ECON 6130 - Problem Set # 1

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Problem 1

(a) For x_t to be considered an element of the Hilbert space L_2, we need

$$\mathbb{E}(x_t^2) = \int_{w\in\Omega} x_t^2(w)dP(w) < \infty$$

Note that for the process x_t, we can rewrite it recursively in the following form

$$\begin{aligned} x_t &= \phi x_{t-1} + \epsilon_t \\ &= \epsilon_t + \phi\epsilon_{t-1} + \phi x_{t-2} \\ &= \epsilon_t + \phi\epsilon_{t-1} + \cdots + \phi^j\epsilon_{t-j} + \phi^j x_{t-j-1} \\ &\dots \\ &= \sum_{t=0}^{\infty} \phi^j\epsilon_{t-j} = \sum_{t=0}^{\infty} \psi_j\epsilon_{t-j} \end{aligned}$$

where $\psi_j = \phi^j$.

An equivalent way to show that x_t is an element of the Hilbert space L_2, is to show that

$$\sum_{t=0}^{\infty} \psi_j^2 < \infty$$

i.e. $(\psi_n) \in l_2$.

Hence,

$$\begin{aligned} \sum_{t=0}^{\infty} \psi_j^2 &= \sum_{t=0}^{\infty} \phi^{2j} \\ &= \begin{cases} \infty & \text{if } |\phi| \geq 1 \\ \frac{1}{1-\phi^2} & \text{if } |\phi| < 1 \end{cases} \end{aligned}$$

Therefore, we need $|\phi| < 1$ for $(\psi_n) \in l_2$.

(b) If $|\phi| < 1$, we have

$$\begin{aligned}
x_t &= \phi x_t + \epsilon_t \\
(1-\phi L)x_t &= \epsilon_t \\
x_t &= \frac{1}{1-\phi L}\epsilon_t \\
&= (1 + \phi L + \phi^2 L^2 + \dots)\epsilon_t \\
\Rightarrow x_t &= \epsilon_t + \phi\epsilon_{t-1} + \phi^2\epsilon_{t-2} + \dots
\end{aligned}$$

This implies that x_t is a linear combination of $\{\epsilon_t, \epsilon_{t-1}, \dots\}$.

Moreover, by definition we have

$$\langle \epsilon_i, \epsilon_j \rangle = \int_\Omega \epsilon_i(w)\epsilon_j(w)dP(w) = \mathbb{E}(\epsilon_i\epsilon_j) = \begin{cases} \sigma_\epsilon^2 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Hence, $\{\epsilon_t, \epsilon_{t-1}, \dots\}$ is a set of orthogonal vector such that $x_t \in span(\{\epsilon_t, \epsilon_{t-1}, \dots\})$. Finally, we let $v_t = \frac{\epsilon_t}{\sigma_\epsilon}$, then

$$\langle v_i, v_j \rangle = \mathbb{E}(v_i v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and $\{v_t, v_{t-1}, \dots\}$ is a also basis for x_t.

(c) By projection theorem, since our variables are normally distributed, we can find $\hat{x} = \beta x_t$ such that $\langle x_{t+s} - \beta x_t, x_t \rangle = 0$ and therefore $\mathbb{E}(x_{t+s} \mid x_t) = \beta x_t$.

Thus,

$$\begin{aligned}
0 &= \langle x_{t+s} - \beta x_t, x_t \rangle \\
0 &= \mathbb{E}[(x_{t+s} - \beta x_t)(x_t)] \\
\Rightarrow \beta &= \frac{\mathbb{E}(x_{t+s}x_t)}{\mathbb{E}(x_t x_t)} \\
&= \frac{\mathbb{E}[(\epsilon_{t+s} + \phi\epsilon_{t+s-1} + \dots)(\epsilon_t + \phi\epsilon_{t-1} + \dots)]}{\mathbb{E}(x_t x_t)} \\
&= \frac{\mathbb{E}[(\epsilon_{t+s} + \dots + \phi^{s-1}\epsilon_{t+1})(\epsilon_t + \phi\epsilon_{t-1} + \dots)] + \mathbb{E}[(\phi^s\epsilon_t + \phi^{s+1}\epsilon_{t-1} + \dots)(\epsilon_t + \phi\epsilon_{t-1} + \dots)]}{\mathbb{E}(x_t x_t)} \\
&= \frac{\phi^s\mathbb{E}[(\epsilon_t + \phi\epsilon_{t-1} + \dots)(\epsilon_t + \phi\epsilon_{t-1} + \dots)]}{\mathbb{E}(x_t x_t)} \\
&= \frac{\phi^s\mathbb{E}(x_t x_t)}{\mathbb{E}(x_t x_t)} \\
&= \phi^s
\end{aligned}$$

This implies that $\mathbb{E}(x_{t+s} \mid x_t) = \beta x_t = \phi^s x_t$.

Problem 2

(a) By projection theorem, since our variables are normally distributed, we can find $\hat{x} = \beta z_t$ such that $\langle x_{t+s} - \beta z_t, z_t \rangle = 0$ and therefore $\mathbb{E}(x_{t+s} \mid z_t) = \beta z_t$.

Thus,

$$\begin{aligned}
\langle x_{t+s} - \beta z_t, z_t \rangle &= 0 \\
\mathbb{E}[(x_{t+s} - \beta z_t)(z_t)] &= 0 \\
\Rightarrow \beta &= \frac{\mathbb{E}(x_{t+s} z_t)}{\mathbb{E}(z_t z_t)} \\
&= \frac{\mathbb{E}(x_{t+s}(x_t + u_t))}{\mathbb{E}[(x_t + u_t)(x_t + u_t)]} \\
&= \frac{\mathbb{E}(x_{t+s} x_t) + \mathbb{E}(x_{t+s} u_t)}{\mathbb{E}(x_t^2) + 2\mathbb{E}(x_t u_t) + \mathbb{E}(u_t^2)} \\
&= \frac{\phi^s \mathbb{E}(x_t^2) + \mathbb{E}(x_{t+s} u_t)}{\mathbb{E}(x_t^2) + 2\mathbb{E}(x_t u_t) + \mathbb{E}(u_t^2)}
\end{aligned}$$

Note that u_t and x_s are uncorrelated, i.e. $E(x_j u_t) = 0$. Additionally,

$$\begin{aligned}
x_t^2 &= \phi^2 x_t^2 + 2x_t \epsilon_t + \epsilon_t^2 \\
\mathbb{E}(x_t^2) &= \phi^2 \mathbb{E}(x_t^2) + 2\mathbb{E}(x_t \epsilon_t) + \mathbb{E}(\epsilon_t^2) \\
\sigma_X^2 &= \phi^2 \sigma_X^2 + \sigma_\epsilon^2 \\
\Rightarrow \mathbb{E}(x_t^2) &= \frac{\sigma_\epsilon^2}{1 - \phi^2}
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}(x_{t+s} \mid z_t) &= \frac{\phi^s \mathbb{E}(x_t^2)}{\mathbb{E}(x_t^2) + \mathbb{E}(u_t^2)} z_t \\
&= \frac{\frac{\sigma_\epsilon^2}{1-\phi^2}}{\frac{\sigma_\epsilon^2}{1-\phi^2} + \sigma_u^2} \phi^s z_t \\
&= \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 - \phi^2 \sigma_u^2 + \sigma_u^2} \phi^s z_t
\end{aligned}$$

(b) Recall that

$$\mathbb{E}(x_{t+s} \mid x_t) = \phi^s x_t$$

and

$$\mathbb{E}(x_{t+s} \mid z_t) = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 - \phi^2 \sigma_u^2 + \sigma_u^2} \phi^s z_t$$

Hence, $\mathbb{E}(x_{t+s} \mid z_t) = \mathbb{E}(x_{t+s} \mid x_t)$ if $\sigma_u^2 = 0$. In fact, if $\sigma_u^2 = 0$, we have $u_t = 0$ with certainty.

$$\mathbb{E}(x_{t+s} \mid z_t) = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 - \phi^2 \sigma_u^2 + \sigma_u^2} \phi^s z_t = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2} \phi^s (x_t + 0) = \phi^s x_t = \mathbb{E}(x_{t+s} \mid x_t)$$

(c) Let $\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s z_t = P_z x_{t+s}$,

Then, by definition of P_z, we have

$$\langle x_{t+s} - P_z x_{t+s}, z_i \rangle = 0 \text{ for all } i = 1, \dots, t$$

Let $i = t-1$. We can replace $P_z x_{t+s}$ by $\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s z_t$ to get the following

$$\begin{aligned}
\langle x_{t+s} - P_z x_{t+s}, z_i \rangle &= 0 \\
\langle x_{t+s} - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s z_t, z_i \rangle &= 0 \\
\langle x_{t+s}, z_i \rangle - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s \langle z_t, z_i \rangle &= 0 \\
\langle x_{t+s}, x_i + u_i \rangle - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s \langle x_t + u_t, x_i + u_i \rangle &= 0 \\
\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s \left(\mathbb{E}(x_t x_{t-1}) + \mathbb{E}(u_t x_{t-1}) + \mathbb{E}(x_t u_{t-1}) + \mathbb{E}(u_t u_{t-1})\right) &= \mathbb{E}(x_{t+s} x_{t-1}) + \mathbb{E}(x_{t+s} u_{t-1}) \\
\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s \mathbb{E}(x_t x_{t-1}) &= \mathbb{E}(x_{t+s} x_{t-1}) \\
\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s \mathbb{E}(x_t x_{t-1}) &= \mathbb{E}((\phi\epsilon_{t+s-1} + \cdots + \phi^s\epsilon_t + \phi^s x_t)x_t \\
\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s \mathbb{E}(x_t x_{t-1}) &= \phi^s \mathbb{E}(x_t x_{t-1}) \\
\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2} &= 0
\end{aligned}$$

i.e. contradiction unless $\sigma_\epsilon = 0$.

Problem 3

(a) As shown previously, $\mathbb{E}(x_t \mid x_{t-1}) = \phi x_{t-1}$. Therefore, we let $x_{t|t-1} = \phi x_{t-1|t-1}$.

For the variance, we have

$$\begin{aligned}
x_t &= \phi x_{t-1} + \epsilon_t \\
x_t - x_{t|t-1} &= \phi x_{t-1} - x_{t|t-1} + \epsilon_t \\
&= \phi x_{t-1} - \phi x_{t-1|t-1} + \epsilon_t \\
&= \phi(x_{t-1} - x_{t-1|t-1}) + \epsilon_t \\
\Rightarrow var(x_t - x_{t|t-1}) &= \phi^2 var(x_{t-1} - x_{t-1|t-1}) + \sigma_\epsilon^2
\end{aligned}$$

i.e. $p_{t|t-1} = \phi^2 p_{t-1|t-1} + \sigma_\epsilon^2$.

Now, we need to correct our estimate $x_{t|t-1}$ by combining it with z_t, i.e. $x_{t|t} = (1 - k_t)x_{t|t-1} + k_t z_t$ for some k_t. The optimal k_t is the one that minizes the variance of

$x_{t|t} - x_t$, hence

$$\begin{aligned} x_{t|t} - x_t &= (1-k_t)x_{t|t-1} - x_t + k_t z_t \\ &= (1-k_t)(x_{t|t-1} - x_t) + k_t u_t \\ var(x_{t|t} - x_t) &= (1-k_t)^2 var(x_{t|t-1} - x_t) + k_t^2 var(u_t) \\ 0 &= -2(1-k_t)p_{t|t-1} + 2k_t\sigma_u^2 \\ \Rightarrow k_t &= \frac{p_{t|t-1}}{p_{t|t-1} + \sigma_u^2} \end{aligned}$$

Finally, we need to update our estimate for the variance.

$$\begin{aligned} var(x_{t|t} - x_t) &= (1 - \frac{p_{t|t-1}}{p_{t|t-1} + \sigma_u^2})^2 var(x_{t|t-1} - x_t) + \left(\frac{p_{t|t-1}}{p_{t|t-1} + \sigma_u^2}\right)^2 var(u_t) \\ &= \frac{p_{t|t-1}\sigma_u^4 + p_{t|t-1}^2\sigma_u^2}{\left(p_{t|t-1} + \sigma_u^2\right)^2} \\ &= \frac{p_{t|t-1}\sigma_u^2\left(p_{t|t-1} + \sigma_u^2\right)}{\left(p_{t|t-1} + \sigma_u^2\right)^2} \\ \Rightarrow p_{t|t} &= \frac{p_{t|t-1}\sigma_u^2}{p_{t|t-1} + \sigma_u^2} \end{aligned}$$

To summarize,

$$\begin{aligned} x_{t|t-1} &= \phi x_{t-1|t-1} \\ p_{t|t-1} &= \phi^2 p_{t-1|t-1} + \sigma_\epsilon^2 \\ k_t &= \frac{p_{t|t-1}}{p_{t|t-1} + \sigma_u^2} \\ x_{t|t} &= (1-k_t)x_{t|t-1} + k_t z_t \\ p_{t|t} &= \frac{p_{t|t-1}\sigma_u^2}{p_{t|t-1} + \sigma_u^2} \end{aligned}$$

Now, we need to choose our starting value. We know that $x_t \sim N(0, \frac{\sigma_\epsilon^2}{1-\phi^2})$. Hence, a good choice for the initial value is the expected value of the x_t process, i.e.

$$\begin{aligned} x_{0|0} &= 0 = \mathbb{E}(x_t) \\ p_{0|0} &= \frac{\sigma_\epsilon^2}{1-\phi^2} = \mathbb{E}(x_t^2) = var(x_t^2) \end{aligned}$$

(b) We can rewrite $p_{t|t}$ in the following way

$$\begin{aligned}
p_{t|t} &= \frac{p_{t|t-1}\sigma_u^2}{p_{t|t-1}+\sigma_u^2} \\
&= \frac{\sigma_u^2\left(\phi^2 p_{t-1|t-1}+\sigma_\epsilon^2\right)}{\phi^2 p_{t-1|t-1}+\sigma_\epsilon^2+\sigma_u^2} \\
&= f(p_{t-1|t-1})
\end{aligned}$$

where $f(x) = \frac{\sigma_u^2\left(\phi^2 x+\sigma_\epsilon^2\right)}{\phi^2 x+\sigma_\epsilon^2+\sigma_u^2}$.
Let $p = \lim_{t\to\infty} p_{t|t}$. then

$$\begin{aligned}
p &= \lim_{t\to\infty} p_{t|t} \\
&= \lim_{t\to\infty} f(p_{t-1|t-1}) \\
&= f(\lim_{t\to\infty} p_{t-1|t-1}) \\
&= f(p)
\end{aligned}$$

Therefore, to find $p = \lim_{t\to\infty} p_{t|t}$, we need to solve $p = \frac{\sigma_u^2\left(\phi^2 p+\sigma_\epsilon^2\right)}{\phi^2 p+\sigma_\epsilon^2+\sigma_u^2}$ for p. For $\phi = 0$, we get $p = \frac{\sigma_u^2\sigma_\epsilon^2}{\sigma_\epsilon^2+\sigma_u^2}$ and for $\phi \neq 0$,

$$\begin{aligned}
\frac{\sigma_u^2\left(\phi^2 p+\sigma_\epsilon^2\right)}{\phi^2 p+\sigma_\epsilon^2+\sigma_u^2} &= p \\
\sigma_u^2\phi^2 p+\sigma_u^2\sigma_\epsilon^2 &= \phi^2 p^2+\sigma_\epsilon^2 p+\sigma_u^2 p \\
\phi^2 p^2+(\sigma_\epsilon^2+\sigma_u^2-\sigma_u^2\phi^2)p-\sigma_u^2\sigma_\epsilon^2 &= 0 \\
\Rightarrow p = \lim_{t\to\infty} p_{t|t} &= \frac{-(\sigma_\epsilon^2+\sigma_u^2-\sigma_u^2\phi^2)+\sqrt{(\sigma_\epsilon^2+\sigma_u^2-\sigma_u^2\phi^2)^2+4\phi^2\sigma_u^2\sigma_\epsilon^2}}{2\phi^2}
\end{aligned}$$

(c) Recall that

$$\mathbb{E}(x_{t+s} \mid z_t) = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}\phi^s z_t$$

Therefore,

$$\begin{aligned}
x_t - \mathbb{E}(x_t \mid z_t) &= x_t - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}(x_t+u_t) \\
&= \frac{\sigma_u^2-\phi^2\sigma_u^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}x_t + \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}u_t \\
&= Ax_t + Bu_t
\end{aligned}$$

where $A = \frac{\sigma_u^2-\phi^2\sigma_u^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}$ and $B = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2-\phi^2\sigma_u^2+\sigma_u^2}$. Note that $\mathbb{E}(x_t - \mathbb{E}(x_t \mid z_t)) = \mathbb{E}(Ax_t + Bu_t) = 0$.

Hence,

$$
\begin{aligned}
\mathbb{E}[(x_t - \mathbb{E}(x_t \mid z_t))^2] &= \mathbb{E}[(Ax_t + Bu_t)^2] \\
&= \mathbb{E}[A^2x_t^2 + 2ABx_tu_t + B^2u_t^2] \\
&= A^2\mathbb{E}[x_t] + 2AB\mathbb{E}[x_tu_t] + B^2\mathbb{E}[u_t^2] \\
&= A^2\frac{\sigma_\epsilon^2}{1-\phi^2} + B^2\sigma_u^2 \\
&= \frac{\sigma_\epsilon^2\sigma_u^4 - \sigma_\epsilon^2\sigma_u^4\phi^2 + \sigma_\epsilon^4\sigma_u^2}{(\sigma_\epsilon^2 - \phi^2\sigma_u^2 + \sigma_u^2)^2} \\
&= \frac{\sigma_\epsilon^2\sigma_u^2}{\sigma_\epsilon^2 - \phi^2\sigma_u^2 + \sigma_u^2}
\end{aligned}
$$

Let's assume $\mathbb{E}[(x_t - \mathbb{E}(x_t \mid z_t))^2] = \lim_{t\to\infty} p_{t|t}$, then

$$
\begin{aligned}
&\frac{\sigma_\epsilon^2\sigma_u^2}{\sigma_\epsilon^2 - \phi^2\sigma_u^2 + \sigma_u^2} = \frac{-(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2) + \sqrt{(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^2 + 4\phi^2\sigma_u^2\sigma_\epsilon^2}}{2\phi^2} \\
&2\sigma_\epsilon^2\sigma_u^2\phi^2 = -(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^2 + \sqrt{(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^4 + 4\phi^2\sigma_u^2\sigma_\epsilon^2(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^2} \\
&2\sigma_\epsilon^2\sigma_u^2\phi^2 + (\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^2 = \sqrt{(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^4 + 4\phi^2\sigma_u^2\sigma_\epsilon^2(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^2} \\
&4\sigma_\epsilon^4\sigma_u^4\phi^4 + 4\sigma_\epsilon^2\sigma_u^2\phi^2(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2) + (\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^4 = (\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2\phi^2)^4 + 4\phi^2\sigma_u^2\sigma_\epsilon^2(\sigma_\epsilon^2 + \sigma_u^2 - \sigma_u^2 \\
&4\sigma_\epsilon^4\sigma_u^4\phi^4 = 0
\end{aligned}
$$

i.e. a contradiction for $\phi \neq 0$. In fact, we have that

$$
\mathbb{E}[(x_t - \mathbb{E}(x_t \mid z_t))^2] \geq \lim_{t\to\infty} p_{t|t}
$$

with equality if $\phi = 0$.

(d) Note that the Wold decomposition of x_t is given by

$$
x_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} + \phi^t x_0
$$

where $\psi_j = \begin{cases} \phi^j & \text{if } t-j>0 \\ 0 & \text{otherwise} \end{cases}$ and x_0 is deterministic.

Hence,

$$
\begin{aligned}
z_t &= \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j} + u_t + \phi^t x_0 \\
&= \sum_{j=0}^{\infty} \psi_j w_{t-j} + \phi^t x_0
\end{aligned}
$$

where $\psi_j = \begin{cases} \phi^j & \text{if } t-j>0 \\ 0 & \text{otherwise} \end{cases}$, x_0 is deterministic and $w_{t-j} = \begin{cases} u_t + e_t & \text{if } j=0 \\ e_t & \text{otherwise} \end{cases}$

Problem 4

(a) The code for part a) to f) is given in the appendix

(b) Note that Σ_X is the solution to

$$\begin{aligned}
\mathbb{E}(X_tX_t') &= \mathbb{E}\left[(AX_{t-1} + C\epsilon_t)(AX_{t-1} + C\epsilon_t)'\right] \\
\mathbb{E}(X_tX_t') &= A\mathbb{E}\left[X_{t-1}X_{t-1}'\right]A' + C\mathbb{E}\left[\epsilon_t X_{t-1}'\right]A' + A\mathbb{E}\left[X_{t-1}\epsilon_t'\right]C' + C\mathbb{E}\left[\epsilon_t\epsilon_t'\right]C' \\
\Sigma_X &= A\Sigma_X A' + C\Sigma_\epsilon C' \\
\Rightarrow \Sigma_X &= A\Sigma_X A' + CC'
\end{aligned}$$

where $A = \begin{pmatrix} 0.95 & 0 \\ 0 & 0.5 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \\ -0.5 & 1 \end{pmatrix}$, and $\Sigma_\epsilon = I$.

Thus,

$$\Sigma_X = \begin{pmatrix} 10.2564 & -0.9524 \\ -0.9524 & 1.6667 \end{pmatrix}$$

The sample variance is given by the following

$$\hat{\Sigma}_X = \begin{pmatrix} 4.4256 & -0.4409 \\ -0.4409 & 1.4729 \end{pmatrix}$$

While in our simulation $\Sigma_X \neq \hat{\Sigma}_X$, with $T \to \infty$ we have $\hat{\Sigma}_X \to \Sigma_X$. In fact, with $T = 10000$, we are within two decimal places of the true Σ_X.

Note that Σ_Z is the solution to

$$\begin{aligned}
\mathbb{E}(Z_tZ_t') &= \mathbb{E}\left[(X_t + v_t)(X_t + v_t)'\right] \\
\mathbb{E}(Z_tZ_t') &= \mathbb{E}\left[X_tX_t'\right] + \mathbb{E}\left[v_tX_t'\right] + \mathbb{E}\left[X_tv_t'\right] + \mathbb{E}\left[v_tv_t'\right] \\
\Rightarrow \Sigma_Z &= \Sigma_X + \Sigma_v = \Sigma_X + I
\end{aligned}$$

where $\Sigma_\epsilon = I$.

Thus,

$$\Sigma_Z = \begin{pmatrix} 11.2564 & -0.9524 \\ -0.9524 & 2.6667 \end{pmatrix}$$

$$\hat{\Sigma}_Z = \begin{pmatrix} 5.8345 & -0.2793 \\ -0.2793 & 2.3672 \end{pmatrix}$$

Again the difference between $\hat{\Sigma}_Z$ and Σ_Z is due to the small T.

(c) We plot the Kalman filter estimate $X_{t|t}$ in Figure 1.

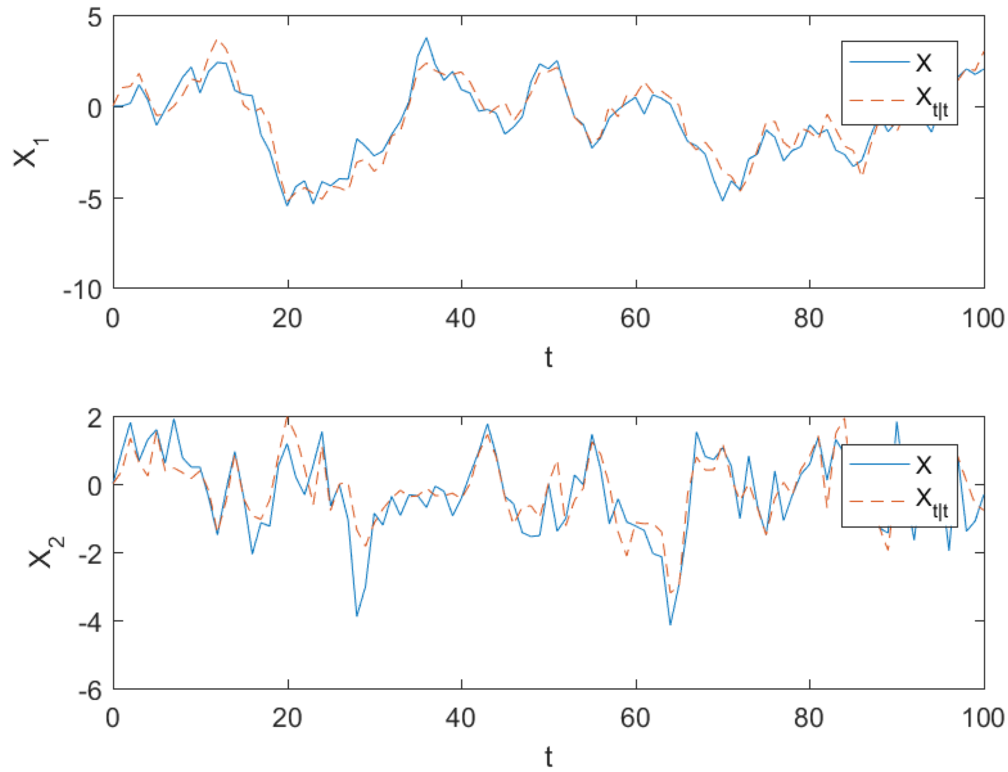

Figure 1: Kalman filter estimate of X_t for $T = 100$.

(d) Note that

$$\Sigma_{X_t - X_{t|t}} = P_{t|t} = \begin{pmatrix} 0.5838 & -0.0949 \\ -0.0949 & 0.5599 \end{pmatrix}$$

where $P_{t|t}$ is defined as in the notes. Note that in our particular case it is time invariant. The sample covariance is given by

$$\hat{\Sigma}_{X_t - X_{t|t}} \begin{pmatrix} 0.5050 & -0.0734 \\ -0.0734 & 0.4794 \end{pmatrix}$$

Note that $\hat{\Sigma}_{X_t - X_{t|t}}$ and $\Sigma_{X_t - X_{t|t}}$ are really close to each other unlike previously with Σ_X, $\hat{\Sigma}_X$, Σ_Z and $\hat{\Sigma}_Z$.

(e) (a) See the code section. Note that we use the previous simulated series X_t.

(b) The variance of Z is given by

$$\sigma_Z^2 = D\Sigma_X D' = 10.0183$$

We can compute the sample variance of Z and we get

$$\hat{\sigma}_Z^2 = 5.0167$$

The discrepancy comes from the low number of data points, i.e. T.

(c) We plot the Kalman filter estimate $X_{t|t}$ in Figure 2.

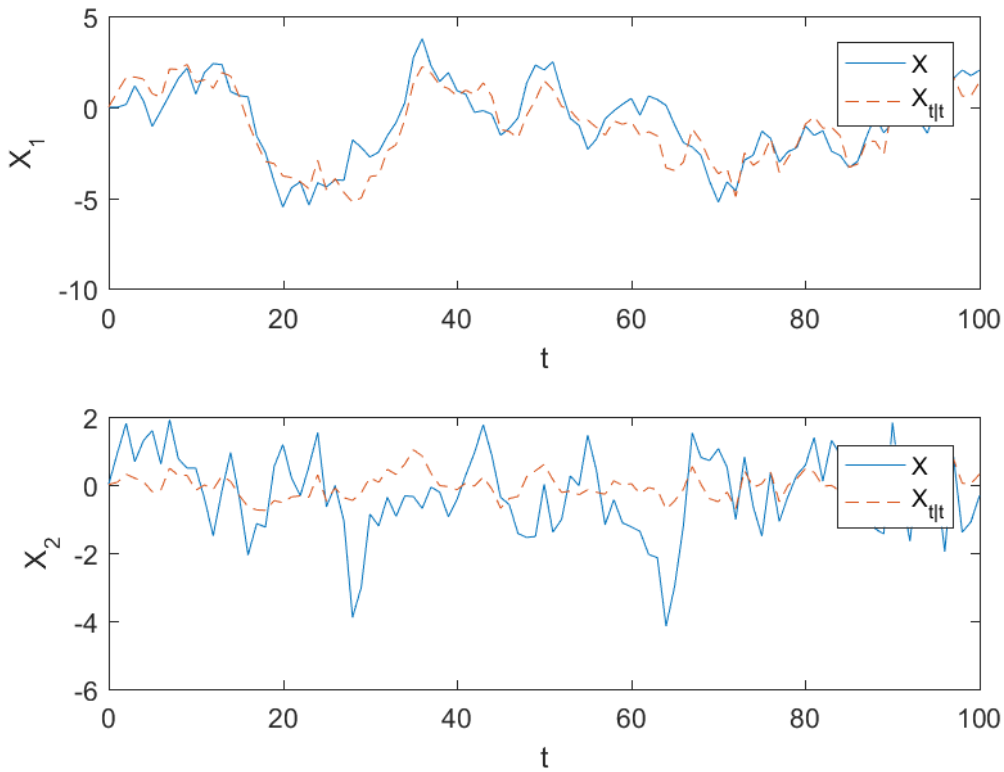

Figure 2: Kalman filter estimate of X_t for $T = 100$.

Note that we now have $Z_t = X_{1t} + X_{2t}$, while we previously had $Z_{1t} = X_{1t} + v_{1t}$ and $Z_2 = X_{2t} + v_{2t}$. This implies that, while the observations of the first part of the problem were plagued with some extra error v_t compare to $Z_t = X_{1t} + X_{2t}$, we did have twice the number of observations. This results in a better estimate of X_t as shown in Figure 1.

But, even with only the sum of X_{1t} and X_{2t}, Figure 2 shows that the Kalman Filter still does a great job at retrieving X_{1t} and X_{2t}.

(f) (a) See the code section. Note that we use the previous simulated series X_t.

(b) The variance of Z is given by

$$\sigma_Z^2 = D\Sigma_X D' = 10.2564$$

We can compute the sample variance of Z and we get

$$\hat{\sigma}_Z^2 = 4.4256$$

The discrepancy comes from the low number of data points, i.e. T.

(c) We plot the Kalman filter estimate $X_{t|t}$ in Figure 3.

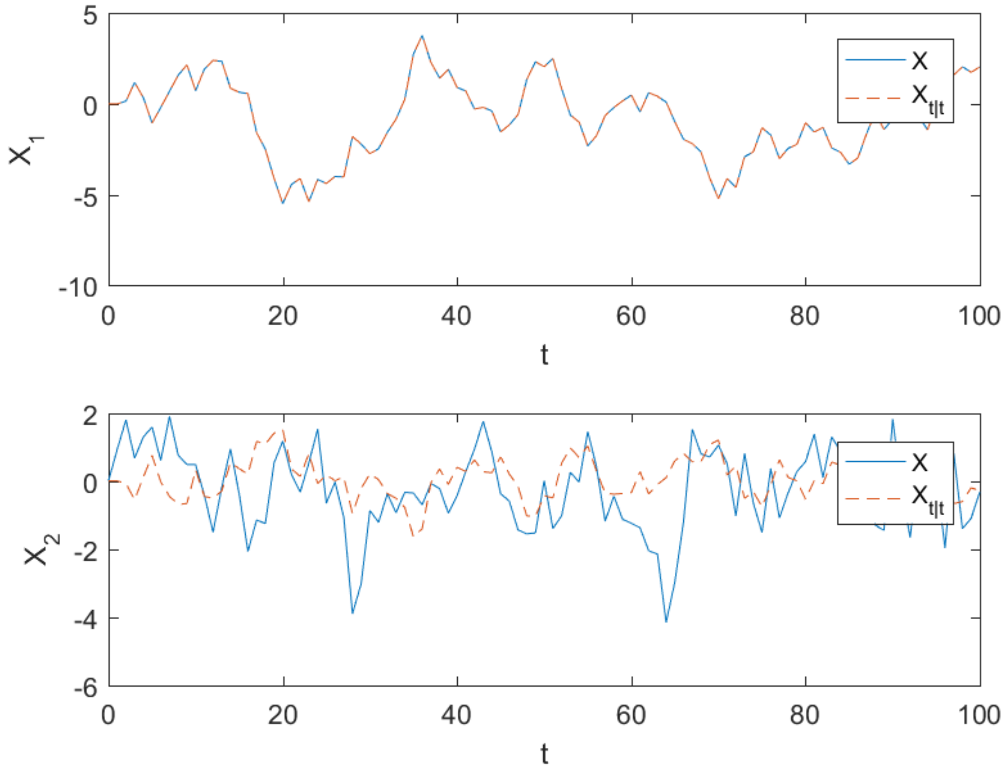

Figure 3: Kalman filter estimate of X_t for $T = 100$.

Note that we now have $Z_t = X_{1t}$, while in part $Z_t = X_{1t} + X_{2t}$. Hence, we can perfectly determined X_{1t} since Z_t is noise free from anything else. Still even with no direct information about X_{2t}, the Kalman filter is still able to implicitly find a good estimate of X_{2t}.

Code

main.m

```
%% Setup
% Set number of periods
T = 100;
time = 0:T;

% Set state AR(1) matrices
A = [0.95 0; 0 0.5];
C = [1 0; -0.5 1];
```

```
%% Part a)-c)
disp('Part a-c')

% Set measurement matrix
D = [1 0; 0 1];

% Set measurement error matrix
S_vv = eye(2);

% Generate random errors
v = randn(2,T+1);
u = randn(2,T+1);

% Set states and observations matrices
X = zeros(2,T+1);
Z = zeros(2,T+1);

% Generate states and observations
for t=1:T
X(:,t+1) = A * X(:,t) + C * u(:,t);
Z(:,t+1) = D * X(:,t+1) + S_vv * v(:,t+1);
end

% Part b)
% Compute state variances
var_x = reshape((eye(4)-kron(A,A))^-1*reshape(C*C',4,1),2,2)
var_x_sample = cov(X')

% Compute observations variances
var_z = D*var_x*D' + S_vv
var_z_sample = cov(Z')

% Part c)
% Set starting values
X0 = zeros(2,1);
P0 = var_x;

% Compute Kalman Filter
[X_post, P_post] = kfilter(Z, A, C, D, S_vv, X0, P0 );

% Plot Kalman Filter estimates
figure
subplot(2,1,1)
plot(time, X(1,:),time, X_post(1,:), '--')
```

```
xlabel('t')
ylabel('X_1')
legend('X','X_{t|t}')
subplot(2,1,2)
plot(time, X(2,:),time, X_post(2,:), '--')
xlabel('t')
ylabel('X_2')
legend('X','X_{t|t}')

print('plot_c','-dpng')

% part d)

var_err = P_post
var_err_sample = cov((X-X_post)')

%% Part e)
disp('Part e')
% Part a)
% Set measurement matrix
D = [1 1];

% Generate observations
Z = D * X;

% Part b)
% Compute observations variances
var_z = D*var_x*D'
var_z_sample = cov(Z')

% Part c)
% Set starting values
X0 = zeros(2,1);
P0 = var_x;

% Compute Kalman Filter
X_post = kfilter(Z, A, C, D, 0, X0, P0 );

% Plot Kalman Filter estimates
figure
subplot(2,1,1)
plot(time, X(1,:),time, X_post(1,:), '--')
xlabel('t')
ylabel('X_1')
legend('X','X_{t|t}')
```

```
subplot(2,1,2)
plot(time, X(2,:),time, X_post(2,:), '--')
xlabel('t')
ylabel('X_2')
legend('X','X_{t|t}')

print('plot_e','-dpng')

%% Part f)
disp('Part f')
% Part a)
% Set measurement matrix
D = [1 0];

% Generate observations
Z = D * X

% Part b)
% Compute observations variances
var_z = D*var_x*D'
var_z_sample = cov(Z')

% Part c)
% Set starting values
X0 = zeros(2,1);
P0 = var_x;

% Compute Kalman Filter
[X_post, P_post] = kfilter(Z, A, C, D, 0, X0, P0 );

% Plot Kalman Filter estimates
figure
subplot(2,1,1)
plot(time, X(1,:),time, X_post(1,:), '--')
xlabel('t')
ylabel('X_1')
legend('X','X_{t|t}')
subplot(2,1,2)
plot(time, X(2,:),time, X_post(2,:), '--')
xlabel('t')
ylabel('X_2')
legend('X','X_{t|t}')

print('plot_f','-dpng')
```

kfilter.m

```
function [ X_post, P_post, X_prior, P_prior, K] = kfilter(Z, A, C,
    D, S_vv, X0, P0 )
%KFILTER Compute the Kalman Filter for a VAR(1) process

% Get size of observations
[~,T] = size(Z);

% Allocate space for estimates
X_prior = cell(1,T);
X_post = cell(1,T);
K = cell(1,T);

% Set starting values
X_post{1} = X0;
P_post = P0;

for t = 1:T-1
    % Compute X_{t+1|t}
     X_prior{t+1}  = A*X_post{t} ;

    % Compute P_{t+1|t}
     P_prior = A * P_post * A' + C * C';

    % Compute K_{t+1}
    K{t+1} = P_prior*D'*(D*P_prior*D'+S_vv)^-1;

    % Compute X_{t+1|t+1}
     X_post{t+1}  = A*X_post{t}  + K{t+1}  * (Z(:,t) - D*X_prior{t
        +1} );

    % Compute P_{t+1|t+1}
     P_post = P_prior- K{t+1}*D*P_prior;

end

% Convert post and prior estimates to matrices
X_prior = cell2mat(X_prior);
X_post = cell2mat(X_post);

end
```