Notes on Numerical Solutions for Optimal Stopping Problems

September 12, 2017

These notes expand on Ben Moll's superb notes in `http://www.princeton.edu/~moll/HACTproject/` and in particular `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` solving option value problems as HJB Variational Inequalities. The approach is based on **?**

TODO:

- We probably want to put most of the `http://www.princeton.edu/~moll/HACTproject/option_simple.pdf` notes in there, but adding in the details on the discretization, boundary values, upwind finite differences, etc. that come from `http://www.princeton.edu/~moll/HACTproject/HACT_Numerical_Appendix.pdf` and `http://www.princeton.edu/~moll/HACTproject/HACT_Additional_Codes.pdf` as required.
- Want to have every formula used in the matlab code explicitly listed, with special care on the boundary values/etc.

The "upwind scheme" is constructed by using X_i Y_i and Z_i , where $i = 1, 2, ..., N_X$. Here the idea is to use forward difference approximation whenever the drift of the state variable is positive and the backward difference approximation whenever it is negative.

$$X_i = -\frac{\min(\mu_i, 0)}{\Delta} + \frac{\sigma_i^2}{2 \times \Delta^2} \tag{1}$$

$$Y = -\frac{\max(\mu_i, 0)}{\Delta} + \frac{\min(\mu_i, 0)}{\Delta} - \frac{\sigma_i^2}{\Delta^2} \tag{2}$$

$$Z = \frac{\max(\mu_i, 0)}{\Delta} + \frac{\sigma_i^2}{2 \times \Delta^2} \tag{3}$$

By using X_i Y_i and Z_i, the matrix A is contructed as

$$A = \begin{bmatrix} Y_1 & Z_1 & 0 & \cdots & \cdots & \cdots & 0 \\ X_2 & Y_2 & Z_2 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & X_{N_X-1} & Y_{N_X-1} & Z_{N_X-1} \\ \vdots & \cdots & \cdots & \cdots & \cdots & X_{N_X} & Y_{N_X} + \frac{\sigma_{N_X}^2}{2\Delta^2} \end{bmatrix} \tag{4}$$

At the boundaries in the i dimension, the equations become

$$\rho v_i = u_i + \mu_i \frac{v_i - v_{i-1}}{\Delta} + \frac{\sigma_i^2}{2} \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta^2} \tag{5}$$

As $i = 1$, the equations becomes

$$\rho v_1 = u_1 + \mu_1 \frac{v_1}{\Delta} + \frac{\sigma_1^2}{2} \frac{v_2 - 2v_1}{\Delta^2} \tag{6}$$

where we have used $v_0 = 0$. In general, this assumption implies the continuity at v_1 and hence $v_1 = v_0$.
As $i = N_X$, the equation becomes

$$\rho v_{N_X} = u_{N_X} + \mu_{N_X} \frac{v_{N_X} - v_{N_X-1}}{\Delta} + \frac{\sigma_{N_X}^2}{2} \frac{-v_{N_X} + v_{N_X-1}}{\Delta^2} \tag{7}$$

where we have used $\boldsymbol{\partial}_x v_{N_X} = \frac{v_{N_X+1} - v_{N_X}}{\Delta} = 0$ and hence $v_{N_X+1} = v_{N_X}$.
Note that here we defined the boundary conditions in terms of the value of points v_0 and v_{N_X+1}. These points are sometimes called "ghost nodes".

Here, we make some variations to the simple option problem. When $\mu < 0$ is known, non-zero entries in the "upwind scheme" matrix become

$$X_i = -\frac{\mu_i}{\Delta} + \frac{\sigma_i^2}{2\Delta^2} \qquad i = 2, 3, ..., N_X \tag{8}$$

$$Y_i = \frac{\mu_i}{\Delta} - \frac{\sigma_i^2}{\Delta^2} \qquad i = 1, 2, ..., N_X - 1 \tag{9}$$

$$Z_i = \frac{\sigma_i^2}{2\Delta^2} \qquad i = 1, 2, ..., N_X - 1 \tag{10}$$

$$Y_{N_X} = \frac{\mu_{N_X}}{\Delta} - \frac{\sigma_{N_X}^2}{2\Delta^2} \tag{11}$$

The format of this varified matrix is the same as matrix A.