

ECON 6130 - Problem Set # 1

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Problem 1

1.

Bob's maximization problem.

$$\begin{aligned} & \max_{\{c_t\}, \{b_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & \text{such that } c_t + b_t \leq e_t + R_t b_{t-1} \\ & \quad b_t \geq 0 \\ & \quad c_t \geq 0 \\ & \quad b_{-1} \text{ given} \\ & \quad e_t \text{ given, } R_t \text{ given} \end{aligned}$$

with $R_t = 1 + r_t$, $\beta \in (0, 1)$, $u'(c) > 0$, $u'' < 0$, $\lim_{c \rightarrow 0} u'(c) = \infty$ and $\lim_{c \rightarrow \infty} u'(c) = 0$

- (i) β . The discounting factor for consumption in the next period. Since $\beta \in (0, 1)$, this implies that Bob prefers to consume coconuts today instead of tomorrow.
- (ii) $R_t = 1 + r_t$. The growth rate of coconuts seed planted in the previous period.
- (iii) c_t . Consumption of coconuts at time t
- (iv) b_t . Seeds planted at time t
- (v) $e_t = (e_0, 0, \dots, 0)$. Endowment of coconuts. Note that Bob is only endowed coconuts at time $t = 0$.

Hence, $c_t + b_t \leq e_t + R_t b_{t-1}$ states that at the beginning of period t there is $e_t + R_t b_{t-1}$ coconuts and Bob can decide to consume them in the current period, plant them for the next one, or throw them in the sea.

Finally, $b_t \geq 0$ and $c_t \geq 0$ simply impose that Bob can't borrow coconuts or consume a negative amounts of coconuts.

2.

We set up the following Lagrangian to solve Bob's maximization problem.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t (e_t + R_t b_{t-1} - c_t - b_t)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t u'(c_t) - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial b_t} &= -\lambda_t + \lambda_{t+1} R_{t+1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= e_t + R_t b_{t-1} - c_t - b_t = 0 \end{aligned}$$

We can combine the FOCs to get the following Euler equation:

$$\boxed{u'(c_t) = R_{t+1} \beta u'(c_{t+1})}$$

Note that since $u(\cdot)$ is a concave function ($u'(c) > 0$, $u'' < 0$), the solution to the Lagrangian is an interior point and the borrowing constraint $b_t \geq 0$ is not bounding in this problem. If $b_t \geq 0$, was bounding, Bob would not smooth consumption at all and would in fact eat all his coconuts at $t = 0$ and starve the next period.

Now let's assume that Bob could actually borrow coconuts (i.e. $b_t < 0$ for some $t = 0, \dots$). Since $e_t + R_t b_{t-1}$ is the amount of coconuts available at time t , having $b_t < 0$ for some $t = 0, \dots$ would imply that next period the amount of coconuts available would be $R_{t+1} b_t < 0$. This means that to keep consuming a positive quantity, Bob would need to borrow at least $R_{t+1} b_t$. It holds recursively that Bob would keep on borrowing bigger and bigger quantities of coconuts without ever being able to repay its debt. Therefore, given the FOCs $u'(c_t) = R_{t+1} \beta u'(c_{t+1})$, it is not optimal for Bob to ever borrow.

3.

1. $R_{t+1} < \frac{1}{\beta}$.

This implies that $u'(c_t) > u'(c_{t+1})$. Since $u'(c)$ is strictly decreasing, we get that $c_t < c_{t+1}$. Then for c to be constant $\Rightarrow R_{t+1} \not< \frac{1}{\beta}$.

2. $R_{t+1} > \frac{1}{\beta}$.

This implies that $u'(c_t) < u'(c_{t+1})$. Since $u'(c)$ is strictly decreasing, we get that $c_t > c_{t+1}$. Then for c to be constant $\Rightarrow R_{t+1} \not> \frac{1}{\beta}$.

3. $R_{t+1} = \frac{1}{\beta}$.

This implies that $u'(c_t) = u'(c_{t+1})$. Since $u'(c)$ is strictly decreasing, we get that $c_t = c_{t+1}$ (if not we would get either $c_t > c_{t+1} \Rightarrow u'(c_t) < u'(c_{t+1})$ or $c_t < c_{t+1} \Rightarrow u'(c_t) > u'(c_{t+1})$).

Hence, for c to be constant we need $R_{t+1} = \frac{1}{\beta}$.

4.

As shown previously, for $c = c_t = c_{t+1}$, we need $\frac{1}{\beta} = R_t = R_{t+1} = R$

We can rewrite our budget constraint in the following way

$$b_{t-1} = \frac{1}{R}(c + b_t - e_t)$$

and iterate this to get

$$b_{-1} = \frac{c - e_0}{R} + \frac{b_0}{R} = \frac{c - e_0}{R} + \frac{c - e_1}{R^2} + \dots + \frac{c - e_n}{R^{n+1}} + \frac{b_n}{R^{n+1}}$$

We know that $b_{-1} = 0$ and $e_t = (e_0, 0, \dots, 0)$. This implies that

$$\begin{aligned} \frac{e_0}{r} &= \sum_{i=0}^n \frac{c}{R^{i+1}} + \frac{b_n}{R^{n+1}} \\ e_0 &= c \sum_{i=0}^n \frac{1}{R^i} + \frac{b_n}{R^n} \end{aligned}$$

Taking the limit as $n \rightarrow \infty$,

$$\begin{aligned} e_0 &= \lim_{n \rightarrow \infty} c \sum_{i=0}^n \frac{1}{R^i} + \underbrace{\lim_{n \rightarrow \infty} \frac{b_n}{R^n}}_0 \\ &= \frac{c}{1 - \frac{1}{R}} \\ \Rightarrow c &= \left(1 - \frac{1}{R}\right)e_0 = \underbrace{(1 - \beta)}_{\text{Propensity to consume}} \cdot \underbrace{e_0}_{\text{Total endowment}} \end{aligned}$$

5.

$$\lim_{r \rightarrow \infty} c = \lim_{R \rightarrow \infty} \left(1 - \frac{1}{R}\right)e_0 = e_0$$

Hence, as $r \rightarrow \infty$, Bob consumes the totality of the coconuts he arrived with (i.e. $c = e_0$) which is finite.

If we let Bob consume a bit less than e_0 (i.e. $c_0 = e_0 - \epsilon$), then next period Bob would have an infinity of coconuts which results in Bob consuming an infinity of coconuts from here on out (i.e. $c_t = \infty$ for $t = 1, \dots$). This contradicts the fact that we have imposed that c is constant.

Moreover, if $c_0 = e_0$ there is nothing to save for tomorrow, hence $c_t = 0$ for $t = 1, \dots$ which again contradicts c being constant.

Thus, to have a more sensible result, we could relax the assumption that make consumption constant (i.e. $R_t = R_{t+1} = \frac{1}{\beta}$) and Bob could take advantage of this miraculously fertile island.

Problem 2

1.

$$\begin{aligned}
 \lim_{\sigma \rightarrow 1} \frac{c_t^{1-\sigma} - 1}{1 - \sigma} &= \lim_{\sigma \rightarrow 1} \frac{e^{\ln(c_t^{1-\sigma})} - 1}{1 - \sigma} \\
 &= \lim_{\sigma \rightarrow 1} \frac{e^{(1-\sigma)\ln(c_t)} - 1}{1 - \sigma} \\
 &\stackrel{\text{H\^opital's Rule}}{=} \lim_{\sigma \rightarrow 1} \frac{-\ln(c_t)e^{(1-\sigma)\ln(c_t)}}{-1} = \ln(c_t)
 \end{aligned}$$

2.

We have the following

$$U'(c_t) = \frac{(1-\sigma)c_t^{-\sigma}}{1-\sigma} = c_t^{-\sigma}$$

and

$$U''(c_t) = -\sigma c_t^{-\sigma-1}$$

Combining both equations yields

$$\frac{-c_t U''(c_t)}{U'(c_t)} = \frac{-c_t(-\sigma c_t^{-\sigma-1})}{c_t^{-\sigma}} = \frac{\sigma c_t^{-\sigma}}{c_t^{-\sigma}} = \sigma$$

3.

Recall that

$$U'(c_t) = c_t^{-\sigma} \Rightarrow U'(c_t)^{-\frac{1}{\sigma}} = c_t$$

Hence,

$$\begin{aligned}
 \ln\left(\frac{c_{t+1}}{c_t}\right) &= \ln\left(\frac{U'(c_{t+1})^{-\frac{1}{\sigma}}}{U'(c_t)^{-\frac{1}{\sigma}}}\right) \\
 &= -\frac{1}{\sigma} \ln\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)
 \end{aligned}$$

Taking the partial derivative with respect to $\frac{U'(c_{t+1})}{U'(c_t)}$ yields

$$\frac{\partial \ln\left(\frac{c_{t+1}}{c_t}\right)}{\partial \ln\left(\frac{U'(c_{t+1})}{U'(c_t)}\right)} = \frac{1}{\sigma}$$

4.

Since $c_t \geq 0$, we get that for $c_t \neq 0$

$$U'(c_t) = c_t^{-\sigma} > 0$$

Therefore, $U(c_t)$ is strictly increasing for $c_t > 0$.

Again since $c_t \geq 0$, we get that for $c_t \neq 0$

$$U''(c_t) = \underbrace{-\sigma}_{<0} \underbrace{c_t^{-\sigma-1}}_{>0} < 0$$

Thus, $U(c_t)$ is strictly concave for $c_t > 0$.

Finally, we check Inada's conditions, i.e.

$$\begin{aligned} \lim_{c_t \rightarrow 0} U'(c_t) &= \lim_{c_t \rightarrow 0} c_t^{-\sigma} \\ &= \lim_{c_t \rightarrow 0} \left(\frac{1}{c_t} \right)^{\sigma} = \infty \end{aligned}$$

and

$$\begin{aligned} \lim_{c_t \rightarrow 0} U'(c_t) &= \lim_{c_t \rightarrow 0} c_t^{-\sigma} \\ &= \lim_{c_t \rightarrow \infty} \left(\frac{1}{c_t} \right)^{\sigma} = 0 \end{aligned}$$

5.

Note that

$$\frac{\partial u(c)}{\partial c_t} = \beta^t c_t^{-\sigma}$$

This gives us that

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\beta^{t+s} c_{t+s}^{-\sigma}}{\beta^t c_t^{-\sigma}} = \beta^s \left(\frac{c_t}{c_{t+s}} \right)^{\sigma}$$

and

$$MRS(\lambda c_{t+s}, \lambda c_t) = \beta^s \left(\frac{\lambda c_t}{\lambda c_{t+s}} \right)^{\sigma} = \beta^s \left(\frac{c_t}{c_{t+s}} \right)^{\sigma} = MRS(c_{t+s}, c_t)$$

Hence, $u(c)$ is homothetic.

6.

Consumer's optimization problem.

$$\begin{aligned} & \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ & \text{such that } \sum_{t=0}^{\infty} p_t c_t \leq \lambda y \\ & c_t \geq 0 \\ & p_t \text{ given, } \lambda \text{ given, } y \text{ given} \end{aligned}$$

with $\beta \in (0, 1)$ and $\sum_{t=0}^{\infty} \beta^t U(c_t) = u(c)$.

We set up the following Lagrangian to solve the consumer's optimization problem.

$$\mathcal{L} = \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) + \mu (\lambda y - \sum_{t=0}^{\infty} p_t c_t)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t c_t^{-\sigma} - \mu p_t = 0 \\ \frac{\partial \mathcal{L}}{\partial \mu} &= \lambda y - \sum_{t=0}^{\infty} p_t c_t = 0 \end{aligned}$$

We can combine the FOCs to get the following equations:

$$\boxed{\beta^t \frac{c_t^{-\sigma}}{p_t} = \beta^s \frac{c_s^{-\sigma}}{p_s}}$$

Without loss of generality, let $p_0 = 1$. Then, we get

$$\begin{aligned} \beta^t \frac{c_t^{-\sigma}}{p_t} &= \beta^0 \frac{c_0^{-\sigma}}{p_0} \\ c_t^{\sigma} &= \frac{\beta^t}{p_t} c_0^{\sigma} \\ c_t &= c_0 \underbrace{\left(\frac{\beta^t}{p_t} \right)^{\frac{1}{\sigma}}}_{\theta_t} \end{aligned}$$

Then,

$$\begin{aligned}\sum_{t=0}^{\infty} p_t c_t &= \lambda y \\ \sum_{t=0}^{\infty} p_t c_0 \theta_t &= \lambda y \\ c_0 \sum_{t=0}^{\infty} p_t \theta_t &= \lambda y \\ c_0 &= \frac{\lambda y}{\sum_{t=0}^{\infty} p_t \theta_t}\end{aligned}$$

and

$$c_t = c_0 \cdot \theta_t = \frac{\lambda \theta_t y}{\sum_{t=0}^{\infty} p_t \theta_t}$$

Therefore, for $\lambda = 1$, the solution is given by

$$\hat{c}_t = \frac{\theta_t y}{\sum_{t=0}^{\infty} p_t \theta_t}$$

For $\lambda \neq 1$, we have

$$\tilde{c}_t = \lambda \frac{\theta_t y}{\sum_{t=0}^{\infty} p_t \theta_t} = \lambda \hat{c}_t$$

7.

We set up the following Lagrangian to solve the optimization problem.

$$\mathcal{L} = u(c) + \lambda_t(e_t + a_t - c_t - \frac{a_{t+1}}{1 + r_{t+1}})$$

FOCs

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial c_t} &= \frac{\partial u(c)}{\partial c_t} - \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial a_{t+1}} &= -\frac{\lambda_t}{1 + r_{t+1}} + \lambda_{t+1} = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= e_t + a_t - c_t - \frac{a_{t+1}}{1 + r_{t+1}} = 0\end{aligned}$$

We can combine the FOCs to get the following Euler equation:

$$\boxed{MRS(c_{t+1}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{1}{1 + r_{t+1}}}$$

and

$$\boxed{MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_{t+s-1}}} \cdot \frac{\frac{\partial u(c)}{\partial c_{t+s-1}}}{\frac{\partial u(c)}{\partial c_{t+s-2}}} \cdot \dots \cdot \frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \prod_{i=t}^{t+s-1} \frac{1}{1 + r_{i+1}}}$$

8.

Recall that $\frac{1}{1+r_{t+1}} = MRS(c_{t+1}, c_t)$ and thus

$$\begin{aligned}
\frac{1}{1+r_{t+1}} &= \frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) \\
&= \beta \left(\frac{c_t}{c_{t+1}} \right)^\sigma \\
&= \beta \left(\frac{(1+g)^t c_0}{(1+g)^{t+1} c_0} \right)^\sigma \\
&= \beta (1+g)^{-\sigma} \\
\Rightarrow r &= \frac{1}{\beta} (1+g)^\sigma - 1
\end{aligned}$$

9.

As shown in part 8, we have the following relation between r and g

$$r = \frac{1}{\beta} (1+g)^\sigma - 1$$

Taking the derivative with respect to g yields

$$\frac{\partial r}{\partial g} = \frac{\sigma}{\beta} (1+g)^{\sigma-1} > 0$$

i.e. $g \uparrow \Rightarrow r \uparrow$.

Note that by the chain rule, we have

$$\begin{aligned}
\frac{\partial \left(\frac{\partial r}{\partial g} \right)}{\partial \left(\frac{1}{\sigma} \right)} &= \frac{\frac{\partial \left(\frac{\partial r}{\partial g} \right)}{\partial \sigma}}{\frac{\partial \left(\frac{1}{\sigma} \right)}{\partial \sigma}} \\
&= \frac{\frac{1}{\beta} ((1+g)^{\sigma-1} + \sigma(1+g)^{\sigma-1} \ln(1+g))}{-\frac{1}{\sigma^2}} \\
&= -\frac{1}{\beta} (\sigma^2(1+g)^{\sigma-1} + \sigma^3(1+g)^{\sigma-1} \ln(1+g)) < 0
\end{aligned}$$

i.e. $\frac{1}{\sigma} \uparrow \Rightarrow \frac{\partial r}{\partial g} \downarrow$.

Hence, as intertemporal elasticity of substitution (IES) increases, we get that the "sensitivity" of r to g decreases. The intuition behind this is that IES represents the willingness to substitute consumption over time, or simply a measure of the curvature of the utility function. Thus, the more IES increases, the more indifferent we become to swings in consumption over time. Now, the higher the growth rate g , the higher the swings in consumption between period. Hence while r will increase to counter a higher g , it will do so less aggressively with an higher IES.

10.

Recall that

$$MRS(c_t, c_{t-1}) = \frac{1}{1+r_t}$$

If consumption is growing at a constant rate g , then $c_t = (1+g)^t c_0$. Without loss of generality let $s > t$, then

$$\begin{aligned} MRS(c_s, c_{s-1}) &= MRS((1+g)^s c_0, (1+g)^{s-1} c_0) \\ &= MRS(\underbrace{(1+g)^{s-t}}_{\gamma} c_t, \underbrace{(1+g)^{s-t}}_{\gamma} c_{t-1}) \\ &\stackrel{\text{Homothetic}}{=} MRS(c_t, c_{t-1}) \end{aligned}$$

Combining the previous with $MRS(c_t, c_{t-1}) = \frac{1}{1+r_t}$, we get that

$$\begin{aligned} \frac{MRS(c_t, c_{t-1})}{MRS(c_s, c_{s-1})} &= \frac{1+r_s}{1+r_t} \\ 1 &= \frac{1+r_s}{1+r_t} \\ \Rightarrow r_t &= r_s \quad \forall s, t \end{aligned}$$

1.

$$U(c_t) = -e^{-\gamma c_t}$$

Note that

$$U'(c_t) = \gamma e^{-\gamma c_t}$$

and

$$U''(c_t) = -\gamma^2 e^{-\gamma c_t}$$

Hence,

$$-\frac{U''(c_t)}{U'(c_t)} = -\frac{-\gamma^2 e^{-\gamma c_t}}{\gamma e^{-\gamma c_t}} = \gamma$$

and

$$-\frac{c_t U''(c_t)}{U'(c_t)} = -\frac{-c_t \gamma^2 e^{-\gamma c_t}}{\gamma e^{-\gamma c_t}} = c_t \gamma$$

Taking the derivative yields

$$\frac{\partial \left[-\frac{c_t U''(c_t)}{U'(c_t)} \right]}{\partial c_t} = \gamma > 0$$

i.e. increasing relative risk aversion.

2.

We have that

$$\frac{\partial u(c)}{\partial c_t} = \beta^t e^{-\gamma c_t}$$

Therefore,

$$MRS(c_{t+s}, c_t) = \frac{\frac{\partial u(c)}{\partial c_{t+s}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{\beta^{t+s} e^{-\gamma c_{t+s}}}{\beta^t e^{-\gamma c_t}} = \beta^s e^{-\gamma(c_{t+s} - c_t)}$$

Thus,

$$MRS(\lambda c_{t+s}, \lambda c_t) = \beta^s e^{-\gamma(\lambda c_{t+s} - \lambda c_t)} = e^\lambda \beta^s e^{-\gamma(c_{t+s} - c_t)} = e^\lambda MRS(c_{t+s}, c_t) \neq MRS(c_{t+s}, c_t)$$

i.e. for any $\lambda \neq 0$ we have $MRS(\lambda c_{t+s}, \lambda c_t) \neq MRS(c_{t+s}, c_t)$ which implies that the function is not homothetic.

3.

Consumer's optimization problem.

$$\begin{aligned} & \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ & \text{such that } \sum_{t=0}^{\infty} p_t c_t \leq y \\ & c_t \geq 0 \\ & p_t \text{ given, } y \text{ given} \end{aligned}$$

with $\beta \in (0, 1)$ and $\sum_{t=0}^{\infty} \beta^t U(c_t) = u(c)$.

We set up the following Lagrangian to solve the optimization problem.

$$\mathcal{L} = \beta^t - e^{-\gamma c_t} + \mu(\lambda y - \sum_{t=0}^{\infty} p_t c_t)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} &= \beta^t \gamma e^{-\gamma c_t} - \mu p_t = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} &= \lambda y - \sum_{t=0}^{\infty} p_t c_t = 0 \end{aligned}$$

We can combine the FOCs to get the following equations:

$$\beta^t \frac{\gamma e^{-\gamma c_t}}{p_t} = \beta^s \frac{\gamma e^{-\gamma c_s}}{p_s}$$

Without loss of generality, let $p_0 = 1$. Then, we get

$$\begin{aligned}
\beta^t \frac{\gamma e^{-\gamma c_t}}{p_t} &= \beta^0 \frac{\gamma e^{-\gamma c_0}}{p_0} \\
e^{\gamma c_t} &= \frac{\beta^t}{p_t} e^{\gamma c_0} \\
\gamma c_t &= \ln \left(\frac{\beta^t}{p_t} \right) + \gamma c_0 \\
c_t &= c_0 + \underbrace{\frac{1}{\gamma} \ln \left(\frac{\beta^t}{p_t} \right)}_{\theta_t}
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t c_t &= \lambda y \\
\sum_{t=0}^{\infty} p_t (c_0 + \theta_t) &= \lambda y \\
c_0 \sum_{t=0}^{\infty} p_t + \sum_{t=0}^{\infty} p_t \theta_t &= \lambda y \\
c_0 &= \frac{\lambda y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t}
\end{aligned}$$

and

$$c_t = c_0 + \theta_t = \frac{\lambda y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t} + \theta_t$$

Therefore, for $\lambda = 1$,

$$\hat{c}_t = \frac{y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t} + \theta_t$$

for $\lambda \neq 1$

$$\tilde{c}_t = \frac{(\lambda - 1)y}{\sum_{t=0}^{\infty} p_t} + \frac{y - \sum_{t=0}^{\infty} p_t \theta_t}{\sum_{t=0}^{\infty} p_t} + \theta_t = \frac{(\lambda - 1)y}{\sum_{t=0}^{\infty} p_t} + \hat{c}_t$$