Optimal Stopping Problems

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These notes expand on Ben Moll's superb notes in `http://www.princeton.edu/~moll/HACTproject/` on solving option value problems as HJB Variational Inequalities, as discussed Huang and Pang (1998).

1 Optimal Stopping of a Univariate, Time-Homogenous Process

This section outlines the general approach to the problems.[1]

1.1 Variational Inequality Formulation

To set notation:

- x is a stochastic process, with infinitesimal generator $\mathcal{A}$.[2]
- An agent with state x can optimally stop with value $S(x)$ at any time.
- The agent gains utility flow $u(x)$ and discounts the future at rate $\rho > 0$.

Classical Formulation The typical formulation of this is as a free-boundary value problem. Assume that the agent chooses an optimal $\hat{x}$ stopping rule, then it will solve the ODE in the continuation region along with value matching and optimal stopping. That is, find a $v(x)$ continuation value and stopping point $\hat{x}$ that fulfills,[3]

$$\rho v(x) = u(x) + \mathcal{A}v(x) \tag{1}$$

$$v(\hat{x}) = S(\hat{x}) \tag{2}$$

$$v'(\hat{x}) = 0 \tag{3}$$

Additionally another boundary value would be required for either a large $\bar{x}$ or a transversality condition. This formulation requires that there is only a single stopping point.

HJB Variational Inequality Formulation Another formulation is to find a $v(x)$ such that the following holds,

$$0 = \min\left\{\rho v(x) - u(x) - \mathcal{A}v(x),\, v(x) - S(x)\right\} \tag{4}$$

for all x (adding an appropriate boundary conditions x). This is a variational inequality since $\mathcal{A}$ is a differential operator. While the value matching condition is clear here, it can be shown that this implies the smooth pasting condition.

[1] See `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` for the simplest case. The algebra expands on `http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf` and `http://www.princeton.edu/~moll/HACTproject/HACT_Additional_Codes.pdf`.

[2] For example, the infinitesimal generator of the SDE $\mathrm{d}x_t = \mu(x_t)\mathrm{d}t + \sigma(x_t)\mathrm{d}\mathbb{W}_t$ is $\mathcal{A} \equiv \mu(x)\boldsymbol{\partial}_x + \frac{\sigma(x)^2}{2}\boldsymbol{\partial}_{xx}$

[3] Note that if anything was a function of time, this would no longer be time-invariant. For example, if $S(t,x)$, then $\mathcal{A}$ would have a $\boldsymbol{\partial}_t$ term as well, and $\hat{x}(t)$ is the stopping rule at any time

1.2 Discretized Problem

For an arbitrary operator $\mathcal{A}$ with associated boundary values, we will find a solution to (4) by discretizing the $\mathcal{A}$ operator on a grid $\{x_i\}_{i=1}^I$ with $x_1 = \underline{x}$ and $x_I = \bar{x}$ subject to appropriate boundary values on $\underline{x}$ and $\bar{x}$. See `operator_discretization_finite_differences.pdf` notes for details on the discretization of operators.

Linear Operator In the case of a linear operator $\mathcal{A}$, the resulting discretized finite-difference operator is a matrix A. Denote $v \equiv \{v(x_i)\}_{i=1}^I, u \equiv \{u(x_i)\}_{i=1}^I$, and $S \equiv \{s(x_i)\}_{i=1}^I$, then (4) becomes

$$0 = \min\{\rho v - u - Av,\, v - S\} \tag{5}$$

where $A \in \mathbb{R}^{I\times I}$. A solution to this problem is a v fulfilling (5). Note that the boundary values are already in the A operator and do not need to be discussed separately.

1.3 LCP Formulation

Following `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` with only minor variations, we note that (4) is a linear-complementarity problem. Define the following (given the identity matrix $\mathbf{I} \in \mathbb{R}^{I\times I}$),

$$B \equiv \rho\mathbf{I} - A \tag{6}$$

$$z \equiv v - S \tag{7}$$

$$q \equiv -u + BS \tag{8}$$

$$w \equiv Bz + q \tag{9}$$

Substitute (6) to (8) into (5)

$$0 = \min\{Bz + q,\, z\} \tag{10}$$

Which could be written as complementarity slackness conditions,

$$z^T(Bz + q) = 0 \tag{11}$$

$$z \geq 0 \tag{12}$$

$$Bz + q \geq 0 \tag{13}$$

Which can be written succinctly with complementarity constraints as,

$$0 \leq (Bz + q) \perp z \geq 0 \tag{14}$$

Alternatively, some LCP and MCP solvers prefer to have a slack variable introduced. Use (9) to find the LCP problem as finding $w, z \in \mathbb{R}^I$ such that,

$$w \equiv Bz + q \tag{15}$$

$$0 \leq w \perp z \geq 0 \tag{16}$$

where (15) is added to the solver with a linear equality constraint.

In either case, to unpack the results, drop any slack variables to get the z vector, and undo the transformation in (7).

References

Huang, J., and J.-S. Pang (1998): "Option Pricing and Linear Complementarity," in *Journal of Computational Finance*. Citeseer.