Macroeconomic Analysis with Machine Learning and Big Data

# Lecture 5: Structural Models in Macroeconomics (I)

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# 1 The Development of Structural Models in Macroeconomics

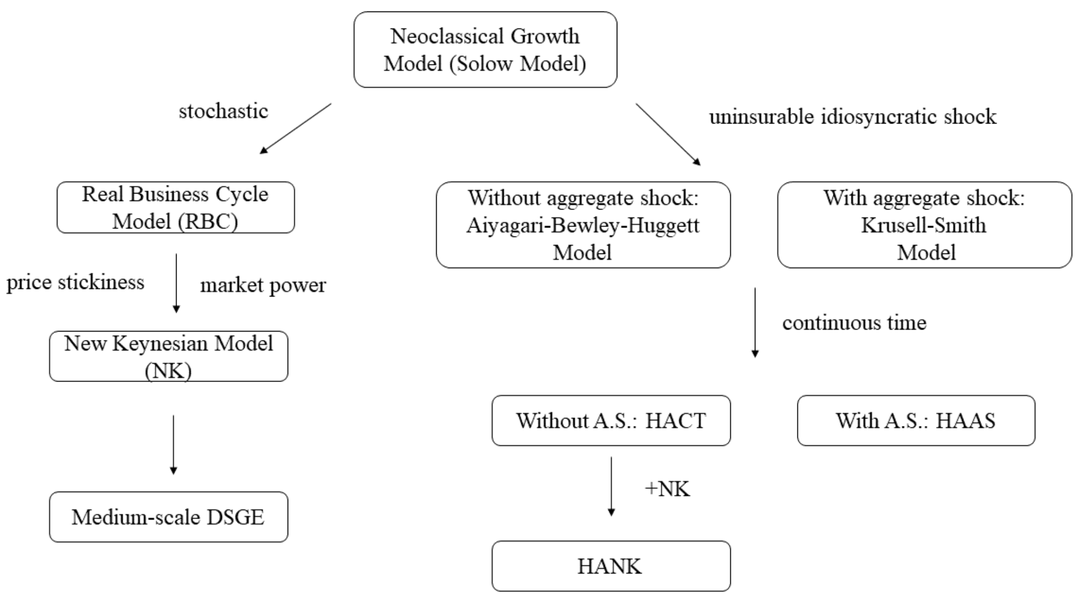


Figure 1: The Development of Structural Models in Macro

# 2 Real Business Cycle (RBC) Model

## 2.1 Planner's problem

$$\max_{\{C_t, K_{t+1}, H_t\}_t} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [logC_t + \psi_t log(1-H_t)]$$

$$\text{s.t.} \qquad K_{t+1} = (1-\delta)K_t + I_t$$

$$C_t + I_t = A_t K_t^{\alpha} H_t^{1-\alpha}$$

where $H_t$ denotes labor input in time $t$. Assume $A_t$ (productivity), $\psi_t$ (preference from leisure) evolve stochastically according to AR(1) processes

$$log\psi_t = log\bar{\psi}(1-\rho) + \rho log\psi_{t-1} + \sigma_\psi \epsilon_t^\psi$$
$$logA_t = \rho logA_{t-1} + \sigma_A \epsilon_t^A$$

## 2.2 The Lagrangian and FOC

Lagrangian:

$$\mathscr{L} : \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{logC_t + \psi_t log(1-H_t) + \lambda_t [A_t K_t^\alpha H_t^{1-\alpha} - C_t - K_{t+1} + (1-\delta)K_t]\}$$

F.O.C:

$$[C_t]: \quad \frac{1}{C_t} = \lambda_t$$

$$[H_t]: \quad \frac{\psi_t}{1-H_t} = \lambda_t (1-\alpha) A_t K_t^\alpha H_t^{1-\alpha}$$

$$[K_{t+1}]: \quad \lambda_t = \beta \mathbb{E}_t [\lambda_{t+1} [A_{t+1} \alpha (\frac{H_{t+1}}{K_{t+1}})^{1-\alpha} + 1 - \delta]]$$

Budget Constraint:

$$[BC]: \quad A_t K_t^\alpha H_t^{1-\alpha} = C_t + K_{t+1} - (1-\delta)K_t$$

We use $\frac{1}{C_t} = \lambda_t$ to substitute $\lambda_t$ and $\lambda_{t+1}$ in equations $[H_t]$ and $[K_{t+1}]$ above.

## 2.3 Steady state

Given initial condition such as $K_0$, can we get $\{C_t, K_{t+1}, H_t\}_{t=0}^{\infty}$? The answer is no, and we will focus on fluctuations around the steady state.

**Definition 2.1.** Steady state: $C_t \equiv C^\star, K_t \equiv K^*, H_t \equiv H^*$.

Using FOCs and BC conditions, we can solve:

1. The steady state value of hours worked:

$$H^* = [1 + \frac{\bar{\psi}}{\alpha} - \frac{\delta\beta\bar{\psi}}{1-\beta(1-\delta)}]^{-1}$$

2. The steady state capital:

$$K^* = H^* [\frac{1-\beta(1-\delta)}{\beta\alpha}]^{-\frac{1}{1-\alpha}}$$

3. The steady state consumption:

$$C^* = (K^*)^\alpha (H^*)^{1-\alpha} - \delta K^*$$

4. The steady state share of output that is invested:

$$S_i \equiv \frac{\delta K^*}{\delta K^* + C^*} = \frac{\delta\beta\alpha}{1-\beta(1-\delta)}$$

5. The steady state share of output that is consumed:

$$S_c = 1 - \frac{\delta\beta\alpha}{1-\beta(1-\delta)}$$

## 2.4 Log linearize

Define $\hat{x}_t = log(\frac{X_t}{X^*}) \approx \frac{X_t - X^*}{X^*}$, then we can rewrite the FOCs and BC as:

$$[H_t]: \quad \hat{\psi}_t + \frac{H^*}{1-H^*}\hat{h}_t = -\hat{c}_t + \hat{a}_t + \alpha\hat{k}_t - \alpha\hat{h}_t$$

$$[K_{t+1}]: \quad -\hat{c}_t = \mathbb{E}_t[-\hat{c}_{t+1} + (1-\beta(1-\delta))(1-\alpha)[\hat{h}_{t+1} - \hat{k}_{t+1}]] + (1-\beta(1-\delta))\mathbb{E}_t\hat{a}_{t+1}$$

$$[BC]: \quad \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t = S_c\hat{c}_t + \frac{S_i}{\delta}[\hat{k}_{t+1} - (1-\delta)\hat{k}_t$$

More details on log-linearization can be found in Eric Sims' notes `https://www3.nd.edu/~esims1/log_linearization_sp17.pdf`

## 2.5 Solving Linear Rational Expectations Models

Plug Equation $[H_t]$ into the other two log-linearized equations $\rightarrow$ 2 x 2 system of linear equations. Matrix form:

$$\mathbf{A}\begin{pmatrix}\mathbb{E}_t[\hat{c}_{t+1}] \\ \hat{k}_{t+1}\end{pmatrix} = \mathbf{B}\begin{pmatrix}\hat{c}_t \\ \hat{k}_t\end{pmatrix} + \mathbf{C}\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix} + \mathbf{Z}\mathbb{E}_t\begin{pmatrix}\hat{a}_{t+1} \\ \hat{\psi}_{t+1}\end{pmatrix}$$

$$\begin{pmatrix}\hat{a}_{t+1} \\ \hat{\psi}_{t+1}\end{pmatrix} = \rho\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix} + \begin{pmatrix}\sigma^A\epsilon_t^A \\ \sigma^\psi\epsilon_t^\psi\end{pmatrix}$$

A general solution method is discussed in Sims (2003) [1]. Here we only present a special case. Define $\mathbf{F} = \mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{G} = [\mathbf{A}^{-1} + \rho\mathbf{Z}]\mathbf{C}$, then

$$\begin{pmatrix}\mathbb{E}_t[\hat{c}_{t+1}] \\ \hat{k}_{t+1}\end{pmatrix} = \mathbf{F}\begin{pmatrix}\hat{c}_t \\ \hat{k}_t\end{pmatrix} + \mathbf{G}\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix} \tag{1}$$

**Next Goal:** Get rid of $\hat{c}_{t+1}$ and write the linearized policy function

$$\hat{k}_{t+1} = M_{kk}\hat{k}_t + M_{ka}\hat{a}_t + M_{k\psi}\hat{\psi}_t$$

$$\hat{c}_t = M_{ck}\hat{k}_t + M_{ca}\hat{a}_t + M_{c\psi}\hat{\psi}_t$$

[1] `http://sims.princeton.edu/yftp/gensys/LINRE3A.pdf`

If $F$ is diagonalizable, we may write

$$F = VDV^{-1}$$

where $D$ is diagonal. Suppose, without loss of generality $D_1 > D_2$. Multiply both sides by $V^{-1}$,we can get

$$V^{-1}\begin{pmatrix}\mathbb{E}_t[\hat{c}_{t+1}] \\ \hat{k}_{t+1}\end{pmatrix} = DV^{-1}\begin{pmatrix}\hat{c}_t \\ \hat{k}_t\end{pmatrix} + V^{-1}G\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}$$

$$\mathbb{E}_t\begin{bmatrix}\Upsilon_{1,t+1} \\ \Upsilon_{2,t+1}\end{bmatrix} = \begin{pmatrix}D_1 & 0 \\ 0 & D_2\end{pmatrix}\begin{bmatrix}\Upsilon_{1,t} \\ \Upsilon_{2,t}\end{bmatrix} + V^{-1}G\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}$$

Take the first equation

$$\mathbb{E}_t\Upsilon_{1,t+1} = D_1\Upsilon_{1,t} + (V^{-1}G)_{1.}\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}$$

Rewrite VAR(1) as VMA: for $\tau$ periods head

$$\mathbb{E}_t\Upsilon_{1,t+\tau} = (D_1)^\tau\Upsilon_{1,t} + \sum_{i=0}^{\tau-1} D_1^{\tau-i-1}(V^{-1}G)_{1.}\mathbb{E}_t\begin{pmatrix}\hat{a}_{t+i} \\ \hat{\psi}_{t+i}\end{pmatrix}$$

Re-arranging this equation:

$$\begin{aligned}\Upsilon_{1,t} &= (D_1)^{-\tau}\mathbb{E}_t\Upsilon_{1,t+\tau} + \sum_{i=0}^{\tau-1} D_1^{-i-1}(V^{-1}G)_{1.}\mathbb{E}_t\begin{pmatrix}\hat{a}_{t+i} \\ \hat{\psi}_{t+i}\end{pmatrix} \\ &= \sum_{i=0}^{\infty} D_1^{-i-1}(V^{-1}G)_{1.}\mathbb{E}_t\begin{pmatrix}\hat{a}_{t+i} \\ \hat{\psi}_{t+i}\end{pmatrix} \\ &= D_1^{-1}(I - D_1^{-1}\rho)^{-1}(V^{-1}G)_{1.}\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}\end{aligned}$$

Need the larger eigenvalue to be bigger than 1 for the sum to converge. By definition of $\Upsilon$:

$$\Upsilon_{1,t} = (V^{-1})_{11}\hat{c}_t + (V^{-1})_{12}\hat{k}_t = D_1^{-1}(I - D_1^{-1}\rho)^{-1}(V^{-1}G)_{1.}\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}$$

Now we can write the policy variable $\hat{c}_t$ as a function of $\hat{k}_t$ and $\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}$. We can also replace the $\hat{c}_t$ in the second line of equation (1) so that we can write $\hat{k}_{t+1}$ as a function of $\hat{k}_t$ and $\begin{pmatrix}\hat{a}_t \\ \hat{\psi}_t\end{pmatrix}$.

# 3 Recursive Method

## 3.1 Model Setup

A simple example of a sequential problem that can be solved by recursive method is

$$\begin{aligned}&\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u(c_t) \qquad (2)\\ &s.t.\ c_t + k_t = f(k_t)\\ &\quad 0 \leq k_{t+1} \leq f(k_t),\ k_0\ given\end{aligned}$$

This can be written as

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) = \max_{\{k_1\}} u(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_{t+1}) - k_{t+2}) \qquad (3)$$

Motivated by this form, we can formulate the Bellman equation

$$V(k) = \max_{k'}\{u(f(k) - k') + \beta V(k')\} \qquad (4)$$

$V(\cdot)$ is the value function of the Bellman equation, $k$ is the state variable, and $k'$ is the control variable. Under certain conditions, the value function of the sequential problem (the total discounted utility) (2) is equivalent to the solution to the functional equation (4).

## 3.2 Value Function Iteration

The functional equation 4 can be solved with various methods. One method is iteration over the value function $V$ by the following algorithm.

- Make a initial guess of $V_0(k)$
- For a given $V_n(k)$, solve the maximization problem

$$V_{n+1}(k) = \max_{k'}\{u(f(k) - k') + \beta V_n(k')\} \qquad (5)$$

- Compare $V_n$ and $V_{n+1}$. If the error falls in the range of tolerance, then $V_{n+1}$ is the solution. Otherwise, go back to step 2 and repeat until convergence.

# 4 Aiyagari-Bewley-Huggett Model

## 4.1 Model Setup

In this model, we only consider the household side, whose income is exogenous and follows a finite state Markov chain. Denote all the possible incomes by $Y = \{y_1, y_2, \cdots y_n\}$. The transitional probability matrix $\Pi = (\pi_{ij})_{N\times N}$, i.e.

$$\mathbb{P}(y_{t+1} = y_j | y_t = i) = \pi_{ij} \qquad (6)$$

In addition, In each period $t$, the household can save and borrow an amount of $a_{t+1}$ at an endogenous interest rate $r_t$. When borrowing, the household faces a debt constraint $a_{t+1} \geq -b$, where $-b$ is a given lower limit. For the whole economy at period $t$, the distribution of households with income $y$ and asset $a$ is denoted by $\Phi_t(a, y)$.
Formally, for a household, the optimization problem can be stated as

$$\begin{aligned} \max\ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ s.t.\ c_t + a_{t+1} = y_t + (1 + r_t)a_t \\ a_{t+1} \geq -b \end{aligned} \tag{7}$$

The corresponding Bellman equation is

$$\begin{aligned} V(a, y, \Phi) = \max_{a'}\{u(c) + \beta \sum_{y' \in Y} \pi(y'|y)V(a', y', \Phi')\} \\ s.t.\ c + a' = y + (1 + r(\Phi))a \\ a' \geq -b \\ \Phi' = H(\Phi) \end{aligned} \tag{8}$$

The law of motion $H$ of the distribution $\Phi_t$ is

$$\Phi'(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y}))\Phi(da, dy) \tag{9}$$

The transitional probability function $Q$ satisfies

$$Q((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \pi(y'|y)\mathbb{I}_{\{a'(a,y) \in \mathcal{A}\}} \tag{10}$$

where $\mathbb{I}$ is an indicator function.

## 4.2 Definition of Recursive Competitive Equilibrium

**Definition** The recursive competition equilibrium (RCE) in this model is:

- Given the interest rate $r_t$. The policy function $a'(a, y, \Phi)$ solves the household optimization problem (8), with the corresponding value function being $V(a, y, \Phi)$.

- Market clearing, i.e., the amount of net saving is 0 and goods market clear for all $\Phi$.

$$\int a'(a, y, \Phi)d\Phi = 0 \tag{11}$$

$$\int c(a, y)d\Phi = \int y d\Phi \tag{12}$$

- Law of motion $H$ for $\Phi_t$ is generated by the exogenous Markov process $\pi$ and the policy function $a'$ as in Equations 9 and 10.

However, since the distribution $\Phi_t$ is time-varying, it is extremely difficult to solve the model even numerically. To address this problem, we further refine our definition of equilibrium to **stationary recursive competitive equilibrium** by imposing the following assumption.

- $\Phi$ is a stationary distribution. That is

$$\Phi_{t+1} = \Phi_t = \Phi^* \tag{13}$$

Given $\Phi$ is unchanged, the interest rate $r_t$ is also time-invariant. Assume $r_t = r^*$

## 4.3 Solving Stationary Equilibrium by Recursive Method

We can solve the stationary equilibrium through the following algorithm

1. Make a initial guess of $r$. Then, solve the Bellman equation of the household problem with standard methods like value function iteration and obtain $V_r$, $a'_r$, and $c_r$.

2. With the policy function $a'$ and the transitional probability matrix $\Pi$, we can get the law of motion $H_r$ for the asset and income joint distribution, and then compute the stationary distribution $\Phi^*_r$.

3. With $\Phi^*_r$, $a'_r$, we can now compute the net saving. If the market clearing condition 11 is satisfied, then the model is solved. If the net saving is great than 0, then we should lower the guess of $r$, and *vice versa*. Go back to step 1 and repeat until convergence.