Proposed Answers to Assignment #4

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(1)

Lagrangean is

$$L \equiv \int_0^1 P_t(i)C_t(i)di - \mu\left(\left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}} - C_t\right).$$

The FOC is given by

$$\begin{aligned} P_t(i) &= \mu\left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} C_t(i)^{\frac{\varepsilon-1}{\varepsilon}-1} \\ &= \mu C_t^{1/\varepsilon}C_t(i)^{-1/\varepsilon}. \end{aligned}$$

for all $i \in [0,1]$. Thus, for any goods (i,j),

$$\frac{C_t(i)}{C_t(j)} = \left(\frac{P_t(i)}{P_t(j)}\right)^{-\varepsilon}$$

holds. Pluging this into the aggregator yields

$$\begin{aligned} \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}di\right)^{\frac{\varepsilon}{\varepsilon-1}} &= \left(\int_0^1 C_t(j)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{P_t(i)}{P_t(j)}\right)^{1-\varepsilon}di\right)^{\frac{\varepsilon}{\varepsilon-1}}, \\ &= C_t(j)P_t(j)^{\varepsilon}\left(\int_0^1 P_t(i)^{1-\varepsilon}di\right)^{\frac{\varepsilon}{\varepsilon-1}}. \end{aligned}$$

By defining the aggregate price index $P_t = \left(\int_0^1 P_t(i)^{1-\varepsilon}di\right)^{\frac{1}{1-\varepsilon}}$, we have

$$C_t = C_t(j)P_t(j)^{\varepsilon}P_t^{-\varepsilon}.$$

(2)

Firm chooses P_t^* so as to maximize

$$\begin{aligned}
&\sum_{k=0}^{\infty}\theta^k E_t\left\{\Lambda_{t,t+k}\left(\frac{P_t^*}{P_{t+k}}-\tilde{\Psi}_{t+k}\right)Y_{t+k|t}\right\},\\
\Leftrightarrow\quad &\sum_{k=0}^{\infty}\theta^k E_t\left\{\Lambda_{t,t+k}\left(\left(\frac{P_t^*}{P_{t+k}}\right)^{1-\varepsilon}Y_{t+k}-\tilde{\Psi}_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}Y_{t+k}\right)\right\}.
\end{aligned}$$

The FOC is

$$\begin{aligned}
&\sum_{k=0}^{\infty}\theta^k E_t\left\{\Lambda_{t,t+k}\left((1-\varepsilon)\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}\frac{Y_{t+k}}{P_{t+k}}+\varepsilon\tilde{\Psi}_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon-1}\frac{Y_{t+k}}{P_{t+k}}\right)\right\}=0,\\
\Leftrightarrow\quad &\sum_{k=0}^{\infty}\theta^k E_t\left\{\Lambda_{t,t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}Y_{t+k}\left((1-\varepsilon)\frac{1}{P_{t+k}}+\varepsilon\tilde{\Psi}_{t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-1}\frac{1}{P_{t+k}}\right)\right\}=0,\\
\Leftrightarrow\quad &\sum_{k=0}^{\infty}\theta^k E_t\left\{\Lambda_{t,t+k}\left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon}Y_{t+k}\left(\frac{P_t^*}{P_{t+k}}-\frac{\varepsilon}{\varepsilon-1}\tilde{\Psi}_{t+k}\right)\right\}=0.
\end{aligned}$$

The last line is obtained by multiplying $(1-\varepsilon)^{-1}P_t^*$.

Next, we will linearize the FOC. Note that $\Lambda_{t,t+k}=\beta^k\left(\frac{C_{t+k}}{C_t}\right)^{-\sigma}$. Then we have

$$\begin{aligned}
\frac{P_t^*}{P_t} &= \frac{\varepsilon}{\varepsilon-1}\frac{\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C_{t+k}^{1-\sigma}\left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon}\tilde{\Psi}_{t+k}\right\}}{\sum_{k=0}^{\infty}(\beta\theta)^k E_t\left\{C_{t+k}^{1-\sigma}\left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon-1}\right\}},\\
&= \frac{\varepsilon}{\varepsilon-1}\frac{S_t}{F_t},
\end{aligned}$$

where

$$\begin{aligned}
S_t &= C_t^{1-\sigma}\tilde{\Psi}_t+\beta\theta E_t C_{t+1}^{1-\sigma}\left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon}\tilde{\Psi}_{t+1}+(\beta\theta)^2E_tC_{t+2}^{1-\sigma}\left(\frac{P_{t+2}}{P_t}\right)^{\varepsilon}\tilde{\Psi}_{t+2},\\
&= C_t^{1-\sigma}\tilde{\Psi}_t+\beta\theta E_t\left\{C_{t+1}^{1-\sigma}\left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon}\tilde{\Psi}_{t+1}+\beta\theta C_{t+2}^{1-\sigma}\left(\frac{P_{t+2}}{P_t}\right)^{\varepsilon}\tilde{\Psi}_{t+2}\right\},\\
&= C_t^{1-\sigma}\tilde{\Psi}_t+\beta\theta E_t\left\{\left(\frac{P_{t+1}}{P_t}\right)^{\varepsilon}\left[C_{t+1}^{1-\sigma}\tilde{\Psi}_{t+1}+\beta\theta C_{t+2}^{1-\sigma}\left(\frac{P_{t+2}}{P_{t+1}}\right)^{\varepsilon}\tilde{\Psi}_{t+2}\right]\right\},\\
&= C_t^{1-\sigma}\tilde{\Psi}_t+\beta\theta E_t\left\{\Pi_{t+1}^{\varepsilon}S_{t+1}\right\}.
\end{aligned}$$

Similarly,

$$F_t = C_t^{1-\sigma} + \beta\theta E_t \left\{ \Pi_{t+1}^{\varepsilon-1} F_{t+1} \right\}.$$

Then we have

$$p_t^* - p_t = \mu + s_t - f_t,$$

where $x_t \equiv \log X_t$. By log-linearization,

$$\begin{aligned}
\hat{s}_t &= (1-\beta\theta)((1-\sigma)\hat{c}_t + \hat{\tilde{\psi}}_t) + \beta\theta E_t \left\{ \varepsilon\pi_{t+1} + \hat{s}_{t+1} \right\}, \\
\hat{f}_t &= (1-\beta\theta)((1-\sigma)\hat{c}_t) + \beta\theta E_t \left\{ (\varepsilon-1)\pi_{t+1} + \hat{f}_{t+1} \right\},
\end{aligned}$$

where $\hat{x}_t \equiv \log(X_t/X)$. Then we have

$$\hat{s}_t - \hat{f}_t = (1-\beta\theta)\hat{\tilde{\psi}}_t + \beta\theta E_t \left\{ p_{t+1} - p_t + \hat{s}_{t+1} - \hat{f}_{t+1} \right\}.$$

Note that $s - f = \tilde{\psi}$ so that

$$s_t - f_t = (1-\beta\theta)\tilde{\psi}_t + \beta\theta E_t \left\{ p_{t+1} - p_t + s_{t+1} - f_{t+1} \right\}.$$

Then we have

$$\begin{aligned}
s_t - f_t &= (1-\beta\theta)\tilde{\psi}_t + \beta\theta E_t \left\{ p_{t+1} - p_t + s_{t+1} - f_{t+1} \right\} \\
&= (1-\beta\theta)\tilde{\psi}_t + \beta\theta E_t(p_{t+1} - p_t) \\
&\quad + \beta\theta(1-\beta\theta)E_t\tilde{\psi}_{t+1} + (\beta\theta)^2 E_t(p_{t+2} - p_{t+1}) \\
&\quad + (\beta\theta)^2(1-\beta\theta)E_t\tilde{\psi}_{t+2} + (\beta\theta)^3 E_t(p_{t+3} - p_{t+2}) + ... \\
&= (1-\beta\theta)\left[\tilde{\psi}_t + \beta\theta E_t\tilde{\psi}_{t+1} + (\beta\theta)^2 E_t\tilde{\psi}_{t+2} + ... \right] \\
&\quad - \beta\theta p_t + (1-\beta\theta)\left[\beta\theta E_t p_{t+1} + (\beta\theta)^2 E_t p_{t+2} + ... \right] \\
&= (1-\beta\theta)\sum_{k=0}^{\infty} (\beta\theta)^k E_t(\tilde{\psi}_{t+k} + p_{t+k} - p_t)
\end{aligned}$$

Then we have

$$p_t^* - p_t = \mu + (1-\beta\theta)\sum_{k=0}^{\infty} (\beta\theta)^k E_t(\tilde{\psi}_{t+k} + p_{t+k} - p_t),$$

$$\Leftrightarrow \quad p_t^* = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t(\tilde{\psi}_{t+k} + p_{t+k}),$$
$$\Leftrightarrow \quad p_t^* = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t \psi_{t+k}.$$

Note that we have used $\psi_{t+k} = \tilde{\psi}_{t+k} + p_{t+k}$.