ECON-GA 1025 Macroeconomic Theory I

Lecture 9

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Today's Lecture

- Job search and monotonicity
- Search with learning
- Search with correlated wage offers

Prequel I: Review of FOSD

Let F and G be CDFs on $\mathbb{R}_+$

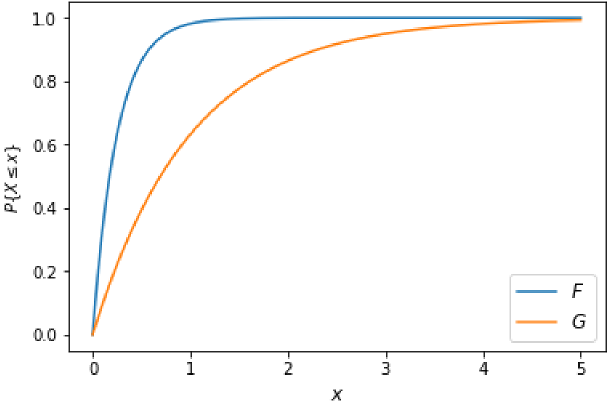
<u>Reminder</u>: F is **first order stochastically dominated** by distribution G (write $F \preceq_{\mathrm{SD}} G$) if

$$\int u(x) F(\mathrm{d}x) \leqslant \int u(x) G(\mathrm{d}x) \text{ for all } u \in ibc\mathbb{R}_+$$

Equivalent to $F \preceq_{\mathrm{SD}} G$:

- $G \leqslant F$ pointwise on $\mathbb{R}_+$
- There exists random variables X and Y with

$$X \stackrel{\mathscr{D}}{=} F, \quad Y \stackrel{\mathscr{D}}{=} G, \quad \mathbb{P}\{X \leqslant Y\} = 1$$



1.0
0.8
0.6
0.4
0.2
0.0
P{X ≤ x}
0
1
2
3
4
5
x
F
G

Prequel II: Monotone Likelihood Ratios

Positive densities (f, g) on interval $I \subset \mathbb{R}$ are said to have a **monotone likelihood ratio** if

$$x, x' \in I \text{ and } x \leqslant x' \implies \frac{f(x)}{g(x)} \leqslant \frac{f(x')}{g(x')}$$

Example. The exponential density is

$$p(x, \lambda) = \lambda e^{-\lambda x} \qquad (x \in \mathbb{R}_+,\ \lambda > 0)$$

Taking $\lambda_1 \leqslant \lambda_2$, we have

$$\frac{p(x, \lambda_1)}{p(x, \lambda_2)} = \frac{\lambda_1}{\lambda_2} \exp((\lambda_2 - \lambda_1)x)$$

Ex. Let (f, g) be given by

$$f = \text{Beta}(4, 2) \quad \text{and} \quad g = \text{Beta}(2, 4)$$

Show that (f, g) has the monotone likelihood ratio property

- Hint: the Gamma function is increasing on $[2, 4]$

Fact. If (f, g) has a monotone likelihood ratio on I, then

$$g \preceq_{\mathrm{SD}} f$$

Proof sketch:

Let F and G be the corresponding CDFS

Course notes show MLR implies $F(y) \leqslant G(y)$ for all $y \in I$

This is equivalent to $G \preceq_{\mathrm{SD}} F$

Job Search Continued: Second Order Stochastic Dominance

How does the **volatility** of the wage process impact on the reservation wage?

Intuitively, greater volatility means

- option value of waiting is larger
- encourages patience — higher reservation wage

But how can we isolate the effect of volatility?

- introduce the notion of a **mean-preserving spread**

Given distribution ψ, we say that φ is a **mean-preserving spread** of ψ if $\exists$ random variables (Y, Z) such that

$$Y \stackrel{\mathscr{D}}{=} \psi, \quad Y + Z \stackrel{\mathscr{D}}{=} \varphi \quad \text{and} \quad \mathbb{E}[Z \,|\, Y] = 0$$

- adds noise without changing the mean

Related definition: ψ **second order stochastically dominates** φ if, with $\mathscr{U}$ as the concave functions in $ibc\mathbb{R}_+$,

$$\int u(x)\varphi(\mathrm{d}x) \leqslant \int u(x)\psi(\mathrm{d}x) \text{ for all } u \in \mathscr{U}$$

Fact. ψ second order stochastically dominates φ if and only if φ is a mean-preserving spread of ψ

Proof that φ is a mean-preserving spread of $\psi \implies \psi$ second order stochastically dominates φ

Let φ be a mean-preserving spread of ψ

Then $\exists$ random pair (Y, Z) such that

$$Y \stackrel{\mathscr{D}}{=} \psi, \quad Y + Z \stackrel{\mathscr{D}}{=} \varphi \quad \text{and} \quad \mathbb{E}[Z \mid Y] = 0$$

Fixing arbitrary $u \in \mathscr{U}$ and applying Jensen's inequality,

$$\mathbb{E}\, u(Y + Z) = \mathbb{E}\, \mathbb{E}[u(Y + Z) \mid Y] \leqslant \mathbb{E}\, u(\mathbb{E}[Y + Z \mid Y]) = \mathbb{E}\, u(Y)$$

$$\therefore \quad \int u(x) \varphi(\mathrm{d}x) = \mathbb{E}\, u(Y + Z) \leqslant \mathbb{E} u(Y) = \int u(x) \psi(\mathrm{d}x)$$

How does the unemployed agent react to a **mean-preserving spread in the offer distribution**?

Prop. If φ is a mean-preserving spread of ψ, then $w^*_\psi \leqslant w^*_\varphi$

Proof: It suffices to show that $h^*_\psi \leqslant h^*_\varphi$ (why?)

Claim: $g(h) = c + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$ increases pointwise with the mean-preserving spread

Equivalently, for all $h \geqslant 0$,

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(\mathrm{d}w') \leqslant \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$$

By definition, there exists a (w', Z) such that $\mathbb{E}[Z \mid w'] = 0$, $w' \stackrel{\mathscr{D}}{=} \psi$ and $w' + Z \stackrel{\mathscr{D}}{=} \varphi$

By this fact and the law of iterated expectations,

$$\begin{aligned}
\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w') &= \mathbb{E}\left[\max\left\{\frac{w'+Z}{1-\beta}, h\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\max\left\{\frac{w'+Z}{1-\beta}, h\right\} \,\middle|\, w'\right]\right]
\end{aligned}$$

Jensen's inequality now produces

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w') \geqslant \mathbb{E} \max\left\{\frac{\mathbb{E}[w'+Z \mid w']}{1-\beta}, h\right\}$$

Using $\mathbb{E}[w' \mid w'] = w'$ and $\mathbb{E}[Z \mid w'] = 0$ leads to

$$\begin{aligned}
\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w') &\geqslant \mathbb{E} \max\left\{\frac{\mathbb{E}[w' + Z \mid w']}{1-\beta}, h\right\} \\
&= \mathbb{E} \max\left\{\frac{w'}{1-\beta}, h\right\} \\
&= \int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(\mathrm{d}w')
\end{aligned}$$

Since h was arbitrary, the function g shifts up pointwise

Since g is isotone and a contraction, this completes the proof

Second Order Stochastic Dominance and Welfare

How does volatility affect **welfare**?

Do mean-preserving spreads have a monotone impact on lifetime value?

More precisely, with

- φ as a mean-preserving spread of ψ
- v_φ and v_ψ as the corresponding value functions

do we have $v_\psi \leqslant v_\varphi$?

Why might this be true?

Prop. If φ is a mean-preserving spread of ψ, then $v_\psi \leqslant v_\varphi$ on $\mathbb{R}_+$

Proof: For a fixed distribution ν, the value function v_ν satisfies

$$v_\nu(w) = \max\left\{\frac{w}{1-\beta}, h_\nu\right\}$$

where the continuation value

$$h_\nu := c + \beta \int v_\nu(w')\nu(\mathrm{d}w')$$

is the fixed point of

$$g_\nu(h) := c + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \nu(\mathrm{d}w')$$

If $h_\psi \leqslant h_\varphi$, then the result is immediate

Let φ be a mean-preserving spread of ψ

Since g_φ is isotone and globally stable on $\mathbb{R}_+$, it suffices to show that

$$g_\psi(h) \leqslant g_\varphi(h) \quad \forall\, h \in \mathbb{R}_+$$

So fix $h \in \mathbb{R}_+$

It is enough to show that

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(\mathrm{d}w') \leqslant \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$$

We already proved this...

Learning the Offer Distribution

Unrealistic assumptions in the previous job search model

- Wage offer distribution never changes
- Unemployed workers know the distribution

More realistic

- The offer distribution shifts around
- Unemployed workers need to learn and re-learn it

Let's study the learning component

- Offer distribution is constant but initially unknown

There are two possible offer distributions, F and G

- with densities f and g on $\mathbb{R}_+$

At the start of time, nature selects q to be either f or g

- entire sequence $\{w_t\}_{t \geqslant 0}$ will be drawn from q

The choice q is not observed by the worker, who puts prior probability $\pi_0 \in (0, 1)$ on f

Thus, the worker's initial guess of q is

$$q_0(w) := \pi_0 f(w) + (1 - \pi_0) g(w)$$

Beliefs update according to **Bayes' rule**

The agent observes w_{t+1}, updates π_t to

$$\pi_{t+1} = \frac{f(w_{t+1})\pi_t}{f(w_{t+1})\pi_t + g(w_{t+1})(1 - \pi_t)}$$

In more intuitive notation, this is

$$\mathbb{P}\{q = f \,|\, w_{t+1}\} = \frac{\mathbb{P}\{w_{t+1} \,|\, q = f\}\mathbb{P}\{q = f\}}{\mathbb{P}\{w_{t+1}\}}$$

We used the law of total probability for the denominator:

$$\mathbb{P}\{w_{t+1}\} = \sum_{\psi \in \{f, g\}} \mathbb{P}\{w_{t+1} \,|\, q = \psi\}\mathbb{P}\{q = \psi\}$$

Dropping time subcripts, let

$$q_\pi := \pi f + (1 - \pi) g$$

- estimate of the offer distribution based on current belief π

In addition, let

$$\kappa(w, \pi) := \frac{\pi f(w)}{\pi f(w) + (1 - \pi) g(w)}$$

- the updated value π' of π having observed draw w

Let $v^*(w, \pi) :=$ maximal lifetime value attainable from state (w, π) conditional on currently being unemployed

Bellman equation:

$$v^*(w, \pi) = \max \left\{ \frac{w}{1-\beta}, c + \beta \int v^*(w', \kappa(w', \pi)) \, q_\pi(w') \, \mathrm{d}w' \right\}$$

Note that π is a state variable

- affects the worker's perception of probabilities for future rewards
- known as the current **belief state**

The optimal policy: select the option that maximizes the RHS

Solution Methods

We can use value function iteration to calculate v^*

1. Introduce a Bellman operator T corresponding to the Bellman equation
2. Choose initial guess v_0
3. Iterate with T

But there is a more efficient approach — allows us to eliminate one state variable

Let $w^*(\pi)$ be the reservation wage at belief state π

- wage at which worker is indifferent between accepting, rejecting
- and therefore satisfies

$$\frac{w^*(\pi)}{1-\beta} = c + \beta \int v^*(w', \kappa(w', \pi))\, q_\pi(w')\ \mathrm{d}w'$$

Note that w^* is a function of **one** argument

So let's try to compute w^* directly

Combine

$$v^*(w,\pi) = \max\left\{\frac{w}{1-\beta},\, c + \beta \int v^*(w', \kappa(w', \pi))\, q_\pi(w')\; \mathrm{d}w'\right\}$$

and

$$\frac{w^*(\pi)}{1-\beta} = c + \beta \int v^*(w', \kappa(w', \pi))\, q_\pi(w')\; \mathrm{d}w'$$

to get

$$v^*(w,\pi) = \max\left\{\frac{w}{1-\beta},\, \frac{w^*(\pi)}{1-\beta}\right\}$$

Ex. Show that these last two equations lead to

$$w^*(\pi) = (1-\beta)c + \beta \int \max\left\{w', w^*[\kappa(w', \pi)]\right\} q_\pi(w')\; \mathrm{d}w'$$

To repeat, the reservation wage satisfies

$$w^*(\pi) = (1-\beta)c + \beta \int \max\left\{w', w^*[\kappa(w', \pi)]\right\} q_\pi(w')\ \mathrm{d}w'$$

Thus, it is a solution to the functional equation in ω given by

$$\omega(\pi) = (1-\beta)c + \beta \int \max\left\{w', \omega[\kappa(w', \pi)]\right\} q_\pi(w')\ \mathrm{d}w'$$

This leads us to introduce the operator

$$(Q\omega)(\pi) = (1-\beta)c + \beta \int \max\left\{w', \omega[\kappa(w', \pi)]\right\}\ q_\pi(w')\ \mathrm{d}w'$$

Fixed points of Q coincide with solutions to the functional equation

Let $\mathscr{C} := bc(0,1)$, paired with the supremum distance d_∞

- a complete metric space?

Assume: f, g are everywhere positive on $[0, M]$ and zero elsewhere

Prop. Under this assumption, the operator

$$(Q\omega)(\pi) = (1-\beta)c + \beta \int \max\left\{w', \omega[\kappa(w',\pi)]\right\} q_\pi(w') \, \mathrm{d}w'$$

is a contraction of modulus β on $\mathscr{C}$

The proof makes use of our max / abs inequality

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Proof: First we need to show that Q is a self-mapping on $\mathscr{C}$

Step 1 (boundedness): Pick any $\omega \in \mathscr{C}$ and consider

$$(Q\omega)(\pi) = (1-\beta)c + \beta \int \max \left\{ w', \omega[\kappa(w', \pi)] \right\} q_{\pi}(w') \, \mathrm{d}w'$$

Observe that, by

- the triangle inequality and
- the fact that q_{π} is a density,

$$|(Q\omega)(\pi)| \leqslant (1-\beta)c + \beta \max\{M, \|\omega\|_{\infty}\}$$

RHS does not depend on π so $Q\omega$ is bounded

Step 2 (continuity): Is $Q\omega$ continuous when $\omega \in \mathscr{C}$?

Suffices to show that $\pi_n \to \pi \in (0,1) \implies$

$$\int \max\left\{w', \omega[\kappa(w', \pi_n)]\right\} q_{\pi_n}(w') \, \mathrm{d}w'$$

$$\to \int \max\left\{w', \omega[\kappa(w', \pi)]\right\} q_{\pi}(w') \, \mathrm{d}w'$$

For fixed w', both $\kappa(w', \pi)$ and $q_\pi(w')$ are continuous in π

Moreover, $H_n(w') := \max\left\{w', \omega[\kappa(w', \pi_n)]\right\} q_{\pi_n}(w')$ satisfies

$$\sup_n |H_n(w')| \leqslant \max\left\{M, \|\omega\|_\infty\right\} (f(w') + g(w'))$$

Now apply the DCT

Step 3 (contractivity): Fixing $\omega, \varphi \in \mathscr{C}$ and $\pi \in (0,1)$, we have

$$|(Q\omega)(\pi) - (Q\varphi)(\pi)| \leqslant \beta \times$$

$$\int \left|\max\left\{w', \omega[\kappa(w', \pi)]\right\} - \max\left\{w', \varphi[\kappa(w', \pi)]\right\}\right| q_\pi(w') \, \mathrm{d}w'$$

Combining this with our max / abs inequality,

$$|(Q\omega)(\pi) - (Q\varphi)(\pi)| \leqslant \beta \int \left|\omega[\kappa(w', \pi)] - \varphi[\kappa(w', \pi)]\right| q_\pi(w') \, \mathrm{d}w'$$

$$\leqslant \beta \|\omega - \varphi\|_\infty$$

Taking the sup over π gives us

$$\|Q\omega - Q\varphi\|_\infty \leqslant \beta \|\omega - \varphi\|_\infty$$

Putting our results together:

- Q is a contraction of modulus β
- on the complete metric space $(\mathscr{C}, d_\infty)$
- Hence a unique solution w^* to the reservation wage functional equation exists in $\mathscr{C}$
- $Q^k \omega \to w^*$ uniformly as $k \to \infty$, for any $\omega \in \mathscr{C}$

Let's compute w^* when

$$f = \text{Beta}(4, 2) \quad \text{and} \quad g = \text{Beta}(2, 4)$$

The other parameters are $c =$ either 0.1 or 0.2 and $\beta = 0.95$

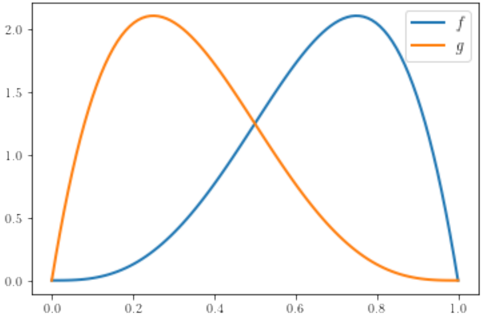


Figure: The two unknown densities f and g

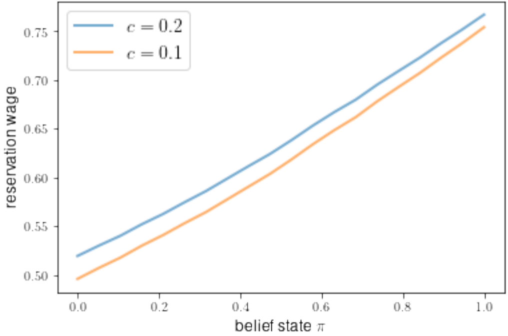


Figure: Reservation wage as a function of beliefs

See the notebook `odu.ipynb`

Note that w^*

(a) shifts upwards when c increases and

(b) is monotonically increasing in π

Ex. Prove that (a) always holds

Result (b) is also intuitive:

- The density f is likely to lead to better draws
- as our belief shifts toward f, we anticipate higher wage offers
- hence our reservation wage should increase

Can we prove this result? If so, what conditions are required on f and g?

Proposition. If (f, g) has a monotone likelihood ratio, then w^* is increasing in π

Proof: Let f and g have the stated property

Let $i\mathscr{C}$ be all increasing functions in $\mathscr{C}$

Ex. Show this is a closed subset of $\mathscr{C}$

Hence it suffices to show that $Q\omega$ is in $i\mathscr{C}$ whenever $\omega \in i\mathscr{C}$

So pick any $\omega \in i\mathscr{C}$

We know that $Q\omega$ is in $\mathscr{C}$

Thus, only need to show that $Q\omega$ is increasing

To repeat, we need to show that

$$(Q\omega)(\pi) = (1-\beta)c + \beta \int \max\left\{w', \omega[\kappa(w', \pi)]\right\} q_\pi(w') \, \mathrm{d}w'$$

is increasing in π when ω is increasing

For $Q\omega$ to be increasing, it suffices that, with

$$h(w', \pi) := \omega \left[\frac{\pi f(w')}{\pi f(w') + (1 - \pi) g(w')} \right]$$

the function

$$\pi \mapsto \int \max \left\{ w', h(w', \pi) \right\} q_\pi(w') \, \mathrm{d}w'$$

is increasing

This will be true if we can establish that

1. $\pi \mapsto q_\pi$ is isotone with respect to $\preceq_{\mathrm{SD}}$
2. h is increasing in both π and w' and

The fact that $\pi \mapsto q_\pi$ is isotone with respect to $\preceq_{\mathrm{SD}}$ follows from the next exercise

Ex. Let

- f and g be two densities on $\mathbb{R}$ with $g \preceq_{\mathrm{SD}} f$
- ν_α be the convex combination defined by

$$\nu_\alpha := \alpha f + (1 - \alpha) g \qquad (0 \leqslant \alpha \leqslant 1)$$

Show that $\alpha \leqslant \beta$ implies $\nu_\alpha \preceq_{\mathrm{SD}} \nu_\beta$

Conclude that $\pi \mapsto q_\pi$ is isotone with respect to $\preceq_{\mathrm{SD}}$

Remains to show that

$$h(w', \pi) := \omega \left[\frac{\pi f(w')}{\pi f(w') + (1 - \pi) g(w')} \right]$$

is increasing in both π and w' and

To see this, write h as

$$h(w', \pi) = \omega \left[\frac{1}{1 + [(1 - \pi)/\pi][g(w')/f(w')]} \right]$$

Increasing in both args because ω is increasing, $g(w')/f(w')$ is decreasing in w'

Correlated Wage Draws

Suppose now that

- the wage distribution is known
- wages = **persistent** + **transient component**

In particular,

$$w_t = \exp(z_t) + \exp(\mu + \sigma \zeta_t)$$

where

- $\{\zeta_t\}_{t \geqslant 1} \overset{\text{IID}}{\sim} N(0, 1)$ and
- $z_{t+1} = \rho z_t + d + s\epsilon_{t+1}$ with $\{\epsilon_t\}_{t \geqslant 1} \overset{\text{IID}}{\sim} N(0, 1)$

Regarding the state process

$$z_{t+1} = \rho z_t + d + s\epsilon_{t+1}, \quad \{\epsilon_t\}_{t \geqslant 1} \overset{\text{IID}}{\sim} N(0,1)$$

- Assume that $-1 < \rho < 1$
- Hence globally stable

The unique stationary density on $\mathbb{R}$ is

$$\psi := N\left(\frac{d}{1-\rho}, \frac{s^2}{1-\rho^2}\right)$$

Otherwise the model is unchanged

The value function satisfies the Bellman equation

$$v(w,z) = \max\left\{\frac{w}{1-\beta}, c + \beta\, \mathbb{E}_z v(w', z')\right\}$$

Here $\mathbb{E}_z$ is expectation conditional on z

For example, given g and $z \in \mathbb{R}$,

$$\mathbb{E}_z g(w', z') =$$

$$\int g\left[\exp(\rho z + d + s\epsilon) + \exp(\mu + \sigma\zeta), \rho z + d + s\epsilon\right] \varphi(\mathrm{d}\epsilon, \mathrm{d}\zeta)$$

where $\varphi := N(0, I)$ on $\mathbb{R}^2$

Solution methods:

1. Introduce a Bellman operator corresponding to the Bellman eq.
2. Reduce dimensionality by refactoring

Second, method, first step: let

$$
\begin{aligned}
h(z) &:= \text{ continuation value associated with state } z \\
&= c + \beta\, \mathbb{E}_z v(w', z')
\end{aligned}
$$

Here

- v can be thought of as a candidate value function
- continuation val depends on z because we use it to forecast

Given $h(z)$, the Bellman equation can be written as

$$v(w, z) = \max\left\{\frac{w}{1-\beta}, h(z)\right\}$$

Combining this with the definition of h, we see that

$$h(z) = c + \beta\, \mathbb{E}_z \max\left\{\frac{w'}{1-\beta}, h(z')\right\} \qquad (z \in \mathbb{R})$$

With a solution h^*, we can act optimally via the policy

$$\sigma^*(w, z) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*(z)\right\}$$

- $\iff$ stop when $w \geqslant w^*(z) := h^*(z)(1-\beta)$

How to solve the functional equation?

$$h(z) = c + \beta \, \mathbb{E}_z \max \left\{ \frac{w'}{1-\beta}, h(z') \right\} \qquad (z \in \mathbb{R})$$

We introduce the operator $h \mapsto Qh$ defined by

$$Qh(z) = c + \beta \, \mathbb{E}_z \max \left\{ \frac{w'}{1-\beta}, h(z') \right\}$$

- Any solution to the functional equation is a fixed point of Q and vice versa

But does such a fixed point exist? Is it unique?

Our last few contraction arguments have used distance d_∞

- requires Q maps bounded functions to bounded functions

Fails here because, even if h is bounded,

$$
\begin{aligned}
Qh(z) &= c + \beta\, \mathbb{E}_z \max\left\{\frac{w'}{1-\beta}, h(z')\right\} \\
&= c + \beta\, \mathbb{E} \max\left\{\frac{\exp(\rho z + d + s\epsilon_{t+1}) + \exp(\mu + \sigma \zeta_{t+1})}{1-\beta}, h(z')\right\} \\
&\geqslant \beta\, \mathbb{E}\, \exp(\rho z + d + s\epsilon_{t+1})
\end{aligned}
$$

is unbounded in z

This means that

- The solution we seek is unbounded
- We need to use a different metric space

The metric space must

- admit unbounded functions
- be complete, so we can use a contraction argument

Let $L_1(\psi) :=$ all Borel measurable functions g from $\mathbb{R}$ to itself satisfying

$$\int |g(x)|\psi(x)\,\mathrm{d}x < \infty$$

- ψ is the stationary density of $\{z_t\}$
- Equivalent: $g(z_t)$ has finite first moment when $z_t \stackrel{\mathscr{D}}{=} \psi$

The distance between f, g in $L_1(\psi)$ is given by

$$d_1(f,g) := \int |f(x) - g(x)|\psi(x)\,\mathrm{d}x$$

- the space $(L_1(\psi), d_1)$ is complete

Lemma. Q is a self-mapping on $L_1(\psi)$

Proof: Fix $h \in L_1(\psi)$

We need to show that $Qh \in L_1(\psi)$

Suffices to show that

$$\kappa(z) := \mathbb{E}_z \max\left\{\frac{w'}{1-\beta}, h(z')\right\}$$

lies in $L_1(\psi)$

In other words, we need to show that

$$\mathbb{E}\,|\kappa(z_t)| = \int |\kappa(z)|\psi(z)\,\mathrm{d}z < \infty$$

For nonnegative numbers a, b, we have $a \vee b \leqslant a + b$, and hence, for any $z \in \mathbb{R}$,

$$\kappa(z) \leqslant \frac{1}{1-\beta} \mathbb{E}_z \left[\exp(z') + \exp(\mu + \sigma\zeta) + |h(z')| \right]$$

Let z_t be a draw from ψ, the preceding inequality yields

$$\mathbb{E}\kappa(z_t) \leqslant \frac{1}{1-\beta} \mathbb{E}\, \mathbb{E}_{z_t} [\exp(z_{t+1}) + \exp(\mu + \sigma\zeta_{t+1}) + |h(z_{t+1})|]$$

$$= \frac{1}{1-\beta} \mathbb{E} \left[\exp(z_{t+1}) + \exp(\mu + \sigma\zeta_{t+1}) + |h(z_{t+1})| \right]$$

$$\propto \mathbb{E} \exp(z_{t+1}) + \mathbb{E} \exp(\mu + \sigma\zeta_{t+1}) + \mathbb{E}\, |h(z_{t+1})|$$

Hence the proof will be done if

$$\mathbb{E}\,\exp(z_{t+1}) + \mathbb{E}\,\exp(\mu + \sigma\zeta_{t+1}) + \mathbb{E}\,|h(z_{t+1})| < \infty$$

Here $z_{t+1} = \rho z_t + d + s\epsilon_{t+1}$

- $\mathbb{E}\,\exp(z_{t+1}) < \infty$ because ?
- $\mathbb{E}\,\exp(\mu + \sigma\zeta_{t+1}) < \infty$ because ?
- $\mathbb{E}\,|h(z_{t+1})| < \infty$ because ?

Prop. Q is a contraction of modulus β on $L_1(\psi)$

Proof: By the inequality $|\alpha \vee x - \alpha \vee y| \leqslant |x - y|$ we have

$$|Qg(z) - Qh(z)| \leqslant \beta \, \mathbb{E}_z \left| \max \left\{ \frac{w'}{1-\beta}, g(z') \right\} - \max \left\{ \frac{w'}{1-\beta}, h(z') \right\} \right|$$

$$\leqslant \beta \, \mathbb{E}_z \left| g(z') - h(z') \right|$$

Let z_t be drawn from ψ

By the last inequality, for any t,

$$|Qg(z_t) - Qh(z_t)| \leqslant \beta \, \mathbb{E}_{z_t} |g(z_{t+1}) - h(z_{t+1})|$$

Taking expectations gives

$$\mathbb{E}\,|Qg(z_t) - Qh(z_t)| \leqslant \beta\,\mathbb{E}\,\mathbb{E}_{z_t}\,|g(z_{t+1}) - h(z_{t+1})|$$
$$= \beta\,\mathbb{E}\,|g(z_{t+1}) - h(z_{t+1})|$$

Since $z_t \stackrel{\mathscr{D}}{=} \psi$, we have $z_{t+1} \stackrel{\mathscr{D}}{=} \psi$, so the last inequality becomes

$$\int |Qg(z) - Qh(z)| \psi(z)\,\mathrm{d}z \leqslant \beta \int |g(z) - h(z)|\,\psi(z)\,\mathrm{d}z$$

or

$$\|Qg - Qh\| \leqslant \beta\,\|g - h\|$$

Ex. Let $c_a \leqslant c_b$ be two levels of unemployment compensation satisfying

Show that $h_a^* \leqslant h_b^*$ pointwise on $\mathbb{R}$, where h_i^* is the continuation value corresponding to c_i

Ex. Give a condition under which the reservation wage

$$w^*(z) := (1 - \beta) h^*(z)$$

is increasing in z

Show that your condition is sufficient

Interpret your result, provide economic intuition

Ex. Suppose the agent seeks to maximize lifetime value

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(y_t)$$

where y_t is earnings at time t and u is a utility function

Letting $u(c) = \ln c$, write down the modified Bellman equation and the Q operator

How does the reservation wage change?