Discretizing Continuous-Time Operators with Finite Differences
Time-Homogenous and Stationary Examples

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These notes expand on Ben Moll's superb notes in `http://www.princeton.edu/~moll/HACTproject/`. This set of notes gives examples on how to discretize linear and nonlinear operators. The discretized operators are used for various methods. The time-homogeneity maintained throughout.

Notation To set some general notation, given an operator (or infinitesimal generator) associated with a particular stochastic process, $\mathcal{A}$. The purpose of these notes is to discretize $\mathcal{A}$ on this grid using using finite differences. Crucially, it is necessary to put boundary-values into the discretized operator as well.

In the univariate case, $\{x_i\}_{i=1}^I$ with $x_1 = \underline{x}$ and $x_I = \bar{x}$ when $x \in [\underline{x}, \bar{x}]$. In the case of a linear $\mathcal{A}$, the resulting discretized operator is a matrix $A \in \mathbb{R}^{I \times I}$, but otherwise may be a nonlinear function.

For a given variable q, define the notation $\mu^- \equiv \min\{q, 0\}$ and $q^+ \equiv \max\{q, 0\}$, which will be useful for defining finite-differences with an upwind scheme.[1] This can apply to vectors as well. For example, $q_i^- = q_i$ if $q_i < 0$ and 0 if $q_i > 0$, and $q_i^- \equiv \left\{q_i^-\right\}_{i=1}^I$

Finally, derivatives are denoted by the operator $\boldsymbol{\partial}$ and univariate derivatives such as $\boldsymbol{\partial}_x v(x) \equiv v'(x)$.

1 Univariate Diffusions

1.1 Stochastic Process and Boundary Values

Take a diffusion process for x_t according to the following stochastic difference equation,

$$\mathrm{d}x_t = \mu(x_t)\mathrm{d}t + \sigma(x_t)\mathrm{d}\mathbb{W}_t \tag{1}$$

where $\mathbb{W}_t$ is Brownian motion and $x \in (\underline{x}, \bar{x})$ where $-\infty \leq \underline{x} < \bar{x} \leq \infty$.[2] The functions $\mu(\cdot)$ and $\sigma(\cdot)$ are left general, but it is essential that they are time-homogeneous.

We will consider several types of boundary values for an operator on a generic function $v(x)$. Because the process is univariate, we require boundary conditions at both sides of x, and assume that the boundary and $\mu(\cdot), \sigma(x)^2$ are independent of time.

- **Absorbing Barrier:** $v(\underline{x}) = \underline{v}$ and/or $v(\bar{x}) = \bar{v}$ for constants $\underline{v}$ and/or $\bar{v}$. This is called a "Dirichlet" boundary condition in PDEs.

- **Reflecting Barrier:** $\boldsymbol{\partial}_x v(\underline{x}) = v'(\underline{x}) = 0$ and/or $v'(\bar{x}) = 0$. This is called a "Neumann" boundary condition in PDEs.

[1] For more details on the notation and the upwind scheme, see `http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf`.

[2] We are being a little sloppy with x_t being exactly at the bounds, because it is a measure 0 event with a diffusion.

- **Transversality:** Problem specific? But the hope is that we can use a reflecting barrier in many cases without effecting most of the results.

The infinitesimal generator associated with this stochastic process is

$$\mathcal{A} \equiv \mu(x)\boldsymbol{\partial}_x + \frac{\sigma(x)^2}{2}\boldsymbol{\partial}_{xx} \tag{2}$$

1.2 Finite Difference Discretization for A with a Uniform Grid

For a uniform grid, define $\Delta \equiv x_{i+1} - x_i$, which is identical for all i. Define the vectors, $X, Y, Z \in \mathbb{R}^I$ such that,

$$X = -\frac{\mu^-}{\Delta} + \frac{\sigma^2}{2 \times \Delta^2} \tag{3}$$

$$Y = -\frac{\mu^+}{\Delta} + \frac{\mu^-}{\Delta} - \frac{\sigma^2}{\Delta^2} \tag{4}$$

$$Z = \frac{\mu^+}{\Delta} + \frac{\sigma^2}{2 \times \Delta^2} \tag{5}$$

$$BV_1 = \begin{cases} 0, & \text{for absorbing barrier } v(\underline{x}) = \underline{v} \\ X_1, & \text{for reflecting barrier } v'(\underline{x}) = 0 \end{cases} \tag{6}$$

$$BV_I = \begin{cases} 0, & \text{for absorbing barrier } v(\bar{x}) = \bar{v} \\ Z_I, & \text{for reflecting barrier } v'(\bar{x}) = 0 \end{cases} \tag{7}$$

With these, the matrix A is constructed as.

$$A \equiv \begin{bmatrix} Y_1 + BV_1 & Z_1 & 0 & \cdots & \cdots & \cdots & 0 \\ X_2 & Y_2 & Z_2 & 0 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & X_{I-1} & Y_{I-1} & Z_{I-1} \\ 0 & \cdots & \cdots & \cdots & 0 & X_I & Y_I + BV_I \end{bmatrix} \in \mathbb{R}^{I \times I} \tag{8}$$

$$b \equiv \begin{bmatrix} X_1\underline{v} & 0 & \cdots & 0 & Z_I\bar{v} \end{bmatrix}^T \in \mathbb{R}^I \tag{9}$$

where,

$$\mathcal{A}v(\{x_i\}) \approx Av + b \tag{10}$$

Note that the special cases for the 1st and Ith row correspond to the boundary values and are adjusted by the BV_1, BV_2 terms. Also, note that the b vector is all zeros if the boundary values are reflective, or the boundary value at the minimum is 0. To better understand this construction, look at individual rows of A with the ODE.[3]

[3] Note that here we defined the boundary conditions in terms of the value of points v_0 and v_{I+1}. These points are sometimes called "ghost nodes".

Interior of A: In the interior ($1 < i < I$), the discretization of (2)

$$\mathcal{A}v(x_i) \approx \underbrace{\frac{v_i - v_{i-1}}{\Delta}\mu_i^- + \frac{v_{i+1} - v_i}{\Delta}\mu_i^+}_{\text{Upwind Scheme}} + \frac{\sigma_i^2}{2}\frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2} \tag{11}$$

The upwind scheme chooses either forward or backward differences, depending on the sign of the drift. Collecting terms, we see the derivation for the definitions in (3) to (5)

$$= \underbrace{\left(-\frac{\mu_i^-}{\Delta} + \frac{\sigma_i^2}{2\Delta^2}\right)}_{\equiv X_i} v_{i-1} + \underbrace{\left(\frac{\mu_i^-}{\Delta} - \frac{\mu_i^+}{\Delta} - \frac{\sigma_i^2}{\Delta^2}\right)}_{\equiv Y_i} v_i + \underbrace{\left(\frac{\mu_i^+}{\Delta} + \frac{\sigma_i^2}{2\Delta^2}\right)}_{\equiv Z_i} v_{i+1} \tag{12}$$

Boundary Value at $\underline{x}$: As $i = 1$, the discretized operator from (12) is

$$\mathcal{A}v(x_1) \approx \left(-\frac{\mu_1^-}{\Delta} + \frac{\sigma_1^2}{2\Delta^2}\right) v_0 + \left(\frac{\mu_1^-}{\Delta} - \frac{\mu_1^+}{\Delta} - \frac{\sigma_1^2}{\Delta^2}\right) v_1 + \left(\frac{\mu_1^+}{\Delta} + \frac{\sigma_1^2}{2\Delta^2}\right) v_2 \tag{13}$$

In the case of the boundary value $v(\underline{x}) = \underline{v}$, subsitute for $v_0 = \underline{v}$ to find,

$$\approx \underbrace{\left(-\frac{\mu_1^-}{\Delta} + \frac{\sigma_1^2}{2\Delta^2}\right)\underline{v}}_{\equiv b_1} + \underbrace{\left(\frac{\mu_1^-}{\Delta} - \frac{\mu_1^+}{\Delta} - \frac{\sigma_1^2}{\Delta^2}\right)}_{\equiv Y_1} v_1 + \underbrace{\left(\frac{\mu_1^+}{\Delta} + \frac{\sigma_1^2}{2\Delta^2}\right)}_{\equiv Z_1} v_2 \tag{14}$$

Alternatively, if the boundary value is $v'(\underline{x}) = 0$, take (13) and use $v'(\underline{x}) \approx \dfrac{v_1 - v_0}{\Delta} = 0, \implies v_1 = v_0$, so

$$\mathcal{A}v(x_1) \approx \left(\underbrace{\left(-\frac{\mu_1^-}{\Delta} + \frac{\sigma_1^2}{2\Delta^2}\right)}_{\equiv BV_1 = X_1} + \underbrace{\left(\frac{\mu_1^-}{\Delta} - \frac{\mu_1^+}{\Delta} - \frac{\sigma_1^2}{\Delta^2}\right)}_{\equiv Y_1}\right) v_1 + \underbrace{\left(\frac{\mu_1^+}{\Delta} + \frac{\sigma_1^2}{2\Delta^2}\right)}_{\equiv Z_1} v_2 \tag{15}$$

$$\tag{16}$$

Boundary Value at $\bar{x}$: As $i = I$, from (12)

$$\mathcal{A}v(x_I) \approx \left(-\frac{\mu_I^-}{\Delta} + \frac{\sigma_I^2}{2\Delta^2}\right) v_{I-1} + \left(\frac{\mu_I^-}{\Delta} - \frac{\mu_I^+}{\Delta} - \frac{\sigma_I^2}{\Delta^2}\right. \tag{17}$$

For the absorbing barrier, substitute for $v(x_{I+1}) = \bar{v}$,

$$\approx \left(-\frac{\mu_I^-}{\Delta} + \frac{\sigma_I^2}{2\Delta^2}\right) v_{I-1} + \left(\frac{\mu_I^-}{\Delta} - \frac{\mu_I^+}{\Delta} - \frac{\sigma_I^2}{\Delta^2}\right) v_I + \left(\frac{\mu_I^+}{\Delta} + \frac{\sigma_I^2}{2\Delta^2}\right) v_{I+1} \tag{18}$$

For a reflecting barrier, the boundary value $v'(\bar{x}) \approx \frac{v_{I+1} - v_I}{\Delta} = 0, \implies v_{I+1} = v_I$,

$$= \underbrace{\left(-\frac{\mu_I^-}{\Delta} + \frac{\sigma_I^2}{2\Delta^2}\right)}_{\equiv X_I} v_{I-1} + \underbrace{\left(\frac{\mu_I^-}{\Delta} - \frac{\mu_I^+}{\Delta} - \frac{\sigma_I^2}{\Delta^2}\right.}_{\underbrace{Y_I + BV_I}} \tag{19}$$

TODO: FINISH In the special case of $\mu(\bar{x}) < 0$,

$$\rho v_I = u_I + \frac{v_I - v_{I-1}}{\Delta}\mu_I + \frac{\sigma_I^2}{2}\frac{v_{I-1} - v_I}{\Delta^2} \tag{20}$$

Also note that for $\mu > 0$, the upwind drift term drops out entirely.

Simplifications with $\mu < 0$: In the simple case where $\mu(x) < 0$ for all x, this simplifies to using backward differences,

$$\mathcal{A}v(x_i) = \frac{v_i - v_{i-1}}{\Delta}\mu_i + \frac{\sigma_i^2}{2}\frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2} \tag{21}$$

Here, we make some variations to the simple option problem. When $\mu < 0$ is known, non-zero entries in the "upwind scheme" matrix become

$$X_i = -\frac{\mu_i}{\Delta} + \frac{\sigma_i^2}{2\Delta^2} \qquad i = 2, 3, ..., I \tag{22}$$

$$Y_i = \frac{\mu_i}{\Delta} - \frac{\sigma_i^2}{\Delta^2} \qquad i = 1, 2, ..., I-1 \tag{23}$$

$$Z_i = \frac{\sigma_i^2}{2\Delta^2} \qquad i = 1, 2, ..., I-1 \tag{24}$$

$$Y_I = \frac{\mu_I}{\Delta} - \frac{\sigma_I^2}{2\Delta^2} \tag{25}$$

The format of this verified matrix is the same as matrix A.

References