Price-Level Uncertainty and Stability in the UK*

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Abstract

Was the UK price level more stable and predictable before World War I or after World War II? We answer these questions by estimating changes in volatilities of transient and permanent components of inflation over the period 1791-2011. For much of the 19th century, trend inflation and its innovation variance were both close to zero, but transient inflation was variable. After World War II, transient volatility was substantially lower, but trend inflation was higher and more variable. On balance, future price levels were least *uncertain* on the eve of World War I under the classical gold standard, before the Great Inflation under Bretton Woods, and in the 2000s after the Bank of England adopted a policy of inflation targeting. But they were most *stable* just before World War I. For price-level stability, the decline in transient volatility was more than counterbalanced by the rise in the level and volatility of trend inflation. For price-level uncertainty, the two forces roughly offset.

JEL CLASSIFICATION: E31, C22

KEY WORDS: Inflation, price stability, price-level uncertainty, nonlinear state-space model

1 Introduction

Figure 1 portrays annual data on the price level and inflation rate in the United Kingdom for the period 1791-2011. To us, the figure shows contending patterns of price stability and predictability. Data on the logarithm of the price level, shown in the left panel, convey the impression that the gold and silver commodity standards

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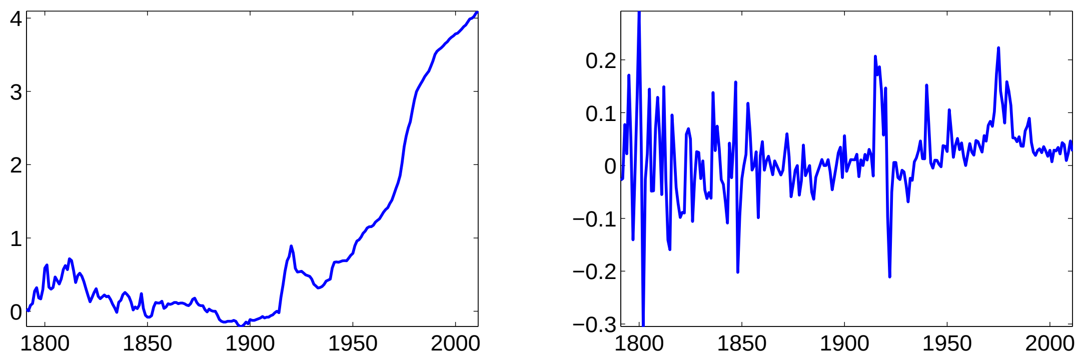

Figure 1: UK Price Level and Inflation

that prevailed before 1914 had produced a century of long-term predictability of price levels, while the years after 1914 witnessed a struggle to prove that a well-managed fiat standard could deliver as much price stability as had been achieved under the gold standard. But data on inflation, exhibited in the right panel, suggest that volatility peaked early in the 1800s and declined for much of the following two centuries, albeit with momentous interruptions during and after World War I and amidst the Great Inflation of the 1970s.

These countervailing visual impressions are associated with different features of the inflation process. The left panel is dominated by variation in a stochastic trend for inflation, which was close to zero for much of the period between 1791 and 1913 and then turned positive around the time of World War I. Variation in the right panel is dominated by transient volatility, especially in the first-half of the 19th century. Both features make future price levels difficult to predict and contribute to price level instability. Because transient volatility has diminished while persistent variation has increased, it is not obvious whether future price levels were more predictable before World War I or after World War II.[1]

In addition, because persistent variation gives rise to predictable movements in inflation, the two features have different implications for nominal *stability* as opposed to nominal *uncertainty*. To the extent that trend inflation is forecastable, it matters less for price-level uncertainty than for price-level instability. The price level might have been less stable after World War II even though it was more predictable. The two features also operate differently at different forecast horizons, with persistent

[1]It is important to account for the decline in transient volatility when comparing the two eras, which is part of our interpretation of some of the comments by Christopher Sims that set us off on this paper.

variation mattering more in the long run. Thus, price levels may have been more predictable over short horizons after World War II and less predictable over long horizons.

In this paper, we roll up our sleeves and estimate a statistical model simple enough and flexible enough to let us evaluate evidence about movements in conditional volatilities for future price levels. At a minimum, we think that four features are required for an adequate statistical representation. First, to fit both the 19th century, when average inflation was close to zero, and the 1970s, when average inflation was in double digits, the model must include a stochastic trend in inflation. Second, to fit the short-term volatility seen near the beginning of the sample, the model must also include a transient component. Third, so that volatility can change, innovations to the two components must have time-varying variances. These three features are captured in a simple and elegant model developed by Stock and Watson (2007), which we adapt below.

The fourth feature is that the model must acknowledge measurement error in older data. Christina Romer (1986a,b) warns about hazards involved in comparing data from before and after World War II. She points out that pre-war data were constructed differently and measured less accurately, and she contends that much of the apparent decline in volatility that followed the war could be due to improved measurements. To respect that possibility, we append a measurement equation to Stock and Watson's model and allow pre-World War II data to be measured with noise. To estimate the magnitude of measurement errors and to purge older data of noise, we solve a nonlinear signal-extraction problem.

We estimate the model using Bayesian methods, then use it to quantify the degree of price-level uncertainty and instability. We measure uncertainty and instability, respectively, by the conditional variance and second moments of cumulative inflation. By comparing conditional variances and second moments at various dates over various forecast horizons, we trace the rise and fall of price stability and predictability in the UK.[2]

For much of the 19th century, trend inflation and its innovation variance were both close to zero, but there was appreciable transient inflation variability. After

[2]Several aspects of British economic history make the UK a particularly interesting case for our purposes. First, for much of our long historical sample, the British empire played a very important role in the global economy, as exemplified by the central role played by the British pound during the gold standard. Second, the establishment of economic and statistical institutions at the national level in the UK preceded by many decades similar developments in other advanced economies, meaning that the coverage and quality of British historical data is probably better than in the rest of the industrialized world. Third, British monetary history has been characterized by a number of sharp changes in policy regimes, ranging from the silver and gold standard, to Bretton Woods, to money and exchange-rate targeting, and, finally, to inflation targeting.

World War II, transient volatility was substantially lower, but trend inflation was higher and more variable. On balance, future price levels were least uncertain on the eve of World War I under the classical gold standard, under Bretton Woods before the Great Inflation, and in the 2000s after the Bank of England adopted a policy of flexible inflation targeting and achieved operational independence. But they were most stable under the gold standard just before World War I. For price-level uncertainty, the decline in transient volatility roughly offset the rise in persistent volatility. For price-level stability, however, the level of trend inflation also matters, and this swings the balance to the classical gold standard.

2 An unobserved-components, stochastic-volatility model for inflation

Our statistical representation extends Stock and Watson's (2007) unobserved components model for inflation:

$$\begin{aligned} \pi_t &= \mu_t + \sqrt{r_t}\varepsilon_{\pi t}, \\ \mu_t &= \mu_{t-1} + \sqrt{q_t}\varepsilon_{\mu t}, \\ \ln r_t &= \ln r_{t-1} + \sigma_r \eta_{rt}, \\ \ln q_t &= \ln q_{t-1} + \sigma_q \eta_{qt}, \end{aligned} \tag{1}$$

where π_t is inflation, μ_t is trend inflation, and r_t and q_t are stochastic volatilities that evolve as geometric random walks. The innovations $\varepsilon_{\pi t}, \varepsilon_{\mu t}, \eta_{rt}$, and η_{qt} are standard normal, serially uncorrelated, and mutually independent. This representation has both transient and persistent components in inflation as well as stochastic volatility in their innovations, so it activates the competing forces that we described in connection with figure 1.[3]

Our main extension of Stock and Watson's model is to confront the measurement issues raised by Romer. To address her concern, we regard (1) as the transition equation for a nonlinear state-space model, and we add measurement error to π_t. Thus, measured inflation y_t is

$$y_t = \pi_t + \sigma_{mt}\varepsilon_{mt}, \tag{2}$$

where the measurement error ε_{mt} is iid standard normal and independent of the state innovations.

[3] Our model is slightly more general than Stock and Watson's. While they restrict $\sigma_r = \sigma_q$ and calibrate the common variance parameter, we allow σ_r and σ_q to differ, and we estimate them. However, our posteriors suggest that Stock and Watson's restrictions are quite reasonable for UK data.

We study inflation data for the period 1721-2011. Data for 1721-1790 are taken from Phelps-Brown and Hopkins (1956) and are used only as a training sample to calibrate aspects of the prior. The sample used for estimation therefore covers the period 1791-2011. A price index for this period was constructed by splicing data from four sources. The first three were taken from B.R. Mitchell's (1988) compilation *British Historical Statistics*. For the period 1791-1850, we used the revised estimates of Lindert and Williamson (1983). For the period 1851-1914, data were taken from Bowley (1937), while those for 1915-1947 came from the U.K. Ministry of Labor.[4] Last but not least, the sample was extended from 1947 through 2011 by appending data from the Global Financial Database. These are the data shown in figure 1.

We assume that the measurement error variance varies across subsamples but is constant within each subsample. We also assume that inflation is correctly measured after 1947. The measurement error variance therefore breaks at the following dates,

$$\begin{aligned} \sigma_{mt} &= \sigma_{1m} \quad t \leq 1850, \\ &= \sigma_{2m} \quad 1851 \leq t \leq 1914, \\ &= \sigma_{3m} \quad 1915 \leq t \leq 1947, \\ &= 0 \quad t \geq 1948. \end{aligned} \tag{3}$$

Equations (2) and (3) define the measurement equation for a nonlinear state-space model.

2.1 Priors

Our first task is to estimate the latent states π_t, μ_t, r_t, and q_t as well as the variance parameters σ_r, σ_q, and σ_m. We do this via Bayesian methods.

We adopt informative priors for the variance parameters σ_r, σ_q, and σ_{im}. Our prior for both σ_r and σ_q is $IG_1(0.224, 10)$. We set the scale parameter by adjusting Stock and Watson's quarterly calibration to account for time aggregation, and we set the degree of freedom parameter to deliver a plausible prior credible set. For this scaling and choice of degrees of freedom, a prior 95 percent credible set ranges from 0.15 to 0.36. The dashed line in figure 2 depicts this prior.

For $\sigma_{im}, i = 1, ..., 3$, our prior is $IG_1(0.0325, 7)$. We chose the scale parameter by dividing the sample standard deviation for inflation for the pre-sample period 1721-1790 by 2. The scale parameter therefore reflects an assumption that half of the pre-sample standard deviation is due to measurement error. Given that scaling, we set the degree of freedom parameter so that a prior 95 percent credible set covers the

[4]See Mitchell 1988, tables 8-10, pp. 737-739.

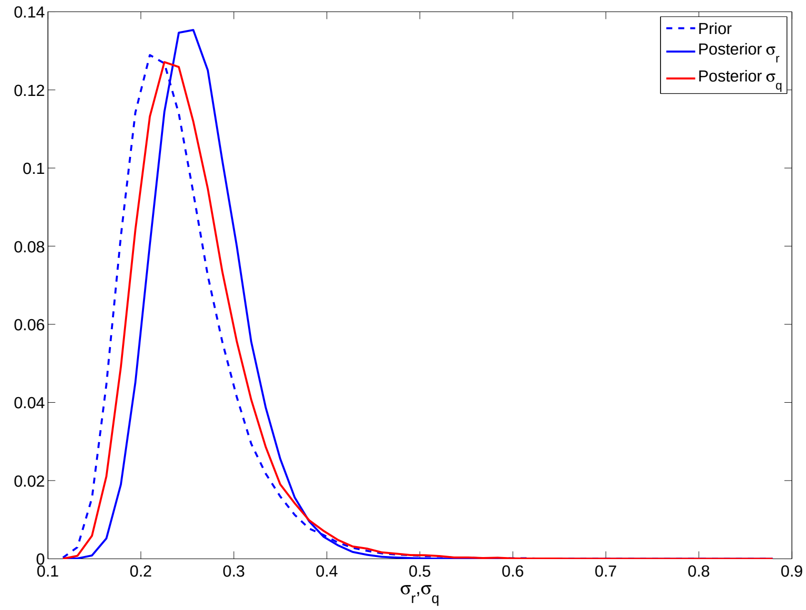

Figure 2: Prior and Posteriors for σ_r and σ_q

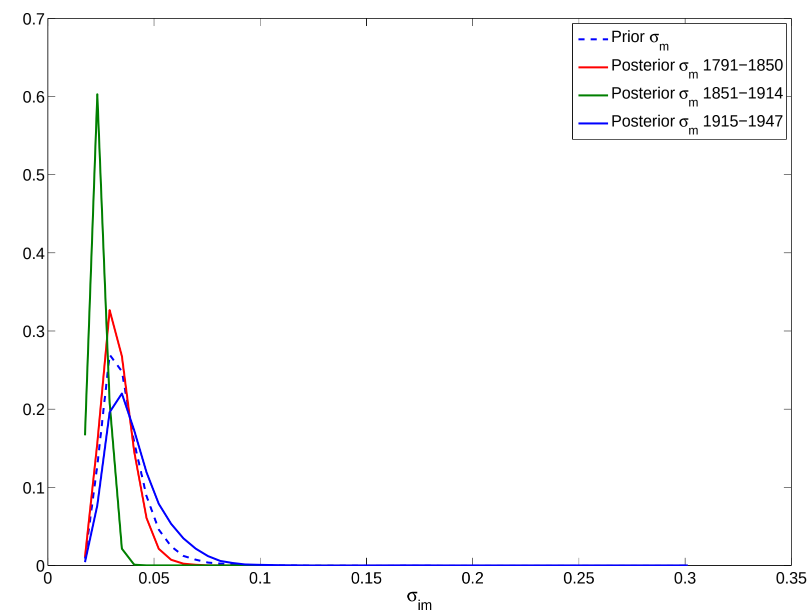

Figure 3: Prior and Posterior for σ_m

interval (0.021,0.058). In other words, the credible set attributes between 30 and 90 percent of pre-sample volatility to measurement error. The dashed line in figure 3 portrays this prior.

Our prior for the initial state (π_0, μ_0) is normal with means equal to sample averages from our training sample 1721-1790 and variances equal to

$$P_0 = \begin{bmatrix} 0.15^2 & 0 \\ 0 & 0.025^2 \end{bmatrix}. \tag{4}$$

Since prior credible sets for π_0 and μ_0 are roughly (-0.3,0.3) and (-0.05,0.05), respectively, this prior is weakly informative for the initial state.

Our prior for $\ln r_0$ is normal with mean equal to the log of the presample variance of inflation and a standard deviation equal to 5. Similarly, the prior for $\ln q_0$ is normal with mean equal to the log of 1/25 times the presample variance of inflation and a standard deviation equal to 5. Since the prior standard deviation is enormous on a log scale, these priors very weakly informative.

We approximate the posterior via a MCMC algorithm. Except for the pre-1948 measurement error and assumptions about the prior, the model has the same structure as that of Stock and Watson (2007), and so is closely related to that of Cogley and Sargent (2005). The MCMC algorithm is therefore also very similar to Cogley and Sargent's. Appendix A describes the details.

3 Features of the posterior

Figures 2-4 record various features of the posterior probability distribution that conditions on our full sample 1791-2011. The solid lines in figures 2 and 3 depict posterior distributions for the log-volatility-innovation variances σ_r, σ_q and the measurement-error variances σ_{im}. Posteriors for the standard deviations of log-volatility innovations overlap substantially with the prior but are shifted slightly to the right (see figure 2). Hence the data want a bit more time variation in log volatilities than is encoded in the prior. Moreover, the posterior for σ_r is shifted further to the right than that for σ_q, implying that the innovation variance for the transient component of inflation varies more than that of the permanent component.

Posteriors for the three measurement-error variances are portrayed in figure 3. The posterior for the first subsample (1791-1850) resembles the prior. To the extent that they differ, it is because the posterior has less mass in the upper tail. Hence there is no strong evidence that σ_{1m} is understated a priori. In the second subsample (1851-1914), the measurement error variance is identified more sharply, and the posterior shifts to the left of the prior, concentrating tightly around a mode of 0.024. The posterior for the third subsample (1915-1947) also resembles the prior but shifts slightly to the right. The posterior mode for σ_{3m} is roughly 0.034, and the upper tail becomes fatter.

Figure 4 portrays median and interquartile ranges for the latent states μ_t (the stochastic trend in inflation), $\pi_t - \mu_t$ (the transitory component of inflation), and $\sqrt{q_t}$ and $\sqrt{r_t}$ (the standard deviations of the log-volatility innovations). A number of salient points emerge.

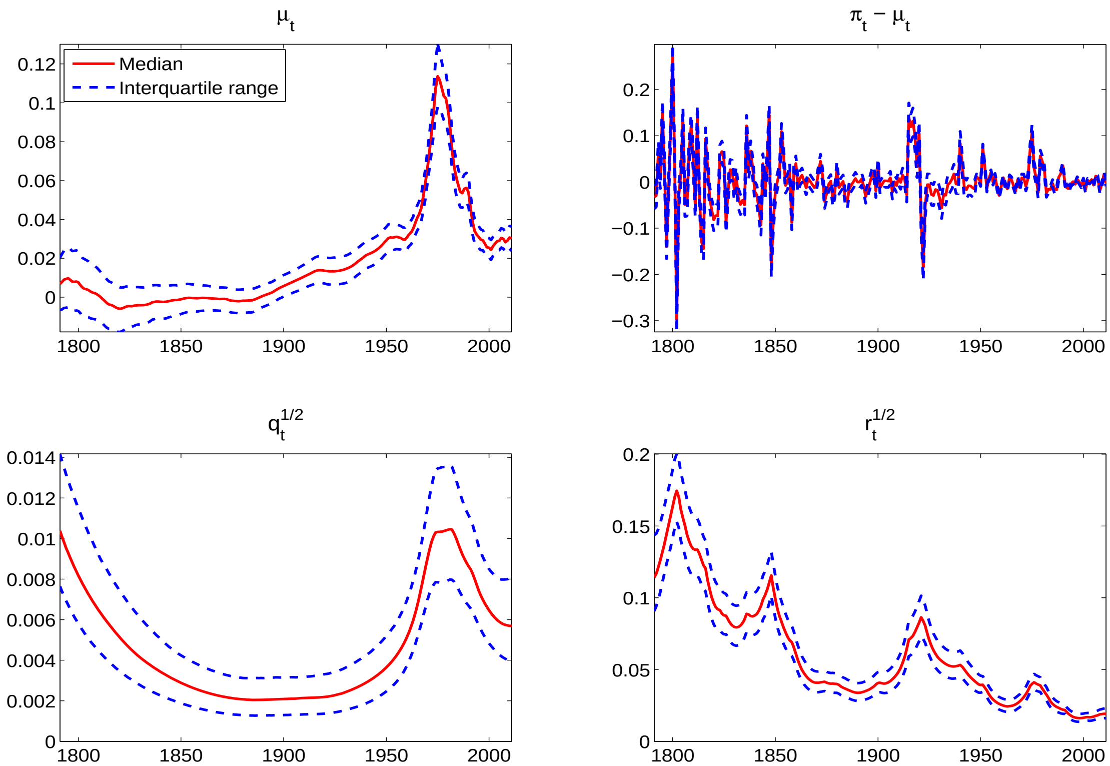

Figure 4: Posteriors of hidden states (conditioned on sample 1791-2011).

First, much of the long-term decline in inflation volatility is due to a decline in the volatility r_t of transient shocks to inflation. Transient volatility was highest during the Napoleonic Wars when convertibility to gold was suspended, but $\sqrt{r_t}$ declined by about 75 percent during the first three quarters of the 19th century and then continued to fall – albeit more gradually – through the end of the century. Transient volatility increased sharply during and after World War I, when convertibility to gold was again suspended, and then declined again throughout much of the remainder of the sample. r_t reached its lowest point toward the end of the sample, after the Bank of England adopted an inflation-targeting framework. By 2011, $\sqrt{r_t}$ had declined to about 1.9 percent per annum, a bit more than one-tenth of its Napoleonic peak.

Second, the volatility q_t of permanent shocks to inflation is smaller than r_t by an order of magnitude, and it evolves more smoothly. q_t declined gradually throughout the 19th century, then remained steady until the 1940s. It rose in the 1950s and 1960s and peaked in the early 1980s, when the Thatcher government began to bring inflation down. Indeed, permanent volatility was about as high at the end of the Great Inflation as it was during the Napoleonic Wars. The volatility q_t of permanent shocks to inflation has been declining since the early 1980s, but it remains high by historical standards. Thus, while transient volatility has been tamed, our estimates suggest that persistent volatility has not.

As we shall see, the conditional variance for the log price level depends on both q_t and r_t, and both contribute terms that make the conditional variance increase with the forecast horizon. For our estimates, however, the term involving the volatility q_t of permanent shocks increases at a faster rate as the horizon is extended. Thus, although q_t is smaller, it becomes important at long horizons.

Third, trend inflation μ_t hovered around zero throughout the 19th century and increased gradually to about 1.35 percent by 1925. Trend inflation rose sharply during the Great Inflation and peaked above 10 percent. μ_t declined throughout the 1980s and 1990s, and the median estimate settled between 2.5 and 3 percent after the Bank of England was granted operational independence in 1997.

Fourth, the transient component $\pi_t - \mu_t$ is centered near zero (by design) throughout the sample. It was enormously volatile during the Napoleonic wars and after World War I and much less volatile after World War II. Because measurement error has been purged, the model asserts that this decline in volatility is genuine.

3.1 Informal account of how measurement-error variances are identified

Measured inflation has two transient components, one that is genuine $\sqrt{r_t}\varepsilon_{\pi t}$ and another due to measurement error $\sigma_{mt}\varepsilon_{mt}$. Because they are independent, the variance of the combined transient component is $r_t + \sigma_{mt}^2$. The measurement error variances σ_{im}^2 must be consistent with movements in total transient variance when innovations to $\ln r_t$ have variance σ_r^2.[5]

Since r_t must be positive, σ_{im} cannot exceed the smallest amount of combined transient variance in the relevant subperiods before 1948. This occurs in 1831 for σ_{1m}, 1891 for σ_{2m}, and just after World War II for σ_{3m}. Furthermore, σ_{im} must be far

[5]Recall that σ_r is identified in part by putatively error-free observations after 1948.

enough below these upper bounds for r_t to move from its peaks to its troughs. For instance, if one of the σ_{im} were close to its upper bound, r_t would be close to zero at that date and $\ln r_t \approx -\infty$, putting the trough too far below the peaks and making them unreachable. In other words, if r_t were close to 0, the transient component would all but vanish, and the model would be left to fit inflation via the permanent component alone. This sets an upper bound on σ_{im}.

On the other hand, if σ_{im} were too small, r_t would have to fit not only genuine movements in transient volatility but also drops in measured volatility due to improved measurement. Presumably this would increase σ_r, thereby worsening the fit of movements in transient volatility after 1948. This puts a lower bound on σ_{im}.

For the 1851-1914 subperiod, these considerations sharply identify σ_{2m} at 0.024, roughly 35 percent below its upper bound. Alas σ_{1m} and σ_{3m} are weakly identified, their posteriors being similar to the prior.

Because measurement-error variances for two subsamples are weakly identified, we also explore whether our results are sensitive to assumptions about the prior. We do this by estimating the model under an alternative 'high-noise' prior $IG_1(0.0488, 2)$ that increases both the prior mode and variance. The prior mode for σ_{im} now equals three-quarters of training sample inflation volatility, and a centered 95 percent credible set covers the interval (0.027,0.18). Happily, the results for this prior are essentially the same as those for the baseline prior (see appendix C). Hence, our conclusions are robust to alternative calibrations of this part of the model.

3.2 Orders of integration

Our model implies that inflation is $I(1)$ and that the log price level is $I(2)$, a common specification for post-World War II data. However, whether these orders of integration are consistent with pre-World War I data is not obvious. To examine this issue, we estimate an augmented Dickey-Fuller regression,

$$y_t = \mu + \rho y_{t-1} + \sum\nolimits_{j=1}^{2} \zeta_j \Delta y_{t-j} + u_t, \tag{5}$$

where y_t is measured inflation for the period 1791-1913, and we calculate the t-statistic for $\rho - 1$. This augmented Dickey-Fuller statistic is -8.7, and its 1-percent asymptotic critical value is -3.43. The test therefore seems very strongly to reject a unit root in inflation.

However, the estimates shown in figure 4 suggest that the random-walk component of inflation was small prior to World War I and that the transient component plus measurement error was large. It is tenuous whether an augmented Dickey-Fuller test can detect a small random-walk component hidden under substantial noise. To check the size of the test for a specification like ours, we simulate our state-space model, generating artificial data on measured inflation by drawing from the posterior for π_t and σ_{1m} and then calculating the implied distribution for the augmented Dickey-Fuller statistic. It turns out that the null distribution is shifted well to the left of the asymptotic distribution and that the correct 10 percent critical value is -9.08, implying that a unit root in inflation is not rejected. Indeed, the p-value for a sample statistic of -8.7 is 0.24.[6]

None of this proves that inflation was $I(1)$, but it does establish that our $I(1)$ representation is not grossly at odds with the data. We are sufficiently reassured that our representation for inflation is good enough for pre-20th century data to allow us to proceed.

4 Price-level uncertainty

To assess price-level uncertainty, we calculate the posterior conditional variance of cumulative inflation, $var(p_{t+h}-p_t|y^t)$. We do this by first conditioning on a hypothetical large information set (y^t, I^t), where $I_t = (\pi^t, \mu^t, r^t, q^t, \sigma_r, \sigma_q)$ and then marginalizing with respect to the additional conditioning variables.[7] In appendix D, we show that the conditional mean and variance with respect to the large information set are

$$\begin{aligned} E(p_{t+h} - p_t|y^t, I_t) &= h\mu_t, & (6)\\ var(p_{t+h} - p_t|y^t, I_t) &= r_t \sum\nolimits_{j=1}^{h} \exp(j\sigma_r^2/2) + q_t \sum\nolimits_{j=1}^{h} j^2 \exp(j\sigma_q^2/2), & (7) \end{aligned}$$

respectively. Notice that the transient and permanent components both contribute terms that make this conditional variance increase with the forecast horizon h. The increment to the term involving the volatility in the permanent component q_t is $h^2 \exp(h\sigma_q^2/2)$ while the increment to the term involving the transient volatility r_t is $\exp(h\sigma_r^2/2)$, so the term involving the permanent component increases more rapidly with the forecast horizon. Since $\sigma_r \approx \sigma_q$, the term involving the permanent component is more important in shaping outcomes for the longer horizons that we study. Finally, the conditional second moment is the sum of the conditional variance plus the square of the conditional mean,

$$E((p_{t+h} - p_t)^2|y^t, I_t) = (h\mu_t)^2 + r_t \sum\nolimits_{j=1}^{h} \exp(j\sigma_r^2/2) + q_t \sum\nolimits_{j=1}^{h} j^2 \exp(j\sigma_q^2/2). \quad (8)$$

[6] A specification with additive white-noise measurement error in the log price level passed this test at the 5 percent level but failed at the 10 percent level. Hence, we prefer the inflation-plus-noise specification.

[7] Notice that (y^t, I^t) implies knowledge of σ_{im}. The omission of the latter just condenses notation.

To find conditional moments with respect to the data history y^t, we integrate across the posterior for I_t,

$$E(p_{t+h} - p_t|y^t) = \int E(p_{t+h} - p_t|y^t, I_t)p(I_t|y^t)dI_t, \tag{9}$$

$$E((p_{t+h} - p_t)^2|y^t) = \int E((p_{t+h} - p_t)^2|y^t, I_t)p(I_t|y^t)dI_t. \tag{10}$$

Our Markov-chain monte carlo algorithm produces a posterior sample for I_t, and we approximate these integrals by averaging across this sample. After calculating conditional first- and second-moments in this way, we calculate the condition variance $var(p_{t+h} - p_t|y^t)$ by subtracting the square of the conditional mean $(E(p_{t+h} - p_t|y^t))^2$ from the conditional second moment $E((p_{t+h} - p_t)^2|y^t)$.

The blue lines in Figure 5 portray conditional standard deviations for forecasts of the log price level over horizons out to 10 years (we will explain the red lines later). For each panel, posteriors are estimated using data through period t; these posteriors are depicted in appendix B.

Uncertainty was high during the Napoleonic wars and shortly after the resumption of convertibility (represented by the years 1800 and 1825, respectively). At those times, both volatilities – r_t and q_t – were near their peak values. However, r_t and q_t both declined throughout the rest of the 19th century, and future price levels became less uncertain as they fell. On the eve of World War I, through some combination of luck and practice, monetary and fiscal authorities had somehow reduced price-level uncertainty by remarkable amounts. For 1913, our estimates of the conditional standard deviation 5 and 10 years ahead are 0.096 and 0.163, respectively, approximately one-sixth of what they were in 1800. The 19th century therefore witnessed a long and gradual conquest of price-level uncertainty.

Alas, the first World War profoundly altered monetary arrangements, and price-level uncertainty increased as war finance became a paramount concern of economic policy. Authorities struggled to reestablish the gold standard after the war but failed to restore the degree of price-level predictability that had prevailed in 1913. For instance, in 1930 conditional standard deviations 5 and 10 years ahead were 0.20 and 0.31, respectively, values roughly double those of 1913. Once shattered, the economic conditions on which prewar price predictability were based proved difficult to reconstitute.

Progress toward reducing price-level uncertainty resumed after World War II, especially under Bretton Woods. By 1960, conditional standard deviations of the log price level had declined to levels almost identical to those of 1913. Furthermore, at long horizons, there was less price level uncertainty than in any of the other post-World War II years that we studied.

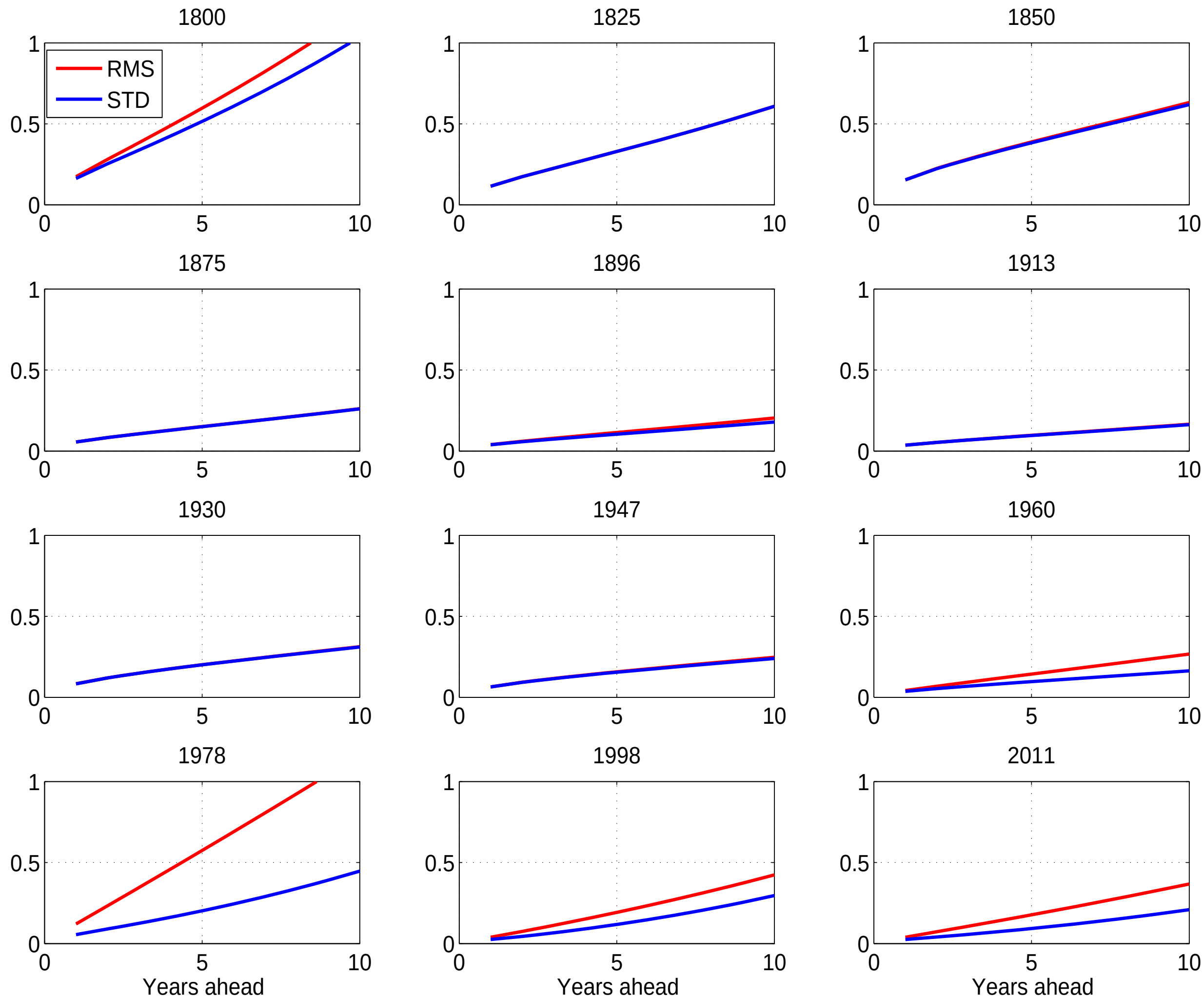

Figure 5: Blue lines represent conditional standard deviations for $p_{t+h} - p_t$. Red lines are root-mean-square statistics.

The Bretton Woods system also proved to be unsustainable after monetary and fiscal pressures in the U.S. and other countries initiated the Great Inflation. By 1978, the year of England's winter of discontent, conditional standard deviations 5 and 10 years ahead had risen to 0.202 and 0.447, greater than in any of the representative years since 1850 and more than twice those of 1913 or 1960. This backsliding was substantially reversed in the 1990s and 2000s. By 1998, conditional standard deviations 5 and 10 years ahead had declined by 34 and 42 percent, respectively, and they

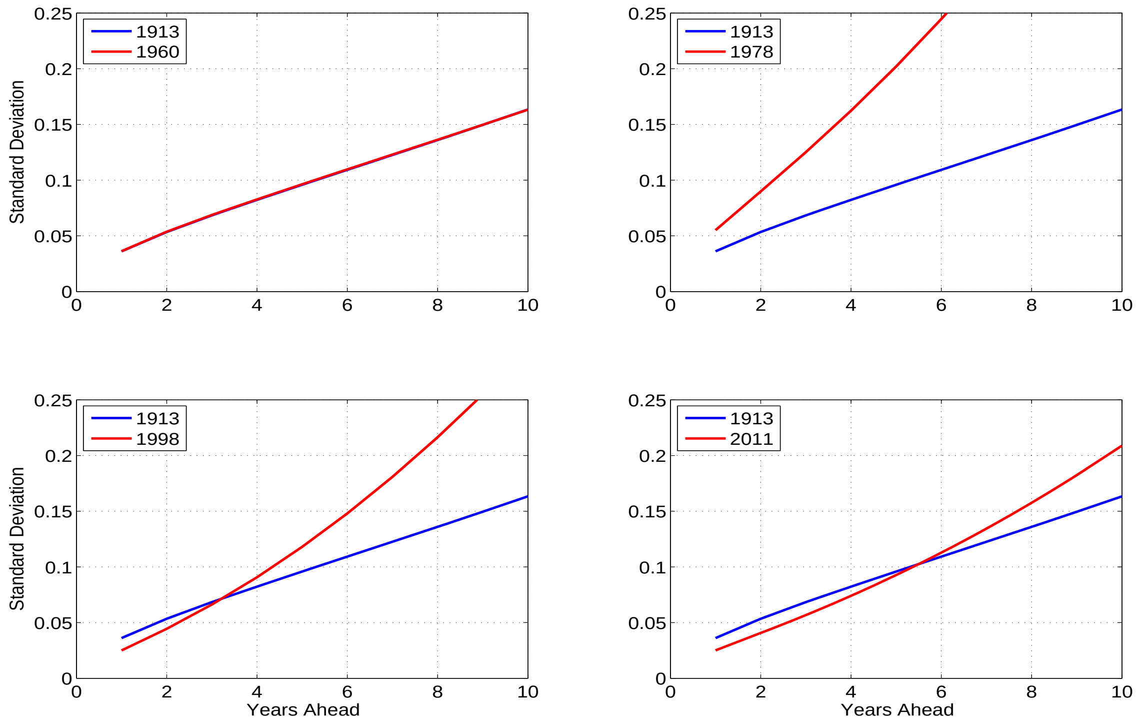

Figure 6: Conditional standard deviations for $p_{t+h} - p_t$

fell by another 20-30 percent by 2011.

Figures 6 and 7 provide two perspectives on whether the price level was more predictable before World War I or after World War II. Since price-level uncertainty has varied a lot, the answer depends on the dates being compared. Figure 6 compares the year 1913 with those after World War II. We set aside 1947 on the grounds that it too closely follows the end of the war and focus on the years 1960, 1978, 1998, and 2011. The blue lines reproduce estimates for 1913, while red lines depict estimates for the years after World War II.

As noted above, conditional standard deviations in 1960 were essentially the same as in 1913 (the red line in the northwest panel overlays the blue line). By 1978, however, price-level uncertainty had increased substantially, and conditional standard deviations 5 and 10 years ahead were roughly 110 and 175 percent higher than in 1913. For 1998 and 2011, conditional standard deviations were lower than in 1913 at short horizons but higher at long horizons, crossing those for 1913 at the 4 and 6 year horizons, respectively.

The year 1913 was the best of those before World War I, and different pictures

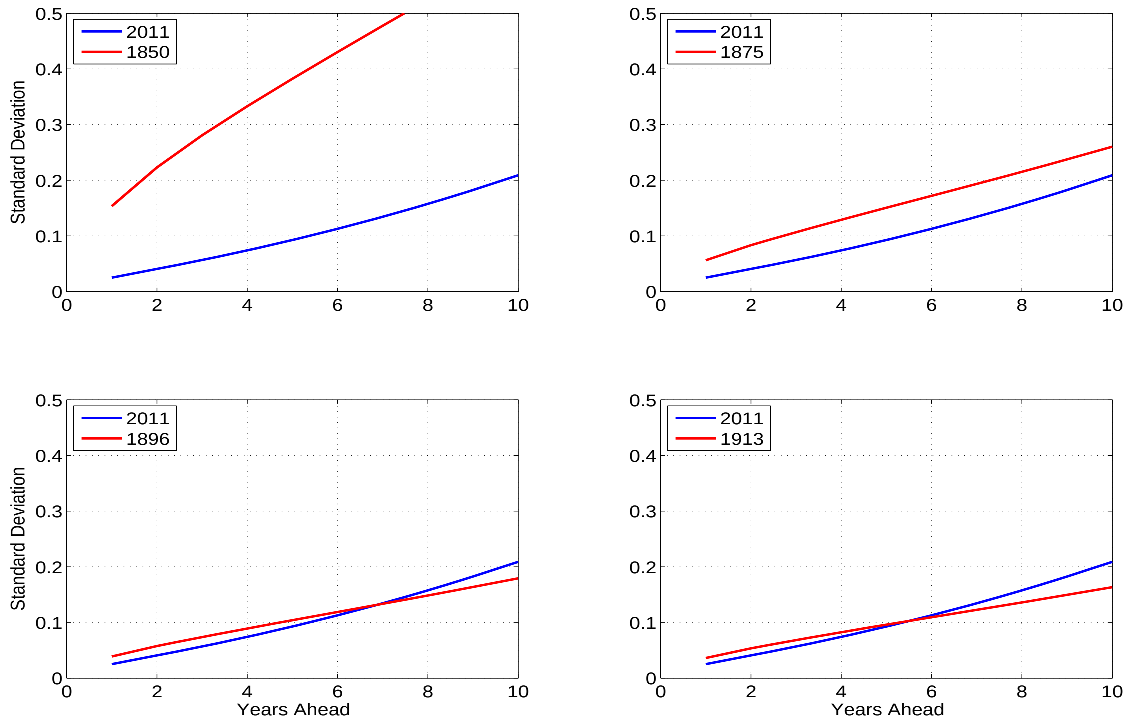

Figure 7: Conditional standard deviations for $p_{t+h} - p_t$

emerge when comparing the others with one of the better post-World War II years. For example, figure 7 compares outcomes in 2011 with those in 1850, 1875, 1896, and 1913. The year 2011 was selected because it represents a fiat regime and the average conditional standard deviation across horizons is among the lowest in the post-World War II years. Blue lines in figure 7 represent 2011, and red lines represent years before World War I. The panel for 1896 is similar to that for 1913, with conditional standard deviations being lower than in 2011 at long horizons and higher at short horizons. For 1875, conditional standard deviations are approximately twice those of 2011 at short horizons and roughly 25 to 30 percent higher at long horizons. For 1850 and earlier, estimates exceed those for 2011 by factors between 3 and 6. Hence there was a lot more uncertainty at mid century and a comparable amount – averaging across horizons – at the end.

That the price level is about as predictable under the current fiat regime than it was under the classical gold standard may seem to contradict the visual impression conveyed by the left panel of figure 1. As shown in figure 4, the permanent component of inflation was both higher and more volatile in 2011 than at the beginning of the

20th century, and one might have guessed that this change would dominate the change in conditional variances. But two facts should be kept in mind. The first is that conditional variances do not depend on the *level* of trend inflation μ_t, only on its *volatility* q_t. Second, as shown in figure 4, the variance of the transient component r_t was much lower in 2011, and the decline in r_t more than offset the increase q_t. The conquest of transient volatility explains why price-level uncertainty now is about the same as at the beginning of the 20th century. Changes in the relative importance of permanent and transient volatilities also explains why the conditional variance curve was more steeply sloped in 2011.

Ricardo and Keynes asserted that a well-managed fiat regime could improve on a commodity standard.[8] For limiting price-level uncertainty, our analysis suggests that the key words are 'well managed', not 'fiat' or 'commodity.' Examples of well-managed systems can be found both before World War I and after World War II and include both fiat and commodity standards. Examples of less-well-managed systems can also be found in both periods.

5 Price-level stability

Conditional variances are suitable for measuring uncertainty, but the concept of price stability seems different. 'Stability' describes *total* variation, not just *unpredictable* variation. For assessing price stability, we therefore compare conditional second moments across dates. Since the conditional second moment is the conditional variance plus the square of the conditional mean, we just need to add the latter to the numbers already reported.

While the conditional variance for the log price level is the same as that for cumulative inflation, $p_{t+h} - p_t$, conditional second moments are not, because the conditional means differ. For our model and the large conditioning set (y^t, I_t), the conditional mean of cumulative inflation is $h\mu_t$, so we just need to add $h^2\mu_t^2$ to the conditional variance to find the conditional second moment. On the other hand, the conditional mean for the log price level h periods ahead is $p_t + h\mu_t$, so $(p_t + h\mu_t)^2$ must be added to the conditional variance. Conditional second moments of the log price level therefore penalize both past and future cumulative inflation. Why past cumulative inflation should be penalized is not obvious, however, so we prefer to measure price (in)stability by conditional second moments for cumulative inflation.[9]

Red lines in figure 5 portray the conditional root-mean-square (the square root of

[8]See Sargent (2008) for accounts of their ideas and proposals.

[9]Since variation in p_t dominates everything else, conditional second moments for the log price level increase with time and are greatest at the end of our sample.

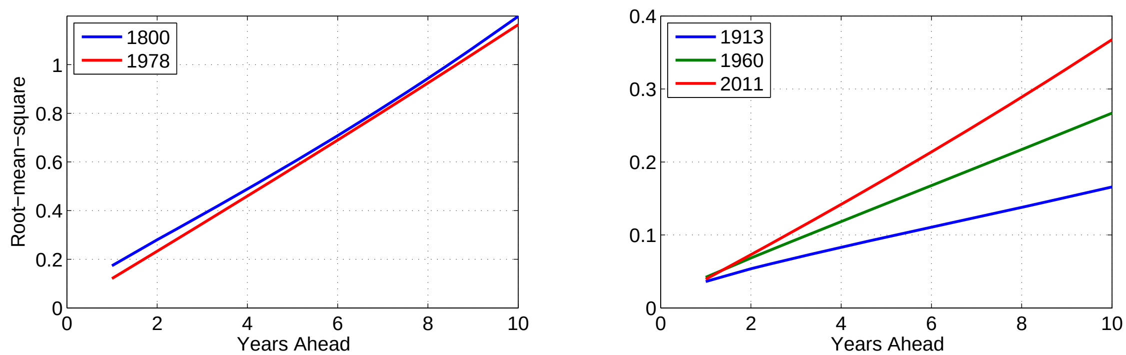

Figure 8: Conditional root mean square for $p_{t+h} - p_t$

the conditional second moment) of $p_{t+h} - p_t$ for various years. Because μ_t was close to zero in many of the selected periods, the red lines frequently hide behind the blue lines. This change of perspective matters when μ_t differs from zero, as in 1800 or after 1960, and it matters most for 1978, when trend inflation was highest. As shown in the left panel of figure 8, conditional second moments in 1978 were almost as large as those in 1800. Not only that, they are a lot higher than in any other year. For instance, the conditional root mean square for cumulative inflation over the next 5 or 10 years was between 1.5 to 3.75 times greater in 1978 than in 1850 or 1930, two other highly volatile years. Along with the Napoleonic era, the Great Inflation was the time of greatest price instability.

Perhaps less obvious from figure 5 is that focusing on price stability rather than uncertainty also changes the section 4 ordering of the best of pre-World War I and post-World War II years. The right panel of figure 8 clarifies this point by comparing outcomes for 1913, 1960, and 2011.[10] At the 1-year horizon, the conditional root-mean-square is the same in all three years, but it increases more rapidly in 1960 and 2011 and is substantially higher at long horizons. For instance, at horizons of 5 and 10 years, the conditional root-mean-square statistics are 48 and 61 percent higher in 1960 and 83 and 121 percent higher in 2011. For delivering price stability, 1913 emerges as a clear winner.

Other pre-World War I years also compare more favorably. As shown in figure 9, the price level was less stable in 2011 than in 1875 or 1896 and also less stable in 1960 than in 1896. Indeed, the best of the post-World War II years now appears to be roughly equivalent to 1875. However, the price level was more stable in both postwar years than in 1850.

[10]Notice that the scale in the right panel is smaller than in the left panel.

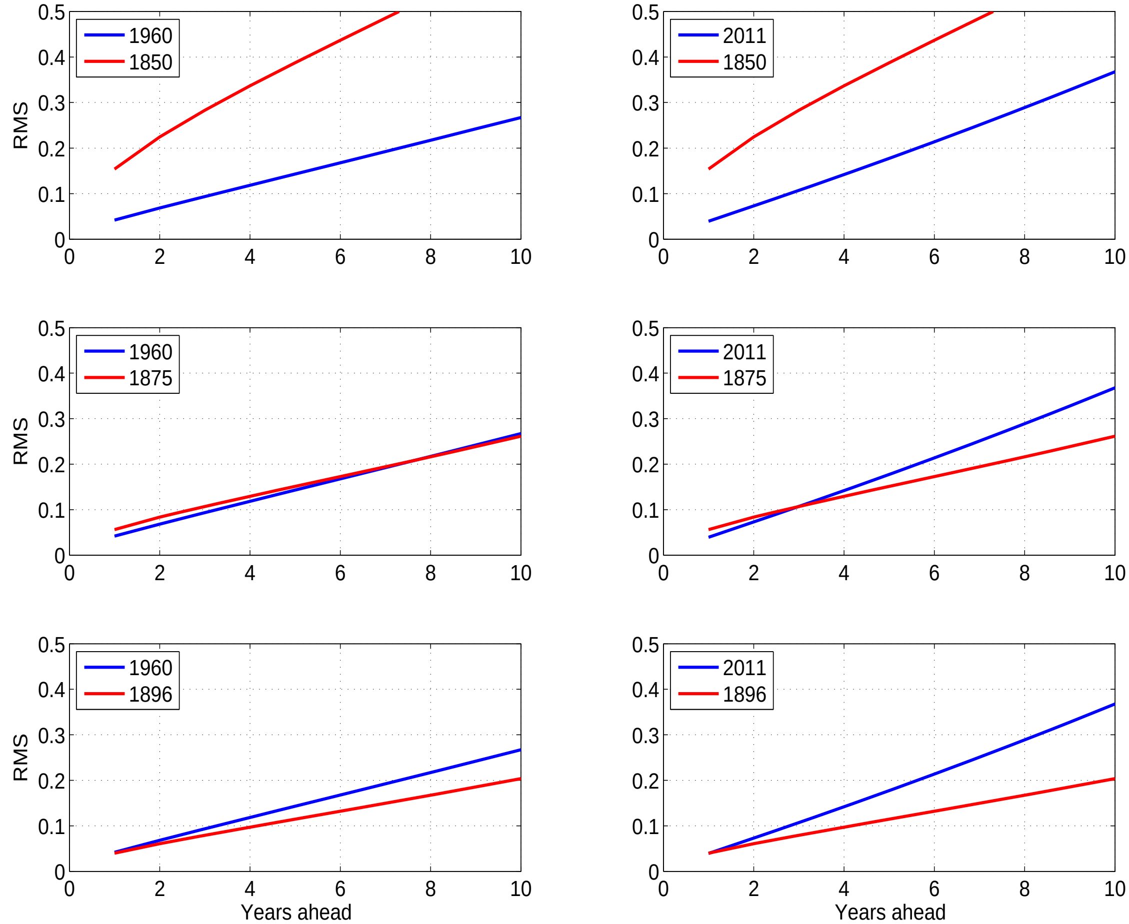

Figure 9: Conditional root mean square for $p_{t+h} - p_t$

6 Concluding remarks

Our analysis of UK inflation data evinces recurring episodes of rising and falling price-level uncertainty. Big shocks such as the French revolution and Napoleonic wars, the two World Wars and Great Depression, and the Great Inflation disrupted monetary arrangements and created appreciable uncertainty about future price levels. In each instance, authorities found a path back to price-level predictability, but sometimes it took a long time.

For limiting price-level uncertainty, well-managed systems sometimes involved a link to gold – as in 1913 or under Bretton Woods – but the inflation-targeting regime that prevails today proves how a well-managed fiat regime can achieve the same end.

Furthermore, commodity standards provide no permanent guarantee of future price predictability. Big shocks plague commodity as well as fiat regimes, and convertibility can and perhaps should be suspended in times of crisis. The classical gold standard and Bretton Woods system ended not because the authorities thought they had discovered better methods for maintaining price predictability but because other economic objectives supervened.

The history of price stability differs from the history of price predictability. According to our estimates, future price levels were least variable just before World War I, and the authorities have still yet to restore the degree of price stability which prevailed then. Although the volatility of a transient component of inflation is lower than ever, trend inflation and its volatility have not been fully tamed. According to our estimates, the latter more than offsets the former, explaining why the price level remains less stable now than at the beginning of the 20th century.

Our analysis is cast in terms of a purely *statistical* model, so we have nothing formal to say about why someone might *care* about the uncertainty and/or the stability of inflation and how they have changed over time. But there are plenty of economic models that do point to why we should care. For example, models in the tradition of Lucas (1973) make social welfare depend on the uncertainty of the log price level, but not its stability. Other models in the tradition of Lucas (2000) focus on welfare effects of the stability of inflation, not its uncertainty. In more general macroeconomic models, people might care about both price level stability and price level uncertainty. Our statistical characterizations of price level stability and price level uncertainty set the stage for a quantitative economic analysis of their welfare consequences.

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A A Markov-chain Monte Carlo algorithm for simulating the posterior

The posterior is simulated by a Metropolis-within-Gibbs algorithm. We use a superscript T to signify a history $\{x_t\}_{t=1}^T$. With this notation, the parameter blocks are

- $S^T = (\pi^T, \mu^T)|r^T, q^T, \sigma_r, \sigma_q, \sigma_m$
- $r_t|r_{\neq t}, q^T, S^T, \sigma_r, \sigma_q, \sigma_m$
- $q_t|q_{\neq t}, r^T, S^T, \sigma_r, \sigma_q, \sigma_m$
- $\sigma_r|r^T, q^T, S^T, \sigma_q, \sigma_m$
- $\sigma_q|r^T, q^T, S^T, \sigma_r, \sigma_m$
- $\sigma_m|r^T, q^T, S^T, \sigma_r, \sigma_q$.

For a Metropolis-within-Gibbs algorithm, we want posterior conditional densities or kernels for each of these blocks.

A.1 $p(S^T|r^T, q^T, \sigma_r, \sigma_q, \sigma_m, y^T)$

Conditional on $r^T, q^T, \sigma_r, \sigma_q, \sigma_m$, and y^T, equations (1)-(2) form a linear-Gaussian state-space system with known innovation variances. It follows that the posterior conditional density for the history of states S^T is Gaussian. To simplify computations, we set $\sigma_{mt} = 0.0001$ (1 basis point) after 1948. With this simplification, S^T can be sampled via Carter and Kohn's (1994) forward-filter, backward sampler.[11]

A.2 $p(r_t|r_{\neq t}, q^T, S^T, \sigma_r, \sigma_q, \sigma_m, y^T)$

Conditional on S^T, the innovations $\sqrt{r_t}\varepsilon_{\pi t}$ are observable. Because measurement errors are independent of other innovations in the model, knowledge of y^T is redundant given knowledge of S^T. Similarly, knowledge of σ_q is redundant given knowledge of q^T. Furthermore, since innovations to lnr_t and lnq_t are independent, knowledge of q^T is irrelevant for inference about r^T. Hence the conditional posterior for this block simplifies to $p(r_t|r_{\neq t}, S^T, \sigma_r)$. The problem therefore reduces to simulating the posterior for a univariate stochastic-volatility process. This problem was studied by Jacquier, et al. (1994), and we adopt their algorithm for sampling from the conditional posterior, modifying their proposal density as in Cogley and Sargent (2005).

A.3 $p(q_t|q_{\neq t}, r^T, S^T, \sigma_r, \sigma_q, \sigma_m, y^T)$

Mutatis mutandis, block 3 has the same structure as block 2. We use the Jacquier, et al. algorithm to simulate this block as well.

A.4 $p(\sigma_r|r^T, q^T, S^T, \sigma_q, \sigma_m, y^T)$

Given r^t, the log-volatility innovation $\sigma_r\eta_{rt}$ is observable. All other conditioning information is irrelevant because η_{rt} is independent of all other shocks in the model. The conditional posterior therefore reduces to $p(\sigma_r|r^T$. The innovations $\sigma_r\eta_{rt}$ are *iid* normal with mean zero and variance σ_r^2. Since the prior on σ_r is inverse-gamma and the likelihood function is gaussian, the posterior is also IG. Hence we sample σ_r by drawing from an IG density.

[11]Carter and Kohn's algorithm can be modified to handle $\sigma_{mt} = 0$, but we felt that a 1 basis point improvement in accuracy did not warrant the extra complication.

A.5 $p(\sigma_q|r^T, q^T, S^T, \sigma_r, \sigma_m, y^T)$

Mutatis mutandis, block 5 has the same structure as block 4. The conditional posterior density for this block is also IG.

A.6 $p(\sigma_{im}|r^T, q^T, S^T, \sigma_r, \sigma_q, y^T)$

While analogs to blocks 1-5 appear in Cogley and Sargent (2005), block 6 is new. The model is designed, however, so that inference about measurement error is straightforward. Conditional on inflation data y^T and the latent variable π^T, the measurement error $\sigma_{mt}\varepsilon_{mt}$ is observed. For the respective subperiods before 1948, these innovations are *iid* normal with mean zero and variance σ_{im}. Since the prior on σ_{im} is IG and the conditional likelihood is gaussian, the posterior is also IG. Furthermore, since σ_{im} drops out of the likelihood function for periods other than the relevant subperiod, one can simulate σ_m by sampling from a IG distribution using data only for the relevant subperiod.

A.7 Convergence diagnostics

The Markov chain was run for 2 million steps, and 1 million steps were discarded as burn-in. Convergence was diagnosed by inspecting recursive mean plots and the outcomes of parallel chains with different initial conditions. The output of alternative chains differed in no interesting respects.

B Posteriors conditioned on data for selected years

Here we portray features of posteriors conditioned on data through to the selected dates discussed in the text (1800, 1825, 1850, 1875, 1896, 1913, 1930, 1947, 1960, 1978, and 1998).[12] The conditional variances and second moments reported in the text are based on these estimates.

[12] Posteriors conditioned on data through 2011 are depicted in the text.

B.1 1800

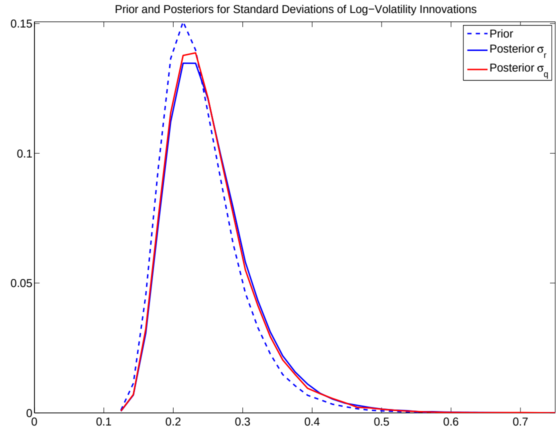

Figure 10: Posteriors for σ_r and σ_q conditioned on data through 1800

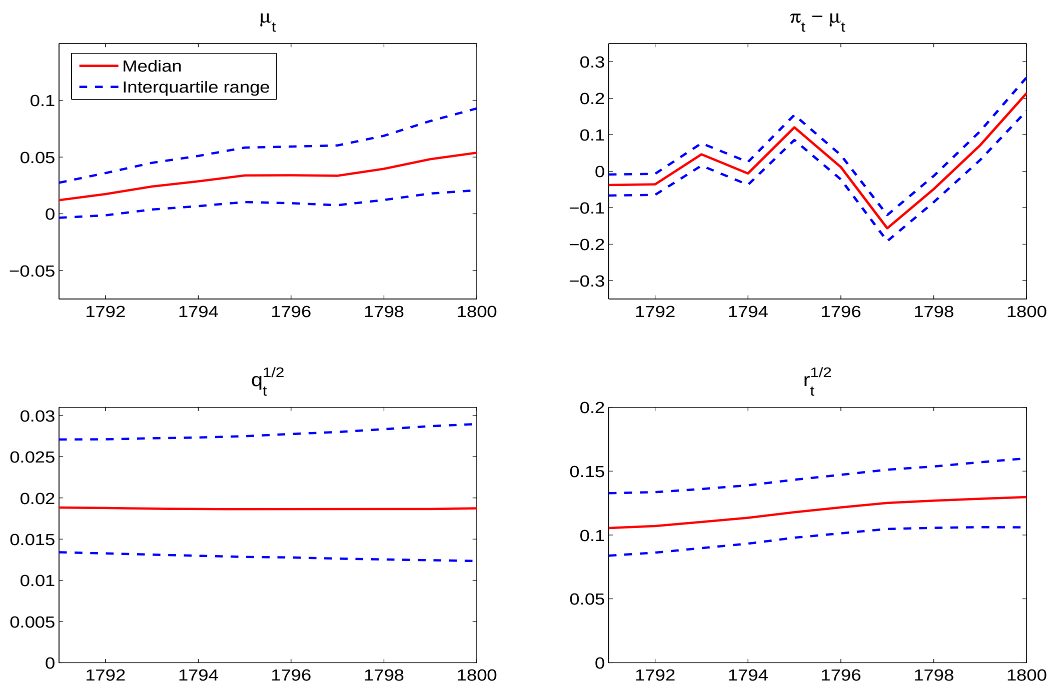

Figure 11: Estimates of hidden states conditioned on data through 1800

B.2 1825

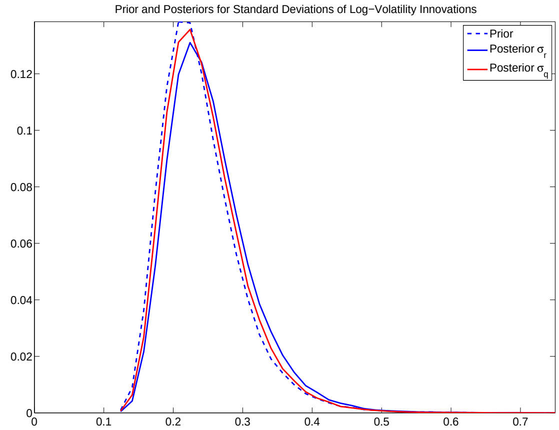

Figure 12: Posteriors for σ_r and σ_q conditioned on data through 1825

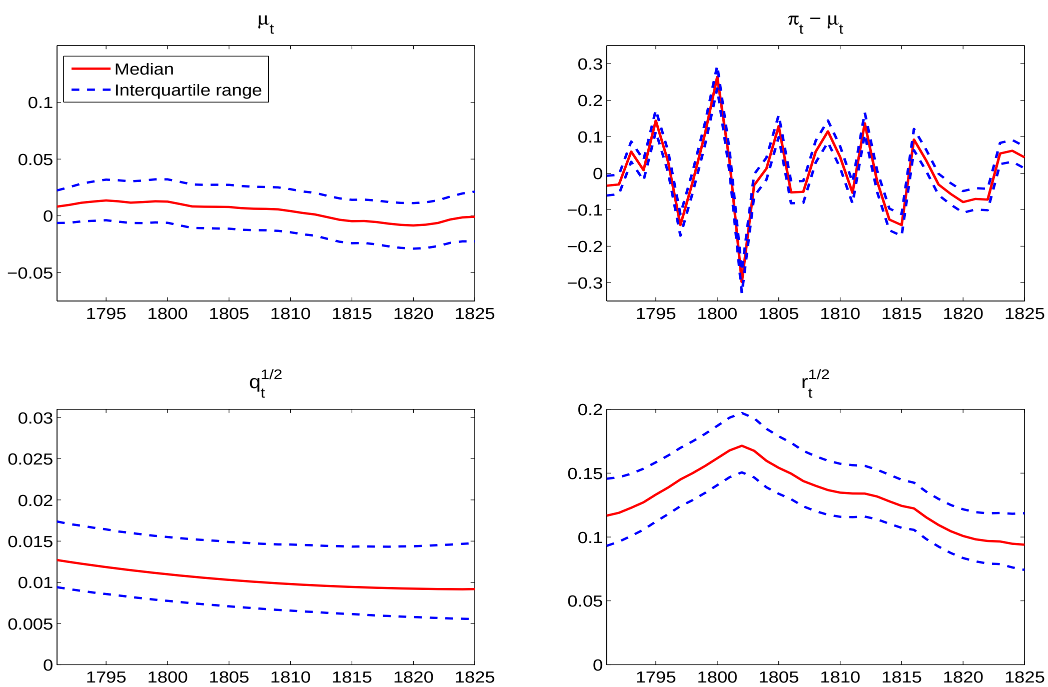

Figure 13: Estimates of hidden states conditioned on data through 1825

B.3 1850

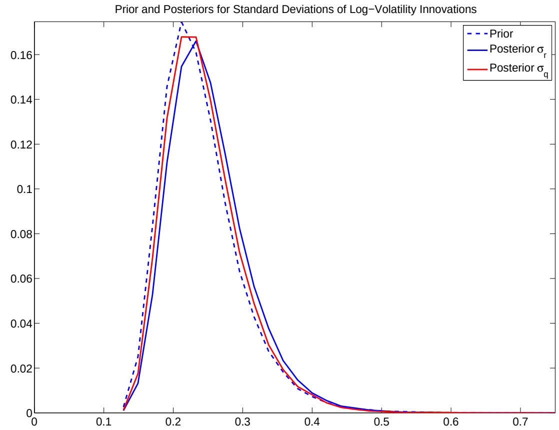

Figure 14: Posteriors for σ_r and σ_q conditioned on data through 1850

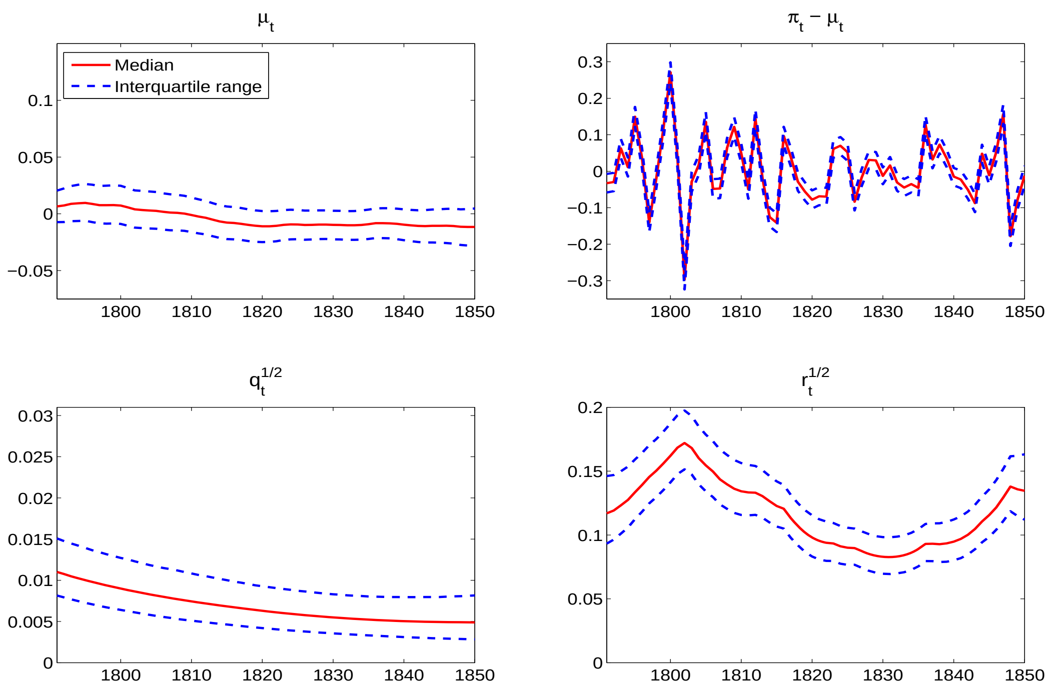

Figure 15: Estimates of hidden states conditioned on data through 1850

B.4 1875

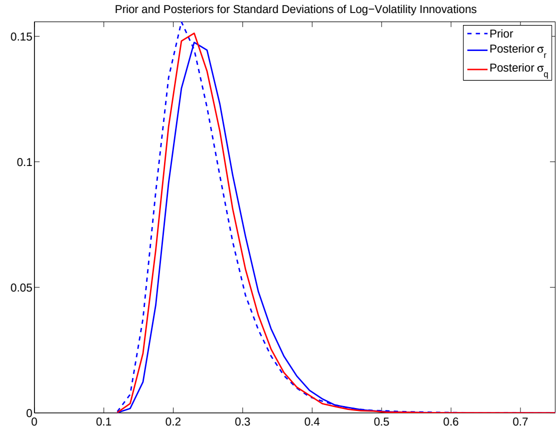

Figure 16: Posteriors for σ_r and σ_q conditioned on data through 1875

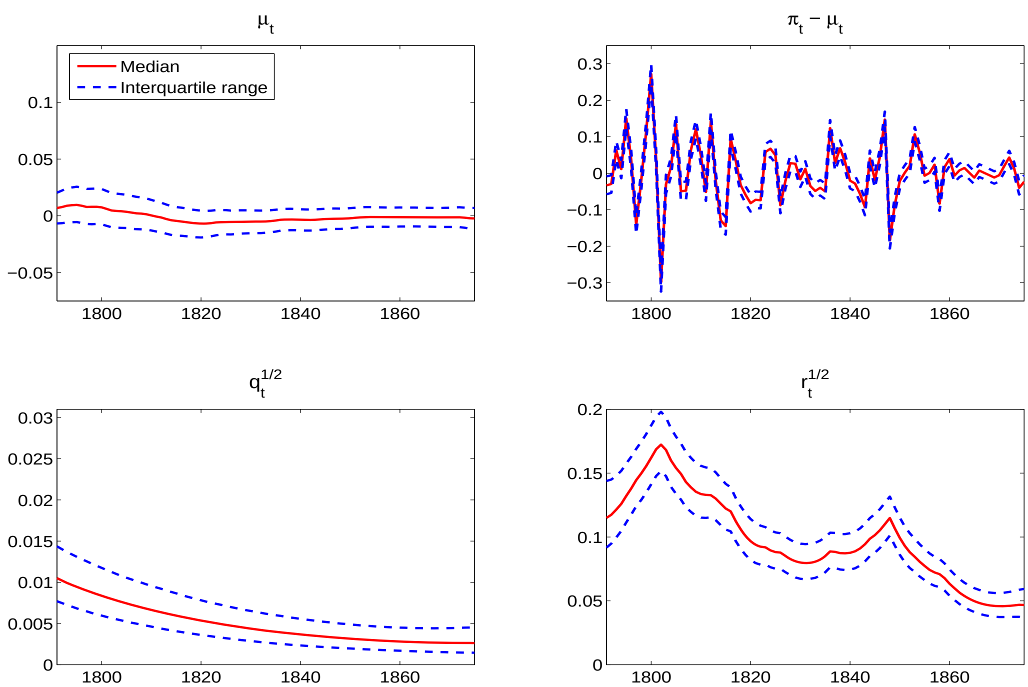

Figure 17: Estimates of hidden states conditioned on data through 1875

B.5 1896

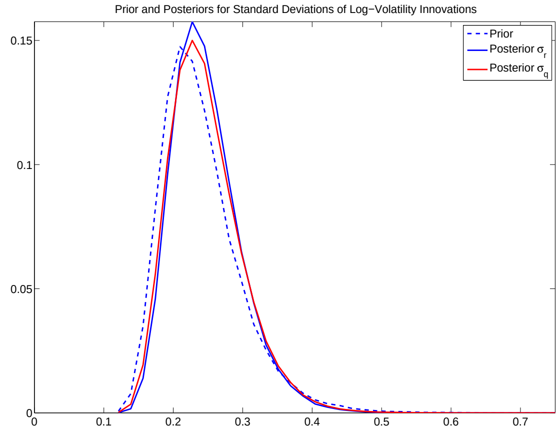

Figure 18: Posteriors for σ_r and σ_q conditioned on data through 1896

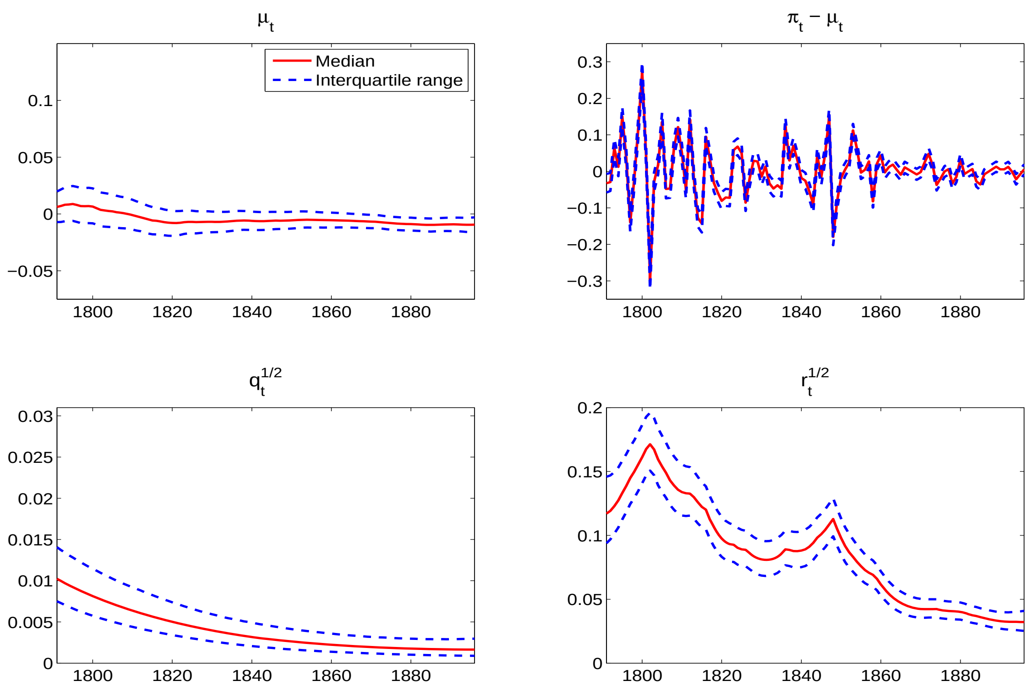

Figure 19: Estimates of hidden states conditioned on data through 1896

B.6 1913

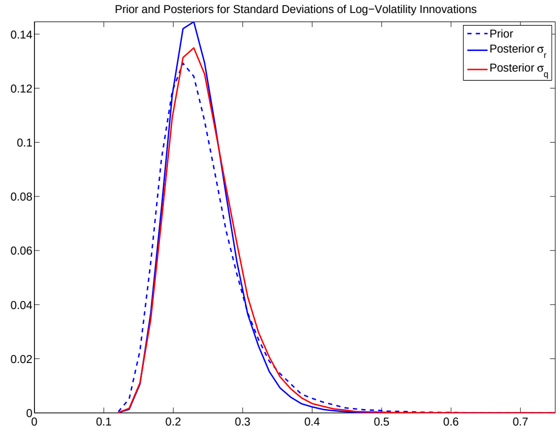

Figure 20: Posteriors for σ_r and σ_q conditioned on data through 1913

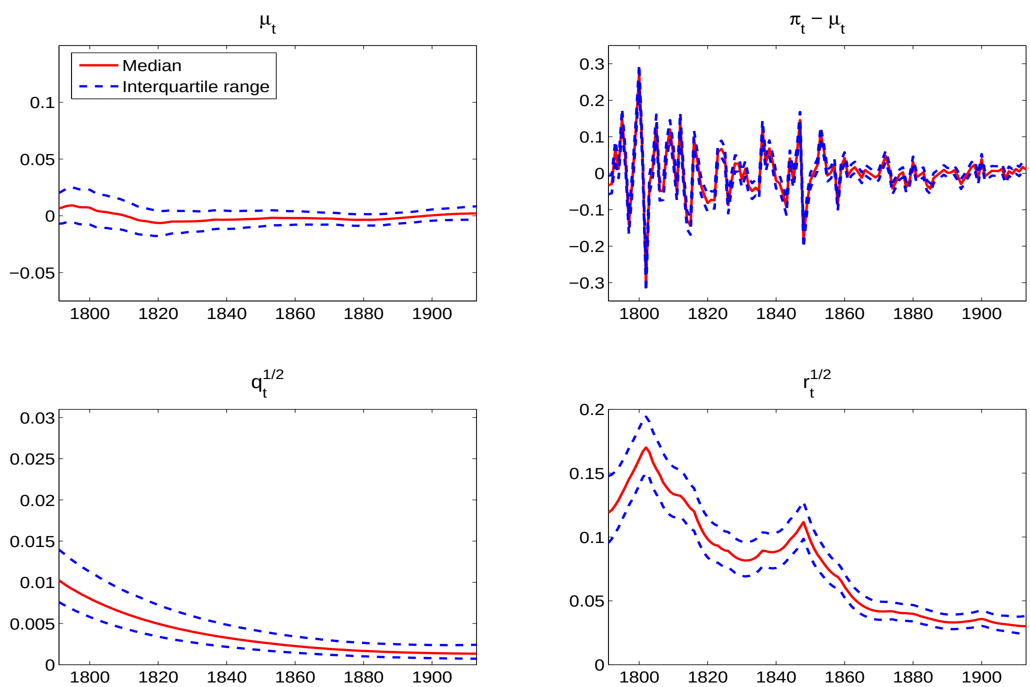

Figure 21: Estimates of hidden states conditioned on data through 1913

B.7 1930

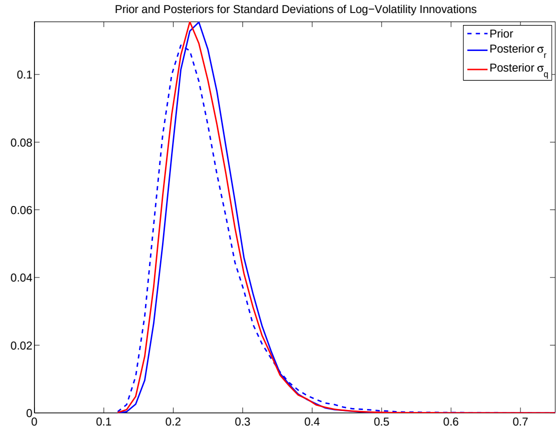

Figure 22: Posteriors for σ_r and σ_q conditioned on data through 1930

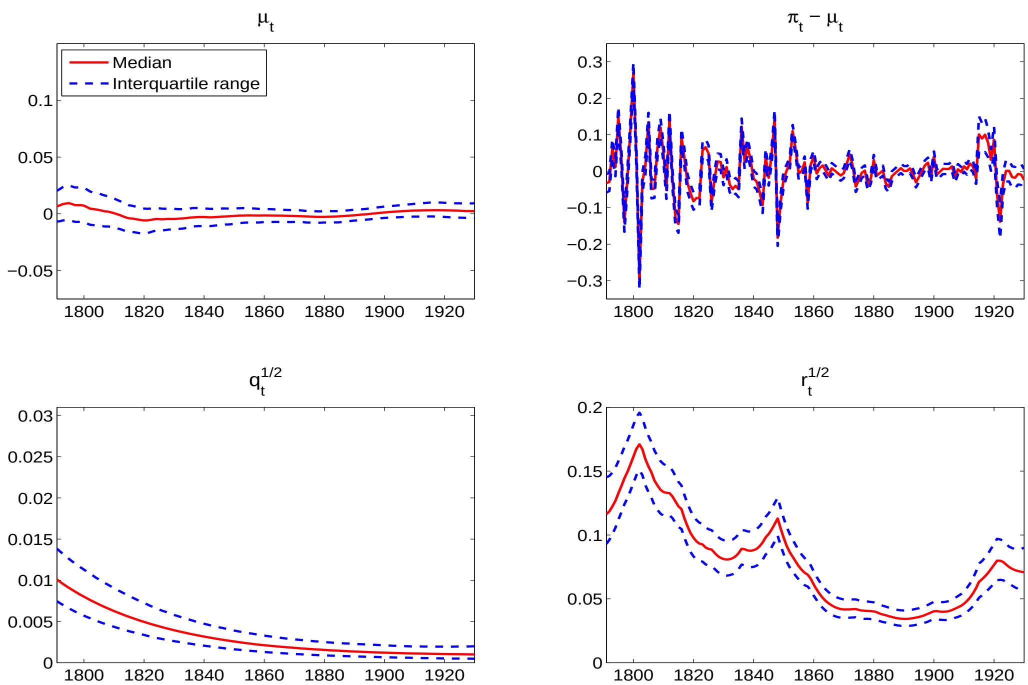

Figure 23: Estimates of hidden states conditioned on data through 1930

B.8 1947

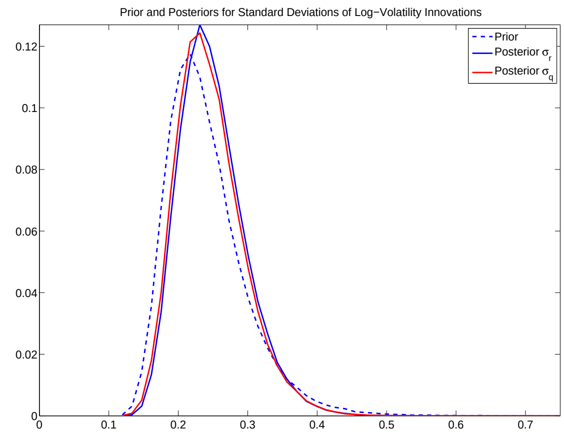

Figure 24: Posteriors for σ_r and σ_q conditioned on data through 1947

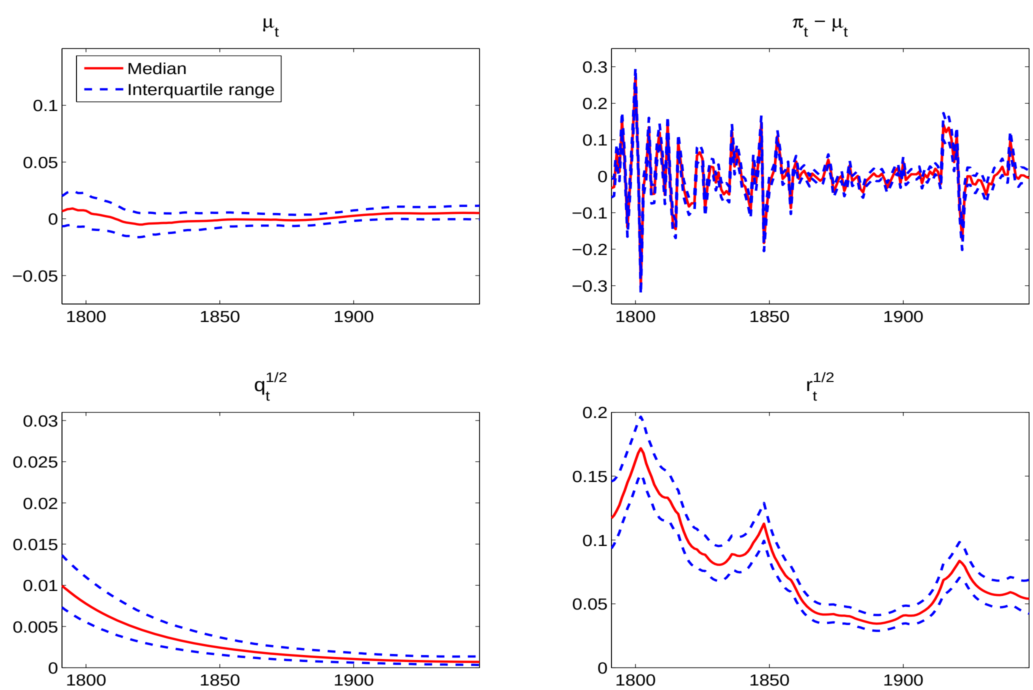

Figure 25: Estimates of hidden states conditioned on data through 1947

B.9 1960

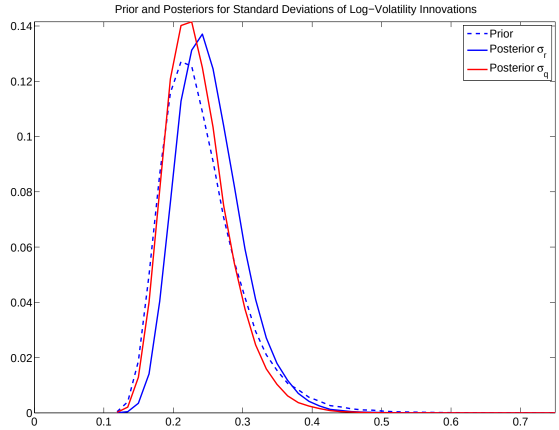

Figure 26: Posteriors for σ_r and σ_q conditioned on data through 1960

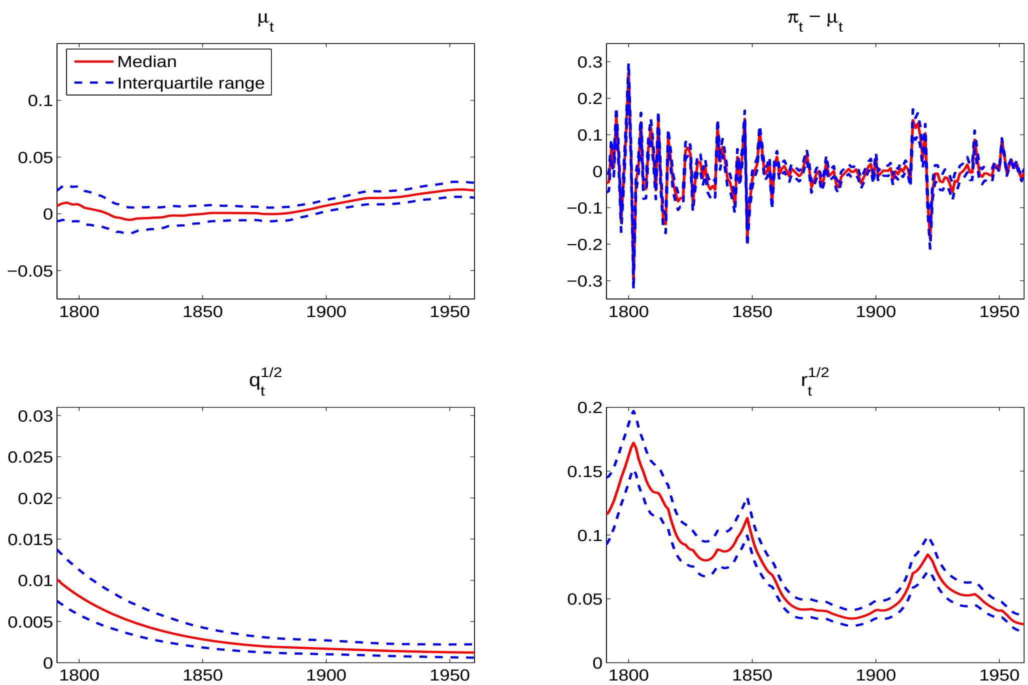

Figure 27: Estimates of hidden states conditioned on data through 1960

B.10 1978

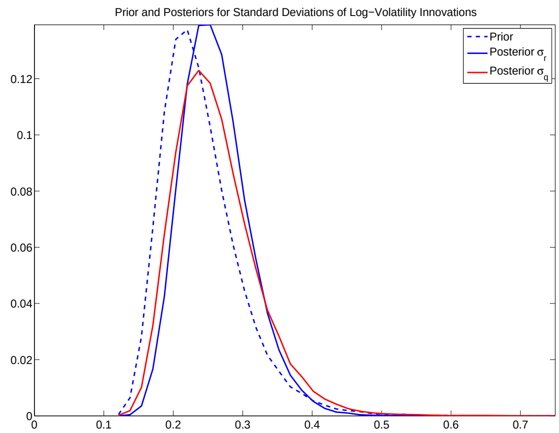

Figure 28: Posteriors for σ_r and σ_q conditioned on data through 1978

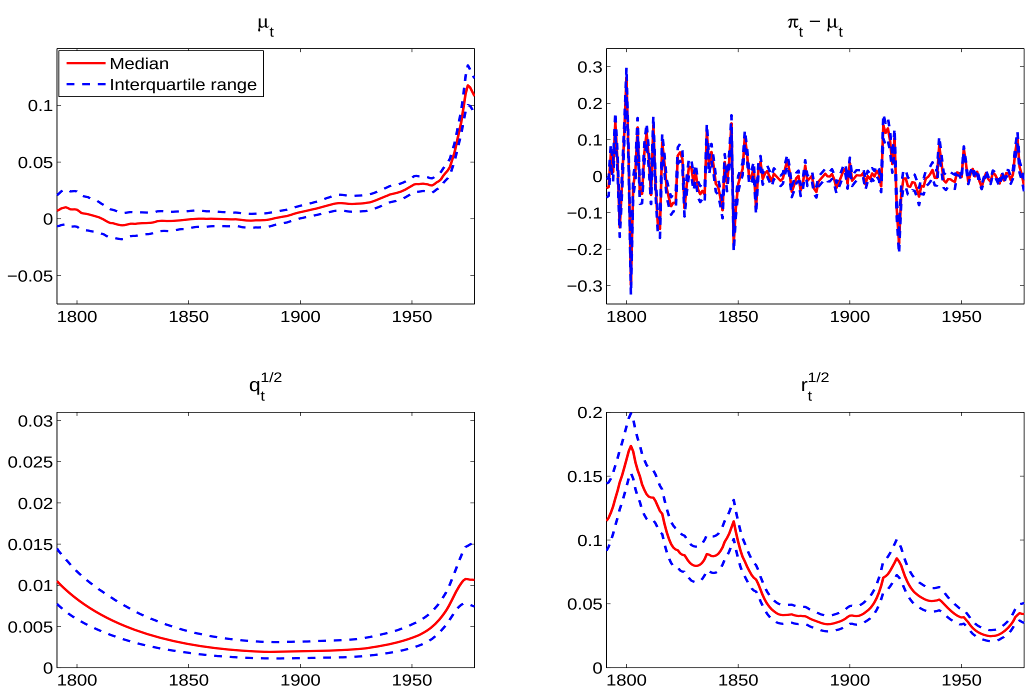

Figure 29: Estimates of hidden states conditioned on data through 1978

B.11 1998

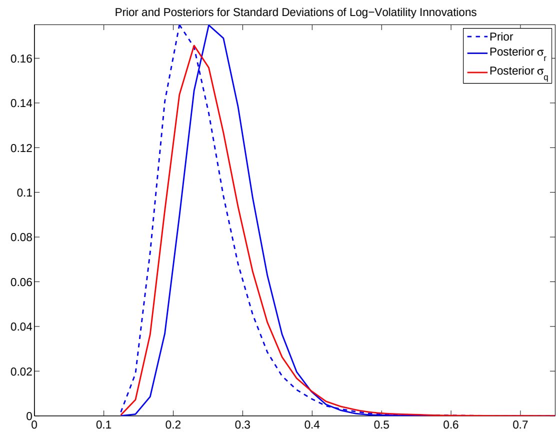

Figure 30: Posteriors for σ_r and σ_q conditioned on data through 1998

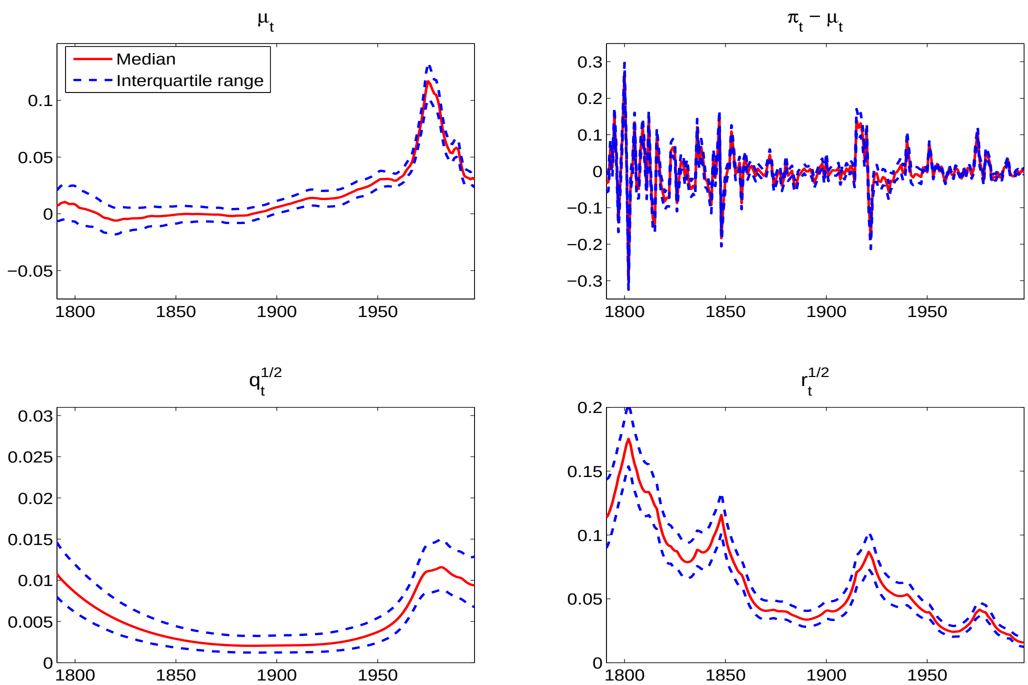

Figure 31: Estimates of hidden states conditioned on data through 1998

C Posteriors under the high-noise prior

The alternative prior for σ_{im} is $IG_1(0.0488, 2)$. We call this a high-noise prior because its mode equals 0.75 times the estimated standard deviation for inflation from the training sample 1721-1790 and because it has a long upper tail. Nevertheless, the posteriors resemble those for the baseline prior. See figures 32-39.

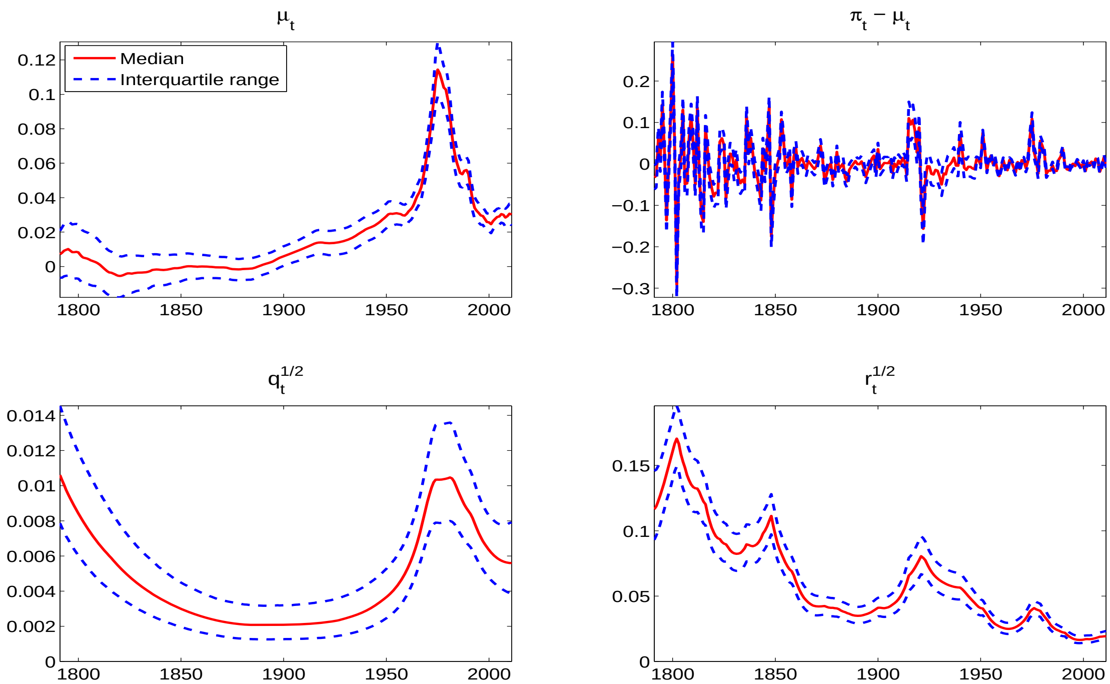

Figure 32: Posteriors of hidden states under the high-noise prior (conditioned on sample 1791-2011).

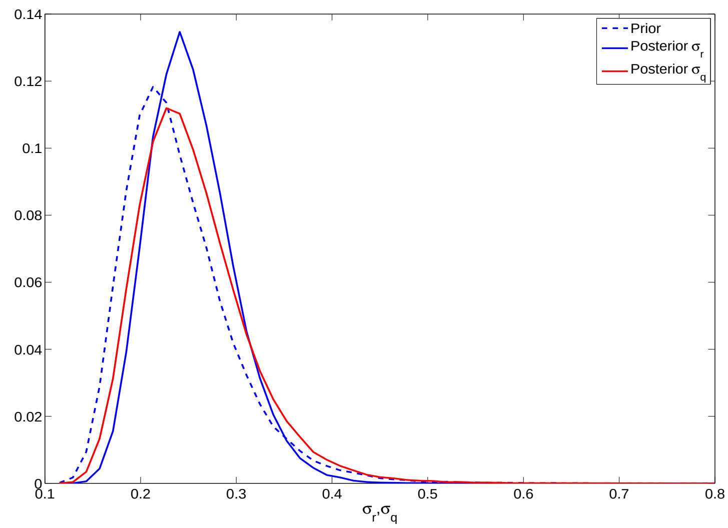

Figure 33: Prior and Posteriors for σ_r and σ_q under the high-noise prior

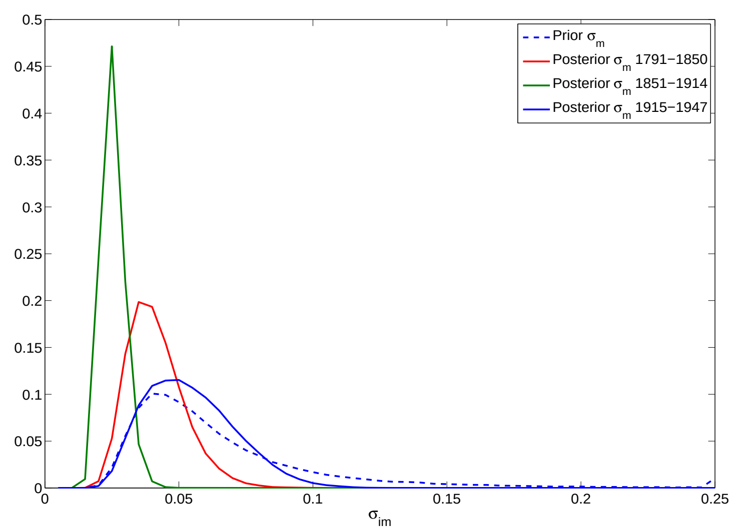

Figure 34: Prior and Posterior for σ_m under the high-noise prior

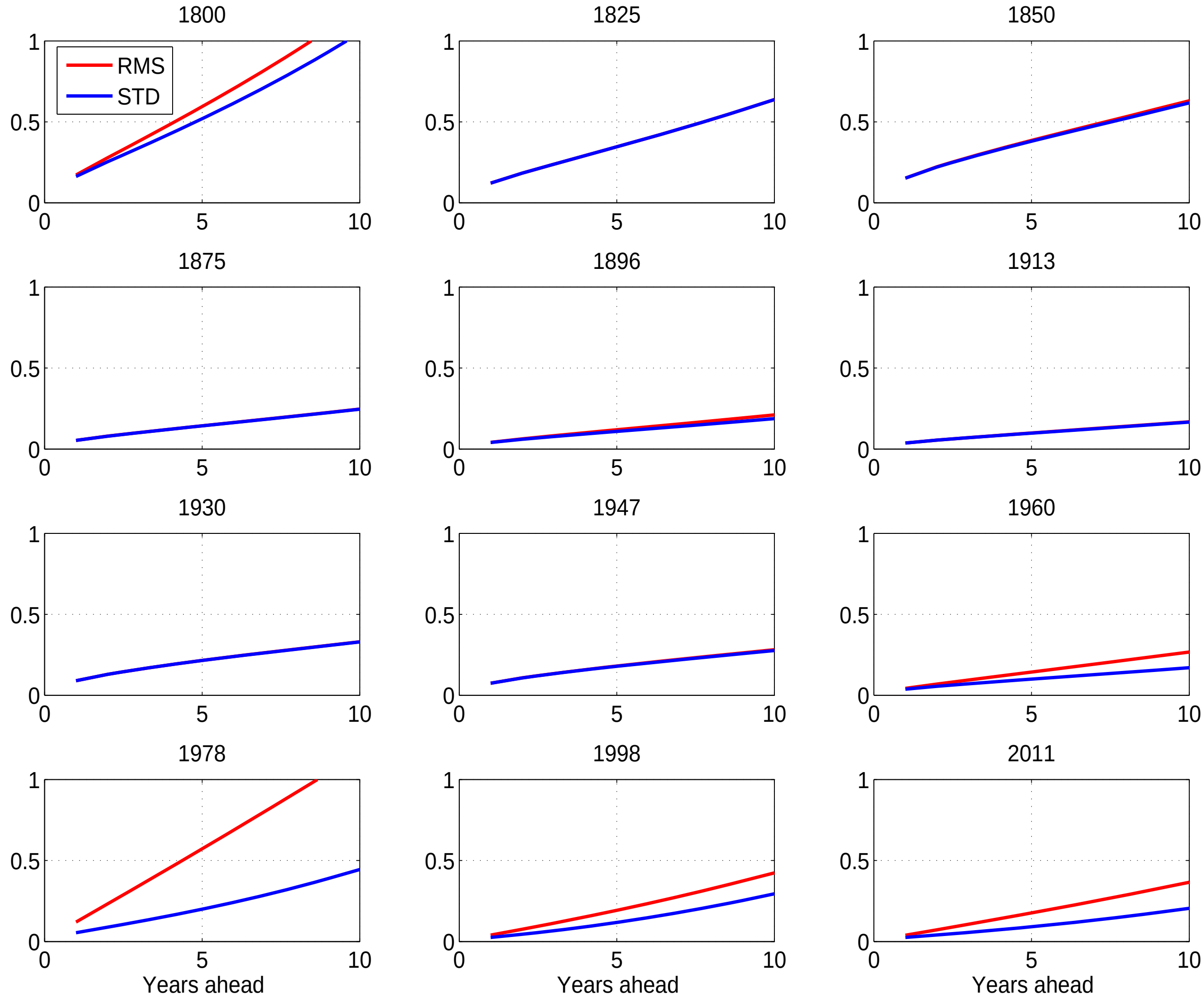

Figure 35: Posterior volatilities under the high-noise prior. Blue lines represent conditional standard deviations for $p_{t+h} - p_t$. Red lines are root-mean-square statistics.

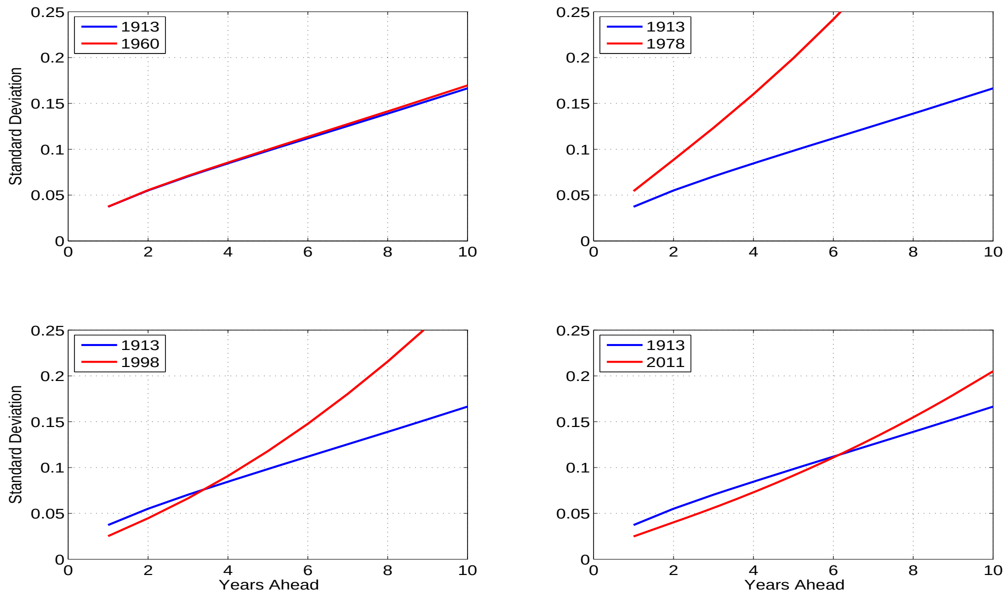

Figure 36: Conditional standard deviations for $p_{t+h} - p_t$ under the high-noise prior.

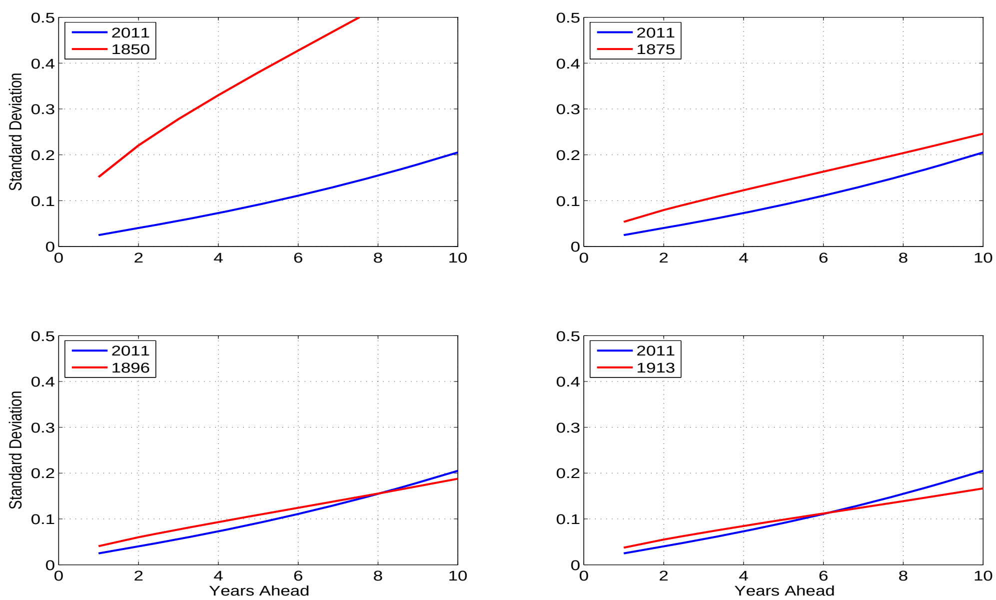

Figure 37: Conditional standard deviations for $p_{t+h} - p_t$ under the high-noise prior

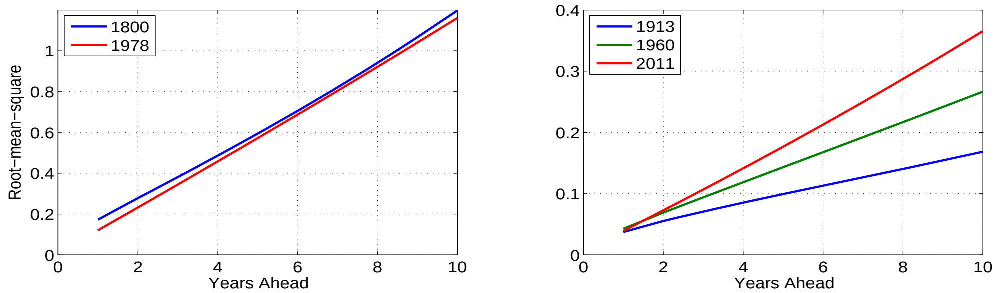

Figure 38: Conditional root mean square for $p_{t+h} - p_t$ under the high-noise prior.

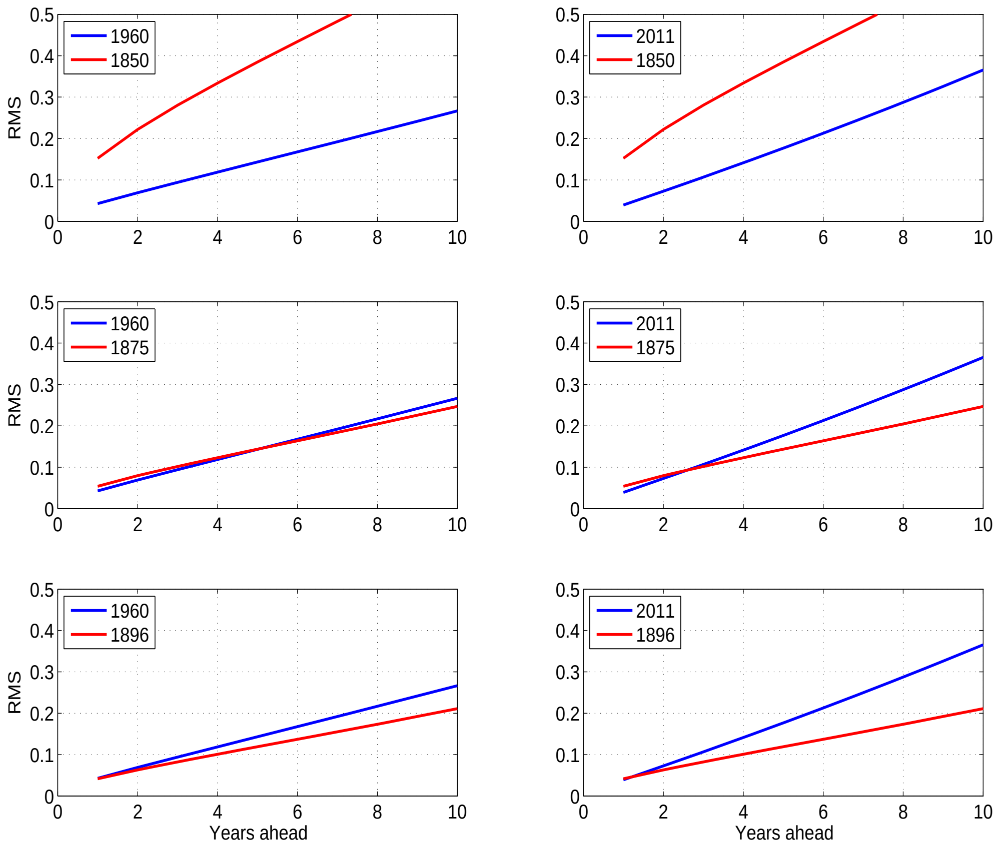

Figure 39: Conditional root mean square for $p_{t+h} - p_t$ under the high-noise prior.

D Deriving the conditional mean and variance of cumulative inflation

Since inflation is

$$\pi_t = \left(\mu_0 + \sum\nolimits_{j=1}^{t} \varepsilon_{\mu j}\right) + \varepsilon_{\pi t}, \tag{11}$$

the log price level can be represented as

$$p_t = p_{t-1} + \left(\mu_0 + \sum\nolimits_{j=1}^{t} \varepsilon_{\mu j}\right) + \varepsilon_{\pi t}. \tag{12}$$

The conditioning set at t is denoted (y^t, I_t) where $I_t = \pi^t, \mu^t, r^t, q^t, \sigma_r, \sigma_q$. Because measurement errors are independent of all other variables, however, y^t is redundant given I_t. Hence moments can be calculated conditional on I_t, with y^t reintroduced at the end, before marginalizing with respect to I_t.

We proceed recursively. Setting the forecast horizon $h = 1$, the conditional mean is

$$E(p_{t+1}|I_t) = p_t + \mu_0 + \sum\nolimits_{j=1}^{t} \varepsilon_{\mu j} = p_t + \mu_t, \tag{13}$$

and the prediction error is

$$p_{t+1} - E(p_{t+1}|I_t) = \varepsilon_{\mu t+1} + \varepsilon_{\pi t+1}. \tag{14}$$

Hence the conditional variance is

$$var(p_{t+1}|I_t) = E\left(\varepsilon^2_{\pi t+1} + \varepsilon^2_{\mu t+1}|I_t\right) = E\left(r_{t+1} + q_{t+1}|I_t\right). \tag{15}$$

The variables r_{t+1} and q_{t+1} are conditionally lognormal one step ahead with means $r_t \cdot \exp(\sigma_r^2/2)$ and $q_t \cdot \exp(\sigma_q^2/2)$, respectively. Hence

$$var(p_{t+1}|I_t) = r_t \cdot \exp(\sigma_r^2/2) + q_t \cdot \exp(\sigma_q^2/2). \tag{16}$$

Now advance to $h = 2$. The conditional mean of p_{t+2} is

$$\begin{aligned} E(p_{t+2}|I_t) &= E(p_{t+1}|I_t) + \mu_0 + \sum\nolimits_{j=1}^{t} \varepsilon_{\mu j}, \\ &= p_t + 2\mu_t, \end{aligned} \tag{17}$$

and the forecast error is

$$\begin{aligned} p_{t+2} - E(p_{t+2}|I_t) &= p_{t+1} - E(p_{t+1}|I_t) + \varepsilon_{\mu t+1} + \varepsilon_{\mu t+2} + \varepsilon_{\pi t+2}, \\ &= (\varepsilon_{\mu t+1} + \varepsilon_{\pi t+1}) + (\varepsilon_{\mu t+1} + \varepsilon_{\mu t+2} + \varepsilon_{\pi t+2}), \\ &= (\varepsilon_{\pi t+1} + \varepsilon_{\pi t+2}) + (2\varepsilon_{\mu t+1} + \varepsilon_{\mu t+2}). \end{aligned} \tag{18}$$

The conditional variance for inflation two steps ahead is

$$\begin{aligned} var(p_{t+2}|I_t) &= var\left(\varepsilon_{\pi t+1} + \varepsilon_{\pi t+2} + 2\varepsilon_{\mu t+1} + \varepsilon_{\mu t+2}|I_t\right), \\ &= r_t \cdot \exp(\sigma_r^2/2) + 2q_t \cdot \exp(\sigma_q^2/2) + E\left(\varepsilon_{\pi t+2}^2 + \varepsilon_{\mu t+2}^2|I_t\right). \end{aligned} \tag{19}$$

By the law of iterated expectations, expectations two steps ahead can be expressed as

$$E(\varepsilon_{xt+2}^2|I_t) = E\left[E(\varepsilon_{xt+2}^2|I_{t+1})|I_t\right]. \tag{20}$$

The inner expectation is

$$E(\varepsilon_{xt+2}^2|I_{t+1}) = x_{t+1}\exp(\sigma_x^2/2), \tag{21}$$

and the outer expectation is

$$\begin{aligned} E\left[x_{t+1}\exp(\sigma_x^2/2)|I_t\right] &= E(x_{t+1}|I_t)\exp(\sigma_x^2/2), \\ &= x_t\exp(\sigma_x^2/2)\exp(\sigma_x^2/2), \\ &= x_t\exp(2\sigma_x^2/2). \end{aligned} \tag{22}$$

It follows that the two-step ahead conditional variance is

$$\begin{aligned} var(p_{t+2}|I_t) &= r_t\left[\exp(\sigma_r^2/2) + \exp(2\sigma_r^2/2)\right] + q_t\left[\exp(\sigma_q^2/2) + 4\exp(2\sigma_q^2/2)\right], \\ &= r_t\sum\nolimits_{j=1}^{2}\exp(j\sigma_r^2/2) + q_t\sum\nolimits_{j=1}^{2} j^2\exp(j\sigma_q^2/2). \end{aligned} \tag{23}$$

Now set $h = 3$. The conditional mean is

$$\begin{aligned} E(p_{t+3}|I_t) &= E(p_{t+2}|I_t) + \mu_0 + \sum\nolimits_{j=1}^{t}\varepsilon_{\mu j}, \\ &= p_t + 3\mu_t, \end{aligned} \tag{24}$$

and the forecast error is

$$\begin{aligned} p_{t+3} - E(p_{t+3}|I_t) &= p_{t+2} - E(p_{t+2}|I_t) + \varepsilon_{\mu t+1} + \varepsilon_{\mu t+2} + \varepsilon_{\mu t+3} + \varepsilon_{\pi t+3}, \\ &= (\varepsilon_{\pi t+1} + \varepsilon_{\pi t+2} + 2\varepsilon_{\mu t+1} + \varepsilon_{\mu t+2}) + (\varepsilon_{\mu t+1} + \varepsilon_{\mu t+2} + \varepsilon_{\mu t+3} + \varepsilon_{\pi t+3}), \\ &= (\varepsilon_{\pi t+1} + \varepsilon_{\pi t+2} + \varepsilon_{\pi t+3}) + (3\varepsilon_{\mu t+1} + 2\varepsilon_{\mu t+2} + \varepsilon_{\mu t+3}). \end{aligned} \tag{25}$$

It follows that the conditional variance is

$$\begin{aligned} var(p_{t+3}|I_t) &= r_t\left[\exp(\sigma_r^2/2) + \exp(2\sigma_r^2/2) + \exp(3\sigma_r^2/2)\right] \\ &\quad + q_t\left[\exp(\sigma_q^2/2) + 4\exp(2\sigma_q^2/2) + 9\exp(3\sigma_q^2/2)\right], \\ &= r_t\sum\nolimits_{j=1}^{3}\exp(j\sigma_r^2/2) + q_t\sum\nolimits_{j=1}^{3} j^2\exp(j\sigma_q^2/2). \end{aligned} \tag{26}$$

By continuing the recursion, one can show that the h-step ahead conditional mean and variance are

$$
\begin{aligned}
E(p_{t+h} - p_t | y^t, I_t) &= h\mu_t, && (27) \\
var(p_{t+h} - p_t | y^t, I_t) &= r_t \sum_{j=1}^{h} \exp(j\sigma_r^2/2) + q_t \sum_{j=1}^{h} j^2 \exp(j\sigma_q^2/2), && (28)
\end{aligned}
$$

respectively.