Shock on Variable or Shock on Distribution with Application to Stress-Tests

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(February, 2012.)

The second author gratefully acknowledges financial support of NSERC, Canada, and of the chair AXA/Risk Foundation : "Large Risks in Insurance". The views expressed in this paper are those of the authors and do not necessarily reflect those of the Banque de France. We thank Jean-Paul Laurent, Martin Schweizer, and the participants at the Banque de France and CREST internal seminars for helpful comments, and J.P. Renne, B. Saes-Escorbiac, and A. Touchais for kindly providing us with the sovereign bond data.

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Abstract

The shocks on a stochastic system can be defined by means of either distribution, or variable. We relate these approaches and provide the link between the global and local effects of both types of shocks. These methodologies are used to perform stress-tests on the portfolio of financial institutions by means of shocks on systematic factors, for which we distinguish the cases of crystallized and optimally updated portfolios. The approach is illustrated by an analysis of the risk of sovereign bonds of the Eurozone.

Keywords : Shock, Copula, Extreme Risk, Stress-Test, Factor Model, Systemic Risk, Portfolio Management, Sovereign Bonds.

1 Introduction

The comparison of risks or the analysis of the effects of shocks on a risky portfolio value are problems concerning a comparison of two distributions. However, these questions are often presented in the economic and finance literature in terms of stochastic variables. This is an abuse of language. It is likely introduced to facilitate the understanding of the notion by the standard reader, but it can also imply misleading interpretations and errors in implementing the notion.
To illustrate this practice, let us consider the notion of second-order stochastic dominance. Rothschild, Stiglitz (1970), Theorem 2, have proposed three equivalent characterizations of this notion. Loosely speaking they consider two variables Y_0, Y_1 with respective cumulative distributions F_0 and F_1. For expository purpose, let us interpret Y_0 and Y_1 as the prices of two assets 0 and 1, or equivalently, as the value of two portfolios completely invested in asset 0 and 1, respectively. The investment Y_0 in asset 0 dominates the investment Y_1 in asset 1 at the second-order if one of the following equivalent conditions i)-iii) is satisfied :

i) $\int_0^K [1 - F_0(y)]\, dy \geq \int_0^K [1 - F_1(y)]\, dy, \; \forall\, K.$

Let us consider the price, under the historical distribution[3], of a European call option written on the investment Y_0 with strike K, that is, the price of a derivative with payoff equal to $Y_0 - K$, if $Y_0 \geq K$, equal to 0, otherwise. We have :

$$
\begin{aligned}
\mathbb{E}\left([Y_0 - K]^+\right) &= \int_K^{+\infty} (y-K) f_0(y) dy \\
&= -\int_K^{+\infty} (y-K) d[1 - F_0(y)] \\
&= -\left[(y-K)(1-F_0(y))\right]_K^{+\infty} + \int_K^{+\infty} [1 - F_0(y)] dy \\
&= \int_K^{+\infty} [1 - F_0(y)] dy \\
&= E(Y_0) - \int_0^K [1 - F_0(y)] dy, \text{ since } Y_0 > 0.
\end{aligned}
$$

Thus, condition i) means that, for a given strike K, a European call option contract written on the price of asset 0 is always cheaper than the similar contract written on asset

[3] Equivalently, this is the price for a risk-neutral investor.

1, whenever Y_0 and Y_1 have the same mean. Therefore, the investment Y_0 dominates Y_1 under i), since the price of the insurance against a drop in investment's value is lower for asset 0.

ii) $\mathbb{E}\left[U\left(Y_0\right)\right] \geq \mathbb{E}\left[U\left(Y_1\right)\right]$, for any bounded concave function U.

Condition ii) means that any risk-averse agent, i.e. with concave utility function, would prefer asset 0 to asset 1.

iii) There exists a variable Z such that : $Y_1 = Y_0 + Z$, with $\mathbb{E}\left[Z|Y_0\right] = 0$.

The first and second characterizations show clearly that the concept of stochastic dominance concerns the distributions. The third characterization seems to be of a different type. It says that we can pass from Y_0 to Y_1 by adding a stochastic shock with zero conditional mean. It concerns variables themselves, but on an extended space, since the comparison of the marginal distributions of Y_0 and Y_1 implicit in i) and ii) is now replaced by a condition, which involves the joint distribution of (Y_0, Y_1) on the product space. As seen from the proof in Rothschild, Stiglitz (1970), the third characterization can only be obtained after having constructed an artificial product space [see also Strassen (1965), Armbruster (2011)].
In Section 2, we consider a parametric family of cumulative distribution functions $(F_\delta, \delta \in I)$, and we show that they can always be considered as the marginal distributions of a family of variable Y_δ defined by a stochastic equation $Y_\delta = h\left(Y_0, \varepsilon; \delta\right)$, where ε is a variable independent of Y_0. As in Rothschild, Stiglitz, the result requires the construction of an artificial product space. The "equivalent" representation of the family of distributions allows to understand why shocks can be defined in terms of either parameter, or distribution, or variable.
The effect of a small change of δ on the distribution is usually studied by considering a Taylor expansion of distribution F_δ with respect to δ. In Section 3 we show how an equivalent expansion can be performed in terms of variables and relate the two types of expansions. Both types of expansions are extended in Section 4 to some functionals of distribution F_δ. More precisely, we consider the effect of shocks on portfolio management, that is, on both the objective function of the investor and on her/his optimal portfolio

allocation.
Some examples of specification of shocks are given in Section 5. While common practices usually perform stress-testing exercises by considering deterministic shock on crystallized portfolio value, we show that our results allow for much richer exercises. As an illustration, the stress-test methodology is applied in Section 6 to portfolios of sovereign bonds. We consider sovereign bonds for different countries on period 2001-2011, and extract the underlying factors by a principal component analysis. The distribution of these factors on periods 2001-2007 and 2007-2011 shows a significant change due to the recent financial crisis. Then we consider period 2001-2007 as a benchmark and shock the first factor by contaminating the benchmark distribution with crisis specific distribution. We analyze the effects of this contamination on a crystallized portfolio and an optimally updated portfolio, both for shocks in distribution and in variable. Section 7 concludes. The technical proofs are gathered in Appendices.

2 Family of Distributions or Family of Variables

The aim of this section is to relate a parametric modeling written in terms of distributions and a parametric modeling written in terms of stochastic variables. For expository purpose, we consider continuous distributions on $\mathbb{R}$ and a scalar parameter δ (the extension to multivariate distributions is given in Appendix 1 i)).
The first type of modeling defines a family of distributions $\{F_\delta, \delta \in I\}$, where I denotes an interval of $\mathbb{R}$ and F_δ the cumulative distribution function. The second modeling is based on some relationship between variables : $Y_\delta = h\left(Y_0, \varepsilon; \delta\right), \delta \in I$, say, where Y_0 and ε are independent variables with Y_0 following F_0 and ε following the uniform distribution on $[0, 1]$. The questions solved in this section are the following ones :

i) Given the parametric family $\{F_\delta, \delta \in I\}$, is it always possible to find a variable ε and a function h such that the marginal distribution of Y_δ is F_δ, for any $\delta \in I$?

ii) Is such a function h unique, if it exists?

2.1 Copula

Whereas the modeling in distribution involves the one-dimensional space $(\mathbb{R}, \mathbb{B}(\mathbb{R}))$, the modeling in variable involves a two dimensional space $\left(\mathbb{R}^2, \mathbb{B}\left(\mathbb{R}^2\right)\right)$. Thus, we have first to introduce such a bidimensional artificial space. Let us denote by (U, V) a pair of variables on this space with marginal distributions which are uniform on $[0, 1]$, and a joint c.d.f. $C(u, v) = \mathbb{P}\left[U < u, V < v\right], u, v \in [0, 1]$. The variables are usually called rank variables and function C is the copula cumulative function.
Let us first consider two values, 0 and 1, say, of parameter δ, and the associated distributions F_0 and F_1. By Sklar theorem [Sklar (1959)], the variables :

$$Y_0 = F_0^{-1}(U), Y_1 = F_1^{-1}(V), \tag{2.1}$$

have the joint c.d.f. :

$$\mathbb{P}\left[Y_0 < y_0, Y_1 < y_1\right] = C\left[F_0(y_0), F_1(y_1)\right]. \tag{2.2}$$

In particular the marginal distribution of Y_0 (resp. Y_1) is F_0 (resp. F_1).

Let us now consider the conditional c.d.f. of Y_1 given Y_0. We have [see Joe (1997), p.245] :

$$F_{1|0}(y_1|Y_0) \equiv \mathbb{P}[Y_1 < y_1|Y_0] = \frac{\partial C}{\partial u}[F_0(Y_0), F_1(y_1)]. \tag{2.3}$$

Thus, by using the inverse transform method, the variable :

$$\varepsilon = F_{1|0}(Y_1|Y_0) = \frac{\partial C}{\partial u}[F_0(Y_0), F_1(Y_1)], \tag{2.4}$$

follows a uniform distribution on $[0,1]$ and is independent of Y_0. We deduce the expected expression in terms of variables :

$$\begin{aligned} Y_1 &= \left(F_{1|0}(\bullet|Y_0)\right)^{-1}(\varepsilon) \\ &= \left(\frac{\partial C}{\partial u}[F_0(Y_0), F_1(\bullet)]\right)^{-1}(\varepsilon) \\ &= h(Y_0, \varepsilon; 1), \text{ say.} \end{aligned} \tag{2.5}$$

Thus we have shown the existence of function h. Moreover, by increasing the dimension of the space, we allow for a variety of choices of copula C and then of function h. Finally, equation (2.4) can also be written as :

$$\varepsilon = \frac{\partial C}{\partial u}(U, V),$$

which shows how the uniform variable ε depends on the basic uniform variables U and V in a complicated nonlinear way.

2.2 Extension to families

The result of Section 2.1 can be applied to any pair of parameter values $(0, \delta)$, associated distributions (F_0, F_δ), and variables (Y_0, Y_δ). We get :

$$\varepsilon_\delta = \frac{\partial C_\delta}{\partial u}[F_0(Y_0), F_\delta(Y_\delta)], \delta \in (0,1),$$

and

$$Y_\delta = \left(\frac{\partial C_\delta}{\partial u}[F_0(Y_0), F_\delta(\bullet)]\right)^{-1}(\varepsilon_\delta), \delta \in (0,1), \tag{2.6}$$

where the copula is now indexed by parameter δ. All variables ε_δ are uniformly distributed on $[0,1]$ and independent of Y_0. In practice the shock defined in terms of variable involves an innovation ε, which is independent of δ. This is easily derived if we consider equality in distribution. Indeed, equation (2.6) implies the equality in distribution

$$Y_\delta \stackrel{d}{=} h\left(Y_0, \varepsilon; \delta\right), \delta \in [0,1], \tag{2.7}$$

where

$$h\left(Y_0, \varepsilon; \delta\right) = \left(\frac{\partial C_\delta}{\partial u}\left[F_0\left(Y_0\right), F_\delta\left(\bullet\right)\right]\right)^{-1}(\varepsilon), \tag{2.8}$$

ε is independent of δ, of Y_0, and uniformly distributed on [0,1].

2.3 Shocks

Let us now discuss the introduction of shocks.

i) The shock can be defined by means of parameter δ. For instance, the parameter can pass from value 0 to value δ, say.

ii) The corresponding effect on distribution will be the change from F_0 to F_δ.

iii) Let us finally consider the variable interpretation. Equality (2.7) in distribution has to be replaced by an equality in terms of variables. In fact, we can define

$$Y_\delta = h\left(Y_0, \varepsilon; \delta\right),$$

with h satisfying (2.8), whenever the following coherency condition is satisfied :

$$Y_0 = h\left(Y_0, \varepsilon; 0\right). \tag{2.9}$$

Then the shock on variables is $Y_\delta - Y_0 = h\left(Y_0, \varepsilon; \delta\right) - Y_0$. For a given Y_0, this is in general a stochastic shock due to the effect of the uniform stochastic variable ε.
The coherency condition (2.9) implies restrictions on the choice of copula in (2.8). Let us for instance consider the Gaussian copula with correlation parameter $\rho(\delta)$, and variables Y_0, Y_δ, whose marginal distributions are $F(y;0)$, $F(y;\delta)$, respectively. We have (see Appendix 4 i)) :

$$\varepsilon = \Phi\left(\frac{\Phi^{-1}\left[F(Y_\delta;\delta)\right] - \rho(\delta)\Phi^{-1}\left[F(Y_0;0)\right]}{\sqrt{1-\rho^2(\delta)}}\right),$$

where Φ is the standard Gaussian c.d.f. Thus :

$$\Phi^{-1}\left[F(Y_\delta;\delta)\right] = \rho(\delta)\Phi^{-1}\left[F(Y_0;0)\right] + \sqrt{1-\rho^2(\delta)}\Phi^{-1}\left(\varepsilon\right). \tag{2.10}$$

Equation (2.10) shows that the coherency condition restricts the Gaussian copula to be such that $\rho(0) = 1$.

Finally note that the definition of the direction of the shock is more accurate with the specification in variable. Indeed, it specifies $Y_\delta - Y_0$ with respect to Y_0. Thus it requires the specification of the joint distribution of Y_0 and Y_δ, whereas the specification in terms of distribution demands the unconditional distributions only. This explains why different specifications of shocks in terms of variable can lead to a same specification in terms of distribution.

3 Local Analysis

We have seen in Section 2 how to link the approaches in distributions and in variables in a global analysis of shocks. However, extreme effects can result from small shocks when the system is nonlinear. In applications to finance, these nonlinearities are due to derivatives (call options, credit derivatives) included in the portfolio as well as the nonlinear portfolio management strategies. This explains why a global analysis has to be completed by a local analysis.
The effect of a small change[4] in δ is usually treated by considering appropriate Taylor expansions. These expansions can be done from the distributions themselves, or from the interpretation in terms of variables. We consider below these expansions in a neighborhood of $\delta = 0$.

3.1 Expansion of the distribution

Let us denote $f(y;\delta)$ the density function corresponding to F_δ. Under standard regularity conditions, we get the Taylor expansion at order p :

$$f(y;\delta) = f(y;0) + \sum_{j=1}^{p} \frac{\delta^j}{j!} \frac{\partial^j f(y;0)}{\partial \delta^j} + o\left(\delta^p\right), \tag{3.1}$$

where $o(.)$ denotes a deterministic negligible term.

In particular, we get at second-order :

$$\begin{aligned} f(y;\delta) &= f(y;0) + \delta \frac{\partial f(y;0)}{\partial \delta} + \frac{\delta^2}{2} \frac{\partial^2 f(y;0)}{\partial \delta^2} + o\left(\delta^2\right) \\ &= f(y;0) \left\{ 1 + \delta \frac{\partial \log f(y;0)}{\partial \delta} + \frac{\delta^2}{2} \left[\frac{\partial^2 \log f(y;0)}{\partial \delta^2} + \left(\frac{\partial \log f(y;0)}{\partial \delta} \right)^2 \right] \right\} + o\left(\delta^2\right), \end{aligned}$$

[see Chesher (1983), (1984) for an application of this expansion for testing neglected heterogeneity].

[4] A small change in δ implies a large change on variable Y_0, if the stochastic direction of the shock concerns extreme risks (see Section 5).

3.2 Expansion in terms of variable

Let us now consider the model :

$$Y_\delta = h\left(Y_0, \varepsilon; \delta\right).$$

We could apply a Taylor expansion at order p to get :

$$Y_\delta = Y_0 + \sum_{j=1}^{p} \left(\frac{\delta^j}{j!} \frac{\partial^j h\left(Y_0, \varepsilon; 0\right)}{\partial \delta^j} \right) + o_{\mathbb{P}}\left(\delta^p\right), \tag{3.2}$$

where $o_{\mathbb{P}}(.)$ denotes a negligible term in probability.

However, such an expansion would be difficult to interpret in terms of distributions. Instead, we consider below the approximate computation of an expectation $\mathbb{E}\left[g(Y_\delta)\right]$, where g is an infinitely differentiable function with compact support. We get :

$$\begin{aligned} \mathbb{E}\left[g(Y_\delta)\right] &= \mathbb{E}\left[g(h\left(Y_0, \varepsilon; \delta\right))\right] \\ &= \mathbb{E}\left[g(Y_0)\right] + \sum_{j=1}^{p} \left\{ \frac{\delta^j}{j!} \mathbb{E}\left[\frac{\partial^j}{\partial \delta^j} g(h\left(Y_0, \varepsilon; \delta\right)) \right]_{\delta=0} \right\} + o\left(\delta^p\right). \end{aligned}$$

Then, we can apply Faa di Bruno's formula [see Faa di Bruno (1855), Johnson (2002), Spindler (2005)], which provides the j-th derivative of a composite function.

Lemma 1 : Faa di Bruno's formula

$$\frac{d^n}{dt^n} g\left[h(t)\right] = \sum_{\mathrm{Dio}_n} \frac{n!}{k_1! k_2! ... k_n!} g^{(k)}\left[h(t)\right] \left(\frac{h^{(1)}(t)}{1!} \right)^{k_1} \left(\frac{h^{(2)}(t)}{2!} \right)^{k_2} ... \left(\frac{h^{(n)}(t)}{n!} \right)^{k_n},$$

where $g^{(k)}$ denotes the k^{th} derivative of function g, and where the sum is over all nonnegative integer solutions of the Diophantine equation : $k_1 + 2k_2 + ... + nk_n = n$, and where $k = k_1 + ... + k_n$.

We deduce that :

$$\mathbb{E}\left[g(Y_\delta)\right] = \mathbb{E}\left[g(Y_0)\right] + \sum_{j=1}^{p} \left\{ \frac{\delta^j}{j!} \sum_{k=1}^{j} \mathbb{E}\left[g^{(k)}(Y_0) A_{j,k}(Y_0, \varepsilon) \right] \right\} + o\left(\delta^p\right),$$

where

$$A_{j,k}\left(Y_0,\varepsilon\right)=\sum_{\mathrm{Dio}_{j,k}}\left\{\frac{j!}{k_1!k_2!...k_j!}\left(\frac{1}{1!}\frac{\partial h(Y_0,\varepsilon,0)}{\partial\delta}\right)^{k_1}...\left(\frac{1}{j!}\frac{\partial^j h(Y_0,\varepsilon,0)}{\partial\delta^j}\right)^{k_j}\right\}, \tag{3.3}$$

and the sum is over $k_1,...,k_j$ such that $k_1+2k_2+...+jk_j=j$ and $k_1+...+k_j=k$.

We can also write by the Iterated Expectation Theorem :

$$\mathbb{E}\left[g(Y_\delta)\right]=\mathbb{E}\left[g(Y_0)\right]+\sum_{j=1}^{p}\left\{\frac{\delta^j}{j!}\sum_{k=1}^{j}\mathbb{E}\left[g^{(k)}(Y_0)a_{j,k}(Y_0)\right]\right\}+o\left(\delta^p\right), \tag{3.4}$$

where

$$a_{j,k}(Y_0)=\mathbb{E}\left[A_{j,k}\left(Y_0,\varepsilon\right)|Y_0\right]. \tag{3.5}$$

Finally the following Lemma is proved in Appendix 2.

Lemma 2 : We have

$$\mathbb{E}\left[g^{(k)}(Y_0)a(Y_0)\right]=(-1)^k\mathbb{E}\left[\frac{g(Y_0)}{f(Y_0;0)}\frac{d^k}{dy^k}\left[a(Y_0)f(Y_0;0)\right]\right],$$

for any k and function a.

Thus, we can rewrite equation (3.4) as :

$$\mathbb{E}\left[g(Y_\delta)\right]=\mathbb{E}\left[g(Y_0)\right]+\sum_{j=1}^{p}\left\{\frac{\delta^j}{j!}\sum_{k=1}^{j}(-1)^k\mathbb{E}\left[\frac{g(Y_0)}{f(Y_0;0)}\frac{d^k}{dy^k}\left[a_{j,k}(Y_0)f(Y_0;0)\right]\right]\right\}+o\left(\delta^p\right). \tag{3.6}$$

The knowledge of the expectation $\mathbb{E}\left[g(Y_\delta)\right]$ for all infinitely differentiable functions with compact support characterizes the distribution of Y_δ. Thus, by comparing expansions (3.1) and (3.6), we see how the Taylor expansion in terms of distribution can be interpreted in terms of variable.

Proposition 1 : We have

$$\frac{\partial^j}{\partial\delta^j}\left[f(y;0)\right]=\sum_{k=1}^{j}(-1)^k\frac{\partial^k}{\partial y^k}\left[a_{j,k}(y)f(y;0)\right],$$

where $a_{j,k}$ is given in (3.5) and (3.3).

For instance, the first and second-order derivatives of a composite function are :

$$\begin{aligned}\tfrac{d}{dt}g[h(t)] &= g^{(1)}[h(t)]h^{(1)}(t),\\ \tfrac{d^2}{dt^2}g[h(t)] &= g^{(2)}[h(t)]\left(h^{(1)}(t)\right)^2+g^{(1)}\left[h(t)\right]h^{(2)}(t).\end{aligned}$$

Thus we get :

$$\begin{aligned}\mathbb{E}\left[g(Y_\delta)\right] &= \mathbb{E}\left[g(Y_0)\right]-\delta\mathbb{E}\left[g^{(1)}(Y_0)\tfrac{\partial h(Y_0,\varepsilon,0)}{\partial\delta}\right]\\ &+ \tfrac{\delta^2}{2}\mathbb{E}\left[g^{(2)}(Y_0)\left(\tfrac{\partial h(Y_0,\varepsilon,0)}{\partial\delta}\right)^2+g^{(1)}(Y_0)\tfrac{\partial^2 h(Y_0,\varepsilon,0)}{\partial\delta^2}\right]+o\left(\delta^2\right).\end{aligned}$$

We deduce the following second-order expansion of the p.d.f :

$$\begin{aligned}f(y;\delta) &= f(y;0)-\delta\frac{d}{dy}\left[f(y;0)\mathbb{E}\left[\frac{\partial h\left(Y_0,\varepsilon,0\right)}{\partial\delta}|Y_0=y\right]\right] \qquad (3.7)\\ &+ \frac{\delta^2}{2}\left\{\frac{d^2}{dy^2}\left[f(y;0)\mathbb{E}\left[\left(\frac{\partial h\left(Y_0,\varepsilon,0\right)}{\partial\delta}\right)^2|Y_0=y\right]\right]-\frac{d}{dy}\left[f(y;0)\mathbb{E}\left[\frac{\partial^2 h\left(Y_0,\varepsilon,0\right)}{\partial\delta^2}|Y_0=y\right]\right]\right\}\\ &+ o\left(\delta^2\right).\end{aligned}$$

The expansion in terms of variable is greatly simplified when the shock in variable is linear in δ.

Corollary 1 : Let us assume that $Y_\delta = Y_0+\delta Z\left(Y_0,\varepsilon\right)$, say, and denote $\mu_p\left(Y_0\right) = \mathbb{E}\left[Z^p\left(Y_0,\varepsilon\right)|Y_0\right]$ the conditional power moments of the stochastic shock Z. We get :

$$f(y;\delta)=f(y;0)+\sum_{j=1}^{p}\left[\frac{\delta^j}{j!}(-1)^j\frac{d^j}{dy^j}\left[f(y;0)\mu_j(y)\right]\right]+o\left(\delta^p\right).$$

This specific expansion has been first derived in the literature by Martin, Wilde (2002), Theorem C, based on the analysis of the moment generating function of variable Y_δ.
For instance, let us assume that $Z(Y_0,\varepsilon)=a(Y_0)U$, with $U=\Phi^{-1}(\varepsilon)$ and Φ is the c.d.f. of the standard normal. Then, we get :

$$\mu_p(Y_0)=a^p(Y_0)\mathbb{E}(U^p)=a^p(Y_0)2^{-p/2}\frac{p!}{(p/2)!},\ \text{for } p=2n, n\in\mathbb{N}-\{0\},\ 0,\ \text{otherwise}.$$

We deduce that :

$$f(y;\delta) \quad = \quad f(y;0) + \sum_{j=1}^{p} \left[\frac{\delta^{2j}}{2^j (j!)} \frac{d^{2j}}{dy^{2j}} \left[f(y;0) a^{2j}(y) \right] \right] + o\left(\delta^{2p}\right).$$

The expansion of the p.d.f for a shock in terms of variable has been derived above in an indirect way. In special cases, it is possible to get it in a direct way, but the computation is rather cumbersome (see Appendix 3).

4 Shock on Risk Management

In this section, we apply the previous characterizations of shocks to risk management problems, and derive the closed-form expressions of the local effects of a shock, both in terms of distribution and variable.

4.1 Risk management

Let us consider a given investor, who allocates his/her wealth W_t at time t among N risky assets and 1 risk-free asset. The vector P_t collects the prices at t of the risky assets, while the risk-free asset worths 1 at t and pays $1 + r_{f,t}$ to the investor in the following period, where $r_{f,t}$ denotes the risk-free rate. The investor's budget constraint at date t is :

$$W_t = \alpha_f + \alpha' P_t, \tag{4.1}$$

where α_f is the amount invested in the risk-free asset and α is the allocation in the risky assets. The portfolio value at the next period is :

$$\begin{aligned} W_{t+1} &= \alpha_f \left(1 + r_{f,t}\right) + \alpha' P_{t+1} \\ &= W_t \left(1 + r_{f,t}\right) + \alpha' \left[P_{t+1} - P_t \left(1 + r_{f,t}\right)\right] \\ &= \tilde{W}_t + \alpha' Y_{t+1}, \end{aligned} \tag{4.2}$$

where Y_{t+1} is the vector of excess gains on the period $[t, t+1]$, and $\tilde{W}_t$ is the future value of a portfolio entirely invested in the risk-free asset.
Let us consider a family of distributions F_δ for the excess gains on asset prices Y_{t+1}, and denote $f_\delta = f(y_{t+1}; \delta)$ the corresponding density function. The results can be written in terms of either distribution, or variable. For this latter approach, we have $Y_{t+1} = h\left(Y_0, \varepsilon; \delta\right)$.
We assume that the investor chooses his/her optimal asset allocation with respect to a general criterion G on the portfolio allocation α and on the distribution f_δ of excess gains Y_{t+1}. We denote $\alpha^*(f_\delta)$ the optimal allocation, corresponding to distribution f_δ :

$$\alpha^*(f_\delta) = \arg\max_{\alpha} G\left(\alpha, f_\delta\right). \tag{4.3}$$

Several criteria have been considered in the literature. Some are written below in terms of both distribution and variable.

i) Expected utility approach :
Distribution based approach :

$$G(\alpha, f_\delta) = \mathbb{E}_\delta\left(U\left[\tilde{W}_t + \alpha' Y_{t+1}\right]\right),$$

where the utility function $U[\bullet]$ is increasing and concave in investor's wealth.
Variable based approach :

$$G(\alpha, f_\delta) = \mathbb{E}\left(U\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta)\right]\right).$$

ii) Mean-Variance approach
Distribution based approach :

$$G(\alpha, f_\delta) = \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right) - \frac{\gamma}{2}\mathbb{V}_\delta\left(\alpha' Y_{t+1}\right),$$

where $\mathbb{E}_\delta(\bullet)$ and $\mathbb{V}_\delta(\bullet)$ are the expectation and variance operators with respect to distribution f_δ, and γ has the interpretation of an absolute risk-aversion..
Variable based approach :

$$G(\alpha, f_\delta) = \mathbb{E}\left[\alpha' h(Y_0, \varepsilon; \delta)\right] - \frac{\gamma}{2}\mathbb{V}\left[\alpha' h(Y_0, \varepsilon; \delta)\right],$$

where the expectation and variance operators are taken with respect to both random variables Y_0 and ε.

iii) Mean-Expected Shortfall approach [mean-ES thereafter] :
Distribution based approach :

$$G(\alpha, f_\delta) = \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right) - \zeta\mathbb{E}_\delta^2\left(\alpha' Y_{t+1} | \alpha' Y_{t+1} < Q_{q,\delta}(\alpha' Y_{t+1})\right),$$

where $Q_{q,\delta}$ is the quantile function of $\alpha' Y_{t+1}$ with respect to distribution f_δ, at risk level q.
Variable based approach :

$$G(\alpha, f_\delta) = \mathbb{E}\left[\alpha' h(Y_0, \varepsilon; \delta)\right] - \zeta\mathbb{E}^2\left[\alpha' h(Y_0, \varepsilon; \delta) | \alpha' h(Y_0, \varepsilon; \delta) < Q_q(\alpha' h(Y_0, \varepsilon; \delta))\right],$$

where Q_q is the quantile function of $\alpha' h(Y_0, \varepsilon; \delta)$.

The mean-ES approach is usually described by means of a constrained optimization problem :

$$\max_\alpha \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right)$$

$$\text{s.t. } \mathbb{E}_\delta\left[\alpha' Y_{t+1} | \alpha' Y_{t+1} < Q_{q,\delta}\left(\alpha' Y_{t+1}\right)\right] < ES_0,$$

where ES_0 is a risk threshold. This problem is equivalent to :

$$\max_\alpha a\left[\mathbb{E}_\delta\left(\alpha' Y_{t+1}\right)\right]$$

$$\text{s.t. } b\left[\mathbb{E}_\delta\left(\alpha' Y_{t+1} | \alpha' Y_{t+1} < Q_{q,\delta}\left(\alpha' Y_{t+1}\right)\right)\right] < b\left[ES_0\right],$$

where a and b are two increasing functions. Therefore, there exist a multiplicity of Lagrangean associated with this constrained optimization, that are :

$$a\left[\mathbb{E}_\delta\left(\alpha' Y_{t+1}\right)\right] - \zeta b\left[\mathbb{E}_\delta\left(\alpha' Y_{t+1} | \alpha' Y_{t+1} < Q_{q,\delta}(\alpha' Y_{t+1})\right)\right],$$

where ζ denotes the Lagrange multiplier. As seen later on, we have chosen $a =$ identity, $b(y) = y^2$ to get a Lagrangean, which is a strictly concave function of the portfolio allocation.

4.2 Expansion in terms of distribution

We consider below the expansion of the optimal allocation and of the criterion function in terms of distribution.

4.2.1 Expansion of the criterion function

As a functional of f, $G(\alpha, f)$ admits under suitable conditions a first-order Taylor expansion :

$$G\left(\alpha, f_\delta\right) = G\left(\alpha, f_0\right) + \int \frac{\partial G\left(\alpha, f(y;0)\right)}{\partial f} df_\delta(y) + o(df_\delta),$$

where $\frac{\partial G(\alpha, f(\bullet;0))}{\partial f}$ is the functional (Hadamard) derivative[5] of $G(\alpha, \bullet)$ at f_0.

[5] Let us consider a functional $h \to \Phi(h)$ defined on a Banach space $\mathcal{H}$ with norm $\|.\|_{\mathcal{H}}$, and taking its values in $(\mathbb{R}, ||.||)$. This functional is Hadamard differentiable if and only if for any vectors h_0, h^* in $\mathcal{H}$, there exists a real $\Phi^{(1)}(h_0, h^*)$ such that :

$$\lim_{\delta \to 0} \left\| \frac{\Phi(h_0 + \delta h^*_\delta) - \Phi(h_0)}{\delta} - \Phi^{(1)}(h_0, h^*) \right\| = 0,$$

for any sequence h^*_δ such that $\lim_{\delta \to 0} \|h^*_\delta - h^*\|_{\mathcal{H}} = 0$.
$\Phi^{(1)}(h_0, h^*)$ is called the Hadamard derivative of functional Φ at h_0 in the direction h^* [see e.g. Van der Vaart (1998)].

4.2.2 Expansion of the optimal allocation

The optimal allocation satisfies the first-order condition : $\frac{\partial G(\alpha^*(f_\delta),f_\delta)}{\partial \alpha} = 0$. This is an implicit condition, which can be expanded to derive the local behavior of the optimal allocation. We get :

$$\frac{\partial^2 G\left(\alpha^*(f_0), f_0\right)}{\partial \alpha \partial \alpha'} d\alpha^*(f_\delta) + \int \frac{\partial^2 G\left(\alpha^*(f_0), f(y;0)\right)}{\partial \alpha \partial f} df_\delta(y) + o(df_\delta) = 0.$$

Therefore,

$$d\alpha^*(f_\delta) = \left[-\frac{\partial^2 G\left(\alpha^*(f_0), f_0\right)}{\partial \alpha \partial \alpha'} \right]^{-1} \int \frac{\partial^2 G\left(\alpha^*(f_0), f(y;0)\right)}{\partial \alpha \partial f} df_\delta(y) + o(df_\delta). \tag{4.4}$$

4.2.3 Expansion of the optimal value of the criterion

Finally, the optimal value of the criterion is :

$$G^*(f_\delta) = G\left(\alpha^*(f_\delta), f_\delta\right).$$

Its expansion is :

$$\begin{aligned} G^*(f_\delta) &= G\left(\alpha^*(f_0), f_0\right) + \frac{\partial G\left(\alpha^*(f_0), f_0\right)}{\partial \alpha} d\alpha^*(f_\delta) + \int \frac{\partial G\left(\alpha^*(f_0), f(y;0)\right)}{\partial f} df_\delta(y) + o(df_\delta) \\ &= G\left(\alpha^*(f_0), f_0\right) + \int \frac{\partial G\left(\alpha^*(f_0), f(y;0)\right)}{\partial f} df_\delta(y) + o(df_\delta), \end{aligned} \tag{4.5}$$

since $\frac{\partial G(\alpha^*(f_0),f_0)}{\partial \alpha} = 0$ at the optimum, which is the envelop theorem [see e.g. Sydsaeter, Hammond (2008)].

4.2.4 Examples

As an illustration, let us consider the objective functions presented in Section 4.1. For each of them, we provide below their relevant derivatives :

When a functional Φ is Hadamard differentiable, we get a first-order Taylor expansion of the type :

$$\Phi(h_0 + dh) = \Phi(h_0) + \left\langle \frac{\partial \Phi(h_0)}{\partial h}, dh \right\rangle + o\left(\|dh\|_{\mathcal{H}}\right),$$

where $\left\langle \frac{\partial \Phi(h_0)}{\partial h}, . \right\rangle$ is a linear form on $\mathcal{H}$. This linear form is characterized by the Hadamard derivative $h^* \to \Phi^{(1)}(h_0, .)$.

i) Expected utility approach :

$$\frac{\partial G\left(\alpha,f(y;0)\right)}{\partial f}=U[\tilde{W}_t+\alpha' y]$$

$$\tfrac{\partial^2 G(\alpha,f(y;0))}{\partial f\partial\alpha}=U^{(1)}[\tilde{W}_t+\alpha' y]y, \qquad \tfrac{\partial^2 G(\alpha^*(f_0),f_0)}{\partial\alpha\partial\alpha'}=\mathbb{E}_0\left(U^{(2)}[\tilde{W}_t+\alpha^{*\prime}(f_0)Y_{t+1}]Y_{t+1}Y'_{t+1}\right).$$

where $U^{(k)}[\bullet]$ is the k-th order derivative of $U[\bullet]$.

ii) Mean-Variance approach :

$$\frac{\partial G\left(\alpha,f(y;0)\right)}{\partial f}=\alpha' y\left\{1-\frac{\gamma}{2}\left[\alpha' y-2\mathbb{E}_0\left(\alpha' Y_{t+1}\right)\right]\right\},$$

$$\tfrac{\partial^2 G(\alpha,f(y;0))}{\partial f\partial\alpha}=y-\gamma\left\{\alpha' y\left[y+\mathbb{E}_0\left(Y_{t+1}\right)\right]-y\mathbb{E}_0\left(\alpha' Y_{t+1}\right)\right\}, \qquad \tfrac{\partial^2 G(\alpha^*(f_0),f_0)}{\partial\alpha\partial\alpha'}=-\gamma\mathbb{V}_0\left[Y_{t+1}\right].$$

iii) Mean-ES approach :

The computation of the relevant derivatives for the mean-ES approach requires the first and second-order derivatives of the expected shortfall :

$$ES_{q,\delta,\alpha}=\mathbb{E}_\delta\left(\alpha' Y_{t+1}|\alpha' Y_{t+1}<Q_{q,\delta}(\alpha' Y_{t+1})\right).$$

They have been derived in Scaillet (2004), and Bertsimas, Lauprete, Samarov (2004) following the approach used for the derivatives of the VaR [Gourieroux, Laurent, Scaillet (2000)]. We have :

$$\begin{aligned}
\frac{\partial ES_{q,\delta,\alpha}}{\partial\alpha} &= \mathbb{E}_\delta\left(Y_{t+1}|\alpha' Y_{t+1}<Q_{q,\delta}(\alpha' Y_{t+1})\right),\\
\frac{\partial^2 ES_{q,\delta,\alpha}}{\partial\alpha\partial\alpha'} &= \frac{1}{q}f_{\alpha' Y_{t+1},\delta}\left(Q_{q,\delta}\left(\alpha' Y_{t+1}\right)\right)\mathbb{V}_\delta\left(Y_{t+1}|\alpha' Y_{t+1}=Q_{q,\delta}\left(\alpha' Y_{t+1}\right)\right),
\end{aligned}$$

where $f_{\alpha' Y_{t+1},\delta}$ denotes the probability density function of $\alpha' Y_{t+1}$. In particular, the Hessian of the expected shortfall is a symmetric positive semi-definite matrix, with a kernel generated by the vector α of portfolio allocation.

We deduce :

$$\begin{aligned}
\frac{\partial G\left(\alpha,f(y;0)\right)}{\partial\alpha} &= \mathbb{E}_0\left(Y_{t+1}\right)-2\zeta ES_{q,0,\alpha}\frac{\partial ES_{q,0,\alpha}}{\partial\alpha},\\
\frac{\partial^2 G\left(\alpha,f(y;0)\right)}{\partial\alpha\partial\alpha'} &= -2\zeta\left\{\frac{\partial ES_{q,0,\alpha}}{\partial\alpha}\frac{\partial ES_{q,0,\alpha}}{\partial\alpha'}+ES_{q,0,\alpha}\frac{\partial^2 ES_{q,0,\alpha}}{\partial\alpha\partial\alpha'}\right\},
\end{aligned}$$

where the Hessian of the objective function is now invertible. Furthermore :

$$\begin{aligned}
\frac{\partial G\left(\alpha, f(y;0)\right)}{\partial f} &= \alpha' y - 2\frac{\zeta}{q} ES_{q,0,\alpha} \mathbb{1}_{\alpha' y \leq Q_{q,0}(\alpha' Y_{t+1})} \left(\alpha' y - Q_{q,0}\left(\alpha' Y_{t+1}\right)\right), \\
\frac{\partial^2 G\left(\alpha, f(y;0)\right)}{\partial f \partial \alpha} &= y - 2\frac{\zeta}{q} \frac{\partial ES_{q,0,\alpha}}{\partial \alpha} \mathbb{1}_{\alpha' y \leq Q_{q,0}(\alpha' Y_{t+1})} \left(\alpha' y - Q_{q,0}\left(\alpha' Y_{t+1}\right)\right) \\
&- 2\frac{\zeta}{q} ES_{q,0,\alpha} \mathbb{1}_{\alpha' y \leq Q_{q,0}(\alpha' Y_{t+1})} \left(\alpha\alpha'\right)^{-1} \alpha \left(\alpha' y - Q_{q,0}(\alpha' Y_{t+1})\right).
\end{aligned}$$

4.3 Expansion in terms of variable

The main difficulty in the variable based approach is to find a specification of the criterion function that includes the main examples encountered in the literature. We consider below a criterion equal to an optimum of expected utilities :

$$\begin{aligned}
G\left(\alpha, f_\delta\right) &= \underset{\theta}{\operatorname{opt}} \, \mathbb{E}_\delta \left(U\left[\tilde{W}_t + \alpha' Y_{t+1}; \theta\right]\right) \\
&= \underset{\theta}{\operatorname{opt}} \, \mathbb{E} \left(U\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta); \theta\right]\right), \qquad (4.6)
\end{aligned}$$

with $h(Y_0, \varepsilon, 0) = Y_0$.
A more general criterion would be :

$$G\left(\alpha, f_\delta\right) = a\left\{\mathbb{E}\left(U_0\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta)\right]\right)\right\} - b\left\{\underset{\theta}{\operatorname{opt}} \, \mathbb{E}\left(U_1\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta); \theta\right]\right)\right\}, \qquad (4.7)$$

where a and b are increasing functions. This latter criterion provides a trade off between two types of expected utility criteria. Then :

i) The expected utility criterion is obtained, when no parameter θ is introduced.

ii) The mean-variance criterion can be written as [see Schweizer, p.1 (2010)] :

$$\begin{aligned}
G\left(\alpha, f_\delta\right) &= \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right) - \frac{\gamma}{2} \mathbb{V}_\delta\left(\alpha' Y_{t+1}\right) \\
&= \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right) - \frac{\gamma}{2} \mathbb{E}_\delta\left(\alpha' Y_{t+1} - \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right)\right)^2 \\
&= \max_\theta \mathbb{E}_\delta\left(\alpha' Y_{t+1} - \frac{\gamma}{2}\left[\alpha' Y_{t+1} - \theta\right]^2\right), \qquad (4.8)
\end{aligned}$$

that is, an optimum of expected quadratic utility functions.

iii) The general criterion in (4.6) allows for robust management strategies, in which the preferences are represented by a fixed utility function u, while different scenarios are considered by the investor. These scenarios can be introduced by means of a parametric family $g\left(y_0, \epsilon; \theta\right)$ of possible distributions of the pair (Y_0, ε). Then the objective function becomes :

$$\begin{aligned}
G\left(\alpha, f_\delta\right) &= \min_\theta \mathbb{E}_\theta \left(u\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta) \right] \right) \qquad (4.9) \\
&= \min_\theta \int \int u\left[\tilde{W}_t + \alpha' h(y_0, \epsilon; \delta) \right] g(y_0, \epsilon; \theta) dy_0 d\epsilon \\
&= \min_\theta \int \int u\left[\tilde{W}_t + \alpha' h(y_0, \epsilon; \delta) \right] \frac{g(y_0, \epsilon; \theta)}{g(y_0, \epsilon; 0)} g(y_0, \epsilon; 0) dy_0 d\epsilon \\
&= \min_\theta \mathbb{E} \left(U\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta); \theta \right] \right),
\end{aligned}$$

where $U\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta); \theta \right] = u\left[\tilde{W}_t + \alpha' h(Y_0, \varepsilon; \delta) \right] \frac{g(Y_0, \varepsilon; \theta)}{g(Y_0, \varepsilon; 0)}$.

Thus the manager follows a maximin procedure by maximizing with respect to the portfolio allocation the expected utility associated with the least favorable scenario. This interpretation in terms of scenario is in the spirit of the representation of coherent risk measures derived in Artzner, Delbaen, Eber and Heath (1999), Proposition 4.1.
In this framework, θ stands for a given scenario considered by the investor, whereas δ indexes the shock. A shock on the distribution (or on the variable) modifies the distribution of excess gains, whatever the scenario chosen by the investor. In particular, the robustness of an optimal portfolio to the least favorable scenario says nothing about the sensitivity of the optimal allocation, or of the optimized criterion to a perturbation on the true distribution of excess gains.

iv) Let us finally consider the mean-ES criterion. We have [see Koenker, p.289 (2005)] :

$$Q_{q,\delta}(\alpha' Y_{t+1}) = \arg\min_\theta \mathbb{E}_\delta \left(\left(\alpha' Y_{t+1} - \theta \right) \left(q - \mathbb{1}_{\alpha' Y_{t+1} < \theta} \right) \right),$$

where $\mathbb{1}_\bullet$ is the indicator function, and :

$$\begin{aligned}
\min_\theta \mathbb{E}_\delta \left(\left(\alpha' Y_{t+1} - \theta \right) \left(q - \mathbb{1}_{\alpha' Y_{t+1} < \theta} \right) \right) &= \mathbb{E}_\delta \left(\left(\alpha' Y_{t+1} - Q_{q,\delta}(\alpha' Y_{t+1}) \right) \left(q - \mathbb{1}_{\alpha' Y_{t+1} < Q_{q,\delta}(\alpha' Y_{t+1})} \right) \right) \\
&= q \mathbb{E}_\delta \left(\alpha' Y_{t+1} \right) - q ES_{q,\delta,\alpha}
\end{aligned}$$

We deduce that :

$$\begin{aligned} G(\alpha, f_\delta) &= \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right) - \zeta \mathbb{E}_\delta^2\left(\alpha' Y_{t+1} | \alpha' Y_{t+1} < Q_{q,\delta}(\alpha' Y_{t+1})\right) \qquad (4.10) \\ &= \mathbb{E}_\delta\left(\alpha' Y_{t+1}\right) - \zeta\left\{\min_\theta\left(-\frac{1}{q}\mathbb{E}_\delta\left[(\alpha' Y_{t+1} - \theta)(q - \mathbb{1}_{\alpha' Y_{t+1} < \theta})\right] + \mathbb{E}_\delta(\alpha' Y_{t+1})\right)\right\}^2 . \end{aligned}$$

4.3.1 Expansion of the criterion function

Let us focus on the criterion function given in (4.6) and expand it with respect to δ in a neighborhood of $\delta = 0$. The case of the mean-ES is treated in Appendix 5. We get :

$$\begin{aligned} G(\alpha, f_\delta) &= \operatorname*{opt}_\theta \mathbb{E}\left(U\left[\tilde{W}_t + \alpha'\left(Y_0 + dh(Y_0, \varepsilon; \delta)\right); \theta\right]\right) \\ &\cong \operatorname*{opt}_\theta \mathbb{E}\left(U\left[\tilde{W}_t + \alpha' Y_0\right] + \frac{\partial U}{\partial W}\left[\tilde{W}_t + \alpha' Y_0; \theta\right]\alpha' dh(Y_0, \varepsilon; \delta)\right), \qquad (4.11) \end{aligned}$$

where $dh(Y_0, \varepsilon; \delta) = \frac{\partial h}{\partial \delta}(Y_0, \varepsilon, 0)\delta$.

The first-order condition providing the optimal value of θ, denoted $\theta^*(\alpha, f_\delta)$, is :

$$\mathbb{E}\left(\frac{\partial U}{\partial \theta}\left[\tilde{W}_t + \alpha' h\left(Y_0, \varepsilon; \delta\right); \theta^*(\alpha, f_\delta)\right]\right) = 0 \qquad (4.12)$$

We deduce the expansion of the first-order condition :

$$\mathbb{E}\left(\frac{\partial^2 U}{\partial \theta \partial W}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\alpha' dh(Y_0, \varepsilon; \delta)\right) + \mathbb{E}\left(\frac{\partial^2 U}{\partial \theta \partial \theta'}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\right) d\theta^*(\alpha, f_\delta) \cong 0,$$

which is equivalent to :

$$\begin{aligned} d\theta^*(\alpha, f_\delta) &= -\left\{\mathbb{E}\left(\frac{\partial^2 U}{\partial \theta \partial \theta'}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\right)\right\}^{-1} \\ &\quad \mathbb{E}\left(\frac{\partial^2 U}{\partial \theta \partial W}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\alpha' dh(Y_0, \varepsilon; \delta)\right) + o(\delta) \qquad (4.13) \end{aligned}$$

Finally, we deduce :

$$\begin{aligned} G(\alpha, f_\delta) &= \mathbb{E}\left(U\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_\delta)\right]\right) + \mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_\delta)\right]\alpha' dh(Y_0, \varepsilon; \delta)\right) + o(\delta) \\ &= \mathbb{E}\left(U\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\right) + \mathbb{E}\left(\frac{\partial U}{\partial \theta'}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\right) d\theta^*(\alpha, f_\delta) \\ &+ \mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\alpha' dh(Y_0, \varepsilon; \delta)\right) + o(\delta), \end{aligned}$$

that is,

$$
\begin{aligned}
G\left(\alpha, f_\delta\right) &= \mathbb{E}\left(U\left[\tilde{W}_t+\alpha' Y_0; \theta^*(\alpha, f_0)\right]\right) \\
&+ \mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha' Y_0; \theta^*(\alpha, f_0)\right]\alpha' dh(Y_0, \varepsilon; \delta)\right)+o(\delta),
\end{aligned} \tag{4.14}
$$

since $\mathbb{E}\left(\frac{\partial U}{\partial \theta}\left[\tilde{W}_t+\alpha' Y_0; \theta^*(\alpha, f_0)\right]\right)=0$ by the optimality of $\theta^*(\alpha, f_0)$. This latter simplification is just the envelop theorem.

4.3.2 Expansion of the optimal allocation

In the variable based approach, the optimal allocation $\alpha^*(f_\delta)$ is defined as :

$$
\alpha^*(f_\delta)=\arg\max_{\alpha}\operatorname*{opt}_{\theta}\mathbb{E}_\delta\left(U\left[\tilde{W}_t+\alpha' Y_{t+1}; \theta\right]\right),
$$

or :

$$
\alpha^*(f_\delta)=\arg\max_{\alpha}\mathbb{E}_\delta\left(U\left[\tilde{W}_t+\alpha' Y_{t+1}; \theta^*(\alpha, f_\delta)\right]\right).
$$

The first-order condition is :

$$
\mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha^{*'}(f_\delta); \theta^*(\alpha^*(f_\delta), f_\delta)\right]h(Y_0, \varepsilon; \delta)\right)=0,
$$

by the envelop theorem applied to $\theta^*(\alpha^*(f_\delta), f_\delta)$.

We deduce the expansion of the first-order condition :

$$
\begin{aligned}
0 &\cong \mathbb{E}\left(\frac{\partial^2 U}{\partial W^2}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0; \theta^*(\alpha^*(f_0), f_0)\right]Y_0 Y_0'\right)d\alpha^*(f_\delta) \\
&+ \mathbb{E}\left(\frac{\partial^2 U}{\partial W^2}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0; \theta^*(\alpha^*(f_0), f_0)\right]Y_0\alpha^{*'}(f_0)dh(Y_0, \varepsilon; \delta)\right) \\
&+ \mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha^{*'}(f_0); \theta^*(\alpha^*(f_\delta), f_\delta)\right]dh(Y_0, \varepsilon; \delta)\right) \\
&+ \mathbb{E}\left(\frac{\partial^2 U}{\partial W\partial\theta'}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0; \theta^*(\alpha^*(f_0), f_0)\right]Y_0\left[\frac{\partial\theta^*(\alpha^*(f_\delta), f_\delta)}{\partial\alpha'}d\alpha^*(f_\delta)+d\theta_f^*(\alpha^*(f_\delta), f_\delta)\right]\right)
\end{aligned}
$$

where $d\theta_f^*(\alpha^*(f_\delta), f_\delta)$ is the partial Hadamard differential of $\theta^*(\alpha, f)$ with respect to f.

As a consequence :

$$\begin{aligned} d\alpha^*(f_\delta) &= -\Big\{\mathbb{E}\left(\frac{\partial^2 U}{\partial W^2}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]Y_0Y_0'\right) \\ &\quad +\mathbb{E}\left(\frac{\partial^2 U}{\partial W\partial\theta'}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]Y_0\frac{\partial\theta^*(\alpha^*(f_0),f_0)}{\partial\alpha'}\right)\Big\}^{-1} \\ &\quad \Big\{\mathbb{E}\left(\frac{\partial^2 U}{\partial W^2}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]Y_0\alpha^{*'}(f_0)dh(Y_0,\varepsilon;\delta)\right) \\ &\quad +\mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]dh(Y_0,\varepsilon;\delta)\right) \\ &\quad +\mathbb{E}\left(\frac{\partial^2 U}{\partial W\partial\theta'}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]Y_0d\theta^*_f(\alpha^*(f_\delta),f_\delta)\right)\Big\}+o(\delta) \end{aligned} \tag{4.15}$$

4.3.3 Expansion of the optimal value of the criterion

Finally, the optimal value of the criterion is :

$$\begin{aligned} G\left(\alpha^*(f_\delta),f_\delta\right) &= \mathbb{E}\left(U\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]\right) \\ &+ \mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]\alpha^{*'}(f_0)dh(Y_0,\varepsilon;\delta)\right) \\ &+ \mathbb{E}\left(\frac{\partial U}{\partial\theta'}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]d\theta^*_f(\alpha^*(f_\delta),f_\delta)\right)+o(\delta), \end{aligned} \tag{4.16}$$

since both $\mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]Y_0\right)$ and $\mathbb{E}\left(\frac{\partial U}{\partial\theta'}\left[\tilde{W}_t+\alpha^{*'}(f_0)Y_0;\theta^*(\alpha^*(f_0),f_0)\right]\right)$ equal zero, due to the optimality of $\alpha^*(f_0)$ and $\theta^*(\alpha^*(f_0),f_0)$.

4.3.4 Examples

Let us now compute the relevant derivatives for the criteria of Section 4.1. For the sake of exposition, we will denote in the following examples :

$$U^*_\delta=U\left[\tilde{W}_t+\alpha^{*'}(f_\delta)h(Y_0,\varepsilon;\delta);\theta^*(\alpha^*(f_\delta),f_\delta)\right],\quad \alpha^*_\delta=\alpha^*(f_\delta),\quad \theta^*_\delta=\theta^*(\alpha^*(f_\delta),f_\delta).$$

i) Expected utility approach :

The expected utility criterion does not require any parameter θ. Thus :

$$G\left(\alpha^*(f_\delta),f_\delta\right)=\mathbb{E}\left(U\left[\tilde{W}_t+\alpha^{*'}_0Y_0\right]\right)+\mathbb{E}\left(\frac{\partial U}{\partial W}\left[\tilde{W}_t+\alpha^{*'}_0Y_0\right]\alpha^{*'}_0dh(Y_0,\varepsilon;\delta)\right)+o(\delta)$$

$$\begin{aligned} d\alpha^*(f_\delta) &= -\left\{\mathbb{E}\left(\tfrac{\partial^2 U}{\partial W^2}\left[\tilde{W}_t + \alpha_0^{*\prime} Y_0\right] Y_0 Y_0'\right)\right\}^{-1} \\ &\quad \mathbb{E}\left(\tfrac{\partial^2 U}{\partial W^2}\left[\tilde{W}_t + \alpha_0^{*\prime} Y_0\right] Y_0 \alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta) + \tfrac{\partial U}{\partial W}\left[\tilde{W}_t + \alpha_0^{*\prime} Y_0\right] dh(Y_0,\varepsilon;\delta)\right) + o(\delta). \end{aligned}$$

ii) Mean-Variance approach :
The relevant derivatives in the mean-variance approach are :

$$\mathbb{E}\left(\frac{\partial U_0^*}{\partial W}\alpha^*(f_0) dh(Y_0,\varepsilon;\delta)\right) = \mathbb{E}\left(\left[1-\gamma\alpha_0^{*\prime} Y_0\right]\alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta)\right) + \gamma\mathbb{E}\left(\alpha_0^{*\prime} Y_0\right)\mathbb{E}\left(\alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta)\right)$$

$$\mathbb{E}\left(\frac{\partial U_0^*}{\partial \theta'} d\theta_f^*(\alpha^*(f_\delta), f_\delta)\right) = \gamma\mathbb{E}\left(\left[\alpha_0^{*\prime} Y_0 - \mathbb{E}\left(\alpha_0^{*\prime} Y_0\right)\right]\mathbb{E}\left(\alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta)\right)\right) = 0.$$

Regarding the expansion of the optimal allocation, we get :

$$\mathbb{E}\left(\frac{\partial^2 U_0^*}{\partial W^2} Y_0 Y_0'\right) + \mathbb{E}\left(\frac{\partial^2 U_0^*}{\partial W \partial \theta'} Y_0 \frac{\partial \theta_0^*}{\partial \alpha'}\right) = -\gamma\mathbb{V}\left(Y_0\right),$$

$$\mathbb{E}\left(\tfrac{\partial^2 U_0^*}{\partial W^2} Y_0 \alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta)\right) = -\gamma\mathbb{E}\left(Y_0 \alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta)\right), \quad \mathbb{E}\left(\tfrac{\partial^2 U_0^*}{\partial W \partial \theta'} Y_0 d\theta_f^*(\alpha^*(f_\delta), f_\delta)\right) = -\gamma\mathbb{E}\left(Y_0\right)\mathbb{E}\left(\alpha_0^{*\prime} dh(Y_0,\varepsilon;\delta)\right),$$

$$\mathbb{E}\left(\frac{\partial U_0^*}{\partial W} dh(Y_0,\varepsilon;\delta)\right) = \mathbb{E}\left(dh(Y_0,\varepsilon;\delta) - \gamma\left[\alpha_0^{*\prime} Y_0 - \mathbb{E}\left(\alpha_0^{*\prime} Y_0\right)\right] dh(Y_0,\varepsilon;\delta)\right).$$

iii) Robust portfolio approach :
Regarding robust portfolio, the criterion G in (4.9) can be expanded in the following way :

$$\begin{aligned} G\left(\alpha, f_\delta\right) &= \mathbb{E}\left(U\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\right) \\ &+ \mathbb{E}\left(\frac{\partial u}{\partial W}\left[\tilde{W}_t + \alpha' Y_0; \theta^*(\alpha, f_0)\right]\frac{g(Y_0,\varepsilon,\theta^*(\alpha,f_0))}{g(Y_0,\varepsilon)}\alpha' dh(Y_0,\varepsilon;\delta)\right) + o(\delta), \end{aligned}$$

where $\theta^*(\alpha, f_0)$ is the optimal scenario minimizing the initial criterion $\mathbb{E}\left(u\left[\tilde{W}_t + \alpha' Y_0\right]\right)$.

5 Systematic Shock

Stress-tests are regularly performed to check the resistance of the financial system. They consist in applying shocks to the balance sheets of the financial institutions (assimilated to risky portfolios). The aim of this section is to discuss the notion of shock by means of a systematic factor.

5.1 Shocks on tails

The calibration of the shocks is an important stage in the implementation of stress-test exercises. In practice, the shocks are calibrated to be extreme, that is, they lead to realizations that can deviate significantly from the usual observations of the variables of interest[6]. The simplest form of shock is obtained through a translation of variable of interest, or equivalently thanks to a shift on its distribution (see Figure 1 in the univariate case). For instance, in its 2011 stress-testing exercise on European banks, the European Banking Authority studied an increase of +1% on the average default rate of loans, equal to about +1.5%, in banks' portfolios (see the EBA 2011 Aggregate Report)[7].

With this definition of shock the approaches in terms of variable and distribution look similar. However this definition shows some deficiencies: the shock is deterministic, and concerns the mean of the distribution, not the higher moments generally used to capture risk.

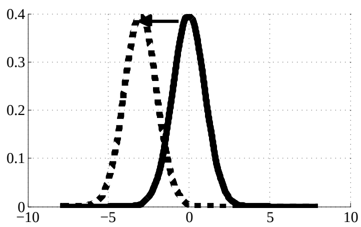


Figure 1: Shift of a distribution. The solid line stands for the baseline univariate distribution.

[6]However, as noted above, small shocks may imply large effects in a nonlinear framework. Thus, it is important to consider the effects as a function of δ, that is, the impulse response function [see the discussion in Section 4.2.3].

[7]Such an increase is rather extreme, and should rather be fixed taking into account the quality of the loans in the balance-sheet of each bank. Typically an increase of +1% corresponds to a downgrade from AAA to BB, but only from BB to B, and can lead to a stable C rating.

In this Section, we investigate more sophisticated forms of shocks that have been considered in the literature, in terms of either variable, or distribution. We present the link between both approaches in the final part of this Section.

5.1.1 Shocks on tails in terms of variable

Richer formulations of the shocks are conceivable. Let us first consider a variable based interpretation, where the shock on Y_0 is denoted $Y_\delta - Y_0 = h\left(Y_0, \varepsilon; \delta\right) - Y_0$. Different forms of shocks on tails are obtained from various dependence structure between the shock and the baseline variable Y_0. Let us consider a shock such that :

$$Y_\delta = h(Y_0, \varepsilon; \delta) = Y_0 + \delta a(Y_0) b(\varepsilon), \tag{5.1}$$

where a (a larger than 0) and b are given functions.
We focus on risks in tails by selecting a function a taking large values in the left and/or right tails of Y_0. The symmetry or asymmetry of extreme shocks can be managed by an appropriate choice of function b. As an illustration, we consider below the following specification :

$$h(Y_0, \varepsilon; \delta) = Y_0 + \delta Y_0^2 \exp\left[-\zeta_y Y_0\right] \Phi^{-1}\left(\varepsilon\right) \exp\left[-\zeta_\varepsilon \Phi^{-1}\left(\varepsilon\right)\right], \tag{5.2}$$

where ε is an independent variable with Uniform distribution (see Section 2), Φ is the c.d.f. of the standard Gaussian distribution, and ζ_y, ζ_ε are two positive scalars.

Four variants are presented in Figure 2 : symmetric extreme shock on both tails of Y_0 ($\zeta_y = 0$ and $\zeta_\varepsilon = 0$, top left panel in Figure 2), symmetric extreme shock concentrated on the left tail of Y_0 ($\zeta_y = 0.5$ and $\zeta_\varepsilon = 0$, bottom left panel), and asymmetric shocks on both tails ($\zeta_y = 0$ and $\zeta_\varepsilon = 2.5$, top right panel), or on the left tail of Y_0 ($\zeta_y = 0.5$ and $\zeta_\varepsilon = 2.5$, bottom right panel). The corresponding unconditional distributions of $Y_\delta = h(Y_0, \varepsilon; \delta)$ are given in Figure 3, which emphasizes the impact of the shock on the thickness of the tails of the distributions, its asymmetry, or its number of modes.

The expansion of the unconditional p.d.f. of Y_δ in (5.2) is derived by Corollary 1. At

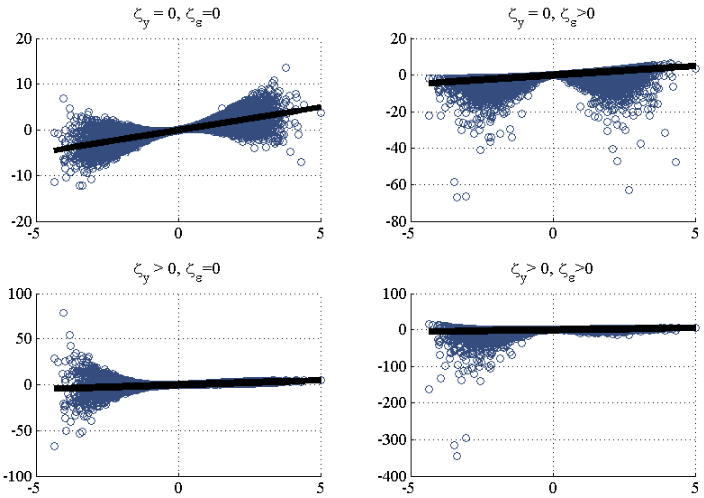


Figure 2: Conditional distributions of the shocked variable $h\left(Y_0, \varepsilon; \delta\right)$ with respect to Y_0 for different a and b functions in (5.2). The solid line stands for $h\left(Y_0, \varepsilon; 0\right) = Y_0$, and $\Phi^{-1}\left(\varepsilon\right)$ has a standard Gaussian distribution.

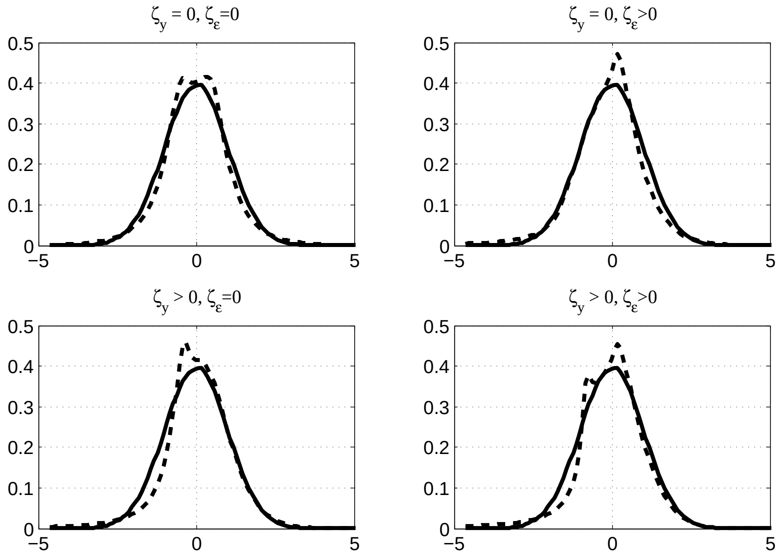


Figure 3: Unconditional distributions of the shocked variable $h\left(Y_0, \varepsilon; \delta\right)$ for different a and b functions in (5.2), where Y_0 and $\Phi^{-1}\left(\varepsilon\right)$ have standard independent Gaussian distributions.

second-order in δ, we get :

$$\begin{aligned} f(y;\delta) &= \phi(y) + \delta\zeta_\varepsilon \exp\left[\zeta_\varepsilon^2\right]\left(2y - y^3 - \zeta_y y\right)\exp\left[-\zeta_y y\right]\phi(y) \\ &+ \frac{\delta^2}{2}\left(1+4\zeta_\varepsilon^2\right)\exp\left[2\zeta_\varepsilon^2\right]\left(2y^3 - \frac{y^5}{2} - \zeta_y y^4\right)\exp\left[-2\zeta_y y\right]\phi(y), \end{aligned}$$

under the assumption of a standard normal variable Y_0, where $\phi(y)$ is the p.d.f. of the standard normal distribution. The first and second-order moments of $U \exp[-\zeta_\varepsilon U]$, where $U = \Phi^{-1}(\varepsilon)$, are derived by considering the first and second-order derivatives of the Laplace transform of the standard Gaussian variable.

5.1.2 Shocks on tails in terms of distribution

Such shocks have been introduced in the literature on robust statistics, which studies the contamination of a baseline distribution by "outliers".

i) Contamination

A standard specification, introduced in Huber (1964), presents the contaminated distribution as a mixture of a baseline c.d.f. $F(y;0)$ and a contaminating c.d.f. $\Xi(y)$:

$$F(y;\delta) = (1-\delta)F(y;0) + \delta\Xi(y), \text{ with } 0 \leq \delta \leq 1. \tag{5.3}$$

This specification is a special case of shocks in terms of distribution, for which the first-order expansion in δ is exact : $f(y;\delta) = f(y;0) + \delta\,[\xi(y) - f(y;0)]$, where $\xi(y)$ is the p.d.f. of the contaminating distribution. A left tail contamination is illustrated in Figure 4, with Gaussian distributions, with different means and the same variance for $F(\bullet;0)$ and $\Xi(\bullet)$.

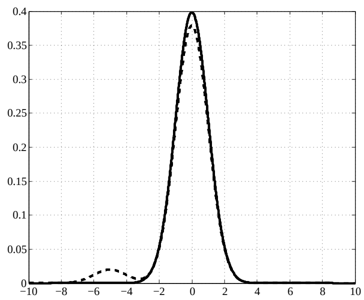


Figure 4: Left tail contaminated distribution. The solid line stands for the baseline distribution, the dashed line for the contaminated distribution.

ii) Contamination in terms of variable

The conversion of contaminated distribution in terms of variable relies on the general link between families of distributions and families of variables presented in Section 2. Let us for instance apply the Gaussian copula of Section 2.3 to the specification (5.3), where the baseline and contaminated variables are denoted Y_0 and Y_δ, respectively. We have (see Appendix 4 ii)) :

$$\Phi\left(\sqrt{1-\rho^2(\delta)}U + F_0(Y_0)\right) = (1-\delta)F_0(Y_\delta) + \delta\Xi(Y_\delta), \tag{5.4}$$

where U is standard normal and $\rho(0) = 1$ to ensure the coherency condition, that is, $Y_\delta = Y_0$, when $\delta = 0$.

5.1.3 How to reconcile shocks on tails in terms of variable and distribution

Let us focus on the previous example of contaminated distribution. Several forms of correlation parameter $\rho(\delta)$, and thus several Gaussian copulas, are consistent with contaminated distribution (5.3), once they satisfy the coherency condition $\rho(0) = 1$. This example stresses the difficulty in identifying a unique variable-based specification of a shock on tails expressed in terms of distribution.
Besides, the linearity in δ of shocks in terms of variable as considered in Section 5.1.1 do not necessarily imply the linearity in δ of the shocks in terms of distributions as in (5.3). Indeed, let us consider the following approximation of the correlation parameter :

$$\rho(\delta) = 1 - \delta^2 r, \text{ where } r = -\left.\frac{\partial\rho(\delta)}{\partial\delta}\right|_{\delta=0},$$

where the first term in the expansion of ρ is of order δ^2 to ensure the constraint $\rho(\delta) \leq 1$, $\forall\delta$. We prove in Appendix 4 iii) that :

$$Y_\delta = Y_0 + \delta Z + o(\delta), \text{ say}, \tag{5.5}$$

where the variable Z is given by :

$$Z \approx \frac{\sqrt{2r}\phi\left(\Phi^{-1}(F(Y_0;0))\right)U + F(Y_0;0) - \Xi(Y_0)}{f(Y_0;0)}. \tag{5.6}$$

The expansion (5.5)-(5.6) highlights the double effect of the contamination of a distribution in terms of variable: i) a drift effect $\frac{F(Y_0;0)-\Xi(Y_0)}{f(Y_0;0)}$, and ii) a conditionally heteroscedatic effect $\frac{\sqrt{2r}\phi\left(\Phi^{-1}(F(Y_0;0))\right)U}{f(Y_0;0)}$ similar to the effects in (5.1), that will impact the tails of the distribution of the contaminated variable. This heteroscedastic effect depends on the

curvature of function $\rho(\bullet)$ in the neighborhood of $\delta = 0$, while the drift effect is negative whenever the contaminating distribution Ξ stochastically dominates at order 1 the baseline distribution $F(Y_0; 0)$. We still have the multiplicity of interpretations of the shock in terms of variable, since the curvature effect r is unconstrained. In terms of variable, larger is this curvature, larger is the weight on the volatility with respect to the drift.
The comparison of (5.6) with the linear interpretation of the shock in the variable-based approach in (5.1), for which drift effects are missing, emphasizes the differences in both specifications, in spite of their common linear in δ formulation.

5.2 Shocks on a systematic factor

For a vast majority of asset classes, statistical analyses of the price dynamics reveal a limited number of linear, or nonlinear, latent factors, which explain most of the variation in asset prices, or asset returns. In a linear dynamic framework, it is known for instance that one factor drives most of the returns on US Treasury bonds [see e.g. Cochrane, Piazzesi (2005)], while the literature identifies few common linear factors for equity returns [see e.g. Fama, French (1992)].
These systematic factors represent the common dynamic patterns among multiple asset prices, and characterize the dependence structure between assets. For regulators, the identification of linear or nonlinear systematic factors is of crucial importance, since their variations have the biggest impact on institutions' portfolios, as opposed to idiosyncratic factors, whose risk can be diversified away by the holding of a large number of assets. As an illustration, let us consider the Basel 2 credit risk model. Inspired from the Value-of-the-Firm model [Merton (1974)], it decomposes in a linear way the log asset/liability Y_j^* of a firm j in common factor X and specific (or idiosyncratic) components η_j :

$$Y_j^* = \sqrt{\rho} X + \sqrt{1-\rho}\,\eta_j,$$

with positive asset correlation ρ, and deduces the default indicator as :

$$Y_j = \mathbb{1}_{Y_j^* < 0} = \mathbb{1}_{\sqrt{\rho} X < -\sqrt{1-\rho}\,\eta_j}.$$

In the basic model, this default indicator is directly related to the payoff of a Credit Default Swap (CDS) written on firm j. Thus, the probability of very large cumulated losses on the institution's portfolio of CDS highly depends in a nonlinear way on the share of common/systematic factors in the distribution of firm's log asset/liability.

We present below the local expansion of the distribution of variables of interest with respect to "systematic shocks", that are shocks on the distribution of systematic factors, both in terms of distribution and variable.

5.2.1 Local analysis of systematic shock in terms of distribution

Let us decompose the joint distribution of variables of interest $f(y;\delta)$ into two parts: one corresponding to the marginal distribution of the common factors, and the other one referring to the distribution of the variables of interest, conditional on the common factors. A systematic shock would hit the marginal distribution of the common factor X without modifying the conditional distribution of Y given X. Thus we get :

$$f(y;\delta) = \int f_1(y|x) f_2(x;\delta) dx. \tag{5.7}$$

A direct application of the results in Section 3.1 gives the following local expansion of the distribution of the variable of interest :

$$f(y;\delta) \quad = \quad f(y;0) + \sum_{j=1}^{p} \frac{\delta^j}{j!} \int f_1(y|x) \frac{\partial^j f_2(x;0)}{\partial \delta^j} dx + o(\delta^p). \tag{5.8}$$

In particular, at second-order, we get :

$$f(y;\delta) = f(y;0) + \delta \int f_1(y|x) \frac{\partial f_2(x;0)}{\partial \delta} dx + \frac{\delta^2}{2} \int f_1(y|x) \frac{\partial^2 f_2(x;0)}{\partial \delta^2} dx + o(\delta^2).$$

5.2.2 Local analysis of systematic shock in terms of variable

Let us now consider the (vector of) variables of interest Y_δ as a function of common factors X_δ, and independent variables η :

$$Y_\delta = b(X_\delta, \eta), \tag{5.9}$$

where the common factors are such that :

$$X_\delta = h(X_0, \varepsilon; \delta). \tag{5.10}$$

The model involves two types of basic impulses, that are the variable η representing the idiosyncratic component and the variable ε used to define shock on the systematic factor. Equation (5.9) is compatible with nonlinear effects of both types of components

and possibly cross-effects of systematic and idiosyncratic components.
From Proposition 1, we get at second-order :

$$\begin{aligned}
f(y;\delta) &= f(y;0) - \delta \int f_1(y|x) \frac{\partial}{\partial x} \left[f_2(x;0) \mathbb{E} \left[\frac{\partial h(X_0, \varepsilon; \delta)}{\partial \delta} | X_0 = x \right] \right] dx \\
&+ \frac{\delta^2}{2} \int f_1(y|x) \left\{ \frac{\partial^2}{\partial x^2} \left[f_2(x;0) \mathbb{E} \left[\left(\frac{\partial h\left(X_0, \varepsilon, 0\right)}{\partial \delta} \right)^2 | X_0 = x \right] \right] \right. \\
&\left. - \frac{\partial}{\partial x} \left[f_2(x;0) \mathbb{E} \left[\frac{\partial^2 h\left(X_0, \varepsilon, 0\right)}{\partial \delta^2} | X_0 = x \right] \right] \right\} dx + o\left(\delta^2\right).
\end{aligned}$$

For instance, let us consider a Value-of-the-Firm model for an homogeneous population of N different firms, where Y_δ^* is the vector of firms' log asset/liability :

$$Y_\delta^* = \sqrt{\rho}\mathbf{1}' X_\delta + \sqrt{1-\rho}\eta,$$

where $\mathbf{1}$ is a Nx1 vector of ones, and η has a standard multivariate normal distribution. Let us assume that the single common factor is such that :

$$X_\delta = X_0 + \delta a(X_0) U,$$

where U is a random variable independent from X_0, whose second-order moments exists.

We get at second-order :

$$\begin{aligned}
f_2(x;\delta) &= f_2(x;0) - \delta \mathbb{E}\left(U\right) \frac{d}{dx} \left[a(x) f_2(x;0) \right] + \frac{\delta^2}{2} \mathbb{E}\left(U^2\right) \frac{d}{dx} \left[a^2(x) f_2(x;0) \right] + o(\delta^2), \\
f(y^*;\delta) &= f(y^*;0) - \delta \mathbb{E}\left(U\right) \int f_1(y^*|x) \left(a^{(1)}(x) f_2(x;0) + a(x) f_2^{(1)}(x;0) \right) dx \\
&+ \frac{\delta^2}{2} \mathbb{E}\left(U^2\right) \int f_1(y^*|x) \left(2a(x) a^{(1)}(x) f_2(x;0) + a^2(x) f_2^{(1)}(x;0) \right) dx + o(\delta^2).
\end{aligned}$$

Then this expansion can be used to deduce the expansion of the distribution of the default indicators.

5.2.3 Systematic shock with extreme effects

This section helps to precise the notion of extreme shock, which remains rather vague in the literature, and calls for richer specifications of shock than the "shock-in-mean" usually considered in stress-test exercises. When performing stress-tests, we are interested in shocks on a systematic factor with extreme impacts on asset portfolio values.

i) Linear dynamic factor model

In a linear dynamic framework, large effects on the tail of the distribution of variable of interest Y are obtained by introducing shocks on the tails of the factor distribution. This is why the literature usually focus on factor tails (see e.g. Section 5.1).

ii) Nonlinear dynamic factor model

The situation is very different in the nonlinear framework encountered for portfolios of derivative assets (see e.g. the example of CDS portfolio), or for portfolios managed in some optimal way, for which the relevance of the chosen shocks depends on the nonlinear link between the distribution of portfolio's value and the distribution of the systematic factor.

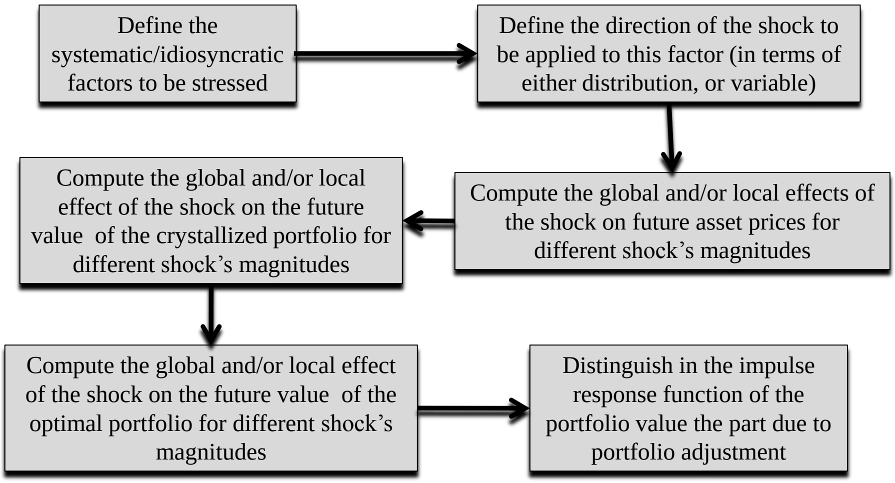
In such nonlinear factor model, a rather small shock on a factor value can sometimes have a large impact on the portfolio. As an illustration, let us consider an investor, who allocates her/his wealth among two assets, when the asset prices are almost perfectly correlated. Such a large correlation would incite her/him to lever up her/his wealth by selling the less profitable asset to buy the other one. Let us now consider a shock on the joint asset distribution changing the asset correlation to a negative value close to -1, whereas keeping the expected returns and volatilities at the same level. Under a Gaussian assumption on returns, the joint Gaussian distribution is modified much more in its central part than in its tails. However, the portfolio allocation crystallized at its previous level will be very sensitive to this change, since now the risks lie in the same direction, and the leverage effect is exactly at the opposite of what has to be done. This is exactly the situation encountered by N. Leeson that implied the failure of the Barings, or in the LTCM default. This example shows that a shock on even a central part of the distribution can have extreme consequences. Therefore, it is important to detect the type of shock on the factor with such huge consequences. For this purpose, it is recommended to define a direction of the shock in terms of either variable, or distribution, and to measure the consequences for different levels of δ, that is, to consider an impulse response function [see e.g. Koop, Pesaran, Potter (1996), and Gourieroux, Jasiak (2005)].

6 Stress-testing the European Sovereign Bond Market

The aim of this section is to avoid the limitations of the current implementation of stress-tests, in which the shocks are assumed deterministic and the portfolio is crystallized. In the application presented below, we stress the portfolio of a financial institution invested in European sovereign bonds. We propose a direction of the shock in terms of distribution and variable and show how the riskiness of investor's portfolio, either crystallized or optimally updated, evolves with the size of the shock on the distribution of excess gains.

6.1 Stress-test

A stress-test requires the application of the different notions introduced in Sections 2, 3, and 5 along the lines of scheme 1.



Scheme 1: A stress-test

In the following parts of the section, we apply these different steps to a portfolio of European sovereign bonds.

6.2 Excess gains on the European sovereign bond market

As an illustration, let us consider an investor with mean-variance objective function, who invests her/his wealth in the European sovereign bond market. For simplicity, we consider zero-coupon bonds, with face value 1, and maturities 10 years. We restrict our sample to six countries representing the variety of the euro area sovereign bond market, that are Germany, France, Italy, Spain, Ireland and Greece. The sample covers the period from July 2001 to June 2011. We assume that the investor has a monthly horizon, and has access to a risk-free asset, which pays the 1M Eonia swap rate[8] after a one-month holding period. The corresponding monthly excess gains in euros[9] are plotted in Figure 5.

We identify from principal components analysis one systematic factor, which explains

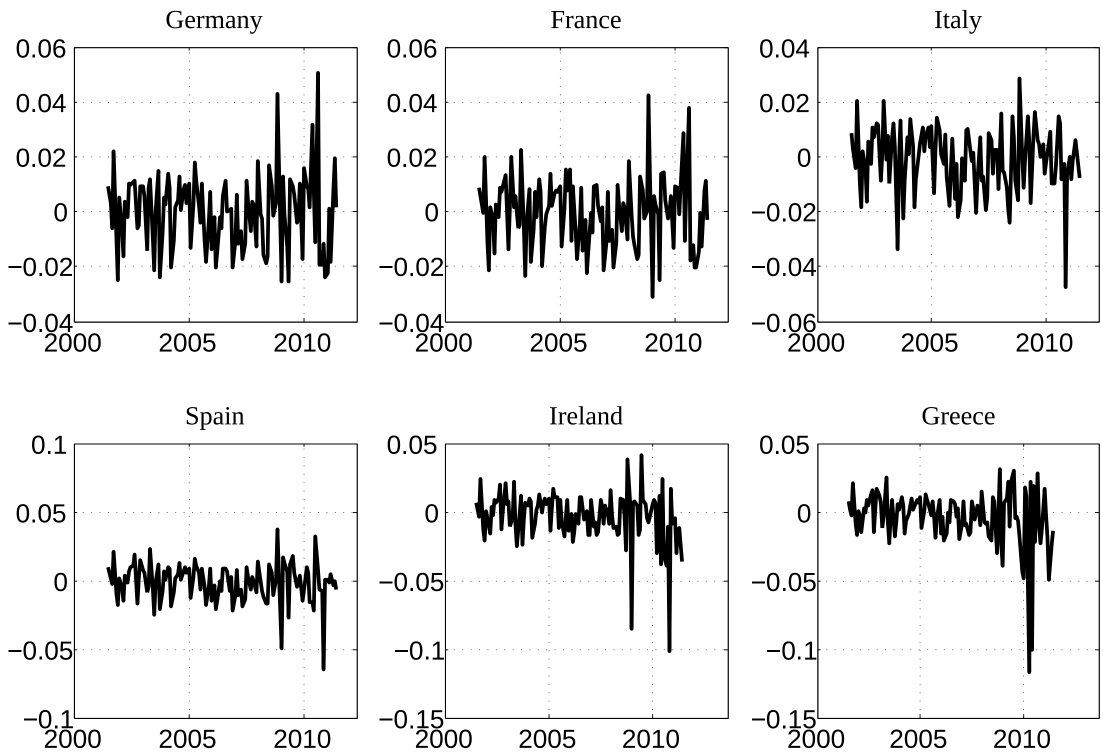


Figure 5: Montly excess gains on 10Y European sovereign bonds.

[8]The Eonia swap rate is the fixed rate in a swap contract, whose variable leg is pegged to the Euro OverNight Index Average.

[9]We did not take into account the 1-month maturity effect on bond prices in the computation of excess gains.

about 95% of the variance of the historical excess gains. The first factor weights uniformly the 10Y rates of all countries (see Appendix 6). Loosely speaking, this factor is a kind of Eurozone systematic risk. We will shock this “Euro factor”[10] .

As put forward in Figure 6, the first factor has different distributions before and after

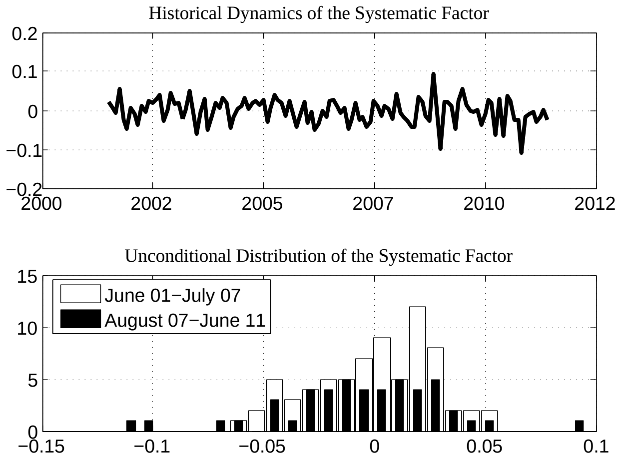


Figure 6: Historical dynamics and histograms of the first systematic factor on periods 2001-2007, and 2007-2011.

July 2007: it features fatter tails, becomes bimodal and seems less asymmetric. We rely on this change to define a systematic shock by contamination.

6.3 Contamination on the systematic factor

More precisely, we identify the distribution on the 2001-2007 period of the systematic factor X as our baseline distribution, denoted $F(x;0)$, while the distribution on 2007-2011 plays the role of the contaminating distribution $\Xi(x)$.

Then we consider a specification of the shock on the factor's distribution as in Section 5.1.2 :

$$F(x;\delta) = (1-\delta)F(x;0) + \delta\Xi(x), \text{ with } 0 \leq \delta \leq 1. \tag{6.1}$$

Let us now consider the contamination in terms of variable. We know that there exists an infinite number of specifications in terms of variable providing the specification (6.1)

[10]It would also be possible to consider specific shocks, for instance on the value of the Greek sovereign debt.

of the unconditional distribution of Y_δ. Instead of selecting one of these specifications, we consider a linear specification providing locally the equivalence. As shown in Section 5.1.3, equation (5.6), we can express at first-order the empirical contaminated distribution in terms of variable as :

$$X_\delta = X_0 + \delta Z + o(\delta), \tag{6.2}$$

where X_0 and X_δ are the baseline and contaminated factors, with

$$Z \approx \frac{\sqrt{2r}\phi\left(\Phi^{-1}(F(X_0;0))\right)U + F(X_0;0) - \Xi(X_0)}{f(X_0;0)}, \tag{6.3}$$

U is standard Gaussian, and the curvature parameter r is set to $r = 2$.
We take advantage of representation (6.3) to simulate a set of 1000 contaminated variables, by drawing independently 1000 times X_0 in the set of realized X_0 and U in the standard normal distribution, and by taking $f(\bullet;0)$, $F(\bullet;0)$, and $\Xi(\bullet)$ at their empirical counterparts $\hat{f}(\bullet;0)$, $\hat{F}(\bullet;0)$, and $\hat{\Xi}(\bullet)$. The corresponding c.d.f. of the simulated contaminated variables are plotted in Figure 7 for different δ values. Even if the contamination models in distribution and variable are equivalent in a neighborhood of $\delta = 0$, the comparison of Figures 7 and 16 shows that the two specifications imply different types of stochastic shocks for larger values of δ. In particular, the effect on tails is more important with the contamination approach (6.2) written in terms of variable.

6.4 Impact of the systematic shock on portfolio characteristics

i) Simulation of contaminated excess-gains

Let us consider a simple factor model for the vector of excess gains :

$$P_\delta = \mu + \beta' X_\delta + \Sigma^{1/2}\eta, \tag{6.4}$$

where Σ is diagonal, η is a standard zero-mean vector, and β collects the systematic factor loadings for each bond's excess gain.
The parameters μ, β, Σ are estimated from the Seemingly Unrelated Regression (SUR) of excess gains P_t on the first factor deduced from the principal component analysis. Then the distribution of η is approximated by the historical distribution of the residuals :

$$\hat{\eta}_t = \hat{\Sigma}^{-1/2}\left(P_t - \hat{\mu} - \hat{\beta}'\hat{X}_t\right).$$

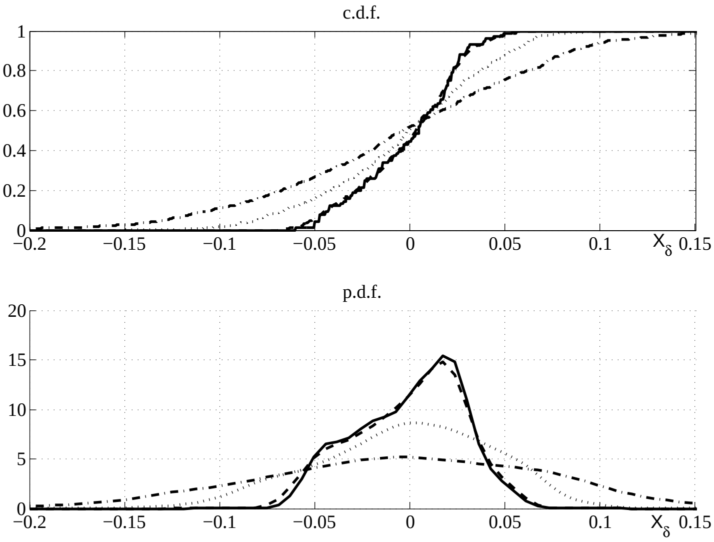


Figure 7: Empirical c.d.f. and p.d.f. of 1000 simulated contaminated factor values X_δ in terms of variable for various δ. The solid line stands for the c.d.f. of the baseline factor X_0, while dashed, dotted, and dash-dotted lines represent the contaminated empirical distribution for $\delta = 0.1, 0.5, 1$.

The simulation of contaminated excess gains is based on model (6.4) after replacement of the parameters by their estimates. Let us consider a contamination in terms of variable (the contamination in terms of distribution is presented with the corresponding results in Appendix 6). We draw independently a value η^s in the empirical distribution of the residuals $\hat{\eta}_t$, and a value X_δ^s in terms of variable as described in Section 6.3. Then the simulated contaminated excess gain is :

$$P_\delta^s = \hat{\mu} + \hat{\beta}' X_\delta^s + \hat{\Sigma}^{1/2} \eta^s.$$

This procedure is repeated $S = 1000$ times. The empirical distribution of the simulated contaminated variables P_δ^s, $s = 1, ..., S$, provides an estimate of the theoretical distribution of the contaminated excess-gains.

ii) The contaminated mean-variance allocation

We can now derive the mean-variance allocation for an investor, who adjusts herself/himself to the contaminated excess gain distribution. From the estimated distribution of the con-

taminated excess-gains derived in 5.4 i), we deduce for each magnitude δ of the shock the mean and variance/covariance matrix of contaminated excess gains, from which we deduce the optimal mean-variance allocation, denoted $\alpha^*(\delta)$[11]. The optimal portfolio allocation as a function of the magnitude of the shock δ is given in Figure 8. This figure highlights the nonlinear effects of δ on the optimal allocation.
This nonlinearity is a direct consequence of the specification of the stochastic shock and of the mean-variance portfolio management. Indeed, we deduce from (6.2)-(6.4) that:

$$\begin{aligned}\mathbb{E}\left(P_\delta\right) &= \mu+\beta'\mathbb{E}\left(X_\delta\right)=\mu+\beta'\mathbb{E}\left(X_0\right)+\beta'\mathbb{E}\left(Z\right),\\ \mathbb{V}\left(P_\delta\right) &= \beta'\mathbb{V}\left(X_\delta\right)\beta+\Sigma=\beta'\left[\mathbb{V}\left(X_0\right)+\delta Cov(X_0,Z)+\delta Cov(Z,X_0)+\delta^2\mathbb{V}\left(Z\right)\right]\beta+\Sigma.\end{aligned}$$

Therefore the mean-variance allocation: $\alpha^*(\delta)=\gamma^{-1}\mathbb{V}\left(P_\delta\right)^{-1}\mathbb{E}\left(P_\delta\right)$, has components, which are ratios of polynomials in δ of degree 12.
Despite this nonlinear pattern, we observe that these optimal portfolios are short in German, French, Italian and Greek bonds [resp. long in Irish, Spanish bonds] for any value of δ.

iii) Impulse responses for crystallized and optimally adjusted portfolios

Let us finally compare the properties of the portfolio value under the systematic shock. We consider two portfolio managements, that are i) the mean-variance portfolio crystallized at its optimal level before contamination, and ii) the mean-variance portfolio adjusted for contamination[12]. The properties of the risky part of these portfolios are represented in Figures 9-12 by their mean, variance (volatility), Sharpe performance, VaR and expected shortfall (at the 1%, 5% and 10% levels for the two last summaries).

Figures 9-12 emphasize the significant impact of the portfolio management on the characteristic of the portfolio. The performance of the optimal portfolio significantly dominates the crystallized portfolio (see the Sharpe ratio of both portfolios in Figure 9). In this example, the risky part of the crystallized portfolio becomes more volatile and features higher VaR and expected shortfall than the optimal portfolio.

[11] We set investor's wealth at 100, and the level of her/his risk aversion, $\gamma = 2$.

[12] As usual in such stress-tests, we assume that the portfolio updating, that is the demand updating by the banks, has no effect on the asset price dynamics.

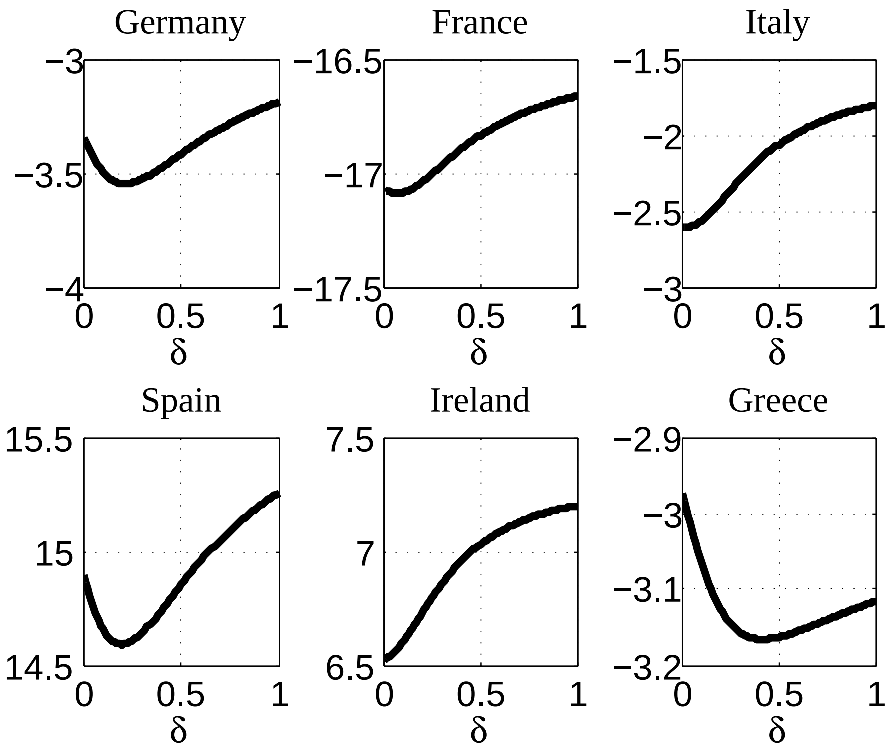


Figure 8: Contaminated mean-variance allocation as a function of δ.

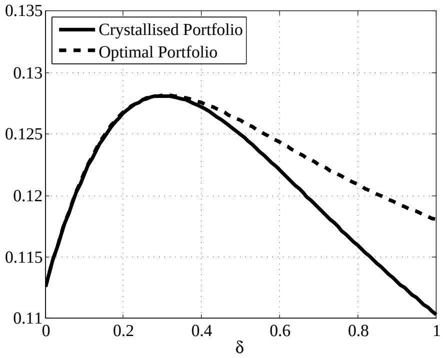


Figure 9: Impulse response of the Sharpe ratio of crystallized and Mean-Variance portfolios (contamination in variable).

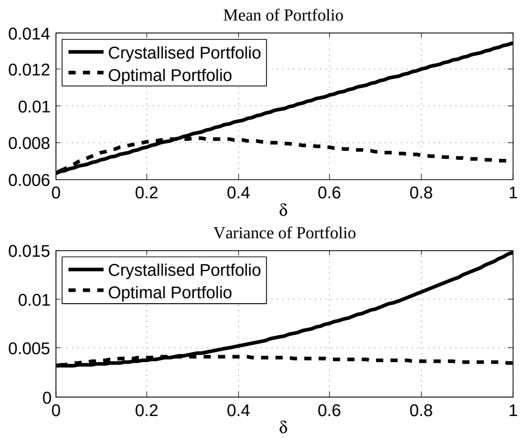


Figure 10: Impulse response of the Mean and Variance of crystallized and Mean-Variance portfolios (contamination in variable).

The local behavior of the VaR for the crystallized and mean-variance portfolios is easily analyzed. The shock in variable is :

$$P_\delta = P_0 + \delta\beta' Z + o(\delta).$$

The mean-variance allocation can be expanded as :

$$\begin{aligned}
\alpha^*(\delta) &= [\mathbb{V}(P_\delta)]^{-1}\,\mathbb{E}(P_\delta) \\
&= \left[\mathbb{V}\left(P_0+\delta\beta' Z\right)\right]^{-1}\mathbb{E}\left(P_0+\delta\beta' Z\right)+o(\delta) \\
&= \left\{\mathbb{V}(P_0)+\delta\left[Cov(P_0,\beta' Z)+Cov(\beta' Z,P_0)\right]\right\}^{-1}\left[\mathbb{E}(P_0)+\delta\mathbb{E}\left(\beta' Z\right)\right]+o(\delta) \\
&= \left\{[\mathbb{V}(P_0)]^{-1}-\delta\,[\mathbb{V}(P_0)]^{-1}\left[Cov(P_0,\beta' Z)+Cov(\beta' Z,P_0)\right][\mathbb{V}(P_0)]^{-1}\right\}\left[\mathbb{E}(P_0)+\delta\mathbb{E}\left(\beta' Z\right)\right]+o(\delta) \\
&= [\mathbb{V}(P_0)]^{-1}\,\mathbb{E}(P_0)+\delta\left\{[\mathbb{V}(P_0)]^{-1}\,\mathbb{E}\left(\beta' Z\right)\right. \\
&- \left.[\mathbb{V}(P_0)]^{-1}\left[Cov(P_0,\beta' Z)+Cov(\beta' Z,P_0)\right][\mathbb{V}(P_0)]^{-1}\,\mathbb{E}(P_0)\right\}+o(\delta) \\
&= \alpha_0+\delta\alpha_1,\ \text{say.}
\end{aligned}$$

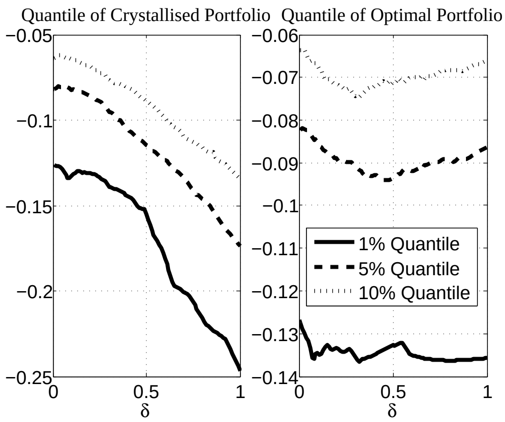


Figure 11: Impulse response of the VaR of crystallized and Mean-Variance portfolios (contamination in variable).

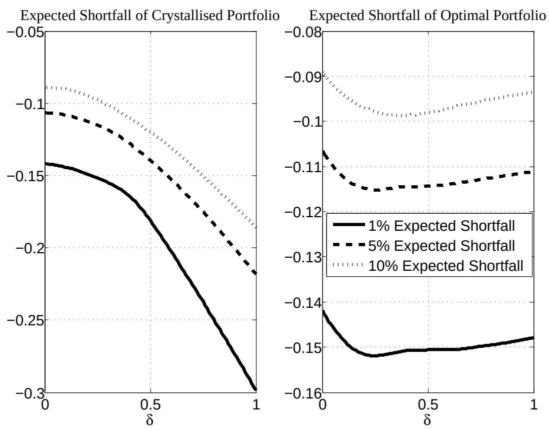


Figure 12: Impulse response of the Expected-shortfall of crystallized and Mean-Variance portfolios (contamination in variable).

The value of the mean-variance portfolio is :

$$Y^*_\delta = \alpha^*(\delta)' P_\delta = \alpha_0' P_0 + \delta\left(\alpha_0' P_1 + \alpha_1' P_0\right) + o(\delta),$$

whereas the value of the crystallized portfolio is equal to :

$$\tilde{Y}_\delta = \alpha_0'\left(P_0 + \delta P_1\right) + o(\delta).$$

We deduce the expansion of the VaR of the mean-variance portfolio :

$$
\begin{aligned}
VaR_q\left(Y_\delta^*\right) &= VaR_q\left(Y_0^*\right) + \delta \mathbb{E}\left[\alpha_0' P_1 + \alpha_1' P_0 | Y_0^* = VaR_q\left(Y_0^*\right)\right] + o(\delta) \\
&= VaR_q\left(\tilde{Y_\delta}\right) + \delta \alpha_1' \mathbb{E}\left[P_0 | Y_0^* = VaR_q\left(Y_0^*\right)\right] + o(\delta),
\end{aligned}
$$

by using the expression of the derivative of the VaR [Gourieroux, Laurent, Scaillet (2000)].

Therefore, the difference between the two VaR's is equivalent to :

$$
\begin{aligned}
\frac{1}{\delta}\left[VaR_q\left(Y_\delta^*\right) - VaR_q\left(\tilde{Y_\delta}\right)\right] \;\approx\;& \mathbb{E}\left[P_0 | Y_0^* = VaR_q\left(Y_0^*\right)\right]' \left[\mathbb{V}\left(P_0\right)\right]^{-1} \beta' \mathbb{E}\left(Z\right) \\
-\;& \mathbb{E}\left[P_0 | Y_0^* = VaR_q\left(Y_0^*\right)\right]' \left[\mathbb{V}\left(P_0\right)\right]^{-1} Cov(P_0, Z) \beta \left[\mathbb{V}\left(P_0\right)\right]^{-1} \mathbb{E}\left(P_0\right) \\
-\;& \mathbb{E}\left[P_0 | Y_0^* = VaR_q\left(Y_0^*\right)\right]' \left[\mathbb{V}\left(P_0\right)\right]^{-1} \beta' Cov(Z, P_0) \left[\mathbb{V}\left(P_0\right)\right]^{-1} \mathbb{E}\left(P_0\right).
\end{aligned}
$$

This difference can be of any sign, especially in our framework in which Z and Y_0 are dependent [see equation (6.3)].

Finally, we present in Figures 13-14 the conditional distribution of the contaminated portfolio value Y_δ with respect to its initial/non-contaminated value Y_0, which can only be derived from a specification of shock in terms of variable. Figures 13-14 emphasize the heteroscedasticity of the shock considered in this exercise: the main effect is concentrated on the central part of the initial distribution. Moreover, the comparison of Figures 13-14 highlights how the optimization of the portfolio allocation circumscribes the shock's impact on the portfolio value, even for large shock's magnitude. This feature is consistent with the low sensitivity to the shock of several risk measures for the optimally adjusted portfolio in this example (see Figures 10-12).

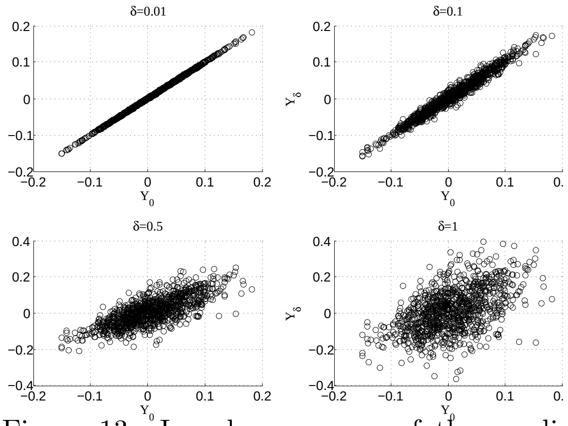


Figure 13: Impulse response of the conditional distribution of the crystallized portfolio $Y_\delta|Y_0$ (contamination in variable).

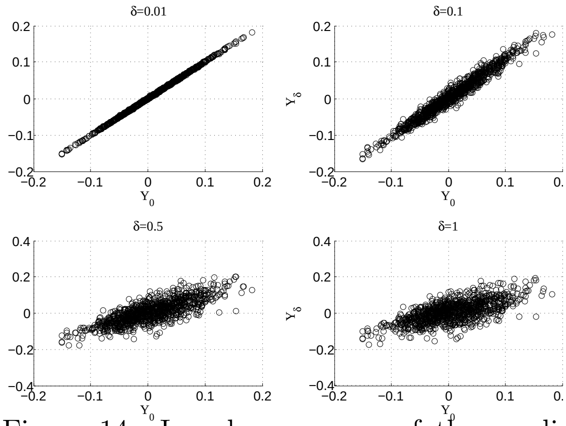


Figure 14: Impulse response of the conditional distribution of the optimally adjusted portfolio$Y_\delta|Y_0$ (contamination in variable).

7 Concluding Remarks

We have discussed the links between a modeling of shocks in terms of distribution and variable, both for global and local shocks. Such shocks can be introduced on systematic factors to perform stress-tests. This methodology of systematic shock has been illustrated for portfolios of European sovereign bonds. This highlights the different sensitivity to systematic shock of crystallized and optimally updated portfolios.

The main message of our paper is the following : we have seen that a multiplicity of specifications of the shock in terms of variable can lead to a same specification of the shock in terms of distribution. Moreover, the link between these specifications is not obvious: for instance, a linear shock in terms of distribution does not imply a linear shock in terms of variable. Therefore, a prudential approach may consist in considering carefully joint interpretations of a shock, both in terms of distribution and variable.

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Appendix 1
Shocks to Multivariate Distribution

i) Multivariate Gaussian copula for bidimensional variables

Definition a.1 : The Gaussian copula for bidimensional variables is the function defined on $[0,1]^4$ by :

$$C(u_1,u_2,v_1,v_2) = \Psi\left[\Phi^{-1}(u_1),\Phi^{-1}(u_2),\Phi^{-1}(v_1),\Phi^{-1}(v_2),R\right], \tag{7.1}$$

where $\Psi(x_1,x_2,y_1,y_2,R)$ is the joint c.d.f. of the four-dimensional Gaussian distribution $\mathcal{N}\left(\mathbf{0}\ ,\begin{pmatrix} Id_2 & R \\ R & Id_2 \end{pmatrix}\right)$. The block matrix R is such that $R=\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$, and $\mathbf{0}$ is a 4x1 vector of zeros.

The block correlation matrix is constrained by the positive definiteness of the variance-covariance matrix, which imposes that :

$$\det\begin{pmatrix} 1 & 0 & \rho_{11} \\ 0 & 1 & \rho_{21} \\ \rho_{11} & \rho_{21} & 1 \end{pmatrix} = 1-\rho_{11}^2-\rho_{21}^2>0,$$

and

$$\begin{aligned}
\det\begin{pmatrix} 1 & 0 & \rho_{11} & \rho_{12} \\ 0 & 1 & \rho_{21} & \rho_{22} \\ \rho_{11} & \rho_{21} & 1 & 0 \\ \rho_{12} & \rho_{22} & 0 & 1 \end{pmatrix} &= \det\left(Id\right)\det\left(Id-R'R\right) \\
&= \det\left(Id-R'R\right) \\
&= \left(1-\rho_{11}^2-\rho_{21}^2\right)\left(1-\rho_{12}^2-\rho_{22}^2\right)-\left(\rho_{11}^2\rho_{12}^2+\rho_{21}^2\rho_{22}^2\right)^2>0,
\end{aligned}$$

by the block description of the determinant [see e.g. Gourieroux, Monfort (1995), property A.5].

This copula defines a joint c.d.f. for the bivariate vectors: $U=\begin{pmatrix} U_1 & , & U_2 \end{pmatrix}'$ and $V=\begin{pmatrix} V_1 & , & V_2 \end{pmatrix}'$, which have the same marginal uniform distribution on $[0,1]^2$.

ii) Derivation of the recursive form

We have now to derive the conditional c.d.f. of $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ given $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$ corresponding to this copula, in order to get interpretations in terms of variables. For this purpose, let us consider two standard Gaussian bivariate variables Y_0 and Y_1 :

$$Y_0 = \begin{pmatrix} Y_{01} \\ Y_{02} \end{pmatrix}, \qquad Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix},$$

such that $Cov(Y_0, Y_1) = R$.
Then the variables $\varepsilon_1 = F(Y_{11}|Y_0)$ and $\varepsilon_2 = F(Y_{12}|Y_0, Y_{11})$ follow uniform distributions on $[0,1]$, are independent of each other, and independent of Y_0. By computing directly the conditional c.d.f., we get:

$$\varepsilon_1 = \Phi\left(\frac{Y_{11} - \mathbb{E}(Y_{11}|Y_0)}{\mathbb{V}^{1/2}(Y_{11}|Y_0)}\right) \tag{7.2}$$

$$\varepsilon_2 = \Phi\left(\frac{Y_{12} - \mathbb{E}(Y_{12}|Y_0, Y_{11})}{\mathbb{V}^{1/2}(Y_{12}|Y_0, Y_{11})}\right). \tag{7.3}$$

The conditional means and variances can be computed explicitly. We get:

$$\mathbb{E}(Y_{11}|Y_0) = \rho_{11}Y_{01} + \rho_{12}Y_{02} \equiv a_1 Y_{01} + a_2 Y_{02}, \tag{7.4}$$

$$\mathbb{V}(Y_{11}|Y_0) = 1 - \rho_{11}^2 - \rho_{12}^2 \equiv \sigma_1^2, \tag{7.5}$$

$$\begin{aligned} \mathbb{E}(Y_{12}|Y_0, Y_{11}) &= Cov\left[Y_{12}, \begin{pmatrix} Y_0 \\ Y_{11} \end{pmatrix}\right]\left[\mathbb{V}\begin{pmatrix} Y_0 \\ Y_{11} \end{pmatrix}\right]^{-1}\begin{pmatrix} Y_0 \\ Y_{11} \end{pmatrix} \\ &= \begin{pmatrix} \rho_{12} & \rho_{22} \end{pmatrix}\left[Id - \begin{pmatrix} \rho_{11} \\ \rho_{21} \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{21} \end{pmatrix}\right]^{-1}\begin{pmatrix} Y_0 \\ Y_{11} \end{pmatrix} \\ &\equiv b_1 Y_{01} + b_2 Y_{02} + +b_3 Y_{11}, \end{aligned} \tag{7.6}$$

by applying the block inversion formula [see e.g. Gourieroux, Monfort (1995) Property A.4]. Moreover :

$$\mathbb{V}(Y_{12}|Y_0, Y_{11}) = 1 - \begin{pmatrix} \rho_{12} & \rho_{22} \end{pmatrix}\left[Id - \begin{pmatrix} \rho_{11} \\ \rho_{21} \end{pmatrix}\begin{pmatrix} \rho_{11} & \rho_{21} \end{pmatrix}\right]^{-1}\begin{pmatrix} \rho_{12} \\ \rho_{22} \end{pmatrix} \equiv \sigma_2^2, \tag{7.7}$$

by applying the block inversion formula.

We deduce that:

$$\begin{cases} Y_{11} &= a_1 Y_{01} + a_2 Y_{02} + \sigma_1 \Phi^{-1}(\varepsilon_1), \\ Y_{12} &= b_1 Y_{01} + b_2 Y_{02} + b_3 Y_{11} + \sigma_2 \Phi^{-1}(\varepsilon_2), \end{cases}$$

or equivalently the conditional distribution of V given U is such that we get the recursive system of equations:

$$\begin{cases} \Phi^{-1}(V_1) &= a_1 \Phi^{-1}(U_1) + a_2 \Phi^{-1}(U_2) + \sigma_1 \Phi^{-1}(\varepsilon_1), \\ \Phi^{-1}(V_2) &= b_1 \Phi^{-1}(U_1) + b_2 \Phi^{-1}(U_2) + b_3 \Phi^{-1}(V_1) + \sigma_2 \Phi^{-1}(\varepsilon_2), \end{cases}$$

where ε_1, ε_2 are independent uniform variables independent of $U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$.

Let us finally consider a bivariate vectors $Y_0 = \begin{pmatrix} Y_{01} \\ Y_{02} \end{pmatrix}$, $Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix}$ with marginal distributions $F_0(y_0)$, $F_1(y_1)$, say. The marginal distribution of the bivariate vector Y_0 can be decomposed into the marginal distribution of Y_{01} with c.d.f. $F_{01}(y_{01})$ and the conditional distribution of Y_{02} given Y_{01} with conditional c.d.f. $F_{02|01}(y_{02}|y_{01})$. Similar notations are introduced for the second bivariate vector, that are $F_{11}(y_{11})$ and $F_{12|11}(y_{12}|y_{11})$.
The basic uniform variable U_1, U_2, V_1, V_2 can be chosen such that:

$$\begin{aligned} U_1 &= F_{01}(Y_{01}), \qquad U_2 = F_{02|01}(Y_{02}|Y_{01}), \\ V_1 &= F_{11}(Y_{11}), \qquad V_2 = F_{12|11}(Y_{12}|Y_{11}). \end{aligned}$$

Then the two bidimensional stochastic variables Y_0 and Y_1 can be linked by a Gaussian copula (7.1).

Proposition a.2 : Let us consider a pair of bidimensional stochastic variables $Y_0 = \begin{pmatrix} Y_{01} \\ Y_{02} \end{pmatrix}$, $Y_1 = \begin{pmatrix} Y_{11} \\ Y_{12} \end{pmatrix}$, such that the marginal c.d.f of Y_{01} is $F_{01}(y_{01})$ and the conditional distribution of Y_{02} given Y_{01} is $F_{02|01}(y_{02}|y_{01})$ [resp. $F_{11}(y_{11})$ and $F_{12|11}(y_{12}|y_{11})$].
Let us denote:

$$\begin{aligned} U_1 &= F_{01}(Y_{01}), \qquad U_2 = F_{02|01}(Y_{02}|Y_{01}), \\ V_1 &= F_{11}(Y_{11}), \qquad V_2 = F_{12|11}(Y_{12}|Y_{11}). \end{aligned}$$

Vectors Y_0 and Y_1 admit a Gaussian copula if and only if the conditional distribution of Y_1 given Y_0 can be represented by the system of equations:

$$\begin{aligned}
\Phi^{-1}[F_{11}(Y_{11})] &= a_1\Phi^{-1}[F_{01}(Y_{01})] + a_2\Phi^{-1}[F_{02|01}(Y_{02}|Y_{01})] + \sigma_1\Phi^{-1}(\varepsilon_1) \\
\Phi^{-1}[F_{12|11}(Y_{12}|Y_{11})] &= b_1\Phi^{-1}[F_{01}(Y_{01})] + b_2\Phi^{-1}[F_{02|01}(Y_{02}|Y_{01})] + b_3\Phi^{-1}[F_{11}(Y_{11})] + \sigma_2\Phi^{-1}(\varepsilon_2),
\end{aligned}$$

where ε_1, ε_2 are independent uniform variables, independent of Y_0, and the coefficients a_1, a_2, b_1, b_2, b_3, σ_1, σ_2 are function of R given in (7.4)-(7.7).

The extension to parametric families of bivariate variables Y_δ is obtained by making the matrix $R(\delta)$ function of δ. The coherency condition (2.9) then implies $R(0) = Id_2$.

ii) Local analysis for multivariate distribution

Let us consider a multivariate variable $Y_\delta = h(Y_0, \varepsilon; \delta)$, where Y_δ, Y_0, ε are vectors of dimension N, $h(Y_0, \varepsilon; \delta) = (h_1(\bullet), ..., h_N(\bullet))'$, and the expectation $\mathbb{E}\left(g[Y_\delta]\right)$, where g is a function of dimension 1. Thus,

$$\begin{aligned}
\mathbb{E}\left(g[Y_\delta]\right) &= \mathbb{E}\left(g[h(Y_0, \varepsilon; \delta)]\right) \\
&= \mathbb{E}\left(g(Y_0)\right) + \delta\mathbb{E}\left(\frac{\partial g(Y_0)}{\partial y'}\frac{\partial h(Y_0, \varepsilon; 0)}{\partial \delta}\right) \\
&+ \frac{\delta^2}{2}\mathbb{E}\left(\frac{\partial h'(Y_0, \varepsilon; 0)}{\partial \delta}\frac{\partial^2 g(Y_0)}{\partial y \partial y'}\frac{\partial h'(Y_0, \varepsilon; 0)}{\partial \delta}\right) + \frac{\delta^2}{2}\mathbb{E}\left(\frac{\partial g(Y_0)}{\partial y'}\frac{\partial^2 h(Y_0, \varepsilon; 0)}{\partial \delta^2}\right) + o(\delta^2) \\
&= \mathbb{E}\left(g(Y_0)\right) + \delta\mathbb{E}\left(\frac{\partial g(Y_0)}{\partial y'}\mathbb{E}\left(\frac{\partial h(Y_0, \varepsilon; 0)}{\partial \delta}|Y_0\right)\right) \\
&+ \frac{\delta^2}{2}\mathbb{E}\left(Tr\left[\frac{\partial^2 g(Y_0)}{\partial y \partial y'}\mathbb{E}\left(\frac{\partial h'(Y_0, \varepsilon; 0)}{\partial \delta}\frac{\partial h'(Y_0, \varepsilon; 0)}{\partial \delta}|Y_0\right)\right]\right) \\
&+ \frac{\delta^2}{2}\mathbb{E}\left(\frac{\partial g(Y_0)}{\partial y'}\mathbb{E}\left(\frac{\partial^2 h(Y_0, \varepsilon; 0)}{\partial \delta^2}|Y_0\right)\right) + o(\delta^2)
\end{aligned}$$

Then, we have :

$$\begin{aligned}
\mathbb{E}\left(\frac{\partial g(Y_0)}{\partial y'}\mathbb{E}\left(\frac{\partial h(Y_0, \varepsilon; 0)}{\partial \delta}|Y_0\right)\right) &= \sum_{j=1}^{N}\mathbb{E}\left[\frac{\partial g(Y_0)}{\partial y_j}\mathbb{E}\left(\frac{\partial h_j(Y_0, \varepsilon; 0)}{\partial \delta}\right)\right] \\
&= -\sum_{j=1}^{N}\mathbb{E}\left[\frac{g(Y_0)}{f(Y_0; 0)}\frac{d}{dy_j}\left[\mathbb{E}\left(\frac{\partial h_j(Y_0, \varepsilon; 0)}{\partial \delta}|Y_0 = y\right)f(y; 0)\right]\right]
\end{aligned}$$

Similarly, we have :

$$\mathbb{E}\left(\frac{\partial g(Y_0)}{\partial y'}\mathbb{E}\left(\frac{\partial^2 h(Y_0,\varepsilon;0)}{\partial \delta^2}|Y_0\right)\right) = -\sum_{j=1}^{N}\mathbb{E}\left(\frac{g(Y_0)}{f(Y_0;0)}\frac{d}{dy_j}\left[\mathbb{E}\left(\frac{\partial^2 h_j(Y_0,\varepsilon;0)}{\partial \delta^2}|Y_0=y\right)f(y;0)\right]\right),$$

and

$$\begin{aligned}&\mathbb{E}\left(Tr\left[\frac{\partial^2 g[Y_0]}{\partial y \partial y'}\mathbb{E}\left(\frac{\partial h'(Y_0,\varepsilon;0)}{\partial \delta}\frac{\partial h'(Y_0,\varepsilon;0)}{\partial \delta}|Y_0=y\right)\right]\right)\\ &=\sum_{j=1}^{N}\sum_{k=1}^{N}\frac{d}{dy_j dy_k}\mathbb{E}\left(\frac{g(Y_0)}{f(Y_0;0)}\left[\mathbb{E}\left(\frac{\partial h_j(Y_0,\varepsilon;0)}{\partial \delta}\frac{\partial h_k(Y_0,\varepsilon;0)}{\partial \delta}|Y_0=y\right)f(y;0)\right]\right)\end{aligned}$$

Thus the multivariate equivalent of Proposition 1 is :

$$\begin{aligned}\frac{\partial}{\partial \delta}[f(y;0)] &= -\sum_{j=1}^{N}\frac{d}{dy_j}\left[\mathbb{E}\left(\frac{\partial h_j(Y_0,\varepsilon;0)}{\partial \delta}|Y_0=y\right)f(y;0)\right]\\ \frac{\partial^2}{\partial \delta^2}[f(y;0)] &= \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{d}{dy_j dy_k}\left[\mathbb{E}\left(\frac{\partial h_j(Y_0,\varepsilon;0)}{\partial \delta}\frac{\partial h_k(Y_0,\varepsilon;0)}{\partial \delta}|Y_0=y\right)f(y;0)\right]\\ &- \sum_{j=1}^{N}\frac{d}{dy_j}\left[\mathbb{E}\left(\frac{\partial^2 h_j(Y_0,\varepsilon;0)}{\partial \delta^2}|Y_0=y\right)f(y;0)\right].\end{aligned}$$

Appendix 2
Proof of Lemma 2

The result is obtained by a sequence of integration by part. Let us denote $(\underline{y},\overline{y})$ the support of the function g. We get :

$$\begin{aligned}\mathbb{E}\left[g^{(k)}(Y_0)a(Y_0)\right] &= \int_{\underline{y}}^{\overline{y}} g^{(k)}(y)a(y)f(y;0)dy\\ &= \left[g^{(k-1)}(y)a(y)f(y;0)\right]_{\underline{y}}^{\overline{y}} - \int_{\underline{y}}^{\overline{y}} g^{(k-1)}(y)\tfrac{d}{dy}\left[a(y)f(y;0)\right]dy\\ &= -\int_{\underline{y}}^{\overline{y}} g^{(k-1)}(y)\tfrac{d}{dy}\left[a(y)f(y;0)\right]dy \text{ (since } g(\underline{y})=g(\overline{y})=0)\\ &= (-1)^k\int_{\underline{y}}^{\overline{y}} g(y)\tfrac{d^k}{dy^k}\left[a(y)f(y;0)\right]dy \text{ (by applying recursively the same argument)}\\ &= (-1)^k\mathbb{E}\left[\tfrac{g(Y_0)}{f(Y_0;0)}\tfrac{d^k}{dy^k}\left[a(Y_0)f(Y_0;0)\right]\right].\end{aligned}$$

Appendix 3
An alternative derivation of the expansion in Corollary 1

Let us consider the specification $Y_\delta = h(Y_0, U; \delta) = Y_0 + \delta a(Y_0)U$, where U is a variable independent of Y_0 with p.d.f. $g(u)$, $\delta \geq 0, a(\bullet) > 0$. We assume that :

Assumption A1 : Given Y_0 and δ, there is an increasing bijective relationship between U and Y_δ;

Assumption A2 : Given Y_δ and δ, there is an increasing bijective relationship between U and Y_0.

Under Assumption A_1, we can write :

$$u = \frac{y - y_0}{\delta a(y_0)},$$

Then the p.d.f. of Y_δ conditional to Y_0 is :

$$f(y|y_0;\delta) = \frac{1}{\delta a(y_0)} g(\frac{y - y_0}{\delta a(y_0)}),$$

and the unconditional p.d.f of Y_δ is :

$$f(y;\delta) = \int \left[\delta a(y_0)\right]^{-1} g\left(\frac{y - y_0}{\delta a(y_0)}\right) f(y_0; 0) dy_0. \tag{7.8}$$

Under Assumption A_2, we have :

$$y = y_0 + \delta a(Y_0)u \Leftrightarrow y_0 = c(y, u; \delta), \text{ say.}$$

Let us now consider the change of variable $y_0 \to u$ in integral (7.8). We get :

$$f(y;\delta) = \int \left[\delta a(c(y,u;\delta))\right]^{-1} g(u) f(c(y,u;\delta); 0) \left|\frac{\partial c(y,u;\delta)}{\partial u}\right| du. \tag{7.9}$$

At first-order in δ we get : $c(y, u; \delta) \approx y - \delta a(y)u$. Thus, (7.9) becomes :

$$\begin{aligned}
f(y;\delta) &\approx \int \frac{a(y)}{a(y - \delta a(y)u)} g(u) f(y - \delta a(y)u; 0) du \\
&\approx \int \left[1 - \delta a^{(1)}(y)u\right]^{-1} g(u) \left[f(y;0) - \delta f^{(1)}(y;0)a(y)u\right] du \\
&\approx \int \left[1 + \delta a^{(1)}(y)u\right] g(u) \left[f(y;0) - \delta f^{(1)}(y;0)a(y)u\right] du \\
&\approx \int g(u) f(y;0) du - \delta \int \left[f^{(1)}(y;0)a(y)u - a^{(1)}(y)u f(y;0)\right] g(u) du, \quad (7.10)
\end{aligned}$$

,which is the first term in expansion of Corollary 1.

Appendix 4
From a distribution-based to a variable-based approach with Gaussian copula

The Gaussian copula is given by :

$$C(u,v) = \Psi\left[\Phi^{-1}(u), \Phi^{-1}(v), \rho\right],$$

where $\Psi(x,y,\rho)$ is the joint c.d.f. of the bivariate Gaussian distribution $\mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$.

i) Derivatives of equation (2.4)

Let us consider a Gaussian copula with correlation parameter ρ and two variables Y_0, Y_1, whose marginal distributions are standard normal. We have from (2.4) :

$$\begin{aligned} \varepsilon &= F_{1|0}\left(Y_1|Y_0\right) \\ &= \frac{\partial C}{\partial u}\left[F_0\left(Y_0\right), F_1\left(Y_1\right)\right] \\ &= \Phi\left(\frac{\Phi^{-1}(V) - \rho\Phi^{-1}(U)}{\sqrt{1-\rho^2}}\right), \end{aligned}$$

where :

$$V = \Phi(Y_1),\ U = \Phi(Y_0).$$

Therefore :

$$\Phi^{-1}(V) = \rho\Phi^{-1}(U) + \sqrt{1-\rho^2}\Phi^{-1}(\varepsilon), \tag{7.11}$$

where ε has a uniform distribution on [0;1]. The result is easily extended to families of distribution by taking Gaussian copulas with correlation parameter $\rho(\delta)$ indexed by δ.

ii) Contamination

Let us now consider the contamination of Section 4.1.2 in the Gaussian copula case :

$$F(y;\delta) = (1-\delta)F(y;0) + \delta\Xi(y).$$

Thus :

$$\begin{aligned}\varepsilon &= \Phi\left(\frac{\Phi^{-1}(F(Y_\delta;\delta)) - \rho(\delta)\Phi^{-1}(F(Y_0;0))}{\sqrt{1-\rho^2(\delta)}}\right)\\ &= \Phi\left(\frac{\Phi^{-1}((1-\delta)F(Y_\delta;0) + \delta\Xi(Y_\delta)) - \rho(\delta)\Phi^{-1}(F(Y_0;0))}{\sqrt{1-\rho^2(\delta)}}\right),\end{aligned}$$

which gives :

$$\Phi\left(\sqrt{1-\rho^2(\delta)}U + \rho(\delta)\Phi^{-1}(F(Y_0;0))\right) = (1-\delta)F(Y_\delta;0) + \delta\Xi(Y_\delta), \tag{7.12}$$

where $U = \Phi^{-1}(\varepsilon)$ is a standard Gaussian variable.

iii) First-order expansion

Let us now assume that $\rho(\delta) = 1 - \delta^2 r + o(\delta^2)$, where $r = -\left.\frac{\partial\rho(\delta)}{\partial\delta}\right|_{\delta=0}$, and approximate Y_δ at first-order :

$$Y_\delta = Y_0 + \delta Z + o(\delta).$$

More precisely, let us consider the two sides of equation (7.12). We get for the left hand side :

$$\begin{aligned}\Phi\left(\sqrt{1-\rho^2(\delta)}U + \rho(\delta)\Phi^{-1}(F(Y_0;0))\right) &\approx \Phi\left(\sqrt{2r\delta^2}U + \Phi^{-1}(F(Y_0;0))\right)\\ &\approx F(Y_0;0) + \delta\sqrt{2r}\phi\left(\Phi^{-1}(F(Y_0;0))\right)U, \end{aligned} \tag{7.13}$$

and

$$\begin{aligned}(1-\delta)F(Y_\delta;0) + \delta\Xi(Y_\delta) &= F(Y_\delta;0) + \delta\left(\Xi(Y_\delta) - F(Y_\delta;0)\right)\\ &\approx F(Y_0;0) + \delta Z f(Y_0;0) + \delta\left(\Xi(Y_0) - F(Y_0;0)\right), \text{ for the right hand side.}\end{aligned}$$

This provides the expression of variable Z, that is (5.6), by identification.

Appendix 5
Application: contamination in terms of variable

GE	FR	IT	SP	IR	GR
−0.9668	−0.8860	−0.9037	−0.7851	−0.9293	−0.7971

Table 1: Mean of excess gains on 2001-2007. All numbers must be divided by 1000.

GE	FR	IT	SP	IR	GR
0.9759	0.3775	−1.2251	−2.1459	−7.3262	−8.9894

Table 2: Mean of excess gains on 2007-2011. All numbers must be divided by 1000.

	GE	FR	IT	SP	IR	GR
GE	0.1253	0.1164	0.1206	0.1195	0.1247	0.1131
FR	0.1164	0.1252	0.1196	0.1235	0.1292	0.1173
IT	0.1206	0.1196	0.1278	0.1215	0.1263	0.1153
SP	0.1195	0.1235	0.1215	0.1274	0.1306	0.1201
IR	0.1247	0.1292	0.1263	0.1306	0.1386	0.1238
GR	0.1131	0.1173	0.1153	0.1201	0.1238	0.1166

Table 3: Covariance matrix of excess gains on 2001-2007. All numbers must be divided by 1000.

	GE	FR	IT	SP	IR	GR
GE	0.2838	0.2346	0.1140	0.1370	0.1491	0.0496
FR	0.2346	0.2224	0.1253	0.1673	0.2023	0.1023
IT	0.1140	0.1253	0.1567	0.1838	0.2239	0.1436
SP	0.1370	0.1673	0.1838	0.2923	0.3545	0.2524
IR	0.1491	0.2023	0.2239	0.3545	0.6431	0.3645
GR	0.0496	0.1023	0.1436	0.2524	0.3645	0.8449

Table 4: Covariance matrix of excess gains on 2007-2011. All numbers must be divided by 1000.

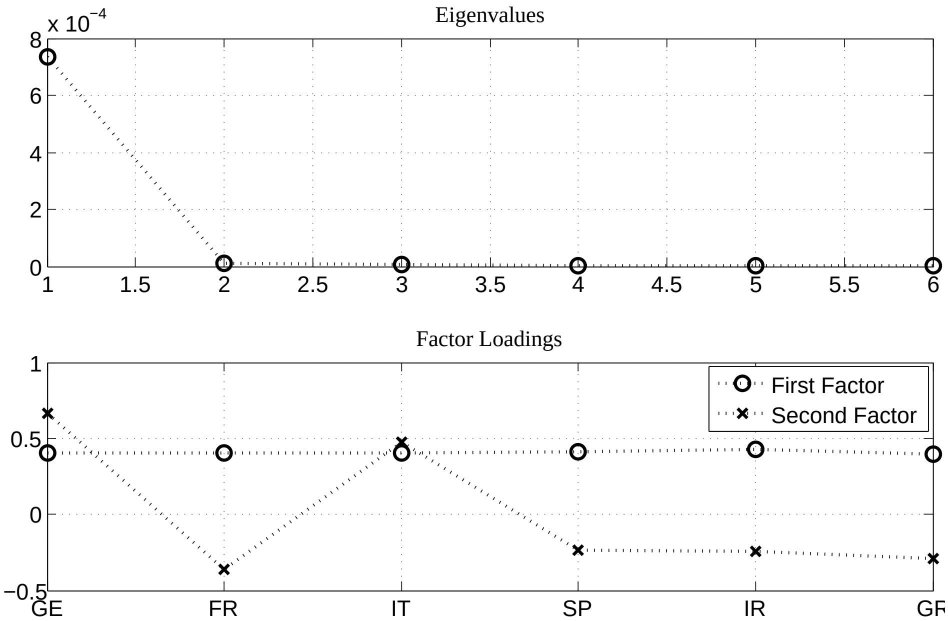

Figure 15: Eigenvalues and factor loadings from the principal components analysis of the 2001-2011 excess gains covariance matrix

Appendix 6
Application: contamination in terms of distribution

Let us perform the same application as in Section 6, when the contamination of the systematic factor is expressed in terms of distribution. In this case, we rely on the specification (5.3) to simulate the contaminated factors from a mixture of the baseline distribution of the factor with c.d.f. $F(x;0)$, which is estimated on the 2001-2007 sample, and a contaminating distribution with c.d.f. $\Xi(x)$ inferred from the 2007-2011 period :

$$F(x;\delta) = (1-\delta)F(x;0) + \delta\Xi(x), \text{ with } 0 \leq \delta \leq 1.$$

More precisely, we proceed in three steps :

1. We draw a set of $S = 1000$ uniform independent variables $(\omega^s)_{s=1\ldots S}$ on [0,1], and two sets of $S = 1000$ non-contaminated and contaminated factors $[(X_0^s)_{s=1\ldots S}$ and $(X_1^s)_{s=1\ldots S}$, respectively] from the realized factors on 2001-2007, and 2007-2011.

2. For a given δ we compute the variable :

$$\begin{cases} Z_\delta^s = & 1, \text{ when } \omega^s \leq \delta, \\ & 0, \text{ otherwise.} \end{cases}$$

3. Finally, we compute the contaminated factor :

$$X_\delta^s = (1 - Z_\delta^s) X_0^s + Z_\delta^s X_1^s.$$

The empirical distribution of the simulated factor is presented in Figure 16 for $\delta = 0$, 0.1, 0.5, and 1, while the properties of the crystallized and optimally adjusted mean-variance portfolios are plotted in Figures 16-20. As in the contamination in terms of variable, the characteristics of the optimal mean-variance portfolio deviates significantly from the crystallized portfolio's ones. As expected the analysis in terms of variable and distribution look similar for small δ, but they can deviate significantly when δ becomes larger.

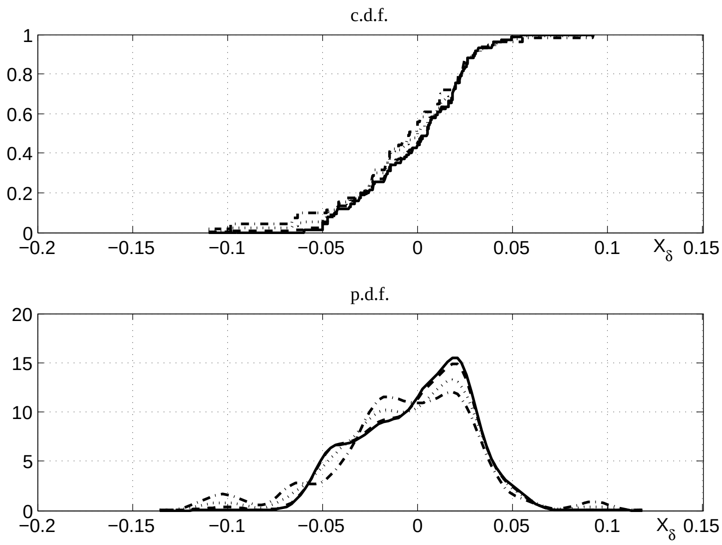

Figure 16: Empirical c.d.f and p.d.f. of 1000 simulated factor X_δ, contaminated in terms of distribution for various δ. The solid line stands for the c.d.f. of the baseline factor X_0, while dashed, dotted, and dash-dotted lines represent the contaminated empirical distribution for $\delta = 0.1, 0.5, 1$.

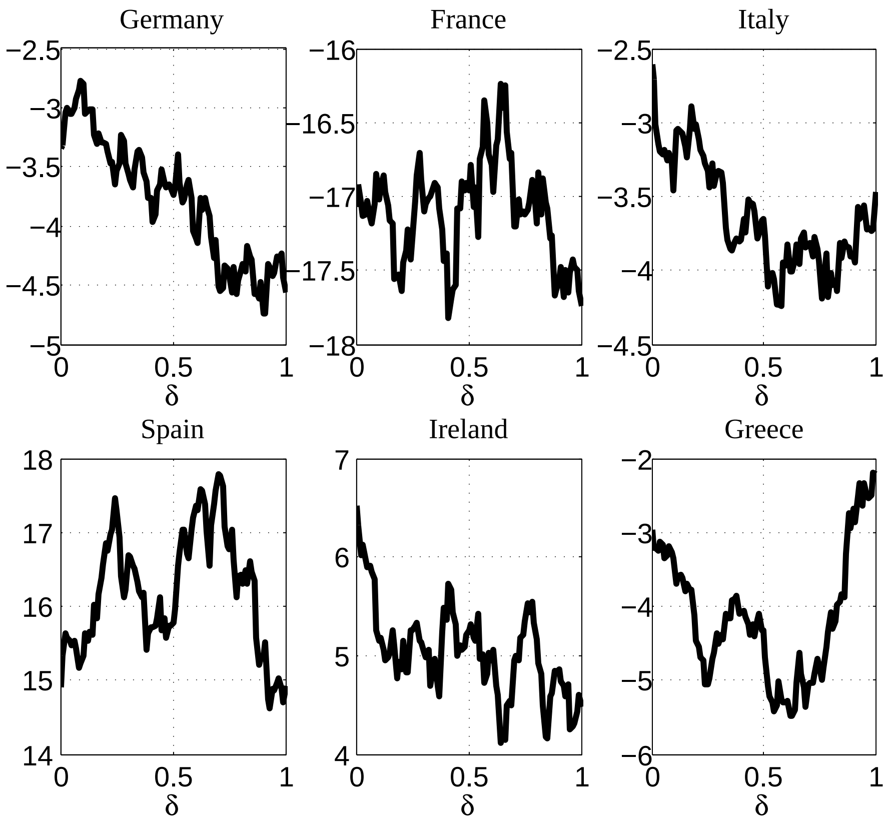

Figure 17: Contaminated mean-variance allocation as a function of δ.

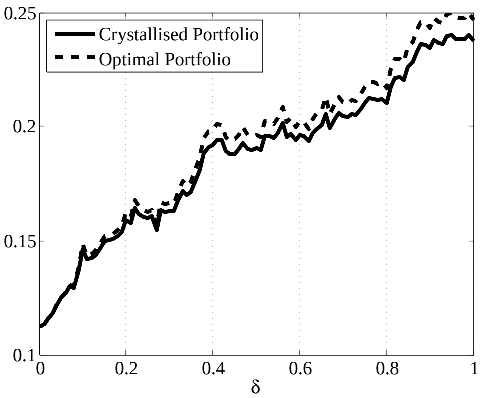

Figure 18: Impulse response of the Sharpe ratio of crystallized and Mean-Variance portfolios (contamination in distribution).

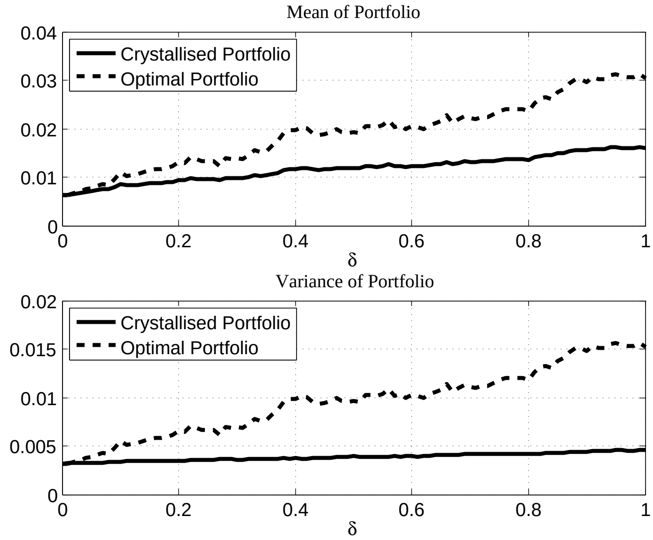

Figure 19: Impulse response of the Mean and Variance of crystallized and Mean-Variance portfolios (contamination in distribution).

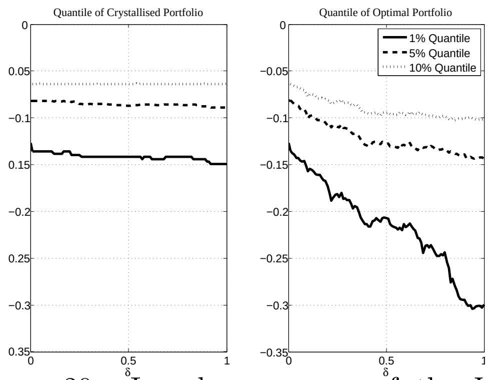

Figure 20: Impulse response of the VaR of crystallized and Mean-Variance portfolios (contamination in distribution).

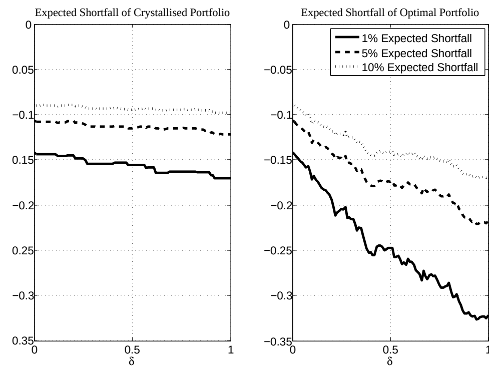

Figure 21: Impulse response of the Expected-shortfall of crystallized and Mean-Variance portfolios (contamination in distribution).