

# Ramsey-Cass-Koopmans model

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# What is the Ramsey-Cass-Koopmans model?

- A.k.a. the neo-classical growth model, a model of long run economic growth.
- Developed by Frank Ramsey (1929) and later extended by David Cass (1969) and Tjalling Koopmans (1969).
- The workhorse model of modern macroeconomics.

# Why is the RCK model useful?

- More reasonable explanation on the household's decision on saving/consumption *à la* Fisher. [The Solow model has an ad-hoc constant saving rate.]
- The model is also a basis for real business cycle (RBC) models of short-run fluctuations, which will be later in the class.

# Recap: Solow model

- In Solow model, we have

$$\begin{aligned}y_t &= A_t k_{t-1}^\theta, \\s_t = i_t &= k_t - (1 - \delta)k_{t-1}, \\s_t &= \sigma y_t,\end{aligned}$$

where  $k_t$  is the capital stock at the *end* of date  $t$ .

- We will relax the assumption of the constant saving rate.

- The representative firm can access to the Cobb-Douglas production function technology:

$$y_t = A_t k_{t-1}^\theta.$$

- As a result of profit maximization,

$$r_t = \theta y_t / k_{t-1},$$

and

$$\pi_t = y_t - r_t k_{t-1} = (1 - \theta) y_t.$$

- Household's saving is equal to investment on capital:

$$s_t = i_t = k_t - (1 - \delta)k_{t-1}.$$

- Household owns capital and firm, and decide how much to save and consume:

$$c_t + s_t = r_t k_{t-1} + \pi_t = y_t.$$

- Decision on consumption and saving is now endogenous.

# Euler equation

- Household choose consumption so as to satisfy the following **the Euler equation**:

$$MU_t = \beta MU_{t+1} (1 + r_{t+1} - \delta),$$

where  $\beta$  is discount factor and  $MU_t$  is the marginal utility from consumption in period  $t$ .

- The LHS is the benefit from one unit of consumption today. The RHS is the benefit from one unit of saving today and consuming the return on saving tomorrow.

- Marginal utility is the benefit from one unit of consumption.
- To link consumption to utility, we need a utility function of the household.  
For example,

$$U_t = \log c_t.$$

Then we have

$$MU_t = 1/c_t.$$



- Now we have the following equilibrium conditions:

$$\begin{aligned}y_t &= A_t k_{t-1}^\theta, \\c_t + k_t - (1 - \delta)k_{t-1} &= y_t, \\1 &= \beta \frac{c_t}{c_{t+1}} (1 + r_{t+1} - \delta), \\r_t &= \theta y_t / k_{t-1}.\end{aligned}$$

- There are four unknowns and four equations, so we can solve the model.

# The two key equations

- The equilibrium conditions are summarized as

$$\frac{c_{t+1}}{c_t} = \beta (1 + \theta A_{t+1} k_t^{\theta-1} - \delta),$$
$$k_t - k_{t-1} = A_t k_{t-1}^{\theta} - \delta k_{t-1} - c_t.$$

# The steady state

- In the steady state,

$$\begin{aligned}1 &= \beta (1 + \alpha A k^{\alpha-1} - \delta), \\ 0 &= A k^{\alpha} - \delta k - c,\end{aligned}$$

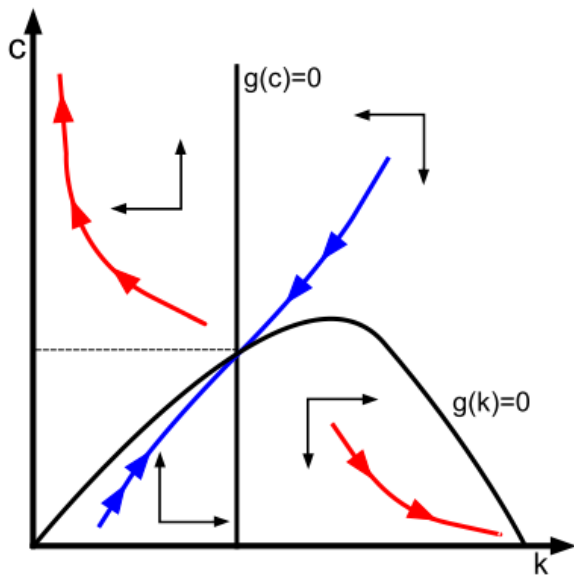
hold.

- The steady-state conditions can be solved for

$$\begin{aligned}\bar{k} &= \left( \frac{\alpha \beta A}{1 - \beta(1 - \delta)} \right)^{\frac{1}{1-\alpha}}, \\ \bar{c} &= A(\bar{k})^{\alpha} - \delta \bar{k}.\end{aligned}$$

These equations are drawn on the  $(k, c)$  plane.

# The Phase diagram



# The transition dynamics

- Let

$$g_{ct} \equiv c_t/c_{t-1} - 1 = \beta (1 + \alpha A_t k_{t-1}^{\alpha-1} - \delta) - 1,$$
$$\tilde{g}_{kt} \equiv k_t - k_{t-1} = A k_{t-1}^{\alpha} - \delta k_{t-1} - c_t.$$

- If  $g_{ct} > 0$ ,  $c_t > c_{t-1}$  holds;  $k_{t-1} < \bar{k}$  implies  $g_c > 0$ .
- If  $\tilde{g}_{kt} > 0$ ,  $k_t > k_{t-1}$  holds;  $c_t > A k_{t-1}^{\alpha} - \delta k_{t-1}$  implies  $\tilde{g}_k < 0$ .
- $c_t$  is called jump variables, whereas  $k_{t-1}$  is state variable.

# Two ways to solve the RCK model

- A Robinson Crusoe (Social Planner) economy.
- A Competitive economy.
- When the allocations and prices in these economies coincide each other?  
The second fundamental theorem of welfare economics.

# A Robinson Crusoe economy

- Consider an economy with only one individual.
- The individual wants to maximize a lifetime utility of the form

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$k_t = (1 - \delta)k_{t-1} + i_t,$$

$$y_t = A_t k_{t-1}^{\alpha} \geq c_t + i_t.$$

where  $c_t$  is consumption,  $k_t$  is capital,  $i_t$  is investment and  $y_t$  is output. Parameters are given as  $\beta \in (0, 1)$  is discount factor,  $\delta \in (0, 1]$  is depreciation rate, and  $\alpha$  is capital share.

- We set up the Lagrangean as

$$L_0 \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t - \lambda_t \left( c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\alpha} \right) \right\}.$$

$\lambda_t$  is called the Lagrange multiplier, which measures the marginal utility of consumption.

- Taking the derivatives of the Lagrangean and set them to zero

$$\partial c_t : \quad \lambda_t = 1/c_t,$$

$$\partial k_t : \quad \lambda_t = \beta \lambda_{t+1} (1 + \alpha A_{t+1} k_t^{\alpha} - \delta),$$

$$\partial \lambda_t : \quad c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\alpha} = 0.$$

These are the necessary conditions for the equilibrium.



# Transversality condition

- The transversality condition is

$$\lim_{t \rightarrow \infty} \beta^t MU_t k_t = 0,$$

where  $k_t$  is the remaining resources and  $MU_t$  converts the value of  $k_t$  to the unit in terms of utility.

- The condition says that the planner must use all the resources and/or have no marginal benefit from consumption.

# A competitive economy

- In a competitive economy, there are consumers who provide labor to the market and firms who hire the labor at wage  $w_t$  and rent capital at rate  $r_t$ .
- All individuals are the same. We consider the “representative” agent’s problem.

- The representative firm can access to the Cobb-Douglas production function technology:

$$y_t = A_t k_{t-1}^\alpha.$$

- As a result of profit maximization,

$$r_t = \alpha y_t / k_{t-1},$$

and

$$\pi_t = y_t - r_t k_{t-1} = (1 - \alpha) y_t.$$

- An individual  $i \in [0, 1]$  maximizes:

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

subject to

$$c_t^i + k_t^i - (1 - \delta)k_{t-1}^i = r_t k_t^i + \pi_t^i,$$

- We set up the Lagrangian as

$$L_0^i \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t^i - \lambda_t \left( c_t^i + k_t^i - (1 - \delta)k_{t-1}^i - r_t k_{t-1}^i - \pi_t^i \right) \right\}.$$

- Taking the derivatives of the Lagrangian and set them to zero,

$$\partial c_t^i : \quad \lambda_t = 1/c_t^i,$$

$$\partial k_t^i : \quad \lambda_t = \beta \lambda_{t+1} (1 + r_{t+1} - \delta),$$

$$\partial \lambda_t : \quad c_t^i + k_t^i - (1 - \delta)k_{t-1}^i - r_t k_{t-1}^i - \pi_t^i = 0.$$

- The aggregation rules are

$$c_t = \int_0^1 c_t^i di, \quad k_t = \int_0^1 k_t^i di,$$
$$y_t = \int_0^1 y_t^i di, \quad \pi_t = \int_0^1 \pi_t^i di,$$

- Note that  $\pi_t + r_t k_{t-1} = y_t$ . Then we have

$$1 = \beta \frac{c_t}{c_{t+1}} (1 + \alpha A_{t+1} k_t^{\alpha-1} - \delta),$$
$$c_t + k_t - (1 - \delta)k_{t-1} = A_t k_{t-1}^{\alpha}.$$

# Second welfare theorem

- **The second fundamental theorem of welfare economics:** If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*

## Second welfare theorem, cont'd

- The first fundamental theorem: Any Competitive Equilibrium allocation (C.E.) is necessarily Pareto Optimum allocation (P.O.).
- The second fundamental theorem: Any P.O. can be achieved as a C.E. with a lump-sum transfer.
- If the second welfare theorem holds, we only need to look at P.O. instead of C.E.
- Counter-example: An economy with distortionary tax.



- ① An economy with variable labor
- ② Incorporating trend
- ③ Notes on Newton's method and deterministic simulation

# An economy with variable labor

- The social planner wants to maximize a lifetime utility

$$\sum_{t=0}^{\infty} \beta^t \log c_t + v(h_t)$$

subject to

$$k_t = (1 - \delta)k_{t-1} + i_t,$$

$$y_t = A_t k_{t-1}^{\alpha} h_t^{1-\alpha} \geq c_t + i_t.$$

where  $c_t$  is consumption,  $k_t$  is capital,  $i_t$  is investment,  $y_t$  is output and  $h_t$  is hours worked. Parameters are given as  $\beta \in (0, 1)$  is discount factor,  $\delta \in (0, 1]$  is depreciation rate, and  $\alpha$  is capital share.

# Labor disutility

- $v(h_t)$  is called labor disutility:  $h_t \in (0, 1)$  and  $v(h_t)$  is a concave function such that  $v(h_t) \rightarrow -\infty$  as  $h_t \rightarrow 1$ .
- There are two forms of labor disutility.
  - Divisible labor:  $v(h_t) = B \log(1 - h_t)$ : Everyone works for  $h_t$  hours.
  - Indivisible labor:  $v(h_t) = -Bh_t$ : Only a fraction of *individuals* works for  $h_0$  hours.

# Indivisibility and lottery

- Labor is indivisible: Individuals can either work full time, denoted by  $h_0$ , or not at all.
- We require individuals to choose lotteries  $\alpha_t$ :

$$\log c_t + A\alpha_t \log(1 - h_0)$$

Note  $\log(1) = 0$ . Total hours worked (per capita) is  $h_t = \alpha_t h_0$ .

# Indivisibility and lottery, cont'd

- It can be viewed as linear disutility from the point of view of “representative” household.
- By substituting out  $\alpha_t$ , we have

$$\begin{aligned} A\alpha_t \log(1 - h_0) &= A \log(1 - h_0)/h_0 h_t, \\ &= -Bh_t \end{aligned}$$

where  $B = -A \log(1 - h_0)/h_0 > 0$ .

- We set up the Lagrangian as

$$L_0 \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + v(h_t) - \lambda_t \left( c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\alpha} h_t^{1-\alpha} \right) \right\}.$$

$\lambda_t$  is the Lagrange multiplier.

- Taking the derivatives of the Lagrangian and set them to zero,

$$\partial c_t : \quad \lambda_t = 1/c_t,$$

$$\partial h_t : \quad \lambda_t (1 - \alpha) A_t k_{t-1}^{\alpha} h_t^{-\alpha} = -v'(h_t),$$

$$\partial k_t : \quad \lambda_t = \beta \lambda_{t+1} \left( 1 + \alpha A_{t+1} k_t^{\alpha-1} h_t^{1-\alpha} - \delta \right),$$

$$\partial \lambda_t : \quad c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\alpha} h_t^{1-\alpha} = 0.$$

- 1 An economy with variable labor
- 2 Incorporating trend
- 3 Notes on Newton's method and deterministic simulation

- Let  $Z_t \equiv A_t^{\frac{1}{1-\alpha}}$  and  $\gamma \equiv Z_{t+1}/Z_t$ . The production function becomes

$$y_t = k_{t-1}^{\alpha} (Z_t h_t)^{1-\alpha}.$$

This is called the Harrod-neutral production function.

- In the steady state,  $y$  and  $k$  exponentially grows at the rate  $\gamma$ .



- The equilibrium conditions are

$$\begin{aligned}y_t &= k_{t-1}^\alpha (Z_t h_t)^{1-\alpha}, \\(1-\alpha) \frac{y_t}{h_t} &= \frac{B c_t}{1-h_t}, \\1 &= \beta \frac{c_t}{c_{t+1}} \left( 1 + \alpha \frac{y_{t+1}}{k_t} - \delta \right), \\c_t + k_t - (1-\delta)k_{t-1} - y_t &= 0.\end{aligned}$$

- $y_t$ ,  $c_t$  and  $k_{t-1}$  grows at  $\gamma$ , whereas the model variables have to be stationary.

- Define  $\tilde{y}_t \equiv y_t/Z_t$ ,  $\tilde{c}_t \equiv c_t/Z_t$  and  $\tilde{k}_{t-1} \equiv k_{t-1}/Z_t$ ,

$$\tilde{y}_t = \tilde{k}_{t-1}^\alpha h_t^{1-\alpha},$$

$$(1 - \alpha) \frac{\tilde{y}_t}{h_t} = \frac{B\tilde{c}_t}{1 - h_t},$$

$$1 = (\beta/\gamma) \frac{\tilde{c}_t}{\tilde{c}_{t+1}} \left( 1 + \alpha \frac{\tilde{y}_{t+1}}{\tilde{k}_t} - \delta \right),$$

$$\tilde{c}_t + \gamma \tilde{k}_t - (1 - \delta) \tilde{k}_{t-1} - \tilde{y}_t = 0.$$

# The steady state

- The steady state conditions are

$$1 = \tilde{\beta} \left( 1 + \alpha \tilde{y} / \tilde{k} - \delta \right),$$

$$0 = \tilde{y} - (\gamma - 1 + \delta) \tilde{k} - \tilde{c},$$

$$(1 - \alpha)(\tilde{y}/h) = B\tilde{c}/(1 - h),$$

$$\tilde{y} = \tilde{k}^{\alpha} h^{1-\alpha},$$

where  $\tilde{\beta} = \beta/\gamma$ .

- Dynare solves for the steady state values numerically with an educated initial guess. In this model, the steady state values are also analytically obtained.

# The steady state

- Let  $h = 1/3$ , then we have

$$\tilde{y}/\tilde{k} = \alpha^{-1}(\tilde{\beta}^{-1} - 1 + \delta),$$

$$\tilde{c}/\tilde{k} = \tilde{y}/\tilde{k} - (\gamma - 1 + \delta),$$

$$\tilde{k} = \left( \frac{\alpha h^{1-\alpha}}{\tilde{\beta}^{-1} - 1 + \delta} \right)^{\frac{1}{1-\alpha}},$$

$$B = (1 - \alpha)(\tilde{y}/\tilde{c})(1 - h)/h.$$

$\tilde{y} = (\tilde{y}/\tilde{k})\tilde{k}$  and  $\tilde{c} = (\tilde{c}/\tilde{k})\tilde{k}$  are also obtained. Note that  $B$  is a normalization parameter.

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# Newton-Raphson method

- We solve an equation  $f(x) = 0$  for  $x$ .
- First-order Taylor expansion:  $f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 0$ .
- The Newton-Raphson method updates  $x_k$

$$x_{k+1} = x_k - f'(x_k)^{-1} f(x_k),$$

until  $\|x_{k+1} - x_k\| \leq \epsilon$ .

# Newton-Raphson method, cont'd

- We solve a system of equations  $f_i(\mathbf{x}) = 0$  for  $i = 1, \dots, n$  for  $\mathbf{x} = [x_1, \dots, x_n]'$ .
- $n$  dimensional First-order Taylor expansion:

$$f_i(\mathbf{x}) \approx f_i(\mathbf{x}_0) + \nabla f_i(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = 0,$$

for  $i = 1, \dots, n$  where  $\nabla f_i(\mathbf{x}_0) = [f'_i(x_1), \dots, f'_i(x_n)]$ .

# Newton-Raphson method, cont'd

- Let  $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \dots, f_n(\mathbf{x})]'$  and

$$\nabla \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \nabla f_1(\mathbf{x}) \\ \nabla f_2(\mathbf{x}) \\ \vdots \\ \nabla f_n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \dots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \dots & \dots & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

is an  $(n \times n)$  matrix called *Jacobian*.

- The Newton-Raphson method updates  $\mathbf{x}_k$

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\nabla \mathbf{f}(\mathbf{x}_k)]^{-1} \mathbf{f}(\mathbf{x}_k),$$

until  $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq \epsilon$ .



# Deterministic simulations

- Dynare can be used for deterministic simulations with the assumption of perfect foresight.
- The numerical problem consists of solving a nonlinear system of simultaneous equations in  $n$  endogenous variables in  $T$  periods.
- To solve the system of  $nT$  equations, Dynare uses a Newton-type method, which is based on the Fair-Taylor (1983) algorithm.
  - My explanation is based on Hollinger (1996).

# A nonlinear system

- A nonlinear dynamic system is of a general form

$$f_{i,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = 0,$$

for  $i = 1, \dots, n$  where  $\mathbf{y}_t = [y_{1,t}, \dots, y_{n,t}]'$ .

- For example, the basic RCK model is expressed as

$$f_{1,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \beta \frac{c_t}{c_{t+1}} (1 + \alpha A_{t+1} k_t^{\alpha-1} - \delta) - 1 = 0,$$

$$f_{2,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = A_t k_{t-1}^\alpha + (1 - \delta) k_{t-1} - k_t - c_t = 0,$$

where  $\mathbf{y}_t = [k_t, c_t]'$ .

- The system is expressed compactly as a  $(n \times 1)$  vector of equations:

$$\mathbf{f}_t(\mathbf{z}_t) = \mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \begin{bmatrix} f_{1,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \vdots \\ f_{n,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \end{bmatrix} = \mathbf{0},$$

where  $\mathbf{f}_t = [f_{1,t}, \dots, f_{n,t}]'$  and  $\mathbf{z}_t = [\mathbf{y}_{t-1}', \mathbf{y}_t', \mathbf{y}_{t+1}']'$ .

# Solving the nonlinear system

- We solve  $\mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \mathbf{0}$  *simultaneously* for  $t = 1, \dots, T$ .
- $\mathbf{y}_{t-1} = \mathbf{y}_0$  is predetermined at  $t = 1$ . Also,  $\mathbf{y}_{t+1} = \mathbf{y}_{T+1}$  is predetermined at  $t = T$  (the boundary conditions).

## Solving the nonlinear system, cont'd

- We stack the system for  $T$  periods as

$$F(Z) = \begin{bmatrix} \mathbf{f}_1(\mathbf{z}_1) \\ \vdots \\ \mathbf{f}_t(\mathbf{z}_t) \\ \vdots \\ \mathbf{f}_T(\mathbf{z}_T) \end{bmatrix} = \mathbf{0},$$

where  $F = [\mathbf{f}'_1, \dots, \mathbf{f}'_t, \dots, \mathbf{f}'_n]'$  and  $Z = [\mathbf{z}'_1, \dots, \mathbf{z}'_t, \dots, \mathbf{z}'_n]'$ . This is a  $(nT \times 1)$  vector of equations.

- We solve  $F(Z)$  by the Newton-Raphson method for  $nT$  variables

$$Z = [\mathbf{y}'_0, \mathbf{y}'_1, \dots, \mathbf{y}'_T, \mathbf{y}'_{T+1}],$$

given  $\mathbf{y}'_0$  and  $\mathbf{y}'_T$ .

# Assignment #2

- Let  $\beta = .96$ ,  $\delta = .1$ ,  $\alpha = .36$ . Consider the basic RCK model without labor.
  - 1 Let  $k_0 = 0.1$ . Compute the initial dynamics converging to the steady state.
  - 2 Assume a distortionary tax on capital income after depreciation,

$$c_t + k_t - (1 - \delta)k_{t-1} \leq r_t k_{t-1} - \tau(r_t - \delta)k_{t-1} + \pi_t.$$

Let  $\tau = .2$ .

- 1 Solve for the equilibrium conditions in the competitive economy.
- 2 Solve for the steady state.
- 3 Compute the initial dynamics with  $k_0 = 0.1$ .
- 4 Compare the results with the ones in the model without distortionary taxes, which you computed in (1).