θ_t, Z_t, v_t, and Π_t^*. Some of these variables inherit unit roots, either from the process (9) for the technology shock or the process (19) for the inflation target. However, the variables $y_t = Y_t/Z_t$, $c_t = C_t/Z_t$, $\pi_t = \Pi_t/\Pi_t^*$, $r_t = R_t/\Pi_t^*$, $q_t = Q_t/Z_t$, x_t, g_t^y, g_t^π, g_t^r, $r_t^{r\pi}$, $\lambda_t = Z_t\Lambda_t$, a_t, θ_t, $z_t = Z_t/Z_{t-1}$, v_t, and $\pi_t^* = \Pi_t^*/\Pi_{t-1}^*$ are all stationary and, in terms of these stationary variables, the system can be rewritten as

$$y_t = c_t + (\phi/2)[\pi_t(\pi_t^*/\pi_{t-1})^\alpha - 1]^2 y_t, \tag{1}$$

$$\ln(a_t) = \rho_a \ln(a_{t-1}) + \sigma_a \varepsilon_{at}, \tag{2}$$

$$\lambda_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{z_{t+1} c_{t+1} - \gamma c_t} \right), \tag{3}$$

$$\lambda_t = \beta r_t E_t[(1/z_{t+1})(1/\pi_{t+1}^*)(\lambda_{t+1}/\pi_{t+1})], \tag{5}$$

$$\ln(\theta_t) = (1 - \rho_\theta)\ln(\theta) + \rho_\theta \ln(\theta_{t-1}) + \sigma_\theta \varepsilon_{\theta t}, \tag{7}$$

$$\ln(z_t) = \ln(z) + \sigma_z \varepsilon_{zt}, \tag{9}$$

$$\begin{aligned} \theta_t - 1 \quad = \quad & \theta_t(a_t/\lambda_t) - \phi[\pi_t(\pi_t^*/\pi_{t-1})^\alpha - 1][\pi_t(\pi_t^*/\pi_{t-1})^\alpha] \\ & +\beta\phi E_t\{(\lambda_{t+1}/\lambda_t)[\pi_{t+1}(\pi_{t+1}^*/\pi_t)^\alpha - 1][\pi_{t+1}(\pi_{t+1}^*/\pi_t)^\alpha](y_{t+1}/y_t)\}, \end{aligned} \tag{11}$$

$$g_t^y = (y_t/y_{t-1})z_t, \tag{12}$$

$$g_t^\pi = (\pi_t/\pi_{t-1})\pi_t^*, \tag{13}$$

$$g_t^r = (r_t/r_{t-1})\pi_t^*, \tag{14}$$

$$r_t^{r\pi} = r_t/\pi_t, \tag{15}$$

$$1 = \frac{z_t}{z_t q_t - \gamma q_{t-1}} - \beta\gamma E_t \left[\left(\frac{a_{t+1}}{a_t} \right) \left(\frac{1}{z_{t+1} q_{t+1} - \gamma q_t} \right) \right], \tag{16}$$

$$x_t = y_t/q_t, \tag{17}$$

$$\ln(r_t) = \ln(r_{t-1}) + \rho_\pi \ln(\pi_t) + \rho_x \ln(x_t/x) + \rho_{gy} \ln(g_t^y/g^y) - \ln(\pi_t^*) + \ln(v_t), \tag{18}$$

$$\ln(\pi_t^*) = \delta_a \varepsilon_{at} - \delta_\theta \varepsilon_{\theta t} - \delta_z \varepsilon_{zt} + \sigma_\pi \varepsilon_{\pi t}, \tag{19}$$

and

$$\ln(v_t) = \rho_v \ln(v_{t-1}) + \sigma_v \varepsilon_{vt} \tag{20}$$

for all $t = 0, 1, 2, ...$

1.8 The Steady State

In the absence of shocks, the economy converges to a steady-state growth path, along which all of the stationary variables are constant, with $y_t = y$, $c_t = c$, $\pi_t = \pi$, $r_t = r$, $q_t = q$, $x_t = x$, $g_t^y = g^y$, $g_t^\pi = g^\pi$, $g_t^r = g^r$, $r_t^{r\pi} = r^{r\pi}$, $\lambda_t = \lambda$, $a_t = a$, $\theta_t = \theta$, $z_t = z$, $v_t = v$, and $\pi_t^* = \pi^*$ for all $t = 0, 1, 2, ...$

The steady-state values $a = 1$, θ, z, $\pi^* = 1$, and $v = 1$ are determined exogenously by (2), (7), (9), (19) and (20). Equation (18) implies that $\pi = 1$ as well.

Equations (1) and (12) imply that $c = y$ and $g^y = z$, while (13) and (14) imply that $g^\pi = 1$ and $g^r = 1$.

Equation (11) implies that

$$\lambda = \theta/(\theta - 1).$$

Equation (3) then implies that

$$y = \left(\frac{\theta - 1}{\theta}\right)\left(\frac{z - \beta\gamma}{z - \gamma}\right).$$

Equation (16) implies that

$$q = \frac{z - \beta\gamma}{z - \gamma},$$

so that (17) implies that

$$x = (\theta - 1)/\theta.$$

Finally, (5) implies that

$$r = z/\beta,$$

so that (15) implies that

$$r^{r\pi} = z/\beta$$

as well.

1.9 The Linearized System

The system consisting of (1)-(3), (5), (7), (9), and (11)-(20) can be log-linearized around the steady state to describe how the economy responds to shocks. Let $\hat{y}_t = \ln(y_t/y)$, $\hat{c}_t = \ln(c_t/c)$, $\hat{\pi}_t = \ln(\pi_t)$, $\hat{r}_t = \ln(r_t/r)$, $q_t = \ln(q_t/q)$, $\hat{x}_t = \ln(x_t/x)$, $\hat{g}_t^y = \ln(g_t^y/g^y)$, $\hat{g}_t^\pi = \ln(g_t^\pi)$, $\hat{g}_t^r = \ln(g_t^r)$, $\hat{r}_t^{r\pi} = \ln(r_t^{r\pi}/r^{r\pi})$, $\hat{\lambda}_t = \ln(\lambda_t/\lambda)$, $\hat{a}_t = \ln(a_t)$, $\hat{\theta}_t = \ln(\theta_t/\theta)$, $\hat{z}_t = \ln(z_t/z)$, $\hat{v}_t = \ln(v_t)$, and $\hat{\pi}_t^* = \ln(\pi_t^*)$ denote the percentage deviation of each stationary variable from its steady-state value. A first-order Taylor approximation to (1) reveals that $\hat{c}_t = \hat{y}_t$, allowing $\hat{c}_t$ to be eliminated from the system. First-order approximations to the remaining 15 equations then imply

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \sigma_a \varepsilon_{at}, \tag{2}$$

$$(z-\gamma)(z-\beta\gamma)\hat{\lambda}_t = \gamma z \hat{y}_{t-1} - (z^2+\beta\gamma^2)\hat{y}_t + \beta\gamma z E_t \hat{y}_{t+1} + (z-\gamma)(z-\beta\gamma\rho_a)\hat{a}_t - \gamma z \hat{z}_t, \tag{3}$$

$$\hat{\lambda}_t = E_t \hat{\lambda}_{t+1} + \hat{r}_t - E_t \hat{\pi}_{t+1}, \tag{5}$$

$$\hat{e}_t = \rho_e \hat{e}_{t-1} + \sigma_e \varepsilon_{et}, \tag{7}$$

$$\hat{z}_t = \sigma_z \varepsilon_{zt}, \tag{9}$$

$$(1+\beta\alpha)\hat{\pi}_t = \alpha\hat{\pi}_{t-1} + \beta E_t \hat{\pi}_{t+1} - \psi\hat{\lambda}_t + \psi\hat{a}_t - \hat{e}_t - \alpha\hat{\pi}_t^* \tag{11}$$

$$\hat{g}_t^y = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t, \tag{12}$$

$$\hat{g}_t^\pi = \hat{\pi}_t - \hat{\pi}_{t-1} + \hat{\pi}_t^*, \tag{13}$$

$$\hat{g}_t^r = \hat{r}_t - \hat{r}_{t-1} + \hat{\pi}_t^*, \tag{14}$$

$$\hat{r}_t^{r\pi} = \hat{r}_t - \hat{\pi}_t, \tag{15}$$

$$0 = \gamma z \hat{q}_{t-1} - (z^2+\beta\gamma^2)\hat{q}_t + \beta\gamma z E_t \hat{q}_{t+1} + \beta\gamma(z-\gamma)(1-\rho_a)\hat{a}_t - \gamma z \hat{z}_t, \tag{16}$$

$$\hat{x}_t = \hat{y}_t - \hat{q}_t, \tag{17}$$

$$\hat{r}_t = \hat{r}_{t-1} + \rho_\pi \hat{\pi}_t + \rho_x \hat{x}_t + \rho_{gy}\hat{g}_t^y - \hat{\pi}_t^* + \hat{v}_t, \tag{18}$$

$$\hat{\pi}_t^* = \delta_a \varepsilon_{at} - \delta_e \varepsilon_{et} - \delta_z \varepsilon_{zt} + \sigma_\pi \varepsilon_{\pi t}, \tag{19}$$

and

$$\hat{v}_t = \rho_v \hat{v}_{t-1} + \sigma_v \varepsilon_{vt} \tag{20}$$

for all $t = 0, 1, 2, ...$, where, in (7), (11), and (19), the cost-push shock $\hat{\theta}_t$ has been renormalized as $\hat{e}_t = (1/\phi)\hat{\theta}_t$ and the new parameters ρ_e, σ_e, ψ and δ_e have been defined as $\rho_e = \rho_\theta$, $\sigma_e = \sigma_\theta/\phi$, $\psi = (\theta-1)/\phi$, and $\delta_e = \delta_\theta$, so that, like $\varepsilon_{\theta t}$, ε_{et} has the standard normal distribution.

As a first step in solving the model, it is helpful to rewrite (18) as

$$\hat{r}_t = \hat{r}_{t-1} + \rho_\pi \hat{\pi}_t + (\rho_x + \rho_{gy})\hat{y}_t - \rho_x \hat{q}_t - \rho_{gy}\hat{y}_{t-1} + \rho_{gy}\hat{z}_t - \hat{\pi}_t^* + \hat{v}_t. \tag{18}$$

As a second step, note that (12)-(15) and (17) can be used to solve for $\hat{g}_t^y$, $\hat{g}_t^\pi$, $\hat{g}_t^r$, $\hat{r}_t^{r\pi}$, and $\hat{x}_t$ in terms of $\hat{y}_t$, $\hat{\pi}_t$, $\hat{r}_t$, $\hat{q}_t$, $\hat{\lambda}_t$, $\hat{a}_t$, $\hat{e}_t$, $\hat{z}_t$, $\hat{v}_t$, and $\hat{\pi}_t^*$. This last observation suggests using the ten equations (2), (3), (5), (7), (9), (11), (16), and (18)-(20) to solve for the ten variables $\hat{y}_t$, $\hat{\pi}_t$, $\hat{r}_t$, $\hat{q}_t$, $\hat{\lambda}_t$, $\hat{a}_t$, $\hat{e}_t$, $\hat{z}_t$, $\hat{v}_t$, and $\hat{\pi}_t^*$ before returning to the five equations (12)-(15) and (17) to solve for the remaining five variables $\hat{g}_t^y$, $\hat{g}_t^\pi$, $\hat{g}_t^r$, $\hat{r}_t^{r\pi}$, and $\hat{x}_t$.

2 Solving the Model

Let

$$s_t^0 = \left[\begin{array}{cccccccc} \hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{q}_{t-1} & \hat{\lambda}_t & \hat{y}_t & \hat{\pi}_t & \hat{q}_t \end{array}\right]'$$

and

$$\xi_t = \left[\begin{array}{ccccc} \hat{a}_t & \hat{e}_t & \hat{z}_t & \hat{v}_t & \hat{\pi}_t^* \end{array}\right]'.$$

Then (3), (5), (11), (16) and (18) can be written as

$$AE_t s_{t+1}^0 = Bs_t^0 + C\xi_t, \tag{21}$$

where

$$A = \left[\begin{array}{cccccccc} z^2+\beta\gamma^2 & 0 & 0 & 0 & 0 & -\beta\gamma z & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1+\beta\alpha & 0 & 0 & 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & z^2+\beta\gamma^2 & 0 & 0 & 0 & -\beta\gamma z \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$B = \left[\begin{array}{cccccccc} \gamma z & 0 & 0 & 0 & -(z-\gamma)(z-\beta\gamma) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & -\psi & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma z & 0 & 0 & 0 & 0 \\ -\rho_{gy} & 0 & 1 & 0 & 0 & \rho_x+\rho_{gy} & \rho_\pi & -\rho_x \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

and

$$C = \left[\begin{array}{ccccc} (z-\gamma)(z-\beta\gamma\rho_a) & 0 & -\gamma z & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \psi & -1 & 0 & 0 & -\alpha \\ \beta\gamma(z-\gamma)(1-\rho_a) & 0 & -\gamma z & 0 & 0 \\ 0 & 0 & \rho_{gy} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Meanwhile, (2), (7), (9), (19), and (20) can be written as

$$\xi_t = P\xi_{t-1} + X\varepsilon_t, \tag{22}$$

where

$$P = \left[\begin{array}{ccccc} \rho_a & 0 & 0 & 0 & 0 \\ 0 & \rho_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_v & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$X = \begin{bmatrix} \sigma_a & 0 & 0 & 0 & 0 \\ 0 & \sigma_e & 0 & 0 & 0 \\ 0 & 0 & \sigma_z & 0 & 0 \\ 0 & 0 & 0 & \sigma_v & 0 \\ \delta_a & -\delta_e & -\delta_z & 0 & \sigma_\pi \end{bmatrix},$$

and

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{at} & \varepsilon_{et} & \varepsilon_{zt} & \varepsilon_{vt} & \varepsilon_{\pi t} \end{bmatrix}'.$$

Equation (21) takes the form of a system of linear expectational difference equations, driven by the exogenous shocks in (22). This system can be solved by uncoupling the unstable and stable components and then solving the unstable component forward. There are a number of algorithms for working through this process; the approach taken here uses the methods outlined by Klein (2000).

Klein's method relies on the complex generalized Schur decomposition, which identifies unitary matrices Q and Z such that

$$QAZ = S$$

and

$$QBZ = T$$

are both upper triangular, where the generalized eigenvalues of B and A can be recovered as the ratios of the diagonal elements of T and S:

$$\lambda(B, A) = \{t_{ii}/s_{ii} | i = 1, 2, ..., 8\}.$$

The matrices Q, Z, S, and T can always be arranged so that the generalized eigenvalues appear in ascending order in absolute value. Note that there are four predetermined variables in the vector s_t^0. Thus, if four of the generalized eigenvalues in $\lambda(B, A)$ lie inside the unit circle and four of the generalized eigenvalues lie outside the unit circle, then the system has a unique solution. If more than four of the generalized eigenvalues in $\lambda(B, A)$ lie outside the unit circle, then the system has no solution. If less than four of the generalized eigenvalues in $\lambda(B, A)$ lie outside the unit circle, then the solution has multiple solutions. For details, see Blanchard and Kahn (1980) and Klein (2000).

Assume from now on that there exactly four generalized eigenvalues that lie outside the unit circle, and partition the matrices Q, Z, S, and T conformably, so that

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

where Q_1 and Q_2 are both 4×8, and

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ 0_{(4\times4)} & S_{22} \end{bmatrix},$$

and

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0_{(4\times4)} & T_{22} \end{bmatrix},$$

where Z_{11}, S_{11}, T_{11}, Z_{12}, S_{12}, T_{12}, Z_{21}, Z_{22}, S_{22}, and T_{22} are all 4×4.

Next, define the vector s_t^1 of auxiliary variables as

$$s_t^1 = Z' s_t^0$$

so that, in particular,

$$s_t^1 = \begin{bmatrix} s_{1t}^1 \\ s_{2t}^1 \end{bmatrix},$$

where

$$s_{1t}^1 = Z'_{11} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + Z'_{21} \begin{bmatrix} \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} \tag{23}$$

and

$$s_{2t}^1 = Z'_{12} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + Z'_{22} \begin{bmatrix} \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} \tag{24}$$

are both 4×1.

Since Z is unitary, $Z'Z = I$ or $Z' = Z^{-1}$ and hence $s_t^0 = Z s_t^1$. Use this fact to rewrite (21) as

$$AZE_t s_{t+1}^1 = BZ s_t^1 + C\xi_t.$$

Premultiply this version of (21) by Q to obtain

$$SE_t s_{t+1}^1 = T s_t^1 + QC\xi_t$$

or, in terms of the matrix partitions,

$$S_{11} E_t s_{1t+1}^1 + S_{12} E_t s_{2t+1}^1 = T_{11} s_{1t}^1 + T_{12} s_{2t}^1 + Q_1 C \xi_t \tag{25}$$

and

$$S_{22} E_t s_{2t+1}^1 = T_{22} s_{2t}^1 + Q_2 C \xi_t. \tag{26}$$

Since the generalized eigenvalues corresponding to the diagonal elements of S_{22} and T_{22} all lie outside the unit circle, (26) can be solved forward to obtain

$$s_{2t}^1 = -T_{22}^{-1} R \xi_t,$$

where the 4×5 matrix R is given by

$$\begin{aligned} vec(R) &= vec \sum_{j=0}^{\infty} (S_{22} T_{22}^{-1})^j Q_2 C P^j = \sum_{j=0}^{\infty} vec[(S_{22} T_{22}^{-1})^j Q_2 C P^j] \\ &= \sum_{j=0}^{\infty} [P^j \otimes (S_{22} T_{22}^{-1})^j] vec(Q_2 C) = \sum_{j=0}^{\infty} [P \otimes (S_{22} T_{22}^{-1})]^j vec(Q_2 C) \\ &= [I_{(20\times 20)} - P \otimes (S_{22} T_{22}^{-1})]^{-1} vec(Q_2 C). \end{aligned}$$

Use this result, along with (24) to solve for

$$\begin{bmatrix} \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} = -(Z'_{22})^{-1} Z'_{12} \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} - (Z'_{22})^{-1} T_{22}^{-1} R \xi_t.$$

Since Z is unitary, $Z'Z = I$ or

$$\begin{bmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} I_{(4\times4)} & 0_{(4\times4)} \\ 0_{(4\times4)} & I_{(4\times4)} \end{bmatrix}.$$

Hence, in particular,

$$Z'_{12} Z_{11} + Z'_{22} Z_{21} = 0$$

or

$$-(Z'_{22})^{-1} Z'_{12} = Z_{21} Z_{11}^{-1}$$

and

$$Z'_{12} Z_{12} + Z'_{22} Z_{22} = I$$

or

$$(Z'_{22})^{-1} = Z_{22} + (Z'_{22})^{-1} Z'_{12} Z_{12} = Z_{22} - Z_{21} Z_{11}^{-1} Z_{12},$$

allowing this solution to be written more conveniently as

$$\begin{bmatrix} \hat{\lambda}_t \\ \hat{y}_t \\ \hat{\pi}_t \\ \hat{q}_t \end{bmatrix} = M_1 \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} + M_2 \xi_t, \tag{27}$$

where

$$M_1 = Z_{21} Z_{11}^{-1}$$

and

$$M_2 = -[Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R.$$

Equation (23) now provides the solution for s_{1t}^1:

$$s_{1t}^1 = (Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1}) \begin{bmatrix} \hat{y}_{t-1} \\ \hat{\pi}_{t-1} \\ \hat{r}_{t-1} \\ \hat{q}_{t-1} \end{bmatrix} - Z'_{21} [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] T_{22}^{-1} R \xi_t.$$

Using

$$Z'_{11} Z_{11} + Z'_{21} Z_{21} = I$$

or

$$Z'_{11} + Z'_{21} Z_{21} Z_{11}^{-1} = Z_{11}^{-1}$$

and

$$Z'_{21} [Z_{22} - Z_{21} Z_{11}^{-1} Z_{12}] = Z'_{21} Z_{22} - Z'_{21} Z_{21} Z_{11}^{-1} Z_{12} = -Z_{11}^{-1} Z_{12},$$

this last result can be written more conveniently as

$$s_{1t}^{1}=Z_{11}^{-1}\begin{bmatrix}\hat{y}_{t-1}\\ \hat{\pi}_{t-1}\\ \hat{r}_{t-1}\\ \hat{q}_{t-1}\end{bmatrix}+Z_{11}^{-1}Z_{12}T_{22}^{-1}R\xi_{t}.$$

Substitute these results into (25) to obtain the solution

$$\begin{bmatrix}\hat{y}_{t}\\ \hat{\pi}_{t}\\ \hat{r}_{t}\\ \hat{q}_{t}\end{bmatrix}=M_3\begin{bmatrix}\hat{y}_{t-1}\\ \hat{\pi}_{t-1}\\ \hat{r}_{t-1}\\ \hat{q}_{t-1}\end{bmatrix}+M_4\xi_{t}, \tag{28}$$

where

$$M_3=Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}$$

and

$$M_4=Z_{11}S_{11}^{-1}(T_{11}Z_{11}^{-1}Z_{12}T_{22}^{-1}R+Q_1C+S_{12}T_{22}^{-1}RP-T_{12}T_{22}^{-1}R)-Z_{12}T_{22}^{-1}RP.$$

To complete the construction of the solution, let

$$s_t=\begin{bmatrix}\hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{q}_{t-1} & \hat{a}_t & \hat{e}_t & \hat{z}_t & \hat{v}_t & \hat{\pi}_t^{*}\end{bmatrix}',$$

$$\Pi=\begin{bmatrix}M_3 & M_4\\ 0_{(5\times 4)} & P\end{bmatrix},$$

and

$$W=\begin{bmatrix}0_{(4\times 5)}\\ X\end{bmatrix},$$

so that (22) and (28) can be written more compactly as

$$s_{t+1}=\Pi s_t+W\varepsilon_{t+1}. \tag{29}$$

Now let

$$f_t^0=\begin{bmatrix}\hat{g}_t^{y} & \hat{g}_t^{\pi} & \hat{g}_t^{r} & \hat{r}_t^{r\pi} & \hat{x}_t\end{bmatrix}$$

and write (12)-(15) and (17) as

$$f_t^0=Ds_{t+1}+Fs_t, \tag{30}$$

where

$$D=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

and

$$F = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Substituting (29) into (30) yields

$$f_t^0 = Gs_t \tag{31}$$

where

$$G = D\Pi + F.$$

Finally, let

$$f_t = \begin{bmatrix} \hat{\lambda}_t & \hat{g}_t^y & \hat{g}_t^\pi & \hat{g}_t^r & \hat{r}_t^{r\pi} & \hat{x}_t \end{bmatrix}$$

and combine (27) and (31) to obtain

$$f_t = Us_t, \tag{32}$$

where

$$U = \begin{bmatrix} H \\ G \end{bmatrix},$$

$$H = \begin{bmatrix} M_{11} & M_{21} \end{bmatrix},$$

and M_{11} and M_{21} denote the first rows of M_1 and M_2. The model's full solution, for all 15 variables, is now described by (29) and (32).

3 Estimating the Model

The model has implications for the behavior of three observable variables: the growth rate of output, the growth rate of inflation, and the ratio of the nominal interest rate to the inflation rate. The empirical model has 19 parameters: z, β, γ, α, ψ, ρ_π, ρ_x, ρ_{gy}, ρ_a, ρ_e, ρ_v, σ_a, σ_e, σ_z, σ_v, σ_π, δ_a, δ_e, and δ_z. Note that values can be assigned to z and β to insure that the steady-state rate of output growth z in the model equals the corresponding average value in the data and so that the steady-state ratio of the nominal interest rate to the inflation rate z/β equals the corresponding average value in the data.

To estimate the remaining 17 parameters via maximum likelihood, let $\{d_t\}_{t=1}^T$ denote the series for the logarithmic deviations of the growth rate of output, the growth rate of inflation, and the ratio of the nominal interest rate to the inflation rate from their average, or steady-state, values:

$$d_t = \begin{bmatrix} \hat{g}_t^y \\ \hat{g}_t^\pi \\ \hat{r}_t^{r\pi} \end{bmatrix} = \begin{bmatrix} \ln(Y_t) - \ln(Y_{t-1}) - \ln(z) \\ \ln(P_t) - 2\ln(P_{t-1}) + \ln(P_{t-2}) \\ \ln(R_t) - \ln(P_t) + \ln(P_{t-1}) - \ln(z) + \ln(\beta) \end{bmatrix},$$

where Y_t is the level of real GDP per-capita, P_t is the level of the GDP deflator, and R_t is the gross nominal interest rate on three-month US Treasury bills.

The empirical model then takes form

$$s_{t+1} = As_t + B\varepsilon_{t+1} \tag{33}$$

and

$$d_t = Cs_t, \tag{34}$$

where $A = \Pi$, $B = W$, C is formed from the rows of U as

$$C = \begin{bmatrix} U_2 \\ U_3 \\ U_5 \end{bmatrix},$$

and the vector of zero-mean, serially uncorrelated innovations ε_{t+1} is normally distributed with diagonal covariance matrix

$$V = E\varepsilon_{t+1}\varepsilon'_{t+1} = I_{(5\times5)}.$$

The model defined by (33) and (34) is in state-space form; hence, the likelihood function for the sample $\{d_t\}_{t=1}^{T}$ can be constructed as outlined by Hamilton (1994, Ch.13). For $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Then, in particular, (33) implies that

$$\hat{s}_{1|0} = Es_1 = 0_{(9\times1)} \tag{35}$$

and

$$vec(\Sigma_{1|0}) = vec(Es_1s_1') = [I_{(81\times81)} - A\otimes A]^{-1}vec(BVB'). \tag{36}$$

Now suppose that $\hat{s}_{t|t-1}$ and $\Sigma_{t|t-1}$ are in hand and consider the problem of calculating $\hat{s}_{t+1|t}$ and $\Sigma_{t+1|t}$. Note first from (34) that

$$\hat{d}_{t|t-1} = C\hat{s}_{t|t-1}.$$

Hence

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

is such that

$$Eu_tu_t' = C\Sigma_{t|t-1}C'.$$

Next, using Hamilton's (p.379, eq.13.2.13) formula for updating a linear projection,

$$\begin{aligned}\hat{s}_{t|t} &= \hat{s}_{t|t-1} + [E(s_t - \hat{s}_{t|t-1})(d_t - \hat{d}_{t|t-1})'][E(d_t - \hat{d}_{t|t-1})(d_t - \hat{d}_{t|t-1})']^{-1}u_t \\ &= \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.\end{aligned}$$

Hence, from (33),

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t-1} + A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Using this last result, along with (33) again,

$$s_{t+1} - \hat{s}_{t+1|t} = A(s_t - \hat{s}_{t|t-1}) + B\varepsilon_{t+1} - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t.$$

Hence,

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t-1}A' - A\Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1}A'.$$

These results can be summarized as follows. Let

$$\hat{s}_t = \hat{s}_{t|t-1} = E(s_t|d_{t-1}, d_{t-2}, ..., d_1)$$

and

$$\Sigma_t = \Sigma_{t|t-1} = E(s_t - \hat{s}_{t|t-1})(s_t - \hat{s}_{t|t-1})'.$$

Then

$$\hat{s}_{t+1} = A\hat{s}_t + K_t u_t$$

and

$$d_t = C\hat{s}_t + u_t,$$

where

$$u_t = d_t - E(d_t|d_{t-1}, d_{t-2}, ..., d_1),$$
$$Eu_t u_t' = C\Sigma_t C' = \Omega_t,$$

the sequences for K_t and Σ_t can be generated recursively using

$$K_t = A\Sigma_t C'(C\Sigma_t C')^{-1}$$

and

$$\Sigma_{t+1} = BVB' + A\Sigma_t A' - A\Sigma_t C'(C\Sigma_t C')^{-1}C\Sigma_t A',$$

and initial conditions $\hat{s}_1$ and Σ_1 are provided by (35) and (36).

The innovations $\{u_t\}_{t=1}^T$ can then be used to form the log likelihood function for $\{d_t\}_{t=1}^T$ as

$$\ln L = -\left(\frac{3T}{2}\right)\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln|\Omega_t| - \frac{1}{2}\sum_{t=1}^{T}u_t'\Omega_t^{-1}u_t.$$

4 Evaluating the Model

4.1 Variance Decompositions

Begin by considering (33), which can be rewritten as

$$s_t = As_{t-1} + B\varepsilon_t,$$

or

$$(I - AL)s_t = B\varepsilon_t,$$

or

$$s_t = \sum_{j=0}^{\infty} A^j B \varepsilon_{t-j}.$$

This last equation implies that

$$s_{t+k} = \sum_{j=0}^{\infty} A^j B \varepsilon_{t+k-j},$$

$$E_t s_{t+k} = \sum_{j=k}^{\infty} A^j B \varepsilon_{t+k-j},$$

$$s_{t+k} - E_t s_{t+k} = \sum_{j=0}^{k-1} A^j B \varepsilon_{t+k-j},$$

and hence

$$\begin{aligned}
\Sigma_k^s &= E(s_{t+k} - E_t s_{t+k})(s_{t+k} - E_t s_{t+k})' \\
&= BVB' + ABVB'A' + A^2 BVB'A^{2\prime} + ... + A^{k-1} BVB'A^{k-1\prime}.
\end{aligned}$$

In addition, (33) implies that

$$\Sigma^s = \lim_{k\to\infty} \Sigma_k^s$$

is given by

$$vec(\Sigma^s) = [I_{(81\times 81)} - A \otimes A]^{-1} vec(BVB').$$

Next, consider (32), which implies that

$$\Sigma_k^f = E(f_{t+k} - E_t f_{t+k})(f_{t+k} - E_t f_{t+k})' = U\Sigma_k^s U'$$

and

$$\Sigma^f = \lim_{k\to\infty} \Sigma_k^f = U\Sigma^s U'.$$

Finally, note that

$$\begin{bmatrix} \ln(Y_t) - \ln(Y_{t-1}) - \ln(z) \\ \ln(\Pi_t) - \ln(\Pi_{t-1}) \\ \ln(R_t) - \ln(R_{t-1}) \end{bmatrix} = \begin{bmatrix} U_2 \\ U_3 \\ U_4 \end{bmatrix} s_t.$$

Hence,

$$\ln(Y_{t+k}) = \ln(Y_t) + k\ln(z) + U_2 \sum_{j=1}^{k} s_{t+j},$$

$$\ln(\Pi_{t+k}) = \ln(\Pi_t) + U_3 \sum_{j=1}^{k} s_{t+j},$$

and

$$\ln(R_{t+k}) = \ln(R_t) + U_4 \sum_{j=1}^{k} s_{t+j}.$$

Consequently,

$$\ln(Y_{t+k}) - E_t \ln(Y_{t+k}) = U_2 \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}),$$

$$\ln(\Pi_{t+k}) - E_t \ln(\Pi_{t+k}) = U_3 \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}),$$

and

$$\ln(R_{t+k}) - E_t \ln(R_{t+k}) = U_4 \sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j}).$$

And hence

$$\Sigma_k^Y = E[\ln(Y_{t+k}) - E_t \ln(Y_{t+k})][\ln(Y_{t+k}) - E_t \ln(Y_{t+k})]' = U_2 \Sigma_k^S U_2',$$
$$\Sigma_k^\Pi = E[\ln(\Pi_{t+k}) - E_t \ln(\Pi_{t+k})][\ln(\Pi_{t+k}) - E_t \ln(\Pi_{t+k})]' = U_3 \Sigma_k^S U_3',$$

and

$$\Sigma_k^R = E[\ln(R_{t+k}) - E_t \ln(R_{t+k})][\ln(R_{t+k}) - E_t \ln(R_{t+k})]' = U_4 \Sigma_k^S U_4',$$

where

$$\begin{aligned}
\Sigma_k^S &= \left[\sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j})\right] \left[\sum_{j=1}^{k} (s_{t+j} - E_t s_{t+j})\right]' \\
&= \left[\sum_{j=1}^{k} \sum_{l=0}^{j-1} A^l B \varepsilon_{t+j-l}\right] \left[\sum_{j=1}^{k} \sum_{l=0}^{j-1} A^l B \varepsilon_{t+j-l}\right]' \\
&= [B\varepsilon_{t+k} + (I + A)B\varepsilon_{t+k-1} + (I + A + A^2)B\varepsilon_{t+k-2} + ... + (I + A + ... + A^{k-1})B\varepsilon_{t+1}] \\
&\quad \times [B\varepsilon_{t+k} + (I + A)B\varepsilon_{t+k-1} + (I + A + A^2)B\varepsilon_{t+k-2} + ... + (I + A + ... + A^{k-1})B\varepsilon_{t+1}]' \\
&= BVB' + (I + A)BVB'(I + A)' + (I + A + A^2)BVB'(I + A + A^2)' \\
&\quad + ... + (I + A + ... + A^{k-1})BVB'(I + A + ... + A^{k-1})'
\end{aligned}$$

Let Θ denote the vector of estimated parameters, and let H denote the covariance matrix of these estimated parameters, so that asymptotically,

$$\Theta \sim N(\Theta^0, H).$$

Note that the elements of Σ_k^s, Σ^s, Σ_k^f, Σ^f, Σ_k^Y, Σ_k^Π, and Σ_k^R can all be expressed as nonlinear functions of Θ:

$$\Sigma = g(\Theta),$$

so that asymptotic standard errors for these elements can be found by calculating

$$\nabla g H \nabla g'.$$

In practice, the gradient ∇g can be evaluated numerically, as suggested by Runkle (1987).

4.2 Producing Smoothed Estimates of the Shocks

Hamilton (Ch.13, Sec.6, pp.394-397) shows how to generate a sequence of smoothed estimates $\{\hat{s}_{t|T}\}_{t=1}^{T}$ of the unobservable state vector, where

$$\hat{s}_{t|T} = E(s_t|d_T, d_{T-1}, ..., d_1).$$

As before, for $t = 1, 2, ..., T$ and $j = 0, 1$, let

$$\hat{s}_{t|t-j} = E(s_t|d_{t-j}, d_{t-j-1}, ..., d_1),$$

$$\Sigma_{t|t-j} = E(s_t - \hat{s}_{t|t-j})(s_t - \hat{s}_{t|t-j})',$$

and

$$\hat{d}_{t|t-j} = E(d_t|d_{t-j}, d_{t-j-1}, ..., d_1).$$

Also as before, let

$$u_t = d_t - \hat{d}_{t|t-1} = C(s_t - \hat{s}_{t|t-1})$$

so that again,

$$Eu_t u_t' = C\Sigma_{t|t-1}C'.$$

Then

$$\hat{s}_{1|0} = Es_1 = 0_{(9\times 1)}$$

and

$$vec(\Sigma_{1|0}) = vec(Ex_1x_1') = [I_{(81\times 81)} - (A \otimes A)]^{-1}vec(BVB').$$

From these starting values, the sequences $\{\hat{s}_{t|t}\}_{t=1}^{T}$, $\{\hat{s}_{t|t-1}\}_{t=1}^{T}$, $\{\Sigma_{t|t}\}_{t=1}^{T}$, and $\{\Sigma_{t|t-1}\}_{t=1}^{T}$ can be generated recursively using

$$u_t = d_t - C\hat{s}_{t|t-1},$$

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}u_t,$$

$$\hat{s}_{t+1|t} = A\hat{s}_{t|t},$$

$$\Sigma_{t|t} = \Sigma_{t|t-1} - \Sigma_{t|t-1}C'(C\Sigma_{t|t-1}C')^{-1}C\Sigma_{t|t-1},$$

and

$$\Sigma_{t+1|t} = BVB' + A\Sigma_{t|t}A'$$

for $t = 1, 2, ..., T$.

Now, to begin, construct a sequence $\{J_t\}_{t=1}^{T-1}$ using Hamilton's equation (13.6.11):

$$J_t = \Sigma_{t|t}A'\Sigma_{t+1|t}^{-1}.$$

Then note that $\hat{s}_{T|T}$ is just the last element of $\{\hat{s}_{t|t}\}_{t=1}^{T}$. From this terminal condition, the rest of the sequence can be generated recursively using Hamilton's equation (13.6.16):

$$\hat{s}_{T-j|T} = \hat{s}_{T-j|T-j} + J_{T-j}(\hat{s}_{T-j+1|T} - \hat{s}_{T-j+1|T-j})$$

for $j = 1, 2, ..., T-1$. Kohn and Ansley (1983) show that in cases where $\Sigma_{t+1|t}$ turns out to be singular, its inverse can be replaced by its Moore-Penrose pseudoinverse in the expression for J_t.