

# The Solow model

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- Solow introduced a model of economic growth that has served as the basis for DSGE models.
- The model is quite simple: There are a constant returns-to-scale production function, a law for the evolution of capital, and a saving rate.
- A first-order difference equation for the evolution of capital per worker is found, and the time path of the economy springs from this equation.

# The production function

- The production function is

$$Y_t = A_t F(K_t, H_t)$$

where

- $Y_t$  is output of the single good in the economy at date  $t$ ,
- $A_t$  is the total factor productivity (TFP),
- $K_t$  is the capital stock at the *beginning* of date  $t$ , and
- $H_t$  is hours worked.

# Constant return to scale

- The production function is homogeneous of degree one; using this property, we get

$$\begin{aligned}y_t = \frac{Y_t}{H_t} &= A_t F\left(\frac{K_t}{H_t}, \frac{H_t}{H_t}\right), \\&= A_t F(k_t, 1), \\&\equiv A_t f(k_t),\end{aligned}\tag{1}$$

where  $y_t = Y_t/H_t$  is output per worker and  $k_t = K_t/H_t$  is capital per worker.

- An example of the constant return-to-scale production function is  $Y_t = A_t K_t^\theta H_t^{1-\theta}$ , where  $\theta$  is capital share.

# The law of motion

- We assume that the labor force grows at a constant net rate  $n$ , so that  $H_{t+1} = (1 + n)H_t$ .

- The capital grows

$$K_{t+1} = (1 - \delta)K_t + I_t,$$

where  $\delta$  is the rate of depreciation and  $I_t$  is investment at time  $t$ .

- By deviding by  $H_{t+1} = (1 + n)H_t$  both side,

$$k_{t+1} = \frac{(1 - \delta)k_t + i_t}{1 + n}, \quad (2)$$

where  $i = I_t/H_t$ .

# Saving rate and closing the model

- Savings is defined as a fraction of output,

$$s_t = \sigma y_t \quad (3)$$

- In equilibrium in a closed economy,  $i_t = s_t$ , from Eqs. (1)-(3),

$$(1 + n)k_{t+1} = (1 - \delta)k_t + \sigma A_t f(k_t),$$

where  $f(k) = k^\theta$ . This equation is called “The fundamental equation of economic growth.”

- A stationary state can be found from this equation for  $k_{t+1} = k_t = \bar{k}$  and  $A_t = \bar{A}$ :

$$(1 + n)\bar{k} = (1 - \delta)\bar{k} + \sigma\bar{A}f(\bar{k}),$$

or when  $f(k) = k^\theta$ , the steady state is given by

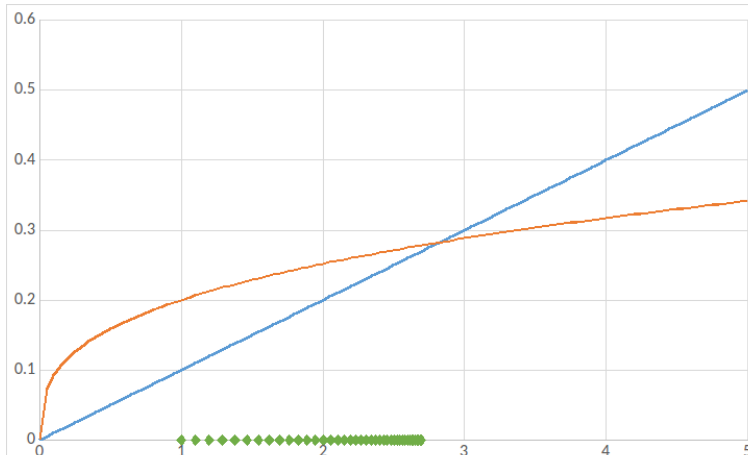
$$\bar{k} = \left( \frac{\sigma\bar{A}}{n + \delta} \right)^{\frac{1}{1-\theta}}.$$

# Equilibrium dynamics

- Let  $n = 0$ . Then

$$\Delta k_{t+1} = \sigma A_t f(k_t) - \delta k_t.$$

- When the first term cuts the second term from the above, the model is convergent and capital per worker converges to the steady state.





- We assume that the TFP follows a stochastic process:

$$\log A_{t+1} = (1 - \rho) \log \bar{A} + \rho \log A_t + \varepsilon_{t+1},$$

where  $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$ .

- Note that

$$A_{t+1} = \bar{A}^{1-\rho} A_t^\rho e^{\varepsilon_{t+1}},$$

holds.

- We approximate the model around the steady state.
- Use the formula of approximation

$$x_t \equiv x \exp \hat{x}_t \approx \bar{x}(1 + \hat{x}_t),$$

where  $\bar{x}$  is the steady state of  $x_t$  and  $\hat{x}_t$  is percent deviation from the steady state.

# Log-linearization: Production function

- Production function:

$$y_t \equiv A_t f(k_t) = A_t k_t^\theta.$$

It can be written as

$$\bar{y} \exp(\hat{y}_t) = \bar{A} \bar{k}^\theta \exp(\hat{a}_t + \alpha \hat{k}_t).$$

In the steady state,  $\bar{y} = \bar{A} \bar{k}^\theta$  holds. Then,

$$\hat{y}_t = \hat{a}_t + \theta \hat{k}_t.$$

Note: This is not approximation.

# Log-linearization: Resource constraint

- Resource constraint:

$$(1 + n)k_{t+1} = (1 - \delta)k_t + \sigma y_t.$$

It can be written as

$$(1 + n)\bar{k} \exp(\hat{k}_{t+1}) = (1 - \delta)\bar{k} \exp(\hat{k}_t) + \sigma \bar{y} \exp(\hat{y}_t).$$

Use the formula of approximation

$$(1 + n)\bar{k}(1 + \hat{k}_{t+1}) = (1 - \delta)\bar{k}(1 + \hat{k}_t) + \sigma \bar{y}(1 + \hat{y}_t).$$

In the steady state,  $(1 + n)\bar{k} = (1 - \delta)\bar{k} + \sigma \bar{y}$  holds. Then we have

$$(1 + n)\bar{k}\hat{k}_{t+1} = (1 - \delta)\bar{k}\hat{k}_t + \sigma \bar{y}\hat{y}_t.$$

- TFP:

$$\log A_{t+1} = (1 - \rho) \log \bar{A} + \rho \log A_t + \varepsilon_{t+1}.$$

It can be written as

$$\log \frac{A_{t+1}}{\bar{A}} = \rho \log \frac{A_t}{\bar{A}} + \varepsilon_{t+1}.$$

Note that  $\hat{a}_t = \log A_t / \bar{A}$ . Then,

$$\hat{a}_{t+1} = \rho \hat{a}_t + \varepsilon_{t+1}.$$

Note: This is not approximation.

# Log-linearization: Summary

- After all, the log-linearized equilibrium conditions are:

$$\begin{aligned}\hat{y}_t &= \hat{a}_t + \theta \hat{k}_t, \\ (1+n)\bar{k}\hat{k}_{t+1} &= (1-\delta)\bar{k}\hat{k}_t + \sigma\bar{y}\hat{y}_t.\end{aligned}$$

Or,

$$(1+n)\bar{k}\hat{k}_{t+1} = (1-\delta)\bar{k}\hat{k}_t + \sigma\bar{y}(\hat{a}_t + \theta\hat{k}_t).$$

# First-order difference equation

- It can be rewritten as the first-order difference equation:

$$\hat{k}_{t+1} = B\hat{k}_t + C\hat{a}_t,$$

where

$$B = \frac{1 - \delta + \sigma\theta(\bar{y}/\bar{k})}{1 + n},$$

$$C = \frac{\sigma(\bar{y}/\bar{k})}{1 + n}.$$

- The model's dynamics is characterized by this equation and the stochastic process of

$$\hat{a}_{t+1} = \rho\hat{a}_t + \varepsilon_{t+1},$$

where  $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$ .

# Analytical solution for the variance

- Assume  $\rho = 0$  so that  $\hat{a}_t = \varepsilon_t$ . Recursively substituting, we have

$$\begin{aligned}\hat{k}_{t+1} &= B \left( B\hat{k}_{t-1} + C\varepsilon_{t-1} \right) + C\varepsilon_t, \\ &= B^2 \left( B\hat{k}_{t-2} + C\varepsilon_{t-2} \right) + BC\varepsilon_{t-1} + C\varepsilon_t, \\ &= B^{i+1}\hat{k}_{t-(i+1)} + B^i C\varepsilon_{t-i} + B^{i-1} C\varepsilon_{t-(i-1)} + \cdots + C\varepsilon_t, \\ &= C\varepsilon_t + BC\varepsilon_{t-1} + B^2 C\varepsilon_{t-2} + \cdots, \\ &= C \sum_{i=0}^{\infty} B^i \varepsilon_{t-i}.\end{aligned}$$



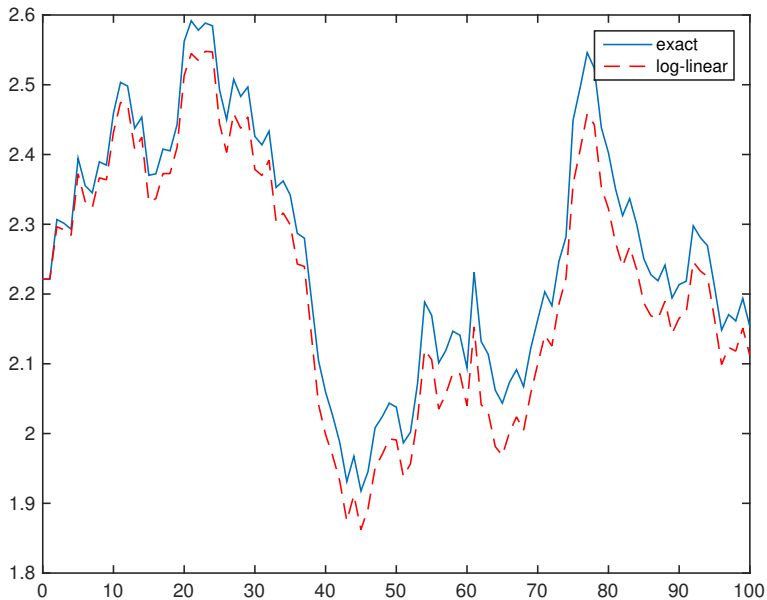
# Analytical solution for the variance, cont'd

- With this expression, the variance of capital around the steady state is given by

$$\begin{aligned}\text{var}(\hat{k}) &= C^2 \text{var}(\varepsilon) + B^2 C^2 \text{var}(\varepsilon) + B^4 C^2 \text{var}(\varepsilon) + \dots, \\ &= C^2 \sigma_\varepsilon^2 (1 + B^2 + B^4 + \dots),\end{aligned}$$

$$\text{var}(\hat{k}) = \frac{C^2 \sigma_\varepsilon^2}{1 - B^2}.$$

# Simulations

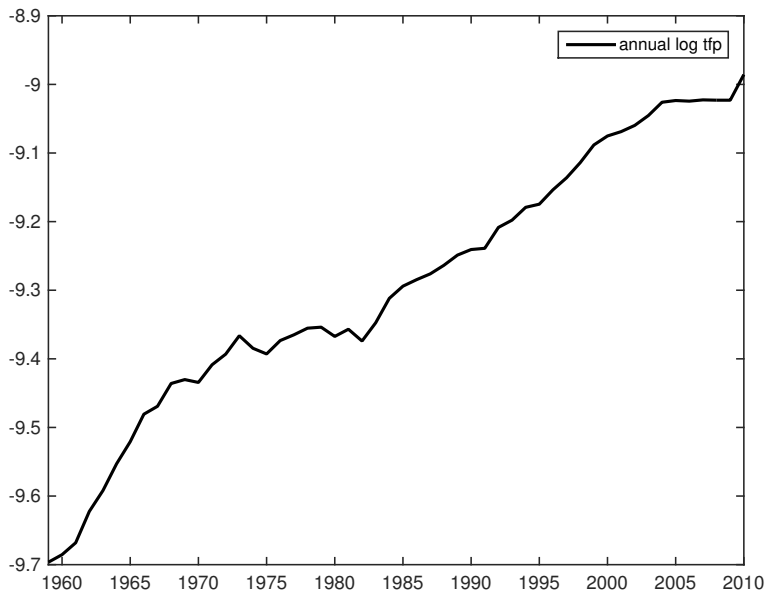


- Identifying the aggregate technology shock with the Solow residual:

$$\log A_t = \log Y_t - \theta \log K_t - (1 - \theta) \log H_t.$$

- $\log A_t$  has a trend. How to remove the trend?

# Solow residual



# Data source (NIPA and CPS)

- GDP, Nominal Capital, and GDP deflator (to deflate nominal capital) are from National Income and Product Accounts (NIPA).
  - GDP: Table 2A. Real Gross Domestic Product > Gross domestic product (Line 1)
  - Nominal Capital: Table 5.9. Changes in Net Stock of Produced Assets (Fixed Assets and Inventories) > Private (Line 2)
  - GDP deflator: Table 1.4.4. Price Indexes for Gross Domestic Product, Gross Domestic Purchases, and Final Sales to Domestic Purchasers > Gross domestic product (Line 1)
- Hours worked is from Consumer Population Survey (CPS).
  - See Cociuba, Prescott and Uberfeldt "U.S. Hours and Productivity Behavior Using CPS Hours Worked Data: 1947-III to 2011-IV"

- Remove linear trend:  $a_t = \log A_t - b_0 - b_1 t$  where  $b_0$  and  $b_1$  are obtained by OLS.

# Hodrick-Prescott filter

- Let  $y_t$  be a time series and

$$y_t = g_t + c_t,$$

where  $g_t$  is trend and  $c_t$  is cyclical component.

- The Hodrick-Prescott filter solves the following problem:

$$\min_{\{g_t\}_{t=1}^T} \left\{ \sum_{t=1}^T (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} [(g_{t+1} - g_t) - (g_t - g_{t-1})]^2 \right\},$$

where  $\lambda$  is smoothing parameter.

- FOCs are

$$\partial g_1 : \quad c_1 = \lambda(g_3 - 2g_2 + g_1),$$

$$\partial g_2 : \quad c_2 = \lambda(g_4 - 2g_3 + g_2) - 2\lambda(g_3 - 2g_2 + g_1),$$

$$\begin{aligned} \partial g_t : \quad c_t &= \lambda(g_{t+2} - 2g_{t+1} + g_t) - 2\lambda(g_{t+1} - 2g_t + g_{t-1}) \\ &\quad + \lambda(g_t - 2g_{t-1} + g_{t-2}) \end{aligned}$$

$$\text{for } t = 3, 4, \dots, T-2,$$

$$\partial g_{T-1} : \quad c_{T-1} = -2\lambda(g_T - 2g_{T-1} + g_{T-2}) + \lambda(g_{T-1} - 2g_{T-2} + g_{T-3}),$$

$$\partial g_T : \quad c_T = \lambda(g_T - 2g_{T-1} + g_{T-2}).$$



- FOCs are

$$\partial g_1 : \quad c_1 = \lambda(g_3 - 2g_2 + g_1),$$

$$\partial g_2 : \quad c_2 = \lambda(g_4 - 4g_3 + 5g_2 - 2g_1),$$

$$\begin{aligned} \partial g_t : \quad c_t &= \lambda(g_{t+2} - 4g_{t+1} + 6g_t - 4g_{t-1} + g_{t-2}) \\ &\text{for } t = 3, 4, \dots, T-2, \end{aligned}$$

$$\partial g_{T-1} : \quad c_{T-1} = \lambda(-2g_T + 5g_{T-1} - 4g_{T-2} + g_{T-3}),$$

$$\partial g_T : \quad c_T = \lambda(g_T - 2g_{T-1} + g_{T-2}).$$

# Matrix form

- In a matrix form,  $\mathbf{c} = \mathbf{y} - \mathbf{g} = \lambda \mathbf{F} \mathbf{g}$  where

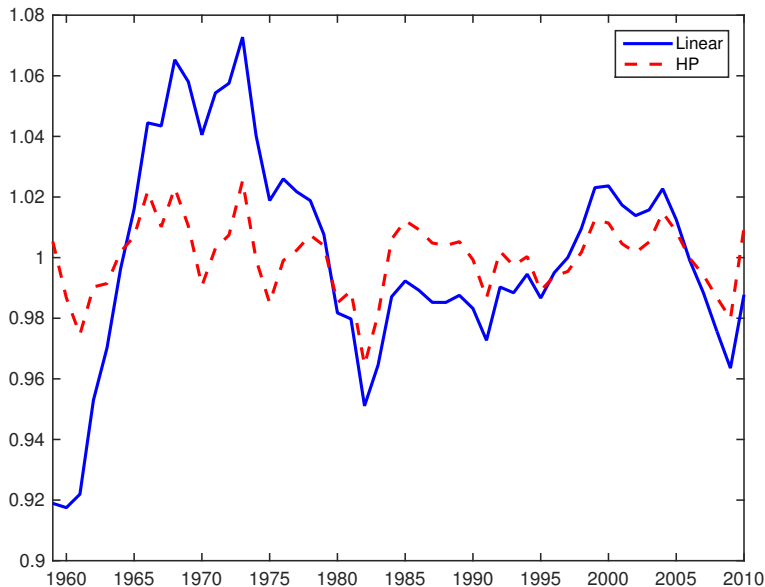
$$\underset{(T \times 1)}{\mathbf{c}} = [c_1, c_2, \dots, c_T]',$$

$$\underset{(T \times 1)}{\mathbf{g}} = [g_1, g_2, \dots, g_T]',$$

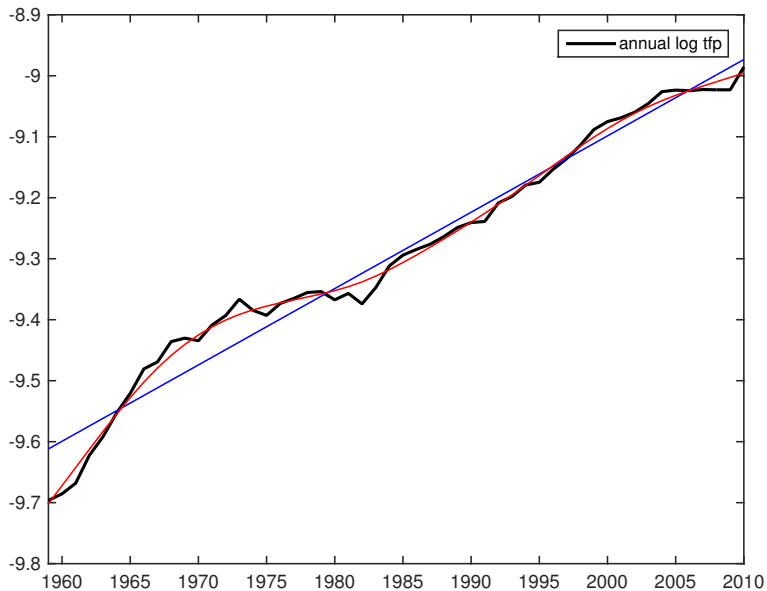
$$\underset{(T \times T)}{\mathbf{F}} = \begin{bmatrix} 1 & -2 & 1 & 0 & \dots & & & & 0 \\ -2 & 5 & -4 & 1 & 0 & \dots & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ \vdots & & & & & & & & \vdots \\ 0 & \dots & & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \dots & & & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \dots & & & & 0 & 1 & -4 & 5 & -2 \\ 0 & \dots & & & & & 0 & 1 & -2 & 1 \end{bmatrix}.$$

Then,  $\mathbf{g} = (\mathbf{I} - \lambda \mathbf{F})^{-1} \mathbf{y}$ .

# Cyclical component



# Trend



# Assignment #1

- Let  $n = .02$ ,  $\delta = .1$ ,  $\theta = .36$  and  $\sigma = .2$ . Also let  $\bar{A} = 1$ ,  $\rho = 0$  and  $\sigma_\varepsilon = .2$ .
  - 1 Simulate the model for 1,000 periods and compute  $\text{var}(k)$ .
  - 2 Compare it with the analytical solution for the variance.
  - 3 Do 1-2 with 100,000 period simulation.
  - 4 What about the case of  $\rho > 0$ ? Try to derive the analytical solution for the variance and compare it with simulation.