CES Production and Assembly with Two Industries, Two Capital Stocks, and Three Final Uses: Lagrangian Problem and First-Order Conditions

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This note contains the Lagrangian problem and period and steady-state equilibrium conditions for a model with CES production functions for two industries, two capital stocks, CES assembly functions for three final uses, costs of moving capital between industries, and adjustment costs for investment. The Lagrangian problem is

$$\mathcal{L} = \sum_{s=t}^{\infty} \beta^{s-t} \langle V(C_s, C_{s-1}) + \Lambda_{Cs} \left[C(C_{Hs}, C_{Ls}) - C_s \right] + \Lambda_{N_s} \left(N - N_{Hs} - N_{Ls} \right)$$

$$+\Lambda_{K_s^E} \left\{ D_{Hs}^E \left[1 - \tfrac{\omega_H^E}{2} \left(\tfrac{K_{Hs}^E}{D_{Hs}^E} - 1 \right)^2 \right] + D_{Ls}^E \left[1 - \tfrac{\omega_L^E}{2} \left(\tfrac{K_{Ls}^E}{D_{Ls}^E} - 1 \right)^2 \right] - K_{Hs}^E - K_{Ls}^E \right\}$$

$$+\Lambda_{K_s^S} \left\{ D_{Hs}^S \left[1 - \tfrac{\omega_H^S}{2} \left(\tfrac{K_{Hs}^S}{D_{Hs}^S} - 1 \right)^2 \right] + D_{Ls}^S \left[1 - \tfrac{\omega_L^S}{2} \left(\tfrac{K_{Ls}^S}{D_{Ls}^S} - 1 \right)^2 \right] - K_{Hs}^S - K_{Ls}^S \right\}$$

$$+\Lambda_{D_H^E s} \left\{ \left(1 - \delta_H^E \right) K_{Hs-1}^E + Z_H^E J_{Hs-1}^E \left[1 - \tfrac{\nu_H^E}{2} \left(\tfrac{J_{Hs-2}^E}{J_{Hs-1}^E} - 1 \right)^2 \right] - D_{Hs}^E \right\}$$

$$+\Lambda_{D_L^E s} \left\{ \left(1 - \delta_L^E \right) K_{Ls-1}^E + Z_L^E J_{Ls-1}^E \left[1 - \tfrac{\nu_L^E}{2} \left(\tfrac{J_{Ls-2}^E}{J_{Ls-1}^E} - 1 \right)^2 \right] - D_{Ls}^E \right\}$$

$$+\Lambda_{D_H^S s} \left\{ \left(1 - \delta_H^S \right) K_{Hs-1}^S + Z_H^S J_{Hs-1}^S \left[1 - \tfrac{\nu_H^S}{2} \left(\tfrac{J_{Hs-2}^S}{J_{Hs-1}^S} - 1 \right)^2 \right] - D_{Hs}^S \right\}$$

$$+\Lambda_{D_L^S s} \left\{ \left(1 - \delta_L^S \right) K_{Ls-1}^S + Z_L^S J_{Ls-1}^S \left[1 - \tfrac{\nu_L^S}{2} \left(\tfrac{J_{Ls-2}^S}{J_{Ls-1}^S} - 1 \right)^2 \right] - D_{Ls}^S \right\}$$

$$+\Lambda_{Y_H s} \left[Y_H(A_{Hs}, N_{Hs}, K_{Hs}^E, K_{Hs}^S) - C_{Hs} - I_{Hs}^E - I_{Hs}^S \right]$$

$$+\Lambda_{Y_L s} \left[Y_L \left(A_{Ls}, N_{Ls}, K_{Ls}^E, K_{Ls}^S \right) - C_{Ls} - I_{Ls}^E - I_{Ls}^S \right]$$

$$+\Lambda_{Js}^E \left[J^E \left(I_{Hs}^E, I_{Ls}^E \right) - J_{Hs}^E - J_{Ls}^E \right] + \Lambda_{Js}^S \left[J^S \left(I_{Hs}^S, I_{Ls}^S \right) - J_{Hs}^S - J_{Ls}^S \right] \rangle$$

where

$$V(C_s, C_{s-1}) = \frac{\left(\frac{C_s - \eta C_{s-1}}{1-\eta} \right)^{1-\gamma} - 1}{1-\gamma}$$

$$C(C_{Hs}, C_{Ls}) = g_C \left[\phi_H^C \left(\frac{C_{Hs}}{\phi_H^C} \right)^{\frac{\sigma_C - 1}{\sigma_C}} + \phi_L^C \left(\frac{C_{Ls}}{\phi_L^C} \right)^{\frac{\sigma_C - 1}{\sigma_C}} \right]^{\frac{\sigma_C}{\sigma_C - 1}}$$

$$Y_H(\cdot) = g_H \left[\alpha_H^N \left(\frac{A_H N_{Hs}}{\alpha_H^N} \right)^{\frac{\sigma_H - 1}{\sigma_H}} + \alpha_H^E \left(\frac{K_{Hs}^E}{\alpha_H^E} \right)^{\frac{\sigma_H - 1}{\sigma_H}} + \alpha_H^S \left(\frac{K_{Hs}^S}{\alpha_H^S} \right)^{\frac{\sigma_H - 1}{\sigma_H}} \right]^{\frac{\sigma_H}{\sigma_H - 1}}$$

$$Y_L(\cdot) = g_L \left[\alpha_L^N \left(\frac{A_L N_{Ls}}{\alpha_L^N} \right)^{\frac{\sigma_L - 1}{\sigma_L}} + \alpha_L^E \left(\frac{K_{Ls}^E}{\alpha_L^E} \right)^{\frac{\sigma_L - 1}{\sigma_L}} + \alpha_L^S \left(\frac{K_{Ls}^S}{\alpha_L^S} \right)^{\frac{\sigma_L - 1}{\sigma_L}} \right]^{\frac{\sigma_L}{\sigma_L - 1}}$$

$$J^E \left(I_{Hs}^E, I_{Ls}^E \right) = g_E \left[\phi_H^E \left(\frac{I_{Hs}^E}{\phi_H^E} \right)^{\frac{\sigma_J - 1}{\sigma_J}} + \phi_L^E \left(\frac{I_{Ls}^E}{\phi_L^E} \right)^{\frac{\sigma_J - 1}{\sigma_J}} \right]^{\frac{\sigma_J}{\sigma_J - 1}}$$

$$J^S\left(I_{Hs}^S, I_{Ls}^S\right) = g_S\left[\phi_H^S\left(\frac{I_{Hs}^S}{\phi_H^S}\right)^{\frac{\sigma_J-1}{\sigma_J}} + \phi_L^S\left(\frac{I_{Ls}^S}{\phi_L^S}\right)^{\frac{\sigma_J-1}{\sigma_J}}\right]^{\frac{\sigma_J}{\sigma_J-1}}$$

The first-order conditions for this problem can be represented by 37 independent equations in 37 variables. The dynamic forms of the conditions are equations (1) through (33) below. The variables are

$$\Lambda_{Cs}, \Lambda_{Ns}, \Lambda_{Y_H s}, \Lambda_{Y_L s}, \Lambda_{D^E s}, \Lambda_{D^S s}, \Lambda_{K_H^E s}, \Lambda_{K_L^E s}, \Lambda_{K_H^S s}, \Lambda_{K_L^S s}, \Lambda_{J^E s}, \Lambda_{J^S s}$$

$$C_s, C_{Hs}, C_{Ls}, N_{Hs}, N_{Ls}, I_{Hs}^E, I_{Ls}^E, I_{Hs}^S, I_{Ls}^S$$
$$K_{Hs}^E, D_{Hs}^E, J_{Hs}^E, K_{Ls}^E, D_{Ls}^E, J_{Ls}^E, K_{Hs}^S, D_{Hs}^S, J_{Hs}^S, K_{Ls}^S, D_{Ls}^S, J_{Ls}^S$$
$$J_s^E, J_s^S, Y_{Hs}, Y_{Ls}$$

where the variables Y_{Hs}, Y_{Ls}, J_s^E, and J_s^S,

$$Y_{Hs} = C_{Hs} + I_{Hs}^E + I_{Hs}^S$$

$$Y_{Ls} = C_{Ls} + I_{Ls}^E + I_{Ls}^S$$
$$J_s^E = J_{Hs}^E + J_{Ls}^E$$
$$J_s^S = J_{Hs}^S + J_{Ls}^S$$

have been introduced for convenience. The steady-state forms of the conditions are equations (34) through (66). Variables with no time subscripts represent steady state values.

Dynamic Equations

$$\Lambda_{Cs} = \frac{1}{1-\eta}\left(\frac{C_s - \eta C_{s-1}}{1-\eta}\right)^{-\gamma} - \frac{\eta}{1-\eta}\left(\frac{C_{s+1} - \eta C_s}{1-\eta}\right)^{-\gamma} \tag{1}$$

$$C_s = g_C\left[\phi_H^C\left(\frac{C_{Hs}}{\phi_H^C}\right)^{\frac{\sigma_C-1}{\sigma_C}} + \phi_L^C\left(\frac{C_{Ls}}{\phi_L^C}\right)^{\frac{\sigma_C-1}{\sigma_C}}\right]^{\frac{\sigma_C}{\sigma_C-1}} \tag{2}$$

$$\phi_H^C + \phi_L^C = 1$$

$$C_{Hs} = (g_C)^{\sigma_C-1}\,\phi_H^C C_s\left(\frac{\Lambda_{Cs}}{\Lambda_{Y_H s}}\right)^{\sigma_C} \tag{3}$$

$$C_{Ls} = (g_C)^{\sigma_C-1}\,\phi_L^C C_s\left(\frac{\Lambda_{Cs}}{\Lambda_{Y_L s}}\right)^{\sigma_C} \tag{4}$$

$$\bar{N} = N_H + N_L \tag{5}$$

$$N_H = \left(\frac{g_H}{A_H}\right)^{\sigma_H-1}\alpha_H^N Y_H\left(\frac{\Lambda_{Y_H}}{\Lambda_N}\right)^{\sigma_H} \tag{6}$$

$$\alpha_H^N + \alpha_H^E + \alpha_H^S = 1$$

$$N_L = \left(\frac{g_L}{A_L}\right)^{\sigma_L-1}\alpha_L^N Y_L\left(\frac{\Lambda_{Y_L}}{\Lambda_N}\right)^{\sigma_L} \tag{7}$$

$$Y_{Hs} = C_{Hs} + I_{Hs}^E + I_{Hs}^S \tag{8}$$

$$Y_{Ls} = C_{Ls} + I_{Ls}^E + I_{Ls}^S \tag{9}$$

$$Y_{Hs} = g_H\left[\alpha_H^N\left(\frac{A_H N_{Hs}}{\alpha_H^N}\right)^{\frac{\sigma_H-1}{\sigma_H}} + \alpha_H^E\left(\frac{K_{Hs}^E}{\alpha_H^E}\right)^{\frac{\sigma_H-1}{\sigma_H}} + \alpha_H^S\left(\frac{K_{Hs}^S}{\alpha_H^S}\right)^{\frac{\sigma_H-1}{\sigma_H}}\right]^{\frac{\sigma_H}{\sigma_H-1}} \tag{10}$$

$$Y_{Ls} = g_L\left[\alpha_L^N\left(\frac{A_L N_{Ls}}{\alpha_L^N}\right)^{\frac{\sigma_L-1}{\sigma_L}} + \alpha_L^E\left(\frac{K_{Ls}^E}{\alpha_L^E}\right)^{\frac{\sigma_L-1}{\sigma_L}} + \alpha_L^S\left(\frac{K_{Ls}^S}{\alpha_L^S}\right)^{\frac{\sigma_L-1}{\sigma_L}}\right]^{\frac{\sigma_L}{\sigma_L-1}} \tag{11}$$

$$K_{Hs}^E + K_{Ls}^E = D_{Hs}^E\left[1 - \frac{\omega_H^E}{2}\left(\frac{K_{Hs}^E}{D_{Hs}^E} - 1\right)^2\right] + D_{Ls}^E\left[1 - \frac{\omega_L^E}{2}\left(\frac{K_{Ls}^E}{D_{Ls}^E} - 1\right)^2\right] \tag{12}$$

$$K_{Hs}^S + K_{Ls}^S = D_{Hs}^S\left[1 - \frac{\omega_H^S}{2}\left(\frac{K_{Hs}^S}{D_{Hs}^S} - 1\right)^2\right] + D_{Ls}^S\left[1 - \frac{\omega_L^S}{2}\left(\frac{K_{Ls}^S}{D_{Ls}^S} - 1\right)^2\right] \tag{13}$$

$$D_{Hs}^{E} = \left(1-\delta_H^E\right) K_{Hs-1}^{E} + Z_H^E J_{Hs-1}^{E} \left[1-\frac{\nu_H^E}{2}\left(\frac{J_{Hs-2}^{E}}{J_{Hs-1}^{E}}-1\right)^2\right] \tag{14}$$

$$D_{Ls}^{E} = \left(1-\delta_L^E\right) K_{Ls-1}^{E} + Z_L^E J_{Ls-1}^{E} \left[1-\frac{\nu_L^E}{2}\left(\frac{J_{Ls-2}^{E}}{J_{Ls-1}^{E}}-1\right)^2\right] \tag{15}$$

$$D_{Hs}^{S} = \left(1-\delta_H^S\right) K_{Hs-1}^{S} + Z_H^S J_{Hs-1}^{S} \left[1-\frac{\nu_H^S}{2}\left(\frac{J_{Hs-2}^{S}}{J_{Hs-1}^{S}}-1\right)^2\right] \tag{16}$$

$$D_{Ls}^{S} = \left(1-\delta_L^S\right) K_{Ls-1}^{S} + Z_L^S J_{Ls-1}^{S} \left[1-\frac{\nu_L^S}{2}\left(\frac{J_{Ls-2}^{S}}{J_{Ls-1}^{S}}-1\right)^2\right] \tag{17}$$

$$\Lambda_{K_s^E}\left[1+\omega_H^E\left(\frac{K_{Hs}^E}{D_{Hs}^E}-1\right)\right] = \beta\Lambda_{D_H^E s+1}\left(1-\delta_H^E\right) + \Lambda_{Y_H s}\left(g_H\right)^{\frac{\sigma_H-1}{\sigma_H}}\left(\frac{\alpha_H^E Y_{Hs}}{K_{Hs}^E}\right)^{\frac{1}{\sigma_H}} \tag{18}$$

$$\Lambda_{K_s^S}\left[1+\omega_H^S\left(\frac{K_{Hs}^S}{D_{Hs}^S}-1\right)\right] = \beta\Lambda_{D_H^S s+1}\left(1-\delta_H^S\right) + \Lambda_{Y_H s}\left(g_H\right)^{\frac{\sigma_H-1}{\sigma_H}}\left(\frac{\alpha_H^S Y_{Hs}}{K_{Hs}^S}\right)^{\frac{1}{\sigma_H}} \tag{19}$$

$$\Lambda_{K_s^E}\left[1+\omega_L^E\left(\frac{K_{Ls}^E}{D_{Ls}^E}-1\right)\right] = \beta\Lambda_{D_L^E s+1}\left(1-\delta_L^E\right) + \Lambda_{Y_L s}\left(g_L\right)^{\frac{\sigma_L-1}{\sigma_L}}\left(\frac{\alpha_L^E Y_{Ls}}{K_{Ls}^E}\right)^{\frac{1}{\sigma_L}} \tag{20}$$

$$\Lambda_{K_s^S}\left[1+\omega_L^S\left(\frac{K_{Ls}^S}{D_{Ls}^S}-1\right)\right] = \beta\Lambda_{D_L^S s+1}\left(1-\delta_L^S\right) + \Lambda_{Y_L s}\left(g_L\right)^{\frac{\sigma_L-1}{\sigma_L}}\left(\frac{\alpha_L^S Y_{Ls}}{K_{Ls}^S}\right)^{\frac{1}{\sigma_L}} \tag{21}$$

$$\Lambda_{K^E s}\left\{1+\frac{\omega_H^E}{2}\left[\left(\frac{K_{Hs}^E}{D_{Hs}^E}\right)^2-1\right]\right\} = \Lambda_{D_H^E s} \tag{22}$$

$$\Lambda_{K^S s}\left\{1+\frac{\omega_H^S}{2}\left[\left(\frac{K_{Hs}^S}{D_{Hs}^S}\right)^2-1\right]\right\} = \Lambda_{D_H^S s} \tag{23}$$

$$\Lambda_{K^E s}\left\{1+\frac{\omega_L^E}{2}\left[\left(\frac{K_{Ls}^E}{D_{Ls}^E}\right)^2-1\right]\right\} = \Lambda_{D_L^E s} \tag{24}$$

$$\Lambda_{K^S s}\left\{1+\frac{\omega_L^S}{2}\left[\left(\frac{K_{Ls}^S}{D_{Ls}^S}\right)^2-1\right]\right\} = \Lambda_{D_L^S s} \tag{25}$$

$$J_s^E = J_{Hs}^E + J_{Ls}^E \tag{26}$$

$$J_s^E = g_E\left[\phi_H^E\left(\frac{I_{Hs}^E}{\phi_H^E}\right)^{\frac{\sigma_E-1}{\sigma_E}} + \phi_L^E\left(\frac{I_{Ls}^E}{\phi_L^E}\right)^{\frac{\sigma_E-1}{\sigma_E}}\right]^{\frac{\sigma_E}{\sigma_E-1}} \tag{27}$$

$$\phi_H^E + \phi_L^E = 1$$

$$I_{Hs}^E = \left(g_E\right)^{\sigma_E-1}\phi_H^E J_s^E\left(\frac{\Lambda_{J^E s}}{\Lambda_{Y_H s}}\right)^{\sigma_E} \tag{28}$$

$$I_{Ls}^E = \left(g_E\right)^{\sigma_E-1}\phi_L^E J_s^E\left(\frac{\Lambda_{J^E s}}{\Lambda_{Y_L s}}\right)^{\sigma_E} \tag{29}$$

$$J_s^S = J_{Hs}^S + J_{Ls}^S \tag{30}$$

$$J_s^S = g_S \left[\phi_H^S \left(\frac{I_{Hs}^S}{\phi_H^S} \right)^{\frac{\sigma_S - 1}{\sigma_S}} + \phi_L^S \left(\frac{I_{Ls}^S}{\phi_L^S} \right)^{\frac{\sigma_S - 1}{\sigma_S}} \right]^{\frac{\sigma_S}{\sigma_S}} \tag{31}$$

$$\phi_H^S + \phi_L^S = 1$$

$$I_{Hs}^S = (g_S)^{\sigma_E - 1} \phi_H^S J_s^S \left(\frac{\Lambda_{J^S s}}{\Lambda_{Y_H s}} \right)^{\sigma_S} \tag{32}$$

$$I_{Ls}^S = (g_S)^{\sigma_E - 1} \phi_L^S J_s^S \left(\frac{\Lambda_{J^S s}}{\Lambda_{Y_L s}} \right)^{\sigma_S} \tag{33}$$

$$\Lambda_{J^E s} = \Lambda_{D_H^E s+1} \beta Z_H^E \left\{ 1 + \frac{\nu_H^E}{2} \left[\left(\frac{J_{Hs-1}^E}{J_{Hs}^E} \right)^2 - 1 \right] \right\} - \Lambda_{D_H^E s+2} \beta^2 Z_H^E \nu_H^E \left(\frac{J_{Hs}^E}{J_{Hs+1}^E} - 1 \right) \tag{34}$$

$$\Lambda_{J^E s} = \Lambda_{D_L^E s+1} \beta Z_L^E \left\{ 1 + \frac{\nu_L^E}{2} \left[\left(\frac{J_{Ls-1}^E}{J_{Ls}^E} \right)^2 - 1 \right] \right\} - \Lambda_{D_L^E s+2} \beta^2 Z_L^E \nu_L^E \left(\frac{J_{Ls}^E}{J_{Ls+1}^E} - 1 \right) \tag{35}$$

$$\Lambda_{J^S s} = \Lambda_{D_H^S s+1} \beta Z_H^S \left\{ 1 + \frac{\nu_H^S}{2} \left[\left(\frac{J_{Hs-1}^S}{J_{Hs}^S} \right)^2 - 1 \right] \right\} - \Lambda_{D_H^S s+2} \beta^2 Z_H^S \nu_H^S \left(\frac{J_{Hs}^S}{J_{Hs+1}^S} - 1 \right) \tag{36}$$

$$\Lambda_{J^S s} = \Lambda_{D_L^S s+1} \beta Z_L^S \left\{ 1 + \frac{\nu_L^S}{2} \left[\left(\frac{J_{Ls-1}^S}{J_{Ls}^S} \right)^2 - 1 \right] \right\} - \Lambda_{D_L^S s+2} \beta^2 Z_L^S \nu_L^S \left(\frac{J_{Ls}^S}{J_{Ls+1}^S} - 1 \right) \tag{37}$$

Limiting Cases

First consider the case in which ω_H and ω_L approach zero. Use equations (**??**) and (**??**) forwarded one period to eliminate $\Lambda_{K_H s+1}$.and in equations (**??**) and (**??**). The limits of the resulting equations as ω_H and ω_L approach zero are

$$\Lambda_{Ks} = \Lambda_{Ds+1}\beta\left(1-\delta_H\right) + \Lambda_{Y_H s}\left(g_H\right)^{\frac{\sigma_H - 1}{\sigma_H}}\left(\frac{\alpha_H Y_{Hs}}{K_{Hs}}\right)^{\frac{1}{\sigma_H}} \tag{38}$$

$$\Lambda_{Ks} = \Lambda_{Ds+1}\beta\left(1-\delta_L\right) + \Lambda_{Y_L s}\left(g_L\right)^{\frac{\sigma_L - 1}{\sigma_L}}\left(\frac{\alpha_L Y_{Ls}}{K_{Ls}}\right)^{\frac{1}{\sigma_L}} \tag{39}$$

It follows from equations (38) and (39) that in the limit as ω_H and ω_L approach zero, the total returns to capital in the two industries are equated:

$$\begin{aligned}\Lambda_{Ds+1}\beta\left(1-\delta_H\right) + \Lambda_{Y_H s}\left(\frac{\alpha_H Y_{Hs}}{K_{Hs}}\right)^{\frac{1}{\sigma_H}} \\ = \Lambda_{Ds+1}\beta\left(1-\delta_L\right) + \Lambda_{Y_L s}\left(\frac{\alpha_L Y_{Ls}}{K_{Ls}}\right)^{\frac{1}{\sigma_L}}\end{aligned} \tag{40}$$

In the special case in which depreciation rates are equal ($\delta_H = \delta_L = \delta$), the marginal revenue products of capital in the two industries are equated:

$$\Lambda_{Y_H s}\left(g_H\right)^{\frac{\sigma_H - 1}{\sigma_H}}\left(\frac{\alpha_H Y_{Hs}}{K_{Hs}}\right)^{\frac{1}{\sigma_H}} = \Lambda_{Y_L s}\left(\frac{\alpha_L Y_{Ls}}{K_{Ls}}\right)^{\frac{1}{\sigma_L}} \tag{41}$$

The capital accumulation equation becomes

$$\begin{aligned}K^E_{Hs} + K^E_{ls} = \left(1-\delta^E_H\right)K^E_{Hs-1} + Z^E_H J^E_{Hs-1}\left[1 - \frac{\nu^E_H}{2}\left(\frac{J^E_{Hs-2}}{J^E_{Hs-1}} - 1\right)^2\right] \\ + \left(1-\delta^E_L\right)K^E_{Ls-1} + Z^E_L J^E_{Ls-1}\left[1 - \frac{\nu^E_L}{2}\left(\frac{J^E_{Ls-2}}{J^E_{Ls-1}} - 1\right)^2\right]\end{aligned} \tag{42}$$

Now consider the case in which ω_H and ω_L approach infinity. It follows from equation (12) that capital is not shifted between industries; that is

$$K_{Hs} = D_{Hs,} \qquad K_{Ls} = D_{Ls} \tag{43}$$

Now use equations (**??**) and (**??**) to eliminate Λ_{Ds} in equations (**??**) and (**??**). The limits of the resulting equations as ω_H and ω_L approach infinity are

$$\Lambda_{K_H s} = \Lambda_{K_H s+1}\beta\left(1-\delta_H\right) + \Lambda_{Y_{Hs}}\left(\frac{\alpha_H Y_{Hs}}{K_{Hs}}\right)^{\frac{1}{\sigma_H}} \tag{44}$$

$$\Lambda_{K_L s} = \Lambda_{K_L s+1}\beta\left(1-\delta_L\right) + \Lambda_{Y_{Ls}}\left(\frac{\alpha_L Y_{Ls}}{K_{Ls}}\right)^{\frac{1}{\sigma_L}} \tag{45}$$

The capital accumulation equations are

$$K^E_{Hs} = \left(1-\delta^E_H\right)K^E_{Hs-1} + Z^E_H J^E_{Hs-1}\left[1 - \frac{\nu^E_H}{2}\left(\frac{J^E_{Hs-2}}{J^E_{Hs-1}} - 1\right)^2\right] \tag{46}$$

$$K^E_{ls} = \left(1-\delta^E_L\right)K^E_{Ls-1} + Z^E_L J^E_{Ls-1}\left[1 - \frac{\nu^E_L}{2}\left(\frac{J^E_{Ls-2}}{J^E_{Ls-1}} - 1\right)^2\right] \tag{47}$$

Steady State Equations

$$\Lambda_C = (C)^{-\gamma} \tag{48}$$

$$C = g_C \left[\phi_H^C \left(\frac{C_H}{h_C} \right)^{\frac{\sigma_C - 1}{\sigma_C}} + \phi_L^C \left(\frac{C_L}{\phi_L^C} \right)^{\frac{\sigma_C - 1}{\sigma_C}} \right]^{\frac{\sigma_C}{\sigma_C - 1}} \tag{49}$$

$$\phi_H^C + \phi_L^C = 1 \tag{50}$$

$$C_H = (g_C)^{\sigma_C - 1} \phi_H^C C \left(\frac{\Lambda_C}{\Lambda_{Y_H}} \right)^{\sigma_C} \tag{51}$$

$$C_L = (g_C)^{\sigma_C - 1} \phi_L^C C \left(\frac{\Lambda_C}{\Lambda_{Y_L}} \right)^{\sigma_C} \tag{52}$$

$$N = N_H + N_L \tag{53}$$

$$N_H = \left(\frac{g_H}{A_H} \right)^{\sigma_H - 1} \alpha_H^N Y_H \left(\frac{\Lambda_{Y_H}}{\Lambda_N} \right)^{\sigma_H} \tag{54}$$

$$N_L = \left(\frac{gL}{A_L} \right)^{\sigma_L - 1} \alpha_L^N Y_L \left(\frac{\Lambda_{Y_L}}{\Lambda_N} \right)^{\sigma_L} \tag{55}$$

$$Y_H = C_H + I_H^E + I_H^S \tag{56}$$

$$Y_L = C_L + I_L^E + I_L^S \tag{57}$$

$$Y_H = g_H \left[\alpha_H^N \left(\frac{A_H N_H^E}{\alpha_H^N} \right)^{\frac{\sigma_H - 1}{\sigma_H}} + \alpha_H^E \left(\frac{K_H^E}{\alpha_H^E} \right)^{\frac{\sigma_H - 1}{\sigma_H}} + \alpha_H^S \left(\frac{K_H^S}{\alpha_H^S} \right)^{\frac{\sigma_H - 1}{\sigma_H}} \right]^{\frac{\sigma_H}{\sigma_H - 1}} \tag{58}$$

$$Y_L = g_L \left[\alpha_L^N \left(\frac{A_L N_L^E}{\alpha_L^N} \right)^{\frac{\sigma_L - 1}{\sigma_L}} + \alpha_L^E \left(\frac{K_L^E}{\alpha_L^E} \right)^{\frac{\sigma_L - 1}{\sigma_L}} + \alpha_L^S \left(\frac{K_L^S}{\alpha_L^S} \right)^{\frac{\sigma_L - 1}{\sigma_L}} \right]^{\frac{\sigma_L}{\sigma_L - 1}} \tag{59}$$

$$D_H^E = K_H^E \tag{60}$$

$$D_L^E = K_L^E \tag{61}$$

$$D_H^S = K_H^S \tag{62}$$

$$D_L^S = K_L^S \tag{63}$$

$$Z_H^E J_H^E = \delta_H^E K_H^E \tag{64}$$

$$Z_L^E J_L^E = \delta_L^E K_L^E \tag{65}$$

$$Z_H^S J_H^S = \delta_H^S K_H^S \tag{66}$$

$$Z_L^S J_L^S = \delta_L^S K_L^S \tag{67}$$

$$\left[1-\beta\left(1-\delta_H^E\right)\right]\Lambda_{D_H^E}=\Lambda_{Y_H}\left(g_H\right)^{\frac{\sigma_H-1}{\sigma_H}}\left(\frac{\alpha_H^E Y_H}{K_H^E}\right)^{\frac{1}{\sigma_H}} \tag{68}$$

$$\left[1-\beta\left(1-\delta_H^S\right)\right]\Lambda_{D_H^S}=\Lambda_{Y_H}\left(g_H\right)^{\frac{\sigma_H-1}{\sigma_H}}\left(g_H\right)^{\frac{\sigma_H-1}{\sigma_H}}\left(\frac{\alpha_H^S Y_H}{K_H^S}\right)^{\frac{1}{\sigma_H}} \tag{69}$$

$$\left[1-\beta\left(1-\delta_L^E\right)\right]\Lambda_{D_L^E}=\Lambda_{Y_L}\left(g_L\right)^{\frac{\sigma_L-1}{\sigma_L}}\left(\frac{\alpha_L^E Y_L}{K_L^E}\right)^{\frac{1}{\sigma_L}} \tag{70}$$

$$\left[1-\beta\left(1-\delta_L^S\right)\right]\Lambda_{D_L^S}=\Lambda_{Y_L}\left(g_L\right)^{\frac{\sigma_L-1}{\sigma_L}}\left(\frac{\alpha_L^S Y_L}{K_L^S}\right)^{\frac{1}{\sigma_L}} \tag{71}$$

$$\Lambda_{D_H^E}=\Lambda_{K^E} \tag{72}$$

$$\Lambda_{D_L^E}=\Lambda_{K^E} \tag{73}$$

$$\Lambda_{D_H^S}=\Lambda_{K^S} \tag{74}$$

$$\Lambda_{D_L^S}=\Lambda_{K^S} \tag{75}$$

$$J^E=J_H^E+J_L^E \tag{76}$$

$$J^E=g_E\left[\phi_H^E\left(\frac{I_H^E}{\phi_J^E}\right)^{\frac{\sigma_E-1}{\sigma_E}}+\phi_L^E\left(\frac{I_L^E}{\phi_L^E}\right)^{\frac{\sigma_E-1}{\sigma_E}}\right]^{\frac{\sigma_E}{\sigma_E-1}} \tag{77}$$

$$I_H^E=\left(g_E\right)^{\sigma_E-1}\phi_H^E J^E\left(\frac{\Lambda_{J^E}}{\Lambda_{Y_H}}\right)^{\sigma_E} \tag{78}$$

$$I_L^E=\left(g_E\right)^{\sigma_E-1}\phi_L^E J^E\left(\frac{\Lambda_{J^E}}{\Lambda_{Y_L}}\right)^{\sigma_E} \tag{79}$$

$$\phi_H^E+\phi_L^E=1$$

$$J^S=J_H^S+J_L^S \tag{80}$$

$$J^S=g_S\left[\phi_H^S\left(\frac{I_H^S}{\phi_J^S}\right)^{\frac{\sigma_S-1}{\sigma_S}}+\phi_L^S\left(\frac{I_L^S}{\phi_L^S}\right)^{\frac{\sigma_S-1}{\sigma_S}}\right]^{\frac{\sigma_S}{\sigma_S-1}} \tag{81}$$

$$I_H^S=\left(g_S\right)^{\sigma_S-1}\phi_H^S J^S\left(\frac{\Lambda_{J^S}}{\Lambda_{Y_H}}\right)^{\sigma_S} \tag{82}$$

$$I_L^S=\left(g_S\right)^{\sigma_S-1}\phi_L^S J^S\left(\frac{\Lambda_{J^S}}{\Lambda_{Y_L}}\right)^{\sigma_S} \tag{83}$$

$$\phi_H^S+\phi_L^S=1$$

$$\Lambda_{J^E} = \beta \Lambda_{D_H^E} Z_H^E \tag{84}$$
$$\Lambda_{J^E} = \beta \Lambda_{D_L^E} Z_L^E$$
$$\Lambda_{J^S} = \beta \Lambda_{D_H^S} Z_H^S \tag{85}$$
$$\Lambda_{J^S} = \beta \Lambda_{D_L^S} Z_L^S$$

The two unnumbered equations are redundant since they imply the conditions

$$\Lambda_{D_H^E} = \Lambda_{D_L^E} \tag{86}$$

$$\Lambda_{D_H^S} = \Lambda_{D_L^S} \tag{87}$$

which are also implied by equations (72), (73), (74), and (75).

Steady State Solution General Version

Strategy is to impose some convenient restrictions and find any further restrictions required for the convenient restrictions to be consistent with steady state equilibrium.

To simplify finding a steady-state solution,we assume that all productivity factors and the consumption constant are equal to one,

$$g_H = g_L = g_C = A_H = A_L = Z_H^E = Z_L^E = Z_H^S = Z_L^S = 1, \tag{88}$$

and that the multipliers satisfy

$$\Lambda_C = \Lambda_N = \Lambda_{Y_H} = \Lambda_{Y_L} = \Lambda_{J^E} = \Lambda_{J^S} \tag{89}$$

Under these assumptions, equations (51) (52), (54), (55), (78), (79), (82) (83), (72), (73), (74), (75), (84), and (85) imply that

$$\alpha_H^N = \tfrac{N_H}{Y_H}, \quad \alpha_L^N = \tfrac{N_L}{Y_L} \tag{90}$$

$$\phi_H^C = \tfrac{C_H}{C}, \quad \phi_L^C = \tfrac{C_L}{C} \tag{91}$$

$$(g_E)^{\sigma_E - 1}\,\phi_H^E = \tfrac{I_H^E}{J^E}, \quad (g_E)^{\sigma_E - 1}\,\phi_L^E = \tfrac{I_L^E}{J^E} \tag{92}$$

$$(g_S)^{\sigma_s - 1}\,\phi_H^S = \tfrac{I_H^S}{J^S}, \quad (g_S)^{\sigma_s - 1}\,\phi_L^S = \tfrac{I_L^S}{J^S} \tag{93}$$

$$\tfrac{\Lambda_C}{\beta} = \Lambda_{D_H^E} = \Lambda_{D_H^S} = \Lambda_{D_L^E} = \Lambda_{D_L^S} = \Lambda_{K^E} = \Lambda_{K^S} \tag{94}$$

By definition, in a steady state

$$D_H^E = K_H^E, \quad D_L^E = K_L^E, \quad D_H^S = K_H^S, \quad D_L^S = K_L^S \tag{95}$$

Equations (68), (70), (69), and (71) imply that

$$\frac{K_H^E}{Y_H} = \left[\frac{\beta}{1-\beta\left(1-\delta_H^E\right)}\right]^{\sigma_H} \alpha_H^E, \quad \frac{K_L^E}{Y_L} = \left[\frac{\beta}{1-\beta\left(1-\delta_L^E\right)}\right]^{\sigma_L} \alpha_L^E \tag{96}$$

$$\frac{K_H^S}{Y_H} = \left[\frac{\beta}{1-\beta\left(1-\delta_H^S\right)}\right]^{\sigma_H} \alpha_H^S, \quad \frac{K_L^S}{Y_L} = \left[\frac{\beta}{1-\beta\left(1-\delta_L^S\right)}\right]^{\sigma_L} \alpha_L^S \tag{97}$$

Therefore, equations (58) and (59) imply

$$1 = \alpha_H^N + \alpha_H^E \left[\frac{\beta}{1-\beta\left(1-\delta_H^E\right)}\right]^{\sigma_H - 1} + \alpha_H^S \left[\frac{\beta}{1-\beta\left(1-\delta_L^E\right)}\right]^{\sigma_H - 1} \tag{98}$$

$$1 = \alpha_L^N + \alpha_L^E \left[\frac{\beta}{1-\beta\left(1-\delta_H^S\right)}\right]^{\sigma_L - 1} + \alpha_L^S \left[\frac{\beta}{1-\beta\left(1-\delta_L^S\right)}\right]^{\sigma_L - 1} \tag{99}$$

α_H^N and α_L^N are given by implications (90) and data. We are free to choose one of α_H^E and α_H^S. The other is implied by equation (98). Similarly, we are free to choose one of α_L^E and α_L^S. The other is implied by equation (99). For example, given α_H^E and α_L^E, α_H^S and α_L^S are given by

$$\alpha_H^S = \frac{1 - \alpha_H^N - \alpha_H^E \left[\frac{\beta}{1-\beta\left(1-\delta_H^E\right)}\right]^{\sigma_H - 1}}{\left[\frac{\beta}{1-\beta\left(1-\delta_H^S\right)}\right]^{\sigma_H - 1}}, \quad \alpha_L^S = \frac{1 - \alpha_L^N - \alpha_L^E \left[\frac{\beta}{1-\beta\left(1-\delta_H^S\right)}\right]^{\sigma_L - 1}}{\left[\frac{\beta}{1-\beta\left(1-\delta_L^S\right)}\right]^{\sigma_L - 1}} \tag{100}$$

Equations (64), (65), (66), and (67) imply

$$\frac{J^E}{K_H^E} = \delta_H^E, \quad \frac{J^E}{K_L^E} = \delta_L^E \tag{101}$$

$$\frac{J^S}{K_H^S} = \delta_H^S, \quad \frac{J^S}{K_L^S} = \delta_L^S \tag{102}$$

Define s_H^E, s_H^S, s_L^E, and s_L^S as

$$s_H^E = \frac{J_H^E}{Y_H} = \frac{J_H^E}{K_H^E}\frac{K_H^E}{Y_H} = \delta_H^E \frac{K_H^E}{Y_H} \tag{103}$$

$$s_H^S = \frac{J_H^S}{Y_H} = \frac{J_H^S}{K_H^S}\frac{K_H^S}{Y_H} = \delta_H^S \frac{K_H^S}{Y_H} \tag{104}$$

$$s_L^E = \frac{J_L^E}{Y_L} = \frac{J_L^E}{K_L^E}\frac{K_L^E}{Y_L} = \delta_L^E \frac{K_L^E}{Y_L} \tag{105}$$

$$s_L^S = \frac{J_L^S}{Y_L} = \frac{J_L^S}{K_L^S}\frac{K_L^S}{Y_L} = \delta_L^S \frac{K_L^S}{Y_L} \tag{106}$$

Define s^E and s^S and make us of (103), (104), (105), and (106) to obtain

$$s^E = \frac{J^E}{Y} = \frac{Y_H}{Y}\frac{J_H^E}{Y_H} + \frac{Y_L}{Y}\frac{J_L^E}{Y_L} = \frac{Y_H}{Y}s_H^E + \frac{Y_L}{Y}s_L^E \tag{107}$$

$$s^S = \frac{J^S}{Y} = \frac{Y_H}{Y}\frac{J_H^S}{Y_H} + \frac{Y_L}{Y}\frac{J_L^S}{Y_L} = \frac{Y_H}{Y}s_H^S + \frac{Y_L}{Y}s_L^S \tag{108}$$

Define the aggregate saving rate s as

$$s = \frac{Y - C}{Y} = \frac{I_H^E + I_L^E + I_H^S + I_L^S}{Y} = (g_E)^{\sigma_E - 1} s^E + (g_S)^{\sigma_S - 1} s^S \tag{109}$$

The definitions above and the resource contraints

$$Y_H = C_H + I_H^E + I_H^S \tag{110}$$

$$Y_L = C_L + I_L^E + I_L^S \tag{111}$$

imply

$$
\begin{aligned}
Y_H &= \phi_H^C (1-s) Y + \phi_H^E (g_E)^{\sigma_E - 1} s^E Y + \phi_H^S (g_S)^{\sigma_{JS} - 1} s^S Y \\
\rightarrow \tfrac{Y_H}{Y} &= \phi_H^C (1-s) + \phi_H^E (g_E)^{\sigma_E - 1} s^E + \phi_H^S (g_S)^{\sigma_S - 1} s^S
\end{aligned} \tag{112}
$$

$$
\begin{aligned}
Y_L &= \phi_L^C (1-s) Y + \phi_L^E (g_E)^{\sigma_E - 1} s^E Y + \phi_L^S (g_S)^{\sigma_S - 1} s^S Y \\
\rightarrow \tfrac{Y_L}{Y} &= \phi_L^C (1-s) + \phi_L^E (g_E)^{\sigma_E - 1} s^E + \phi_L^S (g_S)^{\sigma_S - 1} s^S
\end{aligned} \tag{113}
$$

Equations (107), (108), (109), (112), and (113) can be solved for s^E, s^S, s, $\frac{Y_H}{Y}$ and, $\frac{Y_L}{Y}$. Substituting (112) and (113) into (107) and (108) and making use of (109) yields

$$
\begin{aligned}
s^E &= s_H^E \left\{ \phi_H^C - (g_E)^{\sigma_E - 1} \left(\phi_H^C - \phi_H^E\right) s^E - (g_S)^{\sigma_S - 1} \left(\phi_H^C - \phi_H^S\right) s^S \right\} \\
&\quad + s_L^E \left\{ \phi_L^C - (g_E)^{\sigma_e - 1} \left(\phi_L^C - \phi_L^E\right) s^E - (g_S)^{\sigma_S - 1} \left(\phi_L^C - \phi_L^S\right) s^S \right\}
\end{aligned} \tag{114}
$$

$$
\begin{aligned}
s^E &= s_H^E \left\{ \phi_H^C - (g_E)^{\sigma_E - 1} \left(\phi_H^C - \phi_H^E\right) s^E - (g_S)^{\sigma_S - 1} \left(\phi_H^C - \phi_H^S\right) s^S \right\} \\
&\quad + s_L^E \left\{ \phi_L^C - (g_E)^{\sigma_e - 1} \left(\phi_L^C - \phi_L^E\right) s^E - (g_S)^{\sigma_S - 1} \left(\phi_L^C - \phi_L^S\right) s^S \right\}
\end{aligned} \tag{115}
$$

rearranging yields

$$
\left[1 + (g_E)^{\sigma_E - 1} a_{11})\right] s^E + (g_S)^{\sigma_S - 1} a_{12} s^S = \phi_H^C s_H^E + \phi_L^C s_L^E \tag{116}
$$

$$
(g_E)^{\sigma_E - 1} a_{21} s^E + \left[1 + (g_S)^{\sigma_S - 1} a_{22}\right] s^S = \phi_H^C s_H^S + \phi_L^C s_L^S \tag{117}
$$

where

$$
\begin{aligned}
a_{11} &= s_H^E(\phi_H^C - \phi_H^E) + s_L^E(\phi_L^C - \phi_L^E),\ a_{12} = s_H^E(\phi_H^C - \phi_H^S) + s_L^E(\phi_L^C - \phi_L^S) \\
a_{21} &= s_H^S(\phi_H^C - \phi_H^E) + s_L^S(\phi_L^C - \phi_L^E),\ a_{22} = s_H^S(\phi_H^C - \phi_H^S) + s_L^S(\phi_L^C - \phi_L^S)
\end{aligned} \tag{118}
$$

so

$$
s^E = \frac{\left(\phi_H^C s_H^E + \phi_L^C s_L^E\right)\left(1 + (g_S)^{\sigma_S - 1} a_{22}\right) - \left(\phi_H^C s_H^S + \phi_L^C s_L^S\right)(g_S)^{\sigma_S - 1} a_{12}}{det} \tag{119}
$$

$$
s^S = \frac{\left(1 + (g_E)^{\sigma_E - 1} a_{11}\right)\left(\phi_H^C s_H^S + \phi_L^C s_L^S\right) - \left(\phi_H^C s_H^E + \phi_L^C s_L^E\right)(g_E)^{\sigma_E - 1} a_{21}}{det} \tag{120}
$$

where

$$
\begin{aligned}
det &= \left[1 + (g_E)^{\sigma_E - 1} a_{11})\right]\left[1 + (g_S)^{\sigma_S - 1} a_{22}\right] - (g_E)^{\sigma_E - 1} a_{12} (g_S)^{\sigma_S - 1} a_{21} \\
&= 1 + (g_E)^{\sigma_E - 1} a_{11} + (g_S)^{\sigma_S - 1} a_{22}
\end{aligned} \tag{121}
$$

Now

$$
\frac{N}{Y} = \frac{Y_H}{Y}\frac{N_H}{Y_H} + \frac{Y_L}{Y}\frac{N_L}{Y_L} \tag{122}
$$

Choose a value for N and use expressions for $\frac{Y_H}{Y}, \frac{N_H}{Y_H}, \frac{Y_L}{Y}, \frac{N_L}{Y_L}$ to obtain

$$Y = \frac{N}{denom} \tag{123}$$

where

$$\begin{aligned} denom &= \alpha_H^N \left[\phi_H^C (1-s) + \phi_H^E (g_E)^{\sigma_E - 1} s^E + \phi_H^S (g_S)^{\sigma_S - 1} s^S\right] \\ &+ \alpha_L^N \left[\phi_L^C (1-s) + \phi_L^E (g_E)^{\sigma_E - 1} s^E + \phi_L^S (g_S)^{\sigma_S - 1} s^S\right] \end{aligned} \tag{124}$$

Now all other levels can be determined as a function of N.

There is a pool of capital that the industries can contribute to or draw on and the costs of an industry contributing to the pool or drawing on it depend on the capital stock it has inherited. We seek a formulation for transfer costs that has several desirable properties

- Transfer costs are zero when there are no transfers
- There are tranfer costs both for taking capital from one stock and for adding to the other
- Marginal transfer costs are zero in the steady state in which there are no transfers but they are negative (positive) when $K \lesseqgtr D$
- Marginal costs are increasing at the steady state

At least two candidates meet these criteria. Time subscripts are suppressed for simplicity.

$$C = \frac{\omega}{2} D \left(\frac{K}{D} - 1 \right)^2 \tag{125}$$

$$\frac{\partial C}{\partial K} = \omega \left(\frac{K}{D} - 1 \right) \tag{126}$$

$$\frac{\partial^2 C}{\partial K^2} = \frac{\omega}{D} \tag{127}$$

$$\frac{\partial C}{\partial D} = \frac{\omega}{2} \left(\frac{K}{D} - 1 \right)^2 - \omega \left(\frac{K}{D} - 1 \right) \frac{K}{D} = \frac{\omega}{2} \left[1 - \left(\frac{K}{D} \right)^2 \right] \tag{128}$$

$$\frac{\partial^2 C}{\partial D^2} = \left(\frac{\omega}{D} \right) \left(\frac{K}{D} \right)^2 \tag{129}$$

$$C = \frac{\omega}{2} K \left(1 - \frac{D}{K} \right)^2 \tag{130}$$

$$\frac{\partial C}{\partial K} = \frac{\omega}{2} \left(1 - \frac{D}{K} \right)^2 + \omega \left(1 - \frac{D}{K} \right) \frac{D}{K} = \frac{\omega}{2} \left[1 - \left(\frac{D}{K} \right)^2 \right] \tag{131}$$

$$\frac{\partial C}{\partial D} = \omega \left(\frac{D}{K} - 1 \right) \tag{132}$$

Likewise, we seek a formulation for investment adjustment costs that has several desirable properties

- Adjustment costs are zero when there is no change in investment
- There are adjusment costs both for both increasing and decreasing investment
- Marginal adjusment costs are zero in the steady state in which there is no change in investment but they are negative (positive) when $J \lesseqgtr J_{-1}$.

- Marginal costs are increasing at the steady state

The conventional formulation meets these criteria

$$C = \frac{\nu}{2} J \left(\frac{J}{J_{-1}} - 1 \right)^2 \tag{133}$$

Note that C is total cost of investment and that $\frac{\nu}{2}\left(\frac{J}{J_{-1}} - 1\right)^2$ is cost per unit of investment. Thus marginal cost of a unit of investment is given by average cost plus the increase in average costs times the number of units.

$$\frac{\partial C}{\partial J} = \frac{\nu}{2} \left(\frac{J}{J_{-1}} - 1 \right)^2 + \nu \frac{J}{J_{-1}} \left(\frac{J}{J_{-1}} - 1 \right) \tag{134}$$

$$\frac{\partial^2 C}{\partial J^2} = \frac{\nu}{J_{-1}} \left(\frac{J}{J_{-1}} - 1 \right) + \frac{\nu}{J_{-1}} \left(\frac{J}{J_{-1}} - 1 \right) + \frac{\nu}{J_{-1}} \frac{J}{J_{-1}} \tag{135}$$

$$\frac{\partial C}{\partial J_{-1}} = -\nu \left(\frac{J}{J_{-1}} - 1 \right) \left(\frac{J}{J_{-1}} \right)^2 \tag{136}$$

$$\frac{\partial^2 C}{\partial J_{-1}^2} = \nu \left[\left(\frac{J}{J_{-1}} \right)^2 + 2 \left(\frac{J}{J_{-1}} - 1 \right) \left(\frac{J}{J_{-1}} \right) \right] \frac{J}{(J_{-1})^2} \tag{137}$$

Note that the functional form for adjustment costs is determined by what it is desired to have the expressions

$$H_{Is} \left[1 - \frac{\omega_I}{2} \left(\frac{K_{Is}}{H_{Is}} - 1 \right)^2 \right] \tag{138}$$

$$J_{Is} \left[1 - \frac{\nu_I}{2} \left(\frac{J_{Is}}{J_{Is-1}} - 1 \right)^2 \right] \tag{139}$$

reduce to when adjustment costs approach zero. For this reason we choose

$$C = \frac{\omega}{2} H \left(\frac{K}{H} - 1 \right)^2 \tag{140}$$

for capital transfer costs and the conventional form

$$C = \frac{\nu}{2} J \left(\frac{J_{-1}}{J} - 1 \right)^2 \tag{141}$$

for investment adjustment costs

Consider second-order approximations to two linear homogenous functions. Note that if the functions are approximated about $\bar{Z} = 1$ and $\bar{X} = 1$ respectively and if $\bar{J}dZ = \bar{J}_1 \bar{I}_H dX$, then the two approximations are equal to the first order.

$$\begin{aligned}
2ZJ(I_H, I_L) \approx\ & 2\bar{Z}\bar{J} + 2\bar{J}dZ + 2\bar{Z}\bar{J}_1 dI_H + 2\bar{Z}\bar{J}_2 dI_L \\
& + \bar{J}_1 dZ dI_H + \bar{J}_2 dZ dI_L \\
& + \bar{J}_1 dI_H dZ + \bar{Z}\bar{J}_{11} \left(dI_H\right)^2 + \bar{Z}\bar{J}_{12} dI_H dI_L \\
& + \bar{J}_2 dI_L dZ + \bar{Z}\bar{J}_{21} dI_L dI_H + \bar{Z}\bar{J}_{22} \left(dI_L\right)^2
\end{aligned} \tag{142}$$

$$\begin{aligned}
2J(XI_H, I_L) \approx\ & 2\bar{J} + 2\bar{J}_1 \bar{I}_H dX + 2\bar{J}_1 \bar{X} dI_H + 2\bar{J}_2 dI_L \\
& + \bar{J}_1 dX dI_H + \bar{J}_{11} \left(\bar{I}_H\right)^2 \left(dX\right)^2 + \bar{J}_{11} \bar{I}_H \bar{X} dX dI_H + \bar{J}_{12} \bar{I}_H dX dI_L \\
& + \bar{J}_1 dI_H dX + \bar{J}_{11} \bar{X} \bar{I}_H dI_H dX + \bar{J}_{11} \left(\bar{X}\right)^2 \left(dI_H\right)^2 + \bar{J}_{12} \bar{X} dI_H dI_L \\
& + \bar{J}_{21} I_H dI_L dX + \bar{J}_{21} \bar{X} dI_L dI_H + \bar{J}_{22} \left(dI_L\right)^2
\end{aligned} \tag{143}$$