Ramsey-Cass-Koopmans model

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What is the Ramsey-Cass-Koopmans model?

- A.k.a. the neo-classical growth model, a model of long run economic growth.
- Developed by Frank Ramsey (1929) and later extended by David Cass (1969) and Tjalling Koopmans (1969).
- The workhorse model of modern macroeconomics.

Why is the RCK model useful?

- More reasonable explanation on the household's decision on saving/consumption *à la* Fisher. [The Solow model has an ad-hoc constant saving rate.]
- The model is also a basis for real business cycle (RBC) models of short-run fluctuations, which will be later in the class.

Recap: Solow model

- In Solow model, we have

$$\begin{aligned} y_t &= A_t k_{t-1}^{\theta}, \\ s_t = i_t &= k_t - (1-\delta)k_{t-1}, \\ s_t &= \sigma y_t, \end{aligned}$$

  where k_t is the capital stock at the *end* of date t.

- We will relax the assumption of the constant saving rate.

Firm

- The representative firm can access to the Cobb-Douglas production function technology:

$$y_t = A_t k_{t-1}^{\theta}.$$

- As a result of profit maximization,

$$r_t = \theta y_t / k_{t-1},$$

and

$$\pi_t = y_t - r_t k_{t-1} = (1 - \theta) y_t.$$

Household

- Household's saving is equal to investment on capital:

$$s_t = i_t = k_t - (1-\delta)k_{t-1}.$$

- Household owns capital and firm, and decide how much to save and consume:

$$c_t + s_t = r_t k_{t-1} + \pi_t = y_t.$$

- Decision on consumption and saving is now endogenous.

Euler equation

- Household choose consumption so as to satisfy the following the Euler equation:
$$MU_t = \beta MU_{t+1} \left(1 + r_{t+1} - \delta\right),$$
where β is discount factor and MU_t is the marginal utility from consumption in period t.

- The LHS is the benefit from one unit of consumption today. The RHS is the benefit from one unit of saving today and consuming the return on saving tomorrow.

Marginal utility

- Marginal utility is the benefit from one unit of consumption.

- To link consumption to utility, we need a utility function of the household. For example,

$$U_t = \log c_t.$$

Then we have

$$MU_t = 1/c_t.$$

Equilibrium

- Now we have the following equilibrium conditions:

$$
\begin{gathered}
y_t = A_t k_{t-1}^{\theta}, \\
c_t + k_t - (1-\delta)k_{t-1} = y_t, \\
1 = \beta \tfrac{c_t}{c_{t+1}} \left(1 + r_{t+1} - \delta\right), \\
r_t = \theta y_t / k_{t-1}.
\end{gathered}
$$

- There are four unknowns and four equations, so we can solve the model.

The two key equations

- The equilibrium conditions are summarized as

$$\frac{c_{t+1}}{c_t} = \beta \left(1 + \theta A_{t+1} k_t^{\theta-1} - \delta\right),$$
$$k_t - k_{t-1} = A_t k_{t-1}^{\theta} - \delta k_{t-1} - c_t.$$

The steady state

- In the steady state,

$$1 = \beta \left(1 + \alpha A k^{\alpha-1} - \delta\right),$$
$$0 = A k^{\alpha} - \delta k - c,$$

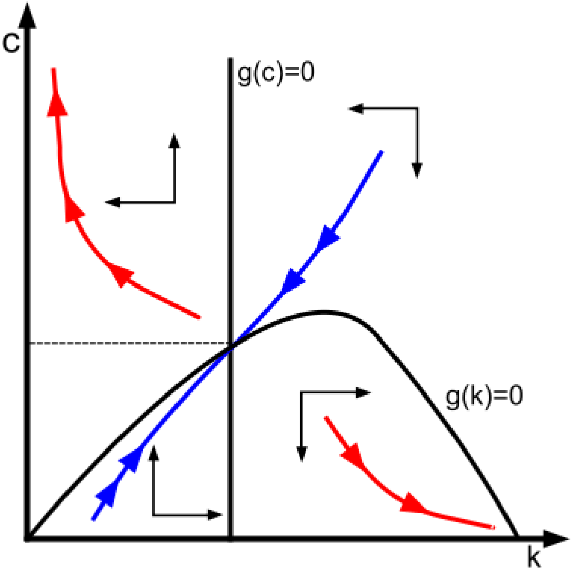
hold.

- The steady-state conditions can be solved for

$$\bar{k} = \left(\frac{\alpha\beta A}{1-\beta(1-\delta)}\right)^{\frac{1}{1-\alpha}},$$
$$\bar{c} = A(\bar{k})^{\alpha} - \delta\bar{k}.$$

These equations are drawn on the (k, c) plane.

The Phase diagram



The transition dynamics

- Let

$$
\begin{aligned}
g_{ct} &\equiv c_t/c_{t-1} - 1 = \beta\left(1 + \alpha A_t k_{t-1}^{\alpha-1} - \delta\right) - 1, \\
\tilde{g}_{kt} &\equiv k_t - k_{t-1} = A k_{t-1}^{\alpha} - \delta k_{t-1} - c_t.
\end{aligned}
$$

  - If $g_{ct} > 0$, $c_t > c_{t-1}$ holds; $k_{t-1} < \bar{k}$ implies $g_c > 0$.
  - If $\tilde{g}_{kt} > 0$, $k_t > k_{t-1}$ holds; $c_t < A k_{t-1}^{\alpha} - \delta k_{t-1}$ implies $\tilde{g}_k > 0$.
- c_t is called jump variables, whereas k_{t-1} is state variable.

Two ways to solve the RCK model

- A Robinson Crusoe (Social Planner) economy.
- A Competitive economy.
- When the allocations and prices in these economies coincide each other?
  The second fundamental theorem of welfare economics.

A Robinson Crusoe economy

- Consider an economy with only one individual.
- The individual wants to maximize a lifetime utility of the form

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$k_t = (1-\delta)k_{t-1} + i_t,$$
$$y_t = A_t k_{t-1}^{\alpha} \geq c_t + i_t.$$

where c_t is consumption, k_t is capital, i_t is investment and y_t is output. Parameters are given as $\beta \in (0,1)$ is discount factor, $\delta \in (0,1]$ is depreciation rate, and α is capital share.

Lagrangean

- We set up the Lagrangean as

$$L_0 \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t - \lambda_t \left(c_t + k_t - (1-\delta)k_{t-1} - A_t k_{t-1}^{\alpha} \right) \right\}.$$

λ_t is called the Lagrange multiplier, which measures the marginal utility of consumption.

- Taking the derivatives of the Lagrangean and set them to zero

$$\begin{aligned} \partial c_t : &\quad \lambda_t = 1/c_t, \\ \partial k_t : &\quad \lambda_t = \beta \lambda_{t+1} \left(1 + \alpha A_{t+1} k_t^{\alpha} - \delta \right), \\ \partial \lambda_t : &\quad c_t + k_t - (1-\delta)k_{t-1} - A_t k_{t-1}^{\alpha} = 0. \end{aligned}$$

These are the necessary conditions for the equilibrium.

Transversality condition

- The transversality condition is

$$\lim_{t\to\infty} \beta^t MU_t k_t = 0,$$

  where k_t is the remaining resources and MU_t converts the value of k_t to the unit in terms of utility.

- The condition says that the planner must use all the resources and/or have no marginal benefit from consumption.

A competitive economy

- In a competitive economy, there are consumers who provide labor to the market and firms who hire the labor at wage w_t and rent capital at rate r_t.

- All individuals are the same. We consider the "representative" agent's problem.

Firms

- The representative firm can access to the Cobb-Douglas production function technology:

$$y_t = A_t k_{t-1}^{\alpha}.$$

- As a result of profit maximization,

$$r_t = \alpha y_t / k_{t-1},$$

and

$$\pi_t = y_t - r_t k_{t-1} = (1 - \alpha) y_t.$$

Households

- An individual $i \in [0,1]$ maximizes:

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

subject to

$$c_t^i + k_t^i - (1-\delta)k_{t-1}^i = r_t k_t^i + \pi_t^i,$$

Lagrangean

- We set up the Lagrangean as

$$L_0^i \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t^i - \lambda_t \left(c_t^i + k_t^i - (1-\delta)k_{t-1}^i - r_t k_{t-1}^i - \pi_t^i \right) \right\}.$$

- Taking the derivatives of the Lagrangean and set them to zero,

$$\begin{aligned}
\partial c_t^i : &\quad \lambda_t = 1/c_t^i, \\
\partial k_t^i : &\quad \lambda_t = \beta \lambda_{t+1} \left(1 + r_{t+1} - \delta\right), \\
\partial \lambda_t : &\quad c_t^i + k_t^i - (1-\delta)k_{t-1}^i - r_t k_{t-1}^i - \pi_t^i = 0.
\end{aligned}$$

Aggregation

- The aggregation rules are

$$c_t = \int_0^1 c_t^i di, \quad k_t = \int_0^1 k_t^i di,$$
$$y_t = \int_0^1 y_t^i di, \quad \pi_t = \int_0^1 \pi_t^i di,$$

- Note that $\pi_t + r_t k_{t-1} = y_t$. Then we have

$$1 = \beta \frac{c_t}{c_{t+1}} \left(1 + \alpha A_{t+1} k_t^{\alpha-1} - \delta\right),$$
$$c_t + k_t - (1-\delta) k_{t-1} = A_t k_{t-1}^{\alpha}.$$

Second welfare theorem

- The second fundamental theorem of welfare economics: If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*

Second welfare theorem, cont'd

- The first fundamental theorem: Any Competitive Equilibrium allocation (C.E.) is necessarily Pareto Optimum allocation (P.O.).
- The second fundamental theorem: Any P.O. can be achieved as a C.E. with a lump-sum transfer.
- If the second welfare theorem holds, we only need to look at P.O. instead of C.E.
- Counter-example: An economy with distortionary tax.

Appendix

1. An economy with variable labor
2. Incorporating trend
3. Notes on Newton's method and deterministic simulation

An economy with variable labor

- The social planner wants to maximize a lifetime utility

$$\sum_{t=0}^{\infty} \beta^t \log c_t + v(h_t)$$

subject to

$$k_t = (1-\delta)k_{t-1} + i_t,$$

$$y_t = A_t k_{t-1}^{\alpha} h_t^{1-\alpha} \geq c_t + i_t.$$

where c_t is consumption, k_t is capital, i_t is investment, y_t is output and h_t is hours worked. Parameters are given as $\beta \in (0,1)$ is discount factor, $\delta \in (0,1]$ is depreciation rate, and α is capital share.

Labor disutility

- $v(h_t)$ is called labor disutility: $h_t \in (0,1)$ and $v(h_t)$ is a concave function such that $v(h_t) \to -\infty$ as $h_t \to 1$.
- There are two forms of labor disutility.
  - Divisible labor: $v(h_t) = B\log(1-h_t)$: Everyone works for h_t hours.
  - Indivisible labor: $v(h_t) = -Bh_t$: Only a fraction of *individuals* works for h_0 hours.

Indivisibility and lottery

- Labor is indivisible: Individuals can either work full time, denoted by h_0, or not at all.
- We require individuals to choose lotteries α_t:

$$\log c_t + A\alpha_t \log(1 - h_0)$$

Note $\log(1) = 0$. Total hours worked (per capita) is $h_t = \alpha_t h_0$.

Indivisibility and lottery, cont'd

- It can be viewed as linear disutility from the point of view of "representative" household.
- By substituting out α_t, we have

$$\begin{aligned} A\alpha_t \log(1-h_0) &= A\log(1-h_0)/h_0 h_t, \\ &= -Bh_t \end{aligned}$$

where $B = -A\log(1-h_0)/h_0 > 0$.

Lagrangean

- We set up the Lagrangean as

$$L_0 \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + v(h_t) - \lambda_t \left(c_t + k_t - (1-\delta)k_{t-1} - A_t k_{t-1}^{\alpha} h_t^{1-\alpha} \right) \right\}.$$

λ_t is the Lagrange multiplier.

- Taking the derivatives of the Lagrangean and set them to zero,

$$\begin{aligned}
\partial c_t : &\quad \lambda_t = 1/c_t, \\
\partial h_t : &\quad \lambda_t (1-\alpha) A_t k_{t-1}^{\alpha} h_t^{-\alpha} = -v'(h_t), \\
\partial k_t : &\quad \lambda_t = \beta \lambda_{t+1} \left(1 + \alpha A_{t+1} k_t^{\alpha-1} h_t^{1-\alpha} - \delta \right), \\
\partial \lambda_t : &\quad c_t + k_t - (1-\delta)k_{t-1} - A_t k_{t-1}^{\alpha} h_t^{1-\alpha} = 0.
\end{aligned}$$

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2. Incorporating trend
3. Notes on Newton's method and deterministic simulation

Incorporating trend

- Let $Z_t \equiv A_t^{\frac{1}{1-\alpha}}$ and $\gamma \equiv Z_{t+1}/Z_t$. The production function becomes

$$y_t = k_{t-1}^{\alpha}(Z_t h_t)^{1-\alpha}.$$

  This is called the Harrod-neutral production function.

- In the steady state, y and k exponentially grows at the rate γ.

Detrending

- The equilibrium conditions are

$$
\begin{aligned}
y_t &= k_{t-1}^{\alpha}(Z_t h_t)^{1-\alpha}, \\
(1-\alpha)\frac{y_t}{h_t} &= \frac{Bc_t}{1-h_t}, \\
1 &= \beta\frac{c_t}{c_{t+1}}\left(1+\alpha\frac{y_{t+1}}{k_t}-\delta\right), \\
c_t + k_t - (1-\delta)k_{t-1} - y_t &= 0.
\end{aligned}
$$

- y_t, c_t and k_{t-1} grows at γ, whereas the model variables have to be stationary.

Detrending, cont'd

- Define $\tilde{y}_t \equiv y_t/Z_t$, $\tilde{c}_t \equiv c_t/Z_t$ and $\tilde{k}_{t-1} \equiv k_{t-1}/Z_t$,

$$\begin{aligned}
\tilde{y}_t &= \tilde{k}_{t-1}^{\alpha} h_t^{1-\alpha}, \\
(1-\alpha)\frac{\tilde{y}_t}{h_t} &= \frac{B\tilde{c}_t}{1-h_t}, \\
1 &= (\beta/\gamma)\frac{\tilde{c}_t}{\tilde{c}_{t+1}}\left(1+\alpha\frac{\tilde{y}_{t+1}}{\tilde{k}_t}-\delta\right), \\
\tilde{c}_t + \gamma\tilde{k}_t &- (1-\delta)\tilde{k}_{t-1} - \tilde{y}_t = 0.
\end{aligned}$$

The steady state

- The steady state conditions are

$$\begin{aligned}
1 &= \tilde{\beta}\left(1 + \alpha\tilde{y}/\tilde{k} - \delta\right), \\
0 &= \tilde{y} - (\gamma - 1 + \delta)\tilde{k} - \tilde{c}, \\
(1-\alpha)(\tilde{y}/h) &= B\tilde{c}/(1-h), \\
\tilde{y} &= \tilde{k}^{\alpha}h^{1-\alpha},
\end{aligned}$$

  where $\tilde{\beta} = \beta/\gamma$.

- Dynare solves for the steady state values numerically with an educated initial guess. In this model, the steady state values are also analytically obtained.

The steady state

- Let $h = 1/3$, then we have

$$\tilde{y}/\tilde{k} = \alpha^{-1}(\tilde{\beta}^{-1} - 1 + \delta),$$
$$\tilde{c}/\tilde{k} = \tilde{y}/\tilde{k} - (\gamma - 1 + \delta),$$
$$\tilde{k} = \left(\frac{\alpha h^{1-\alpha}}{\tilde{\beta}^{-1} - 1 + \delta}\right)^{\frac{1}{1-\alpha}},$$
$$B = (1-\alpha)(\tilde{y}/\tilde{c})(1-h)/h.$$

  $\tilde{y} = (\tilde{y}/\tilde{k})\tilde{k}$ and $\tilde{c} = (\tilde{c}/\tilde{k})\tilde{k}$ are also obtained. Note that B is a normalization parameter.

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Newton-Raphson method

- We solve an equation $f(x) = 0$ for x.
- First-order Taylor expansion: $f(x) \approx f(x_0) + f'(x_0)(x - x_0) = 0$.
- The Newton-Raphson method updates x_k

$$x_{k+1} = x_k - f'(x_k)^{-1} f(x_k),$$

until $\|x_{k+1} - x_k\| \leq \epsilon$.

Newton-Raphson method, cont'd

- We solve a system of equations $f_i(\mathbf{x}) = 0$ for $i = 1, ..., n$ for $\mathbf{x} = [x_1, \cdots, x_n]'$.
- n dimensional First-order Taylor expansion:

$$f_i(\mathbf{x}) \approx f_i(\mathbf{x}_0) + \nabla f_i(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) = 0,$$

for $i = 1, ..., n$ where $\nabla f_i(\mathbf{x}_0) = [f_i'(x_1), ..., f_i'(x_n)]$.

Newton-Raphson method, cont'd

- Let $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), ..., f_n(\mathbf{x})]'$ and

$$\nabla \mathbf{f}(\mathbf{x}) = \begin{bmatrix} \nabla f_1(\mathbf{x}) \\ \nabla f_2(\mathbf{x}) \\ \vdots \\ \nabla f_n(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \cdots & & \vdots \\ \vdots & & & \\ \frac{\partial f_n(\mathbf{x})}{\partial x_1} & \cdots & & \frac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

  is an $(n \times n)$ matrix called *Jacobian*.
- The Newton-Raphson method updates $\mathbf{x}_k$

$$\mathbf{x}_{k+1} = \mathbf{x}_k - [\nabla \mathbf{f}(\mathbf{x}_k)]^{-1} \mathbf{f}(\mathbf{x}_k),$$

  until $\|\mathbf{x}_{k+1} - \mathbf{x}_k\| \leq \epsilon$.

Deterministic simulations

- Dynare can be used for deterministic simulations with the assumption of perfect foresight.
- The numerical problem consists of solving a nonlinear system of simultaneous equations in n endogenous variables in T periods.
- To solve the system of nT equations, Dynare uses a Newton-type method, which is based on the Fair-Taylor (1983) algorithm.
  - My explanation is based on Hollinger (1996).

A nonlinear system

- A nonlinear dynamic system is of a general form

$$f_{i,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = 0,$$

for $i = 1, ..., n$ where $\mathbf{y}_t = [y_{1,t}, \cdots, y_{n,t}]'$.

- For example, the basic RCK model is expressed as

$$\begin{aligned} f_{1,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) &= \beta \frac{c_t}{c_{t+1}} \left(1 + \alpha A_{t+1} k_t^{\alpha-1} - \delta\right) - 1 = 0, \\ f_{2,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) &= A_t k_{t-1}^{\alpha} + (1-\delta) k_{t-1} - k_t - c_t = 0, \end{aligned}$$

where $\mathbf{y}_t = [k_t, c_t]'$.

A nonlinear system, cont'd

- The system is expressed compactly as a $(n \times 1)$ vector of equations:

$$\mathbf{f}_t(\mathbf{z}_t) = \mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \begin{bmatrix} f_{1,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \vdots \\ f_{n,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \end{bmatrix} = \mathbf{0},$$

where $\mathbf{f}_t = [f_{1,t}, \cdots, f_{n,t}]'$ and $\mathbf{z}_t = [\mathbf{y}'_{t-1}, \mathbf{y}'_t, \mathbf{y}'_{t+1}]'$.

Solving the nonlinear system

- We solve $\mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \mathbf{0}$ *simultaneously* for $t = 1, ..., T$.

- $\mathbf{y}_{t-1} = \mathbf{y}_0$ is predetermined at $t = 1$. Also, $\mathbf{y}_{t+1} = \mathbf{y}_{T+1}$ is predetermined at $t = T$ (the boundary conditions).

Solving the nonlinear system, cont'd

- We stack the system for T periods as

$$F(Z) = \begin{bmatrix} \mathbf{f}_1(\mathbf{z}_1) \\ \vdots \\ \mathbf{f}_t(\mathbf{z}_t) \\ \vdots \\ \mathbf{f}_T(\mathbf{z}_T) \end{bmatrix} = \mathbf{0},$$

  where $F = [\mathbf{f}_1', \cdots, \mathbf{f}_t', \cdots, \mathbf{f}_n']'$ and $Z = [\mathbf{z}_1', \cdots, \mathbf{z}_t', \cdots, \mathbf{z}_n']'$. This is a $(nT \times 1)$ vector of equations.

- We solve $F(Z)$ by the Newton-Raphson method for nT variables

$$Z = [\mathbf{y}_0', \mathbf{y}_1', \cdots, \mathbf{y}_T', \mathbf{y}_{T+1}'],$$

  given $\mathbf{y}_0'$ and $\mathbf{y}_T'$.

Assignment #2

- Let $\beta = .96$, $\delta = .1$, $\alpha = .36$. Consider the basic RCK model without labor.
  1. Let $k_0 = 0.1$. Compute the initial dynamics converging to the steady state.
  2. Assume a distortionary tax on capital income after depreciation,
  $$c_t + k_t - (1-\delta)k_{t-1} \leq r_t k_{t-1} - \tau(r_t - \delta)k_{t-1} + \pi_t.$$
  Let $\tau = .2$.
     1. Solve for the equilibrium conditions in the competitive economy.
     2. Solve for the steady state.
     3. Compute the initial dynamics with $k_0 = 0.1$.
     4. Compare the results with the ones in the model without distortionary taxes, which you computed in (1).