ECON 6130 - Problem Set # 1

September 18, 2017

Problem 1

1.

Bob's maximization problem.

$$
\begin{aligned}
&\max_{\{c_t\},\{b_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{such that } & c_t + b_t \le e_t + R_t b_{t-1} \\
& b_t \ge 0 \\
& c_t \ge 0 \\
& b_{-1} \text{ given} \\
& e_t \text{ given, } R_t \text{ given}
\end{aligned}
$$

with $R_t = 1 + r_t$, $\beta \in (0,1)$, $u'(c) > 0$, $u'' < 0$, $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$

(i) β. The discounting factor for consumption in the next period. Since $\beta \in (0,1)$, this implies that Bob prefers to consume coconuts today instead of tomorrow.

(ii) $R_t = 1 + r_t$. The growth rate of coconuts seed planted in the previous period.

(iii) c_t. Consumption of coconuts at time t

(iv) b_t. Seeds planted at time t

(v) $e_t = (e_0, 0, ..., 0)$. Endowment of coconuts. Note that Bob is only endowed coconuts at time $t = 0$.

Hence, $c_t + b_t \le e_t + R_t b_{t-1}$ states that at the beginning of period t there is $e_t + R_t b_{t-1}$ coconuts and Bob can decide to consume them in the current period, plant them for the next one, or throw them in the sea.

Finally, $b_t \ge 0$ and $c_t \ge 0$ simply impose that Bob can't borrow coconuts or consume a negative amounts of coconuts.

2.

We set up the following Lagrangian to solve Bob's maximization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \big(\beta^t u(c_t) + \lambda_t(e_t + R_t b_{t-1} - c_t - b_t)\big)$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial b_t} = -\lambda_t + \lambda_{t+1} R_{t+1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = e_t + R_t b_{t-1} - c_t - b_t = 0$$

We can combine the FOCs to get the following Euler equation:

$$\boxed{u'(c_t) = R_{t+1}\beta u'(c_{t+1})}$$

Note that since $u(\cdot)$ is a concave function ($u'(c) > 0$, $u'' < 0$), the solution to the Lagrangian is an interior point and the borrowing constraint $b_t \geq 0$ is not bounding in this problem. If $b_t \geq 0$, was bounding, Bob would not smooth consumption at all and would in fact eat all his coconuts at $t = 0$ and starve the next period.

Now let's assume that Bob could actually borrow coconuts (i.e. $b_t < 0$ for some $t = 0, ...$). Since $e_t + R_t b_{t-1}$ is the amount of coconuts available at time t, having $b_t < 0$ for some $t = 0, ...$ would imply that next period the amount of coconuts available would be $R_{t+1} b_t < 0$. This means that to keep consuming a positive quantity, Bob would need to borrow at least $R_{t+1} b_t$. It holds recursively that Bob would keep on borrowing bigger and bigger quantities of coconuts without ever being able to repay its debt. Therefore, given the FOCs $u'(c_t) = R_{t+1}\beta u'(c_{t+1})$, it is not optimal for Bob to ever borrow.

3.

1. $R_{t+1} < \frac{1}{\beta}$.

   This implies that $u'(c_t) < u'(c_{t+1})$. Since $u'(c)$ is strictly decreasing, we get that $c_t > c_{t+1}$. Then for c to be constant $\Rightarrow R_{t+1} \not> \frac{1}{\beta}$.

2. $R_{t+1} > \frac{1}{\beta}$.

   This implies that $u'(c_t) > u'(c_{t+1})$. Since $u'(c)$ is strictly decreasing, we get that $c_t < c_{t+1}$. Then for c to be constant $\Rightarrow R_{t+1} \not< \frac{1}{\beta}$.

3. $R_{t+1} = \frac{1}{\beta}$.

   This implies that $u'(c_t) = u'(c_{t+1})$. Since $u'(c)$ is strictly decreasing, we get that $c_t = c_{t+1}$ (if not we would get either $c_t > c_{t+1} \Rightarrow u'(c_t) < u'(c_{t+1})$ or $c_t < c_{t+1} \Rightarrow u'(c_t) > u'(c_{t+1})$).

Hence, for c to be constant we need $R_{t+1} = \frac{1}{\beta}$.

4.

As shown previously, for $c = c_t = c_{t+1}$, we need $\frac{1}{\beta} = R_t = R_{t+1} = R$

We can rewrite our budget constraint in the following way

$$b_{t-1} = \frac{1}{R}(c + b_t - e_t)$$

and iterate this to get

$$b_{-1} = \frac{c - e_0}{R} + \frac{b_0}{R} = \frac{c - e_0}{R} + \frac{c - e_1}{R^2} + ... + \frac{c - e_n}{R^{n+1}} + \frac{b_n}{R^{n+1}}$$

We know that $b_{-1} = 0$ and $e_t = (e_0, 0, ..., 0)$. This implies that

$$\frac{e_0}{r} = \sum_{i=0}^{n} \frac{c}{R^{i+1}} + \frac{b_n}{R^{n+1}}$$
$$e_0 = c\sum_{i=0}^{n} \frac{1}{R^i} + \frac{b_n}{R^n}$$

Taking the limit as $n \to \infty$,

$$e_0 = \lim_{n\to\infty} c\sum_{i=0}^{n} \frac{1}{R^i} + \underbrace{\lim_{n\to\infty} \frac{b_n}{R^n}}_{\text{0if savings does not grow faster than R}}$$
$$= \frac{c}{1 - \frac{1}{R}}$$
$$\Rightarrow c = (1 - \frac{1}{R})e_0 = \underbrace{(1 - \beta)}_{\text{Propensity to consume}} \cdot \underbrace{e_0}_{\text{Total endowment}}$$

5.

$$\lim_{r\to\infty} c = \lim_{R\to\infty}(1 - \frac{1}{R})e_0 = e_0$$

Hence, as $r \to \infty$, Bob consumes the totality of the coconuts he arrived with (i.e. $c = e_0$) which is finite.

If we let Bob consume a bit less than e_0 (i.e. $c_0 = e_0 - \epsilon$), then next period Bob would have an infinity of coconuts which results in Bob consuming an infinity of coconuts from here on out (i.e. $c_t = \infty$ for $t = 1, ...$). This contradicts the fact that we have imposed that c is constant.

Moreover, if $c_0 = e_0$ there is nothing to save for tomorrow, hence $c_t = 0$ for $t = 1, ...$ which again contradicts c being constant.

Thus, to have a more sensible result, we could relax the assumption that make consumption constant (i.e. $R_t = R_{t+1} = \frac{1}{\beta}$) and Bob could take advantage of this miraculously fertile island.

Problem 2

1.

$$\begin{aligned}
\lim_{\sigma\to 1}\frac{c_t^{1-\sigma}-1}{1-\sigma} &= \lim_{\sigma\to 1}\frac{e^{\ln(c_t^{1-\sigma})}-1}{1-\sigma}\\
&= \lim_{\sigma\to 1}\frac{e^{(1-\sigma)\ln(c_t)}-1}{1-\sigma}\\
&\overbrace{=}^{\text{Hôpital's Rule}} \lim_{\sigma\to 1}\frac{-\ln(c_t)e^{(1-\sigma)\ln(c_t)}}{-1} = \ln(c_t)
\end{aligned}$$

2.

We have the following

$$U'(c_t) = \frac{(1-\sigma)c_t^{-\sigma}}{1-\sigma} = c_t^{-\sigma}$$

and

$$U''(c_t) = -\sigma c_t^{-\sigma-1}$$

Combining both equations yields

$$\frac{-c_t U''(c_t)}{U'(c_t)} = \frac{-c_t(-\sigma c_t^{-\sigma-1})}{c_t^{-\sigma}} = \frac{\sigma c_t^{-\sigma}}{c_t^{-\sigma}} = \sigma$$

3.

Recall that

$$U'(c_t) = c_t^{-\sigma} \Rightarrow U'(c_t)^{-\frac{1}{\sigma}} = c_t$$

Hence,

$$\begin{aligned}
\ln\left(\frac{c_{t+1}}{c_t}\right) &= \ln\left(\frac{U'(c_{t+1})^{-\frac{1}{\sigma}}}{U'(c_t)^{-\frac{1}{\sigma}}}\right)\\
&= -\frac{1}{\sigma}\ln\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)
\end{aligned}$$

Taking the partial derivative with respect to $\frac{U'(c_{t+1})}{U'(c_t)}$ yields

$$\frac{\partial \ln\left(\frac{c_{t+1}}{c_t}\right)}{\partial \ln\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)} = \frac{1}{\sigma}$$

4.

Since $c_t \geq 0$, we get that for $c_t \neq 0$

$$U'(c_t) = c_t^{-\sigma} > 0$$

Therefore, $U(c_t)$ is strictly increasing for $c_t > 0$.

Again since $c_t \geq 0$, we get that for $c_t \neq 0$

$$U''(c_t) = \underbrace{-\sigma}_{<0} \underbrace{c_t^{-\sigma-1}}_{>0} < 0$$

Thus, $U(c_t)$ is strictly concave for $c_t > 0$.

Finally, we check Inada's conditions, i.e.

$$\begin{aligned} \lim_{c_t \to 0} U'(c_t) &= \lim_{c_t \to 0} c_t^{-\sigma} \\ &= \lim_{c_t \to 0} \left(\frac{1}{c_t}\right)^{\sigma} = \infty \end{aligned}$$

and

$$\begin{aligned} \lim_{c_t \to 0} U'(c_t) &= \lim_{c_t \to 0} c_t^{-\sigma} \\ &= \lim_{c_t \to \infty} \left(\frac{1}{c_t}\right)^{\sigma} = 0 \end{aligned}$$

5.

Note that

$$\frac{\partial u(c)}{\partial c_t} = \beta^t c_t^{-\sigma}$$

This gives us that

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\beta^{t+s} c_{t+s}^{-\sigma}}{\beta^t c_t^{-\sigma}} = \beta^s \left(\frac{c_t}{c_{t+s}}\right)^{\sigma}$$

and

$$MRS(\lambda c_{t+s}, \lambda c_t) = \beta^s \left(\frac{\lambda c_t}{\lambda c_{t+s}}\right)^{\sigma} = \beta^s \left(\frac{c_t}{c_{t+s}}\right)^{\sigma} = MRS(c_{t+s}, c_t)$$

Hence, $u(c)$ is homothetic.

6.

Consumer's optimization problem.

$$\begin{aligned}
&\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\
\text{such that } &\sum_{t=0}^{\infty} p_t c_t \leq \lambda y \\
&c_t \geq 0 \\
&p_t \text{ given, } \lambda \text{ given, } y \text{ given}
\end{aligned}$$

with $\beta \in (0,1)$ and $\sum_{t=0}^{\infty} \beta^t U(c_t) = u(c)$.

We set up the following Lagrangian to solve the consumer's optimization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) + \mu(\lambda y - \sum_{t=0}^{\infty} p_t c_t)$$

FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t c_t^{-\sigma} - \mu p_t = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu} &= \lambda y - \sum_{t=0}^{\infty} p_t c_t = 0
\end{aligned}$$

We can combine the FOCs to get the following equations:

$$\boxed{\beta^t \frac{c_t^{-\sigma}}{p_t} = \beta^s \frac{c_s^{-\sigma}}{p_s}}$$

Without loss of generality, let $p_0 = 1$. Then, we get

$$\begin{aligned}
\beta^t \frac{c_t^{-\sigma}}{p_t} &= \beta^0 \frac{c_0^{-\sigma}}{p_0} \\
c_t^{\sigma} &= \frac{\beta^t}{p_t} c_0^{\sigma} \\
c_t &= c_0 \underbrace{\left(\frac{\beta^t}{p_t} \right)^{\frac{1}{\sigma}}}_{\theta_t}
\end{aligned}$$

Then,

$$\sum_{t=0}^{\infty} p_t c_t = \lambda y$$
$$\sum_{t=0}^{\infty} p_t c_0 \theta_t = \lambda y$$
$$c_0 \sum_{t=0}^{\infty} p_t \theta_t = \lambda y$$
$$c_0 = \frac{\lambda y}{\sum_{t=0}^{\infty} p_t \theta_t}$$

and

$$c_t = c_0 \cdot \theta_t = \frac{\lambda \theta_t y}{\sum_{t=0}^{\infty} p_t \theta_t}$$

Therefore, for $\lambda = 1$, the solution is given by

$$\hat{c}_t = \frac{\theta_t y}{\sum_{t=0}^{\infty} p_t \theta_t}$$

For $\lambda \neq 1$, we have

$$\tilde{c}_t = \lambda \frac{\theta_t y}{\sum_{t=0}^{\infty} p_t \theta_t} = \lambda \hat{c}_t$$

7.

We set up the following Lagrangian to solve the optimization problem.

$$\mathcal{L} = u(c) + \lambda_t (e_t + a_t - c_t - \frac{a_{t+1}}{1 + r_{t+1}})$$

FOCs

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{\partial u(c)}{\partial c_t} - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial a_{t+1}} = -\frac{\lambda_t}{1 + r_{t+1}} + \lambda_{t+1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = e_t + a_t - c_t - \frac{a_{t+1}}{1 + r_{t+1}} = 0$$

We can combine the FOCs to get the following Euler equation:

$$\boxed{MRS(c_{t+1}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{1}{1 + r_{t+1}}}$$

and

$$\boxed{MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_{t+s-1}}} \cdot \frac{\frac{\partial u(c)}{\partial c_{t+s-1}}}{\frac{\partial u(c)}{\partial c_{t+s-2}}} \cdot \ldots \cdot \frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \prod_{i=t}^{t+s-1} \frac{1}{1 + r_{i+1}}}$$

8.

Recall that $\frac{1}{1+r_{t+1}} = MRS(c_{t+1}, c_t)$ and thus

$$\begin{aligned}
\frac{1}{1+r_{t+1}} &= \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) \\
&= \beta\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \\
&= \beta\left(\frac{(1+g)^t c_0}{(1+g)^{t+1} c_0}\right)^{\sigma} \\
&= \beta(1+g)^{-\sigma} \\
\Rightarrow r &= \frac{1}{\beta}(1+g)^{\sigma} - 1
\end{aligned}$$

9.

As shown in part 8, we have the following relation between r and g

$$r = \frac{1}{\beta}(1+g)^{\sigma} - 1$$

Taking the derivative with respect to g yields

$$\frac{\partial r}{\partial g} = \frac{\sigma}{\beta}(1+g)^{\sigma-1} > 0$$

i.e. $g \uparrow \Rightarrow r \uparrow$.

Note that by the chain rule, we have

$$\begin{aligned}
\frac{\partial\left(\frac{\partial r}{\partial g}\right)}{\partial\left(\frac{1}{\sigma}\right)} &= \frac{\frac{\partial\left(\frac{\partial r}{\partial g}\right)}{\partial \sigma}}{\frac{\partial\left(\frac{1}{\sigma}\right)}{\partial \sigma}} \\
&= \frac{\frac{1}{\beta}\left((1+g)^{\sigma-1} + \sigma(1+g)^{\sigma-1}\ln(1+g)\right)}{-\frac{1}{\sigma^2}} \\
&= -\frac{1}{\beta}\left(\sigma^2(1+g)^{\sigma-1} + \sigma^3(1+g)^{\sigma-1}\ln(1+g)\right) < 0
\end{aligned}$$

i.e. $\frac{1}{\sigma} \uparrow \Rightarrow \frac{\partial r}{\partial g} \downarrow$.

Hence, as intertemporal elasticity of substitution (IES) increases, we get that the "sensitivity" of r to g decreases. The intuition behind this is that IES represents the willingness to substitute consumption over time, or simply a measure of the curvature of the utility function. Thus, the more IES increases, the more indifferent we become to swings in consumption over time. Now, the higher the growth rate g, the higher the swings in consumption between period. Hence while r will increase to counter a higher g, it will do so less aggressively with an higher IES.

10.

Recall that

$$MRS(c_t, c_{t-1}) = \frac{1}{1+r_t}$$

If consumption is growing at a constant rate g, then $c_t = (1+g)^t c_0$. Without loss of generality let $s > t$, then

$$\begin{aligned}
MRS(c_s, c_{s-1}) &= MRS((1+g)^s c_0, (1+g)^{s-1} c_0) \\
&= MRS(\underbrace{(1+g)^{s-t}}_{\gamma} c_t, \underbrace{(1+g)^{s-t}}_{\gamma} c_{t-1}) \\
&\underbrace{=}_{\text{Homothetic}} MRS(c_t, c_{t-1})
\end{aligned}$$

Combining the previous with $MRS(c_t, c_{t-1}) = \frac{1}{1+r_t}$, we get that

$$\begin{aligned}
\frac{MRS(c_t, c_{t-1})}{MRS(c_s, c_{s-1})} &= \frac{1+r_s}{1+r_t} \\
1 &= \frac{1+r_s}{1+r_t} \\
\Rightarrow r_t &= r_s \quad \forall s, t
\end{aligned}$$

1.

$$U(c_t) = -e^{-\gamma c_t}$$

Note that

$$U'(c_t) = \gamma e^{-\gamma c_t}$$

and

$$U''(c_t) = -\gamma^2 e^{-\gamma c_t}$$

Hence,

$$-\frac{U''(c_t)}{U'(c_t)} = -\frac{-\gamma^2 e^{-\gamma c_t}}{\gamma e^{-\gamma c_t}} = \gamma$$

and

$$-\frac{c_t U''(c_t)}{U'(c_t)} = -\frac{-c_t \gamma^2 e^{-\gamma c_t}}{\gamma e^{-\gamma c_t}} = c_t \gamma$$

Taking the derivative yields

$$\frac{\partial \left[-\frac{c_t U''(c_t)}{U'(c_t)}\right]}{\partial c_t} = \gamma > 0$$

i.e. increasing relative risk aversion.

2.

We have that

$$\frac{\partial u(c)}{\partial c_t} = \beta^t e^{-\gamma c_t}$$

Therefore,

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\beta^{t+s} e^{-\gamma c_{t+s}}}{\beta^t e^{-\gamma c_t}} = \beta^s e^{-\gamma(c_{t+s} - c_t)}$$

Thus,

$$MRS(\lambda c_{t+s}, \lambda c_t) = \beta^s e^{-\gamma(\lambda c_{t+s} - \lambda c_t)} = \beta^{s(1-\lambda)}(\beta^s e^{-\gamma(c_{t+s}-c_t)})^\lambda = \beta^{s(1-\lambda)} MRS(c_{t+s}, c_t)^\lambda \neq MRS(c_{t+s}, c_t)$$

i.e. for any $\lambda \neq 0$ we have $MRS(\lambda c_{t+s}, \lambda c_t) \neq MRS(c_{t+s}, c_t)$ which implies that the function is not homothetic.

3.

Consumer's optimization problem.

$$\begin{aligned} \max_{\{c_t\}} & \sum_{t=0}^{\infty} \beta^t U(c_t) \\ \text{such that } & \sum_{t=0}^{\infty} p_t c_t \leq y \\ & c_t \geq 0 \\ & p_t \text{ given, } y \text{ given} \end{aligned}$$

with $\beta \in (0, 1)$ and $\sum_{t=0}^{\infty} \beta^t U(c_t) = u(c)$.

We set up the following Lagrangian to solve the optimization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t - e^{-\gamma c_t} + \mu(\lambda y - \sum_{t=0}^{\infty} p_t c_t)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t \gamma e^{-\gamma c_t} - \mu p_t = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= \lambda y - \sum_{t=0}^{\infty} p_t c_t = 0 \end{aligned}$$

We can combine the FOCs to get the following equations:

$$\beta^t \frac{\gamma e^{-\gamma c_t}}{p_t} = \beta^s \frac{\gamma e^{-\gamma c_s}}{p_s}$$

Without loss of generality, let $p_0 = 1$. Then, we get

$$\begin{aligned}
\beta^t \frac{\gamma e^{-\gamma c_t}}{p_t} &= \beta^0 \frac{\gamma e^{-\gamma c_0}}{p_0} \\
e^{\gamma c_t} &= \frac{\beta^t}{p_t} e^{\gamma c_0} \\
\gamma c_t &= \ln\left(\frac{\beta^t}{p_t}\right) + \gamma c_0 \\
c_t &= c_0 + \underbrace{\frac{1}{\gamma} \ln\left(\frac{\beta^t}{p_t}\right)}_{\theta_t}
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t c_t &= \lambda y \\
\sum_{t=0}^{\infty} p_t \left(c_0 + \theta_t\right) &= \lambda y \\
c_0 \sum_{t=0}^{\infty} p_t + \sum_{t=0}^{\infty} p_t \theta_t &= \lambda y \\
c_0 &= \frac{\lambda y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t}
\end{aligned}$$

and

$$c_t = c_0 + \theta_t = \frac{\lambda y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t} + \theta_t$$

Therefore, for $\lambda = 1$,

$$\hat{c}_t = \frac{y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t} + \theta_t$$

for $\lambda \neq 1$

$$\tilde{c}_t = \frac{(\lambda - 1) y}{\sum_{t=0}^{\infty} p_t} + \frac{y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t} + \theta_t = \frac{(\lambda - 1) y}{\sum_{t=0}^{\infty} p_t} + \hat{c}_t$$