The Solow model

Takeki Sunakawa

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University of Mannheim

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Introduction

- Solow introduced a model of economic growth that has served as the basis for DSGE models.
- The model is quite simple: There are a constant returns-to-scale production function, a law for the evolution of capital, and a saving rate.
- A first-order difference equation for the evolution of capital per worker is found, and the time path of the economy springs from this equation.

The production function

- The production function is

$$Y_t = A_t F(K_t, H_t)$$

where

  - Y_t is output of the single good in the economy at date t,
  - A_t is the total factor productivity (TFP),
  - K_t is the capital stock at the *beginning* of date t, and
  - H_t is hours worked.

Constant return to scale

- The production function is homogeneous of degree one; using this property, we get

$$\begin{aligned} y_t = \frac{Y_t}{H_t} &= A_t F\left(\frac{K_t}{H_t}, \frac{H_t}{H_t}\right), \\ &= A_t F(k_t, 1), \\ &\equiv A_t f(k_t), \end{aligned} \tag{1}$$

where $y_t = Y_t/H_t$ is output per worker and $k_t = K_t/H_t$ is capital per worker.

- An example of the constant return-to-scale profuction function is $Y_t = A_t K_t^\theta H_t^{1-\theta}$, where θ is capital share.

The law of motion

- We assume that the labor force grows at a constant net rate n, so that $H_{t+1} = (1+n)H_t$.
- The capital grows
$$K_{t+1} = (1-\delta)K_t + I_t,$$
where δ is the rate of depreciation and I_t is investment at time t.
- By deviding by $H_{t+1} = (1+n)H_t$ both side,
$$k_{t+1} = \frac{(1-\delta)k_t + i_t}{1+n}, \tag{2}$$
where $i = I_t/H_t$.

Saving rate and closing the model

- Savings is defined as a fraction of output,

$$s_t = \sigma y_t \tag{3}$$

- In equilibrium in a closed economy, $i_t = s_t$, from Eqs. (1)-(3),

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma A_t f(k_t),$$

where $f(k) = k^\theta$. This equation is called "The fundamental equation of economic growth."

Steady state

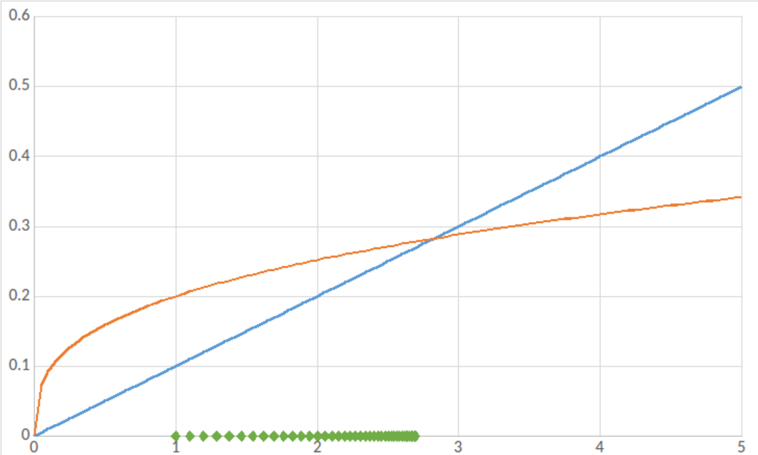
- A stationary state can be found from this equation for $k_{t+1} = k_t = \bar{k}$ and $A_t = \bar{A}$:

$$(1+n)\bar{k} = (1-\delta)\bar{k} + \sigma \bar{A} f(\bar{k}),$$

or when $f(k) = k^{\theta}$, the steady state is given by

$$\bar{k} = \left(\frac{\sigma \bar{A}}{n+\delta} \right)^{\frac{1}{1-\theta}}.$$

- Let $n = 0$. Then
$$\Delta k_{t+1} = \sigma A_t f(k_t) - \delta k_t.$$
- When the first term cuts the second term from the above, the model is convergent and capital per worker converges to the steady state.



Stochastic TFP

- We assume that the TFP follows a stochastic process:

$$\log A_{t+1} = (1 - \rho) \log \bar{A} + \rho \log A_t + \varepsilon_{t+1},$$

  where $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$.

- Note that

$$A_{t+1} = \bar{A}^{1-\rho} A_t^\rho e^{\varepsilon_{t+1}},$$

  holds.

Log-linearization

- We approximate the model around the steady state.
- Use the formula of approximation

$$x_t \equiv x \exp \hat{x}_t \approx \bar{x}(1 + \hat{x}_t),$$

where $\bar{x}$ is the steady state of x_t and $\hat{x}_t$ is percent deviation from the steady state.

Log-linearization: Production function

- Production function:

$$y_t \equiv A_t f(k_t) = A_t k_t^{\theta}.$$

It can be written as

$$\bar{y} \exp(\hat{y}_t) = \bar{A}\bar{k}^{\theta} \exp(\hat{a}_t + \theta \hat{k}_t).$$

In the steady state, $\bar{y} = \bar{A}\bar{k}^{\theta}$ holds. Then,

$$\hat{y}_t = \hat{a}_t + \theta \hat{k}_t.$$

Note: This is not approximation.

Log-linearization: Resource constraint

- Resource constraint:

$$(1+n)k_{t+1} = (1-\delta)k_t + \sigma y_t.$$

It can be written as

$$(1+n)\bar{k}\exp(\hat{k}_{t+1}) = (1-\delta)\bar{k}\exp(\hat{k}_t) + \sigma\bar{y}\exp(\hat{y}_t).$$

Use the formula of approximation

$$(1+n)\bar{k}(1+\hat{k}_{t+1}) = (1-\delta)\bar{k}(1+\hat{k}_t) + \sigma\bar{y}(1+\hat{y}_t).$$

In the steady state, $(1+n)\bar{k} = (1-\delta)\bar{k} + \sigma\bar{y}$ holds. Then we have

$$(1+n)\bar{k}\hat{k}_{t+1} = (1-\delta)\bar{k}\hat{k}_t + \sigma\bar{y}\hat{y}_t.$$

Log-linearization: TFP

- TFP:

$$\log A_{t+1} = (1-\rho)\log \bar{A} + \rho \log A_t + \varepsilon_{t+1}.$$

It can be written as

$$\log \frac{A_{t+1}}{\bar{A}} = \rho \log \frac{A_t}{\bar{A}} + \varepsilon_{t+1}.$$

Note that $\hat{a}_t = \log A_t/\bar{A}$. Then,

$$\hat{a}_{t+1} = \rho \hat{a}_t + \varepsilon_{t+1}.$$

Note: This is not approximation.

Log-linearization: Summary

- After all, the log-linealized equlibrium conditions are:

$$\hat{y}_t = \hat{a}_t + \theta\hat{k}_t,$$
$$(1+n)\bar{k}\hat{k}_{t+1} = (1-\delta)\bar{k}\hat{k}_t + \sigma\bar{y}\hat{y}_t.$$

Or,

$$(1+n)\bar{k}\hat{k}_{t+1} = (1-\delta)\bar{k}\hat{k}_t + \sigma\bar{y}(\hat{a}_t + \theta\hat{k}_t).$$

First-order difference equation

- It can be rewritten as the first-order difference equation:

$$\hat{k}_{t+1} = B\hat{k}_t + C\hat{a}_t,$$

where

$$B = \frac{1-\delta+\sigma\theta(\bar{y}/\bar{k})}{1+n},$$

$$C = \frac{\sigma(\bar{y}/\bar{k})}{1+n}.$$

- The model's dynamics is characterized by this equation and the stochastic process of

$$\hat{a}_{t+1} = \rho\hat{a}_t + \varepsilon_{t+1},$$

where $\varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2)$.

Analytical solution for the variance

- Assume $\rho = 0$ so that $\hat{a}_t = \varepsilon_t$. Recursively substituting, we have

$$
\begin{aligned}
\hat{k}_{t+1} &= B\left(B\hat{k}_{t-1} + C\varepsilon_{t-1}\right) + C\varepsilon_t, \\
&= B^2\left(B\hat{k}_{t-2} + C\varepsilon_{t-2}\right) + BC\varepsilon_{t-1} + C\varepsilon_t, \\
&= B^{i+1}\hat{k}_{t-(i+1)} + B^iC\varepsilon_{t-i} + B^{i-1}C\varepsilon_{t-(i-1)} + \cdots + C\varepsilon_t, \\
&= C\varepsilon_t + BC\varepsilon_{t-1} + B^2C\varepsilon_{t-2} + \cdots, \\
&= C\sum_{i=0}^{\infty} B^i \varepsilon_{t-i}.
\end{aligned}
$$

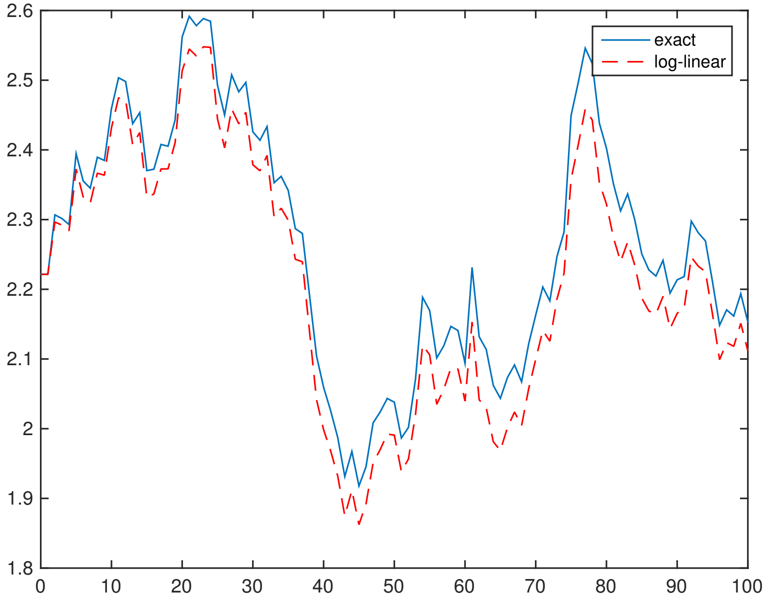
Analytical solution for the variance, cont'd

- With this expression, the variance of capital around the steady state is given by

$$\begin{aligned}\text{var}\left(\hat{k}\right) &= C^2\text{var}\left(\varepsilon\right) + B^2C^2\text{var}\left(\varepsilon\right) + B^4C^2\text{var}\left(\varepsilon\right) + \cdots,\\ &= C^2\sigma_\varepsilon^2\left(1 + B^2 + B^4 + \cdots\right),\end{aligned}$$

$$\text{var}\left(\hat{k}\right) = \frac{C^2\sigma_\varepsilon^2}{1-B^2}.$$

Simulations



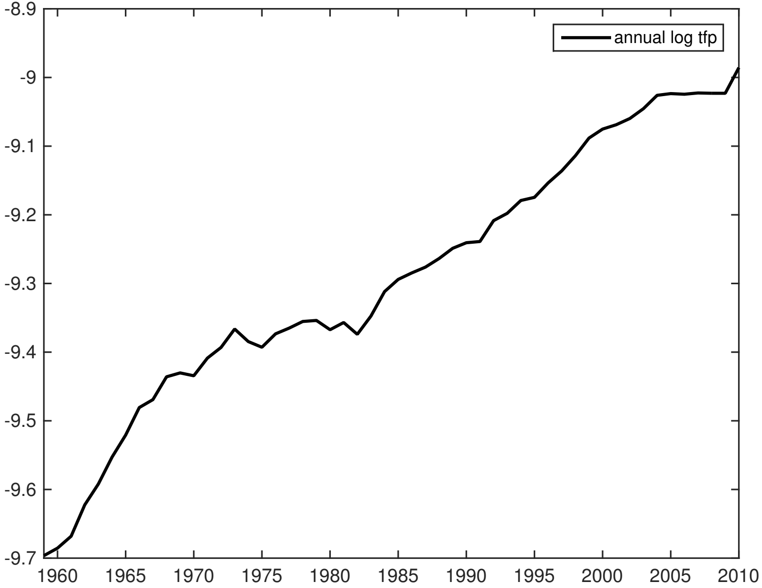
Solow residual

- Identifying the aggregate technology shock with the Solow residual:

$$\log A_t = \log Y_t - \theta \log K_t - (1-\theta) \log H_t.$$

- $\log A_t$ has a trend. How to remove the trend?

Solow residual



Data source (NIPA and CPS)

- GDP, Nominal Capital, and GDP deflator (to deflate nominal capital) are from National Income and Product Accounts (NIPA).
- Hours worked is from Consumer Population Survey (CPS).
    - See Cociuba, Prescott and Uberfeldt "U.S. Hours and Productivity Behavior Using CPS Hours Worked Data: 1947-III to 2011-IV"

Linear trend

- Remove linear trend: $a_t = \log A_t - b_0 - b_1 t$ where b_0 and b_1 are obtained by OLS.

Hodrick-Prescott filter

- Let y_t be a time series and

$$y_t = g_t + c_t,$$

where g_t is trend and c_t is cyclical component.

- The Hodrick-Prescott filter solves the following problem:

$$\min_{\{g_t\}_{t=1}^{T}} \left\{ \sum_{t=1}^{T} (y_t - g_t)^2 + \lambda \sum_{t=2}^{T-1} \left[(g_{t+1} - g_t) - (g_t - g_{t-1}) \right]^2 \right\},$$

where λ is smoothing parameter.

- FOCs are

$$
\begin{aligned}
\partial g_1: &\quad c_1 = \lambda(g_3 - 2g_2 + g_1), \\
\partial g_2: &\quad c_2 = \lambda(g_4 - 2g_3 + g_2) - 2\lambda(g_3 - 2g_2 + g_1), \\
\partial g_t: &\quad c_t = \lambda(g_{t+2} - 2g_{t+1} + g_t) - 2\lambda(g_{t+1} - 2g_t + g_{t-1}) \\
&\quad +\lambda(g_t - 2g_{t-1} + g_{t-2}) \\
&\quad \text{for } t = 3, 4, ..., T-2, \\
\partial g_{T-1}: &\quad c_{T-1} = -2\lambda(g_T - 2g_{T-1} + g_{T-2}) + \lambda(g_{T-1} - 2g_{T-2} + g_{T-3}), \\
\partial g_T: &\quad c_T = \lambda(g_T - 2g_{T-1} + g_{T-2}).
\end{aligned}
$$

Hodrick-Prescott filter, cont'd

- FOCs are

$$
\begin{aligned}
\partial g_1: \quad & c_1 = \lambda(g_3 - 2g_2 + g_1), \\
\partial g_2: \quad & c_2 = \lambda(g_4 - 4g_3 + 5g_2 - 2g_1), \\
\partial g_t: \quad & c_t = \lambda(g_{t+2} - 4g_{t+1} + 6g_t - 4g_{t-1} + g_{t-2}) \\
& \text{for } t = 3, 4, ..., T-2, \\
\partial g_{T-1}: \quad & c_{T-1} = \lambda(-2g_T + 5g_{T-1} - 4g_{T-2} + g_{T-3}), \\
\partial g_T: \quad & c_T = \lambda(g_T - 2g_{T-1} + g_{T-2}).
\end{aligned}
$$

Matrix form

- In a matrix form, $\mathbf{c} = \mathbf{y} - \mathbf{g} = \lambda \mathbf{F} \mathbf{g}$ where

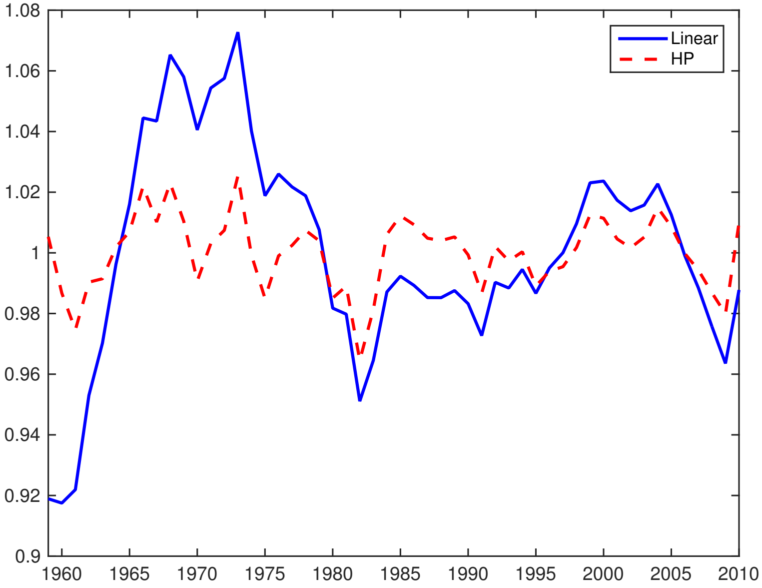
$$\underset{(T\times 1)}{\mathbf{c}} = [c_1, c_2, \cdots, c_T]',$$

$$\underset{(T\times 1)}{\mathbf{g}} = [g_1, g_2, \cdots, g_T]',$$

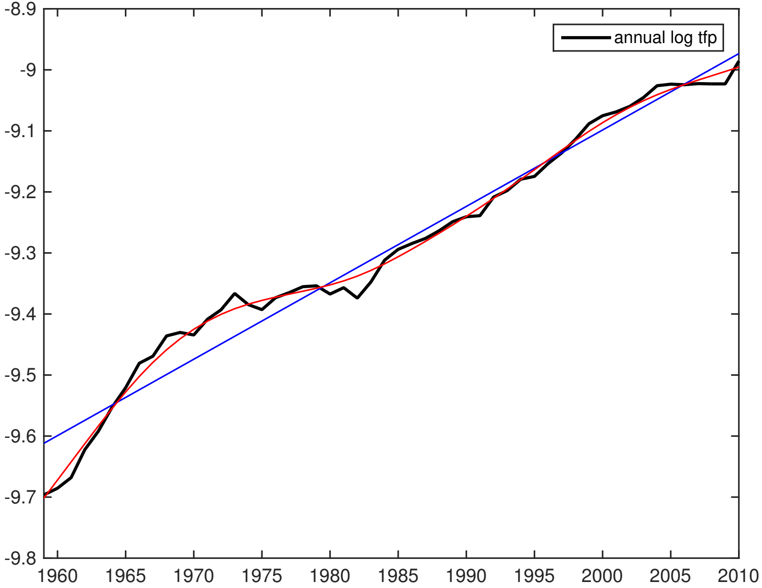
$$\underset{(T\times T)}{\mathbf{F}} = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & & & & & 0 \\ -2 & 5 & -4 & 1 & 0 & \cdots & & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & & 0 \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ 0 & \cdots & & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \cdots & & & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \cdots & & & & 0 & 1 & -4 & 5 & -2 \\ 0 & \cdots & & & & & 0 & 1 & -2 & 1 \end{bmatrix}.$$

Then, $\mathbf{g} = (I + \lambda \mathbf{F})^{-1} \mathbf{y}$.

Cyclical component



Trend



annual log tfp
-8.9
-9
-9.1
-9.2
-9.3
-9.4
-9.5
-9.6
-9.7
-9.8
1960
1965
1970
1975
1980
1985
1990
1995
2000
2005
2010

Assignment #1

- Let $n = .02$, $\delta = .1$, $\theta = .36$ and $\sigma = .2$. Also let $\bar{A} = 1$, $\rho = 0$ and $\sigma_\varepsilon = .2$.
  1. Simulate the model for 1,000 periods and compute $\texttt{var}(k)$.
  2. Compare it with the analytical solution for the variance.
  3. Do 1-2 with 100,000 period simulation.
  4. What about the case of $\rho > 0$? Try to derive the analytical solution for the variance and compare it with simulation.