ECON 6130 - Problem Set # 6

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Problem 1

(1) Given m an Arrow-Debreu equilibrium is an allocation $\hat{c}_1^0$, $\left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^\infty$ and prices $\{p_t\}_{t=1}^\infty$ such that

(i) Given prices, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves

$$\max_{(c_t^t, c_{t+1}^t)} (\log c_t^t + \log c_{t+1}^t)$$
$$\text{such that } p_t c_t^t + p_{t+1} c_{t+1}^t \leq p_t w_1 + p_{t+1} w_2$$

(ii) Given prices, for each $t \geq 1$, $(\hat{c}_1^0)$ solves

$$\max_{c_1^0} \log c_1^0$$
$$\text{such that } p_1 c_1^0 \leq p_1 w_2 + m$$

(iii) For all $t \geq 1$ markets clear

$$\hat{c}_t^{t-1} + \hat{c}_{t+1}^t = e_t^{t-1} + e_t^t$$

(2) Given m, a sequential market equilibrium is an allocation $\hat{c}_1^0$, $\left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^\infty$ and prices $\{r_t\}_{t=1}^\infty$ such that

(i) Given prices, for each $t \geq 1$, $(\hat{c}_t^t, \hat{c}_{t+1}^t)$ solves

$$\max_{(c_t^t, c_{t+1}^t)} (\log c_t^t + \log c_{t+1}^t)$$
$$\text{such that } c_t^t + s_t^t \leq e_t^t$$
$$c_{t+1}^t \leq e_{t+1}^t + (1 + r_{t+1}) s_t^t$$

(ii) Given prices, for each $t \geq 1$, $(\hat{c}_1^0)$ solves

$$\max_{c_1^0} \log c_1^0$$
$$\text{such that } c_1^0 \leq w_2 + (1 + r_1) m$$

(iii) For all $t \geq 1$ markets clear

$$\hat{c}_t^{t-1} + \hat{c}_t^t = e_t^{t-1} + e_t^t$$

(3) Note that for $m = 0$, the budget constraint of generation 0 is $c_1^0 \leq w_2$ in the sequential equilibrium and $p_1 c_1^0 \leq p_1 w_2$ in the Arrow-Debreu equilibrium.

Since $\log(\cdot)$ is strictly increasing, we have $c_1^0 = w_2$ in both cases.

Using the market clearing property, we have

$$\begin{aligned} \hat{c}_1^0 + \hat{c}_1^t &= e_1^0 + e_1^1 \\ \hat{c}_1^1 &= e_1^1 = w_1 \end{aligned}$$

Hence, we can do this recursively and get that $c_t^t = w_1$ and $c_{t+1}^t = w_2$, i.e. autarky.

(4) An allocation c_1^0, $\left\{c_t^t, c_{t+1}^t\right\}_{t=1}^{\infty}$ is Pareto optimal if it is feasible and if there is no other feasible allocation $\hat{c}_1^0$, $\left\{\hat{c}_t^t, \hat{c}_{t+1}^t\right\}_{t=1}^{\infty}$

$$\begin{aligned} u_t(\hat{c}_t^t, \hat{c}_{t+1}^t) &\geq u_t(c_t^t, c_{t+1}^t) \\ u_t(\hat{c}_1^0) &\geq u_t(c_1^0) \end{aligned}$$

with strict inequality for at least one $t \geq 0$.

Note that the price ratio/interest rate that supports the autarky equilibrium is

$$1 + r_{t+1} = \frac{p_t}{p_{t+1}} = \frac{u'(w_1)}{u'(w_2)} = \frac{w_2}{w_1} > 1$$

Hence, this implies that the autarkic interest rate is smaller than 0, i.e. Samuelson case.

As shown in the notes, this implies that the autarkic equilibrium is not Pareto optimal.

We can show it directly by giving δ_0 to generation 0. To offset this, we need to take from generation 1 δ_0. To offset this loss, we need to give δ_1 to generation 1 in the next period, such that

$$\delta_1 u'(w_2) = \delta_0 u'(w_1) \Rightarrow \delta_1 = \delta_0 \frac{w_1}{w_2}$$

If we do this recursively, we have

$$\delta_t = \delta_0 \prod_{\tau=1}^{t} \frac{w_1}{w_2} = \delta_0 \left(\frac{w_1}{w_2}\right)^t$$

Since $\frac{w_1}{w_2} < 1$, $\delta_t < \infty$ and the new $\tilde{c}_t^i$ as define previously is feasible and gives us the same utility for all generation $t \geq 1$, but strictly greater utility for generation 0 compared to autarky. Hence, autarky is not Pareto optimal

(5) Given m, an Arrow-Debreu equilibrium is an allocation $\hat{c}_{1,1}^0, \hat{c}_{2,1}^0, \left\{\hat{c}_{1,t}^t, \hat{c}_{2,t}^t, \hat{c}_{2,t+1}^t, \hat{c}_{2,t+1}^t\right\}_{t=1}^{\infty}$ and prices $\{p_{1,t}, p_{2,t}\}_{t=1}^{\infty}$ such that

(i) Given prices, for each $t \geq 1$, $(\hat{c}^t_{1,t}, \hat{c}^t_{2,t}, \hat{c}^t_{2,t+1}, \hat{c}^t_{2,t+1})$ solves

$$\max_{(\hat{c}^t_{1,t}, \hat{c}^t_{2,t}, \hat{c}^t_{2,t+1}, \hat{c}^t_{2,t+1})} (\log c^t_{1,t} + \log c^t_{2,t} + \log c^t_{1,t+1} + \log c^t_{2,t+1})$$

such that

$$p_{1,t}c^t_{1,t} + p_{2,t}c^t_{1,t} + p_{1,t+1}c^t_{1,t+1} + p_{2,t+1}c^t_{2,t+1} \leq p_{1,t}w^t_{1,t} + p_{2,t}w^t_{2,t} + p_{1,t+1}w^t_{1,t} + p_{2,t+1}w^t_{2,t+1}$$

(ii) Given prices, for each $t \geq 1$, $(\hat{c}^0_{1,1}, \hat{c}^0_{2,1})$ solves

$$\max_{(\hat{c}^0_{1,1}, \hat{c}^0_{2,1})} (\log c^0_{1,1} + \log c^0_{2,1})$$
$$p_{1,1}c^0_{1,1} + p_{2,1}c^0_{2,1} \leq p_{1,1}w^0_{1,1} + p_{2,1}w^0_{2,1} + m$$

(iii) For all $t \geq 1$ and i markets clear

$$\hat{c}^{t-1}_{i,t} + \hat{c}^t_{i,t} = w^{t-1}_{i,t} + w^t_{i,t}$$

Let $m = 0$ and $c^t_{1,t} = 2$, then $c^t_{1,t+1} = 2$ by market clearing.

Note from the generation 0 FOCs, we have

$$\begin{aligned}
\frac{1}{c^0_{i,1}} &= \lambda p_{i,1} \\
\frac{c^0_{2,1}}{c^0_{1,1}} &= \frac{p_{1,1}}{p_{2,1}} \\
c^0_{2,1} &= \frac{p_{1,1}}{p_{2,1}} c^0_{1,1}
\end{aligned}$$

Combining this to the budget constraint yields

$$\begin{aligned}
p_{1,1}c^0_{1,1} + p_{2,1}c^0_{2,1} &= p_{1,1}w^0_{1,1} + p_{2,1}w^0_{2,1} \\
p_{1,1}c^0_{1,1} + p_{1,1}c^0_{1,1} &= 2p_{1,1} + p_{2,1} \\
c^0_{1,1} &= 1 + \frac{1}{2}\frac{p_{2,1}}{p_{1,1}} \\
\frac{1}{2} &= \frac{p_{1,1}}{p_{2,1}}
\end{aligned}$$

Note from the generation 1 FOCs, we have

$$\begin{aligned}
&\frac{1}{c^0_{i,t}} = \lambda p_{i,t} \\
&\frac{c^1_{2,1}}{c^1_{1,1}} = \frac{p_{1,1}}{p_{2,1}} \text{ and } \frac{c^1_{2,2}}{c^1_{1,1}} = \frac{p_{1,1}}{p_{2,2}} \text{ and } \frac{c^1_{1,2}}{c^1_{1,1}} = \frac{p_{1,1}}{p_{1,2}} \\
&c^1_{2,1} = \frac{p_{1,1}}{p_{2,1}} c^1_{1,1} \text{ and } c^1_{2,2} = \frac{p_{1,1}}{p_{2,2}} c^1_{1,1} \text{ and } c^1_{1,2} = \frac{p_{1,1}}{p_{1,2}} c^1_{1,1}
\end{aligned}$$

Combining this to the budget constraint yields

$$\begin{aligned}
p_{1,1}c^1_{1,1} + p_{2,1}c^1_{2,1} + p_{1,2}c^1_{1,2} + p_{2,2}c^1_{2,2} &= p_{1,1}w^1_{1,1} + p_{2,1}w^1_{2,1} + p_{1,2}w^1_{1,2} + p_{2,2}w^1_{2,2} \\
4p_{1,1}c^1_{1,1} &= 2p_{1,1} + 4p_{2,1} + 2p_{1,2} + p_{2,2} \\
8p_{1,1} &= 2p_{1,1} + 4p_{2,1} + 2p_{1,2} + p_{2,2} \\
6 &= 4\frac{p_{2,1}}{p_{1,1}} + 2\frac{p_{1,2}}{p_{1,1}} + \frac{p_{2,2}}{p_{1,1}} \\
-2 &= 2\frac{p_{1,2}}{p_{1,1}} + \frac{p_{2,2}}{p_{1,1}} \\
-2p_{1,1} &= 2p_{1,2} + p_{2,2}
\end{aligned}$$

Thus, if $p_{1,1} \geq 0$, we need $p_{1,2} < 0$ or $p_{2,2} < 0$, which is a contradiction.