

# Ramsey-Cass-Koopmans model

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# What is the Ramsey-Cass-Koopmans model?

- A.k.a. the neo-classical growth model, a model of long run economic growth.
- Developed by Frank Ramsey (1929) and later extended by David Cass (1969) and Tjalling Koopmans (1969).
- The workhorse model of modern macroeconomics.

# Why is the RCK model useful?

- More reasonable explanation on the household's decision on saving/consumption *à la* Fisher. [The Solow model has an ad-hoc constant saving rate.]
- The model is also a basis for real business cycle (RBC) models of short-run fluctuations, which will be later in the class.

# Recap: Solow model

- In Solow model, we have

$$\begin{aligned}y_t &= A_t k_{t-1}^\theta, \\s_t = i_t &= k_t - (1 - \delta)k_{t-1}, \\s_t &= \sigma y_t,\end{aligned}$$

where  $k_t$  is the capital stock at the *end* of date  $t$ .

- We will relax the assumption of the constant saving rate.
- [Show the circulation diagram]

- The representative firm can access to the Cobb-Douglas production function technology:

$$y_t = A_t k_{t-1}^\theta.$$

- As a result of profit maximization,

$$r_t = \theta y_t / k_{t-1},$$

and

$$\pi_t = y_t - r_t k_{t-1} = (1 - \theta) y_t.$$

- Household's saving is equal to investment on capital:

$$s_t = i_t = k_t - (1 - \delta)k_{t-1}.$$

- Household owns capital and firm, and decide how much to save and consume:

$$c_t + s_t = r_t k_{t-1} + \pi_t = y_t.$$

- In Solow model,  $s_t = \sigma y_t$  and hence  $c_t = (1 - \sigma)y_t$ . Here, decision on consumption and saving is endogenous.

# Euler equation

- Household choose consumption so as to satisfy the following **the Euler equation**:

$$MU_t = \beta MU_{t+1} (1 + r_{t+1} - \delta),$$

where  $\beta$  is discount factor and  $MU_t$  is the marginal utility from consumption in period  $t$ .

- The LHS is the benefit from one unit of consumption today. The RHS is the benefit from one unit of saving today and consuming the return on saving tomorrow.

- Marginal utility is the benefit from one unit of consumption.
- To link consumption to utility, we need a utility function of the household.  
For example,

$$U_t = \log c_t.$$

Then we have

$$MU_t = 1/c_t.$$



- Now we have the following equilibrium conditions:

$$\begin{aligned}y_t &= A_t k_{t-1}^\theta, \\c_t + k_t - (1 - \delta)k_{t-1} &= y_t, \\1 &= \beta \frac{c_t}{c_{t+1}} (1 + r_{t+1} - \delta), \\r_t &= \theta y_t / k_{t-1}.\end{aligned}$$

- There are four unknowns and four equations, so we can solve the model.

# The two key equations

- The equilibrium conditions are summarized as

$$\frac{c_{t+1}}{c_t} = \beta \left( 1 + \theta A_{t+1} k_t^{\theta-1} - \delta \right),$$
$$k_t - k_{t-1} = A_t k_{t-1}^{\theta} - \delta k_{t-1} - c_t.$$

# The steady state

- In the steady state,

$$\begin{aligned}1 &= \beta (1 + \alpha A k^{\alpha-1} - \delta), \\ 0 &= A k^{\alpha} - \delta k - c,\end{aligned}$$

hold.

- The steady-state conditions can be solved for

$$\begin{aligned}\bar{k} &= \left( \frac{\alpha \beta A}{1 - \beta(1 - \delta)} \right)^{\frac{1}{1-\alpha}}, \\ \bar{c} &= A(\bar{k})^{\alpha} - \delta \bar{k}.\end{aligned}$$

These equations are drawn on the  $(k, c)$  plane.

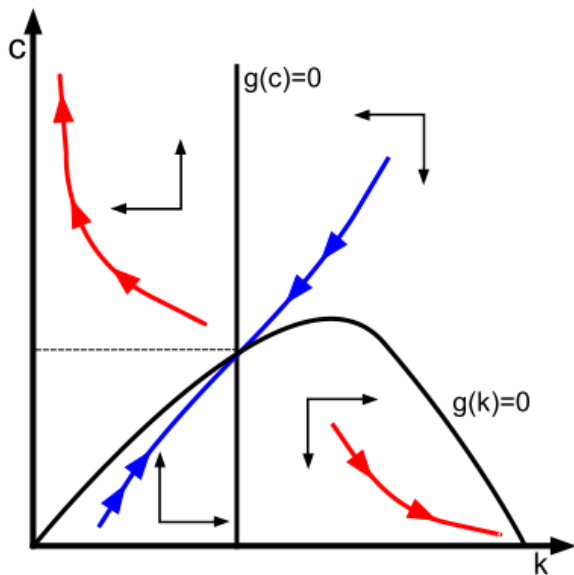
# The transition dynamics

- Let

$$g_{ct} \equiv c_t/c_{t-1} - 1 = \beta (1 + \alpha A_t k_{t-1}^{\alpha-1} - \delta) - 1,$$
$$\tilde{g}_{kt} \equiv k_t - k_{t-1} = A k_{t-1}^{\alpha} - \delta k_{t-1} - c_t.$$

- If  $g_{ct} > 0$ ,  $c_t > c_{t-1}$  holds;  $k_{t-1} < \bar{k}$  implies  $g_c > 0$ .
- If  $\tilde{g}_{kt} > 0$ ,  $k_t > k_{t-1}$  holds;  $c_t < A k_{t-1}^{\alpha} - \delta k_{t-1}$  implies  $\tilde{g}_k > 0$ .
- $c_t$  is called jump variables, whereas  $k_{t-1}$  is state variable.

# The Phase diagram



# Two ways to solve the RCK model

- A Robinson Crusoe (Social Planner) economy.
- A Competitive economy.
- When the allocations and prices in these economies coincide each other?  
The second fundamental theorem of welfare economics.

# A Robinson Crusoe economy

- Consider an economy with only one individual.
- The individual wants to maximize a lifetime utility of the form

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

subject to

$$k_t = (1 - \delta)k_{t-1} + i_t,$$

$$y_t = A_t k_{t-1}^{\theta} \geq c_t + i_t.$$

where  $c_t$  is consumption,  $k_t$  is capital,  $i_t$  is investment and  $y_t$  is output. Parameters are given as  $\beta \in (0, 1)$  is discount factor,  $\delta \in (0, 1]$  is depreciation rate, and  $\theta$  is capital share.

- We set up the Lagrangian as

$$L_0 \equiv \sum_{t=0}^{\infty} \beta^t \{ \log c_t - \lambda_t (c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\theta}) \}.$$

$\lambda_t$  is called the Lagrange multiplier, which measures the marginal utility of consumption.

- Taking the derivatives of the Lagrangian and set them to zero

$$\partial c_t : \quad \lambda_t = 1/c_t,$$

$$\partial k_t : \quad \lambda_t = \beta \lambda_{t+1} (1 + \theta A_{t+1} k_t^{\theta} - \delta),$$

$$\partial \lambda_t : \quad c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\theta} = 0.$$

These are the necessary conditions for the equilibrium.



# The two key equations

- The equilibrium conditions are summarized as

$$1 = \beta \frac{c_t}{c_{t+1}} (1 + \theta A_{t+1} k_t^{\theta-1} - \delta),$$
$$c_t + k_t - (1 - \delta)k_{t-1} = A_t k_{t-1}^{\theta}.$$

# Transversality condition

- The transversality condition is

$$\lim_{t \rightarrow \infty} \beta^t MU_t k_t = 0,$$

where  $k_t$  is the remaining resources and  $MU_t$  converts the value of  $k_t$  to the unit in terms of utility.

- The condition says that the planner must use all the resources and/or have no marginal benefit from consumption.

# A competitive economy

- In a competitive economy, there are consumers who provide labor to the market and firms who hire the labor at wage  $w_t$  and rent capital at rate  $r_t$ .
- All individuals are the same. We consider the “representative” agent’s problem.

- The representative firm can access to the Cobb-Douglas production function technology:

$$y_t = A_t k_{t-1}^\theta.$$

- As a result of profit maximization,

$$r_t = \theta y_t / k_{t-1},$$

and

$$\pi_t = y_t - r_t k_{t-1} = (1 - \theta) y_t.$$

- An individual  $i \in [0, 1]$  maximizes:

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i)$$

subject to

$$c_t^i + k_t^i - (1 - \delta)k_{t-1}^i = r_t k_t^i + \pi_t^i,$$

- We set up the Lagrangean as

$$L_0^i \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t^i - \lambda_t \left( c_t^i + k_t^i - (1 - \delta)k_{t-1}^i - r_t k_{t-1}^i - \pi_t^i \right) \right\}.$$

- Taking the derivatives of the Lagrangean and set them to zero,

$$\partial c_t^i : \quad \lambda_t = 1/c_t^i,$$

$$\partial k_t^i : \quad \lambda_t = \beta \lambda_{t+1} (1 + r_{t+1} - \delta),$$

$$\partial \lambda_t : \quad c_t^i + k_t^i - (1 - \delta)k_{t-1}^i - r_t k_{t-1}^i - \pi_t^i = 0.$$

# The two key equations for households

- The optimality conditions for households are summarized as

$$\frac{c_{t+1}^i}{c_t^i} = \beta (1 + r_{t+1} - \delta),$$
$$c_t^i + k_t^i = (1 - \delta)k_{t-1}^i + r_t k_{t-1}^i + \pi_t^i.$$

- The aggregation rules are

$$c_t = \int_0^1 c_t^i di, \quad k_t = \int_0^1 k_t^i di,$$
$$y_t = \int_0^1 y_t^i di, \quad \pi_t = \int_0^1 \pi_t^i di,$$

- Note that  $\pi_t + r_t k_{t-1} = y_t$ . Then we have

$$1 = \beta \frac{c_t}{c_{t+1}} (1 + \theta A_{t+1} k_t^{\theta-1} - \delta),$$
$$c_t + k_t - (1 - \delta)k_{t-1} = A_t k_{t-1}^{\theta}.$$



# Second welfare theorem

- **The second fundamental theorem of welfare economics:** If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then *any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.*

## Second welfare theorem, cont'd

- The first fundamental theorem: Any Competitive Equilibrium allocation (C.E.) is necessarily Pareto Optimum allocation (P.O.).
- The second fundamental theorem: Any P.O. can be achieved as a C.E. with a lump-sum transfer.
- If the second welfare theorem holds, we only need to look at P.O. instead of C.E.
- Counter-example: An economy with distortionary tax.

- ① An economy with variable labor
- ② Incorporating trend
- ③ Notes on Newton's method and deterministic simulation

# An economy with variable labor

- The social planner wants to maximize a lifetime utility

$$\sum_{t=0}^{\infty} \beta^t \log c_t + v(h_t)$$

subject to

$$k_t = (1 - \delta)k_{t-1} + i_t,$$

$$y_t = A_t k_{t-1}^{\theta} h_t^{1-\theta} \geq c_t + i_t.$$

where  $c_t$  is consumption,  $k_t$  is capital,  $i_t$  is investment,  $y_t$  is output and  $h_t$  is hours worked. Parameters are given as  $\beta \in (0, 1)$  is discount factor,  $\delta \in (0, 1]$  is depreciation rate, and  $\theta$  is capital share.

- $v(h_t)$  is called labor disutility:  $h_t \in (0, 1)$  and  $v(h_t)$  is a concave function such that  $v(h_t) \rightarrow -\infty$  as  $h_t \rightarrow 1$ .
- There are two forms of labor disutility.
  - Divisible labor:  $v(h_t) = B \log(1 - h_t)$ : Everyone works for  $h_t$  hours.
  - Indivisible labor:  $v(h_t) = -Bh_t$ : Only a fraction of *individuals* works for  $h_0$  hours.

# Indivisibility and lottery

- Labor is indivisible: Individuals can either work full time, denoted by  $h_0$ , or not at all.
- We require individuals to choose lotteries  $\alpha_t$ :

$$\log c_t + A\alpha_t \log(1 - h_0)$$

Note  $\log(1) = 0$ . Total hours worked (per capita) is  $h_t = \alpha_t h_0$ .

# Indivisibility and lottery, cont'd

- It can be viewed as linear disutility from the point of view of “representative” household.
- By substituting out  $\alpha_t$ , we have

$$\begin{aligned} A\alpha_t \log(1 - h_0) &= A \log(1 - h_0)/h_0 h_t, \\ &= -Bh_t \end{aligned}$$

where  $B = -A \log(1 - h_0)/h_0 > 0$ .

- We set up the Lagrangian as

$$L_0 \equiv \sum_{t=0}^{\infty} \beta^t \left\{ \log c_t + v(h_t) - \lambda_t \left( c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\alpha} h_t^{1-\alpha} \right) \right\}.$$

$\lambda_t$  is the Lagrange multiplier.

- Taking the derivatives of the Lagrangian and set them to zero,

$$\partial c_t : \quad \lambda_t = 1/c_t,$$

$$\partial h_t : \quad \lambda_t (1 - \alpha) A_t k_{t-1}^{\alpha} h_t^{-\alpha} = -v'(h_t),$$

$$\partial k_t : \quad \lambda_t = \beta \lambda_{t+1} \left( 1 + \theta A_{t+1} k_t^{\theta-1} h_t^{1-\theta} - \delta \right),$$

$$\partial \lambda_t : \quad c_t + k_t - (1 - \delta)k_{t-1} - A_t k_{t-1}^{\alpha} h_t^{1-\alpha} = 0.$$



# Equilibrium conditions

- The equilibrium conditions are

$$y_t = A_t k_{t-1}^\theta h_t^{1-\theta},$$

$$1 = \beta \frac{c_t}{c_{t+1}} \left( 1 + \theta \frac{y_{t+1}}{k_t} - \delta \right),$$

$$c_t + k_t - (1 - \delta)k_{t-1} - y_t = 0,$$

and

$$(1 - \theta) \frac{y_t}{h_t} = \begin{cases} \frac{Bc_t}{1-h_t}, & \text{(divisible labor)} \\ Bc_t. & \text{(indivisible labor)} \end{cases}$$

- There are 4 variables  $\{c_t, k_t, y_t, h_t\}$  and 4 equations, so we can solve the model.

# The steady state

- The steady state conditions are

$$\begin{aligned}y &= k^{\theta} h^{1-\theta}, \\1 &= \beta (1 + \theta y/k - \delta), \\0 &= y - \delta k - c,\end{aligned}$$

and

$$(1 - \theta) \frac{y}{h} = \begin{cases} \frac{Bc}{1-h}, & \text{(divisible labor)} \\ Bc. & \text{(indivisible labor)} \end{cases}$$

- Dynare solves for the steady state values numerically with an educated initial guess. The steady state values are also analytically obtained.

- 1 An economy with variable labor
- 2 Incorporating trend
- 3 Notes on Newton's method and deterministic simulation

- Consider the divisible labor economy. Let  $Z_t \equiv A_t^{\frac{1}{1-\theta}}$  and  $\gamma \equiv Z_{t+1}/Z_t$ . The production function becomes

$$y_t = k_{t-1}^{\theta} (Z_t h_t)^{1-\theta}.$$

This is called the Harrod-neutral production function.

- In the steady state,  $y$  and  $k$  exponentially grows at the rate  $\gamma$ .

- The equilibrium conditions in the divisible labor economy are

$$\begin{aligned}y_t &= k_{t-1}^\theta (Z_t h_t)^{1-\theta}, \\(1-\theta) \frac{y_t}{h_t} &= \frac{B c_t}{1-h_t}, \\1 &= \beta \frac{c_t}{c_{t+1}} \left( 1 + \theta \frac{y_{t+1}}{k_t} - \delta \right), \\c_t + k_t - (1-\delta)k_{t-1} - y_t &= 0.\end{aligned}$$

- $y_t$ ,  $c_t$  and  $k_{t-1}$  grows at  $\gamma$ , whereas the model variables have to be stationary.

- Define  $\tilde{y}_t \equiv y_t/Z_t$ ,  $\tilde{c}_t \equiv c_t/Z_t$  and  $\tilde{k}_{t-1} \equiv k_{t-1}/Z_t$ . Substituting them into the equilibrium conditions,

$$\tilde{y}_t = \tilde{k}_{t-1}^\theta h_t^{1-\theta},$$

$$(1-\theta) \frac{\tilde{y}_t}{h_t} = \frac{B\tilde{c}_t}{1-h_t},$$

$$1 = (\beta/\gamma) \frac{\tilde{c}_t}{\tilde{c}_{t+1}} \left( 1 + \theta \frac{\tilde{y}_{t+1}}{\tilde{k}_t} - \delta \right),$$

$$\tilde{c}_t + \gamma \tilde{k}_t - (1-\delta) \tilde{k}_{t-1} - \tilde{y}_t = 0.$$

# The steady state

- The steady state conditions are

$$\begin{aligned}\tilde{y} &= \tilde{k}^\theta h^{1-\theta}, \\ (1-\theta)(\tilde{y}/h) &= B\tilde{c}/(1-h), \\ 1 &= \tilde{\beta} \left( 1 + \theta\tilde{y}/\tilde{k} - \delta \right), \\ 0 &= \tilde{y} - (\gamma - 1 + \delta)\tilde{k} - \tilde{c},\end{aligned}$$

where  $\tilde{\beta} = \beta/\gamma$ .

# The steady state, cont'd

- Let  $h = 1/3$ , then we have

$$\tilde{y}/\tilde{k} = \theta^{-1}(\tilde{\beta}^{-1} - 1 + \delta),$$

$$\tilde{c}/\tilde{k} = \tilde{y}/\tilde{k} - (\gamma - 1 + \delta),$$

$$\tilde{k} = \left( \frac{\theta h^{1-\theta}}{\tilde{\beta}^{-1} - 1 + \delta} \right)^{\frac{1}{1-\theta}},$$

$$B = (1 - \alpha)(\tilde{y}/\tilde{c})(1 - h)/h.$$

$\tilde{y} = (\tilde{y}/\tilde{k})\tilde{k}$  and  $\tilde{c} = (\tilde{c}/\tilde{k})\tilde{k}$  are also obtained. Note that  $B$  is a normalization parameter.



- 1 An economy with variable labor
- 2 Incorporating trend
- 3 Notes on Newton's method and deterministic simulation

# Newton-Raphson method

- We solve an equation  $f(x) = 0$  for  $x$ .
- First-order Taylor expansion around  $x^{(0)}$ :  
$$f(x) \approx f(x^{(0)}) + f'(x^{(0)})(x - x^{(0)}) = 0.$$
- The Newton-Raphson method updates  $x^{(k)}$  for  $k = 1, 2, \dots$

$$x^{(k)} = x^{(k-1)} - f'(x^{(k-1)})^{-1} f(x^{(k-1)}),$$

until  $\|x^{(k)} - x^{(k-1)}\| \leq \epsilon$ .

- Example:  $f(x) = 4x^3 - 3$ , which has a real root of  $.75^{1/3}$ .

# Newton-Raphson method: The case of 2 variables

- We solve a system of equations

$$f_1(x_1, x_2) = 0, \quad f_2(x_1, x_2) = 0.$$

- First-order Taylor expansion:

$$\begin{aligned} f_1(x_1, x_2) &\approx f_1(x_1^{(0)}, x_2^{(0)}) \\ &\quad + D_1 f_1(x_1^{(0)}, x_2^{(0)})(x_1 - x_1^{(0)}) + D_2 f_1(x_1^{(0)}, x_2^{(0)})(x_2 - x_2^{(0)}), \\ f_2(x_1, x_2) &\approx f_2(x_1^{(0)}, x_2^{(0)}) \\ &\quad + D_1 f_2(x_1^{(0)}, x_2^{(0)})(x_1 - x_1^{(0)}) + D_2 f_2(x_1^{(0)}, x_2^{(0)})(x_2 - x_2^{(0)}). \end{aligned}$$

- It can be written as:

$$\begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} f_1(x_1^{(0)}, x_2^{(0)}) \\ f_2(x_1^{(0)}, x_2^{(0)}) \end{bmatrix} + \begin{bmatrix} D_1 f_1(x_1^{(0)}, x_2^{(0)}) & D_2 f_1(x_1^{(0)}, x_2^{(0)}) \\ D_1 f_2(x_1^{(0)}, x_2^{(0)}) & D_2 f_2(x_1^{(0)}, x_2^{(0)}) \end{bmatrix} \begin{bmatrix} x_1 - x_1^{(0)} \\ x_2 - x_2^{(0)} \end{bmatrix}.$$

# Newton-Raphson method: The case of 2 variables

- Or,

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}^{(0)}) + \nabla \mathbf{f}(\mathbf{x}^{(0)})(\mathbf{x} - \mathbf{x}^{(0)}) = 0.$$

where  $\mathbf{x} = [x_1, x_2]'$  and

$$\nabla \mathbf{f}(\mathbf{x}^{(0)}) = \begin{bmatrix} D_1 f_1(\mathbf{x}^{(0)}) & D_2 f_1(\mathbf{x}^{(0)}) \\ D_1 f_2(\mathbf{x}^{(0)}) & D_2 f_2(\mathbf{x}^{(0)}) \end{bmatrix}$$

is an  $(2 \times 2)$  matrix called *Jacobian*.

# Newton-Raphson method: The case of 2 variables

- The Newton-Raphson method updates  $\mathbf{x}^{(k)}$

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - [\nabla \mathbf{f}(\mathbf{x}^{(k)})]^{-1} \mathbf{f}(\mathbf{x}^{(k)}),$$

until  $\|\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}\| \leq \epsilon$ .

- It is easily extended to the  $N$ -variable case.

# Deterministic simulations

- Dynare can be used for deterministic simulations with the assumption of perfect foresight.
- The numerical problem consists of solving a nonlinear system of simultaneous equations in  $n$  endogenous variables in  $T$  periods.
- To solve the system of  $nT$  equations, Dynare uses a Newton-type method, which is based on the Fair-Taylor (1983) algorithm.
  - My explanation is based on Hollinger (1996).

# A nonlinear system

- A nonlinear dynamic system is of a general form

$$f_{i,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = 0,$$

for  $i = 1, \dots, n$  where  $\mathbf{y}_t = [y_{1,t}, \dots, y_{n,t}]'$ .

- For example, the basic RCK model is expressed as

$$f_{1,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \beta \frac{c_t}{c_{t+1}} (1 + \alpha A_{t+1} k_t^{\alpha-1} - \delta) - 1 = 0,$$

$$f_{2,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = A_t k_{t-1}^\alpha + (1 - \delta) k_{t-1} - k_t - c_t = 0,$$

where  $\mathbf{y}_t = [k_t, c_t]'$ .

# A nonlinear system, cont'd

- The system is expressed compactly as a  $(n \times 1)$  vector of equations:

$$\mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \begin{bmatrix} f_{1,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \vdots \\ f_{n,t}(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \end{bmatrix} = \mathbf{0},$$

where  $\mathbf{f}_t = [f_{1,t}, \dots, f_{n,t}]'$ .



# Solving the nonlinear system

- We solve  $\mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) = \mathbf{0}$  *simultaneously* for  $t = 1, \dots, T$ .
- $\mathbf{y}_{t-1} = \mathbf{y}_0$  is predetermined at  $t = 1$ . Also,  $\mathbf{y}_{t+1} = \mathbf{y}_{T+1}$  is predetermined at  $t = T$  (the boundary conditions).

## Solving the nonlinear system, cont'd

- We stack the system for  $T$  periods as

$$F(Z) = \begin{bmatrix} \mathbf{f}_1(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) \\ \vdots \\ \mathbf{f}_t(\mathbf{y}_{t-1}, \mathbf{y}_t, \mathbf{y}_{t+1}) \\ \vdots \\ \mathbf{f}_T(\mathbf{y}_{T-1}, \mathbf{y}_T, \mathbf{y}_{T+1}) \end{bmatrix} = \mathbf{0},$$

where  $F = [\mathbf{f}'_1, \dots, \mathbf{f}'_t, \dots, \mathbf{f}'_n]'$  and  $Z = [\mathbf{y}'_0, \mathbf{y}'_1, \dots, \mathbf{y}'_T, \mathbf{y}'_{T+1}]'$ . There is a  $(nT \times 1)$  vector of equations.

- We solve  $F(Z) = 0$  by the Newton-Raphson method for  $nT$  variables

$$[\mathbf{y}'_1, \dots, \mathbf{y}'_T]',$$

given  $\mathbf{y}'_0$  and  $\mathbf{y}'_{T+1}$ .

## Assignment #2 (revised on 1 Mar; 11 Mar.)

- Consider the basic RCK model with a distortionary tax on capital income after depreciation. The representative household maximizes the lifetime utility subject to:

$$c_t + k_t \leq (1 - \delta)k_{t-1} + r_t k_{t-1} - \tau(r_t - \delta)k_{t-1} + \pi_t + \tau_t,$$

where  $\pi_t > 0$  is the firm's profit and  $\tau_t$  is the transfer from the government. Note that the household takes both  $\pi_t$  and  $\tau_t$  as given.

- 1 The government budget constraint is given by  $\tau(r_t - \delta)k_{t-1} = \tau_t$ . Derive the resource constraint for the social planner's economy. [Hint: Use the firm's FOC and the balanced government budget in the competitive economy.]
- 2 Solve for the equilibrium conditions in both the social planner's economy and the competitive economy.
- 3 Let  $\beta = .96$ ,  $\delta = .1$ ,  $\alpha = .36$ . and  $\tau = .2$ . Compute the initial dynamics with  $k_0 = 0.1$  in both economies.