Macroeconomic Analysis with Machine Learning and Big Data

Lecture 7: HACT and Mean Field Games (MFG)

Instructor: Jiequn Han *Scribe: Mian Qin, Yunjie Song, Yucheng Yang*

Three advantages of heterogeneous agent model: (1) more realistic micro foundation, (2) better match the micro data, (3) useful for welfare analysis for distributional issues.

1 Recap: Huggett Model (Discrete Time)

- **Model Setup**

$$\min_{c_t \geq 0, a_{t+1} \geq \underline{a}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

$$\text{s.t. } c_t + a_{t+1} = y_t + (1 + r_t)a_t$$

with a_0, y_0 given and y_t follows Markov process with transition matrix Π on possible set Y.

- **Market Clearing Condition**

$$\int_{\underline{a}}^{\infty} a'(a, y, \Phi) d\Phi = B$$

where Φ is the distribution over $\mathcal{A} \times \mathcal{Y}$, $\mathcal{A}$ is $\sigma-$algebra over possible assets, $\mathcal{Y}$ is $\sigma-$algebra over possible set Y. B is fixed supply of asset (bond) in the economy.

- **Recursive Formulation: Bellman Equation**

$$V(a, y, \Phi) = \max_{c \geq 0, a' \geq \underline{a}} u(c) + \beta \sum_{y' \in Y} \pi(y' \mid y) V(a', y', \Phi')$$

$$c + a' = y + (1 + r)a$$

$$\Phi' = H(\Phi)$$

where $H(\cdot)$ is law of motion for the joint distribution Φ, which is determined by the transition matrix Π and optimal policy function $a'(a, y, \Phi)$. Recall from the last lecture, we can only solve for stationary equilibrium where $\Phi' = \Phi$ with recursive guesses on r.

However, if we formulate this problem in continuous time, we can take advantage of advanced mathematical tools of PDEs and bring both new insights and computational convenience to heterogeneous agent modelling.

2 Heterogeneous Agent Model in Continuous Time (HACT)

2.1 Model Setup

$$\max_{c_t \geq 0, a_t \geq \underline{a}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u\left(c_t\right) dt$$
$$\dot{a}_t = y_t + r_t a_t - c_t$$

with a_0, y_0 given. We assume that income follows a two-state Poisson process $y_t \in \{y_1, y_2\}$ with $y_1 < y_2$. y_t jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2. The market clearing condition is

$$\int_{\underline{a}}^\infty a g_1(a,t) da + \int_{\underline{a}}^\infty a g_2(a,t) da = B$$

where $g_j(a,t)$ is the pdf over a, y_j at time t.

2.2 Recursive Formulation: Hamilton-Jacobi-Bellman (HJB) equation and Kolmogorov Forward (KF) equation

In HACT, the problem can be written as a Hamilton-Jacobi-Bellman (HJB) equation for optimization coupled with a Kolmogorov Forward (KF) equation for law of motions. We derive both equations for transition dynamics and obtain the stationary formulation for free.

Deriving HJB equation Consider the problem in discrete time with short periods of length Δt. Individuals discount the future with discount factor $\beta(\Delta t) = e^{-\rho \Delta t}$. Individuals with income y_j keep their income with probability $p_j(\Delta t) = e^{-\lambda_j \Delta t}$ and switch to y_{-j} with probability $1 - p_j(\Delta t)$. As $\Delta t \to 0$,

$$\beta(\Delta t) = e^{-\rho \Delta t} \approx 1 - \rho \Delta t, \quad p_j(\Delta t) = e^{-\lambda_j \Delta t} \approx 1 - \lambda_j \Delta t$$

Therefore, the value function reads

$$v_j(a,t) = \max_c u(c)\Delta t + (1 - \rho \Delta t)[(1 - \lambda_j \Delta t) v_j(a_{t+\Delta t}, t + \Delta t) + (\lambda_j \Delta t) v_{-j}\left(a_{t+\Delta t}, t + \Delta t\right)]$$

Rearranging the equation,

$$0 = \left[v_j\left(a_{t+\Delta t}, t + \Delta t\right) - v_j\left(a, t\right)\right] + \max_c \Delta t u(c) + \Delta t \lambda_j v_{-j}\left(a_{t+\Delta t}, t + \Delta t\right)$$
$$- \Delta t \rho v_j\left(a_{t+\Delta t}, t + \Delta t\right) - \Delta t \lambda_j v_j\left(a_{t+\Delta t}, t + \Delta t\right) + O(\Delta t^2)$$

Dividing by Δt, taking $\Delta t \to 0$, and using that

$$\lim_{\Delta t \to 0} \frac{v_j\left(a_{t+\Delta t}, t + \Delta t\right) - v_j(a,t)}{\Delta t} = \partial_a v_j(a,t)\left(y_j + r_t a - c\right) + \partial_t v_j(a,t)$$

we obtain the following HJB equation:

$$\rho v_j(a,t) = \max_c u(c) + \partial_a v_j(a,t)\left(y_j + r_t a - c\right) + \lambda_j\left(v_{-j}(a,t) - v_j(a,t)\right) + \partial_t v_j(a,t)$$

Deriving KF equation

- First we denote $\tilde{a}_t, \tilde{y}_t$ as random variables, with a_t, y_t as their possible realizations.
- Saving policy function $s_j(a_t, t) = y_j + r_t a_t - c_j(a_t, t)$.
- Wealth evolves as $d\tilde{a}_t = s_j(\tilde{a}_t, t)dt$.
- We define density $g_j(a, t), j = 1, 2$ before, but when deriving KF equation it's easier to work with the cdf $G_j(a, t) = \Pr(\tilde{a}_t \leq a, \tilde{y}_t = y_j)$.

Consider the discrete-time situation[1]:

- Step 1: Individuals make their saving decisions according to $\tilde{a}_t = \tilde{a}_{t+\Delta t} - s_j(\tilde{a}_t)\Delta t$;
- Step 2: After saving decisions are made, next period's income $\tilde{y}_{t+\Delta t}$ is realized, individuals switches from y_j to y_{-j} with probability $\lambda_j \Delta t$. So we have:

$$\begin{aligned}\Pr(\tilde{a}_{t+\Delta t} \leq a, \tilde{y}_{t+\Delta t} = y_j) = (1 - \Delta t \lambda_j) \Pr(\tilde{a}_t \leq a - \Delta t s_j(a, t), \tilde{y}_t = y_j) \\ + \Delta t \lambda_{-j} \Pr(\tilde{a}_t \leq a - \Delta t s_{-j}(a), \tilde{y}_t = y_{-j})\end{aligned}$$

In terms of $G_j(\cdot, \cdot)$, this can be written as:

$$G_j(a, t + \Delta t) = (1 - \Delta t \lambda_j) G_j(a - \Delta t s_j(a, t), t) + \Delta t \lambda_{-j} G_{-j}(a - \Delta t s_{-j}(a), t)$$

Subtracting $G_j(a, t)$ from both sides and dividing by Δt,

$$\begin{aligned}\frac{G_j(a, t + \Delta t) - G_j(a, t)}{\Delta t} =& \frac{G_j(a - \Delta t s_j(a, t), t) - G_j(a, t)}{\Delta t} - \lambda_j G_j(a - \Delta t s_j(a, t), t) \\ &+ \lambda_{-j} G_{-j}(a - \Delta t s_{-j}(a), t)\end{aligned}$$

Taking the limit as $\Delta t \to 0$ gives $\partial_t G_j(a, t) = -s_j(a, t)\partial_a G_j(a, t) - \lambda_j G_j(a, t) + \lambda_{-j} G_{-j}(a, t)$
Take the derivative with respect to a, we obtain KF equation

$$\partial_t g_j(a, t) = -\partial_a [s_j(a, t) g_j(a, t)] - \lambda_j g_j(a, t) + \lambda_{-j} g_{-j}(a, t)$$

Collections of HJB and KF equations for transition dynamics

$$\rho v_j(a, t) = \max_c u(c) + \partial_a v_j(a, t)(y_j + r_t a - c) + \lambda_j (v_{-j}(a, t) - v_j(a, t)) + \partial_t v_j(a, t)$$

$$\partial_t g_j(a, t) = -\partial_a [s_j(a, t) g_j(a, t)] - \lambda_j g_j(a, t) + \lambda_{-j} g_{-j}(a, t)$$

Actually both equations could be derived directly using the infinitesimal generator of the Poisson process.

[1] To note, the KF equation would be the same if the individual realize income first then make saving decisions.

HJB and KF Equations in the Stationary Equilibrium

$$\rho v_j(a) = \max_c u(c) + v_j'(a)\left(y_j + ra - c\right) + \lambda_j\left(v_{-j}(a) - v_j(a)\right)$$

$$0 = -\frac{d}{da}\left[s_j(a)g_j(a)\right] - \lambda_j g_j(a) + \lambda_{-j} g_{-j}(a)$$

We then obtain the following Euler equation with envelope condition and FOC:

$$(\rho - r)u'\left(c_j(a)\right) = u''\left(c_j(a)\right)c_j'(a)s_j(a) + \lambda_j\left(u'\left(c_{-j}(a)\right) - u'\left(c_j(a)\right)\right)$$

Informally using Ito's formula it can be written as

$$\frac{\mathbb{E}_t\left[du'\left(c_j\left(a_t\right)\right)\right]}{u'\left(c_j\left(a_t\right)\right)} = (\rho - r)dt$$

2.3 Borrowing Constraint and Boundary Conditions

In continuous time formulation, the borrowing constraint $a_t \geq \underline{a}$ never binds in the interior of the state space, so FOC $u'(c_j(a)) = v_j'(a)$ always holds for $a > \underline{a}$. The borrowing constraint only requires a boundary condition at $\underline{a}$, which makes the problem much easier to solve. For the stationary equilibrium, the condition is $v_j'(\underline{a}) \geq u_j'(y_j + r\underline{a})$. For transition dynamics, it is $\partial_a v_j(\underline{a}, t) \geq u_j'(y_j + r_t\underline{a})$. The transition dynamics problem also requires initial condition for $g : g_j(a, 0) = g_{j,0}(a)$ and terminal condition for $v : v_j(a, T) = v_{j,\infty}(a)$.

2.4 Analytical Results of HACT

2.4.1 Consumption and Saving Behavior of the Poor

Assumption 1 The coefficient of absolute risk aversion $R(c) := -u''(c)/u'(c)$ when wealth a approaches the borrowing limit $\underline{a}$ is finite, that is

$$\underline{R} := -\lim_{a \to \underline{a}} \frac{u''\left(y_1 + ra\right)}{u'\left(y_1 + ra\right)} < \infty$$

Proposition 1 (MPCs and Saving at Borrowing Constraint) Assume that $r < \rho$, $y_1 < y_2$ and that Assumption 1 holds. Then the solution to the HJB equation and the corresponding policy functions have the following properties:
1) $s_1(\underline{a}) = 0$ but $s_1(a) < 0$ for all $a > \underline{a}$. That is, only individuals exactly at the borrowing constraint are constrained, whereas those with wealth $a > \underline{a}$ are unconstrained and decumulate assets.
2) as $a \to \underline{a}$, the saving and consumption policy function of the low income type and the corresponding instantaneous marginal propensity to consume satisfy

$$s_1(a) \sim -\sqrt{2\nu_1}\sqrt{a - \underline{a}} \qquad (19)$$

$$c_1(a) \sim y_1 + ra + \sqrt{2\nu_1}\sqrt{a - \underline{a}}$$

$$c_1'(a) \sim r + \sqrt{\frac{\nu_1}{2(a - \underline{a})}} \qquad (20)$$

$$\nu_1 := \frac{(\rho - r)u'(\underline{c}_1) + \lambda_1 (u'(\underline{c}_1) - u'(\underline{c}_2))}{-u''(\underline{c}_1)}$$

$$\approx (\rho - r)\,\mathrm{IES}(\underline{c}_1)\,\underline{c}_1 + \lambda_1(\underline{c}_2 - \underline{c}_1) \qquad (21)$$

2.4.2 Consumption and Saving Behavior of the Wealthy

Proposition 2 Assume that $r < \rho$, $y_1 < y_2$ and that relative risk aversion $-cu''(c)/u'(c)$ is bounded above for all c.
1) Then there exists $a_{\max} < \infty$ such that $s_j(a) < 0$ for all $a \geq a_{\max}, j = 1, 2$, and $s_2(a) \sim \zeta_2 (a_{\max} - a)$ as $a \to a_{\max}$ for some constant ζ_2. The wealth of an individual with initial wealth a_0 and successive high income draws y_2 converges to $a_{\max}$ asymptotically (i.e. not in finite time): $a(t) - a_{\max} \sim e^{-\zeta_2 t}(a_0 - a_{\max})$
2) In the special case of CRRA utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ individual policy functions are asymptotically linear in a. As $a \to \infty$, they satisfy

$$s_j(a) \sim \frac{r - \rho}{\gamma} a,\ c_j(a) \sim \frac{\rho - (1-\gamma) r}{\gamma} a \qquad (28)$$

2.5 Numerical Method to Solve the HJB and KF Equations

We present the numerical method to solve the stationary equilibrium with a special case with no income uncertainty $y_1 = y_2 = y$ in this section. The generalization to more complicated cases is straightforward. Similar to the recursive method to solve a discrete-time Huggett model, we first guess a value for r in the stationary equilibrium, then solve the HJB and KF equations respectively, then update r with the market clearing condition and go into the next iteration.

2.5.1 Finite Difference Method for HJB

In the special case with no income uncertainty $y_1 = y_2 = y$, the HJB equation is:

$$\rho v(a) = \max_c u(c) + v'(a)(y + ra - c) \qquad (51)$$

The finite difference method approximates the function v at I discrete points in the space dimension, $a_i, i = 1, \ldots, I$. We use equispaced grids, with distance Δa between grid points, and denote $v_i = v(a_i), v_i' = v'(a_i), c_i = c(a_i), s_i = s(a_i)$. The derivative v_i' is approximated with either a forward or a backward difference approximation

$$v_i' \approx \frac{v_{i+1} - v_i}{\Delta a} =: v_{i,F}' \quad \text{or} \quad v_i' \approx \frac{v_i - v_{i-1}}{\Delta a} =: v_{i,B}'$$

The choice of forward or backward difference approximations is critical to the numerical performance and will be determined via the upwind scheme discussed below. The finite difference approximation to (51) is

$$\rho v_i = u\left(c_i\right) + v_i' s_i \qquad (52)$$

with

$$s_i := y + r a_i - c_i, \quad c_i = \left(u'\right)^{-1}\left(v_i'\right), \quad i = 1, \ldots, I$$

Upwind Scheme The choice of forward or backward difference approximations to v_i' is determined via the so-called "upwind scheme". The rough idea is to use a forward difference approximation whenever the drift of the state variable (here, saving $s_i = y + ra_i - c_i$) is positive and to use a backward difference whenever it is negative. To note, in the Kolmogorov Forward equation, the choice is opposite. The upwind version of (52) is then

$$\rho v_i = u\left(c_i\right) + \frac{v_{i+1} - v_i}{\Delta a} s_i^+ + \frac{v_i - v_{i-1}}{\Delta a} s_i^-, \quad i = 1, \ldots, I \qquad (53)$$

where $s_i^+ = \max\left\{s_i, 0\right\}$ and $s_i^- = \min\left\{s_i, 0\right\}$.

The problem is actually more complicated as calculating s_i requires the value of v_i' as $s_i = y + ra_i - c_i$ with $c_i = \left(u'\right)^{-1}\left(v_i'\right)$. So we have to adopt an iterative algorithm to solve the HJB: with initial guess of $v_i^0, i = 1, ..., I$, for $n = 0, 1, ...$:

- Step 1: compute $(v_{i,F}^n)', (v_{i,B}^n)'$, and compute $c_{i,F}^n, s_{i,F}^n$ with $(v_{i,F}^n)'$, compute $c_{i,B}^n, s_{i,B}^n$ with $(v_{i,B}^n)'$. Then compute

  $$(v_i^n)' = (v_{i,F}^n)' \mathbf{1}_{s_{i,F}^n > 0} + (v_{i,B}^n)' \mathbf{1}_{s_{i,B}^n < 0} + (\bar{v}_i^n)' \mathbf{1}_{s_{i,F}^n \leq 0 \leq s_{i,F}^n}$$

  where $(\bar{v}_i^n)' = u'(y + ra_i)$ is v_i' with zero savings.

- Step 2: with $(v_i^n)'$, compute c_i^n.

- Step 3: update v_i^{n+1} with explicit method or implicit method until v_i^{n+1} is close enough to v_i^n. The implicit method goes as:

  $$\frac{v_i^{n+1} - v_i^n}{\Delta} + \rho v_i^{n+1} = u(c_i^n) + \frac{v_{i+1}^{n+1} - v_i^{n+1}}{\Delta a}(s_{i,F}^n)^+ + \frac{v_i^{n+1} - v_{i-1}^{n+1}}{\Delta a}(s_{i,B}^n)^-$$

  It can be written as a matrix form

  $$\frac{1}{\Delta}(\mathbf{v}^{n+1} - \mathbf{v}^n) + \rho \mathbf{v}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^{n+1}$$

  where $\mathbf{A}$ is a tridiagonal Poission transition matrix with all diagonal entries are negative, all off-diagonal entries are positive and all rows sum to zero.

Boundary Condition Remember we turn the borrowing constraint into boundary condition at $\underline{a}$. For numerics, it is

$$v'_{1,B} = u'\left(y + ra_1\right)$$

only for the backward difference approximation. The forward difference remains $v'_{1,F} = \left(v_2 - v_1\right)/\Delta a$. From (53) we can see that the boundary condition is only imposed if $s_1 < 0$.

2.5.2 Solving the KF Equation

Given v_i's and s_i's, we can solve the KF equation $0 = -\frac{d}{da}[g(a)s(a)]$ directly. It can be discretized as $0 = -[g_i s_i]'$, and the upwind scheme writes

$$0 = -\frac{[(s^n_{i,F})^+ g_i - (s^n_{i-1,F})^+ g_{i-1}]}{\Delta a} - \frac{[(s^n_{i+1,B})^- g_{i+1} - (s^n_{i,B})^- g_i]}{\Delta a}.$$

In the matrix form, it is

$$\mathbf{A}^T \mathbf{g} = 0$$

where $\mathbf{A}^T$ is the transpose of the matrix A in the HJB equation.

3 Extensions of the Models

3.1 Diffusion Income Processes

Our baseline model assumed that income y_t takes one of two values, high and low. We now extend it to a diffusion process. Assume that individual income evolves stochastically over time on a bounded interval $[\underline{y}, \overline{y}]$ with $\overline{y} > \underline{y} \geq 0$, according to the stationary diffusion process:

$$dy_t = \mu\left(y_t\right) dt + \sigma\left(y_t\right) dW_t \qquad (54)$$

W_t is a Wiener process or standard Brownian motion and the functions μ and σ are called the drift and the diffusion of the process. We normalize the process such that its stationary mean equals one.

Similar to Section 1, a *stationary* equilibrium can be written as a system of partial differential equations. The problem of individuals and the joint distribution of income and wealth satisfy stationary HJB and KF equations:

$$\rho v(a,y) = \max_c u(c) + \partial_a v(a,y)(y + ra - c) + \partial_y v(a,y)\mu(y) + \frac{1}{2}\partial_{yy} v(a,y)\sigma^2(y) \qquad (55)$$

$$0 = -\partial_a(s(a,y)g(a,y)) - \partial_y(\mu(y)g(a,y)) + \frac{1}{2}\partial_{yy}\left(\sigma^2(y)g(a,y)\right) \qquad (56)$$

Importantly, the computational algorithm laid out in Section 3 carries over without change: from a computational perspective it is immaterial whether we solve a system of ODEs like (7) and (8) or a system of PDEs like (55) and (56).

3.2 An Alternative Way of Closing the Model: Aiyagari (1994)

Section 1 assumed that wealth takes the form of bonds that are in fixed supply. We can assume as in Aiyagari (1994) that wealth takes the form of productive capital that is used by a representative firm which also hires labor. Each individual's income is the product of an economy-wide wage w_t and her idiosyncratic labor productivity z_t and her wealth follows (2) with $y_t = w_t z_t$. The total amount of capital supplied in the economy equals the total amount of wealth. In a stationary equilibrium it is given by:

$$K = \int_{\underline{z}}^{\overline{z}} \int_{\underline{a}}^{\infty} ag(a,z)dadz := S(r,w) \qquad (60)$$

Capital depreciates at rate δ. There is a representative firm with a constant returns to scale production function $Y = F(K,L)$. Since factor markets are competitive, the wage and the interest rate are given by:

$$r = \partial_K F(K,1) - \delta, \quad w = \partial_L F(K,1) \qquad (61)$$

where we use that the mean of the stationary distribution of productivities z equals one. The computational algorithm is again unchanged except it imposes (60) and (61) rather than (11).

4 Mean Field Games (MFG)

4.1 Background

Mean field game theory is the study of strategic decision making in very large populations of small interacting agents. In HACT, all the agents are small but have impact on the asset prices when making saving decisions, so it is like a "game".

In continuous time a mean field game is typically composed by a Hamilton–Jacobi–Bellman equation that describes the optimal control problem of an individual and a Fokker–Planck equation that describes the dynamics of the aggregate distribution of agents. Under fairly general assumptions it can be proved that a class of mean field games is the limit as $N \to \infty$ of a N player Nash equilibrium.

4.2 Classical Example of MFG in Economics

We consider a large number of oil producers as price-takers in a perfectly competitive market. Each producer has a random initial reserve of oil $R_0 \sim m(0,\cdot)$. The distribution of oil reserve among producers is $m(t,\cdot)$. For each producer, the oil reserve $R(t)$ evolves with the production choice $q(t)$ together with a stochastic term as:

$$dR(t) = -q(t)dt + \nu R(t)dW_t \quad (*)$$

where ν measures the intensity of the noise. This intensity (or volatility) is supposed to be the same for all producers but the noise is proportional to the oil reserve. Then each producer makes the production choices $q(t)$ to optimize the profit:

$$\max_{(q(t))_t} \mathbb{E}\left[\int_0^\infty e^{-rt}(p(t)q(t) - C(q(t)))dt\right] \quad \text{s.t. } q(t) \geq 0, R(t) \geq 0$$

where:

- C is the production cost function: $C(q) = \alpha q + \beta \frac{q^2}{2}$
- the prices $p(t)$ are determined according to the demand-supply equilibrium

$$D(t,p) = We^{\rho t}p^{-\sigma} = \int m(t,R)q(t,R)dR$$

According to assumptions above, we introduce a Bellman function $u(t,R)$ as:

$$u(t,R) = \max_{(q(s))_{s\geq t}, q\geq 0} \mathbb{E}_t\left[\int_t^\infty (p(s)q(s) - C(q(s)))e^{-r(s-t)}ds\right]$$

$$dR(s) = -q(s)ds + \nu R(s)dW_s, \quad R(t) = R, \quad \forall s \geq t, R(s) \geq 0$$

We can discretize the above bellman equation to discrete from:

$$u(t,R) = \max_{(q(t)), q(t)\geq 0} \left[p(t)q - C(q(t))\right]\Delta t + (1 - r\Delta t)\mathbb{E}_t u(t+\Delta t, R_{t+\Delta t})$$

then we can derive:

$$0 = \Delta t\left[\max_{(q(s))_{s\geq t}, q\geq 0} \left[p(t)q - C(q(t))\right] + \frac{E_t u(t+\Delta t, R_{t+\Delta t}) - u(t,R)}{\Delta t} - rE(t+\Delta t, R_{t+\Delta t})\right]$$

and

$$0 = \max_{(q(s))_{s\geq t}, q\geq 0} \left[p(t)q - C(q(t))\right] + \frac{E_t u(t+\Delta t, R_{t+\Delta t}) - u(t,R)}{\Delta t} - rE(t+\Delta t, R_{t+\Delta t})$$

Using Taylor's expansion and Eq.(*), we obtain the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\partial_t u(t,R) + \frac{\nu^2}{2}R^2\partial^2_{RR}u(t,R) - ru(t,R) + \max_{q\geq 0}\left(p(t)q - C(q) - q\partial_R u(t,R)\right) = 0$$

The Hamiltonian of this problem is $\max_{q\geq 0}\left(p(t)q - C(q) - q\partial_R u(t,R)\right)$. Using the quadratic cost function as above, the optimal control is given by:

$$q^*(t,R) = \left[\frac{p(t) - \alpha - \partial_R u(t,R)}{\beta}\right]_+$$

where $q^*(t,R)$ represents the instantaneous production at time t of a producer with an oil reserve R at time t. It's important to notice that R is the reserve at time t, not the initial reserve.

The Hamilton-Jacobi-Bellman equation can be rewritten with the optimal production:

$$\partial_t u(t,R) + \frac{\nu^2}{2} R^2 \partial^2_{RR} u(t,R) - r u(t,R) + \frac{1}{2\beta}\left[(p(t) - \alpha - \partial_R u(t,R))_+\right]^2 = 0$$

For the distribution of oil reserves at time t $m(t,R)$, it is initially given by $m_0(\cdot)$ and then transported by the optimal production decisions of the agents $q^*(t,R)$. The transport equation (Kolmogorov Forward or Fokker–Planck equation) is:

$$\partial_t m(t,R) + \partial_R\left(-q^*(t,R)m(t,R)\right) = \frac{\nu^2}{2}\partial^2_{RR}\left[R^2 m(t,R)\right]$$

with $m(0,\cdot) = m_0(\cdot)$

These HJB and KF equations are the two classical coupled PDEs of the mean field game theory.

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