# Perpetual Learning and Stability in Macroeconomic Models*

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December 19, 2014

Preliminary and incomplete – not for distribution

## 1 Introduction

Increasingly, researchers incorporate adaptive learning into dynamic macroeconomic models. The typical approach involves endowing (public or private) agents with a perceived law of motion (PLM) – a forecasting model – through which they form expectations. These expectations, in turn, affect the aggregate state variables via the self-referential feature of the models. In real-time, the agents update the coefficients of their PLM in light of new data using one of a variety of learning rules. If the asymptotic stability of the model's rational expectations equilibrium is of interest then "decreasing gain" estimation algorithms, like recursive least squares, are employed; if the learning dynamics themselves are the focus then "constant gain" estimation – a version of discounted recursive least squares – may be more appropriate .

For distinct, but related reasons, both decreasing- and constant- gain learning algorithms typically require the use of a "projection facility," which, in a sense, reini-

*We thank our discussant, Cars Hommes, for his patience and exceedingly kind discussion.

tializes the algorithm if the realized data push the estimates into regions of instability. For decreasing gain algorithms, projection facilities are needed to apply the general theorems governing asymptotics: unless global stability can be demonstrated, projection facilities are necessary to guarantee almost-sure convergence. The use of projection facilities with constant gain learning (CGL) algorithms arises more from practical considerations than from formal theory. Because CGL algorithms place constant weight on new data, they are perpetually susceptible to unusual sequences of shocks. In macroeconomic models, these shocks may push the CGL estimates into regions where feedback leads to instability in the form of explosive paths for endogenous state variables. Projection facilities provide a mechanism through which these explosive paths may be tamed and stability recovered.

Applications of constant gain learning have been successful at improving the empirical fit of many business cycle, monetary, and asset pricing models. See, for example, (Branch and Evans 2011); (Sargent 1999); (Orphanides and Williams 2005); (Cho, Williams, and Sargent 2002); (Williams 2004); (Cho and Kasa 2008); (McGough 2006); (Eusepi and Preston 2011). However, in practice, for the reasons just mentioned, many of these constant gain learning models are augmented to include a projection facility – or, even to throw out the results from a particular simulation – to rule out explosive dynamics. The need for a projection facility is particularly acute in serially correlated settings where agents include lagged endogenous variables in their regression model. A robust feature is that for a large enough constant gain, there will be occasional recurring escapes from the stable region unless a sufficiently tight projection facility is imposed.

Applied researchers face a trade-off when specifying an adaptive learning rule. On the one hand, constant gain learning delivers improved empirical fit to macro/asset-pricing models because learning introduces (additional) serial correlation that would not exist if individuals had rational expectations. Moreover, a key feature of learning models is that escapes to (nearly) self-fulfilling equilibria with high degrees of serial correlation occur recurrently under constant gain learning: see, for instance, (Branch and Evans 2011), (Branch and Evans 2014), (Sargent 1999), (Hommes and Zhu 2012), (Lansing 2009). On the other hand, for reasonable gain sizes, the same forces that pull agents' beliefs towards greater serial correlation also can push the dynamics outside the equilibrium's basin of attraction: thus, the need for a projection facility. However, too tight a projection facility may restrict important learning dynamics and too loose of a bound might bias quantitative results with unstable learning dynamics.

One interpretation of a projection facility is that agents ignore data that do not align with their prior belief that the economic system is stable or stationary. However, a projection facility is an *ad hoc* way in which to impose these priors: why wouldn't agents simply abandon their learning model? The potential instability of recursive least-squares in estimating vector auto regressions has been well-known in the statistics and engineering literatures. An alternative procedure for estimating the autoregressive coefficients is to make use of the Yule-Walker equations combined with sample estimates of the autocorrelation coefficients. This approach was first suggested in economics by Hommes and Sorger (1998), and a recursive version implemented in Hommes and Zhu (2013), which they called "sample autocorrelation coefficient learning" (SAC) (Also, applied by Lansing.) In this paper, we generalize sample autocorrelation coefficient learning to multivariate settings and demonstrate how the use of a constant gain Yule-Walker estimator can lead to stable dynamics without the use of a projection facility.

This paper proposes a learning model, based on the Yule-Walker estimator for vector autoregressive models, that is consistent with stability by construction. Because the Yule-Walker estimator is based on recovering parameter estimates from the sample autocorrelation coefficients, the relevant eigenvalues are bounded to lie strictly within the unit circle. We begin by constructing a Yule-Walker estimator for both decreasing and constant gain sequences. Our main theorem is that, for any appropriate stochastic process, the estimated coefficients correspond to stationary belief.[1] Thus, Yule-Walker learning is appropriate for models where agents believe that the underlying process is stationary. This makes explicit the priors held by agents in learning models and avoids making quantitative results sensitive to the precise details of the projection facility.

Our leading example is a New Keynesian model with habit persistence and price-indexation. We show that under constant gain learning, explosive dynamics are a pervasive issue, particularly for constant gain values that are in line with the survey evidence in Branch and Evans (2011). However, under constant gain Yule-Walker learning the system remains dynamically stable with belief coefficients distributed around the rational expectations equilibrium values. Therefore, the learning model under consideration here can yield substantial improvements for applied models of adaptive learning.

This paper proceeds as follows. Section 2 presents a motivating example. Section 3 presents our main theorem on the boundedness of Yule-Walker learning. Section 4 presents applications and examples, while section 5 concludes.

[1]Here, an "appropriate" stochastic process is one which delivers realized time paths almost surely satisfying the conditions of Theorem 1 below. As we argue in the discussion following the theorem, these conditions are very weak.

## 2 Motivating Example

As a means of motivating the Yule-Walker estimator in adaptive learning models, we begin with the following simple, univariate, reduced-form model:

$$y_t = \alpha + \beta E_t y_{t+1} + \delta y_{t-1} + \varepsilon_t, \tag{1}$$

where $\varepsilon_t$ is white noise with variance $\sigma_\varepsilon^2$. Detailed treatment of stability under learning in models of this form are provided by (Evans and Honkapohja 2001) (see Chp. 8.6.2). The minimal state variable (MSV) rational expectations equilibrium solutions to (1) are

$$y_t = \bar{c} + \bar{a} y_{t-1} + \varepsilon_t, \tag{2}$$

where $\bar{c} = \alpha / \left(1 - \beta(1 + \bar{a})\right)$ and $\bar{a}$ satisfies the quadratic equation

$$\beta \bar{a}^2 - \bar{a} + \gamma = 0,$$

i.e. $\bar{a} = \left(1 \pm \sqrt{1 - 4\beta\delta}\right) / 2\beta$.

The adaptive learning approach assumes that agents form expectations via a forecasting model, the perceived law of motion (PLM), that is of the same form as the rational expectations equilibrium of interest. In this case, agents' PLM is

$$y_t = c + a y_{t-1} + \varepsilon_t. \tag{3}$$

Adopting the conventional timing assumption that contemporaneous state variables are not observable ("t-1" dating), expectations are computed as

$$E_t y_{t+1} = a(1 + b) + b^2 y_{t-1}. \tag{4}$$

Imposing these expectations into (1) leads to the actual law of motion (ALM), implied by the PLM,

$$\begin{aligned} y_t &= \alpha + \beta c(1 + a) + \left(\beta a^2 + \delta\right) y_{t-1} + \varepsilon_t & (5) \\ &\equiv T(\theta)' x_t + \varepsilon_t, & (6) \end{aligned}$$

where $x_t' = (1, y_{t-1})$ and $\theta' = (c, a)$. Notice that the fixed points to the map $T$ correspond to the MSV rational expectations equilibria. The T-map, which takes perceived coefficients $(c, a)$ to the actual coefficients under the ALM, plays an important role in analyses of the expectational stability ("E-stability") of rational expectations equilibria. In particular, the T-map can be interpreted as follows. If agents held beliefs in the form of the PLM (3), with coefficients held constant at non-RE values, then their forecasting model would be (4). The resulting stochastic process for $y_t$, (5), leads to a best-fitting linear model of the same form as the PLM with actual coefficients $T(\theta)$. It turns out that one can derive stability conditions under least-squares learning by studying the ode

$$\frac{d\theta}{d\tau} = T(\theta) - \theta$$

where $\tau$ is notional time, which, nonetheless, can be linked to real time $t$. The E-stability Principle states that Lyuapunov stable rest points of the ode are locally stable under least squares learning and other closely related learning algorithms. In this particular case the E-stability conditions are $\delta\beta\,(1-\beta\bar{a})^{-2} < 1$ and $\beta\,(1-\beta\bar{a})^{-1} < 1$.

Under real-time learning the parameters $(c, a)$ are updated over time, e.g. with least squares, in response to new data. Now, rather than assuming $(c, a)$ are fixed we assume that agents enter time $t$ with their last parameter estimates $(c_{t-1}, a_{t-1})$ from which they update their forecasting model (3). The resulting (temporary) equilibrium law of motion then is (5) with $(c, a)$ replaced by $(c_{t-1}, a_{t-1})$. After observing the new data point $y_t$, agents update their parameter estimates using recursive least-squares:

$$\begin{aligned} \theta_t &= \theta_{t-1} + \gamma_t R_t^{-1} x_t \left(y_t - \theta'_{t-1} x_t\right) && (7) \\ R_t &= R_{t-1} + \gamma_t \left(x_t x'_t - R_{t-1}\right) && (8) \end{aligned}$$

The equations in (7)-(8) are the updating equations for recursive least squares where the data are discounted by a "gain" $\gamma_t$. Least-squares updating arises when the gain is a decreasing gain sequence $\gamma_t = t^{-1}$. If the gain is constant over time, i.e. $\gamma_t = \gamma$, then these equations are called constant gain learning, a form of discounted least squares where the data are discounted by $1-\gamma$. (Branch and Evans 2006) find values for $\gamma \in [0.01, 0.05]$ provide a good fit to expectations in the Survey of Professional Forecasters.

In (Evans and Honkapohja 2001), it is shown that if an MSV solution is stationary and E-stable, then it is locally stable under recursive least-squares learning, i.e. when $\gamma_t = t^{-1}$. In particular, one can establish that $(c_t, a_t)$, updated by (7)-(8), converges to $(\bar{c}, \bar{a})$ with probability 1, provided that the associated MSV solution is stationary and E-stable and, importantly, that the learning algorithm is supplemented with a projection facility. The projection facility may be constructed as follows: define $\phi'_t = (\theta_t, vec(R_t))$. Let $D \subset \mathbb{R}^d$ ($d = 6$, in this example) be an open set around the equilibrium of interest. Let $K(r)$ be the closed ball of radius $r > 0$ centered at the REE. The RLS algorithm is augmented so that for some $0 < c_1 < c_2$, with $K(c_2) \subset D$, $(a, b)$ are updated by the RLS algorithm provided $\phi \in int(K(c_2))$; otherwise, $\phi_t$ is projected to some point in $K(c_1)$. A common way this is employed, in simulations, is to restrict parameter estimates so that the forecasting equation is stationary.

In contrast to the decreasing gain algorithms just discussed, constant gain learning (CGL) algorithms geometrically discount past data relative to new data: when $\gamma_t = t^{-1}$ weights on data points are redistributed uniformly each period so that the importance of new information goes quickly to zero; when $\gamma_t = \gamma \in (0, 1)$, the learning algorithm places a cumulative weight of $1 - \gamma$ on past data, and a weight of $\gamma$ on data realized in period $t$. For this reason, and again in contrast to the behavior of decreasing gain algorithms, the variance of estimators associated to constant gain learning do not, in general, converge to zero; and, in particular, these estimators may respond sensitively to period $t$ shocks, even for arbitrarily large $t$.[2]

To motivate the more general work provided in the sequel, we assess the behavior of CGL in the model at hand; and for simplicity, we consider the special case in which agents know $\alpha = 0$ and thus use a forecasting model of the form $y_t = ay_{t-1} + \varepsilon_t$. This assumption implies our estimator is univariate. Given data $\{y_0, \ldots, y_t\}$, the constant gain LS estimator $a_t^{LSC}$ is given by

$$\begin{aligned} R_1 &= \gamma y_0^2 \text{ and } a_1^{LSC} = y_1/y_0, \\ R_t &= R_{t-1} + \gamma \left(y_{t-1}^2 - R_{t-1}\right) \\ a_t^{LSC} &= a_{t-1}^{LSC} + \gamma R_t^{-1} y_{t-1} \left(y_t - a_{t-1}^{LSC} y_{t-1}\right). \end{aligned}$$

Coupled with the model (1) and the PLM just described, the data-generating process (DGP) is given by

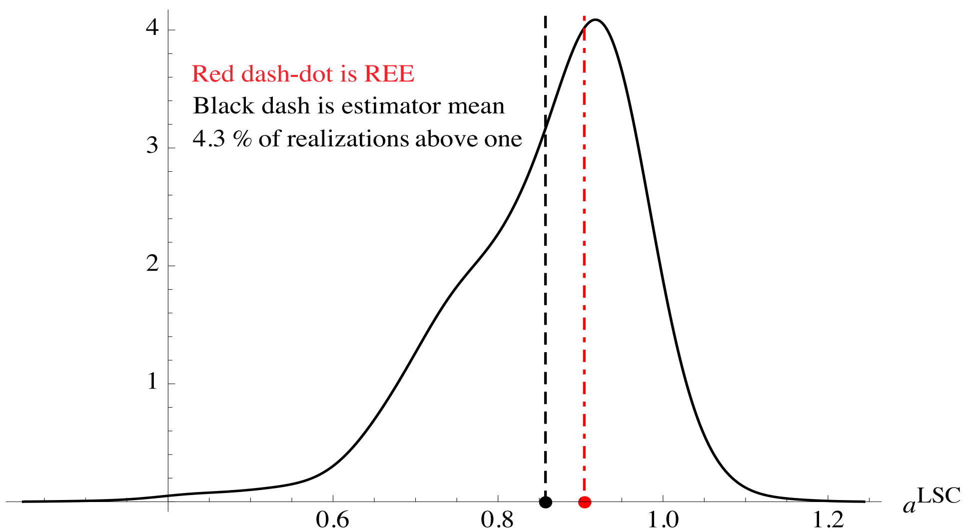
$$y_t = (\beta \left(a_{t-1}^{LSC}\right)^2 + \delta) y_{t-1} + \varepsilon_t,$$

where $a_{t-1}^{LSC}$ is determined using information available in time $t - 1$.

We now simulate the model using parameter values $\beta = .25$ and $\delta = .7$, and with $\gamma = .04$. The corresponding REE correlation coefficient is approximately .91. The learning algorithm is initialized at the REE and no projection facility is employed. After tossing transients, we may plot the estimated distribution for the corresponding realizations of the estimator: see Figure 1. The mean of the distribution is denoted by the black dot and corresponding black, dashed line, and the REE is marked by the red dot and red, dash-dot line. That the mean is below the REE reflects the downward bias of LS in small samples, which is exacerbated in the constant gain environment. Even with this downward bias, we observe a very important phenomenon: persistent realizations of the estimator above one. Indeed for this simulation, 4.3% of the estimator's realizations were above unity. Two conclusions may be drawn: first, if the natural projection facility discussed above, which corresponds to tossing any

[2]Under certain conditions, which include E-stability of the REE, as the gain goes to zero the CGL estimator converges weakly to a normal distribution centered on the REE.

Figure 1: Distribution of $a^{LSC}$




estimate above one, were employed, the realized time-series of this economy would be significantly impacted; and second, while, due in part to its simplicity (quadratic T-map) this model is unusually stable, in more complex models, estimator realizations above one would very likely lead to explosive behavior.

An alternative to LS learning that is particularly appealing in the case under consideration (and that, we will argue, in fact has quite broad appeal) is Hommes' SAC learning as introduced above. Besides its inherent simplicity, an obvious advantage is that, as an estimator, the sample autocorrelation necessarily lies in $(-1, 1)$, and thus precludes the need for a projection facility to prevent explosive estimates.

While the SAC estimator uses uniform weights (and thus may be interpreted as resulting from a decreasing gain algorithm) there is a constant gain counterpart. In the next section we will develop this counterpart in detail, as well as present and analyze a multidimensional version of SAC learning, which, as mentioned above, we refer to as Yule-Walker- or YW-learning. Here, as motivation, we compare constant gain YW-learning to constant gain LS-learning as just presented and detailed in Figure 1.

To promote comparison, we first write the constant gain LS-estimator in the following non-recursive form:

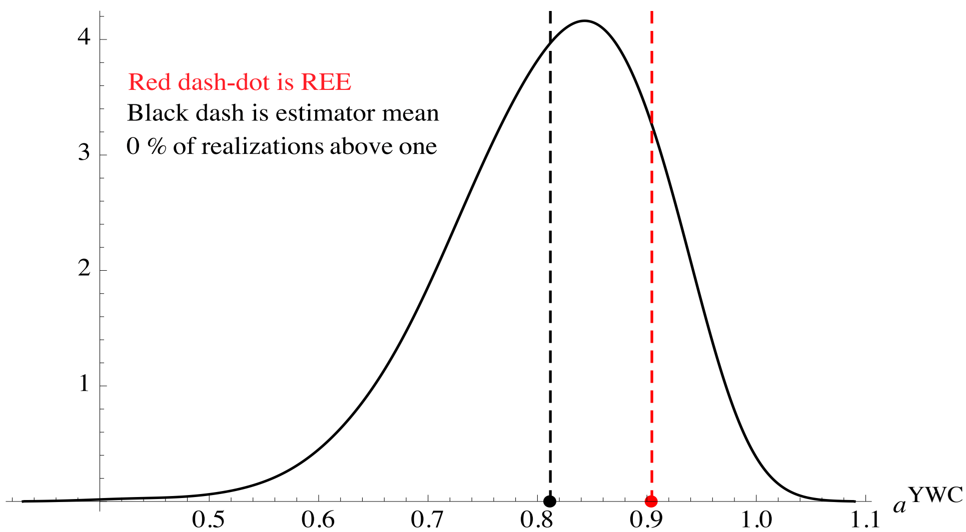
$$a_t^{LSC} = \frac{\sum_{n=1}^{t} \gamma(1-\gamma)^{t-n} y_n y_{n-1}}{\sum_{n=1}^{t} \gamma(1-\gamma)^{t-n} y_{n-1}^2}. \tag{9}$$

As we will discuss in the next section, the constant gain YW estimator may be written

$$a_t^{YWC} = \left(\sqrt{1-\gamma}\right) \frac{\sum_{n=1}^{t} \gamma(1-\gamma)^{t-n} y_n y_{n-1}}{\sum_{n=0}^{t} \gamma(1-\gamma)^{t-n} y_n^2}. \tag{10}$$

There are two principal differences between the estimators: the denominator of the time $t$ LS estimator does not include a dependence on $y_t$; and the YW estimator includes scaling factor $\sqrt{1-\gamma}$, which is analogous to the usual "degrees-of-freedom" correction in the uniform weights case. Theorem 1 shows that $a_t^{YWC} \in (-1, 1)$: the constant gain YW-learning algorithm never provides explosive estimates. The Yule-Walker analog to Figure 1 is given in Figure 2 below. Two observations are warranted:

Figure 2: Distribution of $a^{YWC}$




first, the distribution's mean is below the REE, just as in the LS case, and for the same reason: the YW-estimate is downward biased. In fact, it is somewhat more bias that the LS estimator in small samples, which is reflected in the fact that the distribution's mean is farther below the REE than in the LS case (though, of course, this may also in part reflect that the distributions graphed in Figures 1 and 2 are themselves estimated). Second, the maximum value of the estimator is strictly below one.[3]

The use of projection facilities, in both the theoretical and empirical literature, has been criticized as being *ad hoc*: there is no unique way to augment the learning

[3]The estimated distribution appears to place positive weights on realizations above one because a smooth kernel density estimator was used to create the figure.

rule with a projection facility. In empirical and quantitative applications one concern is that the tightness of the projection facility is a critical parameter in the empirical fit of the learning model. The example just considered suggests constant gain YW-learning, as based on the insights of Hommes and coauthors, may provide a natural alternative. In this paper we explore this suggestion in detail and in the multivariate setting. We find that replacing recursive least-squares with this alternative estimator directly imposes the prior belief that the stochastic process is stationary. Under Yule-Walker learning, the estimated autoregressive coefficient is recovered using sample autocorrelation, and thus has eigenvalues bounded to lie strictly within the unit circle. Further, this result continues to hold in the constant gain case, which makes YW-learning particularly attractive for empirical work.

# 3 The Yule-Walker approach

To generalize the SAC-learning algorithm to multivariate environments, we first consider in isolation (i.e. without reference to an underlying macro model) the VAR model

$$y_t = C + Ay_{t-1} + \varepsilon_t, \tag{11}$$

where $y \in \mathbb{R}^m$ and the matrices are conformable. Provided that the process (11) is stationary, efficient estimates for $C$ and $A$ may be obtained using a least-squares procedure, which has the following recursive form:

$$\begin{aligned} \Theta_t &= \Theta_{t-1} + \gamma_t R_t^{-1} X_t \left(y_t - \Theta_{t-1}' X_t\right)' \\ R_t &= R_{t-1} + \gamma_t \left(X_t X_t' - R_{t-1}\right), \end{aligned}$$

where $\Theta_t' = (C, A)$, and $X_t' = \left(1, y_{t-1}'\right)$.

Least-squares estimation breaks down when $Y_t$ is not stationary, which may be relevant in self-referential models: when the realized values of $Y_t$ depend on expectations formed from the estimated VAR it is possible to enter into feedback loops that drive $Y_t$ to be not-stationary and explosive. As discussed in the previous section, the typical way to deal with these situations is to use a projection facility that says, for example, use least-squares to estimate $A$ so long as the roots of $A$ are inside the unit circle and to project the estimates inside the stability region when the roots are outside the unit circle. Of course, there is no unique way to construct such a projection facility. Moreover, there may be good economic reasons to expect the learning dynamics to be drawn near this boundary, so that a constant gain estimator that discounts past observations may overweight these data and push the estimates outside the stability region. Of particular concern is that quantitative results in adaptive learning models may be influenced by the projection facility.

To provide an alternative to the use of projection facilities, as well as embrace the appealing simplicity of sample estimation, as emphasized by Hommes, we examine Yule-Walker estimation of VAR models. We show how this procedure leads to a direct multivariate extension of SAC-learning and that this extension preserves the desired stability features: the Yule-Walker estimates of $A$ always have eigenvalues with modulus less than one.

## 3.1 Yule-Walker estimation

Consider again the empirical forecasting model (11), which here we take to be the data generating process; and assume further that this process is stationary, so that $\det(I_m - A) \neq 0$. Define $\hat{y}_t = y_t - E(y_t)$, so that $\hat{y}_t = A\hat{y}_{t-1} + \varepsilon_t$. Noting $E(y_t) = (I_n - A)^{-1}C$, it follows that

$$A = Corr(y_t, y_{t-1}) = E\left(\hat{y}_t\hat{y}_{t-1}'\right) E\left(\hat{y}_t\hat{y}_t'\right)^{-1} \tag{12}$$

$$C = \left(I_m - Corr(y_t, y_{t-1})\right) E(y_t) = \left(I_m - E\left(\hat{y}_t\hat{y}_{t-1}'\right) E\left(\hat{y}_t\hat{y}_t'\right)^{-1}\right) E(y_t). \tag{13}$$

These equations may be viewed as identifying the DGP parameters $A$ and $C$ as functions of first and second population moments: $A$ is the first-order autocorrelation and $C$ is determined via this autocorrelation and the population mean. Using (12) and (13), we may apply the method-of-moments to provide estimates of $A$ and $C$: this application is Yule-Walker (YW) estimation of the VAR (11).

We now explicitly develop YW estimation. Here we back off the explicit assumption that the DGP is given by (11), and instead begin with a sequence of data $y^T = \{y_n\}_{n=0}^T$, with $y_n \in \mathbb{R}^m$. We imagine these data are available to an economic agent who needs to forecast $y_{T+k}$. Adopting the view of the literature on adaptive learning, we assume that the agent believes these data are generated by a model of the form (11), and bases his estimation procedure on these beliefs; however, instead of using LS, he anchors his estimation to following standard sample estimates:

$$m_T = \frac{1}{T+1}\sum_{t=0}^{T} y_t, \quad v_T^2 = \frac{1}{T+1}\sum_{i=0}^{T}(y_i - m_T)(y_i - m_T)', \quad \text{and}$$

$$\hat{v}_T^2 = \frac{1}{T+1}\sum_{i=1}^{T}(y_i - m_T)(y_{i-1} - m_T)'.$$

The YW-estimates $(A_T, C_T)$ of $(A, C)$ are given by

$$A_T = \left(\hat{v}_T^2\right)\left(v_T^2\right)^{-1} \text{ and } C_T = (I_m - A_T)m_T. \tag{14}$$

Notice that this estimator is well-defined even if $\det((I_m - A_T)) = 0$. We have the following theorem:

**Theorem 1** *Given data $y^T$ assume $A_T$, as determined by (14), is diagonalizable, and that there exists $n \in \{0, \ldots, T-1\}$ so that $y_n \neq y_{n+1}$. Then the eigenvalues of $A_T$ have modulus less than one.*

Two remarks are in order. First: the conditions needed for this theorem to hold are quite weak. The collection of diagonalizable matrices is open and dense: unless the data are generated by a process that allows for no variation along very specific dimensions, $A_T$ is almost surely diagonalizable. Second, while this result seems to be known, we are unable to find the precise statement in the literature. For this reason, we include our proof of the result in the appendix.

## 3.2 Yule-Walker learning

With the YW-estimator in hand, we now return to a self-referential framework induced by a macroeconomic model. To anchor the discussion, we again consider the model (1), repeated here for convenience, only now we imagine that $y \in \mathbb{R}^m$:

$$y_t = \alpha + \beta E_t y_{t+1} + \delta y_{t-1} + \varepsilon_t. \tag{15}$$

Also as above, the PLM is given by

$$y_t = C + A y_{t-1} + \varepsilon_t,$$

and it will be helpful to write the corresponding ALM as

$$y_t = T(A, C) x_t + \varepsilon_t,$$

where $x_t = (1, y_{t-1}')'$.

Efficient numerical work as well as asymptotic analysis benefits greatly from a recursive formulation, and so we work towards that goal now. Define the following sample second moments:

$$s_t^2 = \frac{1}{t+1}\sum_{i=0}^{t} y_i y_i' \text{ and } \hat{s}_t^2 = \frac{1}{t+1}\sum_{i=1}^{t} y_i y_{i-1}'.$$

These sample moments, together with the sample mean $m_t$, may be written recursively as

$$
\begin{aligned}
m_t &= m_{t-1} + \frac{1}{1+t}\left(y_t - m_{t-1}\right), \quad \text{with } \; m_1 = \frac{1}{2}\left(y_1 + y_0\right) \\
s_t^2 &= s_{t-1}^2 + \frac{1}{1+t}\left(y_t y_t' - s_{t-1}^2\right), \quad \text{with } \; s_1^2 = \frac{1}{2}\left(y_1 y_1' + y_0 y_0'\right) \\
\hat{s}_t^2 &= \hat{s}_{t-1}^2 + \frac{1}{1+t}\left(y_t y_{t-1}' - \hat{s}_{t-1}^2\right), \quad \text{with } \; \hat{s^2}_1 = \frac{1}{2}\left(y_1 y_0'\right).
\end{aligned}
$$

Next, observe that by expanding the various products $(y_i - m_t)(y_i - m_t)'$, we have

$$
v_t^2 = s_t - m_t m_t'.
$$

Similarly, a more involved computation shows

$$
\begin{aligned}
\hat{v}_t^2 &= \frac{1}{t+1}\sum_{n=1}^{t}(y_n - m_t)(y_{i-1} - m_t)' \\
&= \frac{1}{t+1}\sum_{i=0}^{t} y_i y_i' - \left(\frac{t+2}{t+1}\right) m_t m_t' + \left(\frac{1}{t+1}\right)\left(m_t y_t' + y_0 m_t'\right) \\
&= \hat{s}_t^2 - m_t m_t' + \left(\frac{1}{t+1}\right)\rho(\star_t), \qquad (16)
\end{aligned}
$$

where here and in the sequel, $\rho(\star_t)$ will represent a function that depends on $t$-dated variables.[4]

We may use the sample moments just established to compute the time $t$ estimates of the PLM's coefficients:

$$
\begin{aligned}
A_t &= \left(\hat{s}_t^2 - m_t m_t'\right)\left(s_t^2 - m_t m_t'\right)^{-1} + \left(\frac{1}{t+1}\right)\rho(\star_t) = \hat{A}_t + \left(\frac{1}{t+1}\right)\rho(\star_t) \quad (17) \\
C_t &= (I_m - A_t)m_t = (I_m - \hat{A}_t)m_t + \left(\frac{1}{t+1}\right)\rho(\star_t) = \hat{C}_t + \left(\frac{1}{t+1}\right)\rho(\star_t) \quad (18)
\end{aligned}
$$

where the final equalities in each line define notation. Also, note that here we abuse our $\rho$ notation: the $\rho(\star_t)$'s in (16), (17) and (18) are distinct functions. We continue this abuse below. This is for notational convenience: for reasons that will become clear, the terms collected into the various $\rho$ functions have no impact on our asymptotic analysis.

[4] We note that $\rho$ also depends on the initial condition.

Via the T-map, we conclude that the ALM is given by

$$y_t = T(\hat{A}_{t-1}, \hat{C}_{t-1})x_t + \varepsilon_t + \left(\frac{1}{t+1}\right)\hat{\rho}_t(\star_t).$$

Two observations are needed. First, the place-holding function $\rho$ is now modified with a hat. We need to track this particular function as it will have impact on out asymptotic analysis, and thus we give it a special name. Second, $\hat{\rho}$ depends explicitly on time. This is not important, and below various generic place-holding functions will exhibit explicit time dependence.

With this ALM, the recursions for $m$, $s^2$ and $\hat{s}^2$ may be closed. Writing

$$T(\hat{A}_t, \hat{C}_t)x_t = T_C(\hat{A}_t, \hat{C}_t) + T_A(\hat{A}_t, \hat{C}_t)y_{t-1},$$

we have

$$\begin{aligned}
m_t &= m_{t-1} + \frac{1}{1+t}\left(T(\hat{A}_{t-1}, \hat{C}_{t-1})x_t - m_{t-1} + \varepsilon_t\right) + \left(\frac{1}{t+1}\right)^2 \rho_t(\star_t) \\
s_t^2 &= s_{t-1}^2 + \frac{1}{1+t}\left(T(\hat{A}_{t-1}, \hat{C}_{t-1})x_t x_t' T(\hat{A}_{t-1}, \hat{C}_{t-1})' - s_{t-1}^2 + \varepsilon_t\varepsilon_t' + \rho_t(\star_t)\varepsilon_t\right) \\
&\quad + \left(\frac{1}{t+1}\right)^2 \rho_t(\star_t) \\
\hat{s}_t^2 &= \hat{s}_{t-1}^2 + \frac{1}{1+t}\left(T(\hat{A}_{t-1}, \hat{C}_{t-1})x_t y_{t-1}' - \hat{s}_{t-1}^2 + \rho_t(\star_t)\varepsilon_t\right) + \left(\frac{1}{t+1}\right)^2 \rho_t(\star_t) \\
x_{t+1} &= \begin{pmatrix} 1 & 0 \\ T_C(\hat{A}_t, \hat{C}_t) & T_A(\hat{A}_t, \hat{C}_t) \end{pmatrix} x_t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varepsilon_{t+1} + \left(\frac{1}{t+1}\right)\hat{\rho}_t(\star_t),
\end{aligned}$$

where the dependence of $\hat{A}_{t-1}$ and $\hat{C}_{t-1}$ on $s_{t-1}^2, \hat{s}_{t-1}^2$ and $m_{t-1}$ is given in (17) and (18).

With $\Theta = (m, s^2, \hat{s}^2)$, and with $H$, $\mathcal{A}$ and $\mathcal{B}$ appropriately defined, these recursions take the form

$$\Theta_t = \Theta_{t-1} + \frac{1}{1+t}H(\Theta_{t-1}, x_t, \varepsilon_t) + \left(\frac{1}{t+1}\right)^2 \rho_t(\star_t) \tag{19}$$

$$x_{t+1} = \mathcal{A}(\Theta_{t-1})x_t + \mathcal{B}\varepsilon_{t+1} + \left(\frac{1}{t+1}\right)\hat{\rho}_t(\star_t). \tag{20}$$

If $\hat{\rho} = 0$ then the recursive system has the appropriate form needed to apply the usual theorems of Ljung, as found in Evans and Honkapohja (2001). In particular, for (appropriate) fixed $\Theta$, the state dynamics are stationary, and the asymptotic behavior of the recursion (19) is governed by the means dynamics, defined as

$$\frac{d\Theta}{d\tau} = h(\Theta) = \lim_{t\to\infty} EH(\Theta_{t-1}, \varepsilon_t). \tag{21}$$

However, under YW-learning, $\hat{\rho}$ is nonzero, which implies that formal asymptotic analysis will require modification of Ljung's theory. We anticipate that the versions of the theorems will continue to hold: since $\hat{\rho}$ is uniformly bounded the non-stationary term $(t+1)^{-1}\hat{\rho}_t(\star_t)$ vanishes asymptotically; thus it seems likely that the mean dynamics capture the system's asymptotics.

Explicit computation provides that the mean dynamics may be written

$$\begin{aligned}
\frac{dm}{d\tau} &= T_C(\hat{A},\hat{C}) + T_A(\hat{A},\hat{C})\mu - m \\
\frac{ds^2}{d\tau} &= T(\hat{A},\hat{C})\begin{pmatrix} 1 & \mu \\ \mu & \Omega + \mu\mu' \end{pmatrix} T(\hat{A},\hat{C})' + \sigma_\varepsilon^2 - s^2 \\
\frac{d\hat{s}^2}{d\tau} &= T_C(\hat{A},\hat{C})\mu' + T_A(\hat{A},\hat{C})\left(\Omega + \mu\mu'\right) - \hat{s}^2,
\end{aligned} \tag{22}$$

where here we are suppressing the dependence of $\hat{A}$, $\hat{c}$, $\mu$ and $\Omega$ on $(m, s^2, \hat{s}^2)$. Note in particular that $\mu$ and $\Omega$ represent the asymptotic mean and variance, respectively, of $y_t$ under the ALM given beliefs $(m, s^2, \hat{s}^2)$.

Via the notation (21), the right-hand-sides of the equations (22) determines $h$. Now suppose $(A^*, C^*)$ is an REE of (15), and thus fixed point to the T-map. The DGP, then, is $y_t = C^* + A^* y_{t-1} + \varepsilon_t$, from which we may compute the corresponding moments

$$\Theta^* = \left(m^*, (s^2)^*, (\hat{s}^2)^*\right).$$

Straightforward computation shows that $h(\Theta^*) = 0$, as expected: an REE of the model (15) is a rest point of the mean dynamics (22). We conjecture that the stability of this rest point under the mean dynamics governs the local stability of the recursion (19), and thus the stability of YW-learning.

## 3.3 Constant gain Yule-Walker learning

The constant gain YW-estimator, $A^{CG}$, may be developed in a manner analogous to the example of Section 2. For this development, we initially assume the data are known to have zero mean. The non-zero mean case is then provided as a simple generalization. Let $\{y_t\}_{t=0}^T$, be any collection of vectors $y_t \in \mathbb{R}^m$. Let $\gamma \in (0,1)$ and $0 \leq N < T$. Define a sequence $\{\mu_t\}_{t=N}^T$ as follows:

$$\begin{aligned}
\mu_N &= \sum_{n=0}^{N} \gamma(1-\gamma)^{N-n} y_n, \\
\mu_t &= \mu_{t-1} + \gamma(y_t - \mu_{t-1}) \text{ for } t > N.
\end{aligned}$$

We can think of $\mu_N$ as initializing the constant gain $\mu_t$ recursion. Then

$$\mu_t = \sum_{n=0}^{t} \gamma(1-\gamma)^{t-n} y_n$$

This idea can be applied to second-moment matrices. Let

$$\begin{aligned}
Z_N &= \sum_{n=1}^{N} \gamma(1-\gamma)^{N-n} y_n y_{n-1}', \\
Z_t &= Z_{t-1} + \gamma(y_t y_{t-1}' - Z_{t-1}) \text{ for } t > N. \\
V_N &= \sum_{n=0}^{N} \gamma(1-\gamma)^{N-n} y_n y_n' \\
V_t &= V_{t-1} + \gamma(y_t y_t' - V_{t-1}) \text{ for } t > N.
\end{aligned}$$

We can then define the constant-gain YW estimator as $A_T^{CG}(\gamma) = Z_t V_t^{-1}$.

Unfortunately, as written, not much can be said regarding the stability of this estimator. To deal with this concern, we next consider the implications of introducing general unequal weights for the data points. Let $w \in \mathbb{R}^{T+1}$ give the weights and define

$$A(w) = \left( \sum_{i=1}^{T} w_i w_{i-1} y_i y_{i-1}' \right) \left( \sum_{i=0}^{T} w_i^2 y_i y_i' \right)^{-1}. \qquad (23)$$

Since this is simply the YW estimator for the data $\{w_1 y_1, \ldots, w_T y_T\}$, it follows that $A(w)$ is stable. The recursive formulation of the constant gain YW-estimator given above yields

$$A^{CG}(\gamma) = \left( \sum_{i=1}^{T} \gamma(1-\gamma)^{T-i} y_i y_{i-1}' \right) \left( \sum_{i=0}^{T} \gamma(1-\gamma)^{T-i} y_i y_i' \right)^{-1};$$

however, to ensure stability, we adopt a weighted estimator consistent with (23). To this end, let $w_n = \gamma^{\frac{1}{2}}(1-\gamma)^{\frac{T-n}{2}}$. Direct computation provides that

$$A(w) = (1-\gamma)^{\frac{1}{2}} A^{CG}(\gamma) = A^{ACG}(\gamma),$$

where the last equality identifies notation. Therefore, the *adjusted* constant gain estimator $A^{ACG}(\gamma)$ is stable.

For this expression we are implicitly assuming that $y_t$ is zero mean. If not then $y_t$ should have the mean removed if it is known. If the mean is not known then it can be estimated by $\mu_t$ and the YW estimator would be

$$\begin{aligned}
\hat{A}_T^{ACG}(\gamma) &= (1-\gamma)^{\frac{1}{2}}\hat{Z}_T(T)\hat{V}_T(T)^{-1}, \text{ where} \\
\hat{y}_n(T) &= y_n - \mu_T, \\
\hat{Z}_T(T) &= \sum_{n=1}^{T}\gamma(1-\gamma)^{T-n}\hat{y}_n(T)\hat{y}_{n-1}(T)', \\
\hat{V}_T(T) &= \sum_{n=0}^{T}\gamma(1-\gamma)^{T-n}\hat{y}_n(T)\hat{y}_n(T)'.
\end{aligned}$$

## 3.4 Implementing YW-learning

A fully recursive expression for $\hat{A}_T^{ACG}(\gamma)$ is not easily obtained. However, it can be obtained by the following double recursion. The recursions can be started using any $N \geq 1$. We here choose $N = 1$. The outer recursion determines the time $T$ estimate of the mean $\mu_T$:

$$\begin{aligned}
\mu_1 &= \gamma y_1 + \gamma(1-\gamma)y_0, \\
\mu_T &= \mu_{T-1} + \gamma(y_T - \mu_{T-1}) \text{ for } T > 1.
\end{aligned}$$

At each time $T$ the inner recursions are re-initialized and computed using the transformed data $\hat{y}_n(T) = y_n - \mu_T$. The recursions are given by

$$\begin{aligned}
\hat{Z}_1(T) &= \gamma\hat{y}_1(T)\hat{y}_0(T)', \\
\hat{Z}_t(T) &= \hat{Z}_{t-1}(T) + \gamma\left(\hat{y}_t(T)\hat{y}_{t-1}(T)' - \hat{Z}_{t-1}(T)\right) \text{ for } 1 < t \leq T, \\
\hat{V}_1(T) &= \gamma\hat{y}_1(T)\hat{y}_1(T)' + \gamma(1-\gamma)\hat{y}_0(T)\hat{y}_0(T)', \\
\hat{V}_t(T) &= \hat{V}_{t-1}(T) + \gamma\left(\hat{y}_t(T)\hat{y}_t(T)' - \hat{V}_{t-1}(T)\right) \text{ for } 1 < t \leq T.
\end{aligned}$$

Finally, $\hat{A}_T^{ACG}(\gamma) = (1-\gamma)^{\frac{1}{2}}\hat{Z}_T(T)\hat{V}_T(T)^{-1}$.

# 4 Applications

In this section, we compare constant gain least-squares learning with constant gain YW-learning in applied macroeconomic models.

## 4.1 Estimating an exogenous process

In this section, we consider where the data $\{Y_t\}_{t=0}^{N}$ are generated according to the VAR(1):

$$Y_t = C + AY_{t-1} + \varepsilon_t$$

We are interested in the properties of the constant gain YW-learning estimates of $C, A$. To conduct this study, we assume $Y$ is a bivariate VAR(1), set $C' = (0, 0)$ and initially set

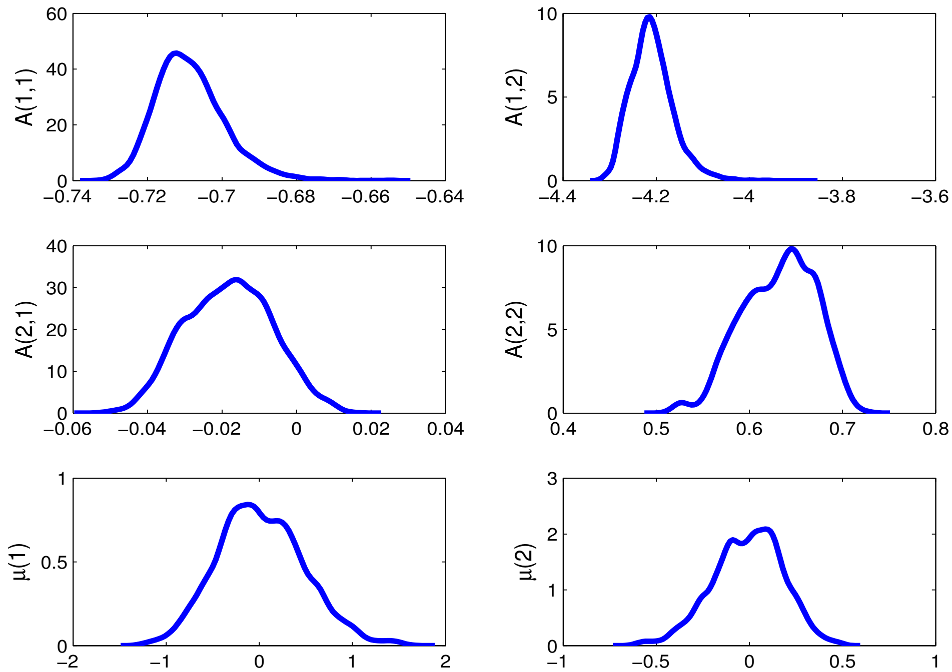
$$A = \begin{pmatrix} -.7125 & -4.2254 \\ -0.0165 & 0.634 \end{pmatrix}$$

These parameters were chosen as they align with the MSV solution to the New Keynesian model considered below. We also consider subsequently the case where $Y$ is non-stationary by setting $A_{22} = 0.99$.

Figures 3-5 plots the kernel density of the estimated YW-coefficients for $C, A$ for each sample-size $T = 1, ..., N$. Notice, in particular, that since this is a weighted estimator the estimates converge in distribution with a mean centered on the true estimates. The top two rows of each figure plot the histogram of the individual elements of $A$, i.e. $A_T(i, j)$. The bottom row plots the histograms of the estimated constants $C_T$. Figure 3 is the case of a small constant gain $\gamma = 0.001$. This figure demonstrates that the parameter estimates are tightly distributed around the true values. In Figure 4, $\gamma = 0.05$, and Figure 5, $\gamma = 0.15$, the parameter estimates are again distributed around the true values but with greater variance. Moreover, in Figures 4-5 the distributions are skewed.

Figures 3-5 show that the constant gain YW-estimator is able to recover the true parameter estimates when the data are generated by and exogenous stationary VAR(1) process. The main theorem, however, shows that the coefficient estimates for $A_T$ remain bounded in the sense that the eigenvalues of $A_T$ have modulus less than one. Figure 6 demonstrates this case by replacing $A_{22} = 0.99$, which produces a maximum eigenvalue of approximately 1.03. In Figure 6, one sample path of $N = 400$ is plotted for the coefficient estimates $C, A$. Additionally, the bottom row plots the largest eigenvalue of the estimated $A_T$. Clearly, the process is explosive which features

Figure 3: YW parameter estimates of exogenous VAR(1): $\gamma = 0.005$.




explosive estimates of the constants $C$ as well as highly volatile estimates for $A$. However, in the southeast quadrant, it is clear that the eigenvalues of $A_T$ remain bounded below one throughout.

## 4.2 New Keynesian Model

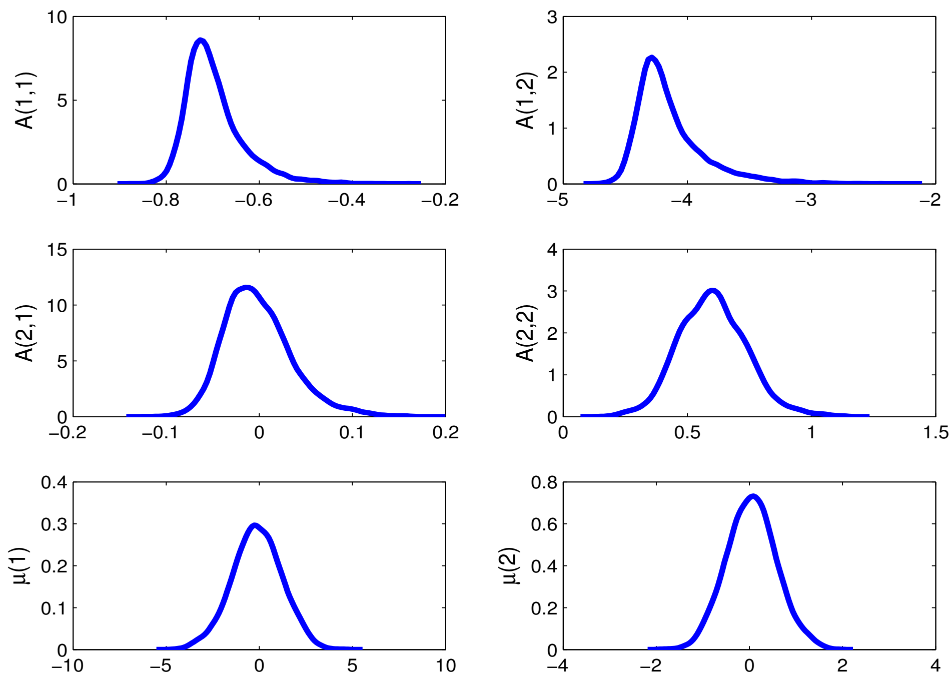
The first example that we consider is the "Hybrid" New Keynesian Model:

$$
\begin{aligned}
x_t &= \delta \hat{E}_t x_{t+1} + (1-\delta)x_{t-1} - \phi\left(i_t - \hat{E}_t \pi_{t+1}\right) + g_t \\
\pi_t &= \beta\left[\vartheta \hat{E}_t \pi_{t+1} + (1-\vartheta)\pi_{t-1}\right] + \lambda x_t + u_t
\end{aligned}
$$

where $x_t$ is the output gap, $\pi_t$ is the inflation rate, $i_t$ is the nominal interest rate, and $g_t, u_t$ are exogenous processes assumed to follow $g_t = \rho_g g_{t-1} + \varepsilon_t$, $u_t = \rho_u u_{t-1} + \eta_t$, $0 \leq \rho_g, \rho_u \leq 1$, and $\sigma_{\varepsilon\eta} = 0$.

Equation (24) is the IS relation derived by log-linearizing the Euler equation for household optimization where households' preferences over consumption display habit

Figure 4: YW parameter estimates of exogenous VAR(1): $\gamma = 0.05$.




persistence. Equation (24) is the hybrid version of the New Keynesian Phillips Curve which is derived from the pricing decisions of firms who face a Calvo price stickiness. The inertial term arises for firms that do not re-set their prices but index them to the past inflation rate. Throughout, we impose $\beta = 1$ so that the weights on the forward- and backward-looking terms in (24) sum to one. As in the simple univariate example, the need for a projection facility with constant gain least-squares is particularly acute in inertial New Keynesian models.

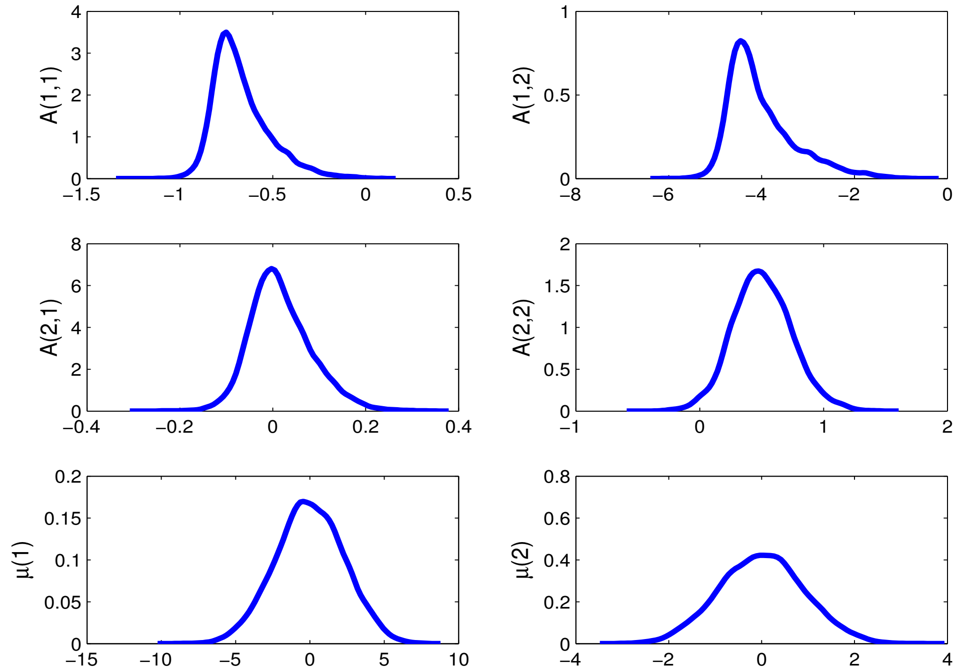
We close the model with a rule for setting the nominal policy rate $i_t$. For illustration purposes, we adopt a backwards-looking Taylor-rule of the form

$$i_t = \alpha_\pi \pi_{t-1} + \alpha_x x_{t-1}$$

Throughout, we choose policy reaction coefficients $\alpha_\pi, \alpha_x$ so that the model is determinate. We specify policy in this way for the same reason that we adopt the Hybrid New Keynesian model: to illustrate the stability of YW-learning *vis a vis* constant gain learning we adopt a specification where the projection facility is invoked frequently in a widely studied framework.

Substituting the policy rule into the IS equation, it is possible to re-write (24)-(24)

Figure 5: YW parameter estimates of exogenous VAR(1): $\gamma = 0.15$.




as,

$$Y_t = M_0 + M_1 \hat{E}_t Y_{t+1} + M_2 Y_{t-1} + M_3 W_t$$

where $Y_t' = (x_t, \pi_t)$ and $W_t' = (g_t, u_t)$. If the model is determinate, then there is a unique non-explosive rational expectations equilibrium of the form

$$Y_t = F + GY_{t-1} + HW_t$$

for appropriately defined matrices $F, G$, and $H$.

Under learning, we specify a VAR(1) PLM. Let $\hat{Y}_t' = (x_{t-1}, \pi_{t-1}, g_t, u_t)$. Then the PLM is

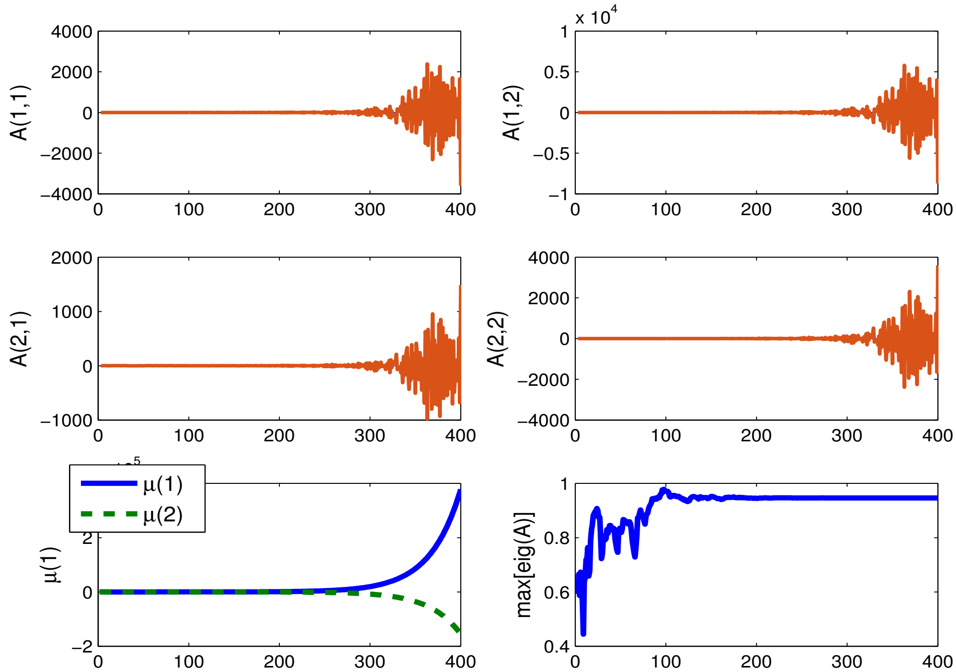
$$\hat{Y}_t = C + A\hat{Y}_{t-1} + \nu_t$$

Then expectations are computed as follows

$$\begin{aligned} E_t \hat{Y}_t &= \hat{Y}_t \\ E_t \hat{Y}_{t+1} &= C + A\hat{Y}_t \\ E_t \hat{Y}_{t+2} &= C + AE_t \hat{Y}_{t+1} \end{aligned}$$

Then expectations of $x, \pi$ can be computed as $E_t Y_{t+1} = e' E_t \hat{Y}_{t+2}$, where $e$ is an appropriately defined selector matrix.

Figure 6: YW parameter estimates of non-stationary exogenous VAR(1): $\gamma = 0.05$.




We compare the learning dynamics under two constant gain estimators of $C, A$. The first, constant gain least-squares. Let $X_t' = (1, \hat{Y}_{t-1})$. The constant gain least-squares recursion equations are given by,

$$
\begin{aligned}
\theta_t &= \theta_{t-1} + \gamma R_t^{-1} X_t \left( \hat{Y}_t - \theta_{t-1}' X_t \right)' \\
R_t &= R_{t-1} + \gamma \left( X_t X_t' - R_{t-1} \right)
\end{aligned}
$$

where $\theta_t' = (C_t, A_t)$. The constant gain least-squares equations are augmented with the projection facility employed by (Orphanides and Williams 2007): update $\theta_t, R_t$ so long as the largest root (in modulus) of $\theta_t$ is less than one. (Orphanides and Williams 2007) further add bounds to $x_t, \pi_t$ so that they cannot be larger than 16 times their standard deviation within a rational expectations equilibrium. We include these bounds as well.

With YW-learning we do not impose a projection facility or bounds on $x_t, \pi_t$. The YW-estimator in this case is

$$
\begin{aligned}
\mu_1 &= \gamma \hat{Y}_1 + \gamma(1-\gamma)\hat{Y}_0, \\
\mu_T &= \mu_{T-1} + \gamma(\hat{Y}_T - \mu_{T-1}) \text{ for } T > 1.
\end{aligned}
$$

At each time $T$ the inner recursions are re-initialized and computed using the transformed data $\tilde{Y}_n(T) = \hat{Y}_n - \mu_T$. The recursions are given by

$$\begin{aligned}
\hat{Z}_1(T) &= \gamma \tilde{Y}_1(T)\tilde{Y}_0(T)', \\
\hat{Z}_t(T) &= \hat{Z}_{t-1}(T) + \gamma\left(\tilde{Y}_t(T)\tilde{Y}_{t-1}(T)' - \hat{Z}_{t-1}(T)\right) \text{ for } 1 < t \leq T, \\
\hat{V}_1(T) &= \gamma \tilde{Y}_1(T)\tilde{Y}_1(T)' + \gamma(1-\gamma)\tilde{Y}_0(T)\tilde{Y}_0(T)', \\
\hat{V}_t(T) &= \hat{V}_{t-1}(T) + \gamma\left(\tilde{Y}_t(T)\tilde{Y}_t(T)' - \hat{V}_{t-1}(T)\right) \text{ for } 1 < t \leq T.
\end{aligned}$$

Finally, $\hat{A}_T^{ACG}(\gamma) = (1-\gamma)^{\frac{1}{2}}\hat{Z}_T(T)\hat{V}_T(T)^{-1}$.

### 4.2.1 Results

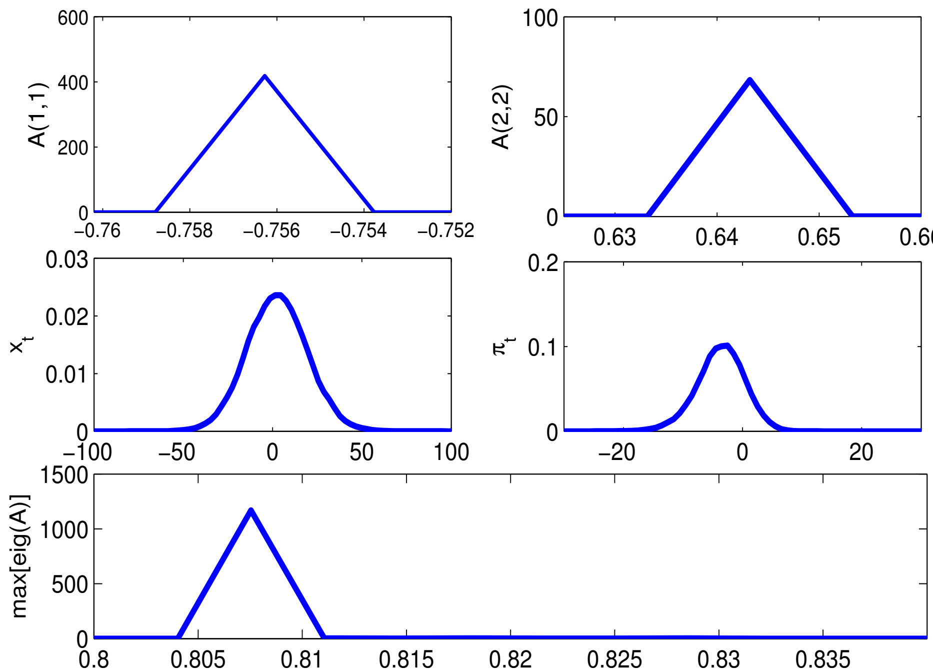
This section demonstrates (i.) the use of the projection facility in the Hybrid New Keynesian model with constant gain learning, and (ii.) the (relative) stability of YW-learning in that model. We proceed via Monte Carlo simulations. In both learning formulations, we initialize the model and simulate for a long transient period with a decreasing gain version of the algorithm. This ensures that when we begin the monte-carlo simulation the model has converged to its limiting distribution. We then simulate the model for 5,000 periods collecting statistics on the frequency with which the projection facility is employed and the distribution of belief coefficients $\mu, A$. We repeat the simulation 1,000 times, average across simulations, and plot the results.

Since the analysis is numerical we require a model parameterization. The main results are robust to the parameterization, however, in the reported exercise we choose the parameters that produces a model with strong expectational feedback in order to highlight the differences between constant gain learning and YW-learning. In subsequent analysis, we focus on an empirical version of the model to assess the quantitative significance of the projection facility. The parameterization we choose is based on (Woodford 2003). We set $\phi = 1/0.157, \lambda = 0.024, \beta = 1, \delta = \gamma = 0.5, \rho_g = 0.2, \rho_u = 0.2, \sigma_g^2 = 0.75, \sigma_u^2 = 1.3$.[5] In addition to the projection facility, we impose bounds that $|x_t| \leq 224, |\pi_t| \leq 50$.

The results for constant gain learning are presented in Figures 7-9. Figure 7 plots the results when $\gamma = 0.001$, the case of a small constant gain, for which one would expect the convergence theorems to apply. The top row plots the histogram for $A_{11}, A_{22}$, the own lag coefficients in agents' perceived law of motion. The middle row plots the histogram for actual output and inflation, while the bottom row is the histogram

[5] The shock variances are taken from (Orphanides and Williams 2007).

Figure 7: Constant Gain Learning: $\gamma = 0.001$.

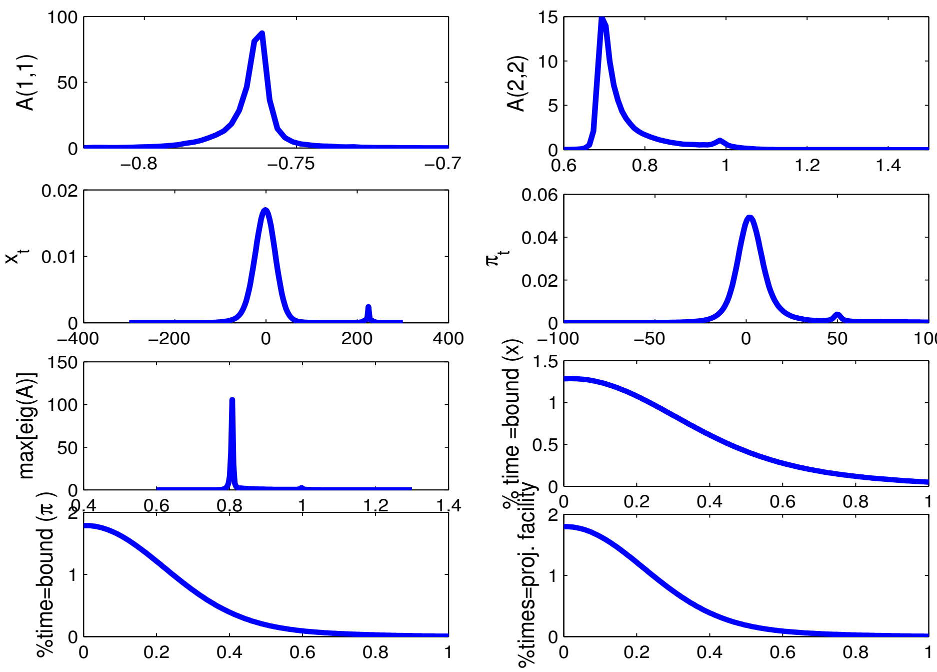



of the largest (absolute) root of $A_t$ at each point in time $t$. Under the calibration, the values for $A_{11}, A_{22}$ that would arise in a rational expectations equilibrium are $-.756$ and $.643$, respectively. Clearly, with a small gain $\gamma$ the belief coefficients are distributed tightly around their rational expectations equilibrium values. Moreover, the largest root of $A$ is tightly distributed at .8075, clearly in the stability region. Along these simulated paths, the projection facility and bounds on $x_t, \pi_t$ are never employed. So the values for $x_t, \pi_t$ are distributed about their mean values.[6]

With slightly higher values of $\gamma$, closer to empirically realistic values, then the qualitative/quantitative nature of the learning dynamics can change considerably. Figure 8 reports the results for a constant gain $\gamma = 0.01$, still a relatively small value. The top row again plots the histograms of the own lag coefficients in $A$, the second row plots the histograms for $x_t, \pi_t$, and the leftmost plot in the third row is, again, the largest (absolute) eigenvalue in $A_t$. The remaining three plots are the histograms of the frequency with which the projection facility/bounds are employed in simulations. The first two are the histograms of the fraction of time spent at the bounds for $x_t, \pi_t$

[6]The histogram for $\pi_t$ as plotted is actually distributed around a non-zero mean. This is just numerical as convergence along a simulated path is quite slow with a gain of 0.001.

Figure 8: Constant Gain Learning: $\gamma = 0.01$.

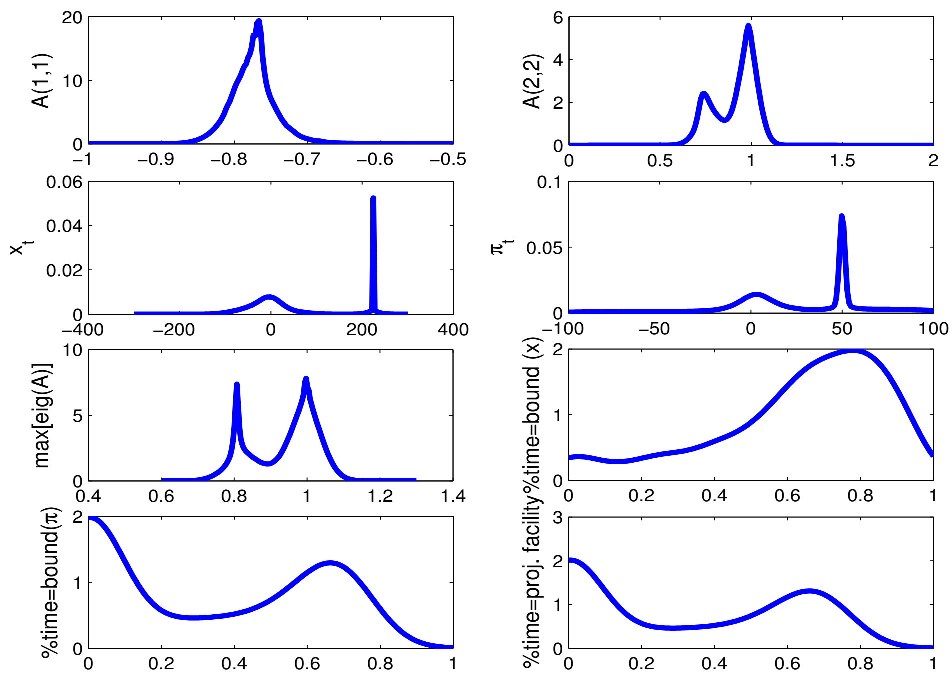



and the last plot is the fraction of the time that the projection facility is employed.

The results in Figure 8 show that with a $\gamma = 0.01$ there is considerably more volatility in $A_t$ as well as $x_t, \pi_t$. The histogram for $A_{11}$ is skewed left, while it is skewed right for $A_{22}$; in both instances, the skew is towards unstable roots of $A$. Notice, in particular, a small mode in the distribution for $A_{22}$ at $A_{22} \approx 1$. This is further reflected in the distribution of largest (absolute) eigenvalues with a small blip right at the projection facility. Furthermore, the histograms for $x_t, \pi_t$ show that a small amount of time is spent at the bounds imposed on $x_t, \pi_t$. The histograms for the projection facilities indicate that most of the time the projection facility is not employed, with a small gain of $\gamma = 0.01$ the learning dynamics on occasion are driven towards the unstable region.

With a modestly higher value for $\gamma$, the learning dynamics are considerably different. Figure 9 reports on the same sets of facts when $\gamma = 0.03$. Notice, now, that most of the time $A_{22} \approx 1$, with occasional values greater than one. These altered constant gain learning dynamics manifest as large spikes in the histograms for $x_t, \pi_t$ right at the bounds. The largest (absolute) eigenvalue of $A$ is bimodal suggesting that the economy recurrently alternates near the rational expectations equilibrium

Figure 9: Constant Gain Learning: $\gamma = 0.03$.




and a value that is at, or near, the projection facility. The histograms for the time spent at the projection facility show that it is not unusual for a simulation to spend 60% or more time employing the projection facility or the bounds.

How do these results differ with constant gain YW-learning? Repeating the same exercise with constant gain least-squares replaced by constant gain YW learning, Figures 10-13 illustrates the intrinsic stability of the alternative learning model. In Figure 10, a very small gain of $\gamma = 0.001$ again has the learning dynamics distributed tightly around the rational expectations equilibrium. Figure 11 plots the results with a small gain of $\gamma = 0.01$. As in Figure 8, the larger gain introduces substantially more volatility into the learning dynamics and the belief coefficients are skewed. However, the histogram for $A_{22}$ is now strongly skewed towards smaller values, i.e. less persistence in inflation. In the bottom row, one can see that skewness in the form of the largest eigenvalue of $A$.

Figure 10: YW Learning: $\gamma = 0.001$.

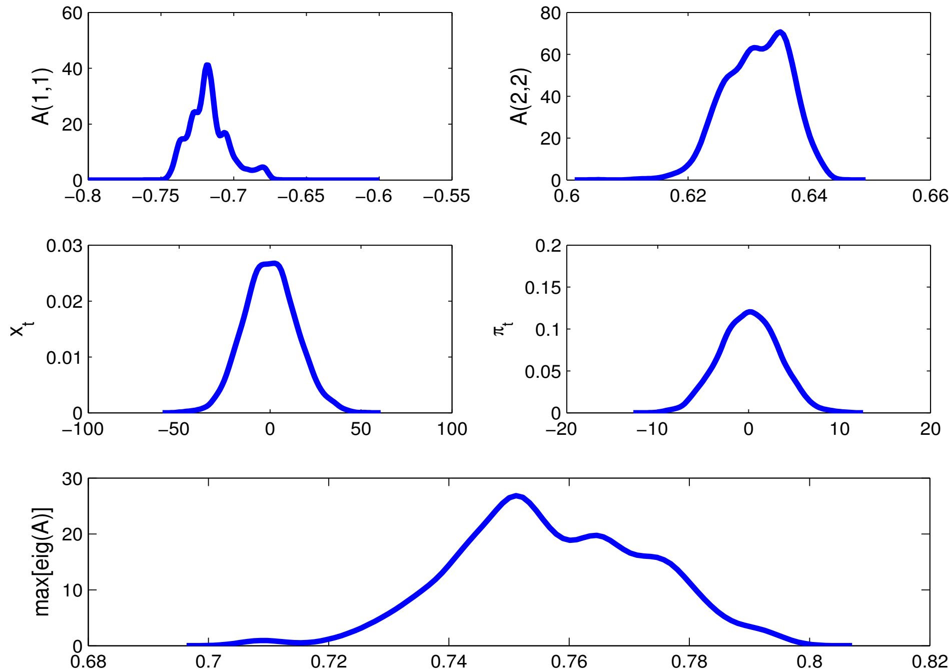



Figure 13: YW Learning: $\gamma = 0.15$.

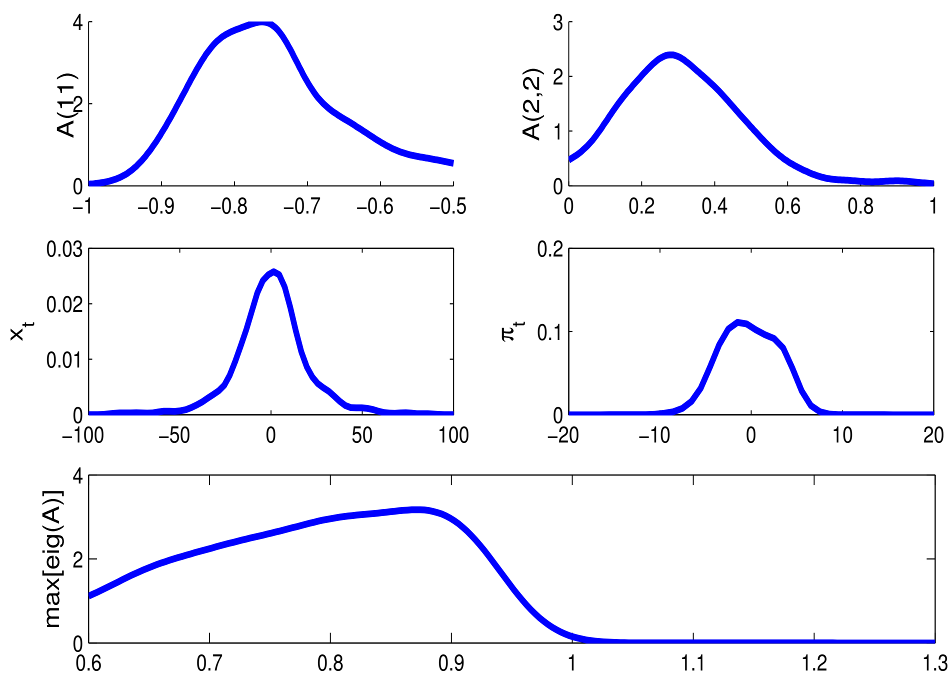


Figure 11: YW Learning: $\gamma = 0.01$.

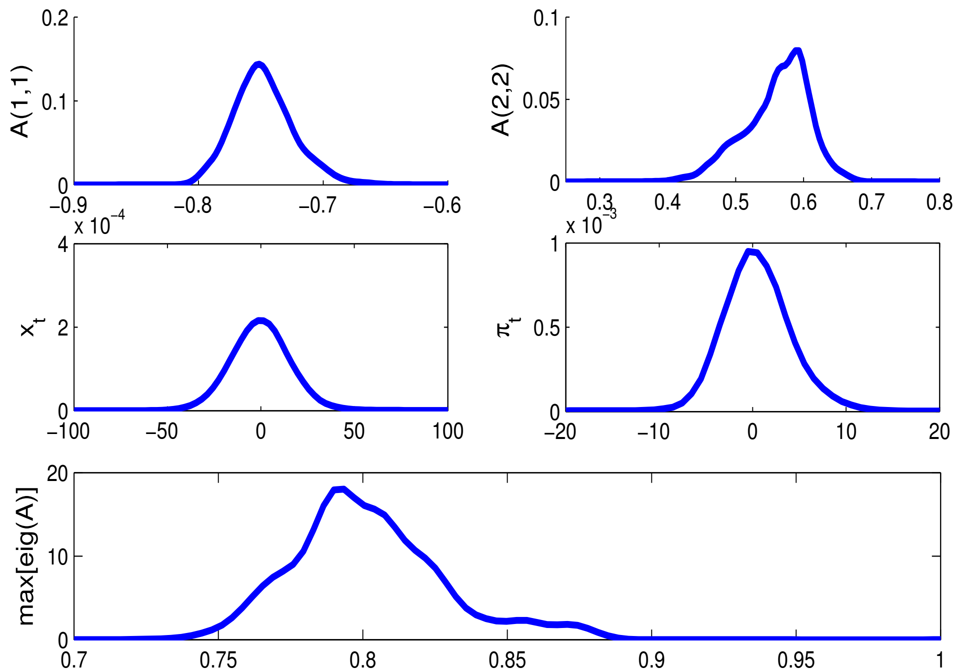
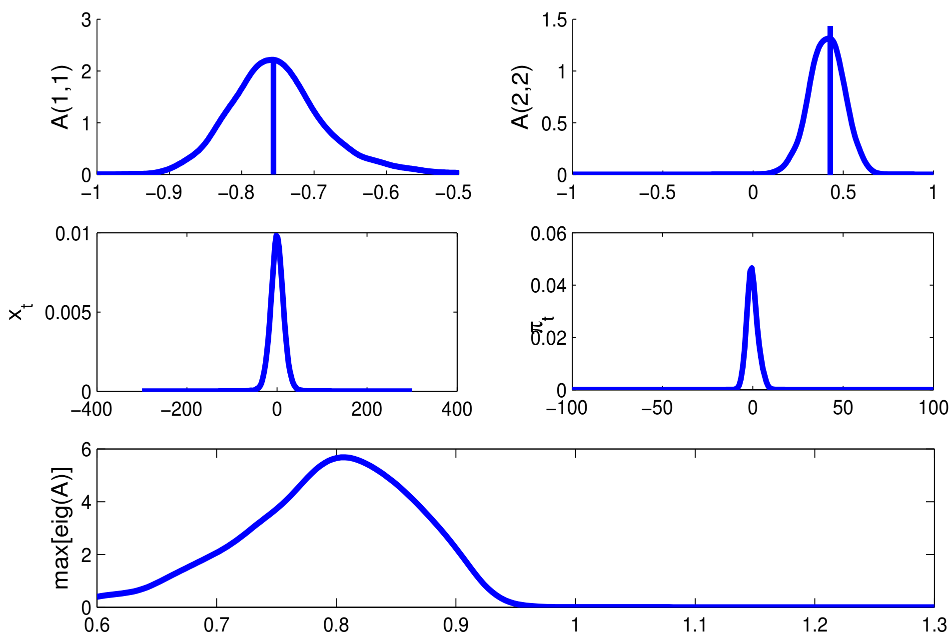



Figure 11 also does not illustrate dynamics being drawn toward the projection facility. This stabilizing feature of YW-learning is most clearly seen in Figures 12-13, which plot the results for constant gains of $\gamma = 0.05$ and $\gamma = 0.15$, respectively. In both cases, one sees, in the bottom row, eigenvalues getting close to the upper bound of one, i.e. the key feature of YW-learning is that these eigenvalues are bounded by one. The histogram for $A_{22}$ is even further skewed and there is modestly more volatility in the output gap and inflation.

The results in Figures 10-13 suggest that the learning dynamics for constant gain least-squares and constant gain YW-learning are very similar for small gains. However, for modest to large gains, the behavior is quite a bit different. With constant gain learning there are frequent, large sustained departures from the rational expectations equilibrium that features the dynamics escaping from the basin of attraction of the rational expectations equilibrium. Without a projection facility, the economic dynamics would be unstable. Under constant gain YW-learning, the beliefs are bounded. Larger values for the constant gain also bias beliefs away from the rational expectations equilibrium, however, the dynamics remain bounded. Larger gains increase economic volatility, but not to the extent that they do with constant gain least-squares. Again, this is because without the projection facility, constant gain

Figure 12: YW Learning: $\gamma = 0.05$.




least-squares can be unstable for some shock realizations when the gain is sufficiently large.

One is tempted to interpret the results in Figures 11-13 as general features of YW-learning, i.e. stable learning dynamics, left-skewed belief coefficient distributions, etc. However, some of these features arise under the particular calibration employed and one cannot expect to find the same qualitative dynamics under an alternative parameterization. This motivates the next section which studies these dynamics in the small quantitative model of (Giannoni and Woodford 2005).

## 4.3 A Quantitative Model

In this section, we take a quantitative model well-known to provide a good empirical fit to U.S. macroeconomic data. We then compare the fit of the model under constant gain least-squares learning to constant gain Yule-Walker learning. The model we adopt here is the simple New Keynesian model estimated in (Giannoni and Woodford 2005) that incorporates wage and price stickiness, inflation inertia, habit persistence, and predetermined pricing and spending decisions. Thus, this section facilitates a comparison of the likely quantitative effects of constant gain learning, with a projection facility, and YW-learning.

Coming Soon

# 5 Conclusion

Adaptive learning models have been successful at replicating key empirical features of business cycles and asset prices. Constant gain, or perpetual, learning models, however, are susceptible to unstable dynamics. The standard approach is to implement a "projection facility" that keeps the learning dynamics bounded within the stability region. However, since there is no unique way to implement a projection facility, it is somewhat difficult to interpret quantitative results in studies that make use of projection facilities. How much of the results are being driven by the projection facility? How much of the calibration/estimation is biased by trying to minimize the implementation of the projection facility?

This paper argues in favor of Yule-Walker learning: the coefficients of a vector autoregression are recovered, in real-time, via the Yule-Walker equations combined with sample autocorrelation coefficients. We extend, and generalize, the sample autocorrelation learning introduced by (Hommes and Sorger 1997) and (Hommes and Zhu 2012), to multivariate settings. The main result in this paper is that Yule-Walker learning coefficients remain bounded. An application is presented in the Hybrid New Keynesian model. The use of the projection facility is frequent under constant gain least squares learning. However, with constant gain Yule-Walker learning, the dynamics remain stable for much higher values of the constant gain than is typically employed in adaptive learning studies.

# 6 Appendix

**Proof of Theorem 1.** Suppose $A_T$ is diagonalizable and write $A_T = S\Lambda S^{-1}$. Let $z_n = S^{-1}y_n$. Next, define

$$\alpha = \left(\sum_{i=1}^{T} z_i z_{i-1}'\right)\left(\sum_{i=0}^{T} z_i z_i'\right)^{-1}. \tag{24}$$

Since $z_i z_{i-1}' = S^{-1} y_i y_{i-1}' (S')^{-1}$ it follows that $\alpha = \Lambda$, so that the eigenvalues of $A_T$ are given by the $\alpha_{kk}$ for $k \in \{1, \ldots, m\}$.

We claim that

$$\alpha_{kk} = \frac{\sum_{i=1}^{T} z_{ki} z_{ki-1}}{\sum_{i=0}^{T} z_{ki}^2}. \tag{25}$$

To see this, note that equation (24) implies

$$\sum_{i=1}^{T} z_i z_{i-1} = \alpha \left(\sum_{i=0}^{T} z_i z_i'\right).$$

Working first on the left and then on the right, we have

$$\left(\sum_{i=1}^{T} z_i z_{i-1}'\right)_{kk} = \sum_{i=1}^{T} z_{ki} z_{ki-1}, \tag{26}$$

and

$$\left(\alpha\left(\sum_{i=0}^{T} z_i z_i'\right)\right)_{kk} = \sum_{j=1}^{m} \alpha_{kj} \left(\sum_{i=0}^{T} z_i z_i'\right)_{jk} = \alpha_{kk} \left(\sum_{i=0}^{T} z_i z_i'\right)_{kk} = \alpha_{kk} \sum_{i=0}^{T} z_{ki}^2, \tag{27}$$

where the second equality follows from the fact that $\alpha = \Lambda \implies \alpha_{jk} = 0$ for $j \neq k$. Equation (25) is then demonstrated by combining (26) and (27) to solve for $\alpha_{kk}$.

Next we recall that, by the Cauchy-Schwartz inequality, if $\{x_0, \ldots, x_T\}$ is any collection of real numbers then

$$\left|\frac{\sum_{i=1}^{T} x_i x_{i-1}}{\sum_{i=0}^{T} x_i^2}\right| \leq 1. \tag{28}$$

Indeed,

$$\begin{aligned}
\left|\frac{\sum_{i=1}^{T} x_i x_{i-1}}{\sum_{i=0}^{T} x_i^2}\right| &\leq \frac{\sum_{i=1}^{T} |x_i x_{i-1}|}{\sum_{i=0}^{T} x_i^2} = \frac{\sum_{i=1}^{T} |x_i x_{i-1}|}{\sqrt{\sum_{i=0}^{T} x_i^2}\sqrt{\sum_{i=0}^{T} x_i^2}} \\
&\leq \frac{\sum_{i=1}^{T} |x_i x_{i-1}|}{\sqrt{\sum_{i=1}^{T} x_i^2}\sqrt{\sum_{i=0}^{T-1} x_i^2}} \leq 1,
\end{aligned}$$

where the last inequality follows by Cauchy-Schwartz.

To complete the proof of the theorem we observe that by (25) and (28), the eigenvalues $\alpha_{kk}$ of $A_T$ have modulus less than or equal to one. Since the Cauchy-Schwartz inequality is strict when the vectors are not identical, and since $\{y_0, \ldots, y_{T-1}\} \neq \{y_1, \ldots, y_T\}$ by assumption, the result follows. ■

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