ECON-GA 1025 Macroeconomic Theory I

Lecture 8

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Today's Lecture

- Comments on assessment
- Finish: Numerical methods for tracking distributions
- Start: Job Search

Shifting office hours to Wed 4:00–5:00

Style of exam:

- broad understanding of all course material
- applications of ideas

Exam prep:

- Weekly assignments
- Other **Ex.** in slides
- Review logic and applications in slides

Wealth Distributions: Estimation by Monte Carlo (Continued)

In the last lecture we studied estimation of the time t distribution Ψ_t using Monte Carlo

Method:

1. Compute sample $\{w_t^m\}$, time t wealth of m independent households
2. Calculate the empirical distribution

$$F_t^m(x) := \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{w_t^i \leqslant x\}$$

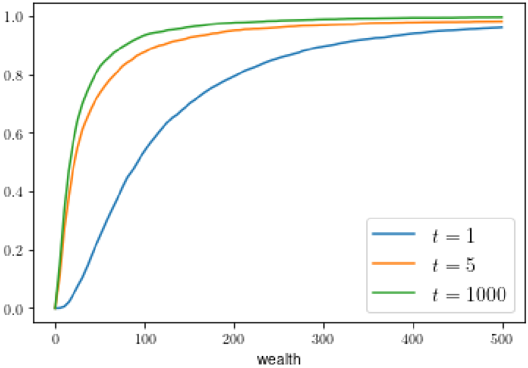

Figure: The empirical distribution F_m^t for different values of t

But we know that Ψ_t can be represented by a density ψ_t

This is structure that we would like to exploit

- helps when we get to high dimensional problems
- helps extract information from the tails

Unfortunately there is no natural estimator of densities that

- works in every setting (like the empirical distribution does)
- is always unbiased and consistent

Why?

- Empirical distributions just reflect the sample
- Density estimates must make statements about probability mass in the **neighborhood** of each observation

Let's look at our options

Option 1. **Nonparametric kernel density estimation**, where

$$\hat{f}_t^m(x) = \frac{1}{h} \sum_{i=1}^{m} K\left(\frac{x - w_t^i}{h}\right)$$

Here

- K is a density, called the **kernel**
- h is the **bandwidth** of the estimator

Idea:

- Put a smooth bump on each data point and then sum
- Larger h means smoother estimate

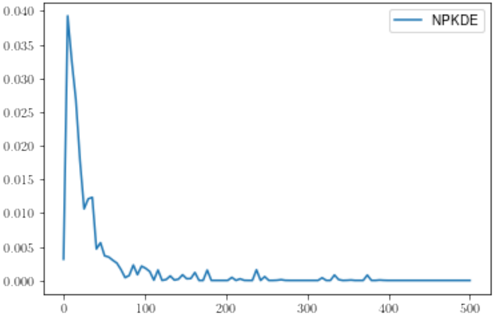

Figure: NPKDE of ψ_t using Scikit Learn

Option 2. The **look ahead estimator**

$$\ell_t^m(w') := \frac{1}{m}\sum_{i=1}^{m} \pi(w_{t-1}^i, w')$$

Note: sample $\{w_{t-1}^i\}$ is from time $t-1$

This makes the estimator **unbiased**

$$\mathbb{E}[\ell_t^m(w')] = \frac{1}{m}\sum_{i=1}^{m} \mathbb{E}[\pi(w_{t-1}^i, w')]$$

$$= \frac{m}{m}\int \pi(w, w')\psi_{t-1}(w)\,\mathrm{d}w = \psi_t(w')$$

From the SLLN, we also have

$$\ell_t^m(w') \to \mathbb{E}[\pi(w_{t-1}^i, w')] = \psi_t(w') \quad \text{as} \quad m \to \infty$$

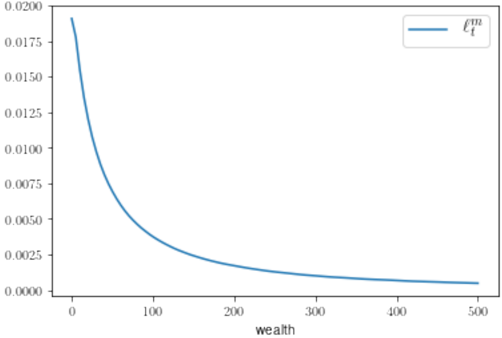

Figure: The look ahead estimate of ψ_t

Stability of the Wealth Process

Lemma. The dynamical system $(\mathcal{D}, \Pi)$ corresponding to the wealth process

$$w_{t+1} = R_{t+1} s(w_t) + y_{t+1}$$

is globally stable whenever

(a) y_t has finite first moment, $\varphi \gg 0$ and

(b) $\mathbb{E}[R_t] s(w) \leqslant \lambda w + L$ for some $\lambda < 1$ and $L < \infty$

If ψ^* is the stationary density and $\int |h(w)| \psi^*(w) \, \mathrm{d}w < \infty$, then, with prob one,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} h(w_t) = \int h(w) \psi^*(w) \, \mathrm{d}w$$

Ex. Apply the last result to the case

$$s(w) = \mathbb{1}\{w > \bar{w}\} s_0 w \qquad (w \geqslant 0)$$

- Here s_0 and $\bar{w}$ are positive parameters
- What conditions do you need to impose on the parameters in the model in order to get global stability?
- Can you give some interpretation?

A **stationary** density look ahead estimator:

$$\ell_n^*(w') := \frac{1}{n} \sum_{t=1}^{n} \pi(w_t, w')$$

- sample is a single time series $\{w_t\}$ generated by simulation

Consistent for $\psi^*(w')$, since, with probability one as $n \to \infty$,

$$\ell_n^*(w') = \frac{1}{n} \sum_{t=1}^{n} \pi(w_t, w') \to \int \pi(w, w') \psi^*(w) \, \mathrm{d}w = \psi^*(w')$$

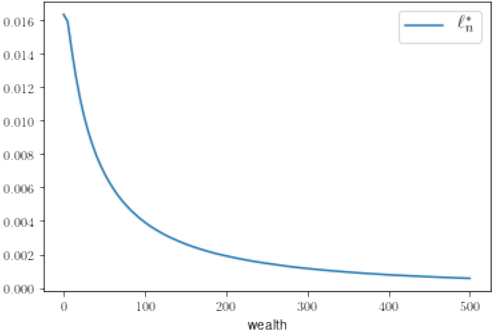

Figure: The stationary density look ahead estimator of the wealth distribution

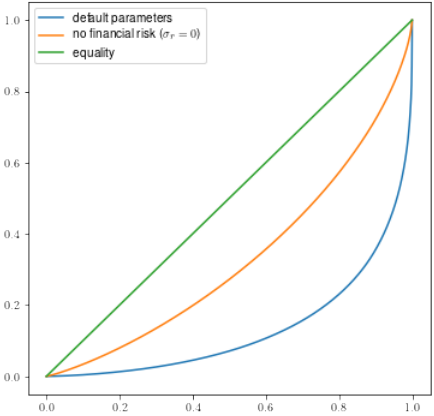

Figure: Lorenz curve, wealth distribution at default parameters

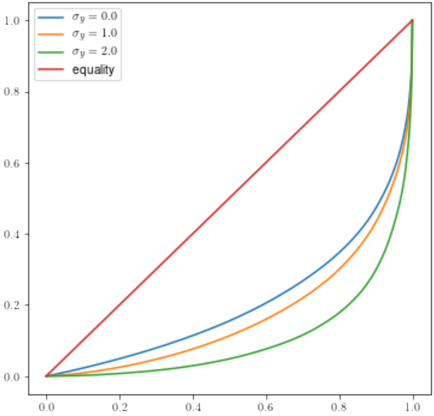

Figure: Lorenz curves with increasing variance in labor income

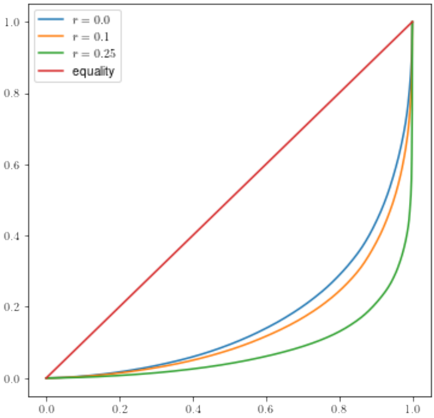

Figure: Lorenz curves with increasing rate of return on wealth

See notebook `wealth_ineq_plots.ipynb`

New Topic: Job Search

Our first deep dive into dynamic programming

- An integral part of labor and macroeconomics
- Relatively simple (binary choice)

Related to

- Optimal stopping
- Firm entry and exit decisions
- Pricing American options
- etc.

As discussed earlier

- Unemployed agent seeks to maximize

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t y_t$$

- Observes an employment opportunity with wage offer w_t
- Wage offers are IID and drawn from distribution φ
- Acceptance means lifetime value $w_t / (1 - \beta)$
- Rejection yields unemployment compensation $c \geqslant 0$ and a new offer next period

Overview

The **value function** $v^*(w) :=$ the maximal value that can be extracted from any given state w

It satisfies the **Bellman equation**

$$v^*(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \int v^*(w')\varphi(\mathrm{d}w')\right\} \qquad (w \in \mathbb{R}_+)$$

Optimal policy is then obtained via

$$\sigma^*(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant c + \beta \int v^*(w')\varphi(\mathrm{d}w')\right\}$$

To calculate the optimal policy we need to evaluate $\int v^*(w')\varphi(\mathrm{d}w')$

To compute v^*, we introduce the **Bellman operator**

$$Tv(w) := \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\varphi(\mathrm{d}w')\right\}$$

Note:

- fixed points of T exactly coincide with solutions to the Bellman equation

Simplifying assumption:

- There exists an $M \in \mathbb{R}_+$ such that $\int_0^M \varphi(\mathrm{d}w) = 1$

Case 1: Continuous Wage Draws

Assumption. The offer distribution φ is a density supported on $[0, M]$

Any w in $[0, M]$ is possible so v^* needs to be defined on $[0, M]$

Leads us to seek a fixed point of T in $\mathscr{C} :=$ all continuous functions on $[0, M]$ paired with

$$d_\infty(f, g) := \|f - g\|_\infty, \qquad \|g\|_\infty := \sup_{x \in [0,M]} |g(x)|$$

- $(\mathscr{C}, d_\infty)$ is a complete metric space

Question: Why restrict ourselves to continuous functions?

Proposition. In this setting, T is a contraction of modulus β on $\mathscr{C}$

In particular,

1. T has a unique fixed point in $\mathscr{C}$,
2. that fixed point is equal to the value function v^* and
3. if $v \in \mathscr{C}$, then $\|T^n v - v^*\|_\infty \leqslant O(\beta^n)$.

For now let's take (2) as given — we'll prove it soon

- Remainder will be verified if we show T is a contraction of modulus β on $(\mathscr{C}, d_\infty)$

We use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Fixing f, g in $\mathscr{C}$ and $w \in [0, M]$,

$$\begin{aligned} |Tf(w) - Tg(w)| &\leqslant \left| \beta \int f(w')\varphi(w')\, \mathrm{d}w' - \beta \int g(w')\varphi(w')\, \mathrm{d}w' \right| \\ &= \beta \left| \int [f(w') - g(w')]\varphi(w')\, \mathrm{d}w' \right| \\ &\leqslant \beta \int |f(w') - g(w')|\varphi(w')\, \mathrm{d}w' \leqslant \|f - g\|_\infty \end{aligned}$$

Taking the supremum over all $w \in [0, M]$ leads to

$$\|Tf - Tg\|_\infty \leqslant \|f - g\|_\infty$$

Ex. Show that T maps the set of increasing convex functions on the interval $[0, M]$ to itself

Ex. Show that v^* is increasing and convex on $[0, M]$

Case 2: Discrete Wage Draws

Let's swap the density assumption for a discrete distribution

Assumption. The offer distribution φ is supported on finite set W with probabilities $\varphi(w)$, $w \in W$

Now v^* need only be defined on these points

Hence we define T on $\mathbb{R}^W$ by

$$Tv(w) = \max\left\{\frac{w}{1-\beta}, c + \beta \sum_{w \in W} v(w)\varphi(w)\right\} \qquad (w \in W)$$

- $(\mathbb{R}^W, d_\infty)$ is a complete metric space

Proposition. In this setting, T is a contraction of modulus β on $\mathbb{R}^W$

In particular,

1. T has a unique fixed point in $\mathbb{R}^W$,
2. that fixed point is equal to the value function v^* and
3. if $v \in \mathbb{R}^W$, then $\|T^n v - v^*\|_\infty \leqslant O(\beta^n)$

Ex. Prove that T is a contraction of modulus β on $(\mathbb{R}^W, d_\infty)$

To compute the optimal policy we can use **value function iteration**

1. Start with arbitrary $v \in \mathbb{R}^W$
2. iterate with T until $v_k := T^k v$ is a good approximation to v^*

Then compute

$$\sigma_k(w) := \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant c + \beta \int v_k(w')\varphi(\mathrm{d}w')\right\}$$

Approximately optimal when v_k is close to v^*

Error bounds available...

Rearranging the Bellman Equation

Actually, for this particular problem, there's an easier solution method

- involves a "rearrangement" of the Bellman equation
- shifts us to a lower dimensional problem

Recall: a function v satisfies the Bellman equation if

$$v(w) := \max\left\{\frac{w}{1-\beta}, c + \beta \int v(w')\varphi(\mathrm{d}w')\right\}$$

Taking v as given, consider

$$h := c + \beta \int v(w')\varphi(\mathrm{d}w')$$

Using h to eliminate v from the Bellman equation yields

$$h = c + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$$

We now seek an $h \in \mathbb{R}_+$ satisfying

$$h = c + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$$

Solution h^* is the continuation value

Optimal policy can be written as

$$\sigma^*(w) = \mathbb{1}\left\{\frac{w}{1-\beta} \geqslant h^*\right\} \qquad (w \in \mathbb{R}_+)$$

Alternatively,

$$\sigma^*(w) = \mathbb{1}\{w \geqslant w^*\} \quad \text{where} \quad w^* := (1-\beta)h^*$$

The term w^* is called the **reservation wage**

To solve for h^* we introduce the mapping

$$g(h) = c + \beta \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(\mathrm{d}w') \qquad (h \in \mathbb{R}_+)$$

Any solution to $h = c + \beta \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(\mathrm{d}w')$ is a fixed point of g and vice versa

Assumption. The distribution φ has finite first moment

Ex. Confirm that

- g is a well defined map from $\mathbb{R}_+$ to itself
- g is a contraction map on $\mathbb{R}_+$ under the usual Euclidean distance

Conclude that g has a unique fixed point in $\mathbb{R}_+$

For computation it is somewhat easier to work with the case where wages are bounded

Ex. Suppose that $\mathbb{P}\{w_t \leqslant M\} = 1$ for some positive constant M

- Confirm that g maps $[0, K]$ to itself, where

$$K := \frac{\max\{M, c\}}{1 - \beta}$$

- Conclude that g has a fixed point in $[0, K]$, which is the unique fixed point of g in $\mathbb{R}_+$

See notebook `iid_job_search.ipynb`

Parametric Monotonicity

Recall this result:

Fact. If (M, g_1) and (M, g_2) are dynamical systems such that

1. g_2 is isotone and dominates g_1 on M
2. (M, g_2) is globally stable with unique fixed point u_2,

then $u_1 \preceq u_2$ for every fixed point u_1 of g_1

Now consider

$$g(h) = c + \beta \int \max\left\{ \frac{w'}{1-\beta}, h \right\} \varphi(\mathrm{d}w')$$

This map is

1. globally stable
2. an isotone self-map on $\mathbb{R}_+$

Hence any parameter that shifts up the function g pointwise on $\mathbb{R}_+$ also shifts up h^*

Ex. Show that

1. the continuation value h^* is increasing in unemployment compensation c
2. the reservation wage w^* is increasing in c

Interpret

Shifting the Offer Distribution

How do shifts in this distribution affect the reservation wage?

Intuition: “more favorable” wage distribution would tend to increase the reservation wage

- the agent can expect better offers

What does “more favorable” mean for offer distributions?

One possible answer: **(first order) stochastic dominance**

First Order Stochastic Dominance

Definition. Distribution φ is **stochastically dominated** by distribution ψ (write $\varphi \preceq_{\mathrm{SD}} \psi$) if

$$\int u(x)\varphi(\mathrm{d}x) \leqslant \int u(x)\psi(\mathrm{d}x) \text{ for all } u \in ibc\mathbb{R}_+$$

With $ibm\mathbb{R}_+$ as the increasing bounded Borel measurable functions, this is equivalent:

$$\int u(x)\varphi(\mathrm{d}x) \leqslant \int u(x)\psi(\mathrm{d}x) \text{ for all } u \in ibm\mathbb{R}_+$$

Interpretation: Anyone with increasing utility likes ψ better

Let φ and ψ be two wage distributions on $\mathbb{R}_+$ with finite first moment

Let

- w^*_φ and w^*_ψ be the associated reservation wages
- h^*_φ and h^*_ψ be the associated continuation values

Assume both are supported on $[0, M]$

Lemma. If $\varphi \preceq_{\text{SD}} \psi$, then $w^*_\varphi \leqslant w^*_\psi$

Proof: Let ψ and φ have the stated properties

It suffices to show that $h^*_\varphi \leqslant h^*_\psi$

We aim to show that

$$g(h) = c + \beta \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(\mathrm{d}w')$$

increases at any h if we shift up the offer distribution in $\preceq_{\mathrm{SD}}$

Sufficient: given $\varphi \preceq_{\mathrm{SD}} \psi$ and $h \geqslant 0$,

$$\int \max \left\{ \frac{w'}{1-\beta}, h \right\} \varphi(\mathrm{d}w') \leqslant \int \max \left\{ \frac{w'}{1-\beta}, h \right\} \psi(\mathrm{d}w')$$

Since $w' \mapsto \max\{w'/(1-\beta), h\}$ is in $ibc[0, M]$, this follows directly from the definition of stochastic dominance

Second Order Stochastic Dominance

What about the **volatility** of the wage process?

How does it impact on the reservation wage?

Intuitively, greater volatility means

- option value of waiting is larger
- encourages patience — higher reservation wage

But how can we isolate the effect of volatility?

- introduce the notion of a **mean-preserving spread**

Given distribution ψ, we say that φ is a **mean-preserving spread** of ψ if $\exists$ random variables (Y, Z) such that

$$\mathbb{E}[Z \mid Y] = 0, \quad Y \stackrel{\mathscr{D}}{=} \psi \quad \text{and} \quad Y + Z \stackrel{\mathscr{D}}{=} \varphi$$

- adds noise without changing the mean

Related definition: ψ **second order stochastically dominates** φ if, with $\mathscr{U}$ as the concave functions in $ib\mathbb{R}_+$,

$$\int u(x)\varphi(\mathrm{d}x) \leqslant \int u(x)\psi(\mathrm{d}x) \text{ for all } u \in \mathscr{U}$$

Fact. ψ second order stochastically dominates φ if and only if φ is a mean-preserving spread of ψ

Proof that φ is a mean-preserving spread of $\psi \implies \psi$ second order stochastically dominates φ

Let φ be a mean-preserving spread of ψ

We can find random variables (Y, Z) such that

$$\mathbb{E}[Z \mid Y] = 0, \quad Y \stackrel{\mathscr{D}}{=} \psi \quad \text{and} \quad Y + Z \stackrel{\mathscr{D}}{=} \varphi$$

By Jensen's inequality,

$$\mathbb{E}u(Y+Z) = \mathbb{E}\mathbb{E}[u(Y+Z) \mid Y] \leqslant \mathbb{E}u(\mathbb{E}[Y+Z \mid Y]) = \mathbb{E}u(Y)$$

$$\therefore \quad \int u(x)\varphi(\mathrm{d}x) = \mathbb{E}u(Y) \leqslant \mathbb{E}u(Y+Z) = \int u(x)\psi(\mathrm{d}x)$$

How does the unemployed agent react to a **mean-preserving spread in the offer distribution**?

Lemma. If φ is a mean-preserving spread of ψ, then $w^*_\psi \leqslant w^*_\varphi$

Proof: Again, it suffices to show that $h^*_\psi \leqslant h^*_\varphi$

Claim: g increases pointwise with the mean-preserving spread

Equivalently, for all $h \geqslant 0$,

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(\mathrm{d}w') \leqslant \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$$

By definition, there exists a (w', Z) such that $\mathbb{E}[Z \mid w'] = 0$, $w' \stackrel{\mathscr{D}}{=} \psi$ and $w' + Z \stackrel{\mathscr{D}}{=} \varphi$

By this fact and the law of iterated expectations,

$$
\begin{aligned}
\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w') &= \mathbb{E}\left[\max\left\{\frac{w'+Z}{1-\beta}, h\right\}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\max\left\{\frac{w'+Z}{1-\beta}, h\right\} \,\middle|\, w'\right]\right]
\end{aligned}
$$

Jensen's inequality now produces

$$
\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w') \geqslant \mathbb{E} \max\left\{\frac{\mathbb{E}[w'+Z \mid w']}{1-\beta}, h\right\}
$$

Using $\mathbb{E}[w' \mid w'] = w'$ and $\mathbb{E}[Z \mid w'] = 0$ leads to

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w') \geqslant \mathbb{E} \max\left\{\frac{\mathbb{E}[w' + Z \mid w']}{1-\beta}, h\right\}$$

$$= \mathbb{E} \max\left\{\frac{w'}{1-\beta}, h\right\}$$

$$= \int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(\mathrm{d}w')$$

Since h was arbitrary, the function g shifts up pointwise

Since g is isotone and a contraction, this completes the proof

Second Order Stochastic Dominance and Welfare

How does volatility impact on **welfare**?

Do mean-preserving spreads have a monotone impact on lifetime value?

More precisely, with

- φ as a mean-preserving spread of ψ
- v_φ and v_ψ as the corresponding value functions

do we have $v_\psi \leqslant v_\varphi$?

Why might this be true?

Prop. If φ is a mean-preserving spread of ψ, then $v_\psi \leqslant v_\varphi$ on $\mathbb{R}_+$

Proof: For a fixed distribution ν, the value function v_ν satisfies

$$v_\nu(w) = \max\left\{\frac{w}{1-\beta}, h_\nu\right\}$$

where the continuation value

$$h_\nu := c + \beta \int v_\nu(w')\nu(\mathrm{d}w')$$

is the fixed point of

$$g_\nu(h) := c + \beta \int \max\left\{\frac{w'}{1-\beta}, h\right\} \nu(\mathrm{d}w')$$

If $h_\psi \leqslant h_\varphi$, then the result is immediate

Let φ be a mean-preserving spread of ψ

Since g_φ is isotone and globally stable on $\mathbb{R}_+$, it suffices to show that

$$g_\psi(h) \leqslant g_\varphi(h) \quad \forall\, h \in \mathbb{R}_+$$

So fix $h \in \mathbb{R}_+$

It is enough to show that

$$\int \max\left\{\frac{w'}{1-\beta}, h\right\} \psi(\mathrm{d}w') \leqslant \int \max\left\{\frac{w'}{1-\beta}, h\right\} \varphi(\mathrm{d}w')$$

Ex. Prove it