

Optimal Stopping Problems

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These notes expand on Ben Moll's superb notes in <http://www.princeton.edu/~moll/HACTproject/> on solving option value problems as HJB Variational Inequalities, as discussed ?.

1 Optimal Stopping of a Univariate, Time-Homogenous Process

This section outlines the general approach to the problems.¹

1.1 Variational Inequality Formulation

To set notation:

- x is a stochastic process, with infinitesimal generator \mathcal{A} .²
- An agent with state x can optimally stop with value $S(x)$ at any time.
- The agent gains utility flow $u(x)$ and discounts the future at rate $\rho > 0$.

Classical Formulation The typical formulation of this is as a free-boundary value problem. Assume that the agent chooses an optimal \hat{x} stopping rule, then it will solve the ODE in the continuation region along with value matching and optimal stopping. That is, find a $v(x)$ continuation value and stopping point \hat{x} that fulfills,³

$$\rho v(x) = u(x) + \mathcal{A}v(x) \tag{1}$$

$$v(\hat{x}) = S(\hat{x}) \tag{2}$$

$$v'(\hat{x}) = 0 \tag{3}$$

Additionally another boundary value would be required for either a large \bar{x} or a transversality condition. This formulation requires that there is only a single stopping point.

HJB Variational Inequality Formulation Another formulation is to find a $v(x)$ such that the following holds,

$$0 = \min \{ \rho v(x) - u(x) - \mathcal{A}v(x), v(x) - S(x) \} \tag{4}$$

for all x (adding an appropriate boundary conditions x). This is a variational inequality since \mathcal{A} is a differential operator. While the value matching condition is clear here, it can be shown that this implies the smooth pasting condition.

¹See http://www.princeton.edu/~moll/HACTproject/option_simple.pdf for the simplest case. The algebra expands on http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf and http://www.princeton.edu/~moll/HACTproject/HACT_Additional_Codes.pdf.

²For example, the infinitesimal generator of the SDE $dx_t = \mu(x_t)dt + \sigma(x_t)dW_t$ is $\mathcal{A} \equiv \mu(x)\partial_x + \frac{\sigma(x)^2}{2}\partial_{xx}$

³Note that if anything was a function of time, this would no longer be time-invariant. For example, if $S(t, x)$, then \mathcal{A} would have a ∂_t term as well, and $\hat{x}(t)$ is the stopping rule at any time

1.2 Discretized Problem

For an arbitrary operator \mathcal{A} with associated boundary values, we will find a solution to (4) by discretizing the \mathcal{A} operator on a grid $\{x_i\}_{i=1}^I$ with $x_1 = \underline{x}$ and $x_I = \bar{x}$ subject to appropriate boundary values on \underline{x} and \bar{x} . See `operator_discretization_finite_differences.pdf` notes for details on the discretization of operators.

Linear Operator In the the case of a linear operator \mathcal{A} , the resulting discretized finite-difference operator is a matrix A . Denote $v \equiv \{v(x_i)\}_{i=1}^I$, $u \equiv \{u(x_i)\}_{i=1}^I$, and $S \equiv \{s(x_i)\}_{i=1}^I$, then (4) becomes

$$0 = \min \{\rho v - u - Av, v - S\} \quad (5)$$

where $A \in \mathbb{R}^{I \times I}$. A solution to this problem is a v fulfilling (5). Note that the boundary values are already in the A operator and do not need to be discussed separately.

1.3 LCP Formulation

Following http://www.princeton.edu/~moll/HACTproject/option_simple.pdf with only minor variations, we note that (4) is a linear-complementarity problem. Define the following (given the identity matrix $\mathbf{I} \in \mathbb{R}^{I \times I}$),

$$B \equiv \rho \mathbf{I} - A \quad (6)$$

$$z \equiv v - S \quad (7)$$

$$q \equiv -u + BS \quad (8)$$

$$w \equiv Bz + q \quad (9)$$

Substitute (6) to (8) into (5)

$$0 = \min \{Bz + q, z\} \quad (10)$$

Which could be written as complementarity slackness conditions,

$$z^T (Bz + q) = 0 \quad (11)$$

$$z \geq 0 \quad (12)$$

$$Bz + q \geq 0 \quad (13)$$

Which can be written succinctly with complementarity constraints as,

$$0 \leq (Bz + q) \perp z \geq 0 \quad (14)$$

Alternatively, some LCP and MCP solvers prefer to have a slack variable introduced. Use (9) to find the LCP problem as finding $w, z \in \mathbb{R}^I$ such that,

$$w \equiv Bz + q \quad (15)$$

$$0 \leq w \perp z \geq 0 \quad (16)$$

where (15) is added to the solver with a linear equality constraint.

In either case, to unpack the results, drop any slack variables to get the z vector, and undo the transformation in (7).