Proposed Answer to Assignment #1

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Note that $E\hat{a}_t^2 = (1-\rho^2)^{-1}\sigma_\varepsilon^2 = \sigma_a^2$ and

$$E\hat{a}_t\hat{a}_{t-j} = E\hat{a}_{t-j}\left(\rho^j\hat{a}_{t-j} + \varepsilon_t + \varepsilon_{t-1} + ... + \varepsilon_{t-(j-1)}\right) = \rho^j\sigma_a^2$$

for $j \geq 1$. Then,

$$\begin{aligned}
\hat{k}_{t+1}^2 &= C^2\left(\hat{a}_t + B\hat{a}_{t-1} + B^2\hat{a}_{t-2} + B^3\hat{a}_{t-3} + ...\right)^2, \\
&= C^2\left(\hat{a}_t^2 + B^2\hat{a}_{t-1}^2 + B^4\hat{a}_{t-2}^2 + ... + 2B\hat{a}_t\hat{a}_{t-1} + 2B^3\hat{a}_{t-1}\hat{a}_{t-2} + 2B^5\hat{a}_{t-2}\hat{a}_{t-3} + ...\right. \\
&\quad \left. + 2B^2\hat{a}_t\hat{a}_{t-2} + 2B^4\hat{a}_{t-1}\hat{a}_{t-3} + 2B^6\hat{a}_{t-2}\hat{a}_{t-4} + ...\right).
\end{aligned}$$

and

$$\begin{aligned}
E\hat{k}_{t+1}^2 &= C^2\left(\sigma_a^2 + B^2\sigma_a^2 + B^4\sigma_a^2 + ... + 2B\rho\sigma_a^2 + 2B^3\rho\sigma_a^2 + 2B^5\rho\sigma_a^2 + ...\right. \\
&\quad \left. + 2B^2\rho^2\sigma_a^2 + 2B^4\rho^2\sigma_a^2 + 2B^6\rho^2\sigma_a^2 + ...\right), \\
&= C^2\left(\sigma_a^2\left[1 + B^2 + B^4 + ...\right] + 2B\rho\sigma_a^2\left[1 + B^2 + B^4 + ...\right]\right. \\
&\quad \left. + 2B^2\rho^2\sigma_a^2\left[1 + B^2 + B^4 + ...\right] + ...\right), \\
&= C^2(1-B^2)^{-1}\sigma_a^2\left(1 + 2B\rho + 2B^2\rho^2 + ...\right).
\end{aligned}$$

Note that $1 + 2B\rho(1 + B\rho + B^2\rho^2 ...) = (1+B\rho)/(1-B\rho)$. Finally, we have

$$\text{var}\left(\hat{k}\right) = \frac{C^2(1+B\rho)\sigma_\varepsilon^2}{(1-B^2)(1-B\rho)(1-\rho^2)}.$$

Note that as $\rho \to 1$ and $\hat{k}_t$ becomes a random walk, the error between the analytical and numerical solutions becomes larger.