Macroeconomic Analysis with Machine Learning and Big Data

Lecture 8: Heterogeneous Agent New Keynesian (HANK) Models

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1 Introduction

Key questions always to keep in mind: Does heterogeneity matters? Why heterogeneity matters?

1.1 RANK (Representative Agent New Keynesian)

1.1.1 Model Setup

$$\max \int_0^\infty e^{-\rho t} U(C_t)\, dt, \quad U(C) = \frac{C^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$
$$s.t.\, \dot{a}_t = Y_t + r_t a_t - c_t$$

where Y_t is the output following the production function $Y_t = N_t$ and N_t is the labor supply. r_t is the interest rate and a_t is the asset holding. C_t is the consumption.

1.1.2 Monetary Transmission in RANK

We can solve this problem to get the Euler equation

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\gamma}(r_t - \rho)$$

Solving this ODE we will get the path of consumption

$$C_t = \overline{C} \exp\left(-\frac{1}{\gamma}\int_t^\infty (r_s - \rho)\, ds\right)$$

To close the model we exogeneously assume a interest rule such that

$$r_t = \rho + e^{-\eta t}(r_0 - \rho), \quad t \geq 0$$

This formula collapses to a simple expression for the elasticity of initial consumption to the initial change in the interest rate

$$\frac{d \log C_0}{d r_0} = -\frac{1}{\gamma \eta}$$

where η is the speed of interesting to decentralized and γ is the elasticity of substitution.
Total differential of $log(C_0)$ is

$$d\log C_0 = -\frac{1}{\gamma}\int_0^\infty e^{-\rho t} dr_t dt - \frac{\rho}{\gamma}\int_0^\infty e^{-\rho t}\int_t^\infty dr_s ds dt$$

where the first term is the direct response to r and second term is the indirect effects due to Y
By giving the interest path we can solving this analytically

$$-\frac{d\log C_0}{dr_0} = \frac{1}{\gamma\eta}\left(\underbrace{\frac{\eta}{\rho+\eta}}_{\text{direct response to } r} + \underbrace{\frac{\rho}{\rho+\eta}}_{\text{indirect effects due to } Y}\right)$$

Normally the first term, direct response, is 99% and the second term, indirect effects, is 1% calculated by the model. However is result is not supportted by the data.

1.2 TANK (Two Agent New Keynesian Models)

1.2.1 Model setting

The setup is identical, except that we assume that a fraction Λ of households consume their entire current income. Therefore they will not directly response to the interest rate since no matter what they will always consume all of their income.

1.2.2 Monetary Transmission in TANK

By the same logic and method we can solve this problem to get

$$-\frac{d\log C_0}{dr_0} = \frac{1}{\gamma\eta}\left(\underbrace{(1-\Lambda)\frac{\eta}{\rho+\eta}}_{\text{direct response to } r} + \underbrace{\left((1-\Lambda)\frac{\rho}{\rho+\eta} + \Lambda\right)}_{\text{indirect due to } Y}\right)$$

A reasonable estimate for the proportion of handto- mouth households in the United States is 0.3 (Kaplan, Violante and Weidner, 2014). Thus, in TANK the share of direct effects is roughly 0.7. The overall effect in TANK is the same as in RANK because the addition of hand-to-mouth households decreases direct effects and increases indirect effects by the same magnitude. However, the results estimated by TANK still do not match the real data well.

2 HANK (Heterogeneous Agent New Keynesian Models)

2.1 Model Setup

The innovation of HANK model is that households face uninsurable idiosyncratic income risk which they can self-insure through two savings instrument with different degree of liquidity and return rates.

2.1.1 Household

$$\max_{\{c_t,l_t,k_t\}} E_0 \int_0^\infty e^{-(\rho+\zeta)t} u\left(c_t, \ell_t\right) dt$$

s.t.

$$\begin{aligned}
\dot{b}_t &= \underbrace{\left(1-\tau_t\right) w_t z_t \ell_t}_{\text{Income}} + \underbrace{r_t^b\left(b_t\right) b_t}_{\text{Capital}} + \underbrace{T_t}_{\text{Transfer}} - \underbrace{d_t}_{\text{Deposit}} - \underbrace{\chi\left(d_t, a_t\right)}_{\text{Transaction cost}} - \underbrace{c_t}_{\text{Consumption}} \\
\dot{a}_t &= r_t^a a_t + d_t \\
b_t &\geq -\underline{b}, \quad a_t \geq 0
\end{aligned}$$

where ζ is rate of an exodogenous death Poisson process,and upon death households give birth to an offspring with wealth a_0. z_t is idiosyncratic labor productivity following a log jump drift process to be specified later. b_t refers to household's liquid asset with interest rate r_t^b, and a_t refers to household's illiquid asset with interest rate r_t^a. Labor earnings are taxed at rate τ_t.

Transaction cost $\chi(d,a)$ is given by

$$\chi(d, a)=\chi_0|d|+\chi_1\left|\frac{d}{a}\right|^{X_2} a$$

$$\chi_0>0, \chi_1>0, \chi_2>1$$

which ensures that $|d_t|<\infty$.

Households maximize life-long utility by choosing decision path $\{c_t, l_t, k_t\}_{t\geq 0}$, taking as given the time path of prices and policies $\{\Gamma_t\}_{t\geq 0} := \left\{r_t^b, r_t^a, w_t, \tau_t, T_t\right\}_{t\geq 0}$. Similar to HACT, we will derive the HJB and KF equations for the household problem.

$$V(a,b,y,t)=\max_{\{c_t,l_t,d_t\}} U(c,l)\Delta t+(1-(\rho+\zeta)\Delta t)(V(a_{t+\Delta t}, b_{t+\Delta t}, y_{t+\Delta t}, t+\Delta t)$$

where $y_{i,t} \equiv log z_{i,t}$ follows a "jumps-drift process":

$$dy_{i,t}=-\beta y_{i,t} dt+dJ_{i,t}.$$

Jumps arrive at a Poisson arrival rate λ. Conditional on a jump, a $y'_{it} \sim \mathcal{N}(0, \sigma^2)$ is drawn. Denote ϕ as density of a normal distribution with variance σ^2. Rearrange and divide both sides by Δt,

$$0 = \max_{c,\ell,d} U(c,l) + \frac{V_{t+\Delta t} - V_t}{\Delta t} - (\rho + \zeta) V_{t+\Delta t}$$

Take $\Delta t \to 0$:

$$0 = \max_{c,\ell,d} U(c,l) + \partial_a V \cdot \dot{a} + \partial_b V \cdot \dot{b} + \partial_y V \cdot \dot{y} + \partial_t V - (\rho + \zeta) V$$

HJB in the steady state equilibrium:

$$\begin{aligned}(\rho+\zeta)V(a,b,y) = & \max_{c,\ell,d} u(c,\ell) + V_b(a,b,y)\left[(1-\tau)we^y\ell + r^b(b)b + T - d - \chi(d,a) - c\right] \\ & + V_a(a,b,y)\left(r^a a + d\right) \\ & + V_y(a,b,y)(-\beta y) + \lambda \int_{-\infty}^{\infty} (V(a,b,x) - V(a,b,y))\phi(x)dx\end{aligned}$$

For the KF equation, define:

$$\begin{aligned} G(a,b,y,t) &\equiv Pr(\tilde{a}_t \leq a, \tilde{b}_t \leq b, \tilde{y}_t \leq y) \\ s^a &= ra + d \\ s^b &= \underbrace{(1-\tau)\omega Zl}_{\text{Income}} + \underbrace{r^b b}_{\text{Capital}} + \underbrace{T}_{\text{Transfer}} - \underbrace{d}_{\text{Deposit}} + \underbrace{\chi(d,a)}_{\text{Transaction cost}} - \underbrace{C}_{\text{Consumption}} \\ dy_{it} &= -\beta y_{it} dt + dJ_{it} \end{aligned}$$

$$\begin{aligned} G(a,b,y,t+\Delta t) &= Pr(\tilde{a}_{t+\Delta t} \leq a, \tilde{b}_{t+\Delta t} \leq b, \tilde{y}_{t+\Delta t} \leq y) \\ &= \zeta \Delta t \Phi(a,b,y) + Pr(\tilde{a}_t \leq a - \Delta t s^a, \tilde{b}_t \leq b - \Delta t s^b, \tilde{y}_{t+\Delta t} \leq y) \cdot (1 - \zeta \Delta t) \\ &= \zeta \Delta t \Phi(a,b,y) + [Pr(\tilde{a}_t \leq a - \Delta t s^a, \tilde{b}_t \leq b - \Delta t s^b, \tilde{y}_t \leq y + \Delta t \beta y_{it}) \cdot (1 - \lambda \Delta t) \\ &\quad + Pr(\tilde{a}_t \leq a - \Delta t s^a, \tilde{b}_t \leq b - \Delta t s^b) \cdot \lambda \Delta t \cdot \int_{-\infty}^{Y} \int_{-\infty}^{+\infty} \varphi(y) g(\tilde{a}_t, \tilde{b}_t, x) dx dy](1 - \zeta \Delta t) \\ &= G(a - \Delta t s^a, b - \Delta t s^b, y + \Delta t \beta y, t) \cdot (1 - \lambda \Delta t) \cdot (1 - \zeta \Delta t) + \zeta \Delta t \Phi(a,b,y) \\ &\quad + \lambda \Delta t \cdot \int_{-\infty}^{Y} \int_{-\infty}^{+\infty} \phi(y) G(a - \Delta t s^a, b - \Delta t s^b, x) dx dy \end{aligned}$$

Subtract $G(a,b,y,t)$ from both sides, take the limit as $\Delta t \to 0$ and take derivative with respect to a, b, y, we obtain the KF equation in the steady state equilibrium:

KF in the steady state equilibrium:

$$\begin{aligned} 0 = & -\partial_a \left(s^a(a,b,y) g(a,b,y)\right) - \partial_b \left(s^b(a,b,y) g(a,b,y)\right) \\ & - \partial_y(-\beta y g(a,b,y)) - \lambda g(a,b,y) + \lambda \phi(y) \int_{-\infty}^{\infty} g(a,b,x) dx \\ & - \zeta g(a,b,y) + \zeta \delta(a - a_0) \delta(b - b_0) g^*(y) \end{aligned}$$

where δ is the Dirac delta function, (a_0, b_0) are starting assets and $g^*(y)$ is the stationary distribution of y. The last term implies that so as to have the equilibrium to be achieved, after the origin household dies, the new-born households, with initial wealth, should have the stationary distribution of income.

2.1.2 Producers

Mathematically, the production side provides conditions to set the prices and close the model.

Final-Good Producers uses a continuum of intermediate inputs indexed by $j \in [0,1]$ and produce with constant elasticity of substitution ε (CES):

$$Y_t = \left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}}$$

Cost minimization[1] gives the demand for intermediate good j :

$$y_{j,t}\left(p_{j,t}\right) = \left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t, \quad \text{where} \quad P_t = \left(\int_0^1 p_{j,t}^{1-\varepsilon} dj\right)^{\frac{1}{1-\varepsilon}}$$

Intermediate Good Producers produce intermediate good j in monopolistically competitive market by:

$$y_{j,t} = k_{i,t}^{\alpha} n_{i,t}^{1-\alpha}$$

Again, cost minimization:

$$\min_{\{n_{j,t}, k_{j,t}\}} \omega_t n_{j,t} + r_t^k k_{j,t}$$

s.t.

$$y_{j,t} = k_{j,t}^{\alpha} n_{j,t}^{1-\alpha}$$
$$y_{j,t} = \left(\frac{p_{j,t}}{P_t}\right)^{-\varepsilon} Y_t$$

from which we can solve for $k_{j,t}, n_{j,t}$ and this also implies that marginal cost m_t is given by:

$$m_t = \left(\frac{r_t^k}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}$$

[1] Construct $\mathcal{L} = \min \sum_{j=0}^{1} P_j y_j - \lambda[\left(\int_0^1 y_{j,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right)^{\frac{\varepsilon}{\varepsilon-1}} - Y_t]$, let $\frac{\partial \mathcal{L}}{\partial y_{j,t}} = 0$ and get $p_j = \lambda Y^{\frac{1}{\varepsilon}} \cdot y_j^{-\frac{1}{\varepsilon}}$. Take two indexes of products j_1 and j_2 and get $y_{j_2} = \left(\frac{p_{j_1}}{p_{j_2}}\right)^{\varepsilon} y_{j_1}$. Taking the $\frac{\varepsilon-1}{\varepsilon}$ power of both sides and integral by dj_2 yields $\left(\int_0^1 y_{j_2}^{\frac{\varepsilon-1}{\varepsilon}} d_{j_2}\right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(y_{j_1}^{\frac{\varepsilon-1}{\varepsilon}} p_{j_1}^{\varepsilon-1}\right)^{\frac{\varepsilon}{\varepsilon-1}} \cdot \left(\int_0^1 p_{j_2}^{1-\varepsilon} d_{j_2}\right)^{\frac{\varepsilon}{\varepsilon-1}}$.

We define cost adjustment cost:

$$\Theta_t\left(\frac{\dot{p}_t}{p_t}\right) = \frac{\theta}{2}\left(\frac{\dot{p}_t}{p_t}\right)^2 Y_t = \frac{\theta}{2}\pi_t^2 Y_t$$

Since producers are in monopolistically competitive market, the profit for representative intermediate good producers can be defined as:

$$\Pi_t(p_t) \equiv (\frac{p_t}{P_t} - m_t)(\frac{p_t}{P_t})^{-\varepsilon} Y_t - \frac{\theta}{2}\pi_t^2 Y_t$$

Households' illiquid assets can be invested into two assets: capital k_t and equity share of intermediate firms s_t, which implies $a_t = k_t + q_t s_t$. The dynamics of capital and equity satisfy:

$$\dot{k}_t + q_t \dot{s}_t = (r_t^k - \delta)k_t + \Pi_t s_t + d_t$$

Since households can switch costlessly between capital and equity, the return of both assets should be equal, which implies

$$\frac{\Pi_t + \dot{q}_t}{q_t} = r_t^k - \delta = r_t^a$$

And **this connects r_t^k with r_t^a.**

2.1.3 Monetary Authority

According to the Taylor Rule:

$$i_t = \bar{r}^b + \phi\pi_t + \epsilon_t$$

where ϵ_t is the monetary shock. And according to the Fisher Equation:

$$r_t^b = i_t - \pi_t$$

And **this connects r_t^b with ϵ_t.**

2.1.4 Government

The governmental budget balance condition gives us:

$$\dot{B}_t^g + G_t + T_t = \tau_t \int w_t z \ell_t(a,b,z) d\mu_t + r_t^b B_t^g$$

where μ_t is the measure of households over (a, b, z). To note, outside of steady state, the fiscal instrument to balance the budget can be either T_t, τ_t or G_t, and the responses of other variables would depend on which fiscal instrument to use.

2.2 Equilibrium

- Decisions: $\{a_t, b_t, c_t, d_t, \ell_t, n_t, k_t\}_{t\geq 0}$
- States: $\{\Gamma_t\}_{t\geq 0} := \left\{\omega_t, r_t^k, r_t^a, r_t^b, q_t, \pi_t, \tau_t, \boldsymbol{T}_t, \boldsymbol{G}_t, \boldsymbol{B}_t, \mu_t\right\}_{t\geq 0}$
  - Liquid asset clears: $B_t^h + B_t^g = 0$
  - Illiquid asset clears: $K_t + q_t = A_t$, with the total number of shares normalized to 1
  - good market clears: $Y_t = C_t + I_t + G_t + \Theta_t + \chi_t + \kappa \int \max\{-b, 0\} d\mu_t$, the last term reflects borrowing costs.

3 Monetary Transmission in HANK

Consider an expansionary monetary shock ($\epsilon_0 < 0$). Recall from Section 2.1.3, ϵ_t is directly connected with liquid assets' interest rate. The monetary shock also affect other state variables in Γ_t.

We define aggregate consumption as

$$C_t\left(\{\Gamma_t\}_{t\geq 0}\right) = \int c_t\left(a, b, z; \{\Gamma_t\}_{t\geq 0}\right) d\mu_t$$

which can be total differentiated into:

$$dC_0 = \underbrace{\int_0^\infty \frac{\partial C_0}{\partial r_t^b} dr_t^b dt}_{\text{direct effect } (\approx 19\%)} + \underbrace{\int_0^\infty \left(\frac{\partial C_0}{\partial w_t} dw_t + \frac{\partial C_0}{\partial r_t^a} dr_t^a + \frac{\partial C_0}{\partial \tau_t} d\tau_t + \frac{\partial C_0}{\partial T_t} dT_t\right) dt}_{\text{indirect effects} (\approx 81\%)}$$

4 Solving the steady state equilibrium

- Step 1: Make initial guess for $\{K, N, r^b\}$ (or guess prices $\{r^a, r^b, \omega\}$ and solve the remaining in Step 2).
- Step 2: Using the relations we get from the producer's problem to solve for $\{\omega, r^a, \Pi, \tau/T/G\}$ as necessary input to solve the HJB.
- Step 3: Given $\omega, r^a, r^b, \Pi, \tau/T/G$, solve the HJB equation to get policy function a, b, c, l, d with the HACT method.
- Step 4: Given policy functions, solve the KF equation for distribution g with the HACT method.
- Step 5: Check if market clear conditions are satisfied. If not, update the values in the Step 1 according to the difference.

5 Summary: Comparison between RANK and HANK

- RANK views:
  - High sensitivity of C to r: intertemporal substitution
  - Low sensitivity of C to Y: the representative agent is permanent income hypothesis (PIH) consumer
- HANK views:
  - Low sensitivity to r: income effect of wealthy offsets substitution effect
  - High sensitivity to Y: sizable share of hand-to-mouth agents

6 The Road Ahead

In both HACT and HANK models, the labor productivity z is idiosyncratic for households, while the aggregation remains unchanged over time. What if the aggregated Y_t and Z_t are stochastic? The problem becomes much more complicated mathematically. Economists have proposed methods like perturbation to solve such problems, and other tools like deep neural networks might be helpful to further explore the non-linear dynamics of the distribution. We will discuss some recent work in the next class.

Reference

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