ECON-GA 1025 Macroeconomic Theory I

Lecture 12

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Today's Lecture

- Optimal savings models
- Optimal growth
- Envelope conditions
- the Euler equation

Notes: Exam Prep, etc.

Exam = Monday 22nd Oct 9:30–11:30am Room 517

- Closed book
- Material covered by TJS is examinable — PS 6 provides practice
- Review lecture slides
- Updated course notes with solved exercises

Office hours: Wed 4pm–5pm

Prequel: Topological Conjugacy

Let M and N be metric spaces

A **homeomorphism** between M and N is a continuous bijection with continuous inverse

Example. The map $\tau(x) = \ln x$ from $(0, \infty)$ to $\mathbb{R}$ is a homeomorphism

Example. Let

- $M = N = \mathbb{R}^n$ with Euclidean distance
- A be an $n \times n$ matrix

When is the map $\tau \colon M \to N$ defined by $\tau(x) = Ax$ a homeomorphism?

Dynamical systems (M, g) and (N, h) are called **topologically conjugate** if $\exists$ a homeomorphism τ from M to N such that

$$g = \tau^{-1} \circ h \circ \tau \quad \text{on} \quad M$$

Visually,

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "\tau"'] & M \\ N \arrow[r, "h"'] & N \arrow[u, "\tau^{-1}"'] \end{tikzcd}$$

Theorem. Let (M, g) and (N, h) be topologically conjugate under homeomorphism τ

In this setting:

1. $g^n = \tau^{-1} \circ h^n \circ \tau$ for all n in $\mathbb{N}$
2. x is a steady state of (M, g) iff $\tau(x)$ is a steady state of (N, h)
3. (M, g) is globally stable iff (N, h) is globally stable

We say that (M, g) and (N, h) have **equivalent dynamics**

For example, let's show that

$$x \text{ fixed for } g \implies \tau(x) \text{ fixed for } h$$

Note that

$$g = \tau^{-1} \circ h \circ \tau \iff \tau \circ g = h \circ \tau$$

Now let x be a fixed point of g in M

We have

$$h(\tau(x)) = \tau(g(x)) = \tau(x)$$

QED

Prequel: Berge's Theorem of the Maximum

Let A and X be metric spaces

Let Γ be a nonempty compact valued correspondence from X to A

- $\Gamma(x)$ is a nonempty compact subset of A for every $x \in \mathsf{X}$

Let q be a real valued function on

$$\mathbb{G} := \{(x, a) \in \mathsf{X} \times \mathsf{A} : a \in \Gamma(x)\}$$

and set

$$v(x) := \max_{a \in \Gamma(x)} q(x, a) \qquad (x \in \mathsf{X})$$

Theorem. If Γ is continuous on X and q is continuous on $\mathbb{G}$, then v is well defined and continuous on X

Note: We omitted the definition of continuity of correspondences

A **sufficient condition** for Γ to be a continuous nonempty compact valued correspondence is that $\mathsf{A} \subset \mathbb{R}^k$ and

$$\Gamma(x) = \{a \in \mathsf{A} : \ell(x) \leqslant a \leqslant m(x)\}$$

where

- ℓ, m are continuous $\mathbb{R}^k$ valued functions on X
- $\ell(x) \leqslant m(x)$ for all x in X

A Generic Optimal Savings Problem

A foundation stone for

- DSGE models
- Bewley / Huggett / Aiyagari heterogeneous agent models

Agent chooses consumption path $\{c_t\}$ to maximize

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$$

where

- $u(c_t)$ is utility of current consumption
- β is a discount factor satisfying $0 < \beta < 1$

Consumption affects a **state process** via the **law of motion**

$$x_{t+1} = g(x_t, c_t, \xi_{t+1}) \qquad \{\xi_t\} \overset{\text{IID}}{\sim} \varphi$$

where

- consumption c_t values in $\mathbb{R}_+$
- the **state** x_t values in metric space X
- x_0 is given
- the **innovation process** $\{\xi_t\}$ takes values in metric space E

(Arbitrary metric spaces so continuous & discrete both possible)

The state restricts consumption via a **feasibility constraint**

$$c_t \in \Gamma(x_t) \subset \mathbb{R}_+$$

- for example, $\Gamma(x) = [0, x]$ when x is assets
- Γ is called the **feasible correspondence**

Consumption also required to be **adapted** to the history

$$\mathcal{H}_t := \{x_j\}_{j \leqslant t}$$

- c_t cannot depend on future realizations of the state

Assumption. The following conditions hold:

1. u is continuous, strictly concave and strictly increasing on $\mathbb{R}_+$
2. g is everywhere continuous
3. Γ is nonempty, compact valued and continuous

Collectively, $(\beta, u, g, \varphi, \Gamma)$ called the **generic optimal savings model**

Interpretations

- Consumption and investment in a DSGE model
- Savings and asset accumulation for a household
- Optimal exploitation of a natural resource

Example. In Brock and Mirman (1972), a representative agent owns capital $k_t \in \mathbb{R}_+$, produces output

$$y_t := f(k_t, z_t)$$

Here f is the production function and $\{z_t\}$ is an **exogenous productivity process**

Consumption is chosen to maximize $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right]$

The resource constraint is

$$0 \leqslant k_{t+1} + c_t \leqslant y_t$$

This combined with the production function leads to the law of motion

$$k_{t+1} = f(k_t, z_t) - c_t$$

The exogenous state process is assumed to follow the Markov law

$$z_{t+1} = G(z_t, \epsilon_{t+1}), \qquad \{\epsilon_t\} \overset{\text{IID}}{\sim} \varphi$$

Maps to the generic optimal savings model $(\beta, u, g, \varphi, \Gamma)$ if we set

- $x = (k, z)$
- law of motion

$$g((k,z), c, \xi) = \begin{pmatrix} f(k,z) - c \\ G(z, \xi) \end{pmatrix}$$

- $\Gamma(x) = [0, f(k,z)]$

What do we need for g to be continuous?

Example. Consider the model of household wealth dynamics

$$w_{t+1} = (1 + r_{t+1})(w_t - c_t) + y_{t+1}$$

- w_t = household assets
- c_t = consumption
- y_{t+1} = non-financial income
- r_{t+1} = the rate of return on financial assets

Assume $y_t = y(z_t, \eta_t)$ and $r_t = r(z_t, \zeta_t)$ where

- $z_{t+1} = G(z_t, \epsilon_{t+1})$
- $\{\eta_t\}$, $\{\zeta_t\}$ and $\{\epsilon_t\}$ are IID

Consumption is chosen to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Maps to the generic optimal savings model $(\beta, u, g, \varphi, \Gamma)$ when

- $x = (w, z)$
- $\varphi =$ distribution of $\xi := (\epsilon, \eta, \zeta)$
- g is set to

$$g((w, z), c, \xi) = \begin{pmatrix} (1 + r(z, \zeta))(w - c) + y(z, \eta) \\ G(z, \epsilon) \end{pmatrix}$$

- $\Gamma((w, z)) = [0, w]$

What do we need for g to be continuous?

Stationary Markov Policies

Recall: Consumption must be adapted to $\mathcal{H}_t := \{x_j\}_{j \leqslant t}$

Means that, at each point in time t, we have

$$c_t = \sigma_t(x_0, x_1, \ldots, x_t)$$

for some suitable function σ_t — called a **policy function**

In what follows we focus exclusively on **stationary Markov policies**

- depend only on the **current** state
- time invariant $(\sigma_t = \sigma)$

(In fact every optimal policy has these properties)

A stationary Markov policy is a function σ mapping X to $\mathbb{R}_+$

Interpretation:

$$c_t = \sigma(x_t) \quad \text{for all } t \geqslant 0$$

We call σ a **feasible consumption policy** if

1. it is Borel measurable and
2. it satisfies

$$\sigma(x) \in \Gamma(x) \quad \text{for all} \quad x \in \mathsf{X}$$

Requires that

- functions nice enough to compute all expectations
- resource constraint is respected

Each $\sigma \in \Sigma$ **closes the loop** for the state process

- determines a first order Markov process $\{x_t\}$ via

$$x_{t+1} = g(x_t, \sigma(x_t), \xi_{t+1})$$

This is important!

Choosing a policy $\sigma \in \Sigma$ chooses a Markov process

Associated value is

$$v_\sigma(x) := \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(\sigma(x_t))$$

- Here $\{x_t\}$ obeys (20) with $x_0 = x$
- Called the **σ-value function**

The **value function** v^* is defined by

$$v^*(x) := \sup_{\sigma \in \Sigma} v_\sigma(x) \qquad (x \in \mathsf{X})$$

A consumption policy σ^* is called **optimal** if it is feasible and

$$v_{\sigma^*}(x) = v^*(x) \;\text{ for all } x \in \mathsf{X}$$

In most settings v^* satisfies the Bellman equation

$$v(x) = \max_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(\mathrm{d}z) \right\} \qquad (x \in \mathsf{X})$$

Intuition: maximal value obtained by trading off current vs expected future rewards possible from next state

Proposition. Let $(\beta, u, f, \varphi, \Gamma)$ be a generic optimal savings model

If u is bounded, then

1. v^* is the unique solution to the Bellman equation in $bc\mathsf{X}$
2. A feasible consumption policy σ is optimal if and only if

$$\sigma(x) \in \operatorname*{argmax}_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v^*(g(x, c, z)) \varphi(\mathrm{d}z) \right\}$$

   for all $x \in \mathsf{X}$

3. At least one such policy exists

Proof: Deferred

Consistent with earlier notation, $\sigma \in \Sigma$ is called **v-greedy** if

$$\sigma(x) \in \operatorname*{argmax}_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(\mathrm{d}z) \right\}$$

for all $x \in \mathsf{X}$

The last proposition states that, for $\sigma \in \Sigma$

$$\sigma \text{ is } v^*\text{-greedy} \iff \sigma \text{ is optimal}$$

This is another version of **Bellman's principle of optimality**

We started with one optimization problem

- choosing an optimal consumption path $c_0, c_1, \ldots$ to maximize expected discounted lifetime utility

and ended up with another one

- finding a greedy policy from the value function

But we are much better off — why?

Of course, being better off is contingent on obtaining the value function

- needed to compute v^*-greedy policies

Standard method:

1. Choose initial guess v
2. iterate from v via the Bellman operator

$$Tv(x) = \max_{c \in \Gamma(x)} \left\{ u(c) + \beta \int v(g(x, c, z)) \varphi(\mathrm{d}z) \right\}$$

Proposition. If u is bounded, then

1. T is a contraction of modulus β on $(bc\mathsf{X}, d_\infty)$
2. Its unique fixed point in $bc\mathsf{X}$ is v^*

- Why is u required to be bounded?

This assumption is not ideal, since it fails in many applications

Unbounded u issues have to be treated case-by-case

For now let's prove part 1 of the proposition

First let's show that T is a self-map on $bc\mathsf{X}$

Is Tv is bounded on X whenever $v \in bc\mathsf{X}$?

Fix any such v and any feasible x

We have

$$|Tv(x)| \leqslant \max_{a \in \Gamma(x)} \left| u(c) + \beta \int v(g(x, c, z)) \varphi(\mathrm{d}z) \right|$$

$$\leqslant \|u\|_\infty + \beta \|v\|_\infty$$

RHS does not depend on x, so Tv is bounded

Next we need to show that Tv is continuous when $v \in bc\mathsf{X}$

We employ **Berge's theorem of the maximum**, which tells us that Tv will be continuous whenever

$$q(x, c) := u(c) + \beta \int v(g(x, c, z)) \varphi(\mathrm{d}z)$$

is continuous on $\mathbb{G} := \{(x, c) \in \mathsf{X} \times \mathbb{R}_+ : c \in \Gamma(x)\}$

The tricky part is to show that

$$\int v(g(x_n, c_n, z)) \varphi(\mathrm{d}z) \to \int v(g(x, c, z)) \varphi(\mathrm{d}z)$$

when $(x_n, c_n) \to (x, c)$

Follows from the DCT (see course notes)

Finally, let v and w be elements of $bc\mathsf{X}$ and fix $x \in \mathsf{X}$

Recalling our sup inequality

$$|\sup_{a \in E} f(a) - \sup_{a \in E} g(a)| \leqslant \sup_{a \in E} |f(a) - g(a)|$$

we have

$$|Tv(x) - Tw(x)| \leqslant \max_{c \in \Gamma(x)} \beta \left| \int v(g(\cdot))\varphi(\mathrm{d}z) - \int w(g(\cdot))\varphi(\mathrm{d}z) \right|$$

$$\leqslant \max_{c \in \Gamma(x)} \beta \int |v(g(x,c,z)) - w(g(x,c,z))| \, \varphi(dz)$$

$$\therefore \quad \|Tv - Tw\|_\infty \leqslant \beta \|v - w\|_\infty$$

Problems with Analytical Solutions

For a small subset of optimal savings problems, both the optimal policy and the value function have known analytical solutions

These models are limited and simplistic!

But helpful for

- building intuition
- testing ideas
- testing numerical algorithms

Let's look at some examples

Cake Eating with Interest

Objective function is $\sum_{t=0}^{\infty} \beta^t u(c_t)$

Utility is

$$u(c) := \frac{c^{1-\gamma}}{1-\gamma} \qquad (\gamma > 0, \ \gamma \neq 1)$$

and

$$w_{t+1} = R\,(w_t - c_t)$$

Here

- $R = 1 + r$ is a gross interest rate
- $0 \leqslant c_t \leqslant w_t$ where w_t is wealth
- $\beta R^{1-\gamma} < 1$ is assumed to hold

Maps to generic savings model $(\beta, u, g, \varphi, \Gamma)$ with

- $x_t = w_t$
- $g(x, c, \xi) = R(x - c)$
- $\Gamma(x) = [0, x]$
- $\varphi = \delta_1$

Fact. There exists a constant $\theta \in (0, 1)$ such that

$$\sigma^*(w) = \theta w$$

is the optimal consumption policy

Let's verify this claim and seek the value of θ

First, observe that if $c_t = \theta w_t$ for all t, then

$$w_t = R^t (1-\theta)^t w \quad \text{when} \quad w_0 = w$$

Hence

$$v^*(w) = \sum_t \beta^t u(\theta w_t) = \sum_t \beta^t u\left(\theta R^t (1-\theta)^t w\right)$$

$$= \sum_t \beta^t \left(\theta R^t (1-\theta)^t\right)^{1-\gamma} u(w)$$

$$= \frac{\theta^{1-\gamma}}{1-\beta\left(R(1-\theta)\right)^{1-\gamma}} u(w)$$

Under the conjecture $\sigma^*(w) = \theta w$, the Bellman equation takes the form

$$v^*(w) = \max_c \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \cdot \frac{\theta^{1-\gamma}}{1 - \beta\left(R\left(1-\theta\right)\right)^{1-\gamma}} \cdot \frac{\left(R\left(w-c\right)\right)^{1-\gamma}}{1-\gamma} \right\}$$

Taking the derivative w.r.t. c yields the first-order condition

$$c^{-\gamma} + \beta m \left(R\left(w-c\right)\right)^{-\gamma}\left(-R\right) = 0$$

where

$$m := \frac{\theta^{1-\gamma}}{1 - \beta\left(R\left(1-\theta\right)\right)^{1-\gamma}}$$

Hence $c^{-\gamma} = \beta m R^{1-\gamma}(w-c)^{-\gamma}$

Substituting the optimal policy $\sigma^*(w) = \theta w$ into this equality gives us

$$(\theta w)^{-\gamma} = \frac{\beta R^{1-\gamma}\theta^{1-\gamma}}{1-\beta\left(R\left(1-\theta\right)\right)^{1-\gamma}}\left(1-\theta\right)^{-\gamma} w^{-\gamma}$$

Solving the above equality for θ yields

$$\theta = 1 - \left(\beta R^{1-\gamma}\right)^{1/\gamma}$$

The value function becomes

$$v^*(w) = \frac{\theta^{1-\gamma}}{1-\beta\left(R\left(1-\theta\right)\right)^{1-\gamma}} u(w) = \theta^{-\gamma} u(w)$$

Log–CD Example

Set $u(c) = \ln c$ and

$$f(k) = Ak^{\alpha}, \quad 0 < A, \quad 0 < \alpha < 1$$

Let $\{z_t\}$ be a lognormal IID sequence, with $\ln z_t \stackrel{\mathscr{D}}{=} N(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$

The state can be set to

$$y_{t+1} = f(y_t - c_t) z_{t+1} = A(y_t - c_t)^{\alpha} z_{t+1}$$

The agent maximizes

$$\mathbb{E} \sum_{t \geqslant 0} \beta^t \ln c_t$$

Ex. Conjecture that the optimal policy is linear in income y

That is, $\exists$ a positive constant θ such that $\sigma^*(y) = \theta y$ is optimal

Following the approach of the CRRA cake eating example

1. find the value of θ
2. obtain an expression for the value function and
3. confirm that the value function satisfies the Bellman equation

CRRA Preferences and Stochastic Financial Returns

Let's look at a recent paper by Alexis Akira Toda (2018, JME)

He studies a heterogeneous agent economy where households optimize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta(z_t)^t u(c_t) = \mathbb{E} \sum_{t=0}^{\infty} \beta(z_t)^t \frac{c_t^{1-\gamma}}{1-\gamma}$$

- u is CRRA as before and $\gamma > 0$
- Note that β is state dependent

Gives conditions for Pareto tails in the wealth distribution

Wealth dynamics are given by

$$w_{t+1} = R(z_t)\,(w_t - c_t)$$

The **state process** $\{z_t\}$ is

- exogenous
- a Markov chain on finite set Z with stochastic kernel Π

We assume that

1. $\Pi(z, z') > 0$ for all z, z' in Z
2. $\beta(z) > 0$ and $R(z) > 0$ for all $z \in$ Z

What does positivity of Π imply?

The Bellman equation is now

$$v(w,z) = \max_{0 \leqslant c \leqslant w} \left\{ u(c) + \beta(z) \sum_{z' \in \mathsf{Z}} v[R(z)(w-c), z'] \Pi(z,z') \right\}$$

for all $(w,z) \in \mathsf{X} := \mathbb{R}_+ \times \mathsf{Z}$.

Let K be the square matrix defined by

$$K(z,z') = \beta(z) R(z)^{1-\gamma} \Pi(z,z') \qquad ((z,z') \in \mathsf{Z} \times \mathsf{Z})$$

In the slides below,

$$Kg(z) := \sum_{z'} g(z') K(z,z') \qquad (z \in \mathsf{Z})$$

(Think of the matrix product with column vector g)

Toda (2018) shows that if $r(K) < 1$, then

1. There exists a g^* in $\mathbb{R}^{\mathsf{Z}}$ satisfying

$$g^*(z) = \left\{ 1 + [K g^*(z)]^{1/\gamma} \right\}^{\gamma} \qquad (z \in \mathsf{Z})$$

2. The optimal consumption policy is

$$\sigma^*(w, z) = g^*(z)^{-1/\gamma} w$$

3. The value function satisfies

$$v^*(w, z) = g^*(z) \frac{w^{1-\gamma}}{1 - \gamma}$$

Let's

1. do the proof of part 1
2. work out how to compute the solution g^*
3. study the impact of parameters

We adopt the standard pointwise partial order $\leqslant$ on $\mathbb{R}^{\mathsf{Z}}$

Recall that

- self-map T on $\mathbb{R}^{\mathsf{Z}}$ is called isotone if $g \leqslant h$ implies $Tg \leqslant Th$
- $g \ll h$ means $g(z) < h(z)$ for all z

Let ψ be the scalar map defined by

$$\psi(t) := (1 + t^{1/\gamma})^{\gamma} \qquad (t \geqslant 0)$$

Consider the operator S mapping

$$\mathcal{C} = \{g \in \mathbb{R}^{\mathsf{Z}} : g \geqslant 0\}$$

to itself via

$$Sg(z) = \psi(Kg(z))$$

Note that, for $g \in \mathcal{C}$,

$$g(z) = \left\{1 + [Kg(z)]^{1/\gamma}\right\}^{\gamma}, \forall z \quad \iff \quad Sg = g$$

Proposition. If $r(K) < 1$, then $(\mathcal{C}, S)$ is globally stable

To prove the proposition we use this result from lecture 4:

(FPT2): Let T be an isotone self-mapping on sublattice L of $\mathbb{R}^d$ such that

1. $\forall\, u \in L$, $\exists$ a point $a \in L$ with $a \leqslant u$ and $Ta \gg a$
2. $\forall\, u \in L$, $\exists$ a point $b \in L$ with $b \geqslant u$ and $Tb \ll b$

Suppose, in addition, that T is either concave or convex

Then (L, T) is globally stable

To apply this result we need to show that

1. $\mathcal{C}$ is a sublattice of $\mathbb{R}^{\mathsf{Z}}$
2. S is an isotone self-map on $\mathcal{C}$
3. For all $g \in \mathcal{C}$,

$$\exists\, \ell \in \mathcal{C} \text{ with } \ell(z) \leqslant g(z) \text{ and } (S\ell)(z) > \ell(z) \text{ for all } z$$

4. For all $g \in \mathcal{C}$,

$$\exists\, m \in \mathcal{C} \text{ with } g(z) \leqslant m(z) \text{ and } (Sm)(z) < m(z) \text{ for all } z$$

5. S is either concave or convex

We already know that $\mathcal{C}$ is a sublattice of $\mathbb{R}^{\mathsf{Z}}$

Ex. Show that S is a self-mapping on $\mathcal{C}$

To see that S is isotone on $\mathbb{R}^{\mathsf{Z}}$, observe that

- $S = \psi \circ K$
- the composition of two isotone maps is isotone

The map $g \mapsto Kg$ is isotone on $\mathbb{R}^{\mathsf{Z}}$ because K is nonnegative

Indeed, if $f \leqslant g$ on $\mathbb{R}^{\mathsf{Z}}$, then

$$K(g - f)(z) = \sum_{z'} [g(z') - f(z')] K(z, z') \geqslant 0$$

Hence $K(g - f) = Kg - Kf \geqslant 0$

Clearly $\psi(t) = (1 + t^{1/\gamma})^{\gamma}$ is also isotone

Ex. Show that

$$\psi(t) = (1 + t^{1/\gamma})^{\gamma}$$

is

1. convex on $\mathbb{R}_+$ whenever $0 < \gamma \leqslant 1$
2. concave on $\mathbb{R}_+$ whenever $\gamma \geqslant 1$

Ex. Show that $S = \psi \circ K$ is

1. convex on C whenever $0 < \gamma \leqslant 1$
2. concave on C whenever $\gamma \geqslant 1$

Ex. Show that $S0 \gg 0$

By the **Perron–Frobenius theorem** and positivity of K,

$$\exists\, e \gg 0 \text{ s.t. } Ke = r(K)e$$

- e is called the **dominant eigenvector** of K
- $\lambda := r(K)$ is called the **dominant eigenvalue**

Ex. Let α be a positive constant and let $\mathbb{1}$ be a vector of ones

Show that

$$\alpha e \gg \left(\frac{1}{1-\lambda^{1/\gamma}}\right)^{\gamma} \mathbb{1} \quad \implies \quad S(\alpha e) \ll \alpha e$$

To complete the proof we need only show that

$$\forall g \in C, \ \exists m \geqslant g \ \text{ s.t. } \ Sm \ll m$$

So fix $g \in C$ and choose α such that

$$\alpha e \gg \left(\frac{1}{1 - \lambda^{1/\gamma}} \right)^{\gamma} \mathbb{1} \quad \text{and} \quad \alpha e \geqslant g$$

For $m := \alpha e$, we have $m \geqslant g$ and

$$Sm = S(\alpha e) \ll \alpha e =: m$$

the proof is now done

See `toda_crra.ipynb`, which solves for

- the unique positive fixed point g^* of S
- the corresponding state contingent savings rate

$$s(z) := 1 - (g^*(z))^{-1/\gamma} \qquad (z \in \mathsf{Z})$$

The simulations suggest that

1. $\beta \leqslant \hat{\beta} \implies s \leqslant \hat{s}$
2. $R \leqslant \hat{R} \implies s \leqslant \hat{s}$ when $0 < \gamma < 1$
3. $R \leqslant \hat{R} \implies \hat{s} \leqslant s$ when $1 < \gamma < \infty$

Ex. Show that this is always true

A Model with Independent Shocks

How can analysis can proceed without analytical solutions?

As a starting point, we consider a model with

- only one source of randomness — exogenous process $\{z_t\}$
- this shock process is IID

Simplifies the problem to one with a single state variable

That state variable is $\{y_t\}$ evolving according to

$$y_{t+1} = f(y_t - c_t) z_{t+1}$$

- Example. stock of a renewable resource

Assumption.

- f is continuous, concave and strictly increasing with $f(0) = 0$
- u is continuous, strictly concave and strictly increasing on $\mathbb{R}_+$

The Bellman equation is now

$$v(y) = \max_{0 \leqslant c \leqslant y} \left\{ u(c) + \beta \int v(f(y-c)z)\varphi(dz) \right\} \qquad (y \in \mathbb{R}_+)$$

The corresponding Bellman operator T is

$$Tv(y) = \max_{0 \leqslant c \leqslant y} \left\{ u(c) + \beta \int v(f(y-c)z)\varphi(dz) \right\}$$

Theorem. T is a contraction of modulus β on $(bc\mathbb{R}_+, d_\infty)$

Moreover,

1. v^* is the unique fixed point of T in $bc\mathbb{R}_+$
2. $\sigma \in \Sigma$ is optimal if and only if it is v^*-greedy
3. Exactly one optimal policy and that policy is continuous

Proof:

Parts 1 and 2 follow from earlier results for the generic optimal savings model

Same for the existence component of part 3

Regarding uniqueness of the optimal policy,

Ex. Let $\mathscr{C}$ be the set of increasing concave functions in $bc\mathbb{R}_+$

- Show that T maps $\mathscr{C}$ into itself
- Show that v^* is concave and increasing

Regarding uniqueness, observe that

$$\operatorname*{argmax}_{0 \leqslant c \leqslant y} \left\{ u(c) + \beta \int v(f(y-c)z)\varphi(dz) \right\}$$

is a singleton

- why?
- why does this imply uniqueness of the optimal policy?

To compute v^* we can use value function iteration

Pick intial v_0 in $bc\mathbb{R}_+$ and iterate with

$$Tv(y) = \max_{0 \leqslant c \leqslant y} \left\{ u(c) + \beta \int v(f(y-c)z)\varphi(dz) \right\}$$

But how to store Tv, T^2v, etc.?

Options:

1. Discretize the whole model
2. Use interpolation over a grid to store $T^k v$ at each k

The second option

- is less susceptible to the curse of dimensionality
- allows us to track errors

We will focus on **piecewise linear interpolation**

Advantages

- preserves monontonicity of interpolant
- preserves shape properties like concavity / convexity
- preserves contractivity of the Bellman operator

For details see the course notes

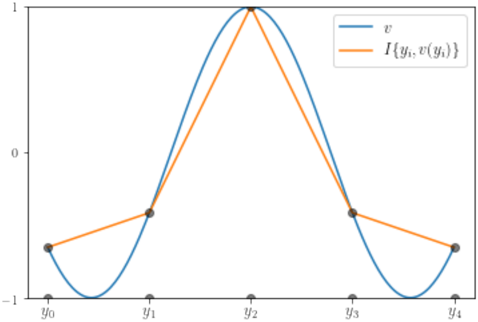

Figure: Approximation by piecewise linear interpolation

```
draw {z_j} ~IID φ ;
input grid G_n := {y_i}_{i=0}^{n-1} ⊂ R_+ ;
input {v_0(y_i)}_{i=0}^{n-1}, an initial guess of v* evaluated on G_n ;
input error tolerance τ and set ϵ ← τ + 1 ;
k ← 0 ;
while ϵ > τ do
    v_k ← I{y_i, v_k(y_i)} ;                    // interpolated function
    for i ∈ {0, ..., n − 1} do
        v_{k+1}(y_i) ← max_{0 ≤ c ≤ y_i} { u(c) + β (1/m) Σ_{j=1}^{m} v_k(f(y_i − c) z_j) } ;
    end
    ϵ ← max_i |v_k(y_i) − v_{k+1}(y_i)| ;
    k ← k + 1 ;
end
return v_k
```

Draw $\{z_j\} \overset{\text{IID}}{\sim} \varphi$; input grid $G_n := \{y_i\}_{i=0}^{n-1} \subset \mathbb{R}_+$; input $\{v_0(y_i)\}_{i=0}^{n-1}$; update $v_{k+1}(y_i) \leftarrow \max_{0 \leqslant c \leqslant y_i} \left\{ u(c) + \beta \frac{1}{m} \sum_{j=1}^{m} v_k(f(y_i - c) z_j) \right\}$; $\epsilon \leftarrow \max_i |v_k(y_i) - v_{k+1}(y_i)|$.

See `opt_growth.ipynb`

The Envelope Condition

We can get additional characterizations of optimality if we impose more conditions

Assumption. (INA) Both f and u are strictly increasing, continuously differentiable and strictly concave

In addition,

$$f(0) = 0, \quad \lim_{k \to 0} f'(k) > 0$$

and

$$u(0) = 0, \quad \lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0$$

Remark. We ignore the restriction $u(0) = 0$ in some applications below — I'm aiming to remove it

Proposition. Let

- v be an increasing concave function in $bc\mathbb{R}_+$
- σ be the unique v-greedy policy in Σ

If assumption (INA) holds, then

1. σ is strictly increasing and interior, while
2. Tv is strictly concave, strictly increasing, continuously differentiable and satisfies

$$(Tv)' = u' \circ \sigma$$

Corollary If σ^* is the optimal consumption policy, then

$$(v^*)' = u' \circ \sigma^*$$

Proof that $(Tv)' = u' \circ \sigma$ when σ is v-greedy:

Evaluating the RHS of the Bellman operator at its maximum gives

$$Tv(y) = u(\sigma(y)) + \beta \int v(f(y - \sigma(y))z)\varphi(dz)$$

By the **envelope theorem**,

$$(Tv)'(y) = \beta \int (v)'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz)$$

The FOC from the Bellman equation yields

$$u'(\sigma(y)) = \beta \int (v)'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz)$$

Combining the last two equations gives $(Tv)' = u' \circ \sigma$

Let $\mathscr{C}$ be all continuous strictly increasing functions on $\mathbb{R}_+$ satisfying $0 < \sigma(y) < y$

We say that $\sigma \in \mathscr{C}$ **satisfies the Euler equation** if

$$(u' \circ \sigma)(y) = \beta \int (u' \circ \sigma)(f(y - \sigma(y))z) f'(y - \sigma(y)) z \varphi(dz)$$

for all $y > 0$

Let's introduce an operator K corresponding to this functional equation

For each $\sigma \in \mathscr{C}$ and each $y > 0$, the value $K\sigma(y)$ is the c in $(0, y)$ that solves

$$u'(c) = \beta \int (u' \circ \sigma)(f(y - c)z) f'(y - c) z \varphi(dz)$$

We call K the **Coleman–Reffett** operator

Proof that K well defined:

For any $\sigma \in \mathscr{C}$, the RHS of

$$u'(c) = \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c) z \varphi(dz)$$

is continuous and strictly increasing in c on $(0, y)$, diverges to $+\infty$ as $c \uparrow y$

The LHS is continuous and strictly decreasing in c on $(0, y)$, diverges to $+\infty$ as $c \downarrow 0$

Hence

$$H(y, c) := u'(c) - \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c) z \varphi(dz)$$

when regarded as a function of c, has exactly one zero

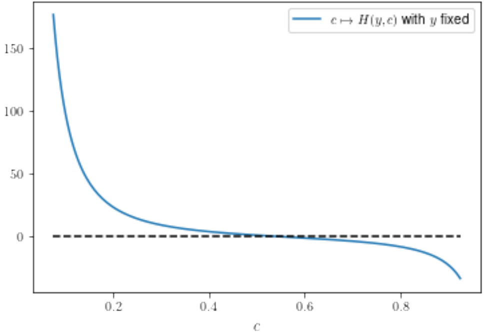

Figure: Solving for the c that satisfies $H(y,c) = 0$.

Necessity and Sufficiency of the Euler Equation

Ex. Show σ in $\mathscr{C}$ is a fixed point of K if and only if it satisfies the Euler equation

Proposition. If assumption (INA) holds and σ^* is the unique optimal policy, then

1. $(\mathscr{C}, K)$ is globally stable and
2. the unique fixed point of K in $\mathscr{C}$ is σ^*

In particular, $\sigma \in \mathscr{C}$ is optimal if and only it satisfies the Euler equation

Sketch of proof:

Let $\mathscr{V}$ be all strictly concave, continuously differentiable v mapping $\mathbb{R}_+$ to itself and satisfying $v(0) = 0$ and $v'(y) > u'(y)$ whenever $y > 0$

As before, let $\mathscr{C}$ be all a continuous, strictly increasing functions on $\mathbb{R}_+$ satisfying $0 < \sigma(y) < y$

For $v \in \mathscr{V}$ let Mv be defined by

$$(Mv)(y) = \begin{cases} m(v'(y)) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases} \tag{1}$$

where $m(y) := (u')^{-1}(y)$

The course notes show that

1. M is a homeomorphism from $\mathscr{V}$ to $\mathscr{C}$
2. for every increasing concave function in $bc\mathbb{R}_+$, the function MTv is the unique v-greedy policy
3. The Bellman operator and Coleman–Reffett operator are related by
$$T = M^{-1} \circ K \circ M \text{ on } \mathscr{V}$$

Ex. Use 1–3 above to show that $(\mathscr{C}, K)$ is globally stable with unique fixed point σ^*

Remark. The Euler equation is often paired with the **transversality condition**

$$\lim_{t \to \infty} \beta^t \mathbb{E} u'(c_t) k_t = 0$$

Standard results (see, e.g., Stokey and Lucas) tell us that

$$\text{Euler} + \text{transversality condition} \implies \text{optimality}$$

Our last result shows transversality is not needed under our assumptions

Ex. Following the basic CRRA cake eating model, set

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad f(k)z = Rk$$

Insert the conjecture $\sigma^*(y) = \theta y$ into the Euler equation

Recover our earlier result that this policy is optimal when

$$\theta = 1 - \left(\beta R^{1-\gamma}\right)^{1/\gamma}$$

Ex. Repeat for the log / CD model, where $u(c) = \ln c$ and

$$f(k)z = Ak^{\alpha} z, \quad 0 < A, \quad 0 < \alpha < 1$$

Insert the conjecture $\sigma^*(y) = \theta y$ into the Euler equation and recover your earlier result for the optimal policy