Real Business Cycle model

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Stochastic neoclassical growth model

- A stochastic version of the RCK (neoclassical growth) model is originally developed by Brock and Mirman (1972).
- The model is adapted to the analysis of economic fluctuations by Kydland and Prescott (1982).
- There are only two differences from the basic RCK model: Stochastic TFP process and Euler equation.

Recap: Stochastic TFP

- We assume that the TFP follows a stochastic process:

$$\log A_{t+1} = (1-\rho)\log \bar{A} + \rho \log A_t + \varepsilon_{t+1},$$

where $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$.

- Note that

$$A_{t+1} = \bar{A}^{1-\rho} A_t^{\rho} e^{\varepsilon_{t+1}},$$

holds.

- The stochastic process of A_t is estimated by OLS using a time series of TFP. A typical quarterly estimate for the U.S. economy is $\rho = 0.95$ and $\sigma_\varepsilon = 0.008$.

Stochastic Euler equation

- The Euler equation becomes

$$c_t^{-1} = \beta E_t \left\{ c_{t+1}^{-1} \left(1 + r_{t+1} - \delta \right) \right\}.$$

  Recall: The LHS is the benefit from one unit of consumption today. The RHS is the benefit from one unit of saving today and consuming the return on saving tomorrow.

- We, modelers, are uncertain about the return on saving and hence consumption tomorrow. How do we estimate them?

- We assume rational expectations: Agents in the model have a perfect knowledge of the model economy, and correctly estimate the expected values of future variables.

Equilibrium

- Now we have the following equilibrium conditions:

$$
\begin{gathered}
1 = \beta E_t \left\{ \frac{c_t}{c_{t+1}} (1 + r_{t+1} - \delta) \right\}, \\
r_t = \theta y_t / k_{t-1}, \\
y_t = A_t k_{t-1}^{\theta}, \\
c_t + k_t - (1-\delta) k_{t-1} = y_t, \\
\log A_t = \rho \log A_{t-1} + \varepsilon_t.
\end{gathered}
$$

- There are five unknowns and five equations, so we can solve the model.

The steady state

- Recall: In the steady state,

$$1 = \beta \left(1 + \theta A k^{\theta-1} - \delta\right),$$
$$0 = A k^{\theta} - \delta k - c,$$

hold.

- The steady-state conditions can be solved for

$$\bar{k} = \left(\frac{\theta \beta A}{1 - \beta(1-\delta)}\right)^{\frac{1}{1-\theta}},$$
$$\bar{c} = A \bar{k}^{\theta} - \delta \bar{k}.$$

Log-linearization

- To solve the rational expectation equilibrium of the model, we approximate the model around the steady state.
- Use the formula of approximation

$$x_t \equiv x \exp \hat{x}_t \approx x(1 + \hat{x}_t),$$

where x is the steady state of x_t and $\hat{x}_t$ is percent deviation from the steady state.

Log-linearization, cont'd

- Other useful formulae:

$$
\begin{aligned}
x_t y_t &= xy \exp(\hat{x}_t + \hat{y}_t) \\
&\approx xy(1 + \hat{x}_t + \hat{y}_t), \\
x_t / y_t &= (x/y) \exp(\hat{x}_t - \hat{y}_t) \\
&\approx (x/y)(1 + \hat{x}_t - \hat{y}_t), \\
y_t^a &= y^a \exp(a\hat{y}_t) \\
&\approx y^a (1 + a\hat{y}_t).
\end{aligned}
$$

$$\hat{x}_t^n = 0 \text{ for } n > 1, \qquad \hat{x}_t \hat{y}_t = 0,$$

$$
\begin{aligned}
E_t y_{t+1}^a &= E_t y^a \exp(a\hat{y}_{t+1}). \\
&\approx y^a (1 + a E_t \hat{y}_{t+1}).
\end{aligned}
$$

Log-linearization: Production function

- Production function:

$$y_t = A_t k_{t-1}^{\theta}.$$

It can be written as

$$y \exp(\hat{y}_t) = Ak \exp(\hat{a}_t + \theta \hat{k}_{t-1}).$$

In the steady state, $y = Ak$ holds. Then,

$$\hat{y}_t = \hat{a}_t + \theta \hat{k}_{t-1}.$$

[Note: This is not approximation.]

Log-linearization: Euler equation

- Euler equation:

$$1 = \beta E_t \left\{ \frac{c_t}{c_{t+1}} R_{t+1} \right\},$$

where $R_{t+1} \equiv 1 + r_{t+1} - \delta$. It can be written as

$$1 = \beta R E_t \left\{ \exp(\hat{c}_t - \hat{c}_{t+1} + \hat{R}_{t+1}) \right\}.$$

Use the formula of approximation

$$1 = \beta R E_t \left\{ 1 + \hat{c}_t - \hat{c}_{t+1} + \hat{R}_{t+1} \right\}.$$

In the steady state, $1 = \beta R$ holds. Then,

$$\hat{c}_t - E_t \hat{c}_{t+1} + E_t \hat{R}_{t+1} = 0.$$

Log-linearization: Gross interest rate

- Gross interest rate (rate of return from capital):

$$R_{t+1} = 1 + \theta A_{t+1} k_t^{\theta-1} - \delta.$$

It can be written as

$$R \exp(\hat{R}_{t+1}) = 1 + \theta(y/k) \exp(\hat{a}_{t+1} + (\theta-1)\hat{k}_t) - \delta.$$

Use the formula of approximation

$$R(1 + \hat{R}_{t+1}) = 1 + \theta(y/k)(1 + \hat{a}_{t+1} + (\theta-1)\hat{k}_t) - \delta.$$

In the steady state, $R = \beta^{-1} = 1 + \theta(y/k) - \delta$ holds. Then we have

$$R\hat{R}_{t+1} = \theta(y/k)(\hat{a}_{t+1} + (\theta-1)\hat{k}_t),$$

or,

$$\hat{R}_{t+1} = \beta\theta(y/k)(\hat{a}_{t+1} + (\theta-1)\hat{k}_t).$$

Log-linearization: Resource constraint

- Resource constraint:
$$c_t + k_t - (1-\delta)k_{t-1} = y_t.$$
It can be written as
$$c\exp(\hat{c}_t) + k\exp(\hat{k}_t) - (1-\delta)k\exp(\hat{k}_{t-1}) = y\exp(\hat{y}_t).$$
Use the formula of approximation
$$c(1+\hat{c}_t) + k(1+\hat{k}_t) - (1-\delta)k(1+\hat{k}_{t-1}) = y(1+\hat{y}_t).$$
In the steady state, $c + k - (1-\delta)k = y$ holds. Then we have
$$c\hat{c}_t + k\hat{k}_t - (1-\delta)k\hat{k}_{t-1} = y\hat{y}_t.$$

Log-linearization: Summary

- After all, the log-linearized equilibrium conditions are:

$$
\begin{aligned}
&\hat{c}_t - E_t\hat{c}_{t+1} + E_t\hat{R}_{t+1} = 0,\\
&\hat{R}_{t+1} = \beta\theta(y/k)(\hat{a}_{t+1} + (\theta-1)\hat{k}_t),\\
&\hat{y}_t = \hat{a}_t + \theta\hat{k}_{t-1},\\
&c\hat{c}_t + k\hat{k}_t - (1-\delta)k\hat{k}_{t-1} = y\hat{y}_t.
\end{aligned}
$$

- Or,

$$
\begin{aligned}
&\hat{c}_t - E_t\hat{c}_{t+1} + \beta\theta(y/k)(E_t\hat{a}_{t+1} + (\theta-1)\hat{k}_t) = 0,\\
&(c/k)\hat{c}_t + \hat{k}_t = (\theta(y/k) + 1 - \delta)\hat{k}_{t-1} + (y/k)\hat{a}_t.
\end{aligned}
$$

Summary

- We have

$$
\begin{aligned}
&\hat{c}_t - E_t\hat{c}_{t+1} + c_{ck}\hat{k}_t + c_{ca}E_t\hat{a}_{t+1} = 0, \\
&c_{kc}\hat{c}_t + \hat{k}_t = c_{kk}\hat{k}_{t-1} + c_{ka}\hat{a}_t.
\end{aligned}
$$

where

$$
\begin{aligned}
c_{ck} &= -\beta\theta(1-\theta)(y/k), & c_{kc} &= c/k, \\
c_{ca} &= \beta\theta(y/k), & c_{kk} &= 1/\beta, \\
& & c_{ka} &= y/k.
\end{aligned}
$$

A brute force method

- We use a brute force method, which is a variant of undetermined coefficient methods.
- Conjecture the solution

$$\begin{aligned}\hat{c}_t &= \gamma_{ck}\hat{k}_{t-1} + \gamma_{ca}\hat{a}_t, \\ \hat{k}_t &= \gamma_{kk}\hat{k}_{t-1} + \gamma_{ka}\hat{a}_t.\end{aligned}$$

- Substitute them into the equilibrium conditions,

$$\gamma_{ck}\hat{k}_{t-1} + \gamma_{ca}\hat{a}_t - (\gamma_{ck}\hat{k}_t + \gamma_{ca}E_t\hat{a}_{t+1}) + c_{ck}\hat{k}_t + c_{ca}E_t\hat{a}_{t+1} = 0,$$
$$c_{kc}(\gamma_{ck}\hat{k}_{t-1} + \gamma_{ca}\hat{a}_t) + \hat{k}_t = c_{kk}\hat{k}_{t-1} + c_{ka}\hat{a}_t.$$

[substitute $\hat{k}_t$ in the first equation out]

A brute force method, cont'd

- Terms related to $\hat{k}_{t-1}$ and $\hat{a}_t$ each are put together

$$
\begin{aligned}
&[\gamma_{ck} + (c_{ck} - \gamma_{ck})\gamma_{kk}]\hat{k}_{t-1} \\
&+[\gamma_{ca} + (c_{ck} - \gamma_{ck})\gamma_{ka} + (c_{ca} - \gamma_{ca})\rho]\hat{a}_t = 0, \\
&[c_{kc}\gamma_{ck} + \gamma_{kk}]\hat{k}_{t-1} + [c_{kc}\gamma_{ca} + \gamma_{ka}]\hat{a}_t = c_{kk}\hat{k}_{t-1} + c_{ka}\hat{a}_t.
\end{aligned}
$$

- These equations must hold for any $\hat{k}_{t-1}$ and $\hat{a}_t$, which implies

$$
\begin{aligned}
&\gamma_{ck} + c_{ck}\gamma_{kk} - \gamma_{ck}\gamma_{kk} = 0, \\
&c_{kc}\gamma_{ck} + \gamma_{kk} = c_{kk}, \\
&\gamma_{ca} + (c_{ck} - \gamma_{ck})\gamma_{ka} + (c_{ca} - \gamma_{ca})\rho = 0, \\
&c_{kc}\gamma_{ca} + \gamma_{ka} = c_{ka}.
\end{aligned}
$$

There are four equations and four unknowns $(\gamma_{ck}, \gamma_{ca}, \gamma_{kk}, \gamma_{ka})$.

Solving for the dynamics

- The first two equations become

$$
\begin{aligned}
& (1-\gamma_{kk})(-\gamma_{kk}+c_{kk})/c_{kc}+c_{ck}\gamma_{kk}=0, \\
\Leftrightarrow \quad & -(1-\gamma_{kk})\gamma_{kk}+(1-\gamma_{kk})c_{kk}+c_{kc}c_{ck}\gamma_{kk}=0, \\
\therefore \quad & \gamma_{kk}^2-(1+c_{kk}-c_{kc}c_{ck})\gamma_{kk}+c_{kk}=0.
\end{aligned}
$$

- Applying the quadratic formula,

$$
\gamma_{kk}=\frac{1+c_{kk}-c_{kc}c_{ck}\pm\sqrt{(1+c_{kk}-c_{kc}c_{ck})^2-4c_{kk}}}{2}
$$

This equation has two roots. We pick the stable one $\gamma_{kk}\in(-1,1)$.

Solving for the dynamics, cont'd

- Once we solve for γ_{kk}, we also obtain

$$\begin{aligned}
\gamma_{ck} &= -c_{ck}\gamma_{kk}/(1-\gamma_{kk}), \\
\gamma_{ca} &= \frac{-\rho c_{ca} - c_{ck}c_{ka} + c_{ka}\gamma_{ck}}{1-\rho - c_{ck}c_{kc} + c_{kc}\gamma_{ck}}, \\
\gamma_{ka} &= c_{ka} - c_{kc}\gamma_{ca},
\end{aligned}$$

  for the conjectured solution $\hat{c}_t = \gamma_{ck}\hat{k}_{t-1} + \gamma_{ca}\hat{a}_t$ and $\hat{k}_t = \gamma_{kk}\hat{k}_{t-1} + \gamma_{ka}\hat{a}_t$.

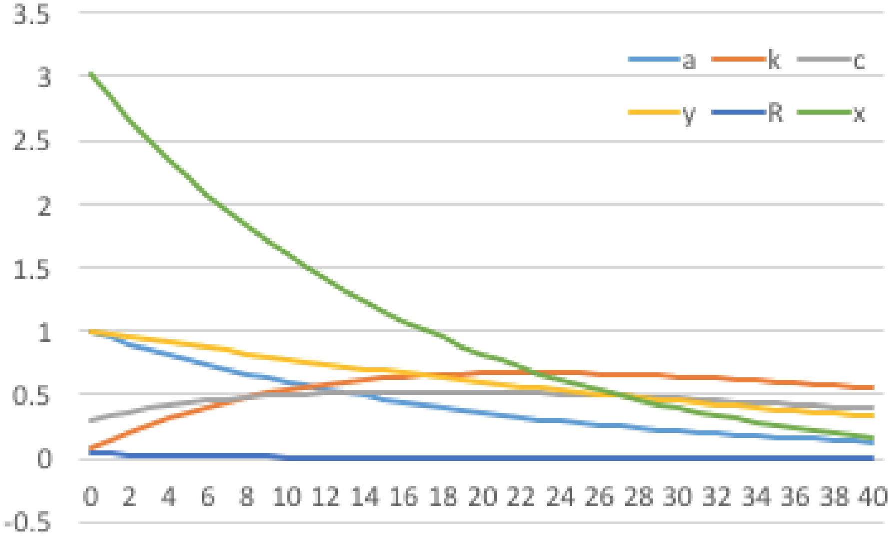
Calibration

- We set $\beta = 0.99$, $\theta = 0.36$ and $\delta = 0.025$. For the stochastic process, we use $\rho = 0.95$ and $\sigma = 0.008$.
- With these parameter values, we obtain

$$\begin{aligned}
\hat{k}_t &= 0.9653\hat{k}_{t-1} + 0.0754\hat{a}_t, \\
\hat{c}_t &= 0.6182\hat{k}_{t-1} + 0.3052\hat{a}_t, \\
\hat{a}_t &= 0.95\hat{a}_{t-1} + \varepsilon_t.
\end{aligned}$$

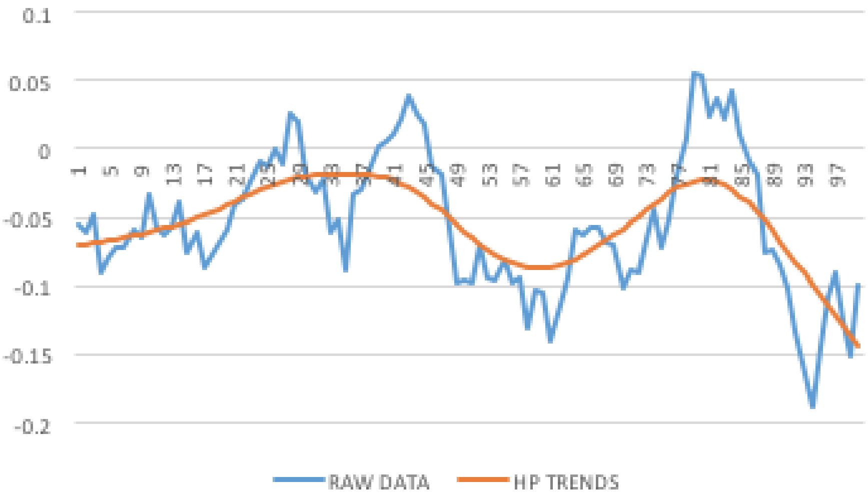
Impulse responses

- We set $\varepsilon_1 = 1.0$ and $\varepsilon_t = 0$ for $t \geq 2$.



Stochastic simulation

- We set $\varepsilon_t \sim N(0, \sigma^2)$ for all t. Detrending is done by HP filter.



Two methods to solve linearized models

- Blanchard and Khan's method
- Uhlig's method of undetermined coefficients

A general form

- A linear model can be written as

$$\underset{((n+m)\times(n+m))}{B}\begin{bmatrix}\underset{(n\times 1)}{x_{t+1}}\\ \underset{(m\times 1)}{E_t y_{t+1}}\end{bmatrix} = \underset{((n+m)\times(n+m))}{A}\begin{bmatrix}\underset{(n\times 1)}{x_t}\\ \underset{(m\times 1)}{y_t}\end{bmatrix} + \underset{((n+m)\times k)}{G}\underset{(k\times 1)}{\varepsilon_t},$$

where

  - x_t: vector of predetermined variables
  - $E_t y_{t+1}$: vector of expectations for non-predetermined (jump) variables
  - ε_t: vector of stochastic shocks

- Predetermined variables do not depend on shocks in $t+1$, while jump variables do. That is why we have an expectational operator on y_{t+1}.

Blanchard and Khan

- If B is invertible,

$$\begin{bmatrix} x_{t+1} \\ E_t y_{t+1} \end{bmatrix} = B^{-1}A \begin{bmatrix} x_t \\ y_t \end{bmatrix} + B^{-1}G\varepsilon_t,$$

- $Z = B^{-1}A$ can be decomposed into $Z = M\Lambda M^{-1}$, where diagonal elements of Λ are the eigenvalues of Z and M is a matrix of the right eigenvectors (eigenvalue decomposition).

- In Matlab, the command `[M Lambda] = eig(Z)` computes the eigenvalues and eigenvectors.

Notes on Blanchard and Khan

- Reorder the eigenvalues from smallest to largest as $\bar{\Lambda}$ and the corresponding $\bar{M}$.
- The Blanchard and Khan condition: (# of eigenvalues that have an absolute value greater than one) = (# expectational variables), m.
- When the BK condition is satisfied, one can impose conditions so that there exists a stable solution.

Deterministic case

- Consider a deterministic case with $E_t y_{t+1} = y_{t+1}$,

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \bar{M}\bar{\Lambda}\bar{M}^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

$$\bar{M}^{-1} \begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \bar{\Lambda}\bar{M}^{-1} \begin{bmatrix} x_t \\ y_t \end{bmatrix},$$

and partition matrices as

$$\bar{M}^{-1} = \begin{bmatrix} \underset{(n\times n)}{\hat{M}_{11}} & \underset{(n\times m)}{\hat{M}_{12}} \\ \underset{(m\times n)}{\hat{M}_{21}} & \underset{(m\times m)}{\hat{M}_{22}} \end{bmatrix}, \qquad \bar{\Lambda} = \begin{bmatrix} \underset{(n\times n)}{\bar{\Lambda}_{11}} & \underset{(n\times m)}{0_{12}} \\ \underset{(m\times n)}{0_{21}} & \underset{(m\times m)}{\bar{\Lambda}_{22}} \end{bmatrix}.$$

Note that the BK condition is satisfied, i.e., $m =$(# of the explosive roots).

Deterministic case, cont'd

- Using this partition, there are two matrix equations

$$
\begin{aligned}
\underset{(n\times n)}{\hat{M}_{11}}\underset{(n\times 1)}{x_{t+1}} + \underset{(n\times m)}{\hat{M}_{12}}\underset{(m\times 1)}{y_{t+1}} &= \underset{(n\times n)}{\bar{\Lambda}_{11}}\left[\underset{(n\times n)}{\hat{M}_{11}}\underset{(n\times 1)}{x_t} + \underset{(n\times m)}{\hat{M}_{12}}\underset{(m\times 1)}{y_t}\right], \\
\underset{(m\times n)}{\hat{M}_{21}}\underset{(n\times 1)}{x_{t+1}} + \underset{(m\times m)}{\hat{M}_{22}}\underset{(m\times 1)}{y_{t+1}} &= \underset{(m\times m)}{\bar{\Lambda}_{22}}\left[\underset{(m\times n)}{\hat{M}_{21}}\underset{(n\times 1)}{x_t} + \underset{(m\times m)}{\hat{M}_{22}}\underset{(m\times 1)}{y_t}\right].
\end{aligned}
$$

- Given that the elements of $\bar{\Lambda}_{22}$ are greater than one, iff $\hat{M}_{21}x_t + \hat{M}_{22}y_t = 0$, the solution is stable. Then we have

$$
\begin{aligned}
y_t &= -\left(\hat{M}_{22}\right)^{-1}\hat{M}_{21}x_t, \\
x_{t+1} &= \left[\hat{M}_{11} - \hat{M}_{12}\left(\hat{M}_{22}\right)^{-1}\hat{M}_{21}\right]^{-1}\bar{\Lambda}_{11}\left[\hat{M}_{11} - \hat{M}_{12}\left(\hat{M}_{22}\right)^{-1}\hat{M}_{21}\right]x_t.
\end{aligned}
$$

Stochastic case

- Consider a stochastic case with $E_t y_{t+1}$:

$$\left[\begin{array}{c} x_{t+1} \\ E_t y_{t+1} \end{array}\right] = \bar{M}\bar{\Lambda}\bar{M}^{-1}\left[\begin{array}{c} x_t \\ y_t \end{array}\right] + B^{-1}G\varepsilon_t,$$

$$\Leftrightarrow \quad \bar{M}^{-1}\left[\begin{array}{c} x_{t+1} \\ E_t y_{t+1} \end{array}\right] = \bar{\Lambda}\bar{M}^{-1}\left[\begin{array}{c} x_t \\ y_t \end{array}\right] + \bar{M}^{-1}B^{-1}G\varepsilon_t,$$

or,

$$\left[\begin{array}{cc} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{array}\right]\left[\begin{array}{c} x_{t+1} \\ E_t y_{t+1} \end{array}\right] = \left[\begin{array}{cc} \bar{\Lambda}_{11} & 0 \\ 0 & \bar{\Lambda}_{22} \end{array}\right]\left[\begin{array}{cc} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{array}\right]\left[\begin{array}{c} x_t \\ y_t \end{array}\right] + \left[\begin{array}{c} \hat{G}_1 \\ \hat{G}_2 \end{array}\right]\varepsilon_t,$$

where $\left[\begin{array}{c} \underset{(n\times k)}{\hat{G}_1} \\ \underset{(m\times k)}{\hat{G}_2} \end{array}\right] = \bar{M}^{-1}B^{-1}G.$

Stochastic case, cont'd

- The lower partition is written as

$$\underset{(m\times n)}{\hat{M}_{21}}\underset{(n\times 1)}{x_{t+1}} + \underset{(m\times m)}{\hat{M}_{22}}\underset{(m\times 1)}{y_{t+1}}$$

$$= \underset{(m\times m)}{\bar{\Lambda}_{22}}\left[\underset{(m\times n)}{\hat{M}_{21}}\underset{(n\times 1)}{x_t} + \underset{(m\times m)}{\hat{M}_{22}}\underset{(m\times 1)}{y_t}\right] + \underset{(m\times k)}{\hat{G}_2}\underset{(k\times 1)}{\varepsilon_t},$$

or, letting $\lambda_t = \hat{M}_{21}x_t + \hat{M}_{22}y_t$,

$$E_t\lambda_{t+1} = \bar{\Lambda}_{22}\lambda_t + \hat{G}_2\varepsilon_t.$$

Stochastic case, cont'd

- It can be solved forward:

$$\begin{aligned}\lambda_t =& \bar{\Lambda}_{22}^{-1} E_t \lambda_{t+1} - \bar{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t, \\ =& \bar{\Lambda}_{22}^{-1} E_t \left(\bar{\Lambda}_{22}^{-1} E_{t+1} \lambda_{t+2} - \bar{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_{t+1} \right) - \bar{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t, \\ =& \bar{\Lambda}_{22}^{-1} \bar{\Lambda}_{22}^{-1} E_t \lambda_{t+2} - \bar{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t, \\ =& \ldots \\ =& - \bar{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t.\end{aligned}$$

- Then we have

$$y_t = -\hat{M}_{22}^{-1} \hat{M}_{21} x_t - \hat{M}_{22}^{-1} \bar{\Lambda}_{22}^{-1} \hat{G}_2 \varepsilon_t.$$

Stochastic case, cont'd

- The upper partition is written as

$$
\begin{aligned}
& \underset{(n\times n)}{\hat{M}_{11}} \underset{(n\times 1)}{x_{t+1}} + \underset{(n\times m)}{\hat{M}_{12}} \underset{(m\times 1)}{y_{t+1}} \\
= & \underset{(n\times n)}{\bar{\Lambda}_{11}} \left[\underset{(n\times n)}{\hat{M}_{11}} \underset{(n\times 1)}{x_t} + \underset{(n\times m)}{\hat{M}_{12}} \underset{(m\times 1)}{y_t} \right] + \underset{(n\times k)}{\hat{G}_1} \underset{(k\times 1)}{\varepsilon_t},
\end{aligned}
$$

Substitute y_t and $E_t y_{t+1} = -\hat{M}_{22}^{-1}\hat{M}_{21}x_{t+1}$, we have

$$
\begin{aligned}
x_{t+1} = & \left[\hat{M}_{11} - \hat{M}_{12}\left(\hat{M}_{22}\right)^{-1}\hat{M}_{21}\right]^{-1}\bar{\Lambda}_{11}\left[\hat{M}_{11} - \hat{M}_{12}\left(\hat{M}_{22}\right)^{-1}\hat{M}_{21}\right]x_t \\
& - \left[\hat{M}_{11} - \hat{M}_{12}\left(\hat{M}_{22}\right)^{-1}\hat{M}_{21}\right]^{-1}\left[\bar{\Lambda}_{11}\hat{M}_{12}\left(\hat{M}_{22}\right)^{-1}\bar{\Lambda}_{22}^{-1}\hat{G}_2 - \hat{G}_1\right]\varepsilon_t.
\end{aligned}
$$

Example

- Recall the stochastic neoclassical growth model

$$\begin{aligned}
&\hat{c}_t - E_t\hat{c}_{t+1} + c_{ck}\hat{k}_t + c_{ca}\rho\hat{a}_t = 0,\\
&c_{kc}\hat{c}_t + \hat{k}_t = c_{kk}\hat{k}_{t-1} + c_{ka}\hat{a}_t,\\
&\hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t.
\end{aligned}$$

- Consider the following matrix difference equation

$$\begin{bmatrix} 1 & 0 & 0 \\ -c_{ca}\rho & -c_{ck} & 1 \\ -c_{ka} & 1 & 0 \end{bmatrix}\begin{bmatrix} a_t \\ k_t \\ E_t c_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 1 \\ 0 & c_{kk} & -c_{kc} \end{bmatrix}\begin{bmatrix} a_{t-1} \\ k_{t-1} \\ c_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\varepsilon_t.$$

- It can be solved for

$$\begin{aligned}
\hat{k}_t &= 0.9653\hat{k}_{t-1} + 0.0716\hat{a}_{t-1} + 0.0754\varepsilon_t,\\
\hat{c}_t &= 0.6182\hat{k}_{t-1} + 0.2900\hat{a}_{t-1} + 0.3052\varepsilon_t.
\end{aligned}$$

Two methods to solve linearized models

- Blanchard and Khan's method
- Uhlig's method of undetermined coefficients

Undetermined coefficient methods

- The model is written as

$$
\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t, \\
0 &= E_t \left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t \right],
\end{aligned}
$$

  and stochastic process $z_{t+1} = Nz_t + \varepsilon_{t+1}$.

- Conjecture the solution

$$
\begin{aligned}
x_t &= Px_{t-1} + Qz_t, \\
y_t &= Rx_{t-1} + Sz_t.
\end{aligned}
$$

- The problem is to find the values for the matrices P, Q, R and S.

Undetermined coefficient methods, cont'd

- Substitute the conjectured solution,

$$
\begin{aligned}
0 &= [AP + B + CR]\, x_{t-1} + [AQ + CS + D]\, z_t, \\
0 &= [FPP + GP + H + JRP + KR]\, x_{t-1} \\
&\quad + [FPQ + FQN + GQ + JRQ + JSN + KS + LN + M]\, z_t,
\end{aligned}
$$

Both equation must equal zero for any x_{t-1} and z_t. Then we have four equations to find the four matrices:

$$
\begin{aligned}
0 &= AP + B + CR, \\
0 &= AQ + CS + D, \\
0 &= FP^2 + GP + H + JRP + KR, \\
0 &= FPQ + FQN + GQ + JRQ + JSN + KS + LN + M.
\end{aligned}
$$

Undetermined coefficient methods, cont'd

- The solution is given by

$$R = -C^{-1}\left[AP + B\right], \quad S = -C^{-1}\left[AQ + D\right],$$

P is the solution of the matrix quadratic equation

$$0 = \left[F - JC^{-1}A\right]P^2 - \left[JC^{-1}B - G + KC^{-1}A\right]P - KC^{-1}B + H,$$

and Q is constructed from vec(Q), where

$$\begin{aligned} \text{vec}(Q) \quad &= \quad \left(N' \otimes (F - JC^{-1}A) + I_k \otimes (FP + G + JR - KC^{-1}A)\right)^{-1} \\ &\quad \times \text{vec}\left((JC^{-1}D - L)N + KC^{-1}D - M\right), \end{aligned}$$

where $\otimes$ is a kronecker product. [See the text for derivation.]

Example

- Again, recall the stochastic neoclassical growth model

$$
\begin{aligned}
&\hat{c}_t - E_t\hat{c}_{t+1} + c_{ck}\hat{k}_t + c_{ca}\rho\hat{a}_t = 0,\\
&c_{kc}\hat{c}_t + \hat{k}_t = c_{kk}\hat{k}_{t-1} + c_{ka}\hat{a}_t,\\
&\hat{a}_t = \rho\hat{a}_{t-1} + \varepsilon_t.
\end{aligned}
$$

- The model is written as

$$
\begin{aligned}
0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t,\\
0 &= E_t\left[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t\right],
\end{aligned}
$$

where $A = 1$, $B = -c_{kk}$, $C = c_{kc}$, $D = -c_{ka}$, $F = 0$, $G = c_{ck}$, $H = 0$, $J = -1$, $K = 1$, $L = c_{ca}$, $M = 0$ and $N = \rho$.

Example, cont'd

- The solution is

$$\begin{aligned}\hat{k}_t &= 0.9653\hat{k}_{t-1} + 0.0754\hat{a}_t,\\ \hat{c}_t &= 0.6182\hat{k}_{t-1} + 0.3052\hat{a}_t.\end{aligned}$$

Exactly the same as in the other methods.

Hansen (1985) "Indivisible labor and the business cycle"

- The original studies of the real business cycle theory (e.g., Kydland and Prescott, 1982) fail to account for some important labor market phenomena.
- In the existing papers,
  - There is no unemployment (only the "intensive margin")
  - Fluctuations in hours worked are small relative to productivity fluctuations
- In micro studies using panel data, low intertemporal substitution of leisure is detected.

Business cycle statistics

- Output is more volatile than consumption and less volatile than investment.
- Correlation between output, consumption and investment is high.
- Hours are as much volatile as output is.
- Hours and labor productivity are less correlated with output.

Table 1

Standard deviations in percent (a) and correlations with output (b) for U.S. and artificial economies.

	Quarterly U.S. time series[a] (55,3–84,1)		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

Fluctuations in hours worked

- Fluctuations in total hours worked is largely explained by the "extensive margin", not the intensive margin

$$
\begin{aligned}
\text{var}(\log H_t) &= \text{var}(\log h_t) + \text{var}(\log N_t) + 2\text{cov}(\log h_t, \log N_t), \\
&= 20\% + 55\% + 25\%.
\end{aligned}
$$

- Most people either work full time or not at all.

- This paper: Large fluctuations of total hours worked are explained by the number of employed and low elasticity of substitution by individuals.

Hours worked (from Shimer, 2010)

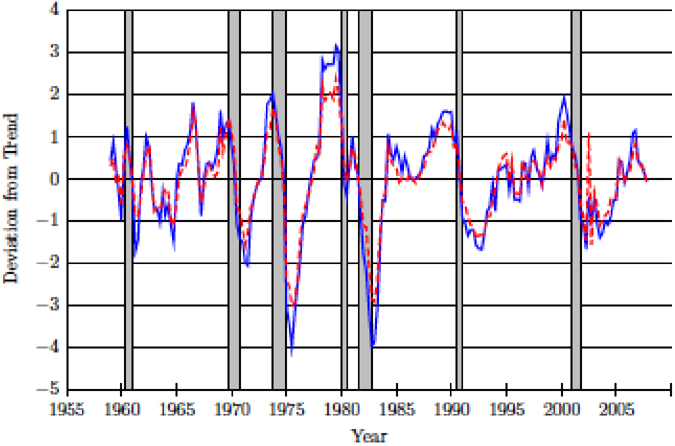


Figure 1.5: Deviation of per capita hours and the e-pop ratio from log trend, HP filter with parameter 1600. The solid blue line shows the deviation of hours from log trend and the dashed red line shows the deviation of employment. The gray bands show NBER recession dates.

Correlation between per capita hours and the e-pop ratio is 0.97.

- The social planner has the problem of:

$$\sum_{t=0}^{\infty} \beta^t \log c_t + A\alpha_t \log(1-h_0)$$

subject to

$$k_t = (1-\delta)k_{t-1} + i_t,$$
$$y_t = \lambda_t k_{t-1}^{\theta} h_t^{1-\theta} \geq c_t + i_t,$$
$$h_t = \alpha_t h_0$$

where h_t is total hours worked and h_0 is hours worked by individuals. α_t is the probability of being employed.

- The technology shock follows an AR(1) process:

$$\lambda_t = (1-\gamma) + \gamma\lambda_{t-1} + \varepsilon_t, \qquad \varepsilon \sim N(0, \sigma_\varepsilon^2).$$

Linear disutility of labor

- The problem can be rewritten as

$$\sum_{t=0}^{\infty} \beta^t \log c_t + B(1 - h_t)$$

subject to

$$k_t = (1-\delta)k_{t-1} + i_t,$$

$$y_t = \lambda_t k_{t-1}^{\theta} h_t^{1-\theta} \geq c_t + i_t,$$

where $B = -A \log(1 - h_0)/h_0 > 0$.

Lagrangean

- We set up the Lagrangean as

$$
\begin{aligned}
L_0 \quad \equiv \quad & E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + B(1-h_t) \right. \\
& \left. -\phi_t \left(c_t + k_t - (1-\delta)k_{t-1} - \lambda_t k_{t-1}^{\theta} h_t^{1-\theta} \right) \right\}.
\end{aligned}
$$

ϕ_t is the Lagrange multiplier.

- Taking the derivatives of the Lagrangean and set them to zero,

$$
\begin{aligned}
\partial c_t : \quad & \phi_t = 1/c_t, \\
\partial h_t : \quad & \phi_t (1-\theta) \lambda_t k_{t-1}^{\theta} h_t^{-\theta} = B, \\
\partial k_t : \quad & \phi_t = \beta \phi_{t+1} \left(1 + \theta \lambda_{t+1} k_t^{\theta-1} h_t^{1-\theta} - \delta \right), \\
\partial \phi_t : \quad & c_t + k_t - (1-\delta) k_{t-1} - A_t k_{t-1}^{\theta} h_t^{1-\theta} = 0.
\end{aligned}
$$

Equilibrium

- The equilibrium conditions are

$$
\begin{aligned}
&1 = \beta E_t \left\{ \frac{c_t}{c_{t+1}} \left(1 + \theta \frac{y_{t+1}}{k_t} - \delta \right) \right\}, \\
&(1-\theta)\frac{y_t}{h_t} = Bc_t, \\
&y_t = \lambda_t k_{t-1}^{\theta} h_t^{1-\theta}, \\
&c_t + k_t - (1-\delta)k_{t-1} - y_t = 0, \\
&\lambda_t = (1-\gamma) + \gamma\lambda_{t-1} + \varepsilon_t,
\end{aligned}
$$

There are five unknowns and five equations, so we can solve the model.

The steady state

- The equilibrium conditions are

$$
\begin{aligned}
1 &= \beta\left(1+\theta\frac{y}{k}-\delta\right), \\
(1-\theta)\frac{y}{h} &= Bc, \\
y &= k^{\theta}h^{1-\theta}, \\
c+\delta k &= y.
\end{aligned}
$$

These equations can be solved for (k, h, y, c). Note h is not normalized in Hansen (1985).

Assignment #3

- Read Hansen (1985) and use the model with indivisible labor. Also use his calibration: $\theta = 0.36$, $\delta = 0.025$, $\beta = 0.99$, $A = 2$, $h_0 = 0.53$, $\gamma = 0.95$ and $\sigma_\varepsilon = 0.00712$.
  1. Derive the steady state values of (k, h, y, c) and the log-linearized equilibrium conditions.
  2. Replicate Table 1 in the paper (column of "the economy with indivisible labor").
  3. Compare the result with the one in the basic stochastic growth model without labor. How different are business cycle statistics?