ECON-GA 1025 Macroeconomic Theory I

Lecture 13

John Stachurski

Fall Semester 2018

Today's Lecture

- Lots of new theory
- Just joking
- Euler equation + revision
- See "Lecture 14" for general DP results (not examinable)

Optimal Savings: The Envelope Condition

Recall the IID model with Bellman equation

$$v(y) = \max_{0 \leqslant c \leqslant y} \left\{ u(c) + \beta \int v(f(y-c)z)\varphi(dz) \right\} \qquad (y \in \mathbb{R}_+)$$

We know that

$$\sigma \text{ is optimal } \iff \sigma \text{ is } v^*\text{-greedy}$$

We can get additional characterizations of optimality if we impose more conditions

Assumption. (INA) Both f and u are strictly increasing, continuously differentiable and strictly concave

In addition,

$$f(0) = 0, \quad \lim_{k \to 0} f'(k) > 0$$

and

$$u(0) = 0, \quad \lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0$$

Remark. We ignore the restriction $u(0) = 0$ in some applications below — I'm aiming to remove it

Proposition. Let

- v be an increasing concave function in $bc\mathbb{R}_+$
- σ be the unique v-greedy policy in Σ

If assumption (INA) holds, then

1. σ is interior, while
2. Tv is continuously differentiable and satisfies

$$(Tv)' = u' \circ \sigma$$

Corollary If σ^* is the optimal consumption policy, then

$$(v^*)' = u' \circ \sigma^*$$

Proof that $(Tv)' = u' \circ \sigma$ when σ is v-greedy:

Since σ is v-greedy,

$$Tv(y) = u(\sigma(y)) + \beta \int v(f(y - \sigma(y))z)\varphi(dz)$$

By the **envelope theorem**,

$$(Tv)'(y) = \beta \int (v)'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz)$$

The FOC from the Bellman equation yields

$$u'(\sigma(y)) = \beta \int (v)'(f(y - \sigma(y))z)f'(y - \sigma(y))z\varphi(dz)$$

Combining the last two equations gives $(Tv)' = u' \circ \sigma$

Let $\mathscr{C} :=$ all continuous strictly increasing $\sigma \in \Sigma$ satisfying

$$0 < \sigma(y) < y \quad \text{for all} \quad y > 0$$

We say that $\sigma \in \mathscr{C}$ **satisfies the Euler equation** if

$$(u' \circ \sigma)(y) = \beta \int (u' \circ \sigma)(f(y - \sigma(y))z) f'(y - \sigma(y)) z \varphi(dz)$$

for all $y > 0$

- A functional equation in **policies**

In sequence notation, $u'(c_t) = \beta \mathbb{E} u'(c_{t+1}) f'(k_t) z_{t+1}$

Let's introduce an operator K corresponding to the Euler equation

Fix $\sigma \in \mathscr{C}$ and $y > 0$

The value $K\sigma(y)$ is the c in $(0, y)$ that solves

$$u'(c) = \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c) z \varphi(dz)$$

We call K the **Coleman–Reffett** operator

Ex. Show σ in $\mathscr{C}$ is a fixed point of K if and only if it satisfies the Euler equation

Proof that K is well defined:

For any $\sigma \in \mathscr{C}$, the RHS of

$$u'(c) = \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c) z \varphi(dz)$$

is continuous, strictly increasing in c, diverges to $+\infty$ as $c \uparrow y$

The LHS is continuous, strictly decreasing in c, diverges to $+\infty$ as $c \downarrow 0$

Hence

$$H(y,c) := u'(c) - \beta \int (u' \circ \sigma)(f(y-c)z) f'(y-c) z \varphi(dz)$$

when regarded as a function of c, has exactly one zero

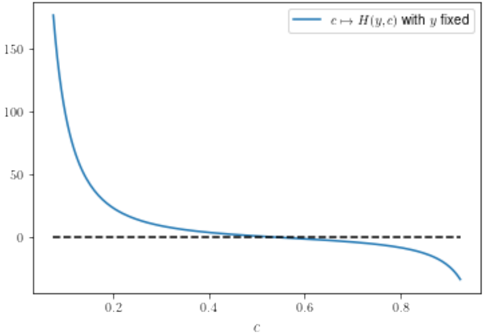

Figure: Solving for the c that satisfies $H(y, c) = 0$.

The Euler Equation and Optimality

Proposition. If assumption (INA) holds and σ^* is the unique optimal policy, then

1. $(\mathscr{C}, K)$ is globally stable and
2. the unique fixed point of K in $\mathscr{C}$ is σ^*

In particular, $\sigma \in \mathscr{C}$ is optimal if and only it satisfies the Euler equation

Sketch of proof:

Let $\mathscr{V}$ be all strictly concave, continuously differentiable v mapping $\mathbb{R}_+$ to itself and satisfying $v(0) = 0$ and $v'(y) > u'(y)$ whenever $y > 0$

As before, let $\mathscr{C}$ be all a continuous, strictly increasing functions on $\mathbb{R}_+$ satisfying $0 < \sigma(y) < y$

For $v \in \mathscr{V}$ let Mv be defined by

$$(Mv)(y) = \begin{cases} m(v'(y)) & \text{if } y > 0 \\ 0 & \text{if } y = 0 \end{cases} \tag{1}$$

where $m(y) := (u')^{-1}(y)$

The course notes show that

1. M is a homeomorphism from $\mathscr{V}$ to $\mathscr{C}$
2. for every increasing concave function in $bc\mathbb{R}_+$,

$$\sigma := MTv$$

   is the unique v-greedy policy

3. The Bellman operator and Coleman–Reffett operator are related by

$$T = M^{-1} \circ K \circ M \text{ on } \mathscr{V}$$

Ex. Use 1–3 above to show that $(\mathscr{C}, K)$ is globally stable with unique fixed point σ^*

Remark. The Euler equation is often paired with the **transversality condition**

$$\lim_{t \to \infty} \beta^t \mathbb{E} u'(c_t) k_t = 0$$

Standard results (see, e.g., Stokey and Lucas) tell us that

$$\text{Euler} + \text{transversality condition} \implies \text{optimality}$$

Our last result shows transversality is not needed under our assumptions

Ex. Following the basic CRRA cake eating model, set

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma} \quad \text{and} \quad f(k)z = Rk$$

Insert the conjecture $\sigma^*(y) = \theta y$ into the Euler equation

Recover our earlier result that this policy is optimal when

$$\theta = 1 - \left(\beta R^{1-\gamma}\right)^{1/\gamma}$$

Ex. Repeat for the log / CD model, where $u(c) = \ln c$ and

$$f(k)z = Ak^{\alpha}z, \quad 0 < A, \quad 0 < \alpha < 1$$

Insert the conjecture $\sigma^*(y) = \theta y$ into the Euler equation and recover your earlier result for the optimal policy

Exercise: Predicting Quadratics

Let

- $\{\mathscr{G}_t\}_{t \geqslant 1}$ be a filtration
- $\{w_t\}_{t \geqslant 1}$ be a stochastic process in $\mathbb{R}^j$

Recall: $\{w_t\}_{t \geqslant 1}$ is called a **martingale difference sequence** (MDS) with respect to $\{\mathscr{G}_t\}$ if

- $\mathbb{E}\|w_t\|_1 < \infty$ and
- $\{w_t\}_{t \geqslant 1}$ is adapted to $\{\mathscr{G}_t\}$
- and

$$\mathbb{E}[w_{t+1} \,|\, \mathscr{G}_t] = 0, \quad \forall \, t \geqslant 1$$

Suppose that

- $x_{t+1} = Ax_t + C\xi_{t+1}$ in $\mathbb{R}^n$ with x_0 given
- $\mathscr{G}_t = \{x_0, \xi_0, \xi_1, \ldots, \xi_t\}$
- $\{\xi_t\}_{t \geqslant 1}$ is an $\mathbb{R}^j$-valued MDS with respect to $\mathscr{G}_t$ satisfying

$$\mathbb{E}[\xi_t \xi_t'] = I$$

Question: Is $\{x_t\}$ adapted to $\mathscr{G}_t$?

Ex. Let $\mathbb{E}_t := \mathbb{E}[\cdot \,|\, \mathscr{G}_t]$

Show that if $H \in \mathcal{M}(n \times n)$, then

$$\mathbb{E}_t[x_{t+1}' H x_{t+1}] = x_t' A' H A x_t + \operatorname{trace}(C' H C)$$

Solution: We have

$$\mathbb{E}_t[x_{t+1}' H x_{t+1}] = \mathbb{E}_t[(Ax_t + Cw_{t+1})' H (Ax_t + Cw_{t+1})]$$

The RHS expands to

$$\begin{aligned}\mathbb{E}_t[x_t' A' H A x_t] + 2\mathbb{E}_t[x_t' A' H C w_{t+1}] + \mathbb{E}_t[w_{t+1}' C' H C w_{t+1}] \\ = I + II + III\end{aligned}$$

Since x_t is known at t we have

$$I = \mathbb{E}_t[x_t' A' H A x_t] = x_t' A' H A x_t$$

Since $\{w_t\}$ is an MDS,

$$II = 2\mathbb{E}_t[x_t' A' H C w_{t+1}] = 2 x_t' A' H C \, \mathbb{E}_t[w_{t+1}] = 0$$

Finally,

$$III = \mathbb{E}_t[w_{t+1}' C' H C w_{t+1}] = \operatorname{trace}(C' H C)$$

Hence

$$\mathbb{E}_t[x_{t+1}' H x_{t+1}] = x_t' A' H A x_t + \operatorname{trace}(C' H C)$$

Application: LQ Risk Neutral Asset Pricing

Recall the risk neutral asset pricing formula

$$p_t = \beta \, \mathbb{E}_t [d_{t+1} + p_{t+1}]$$

Here

- $\{d_t\}$ is a cash flow
- p_k is asset price at time k
- $\beta \in (0, 1)$ discounts values
- $\mathbb{E}_t$ is time t conditional expectation

Aim: solve for $\{p_t\}$

Assume that

$$d_t = x_t' D x_t \text{ for some positive definite } D$$

Here

- $x_{t+1} = A x_t + C \xi_{t+1}$ in $\mathbb{R}^n$ with x_0 given
- $\mathscr{G}_t = \{x_0, \xi_0, \xi_1, \ldots, \xi_t\}$
- $\{\xi_t\}_{t \geqslant 1}$ is an $\mathbb{R}^j$-valued MDS with respect to $\mathscr{G}_t$ satisfying

$$\mathbb{E}[\xi_t \xi_t'] = I$$

Prices as Functions of the State

We conjecture that

$$p_t = p(x_t) \quad \text{for some function } p$$

Another leap: guess that prices are a **quadratic** in x_t

In particular, we guess that

$$p(x) = x' P x + \delta$$

for some positive definite P and scalar δ

Substituting

$$p_t = x_t' P x_t + \delta \quad \text{and} \quad d_t = x_t' D x_t$$

into

$$p_t = \beta \mathbb{E}_t [d_{t+1} + p_{t+1}]$$

gives

$$\begin{aligned} x_t' P x_t + \delta &= \beta \mathbb{E}_t [x_{t+1}' D x_{t+1} + x_{t+1}' P x_{t+1} + \delta] \\ &= \beta \mathbb{E}_t [x_{t+1}' (D + P) x_{t+1}] + \beta \delta \\ &= \beta x_t' A' (D + P) A x_t + \beta \operatorname{trace}(C' (D + P) C) + \beta \delta \end{aligned}$$

So, we seek a pair $P \in \mathcal{M}(n \times n)$, $\delta \in \mathbb{R}$ such that

$$x'Px + \delta = \beta x'A'(D + P)Ax + \beta \operatorname{trace}(C'(D + P)C) + \beta\delta$$

for any $x \in \mathbb{R}^n$

Claim: If P^* satisfies

$$P^* = \beta A'(D + P^*)A$$

and

$$\delta^* := \frac{\beta}{1 - \beta} \operatorname{trace}(C'(D + P^*)C)$$

then P^*, δ^* solves the above equation for any x

Proof: By hypothesis, $P^* = \beta A'(D + P^*)A$

$$\therefore \quad x'P^*x = \beta x'A'(D + P^*)Ax$$

$$\therefore \quad x'P^*x + \delta^* = \beta x'A'(D + P^*)Ax + \delta^*$$

To complete the proof, suffices to show that

$$\delta^* = \beta \operatorname{trace}(C'(D + P^*)C) + \beta\delta^*$$

True by definition of δ^*

Summary: With such a P^*,

$$p_t := x_t' P^* x_t + \delta^*$$

is an equilibrium price sequence

But does there exist a $P \in \mathcal{M}(n \times n)$ that solves

$$P = \beta A'(D + P)A$$

Ex. Under what condition does a unique solution exist?

Solution: Write $P = \beta A'(D + P)A$ as the discrete Lyapunov equation

$$P = \Lambda' P \Lambda + M$$

where

- $\Lambda := \sqrt{\beta} A$
- $M := \Lambda' D \Lambda$

The solution P^* is the fixed point of operator ℓ defined by

$$\ell P = \Lambda' P \Lambda + M$$

As shown previously, ℓ is globally stable when $r(\Lambda) < 1$

Equivalent: $r(A) \leqslant 1/\sqrt{\beta}$

Recall that P^* is the solution to

$$P = \beta A'(D + P)A$$

Ex. Show that P is positive semidefinite

We need to show that $x'P^*x \geqslant 0$ for all $x \in \mathbb{R}^n$

Let $\mathcal{M}_P$ be the set of positive semidefinite matrices in $\mathcal{M}(n \times n)$

This set is closed in $\mathcal{M}(n \times n)$ under the matrix norm

To see this, pick

- any $\{E_n\} \subset \mathcal{M}_P$ and $E \in \mathcal{M}(n \times n)$ with $E_n \to E$
- any $x \in \mathbb{R}^n$

We showed in another context that $x'E_nx \to x'Ex$ in $\mathbb{R}$

Since $x'E_nx \geqslant 0$ for all n, we have $x'Ex \geqslant 0$

Since x was arbitrary we have $E \in \mathcal{M}_P$

Hence $\mathcal{M}_P$ is closed

Since

1. $\mathcal{M}_P$ is closed in $\mathcal{M}(n \times n)$
2. P^* is the fixed point of $\ell P = \Lambda' P \Lambda + M$

it suffices to show that ℓ maps $\mathcal{M}_P$ to itself

So pick any $P \in \mathcal{M}_P$ and any $x \in \mathbb{R}^n$

We have, with $y := Ax$,

$$\begin{aligned} x'(\ell P)x &= x'(\Lambda' P \Lambda + M)x \\ &= x'(\beta A'(D + P)A)x \\ &= \beta x' A' D A x + \beta x' A' P A x \\ &= \beta y' D y + \beta y' P y \geqslant 0 \end{aligned}$$

Application: Firm Entry and Exit

Let

- Z be a finite subset of $\mathbb{R}$
- Π a stochastic kernel on Z
- q be a distribution in $\mathcal{P}(\mathsf{Z})$

An individual firm's productivity $\{z_t\}$ obeys Π

- $z_{t+1} \sim \Pi(z_t, \cdot)$ for all t

When a firm's productivity falls below $\bar{z} \in \mathsf{Z}$, the firm exits

Replaced by a new firm with productivity $z_{t+1} \sim q$

Ex. How does the distribution of firms (i.e., cross-section) evolve?

Solution: A randomly selected firm has

$$\mathbb{P}\{z_{t+1} = z' \mid z_t = z\} = \begin{cases} \Pi(z, z') & \text{if } z > \bar{z} \\ q(z') & \text{if } z \leqslant \bar{z} \end{cases}$$

The cross-sectional firm distribution sequence $\{\psi_t\}$ satisfies

$$\begin{aligned} \psi_{t+1}(z') &= q(z') \sum_{z \leqslant \bar{z}} \psi_t(z) + \sum_{z > \bar{z}} \Pi(z, z') \psi_t(z) \\ &= \sum_{z \in \mathsf{Z}} \left[\mathbb{1}\{z \leqslant \bar{z}\} q(z') + \mathbb{1}\{z > \bar{z}\} \Pi(z, z') \right] \psi_t(z) \end{aligned}$$

We can write

$$\psi_{t+1}(z') = \sum_{z \in \mathsf{Z}} \left[\mathbb{1}\{z \leqslant \bar{z}\} q(z') + \mathbb{1}\{z > \bar{z}\} \Pi(z, z') \right] \psi_t(z)$$

as

$$\psi_{t+1}(z') = \sum_{z \in \mathsf{Z}} Q(z, z') \psi_t(z)$$

where

$$Q(z, z') := \mathbb{1}\{z \leqslant \bar{z}\} q(z') + \mathbb{1}\{z > \bar{z}\} \Pi(z, z')$$

is the stochastic kernel for the "rejuvenating firm"

Ex.

- Under what conditions does

$$Q(z, z') := \mathbb{1}\{z \leqslant \bar{z}\} q(z') + \mathbb{1}\{z > \bar{z}\} \Pi(z, z')$$

have a stationary distribution?

- What's a simple condition on q, Π under which $(Q, \mathcal{P}(\mathsf{Z}))$ is globally stable?

- How would you go about computing that stationary distribution when it the condition is satisfied?

Application: Optimal Firm Exit

An incumbent within an industry has current profits

$$\pi_t = \pi(z_t, p_t)$$

- $\{z_t\} \overset{\text{IID}}{\sim} \varphi$ is a firm-specific productivity process
- $\{p_t\} \overset{\text{IID}}{\sim} \nu$ is an exogenous price process
- (z_t, p_t) takes values in $\mathsf{X} \subset \mathbb{R}^k$

Decision problem: Continue or exit? (optimal stopping)

Timing runs as follows

At the start of time t, observe z_t and p_t

If decision = continue, then

1. receive π_t now
2. start next period as an incumbent

If decision = exit, then

1. receive scrap value $s \in \mathbb{R}$
2. receive 0 in all subsequent periods

Maximizes

$$\mathbb{E} \sum_{t \geqslant 0} \beta^t r_t$$

Here

$$r_t = \begin{cases} \pi_t & \text{if still incumbent} \\ s & \text{upon exit} \\ 0 & \text{after exit} \end{cases}$$

Assume that $0 < \beta < 1$ and π is continuous and bounded

Ex. Without looking at the next slide, try to write down the Bellman equation for an incumbent firm

Bellman equation:

$$v(z,p) = \max\left\{s,\ \pi(z,p) + \beta \int v(z',p')\varphi(\mathrm{d}z')\nu(\mathrm{d}p')\right\}$$

Bellman operator:

$$Tv(z,p) = \max\left\{s,\ \pi(z,p) + \beta \int v(z',p')\varphi(\mathrm{d}z')\nu(\mathrm{d}p')\right\}$$

Ex. Without looking ahead, show that T is a self-mapping on bcX

If $v \in bc\mathsf{X}$, then

$$|Tv(z,p)| = \left| \max \left\{ s,\ \pi(z,p) + \beta \int v(z',p') \varphi(\mathrm{d}z') \nu(\mathrm{d}p') \right\} \right|$$

$$\leqslant |s| + \|\pi\|_\infty + \|v\|_\infty$$

Moreover,

$$Tv(z,p) = \max \left\{ s,\ \pi(z,p) + \beta \int v(z',p') \varphi(\mathrm{d}z') \nu(\mathrm{d}p') \right\}$$

$$= \max \{ s,\ \pi(z,p) + \text{constant} \}$$

is clearly continuous in (z,p)

Without looking ahead, show that T is a contraction of mod β on $(bc\mathsf{X}, d_\infty)$

We use the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

Fix v, w in $bc\mathsf{X}$ and $(z, p) \in \mathsf{X}$

By this bound and the triangle inequality (check details),

$$|Tv(z,p) - Tw(z,p)| \leqslant \beta \int |v(z',p') - w(z',p')| \varphi(\mathrm{d}z')\nu(\mathrm{d}p')$$

$$\leqslant \beta \|v - w\|_\infty$$

Taking the supremum over all $(z, p) \in \mathsf{X}$ leads to

$$\|Tv - Tw\|_\infty \leqslant \beta \|v - w\|_\infty$$

The value function v^* is the unique fixed point of T in bcX

- proof is in lecture 14

Suppose that p and z are real valued

Ex. Under what conditions is v^* increasing in (z, p)?

Hint: Look at

$$Tv(z,p) = \max \left\{ s,\ \pi(z,p) + \beta \int v(z', p') \varphi(\mathrm{d}z') \nu(\mathrm{d}p') \right\}$$

Solution: If π is increasing in (z, p), then so is v^*

Indeed, under this condition

$$Tv(z,p) = \max\left\{s,\ \pi(z,p) + \beta \int v(z',p')\varphi(\mathrm{d}z')\nu(\mathrm{d}p')\right\}$$

maps $ibc\mathsf{X}$ into itself (increasing bounded continuous)

Moreover, $ibc\mathsf{X}$ is closed in $(bc\mathsf{X}, d_\infty)$

Hence the fixed point v^* lies in $ibc\mathsf{X}$

Ex. Can you suggest an easier way to solve the Bellman equation

$$v(z, p) = \max \left\{ s,\ \pi(z, p) + \beta \int v(z', p') \varphi(\mathrm{d}z') \nu(\mathrm{d}p') \right\}$$

Can you map it to a lower dimensional problem?

Hint: We only need to find the **expectation** of the value function

One solution: set $h := \int v(z', p')\varphi(\mathrm{d}z')\nu(\mathrm{d}p')$

The Bellman equation is

$$v(z, p) = \max\left\{s,\ \pi(z, p) + \beta h\right\}$$

Now shift forward in time and take expectations to get

$$h = \int \max\left\{s,\ \pi(z', p') + \beta h\right\} \varphi(\mathrm{d}z')\nu(\mathrm{d}p')$$

Ex. Show that

$$F(h) := \int \max\left\{s,\ \pi(z', p') + \beta h\right\} \varphi(\mathrm{d}z')\nu(\mathrm{d}p')$$

is a contraction on $\mathbb{R}_+$

Solution: The triangle inequality for integrals gives

$$|F(g) - F(h)| \leqslant$$

$$\int \left|\max\left\{s,\ \pi(z',p') + \beta g\right\} - \max\left\{s,\ \pi(z',p') + \beta h\right\}\right| \varphi(\mathrm{d}z')\nu(\mathrm{d}p')$$

From the elementary bound

$$|\alpha \vee x - \alpha \vee y| \leqslant |x - y| \qquad (\alpha, x, y \in \mathbb{R})$$

this leads to

$$|F(g) - F(h)| \leqslant \int |\beta g - \beta h| \, \varphi(\mathrm{d}z')\nu(\mathrm{d}p') \leqslant \beta \, |g - h|$$

Final comments

Exam might ask you to provide algorithms

- Describe steps of a computer solution, why the procedure works
- You don't need to write code by hand

All lemmas / thms / facts from the slides can be used freely in the exam

Example. "From the lectures we have $|\alpha \vee x - \alpha \vee y| \leqslant |x - y|$, from which it follows that ..."

Good luck :-)