A Primer for Difference Equations and Time Series

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First-order difference equations

- Consider a first-order difference equation:

$$y_t = \phi y_{t-1} + w_t.$$

- w_t is a random variable, or a sequence of deterministic numbers.

Solving First-order difference equations

- Let

time	equation
0	$y_0 = \rho y_{-1} + w_0,$
1	$y_1 = \rho y_0 + w_1,$
2	$y_2 = \rho y_1 + w_2,$
$\vdots$	$\vdots$
t	$y_t = \rho y_{t-1} + w_t.$

- If we know y_{-1} and $\{w_0, w_1, w_2, ...\}$, we can find y_t for any date.
- For example, we can calculate y_1 as

$$y_1 = \rho y_0 + w_1 = \rho(\rho y_{-1} + w_0) + w_1$$

or

$$y_1 = \rho^2 y_{-1} + \rho w_0 + w_1.$$

Solving First-order difference equations

- Given this value of y_1, we can calculate y_2 as

$$y_2 = \rho y_1 + w_2 = \rho(\rho^2 y_{-1} + \rho w_0 + w_1) + w_2$$

  or

$$y_2 = \rho^3 y_{-1} + \rho^2 w_0 + \rho w_1 + w_2.$$

- Continuing recursively,

$$y_t = \rho^{t+1} y_{-1} + \rho^t w_0 + \rho^{t-1} w_1 + \rho^{t-2} w_2 + \cdots + \rho w_{t-1} + w_t.$$

  This procedure is called recursive substitution.

pth-order difference equations

- Generalize the previous equation and consider pth-order difference equation:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \cdots + \rho_p y_{t-p} + w_t.$$

- It can be rewritten as a first-order difference equation in a vector ξ_t.
- Define

$$\underset{(p\times 1)}{\xi_t} = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix}.$$

pth-order difference equations

- Also define

$$\underset{(p\times p)}{F} = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{p-1} & \rho_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \qquad \underset{(p\times 1)}{v_t} = \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- Then

$$\underset{(p\times 1)}{\xi_t} = \underset{(p\times p)}{F}\underset{(p\times 1)}{\xi_{t-1}} + \underset{(p\times 1)}{v_t}$$

or

$$\begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_{p-1} & \rho_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ y_{t-3} \\ \vdots \\ y_{t-p} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

pth-order difference equations

- For example, let $p = 2$,

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-1} + w_t,$$

or

$$\begin{bmatrix} y_t \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_{t-2} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \end{bmatrix}.$$

Eigenvalues and eigenvectors

- Given F, there are scalars and vectors satisfying the following equation

$$Fx_t = \lambda x_t.$$

  λ is eigenvalue and x_t is eigenvector of a matrix F.

- The eigenvalues λ satisfies

$$|F - \lambda I_p| = 0.$$

- There are p eigenvalues and eigenvectors associated with F.

Stability

- If the eigenvalues are real,
  - If all of the eigenvalues are less than 1 in absolute value, the system is stable.
  - If one of the eigenvalues are greater than unity in absolute value, the system is explosive.

Solving for the eigenvalues

- For $p = 2$, the eigenvalues are the solutions to

$$\left| \begin{bmatrix} \rho_1 & \rho_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

or

$$\begin{vmatrix} \rho_1 - \lambda & \rho_2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \rho_1 \lambda - \rho_2 = 0.$$

The two eigenvalues of F for a second-order difference equation are

$$\lambda_1 = \frac{\rho_1 + \sqrt{\rho_1^2 + 4\rho_2}}{2}, \qquad \lambda_2 = \frac{\rho_1 - \sqrt{\rho_1^2 + 4\rho_2}}{2}.$$

MA(1) process

- $\{\varepsilon_t\}$ is a mean-zero and covariance-stationary stochastic process, which is serially uncorrelated at all leads and lags: $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2$ and $E(\varepsilon_t \varepsilon_s) = 0$ for $t \neq s$.

- y_t is constructed as

$$\begin{aligned} y_t &= \varepsilon_t + \theta\varepsilon_{t-1}, \\ &= (1 + \theta L)\varepsilon_t \end{aligned}$$

  which is called a moving average process of order 1, or MA(1) process.

- L is called lag operator, $L^j \varepsilon_t = \varepsilon_{t-j}$ for $j = 1, 2, ...$

MA(1) process, cont'd

- The variance of y_t, denoted as σ_y^2 or $\gamma(0)$, is given by

$$\begin{aligned}\sigma_y^2 = \gamma(0) &= E(y_t^2) \\ &= E(\varepsilon_t^2 + 2\theta\varepsilon_t\varepsilon_{t-1} + \theta^2\varepsilon_{t-1}^2) \\ &= (1+\theta^2)\sigma^2.\end{aligned}$$

- Also,

$$\begin{aligned}\gamma(1) &= E(y_t y_{t-1}) \\ &= E(\varepsilon_t + \theta\varepsilon_{t-1})(\varepsilon_{t-1} + \theta\varepsilon_{t-2}) \\ &= \theta\sigma^2 \\ \gamma(s) &= E(y_t y_{t-s}) = 0, \quad s > 1.\end{aligned}$$

- $\varphi(s) = \gamma(s)/\gamma(0)$ is the sth-order autocorrelation of y_t, and given by

$$\begin{aligned}\varphi(1) &= \theta/(1+\theta^2) \\ \varphi(s) &= 0, \quad s > 1.\end{aligned}$$

MA(q) process

- An MA(q) process genelizes to

$$\begin{aligned} y_t &= \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q}, \\ &= \sum_{j=0}^{q} \theta_j \varepsilon_{t-j}, \\ &= \sum_{j=0}^{q} (\theta_j L^j) \varepsilon_t, \\ &= \theta(L) \varepsilon_t, \end{aligned}$$

  where $\theta_0 = 1$ and $\theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$.

- The variance in this case is given by $\sigma_y^2 = \sigma^2 \sum_{j=0}^{q} \theta_j^2$. Check the covariance pattern by yourself.

MA(∞) process

- Finally, an infinite-order MA is given by

$$\begin{aligned} y_t &= \sum_{j=0}^{\infty} \psi_j \varepsilon_{t-j}, \\ &= \psi(L)\varepsilon_t. \end{aligned}$$

- The Wold decomposition theorem: When

$$\sum_{j=0}^{\infty} \psi_j^2 < \infty$$

holds, then the sequence of y_t satisfies

$$\varepsilon_t = y_t - E_{t-1}(y_t | y_{t-1}, y_{t-2}, ...),$$

which is a one-step-ahead forecasting error. The condition is referred to as square summability.

AR(1) process

- Now consider

$$y_t = \rho y_{t-1} + \varepsilon_t, \quad |\rho| < 1.$$

y_t is an autoregressive process of order 1, or AR(1) process.

- By recursive substitution,

$$\begin{aligned} y_t &= \rho(\rho y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t, \\ &= \varepsilon_t + \rho\varepsilon_{t-1} + \rho^2 y_{t-2}, \\ &= \dots \\ &= \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}. \end{aligned}$$

This is a Wold representation of AR(1) process.

- In the case of $\rho = 1$, square summability of MA(∞) pricess is violated.

AR(1) process, cont'd

- By using lag operator,

$$
\begin{aligned}
(1-\rho L)y_t &= \varepsilon_t, \\
y_t &= \frac{1}{1-\rho L}\varepsilon_t, \\
&= \sum_{j=0}^{\infty}(\rho^j L^j)\varepsilon_t, \\
&= \sum_{j=0}^{\infty}\rho^j \varepsilon_{t-j}.
\end{aligned}
$$

- The root of the polynomial $(1-\rho L)$ is $1/\rho$, so the system is stable when $|\rho|<1$ (the root of the polyminal is greater than 1 in absoluete value).

AR(2) process

- An AR(2) process is given by

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \varepsilon_t,$$

or

$$(1 - \rho_1 L - \rho_2 L^2) y_t \quad = \quad \varepsilon_t.$$

- The process of y_t is stable when all the roots of the polynominal are greater than 1 in absolute value.

- Recall: The process of y_t is stable when all the eigenvalues of $\lambda^2 - \rho_1 \lambda - \rho_2 = 0$ are less than 1 in absolute value.