ECON 6130 - Problem Set # 3

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September 27, 2017

Problem 1

1.

Maximization problems:

(i) Households.

$$
\begin{aligned}
\max_{\{c_t,k_{t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty &\beta^t u(c_t, s_t)\\
&\text{such that}\\
&c_t + k_{t+1} - (1-\delta)k_t \leq w_t n_t + r_t k_t\\
&c_t, k_{t+1} \geq 0, n_t = 1\\
&n_t = 1\\
&s_t, k_0 \text{ given}
\end{aligned}
$$

(ii) Firms.

$$
\begin{aligned}
\max_{\{k_t,n_t\}_{t=0}^\infty} &f(k_t, n_t) - w_t n_t - r_t k_t\\
&\text{such that}\\
&n_t, k_t \geq 0\\
&k_0 \text{ given}
\end{aligned}
$$

Note that the representative consumer does not understand its own impact on s_t, therefore, in the household problem, s_t is taken as given even though $s_t = \theta f(k_t, n_t)$.

A sequential markets equilibrium is a sequence of prices $\{w_t, r_t\}_{t=0}^\infty$, allocations for the representative household $\left\{c_t, k_{t+1}^s\right\}_{t=0}^\infty$ and allocations for the representative $\left\{n_t^d, k_t^d\right\}_{t=0}^\infty$ such that

(i) Given k_0 and $\{w_t, r_t\}_{t=0}^\infty$, allocations for the representative household $\left\{c_t, k_{t+1}^s\right\}_{t=0}^\infty$ solves the household problem.

(ii) For each $t \geq 0$, given (w_t, r_t) the firm allocation (n_t^d, k_t^d) solves the firms' maximization problem.

(iii) Markets clears.

$$\begin{aligned} n_t^d &= 1 \\ k_t^d &= k_t^s \\ f(k_t^d, n_t^d) &= c_t + k_{t+1}^s - (1-\delta)k_t^s \end{aligned}$$

2.

Firms FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial n_t} &= \frac{\partial f(k_t, n_t)}{\partial n_t} - w_t = 0 \\ \frac{\partial \mathcal{L}}{\partial k_t} &= \frac{\partial f(k_t, n_t)}{\partial k_t} - r_t = 0 \end{aligned}$$

Euler theorem and constant return to scale, we have $f(k_t, n_t) = \frac{\partial f(k_t,n_t)}{\partial n_t} n_t + \frac{\partial f(k_t,n_t)}{\partial k_t} k_t$. Accordingly, the households problem becomes

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} & \beta^t u(c_t, s_t) \\ & \text{such that} \\ & c_t + k_{t+1} - (1-\delta)k_t = f(k_t, 1) \\ & c_t, k_{t+1} \geq 0 \\ & s_t, k_0 \text{ given} \end{aligned}$$

We set up the following Lagrangian to solve the maximization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, s_t) + \lambda_t \left(f(k_t, 1) - c_t - k_{t+1} + (1-\delta)k_t \right)$$

Houshold FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t u_c(c_t, s_t) - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \lambda_{t+1} \left(f_k(k_{t+1}, 1) + (1-\delta) \right) - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= f(k_t, 1) - c_t - k_{t+1} + (1-\delta)k_t = 0 \end{aligned}$$

Combining these equations, we get

$$u_c(c_t, s_t) = \beta u_c(c_{t+1}, s_{t+1}) \left[f_k(k_{t+1}, 1) + (1-\delta) \right]$$

or

$$\frac{u_c(f(k_t,1) - k_{t+1} + (1-\delta)k_t, \theta f_k(k_t,1))}{u_c(f(k_{t+1},1) - k_{t+2} + (1-\delta)k_{t+1}, \theta f_k(k_{t+1},1))} = \beta \left[f_k(k_{t+1},1) + (1-\delta)\right]$$

In general, the ratio of $u_c(c_t, s_t)$ and $u_c(c_{t+1}, s_{t+1})$ will not be equal to one and will depend on θ. Therefore, while the household does not account for its impacts on s_t, the path of k_t will depend on θ since marginal utility relies on $s_t = \theta f_k(k_t, 1)$.

For the steady state, we have $c_t = c_{t+1} = c$ and $k_t = k_{t+1} = k$ which implies that $s_t = s = \theta f(k, 1)$ is constant. Then,

$$\begin{aligned} u_c(c,s) &= \beta u_c(c,s)\left[f_k(k,1) + (1-\delta)\right] \\ \frac{1}{\beta} &= f_k(k,1) + (1-\delta) \\ \Rightarrow f_k(k,1) &= 1 + \frac{1}{\beta} - \delta \end{aligned}$$

i.e. k does not depend on θ.

Note that the problem is designed in such a way that θ is not part of the information set for the representative consumer. Therefore, since the households find a steady state that maximizes their utility given any s_t and as such any θ, k will not depend on θ.

Hence, while the steady state is independent of θ, the law of motion of s_t is not. This will impact marginal utility and as such how the representative will adjust k_t to maximize utility and reach the steady state. Thus, even if the representative agent does not understand its impact on s_t, it will still be reflected on how k_t changes, but not on the steady state per se.

3.

Social planner maximization problem:

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} & \beta^t u(c_t, s_t) \\ & \text{such that} \\ & c_t + k_{t+1} - (1-\delta)k_t = f(k_t, n_t) \\ & s_t = \theta f(k_t, n_t) \\ & c_t, k_{t+1} \geq 0, n_t = 1 \\ & k_0 \text{ given} \end{aligned}$$

We set up the following Lagrangian to solve the maximization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, s_t) + \lambda_t \left(f(k_t,1) - c_t - k_{t+1} + (1-\delta)k_t\right) + \mu_t \left(\theta f(k_t,1) - s_t\right)$$

FOCs:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t u_c(c_t, s_t) - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial s_t} &= \beta^t u_s(c_t, s_t) - \mu_t = 0 \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} &= \lambda_{t+1}\left(f_k(k_{t+1}, 1) + (1-\delta)\right) + \mu_{t+1}\theta f_k(k_{t+1}, 1) - \lambda_t = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_t} &= f(k_t, 1) - c_t - k_{t+1} + (1-\delta)k_t = 0 \\
\frac{\partial \mathcal{L}}{\partial \mu_t} &= f(k_t, 1) - s_t = 0
\end{aligned}
$$

Combining these equations, we get

$$
u_c(c_t, s_t) = \beta\left\{u_c(c_{t+1}, s_{t+1})\left[f_k(k_{t+1}, 1) + (1-\delta)\right] + u_s(c_{t+1}, s_{t+1})\theta f_k(k_{t+1}, 1)\right\}
$$

Therefore at the steady state,

$$
\begin{aligned}
u_c(c, s) &= \beta\left\{u_c(c, s)\left[f_k(k, 1) + (1-\delta)\right] + u_s(c, s)\theta f_k(k, 1)\right\} \\
\frac{1}{\beta} &= f_k(k, 1)\left[1 + \theta\frac{u_s(c, s)}{u_c(c, s)}\right] + (1-\delta) \\
\Rightarrow f_k(k, 1) &= \left[1 + \frac{1}{\beta} - \delta\right] \cdot \left[1 + \theta\frac{u_s(c, s)}{u_c(c, s)}\right]^{-1}
\end{aligned}
$$

Thus, the solution for the social planner problem and the sequential market equilibrium coincide if and only if

$$
\begin{aligned}
1 + \theta\frac{u_s(c, s)}{u_c(c, s)} &= 1 \\
\theta\frac{u_s(c, s)}{u_c(c, s)} &= 0
\end{aligned}
$$

Hence, both problems yield the same steady state if either $\theta = 0$ (i.e. no pollution) or $u_s(c, s) = 0$ (i.e. no marginal utility for pollution).

4.

Recall the sequential market equilibrium problem. We can modify the household problem by taxing both wages and capital gains at a rate τ. In order to keep the budget-neutral, we will transfer tax revenues to the consumer in the form of a lump-sum transfer T_t where

$$
(1-\tau)w_t n_t + (1-\tau)r_t k_t = (1-\tau)f(k_t, n_t) \equiv T_t
$$

Accordingly, the households problem becomes

$$\max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, s_t)$$
$$\text{such that}$$
$$c_t + k_{t+1} - (1-\delta)k_t = (1-\tau)f(k_t, n_t) + T_t$$
$$c_t, k_{t+1} \geq 0$$
$$T_t, s_t, k_0 \text{ given}$$

Note that we assume that the representative agent act as if T_t is simply given. If the agent understood its impact on T_t, we would get the exact same equilibrium as before since we are dealing with a budget-neutral tax.

We set up the following Lagrangian to solve the maximization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t, s_t) + \lambda_t \left((1-\tau)f(k_t, 1) + T_t - c_t - k_{t+1} + (1-\delta)k_t\right)$$

Houshold FOCs:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \beta^t u_c(c_t, s_t) - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial k_{t+1}} = \lambda_{t+1}\left[(1-\tau)f_k(k_{t+1}, 1) + (1-\delta)\right] - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda_t} = (1-\tau)f(k_t, 1) + T_t - c_t - k_{t+1} + (1-\delta)k_t = 0$$

Combining these equations, we get

$$u_c(c_t, s_t) = \beta u_c(c_{t+1}, s_{t+1})\left[(1-\tau)f_k(k_{t+1}, 1) + (1-\delta)\right]$$

Therefore at the steady state,

$$u_c(c, s) = \beta u_c(c, s)\left[(1-\tau)f_k(k, 1) + (1-\delta)\right]$$
$$\frac{1}{\beta} = (1-\tau)f_k(k, 1) + (1-\delta)$$
$$\Rightarrow f_k(k, 1) = \left[1 + \frac{1}{\beta} - \delta\right] \cdot [1-\tau]^{-1}$$

Hence, we obtain the exact same solution as the social planner problem if we set τ such that

$$\tau = -\theta \frac{u_s(c, s)}{u_c(c, s)}$$

Problem 2

Proof.

$$\begin{aligned}
\rho(T^n v_0, v) &= \rho(T^n v_0, Tv) && \text{(By definition of fixed point)} \\
&\leq \rho(T^n v_0, T^{n+1} v_0) + \rho(T^{n+1} v_0, Tv) && \text{(By triangle inequality)} \\
&\leq \rho(T^n v_0, T^{n+1} v_0) + \beta \rho(T^n v_0, v) && \text{(By definition of contraction mapping)} \\
(1-\beta)\rho(T^n v_0, v) &\leq \rho(T^n v_0, T^{n+1} v_0) \\
\Rightarrow \rho(T^n v_0, v) &\leq \frac{1}{1-\beta} \rho(T^n v_0, T^{n+1} v_0)
\end{aligned}$$

□