A Online Appendix Material

A.1 State Space Formulation

Based on the system of equations in section 2 of the paper, substituting the formula for r_t^* into the output gap equation and expanding, we can come to a version of these equations that can be expressed in the traditional observation/transition equation style of the standard state space model. Following some algebraic manipulation, these equations are given as follows. First, the observation equations on real GDP and inflation.

$$\begin{aligned} y_t = y_t^* - a_1 y_{t-1}^* - a_2 y_{t-2}^* - 2a_r\rho_g g_{t-1} - 2a_r\rho_g g_{t-2} \\ -\frac{a_r}{2}\rho_z z_{t-1} - \frac{a_r}{2}\rho_z z_{t-2} - 4a_r\mu_g(1-\rho_g) + a_1 y_{t-1} \\ +a_2 y_{t-2} + \frac{a_r}{2} r_{t-1} + \frac{a_r}{2} r_{t-2} + \sigma_1\varepsilon_{1,t} \end{aligned} \tag{A.1}$$

$$\pi_t = -b_Y y_{t-1}^* + b_Y y_{t-1} + b_1\pi_{t-1} + (1-b_1)\sum_{i=2}^{4}\frac{\pi_{t-i}}{3} + \sigma_2\varepsilon_{2,t} \tag{A.2}$$

Then, the transition equations for unobserved potential real GDP, its growth rate, and the z process.

$$y_t^* = y_{t-1}^* + \mu_g(1-\rho_g) + \rho_g g_{t-2} + \sigma_5\varepsilon_{5,t-1} + \sigma_4\varepsilon_{4,t} \tag{A.3}$$

$$z_{t-1} = \rho_z z_{t-2} + \sigma_3\varepsilon_{3,t-1} \tag{A.4}$$

$$g_{t-1} = \rho_g g_{t-2} + \mu_g(1-\rho_g) + \sigma_5\varepsilon_{5,t-1} \tag{A.5}$$

These equations can be represented in state space form using the standard structure:

$$s_t = As_{t-1} + Bu_t + Cw_t \tag{A.6}$$

$$x_t = Ds_t + Fu_t + Gw_t \tag{A.7}$$

where:

$$s_t = \begin{bmatrix} y_t^* \\ y_{t-1}^* \\ y_{t-2}^* \\ g_{t-1} \\ g_{t-2} \\ z_{t-1} \\ z_{t-2} \end{bmatrix}, \quad x_t = \begin{bmatrix} y_t \\ \pi_t \end{bmatrix}, \quad u_t = \begin{bmatrix} 1 \\ y_{t-1} \\ y_{t-2} \\ r_{t-1} \\ r_{t-2} \\ \pi_{t-1} \\ \sum_{i=2}^{4}\frac{\pi_{t-i}}{3} \end{bmatrix}, \quad w_t = \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t-1} \\ \varepsilon_{4,t} \\ \varepsilon_{5,t-1} \end{bmatrix},$$

and

$$A = \begin{bmatrix} 1 & 0 & 0 & \rho_g & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \mu_g(1-\rho_g) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mu_g(1-\rho_g) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 0 & \sigma_4 & \sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -a_1 & -a_2 & -2a_r\rho_g & -2a_r\rho_g & -\frac{a_r\rho_z}{2} & -\frac{a_r\rho_z}{2} \\ 0 & -b_Y & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} -4a_r(1-\rho_g)\mu_g & a_1 & a_2 & \frac{a_r}{2} & \frac{a_r}{2} & 0 & 0 \\ 0 & b_Y & 0 & 0 & 0 & b_1 & (1-b_1) \end{bmatrix}, \quad G = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \end{bmatrix}$$

The ε's are all assumed to be *i.i.d.* $N(0,1)$ variables, with the standard deviation of the processes controlled by the σ_i's.

A.2 Data

The data used in this analysis is the same as the US data used in Holston, Laubach, and Williams (2017), henceforth HLW, and it is transformed in the same way. See the data appendix in HLW for additional specifics on obtaining the data. Real GDP data are obtained from the BEA, inflation is calculated as the annualized quarterly growth rate of the price index for personal consumption expenditures excluding food and energy (commonly referred to as "core PCE inflation"). We follow HLW in using a 4-quarter moving average of inflation in period t as a proxy for inflation expectations in that period. The short-term interest rate is the annualized nominal effective federal funds rate, where the quarterly value is constructed as the average of the monthly values. Prior to 1965, we use the Federal Reserve Bank of New York's discount rate.

A.3 Bayesian estimation details

Some additional details:

- Restrictions on parameters (primarily inherited from HLW):
  - We enforce that a_r be negative and b_Y positive, following HLW in using the actual restrictions $a_r < -0.0025$ and $b_Y > 0.025$.
  - As the sum of the coefficients on lags of inflation must sum to 1, we restrict b_1 to be between 0 and 1.
  - Because of our expectation of a positive autocovariance for both g_t and z_t in the event of stationarity, we restrict ρ_g and ρ_z to be positive.

The initialization for the states was duplicated from the process used in HLW: the initial values for potential output y^* were constructed by HP filtering the GDP series beginning in 1960Q1, then using the trend component of the filtered output for the observations just before the beginning of the data used in the estimation (1960 Q2, Q3 and Q4); the initial values for g were the changes of that trend component in the second half of 1960. The initial values for z were set to zero, as in HLW.

The estimation is performed in MATLAB using our own code to implement the random-walk Metropolis-Hastings algorithm (see, e.g., Herbst and Schorfheide, 2015). The filter code was written to execute the forward-filter, backward sample methodology of Frühwirth-Schnatter (1994) and Carter and Kohn (1994, 1996) to obtain samples of the unobserved states. We used a burn-in period of 250,000 draws before accepting every tenth draw for a total of 500 kept draws from each of 20 chains, for a total sample of 10,000 draws from the posterior distribution.

A.4 A Flexible Specification Where g and z Are Both AR(1)

Another specification which we tested was to allow both z and g to be estimated as AR(1) processes without forcing either to be a random walk. Allowing rho_g and μ_g to be estimated along with ρ_z did not dramatically alter the median path of r^* that was estimated as the alternative specification in the paper, as can be seen below in Figure A.1. This is because the posterior distributions provide considerable evidence that the persistence parameter, ρ_g, can be reasonably assumed to be one for the purposes of parsimony, see Figures A.2 and A.3. In fact, when we conduct the same Savage-Dickey density ratio test on ρ_g in this specification that we conduct on ρ_z in the alternative specification of the main text, we find that the data adds weight to the posterior at the value $\rho_g = 1$, see Figure A.4. The posterior distributions are described in Table A.1 and are generally similar to the alternative specification except for the new parameters of g. The Newton and Raftery (1994) log marginal likelihood value is -526, the same as that of the alternative specification.

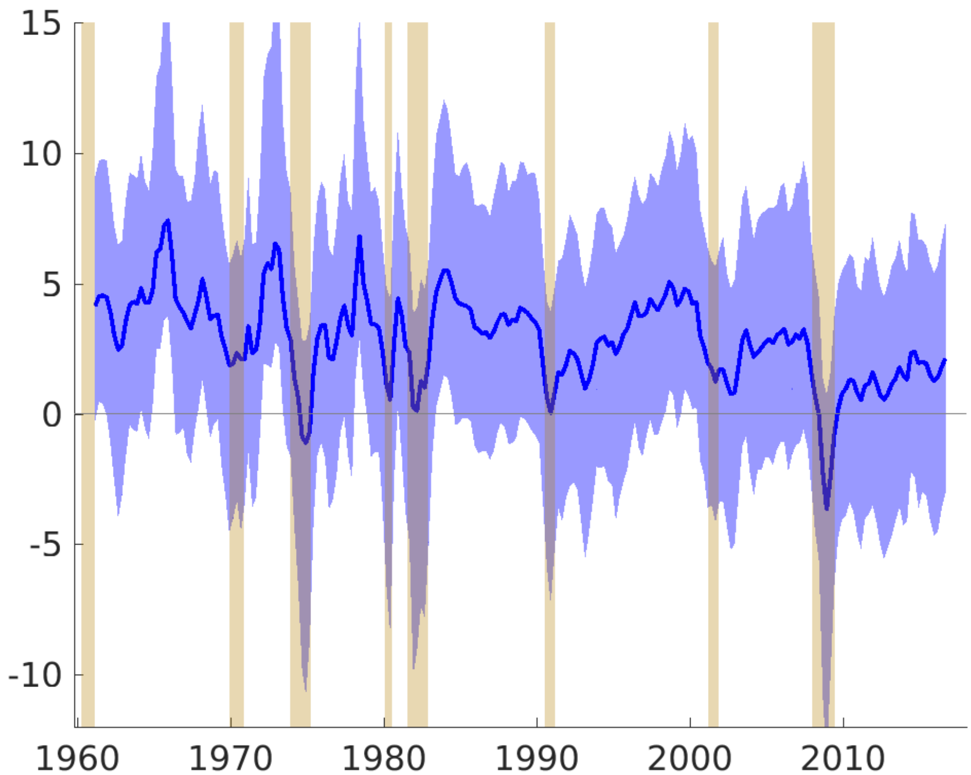

Figure A.1: r^* Path

NOTES: The path of r^* under the a specification in which both g and z are estimated as AR(1) processes. The solid blue line shows the median path of the smoothed (two-sided) estimate and the blue-shaded area is bounded by the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles of the estimated path. The vertical shaded bars represent NBER-dated recessions.

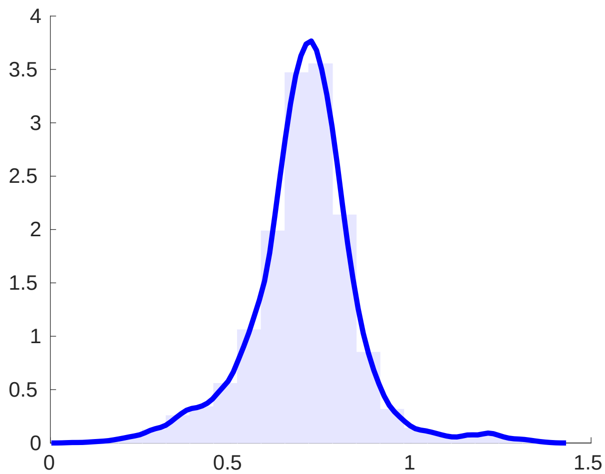

Figure A.2: μ_g from a flexible AR(1) specification for g

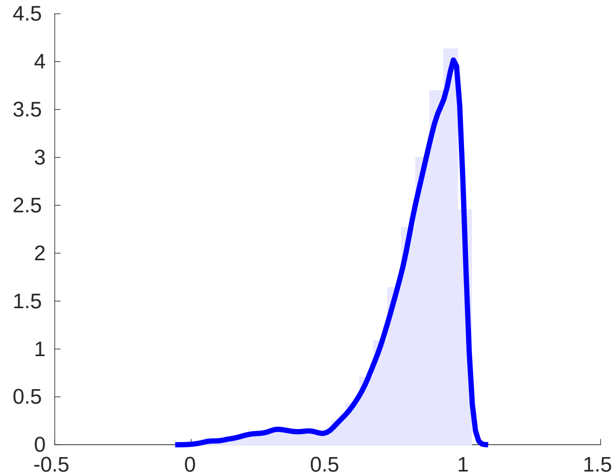

Figure A.3: ρ_g from a flexible AR(1) specification for g

NOTES: The posterior distributions for the parameters of the AR(1) process for g in a specification in which both g and z are allowed to be freely estimated.

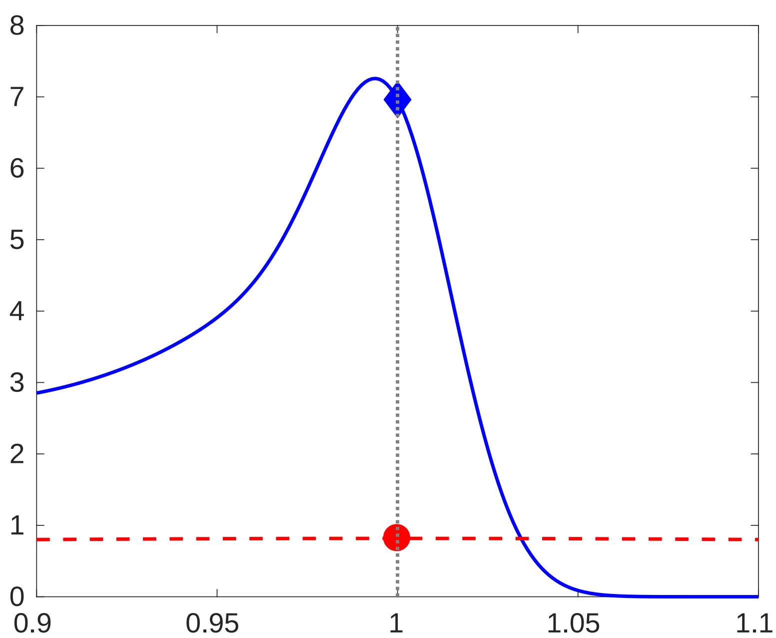

Figure A.4: SDDR for ρ_g

NOTES: The illustration of the Savage-Dickey density ratio for ρ_g in a specification in which g and z were both estimated AR(1) processes, accounting for the pileup problem via priors. Evidence suggests that the assumption that $\rho_g = 1$ is valid.

	Fully Flexible Specificaton		
	10^{th} Perc.	Median	90^{th} Perc.
a_1	0.85	1.17	1.43
a_2	-0.53	-0.30	0.00
a_r	-0.14	-0.08	-0.04
b_1	0.59	0.68	0.76
b_Y	0.05	0.10	0.17
σ_1	0.05	0.24	0.49
σ_2	0.75	0.80	0.86
σ_3	0.62	2.23	5.24
σ_4	0.31	0.55	0.63
σ_5	0.06	0.17	0.35
ρ_g	0.64	0.87	0.98
ρ_z	0.15	0.53	0.81
μ_g	0.53	0.71	0.85
λ_g	0.11	0.33	1.01
λ_z	0.16	0.80	4.73

Table A.1: Information from posterior distributions of the parameters from the fully flexible specification

A.5 Prior Distributions of λ_g and λ_z and Pile-Up

The prior distributions for the σ_i's were chosen to reflect the high degree of uncertainty about the volatility of the hidden processes. Using uniform distributions gave us a simple way to allow for significant mass across potentially larger values without significantly under-weighting the region close to zero. We use restrictions on λ_g and λ_z, requiring that they have properties that limit the risk of pileup. Indeed, Figures A.5 and A.6 compare our implied prior distributions for λ_g and λ_z to those used by Holston et al. (2017) and Pescatori and Turunen (2016). Our priors on λ_g and λ_z are less informative than others used in the literature, this is especially true for the case of λ_z, where inverse gamma distributions with means near the HLW point estimates actually place more mass to the left of that estimate, very close to zero.

Figures A.7 and A.8 show that the unobserved volatility parameters display no signs of pileup. The signals from our analysis line up with a finding from Clark and Kozicki (2005) that λ_z and λ_g may be higher than estimated by Laubach and Williams (2003).

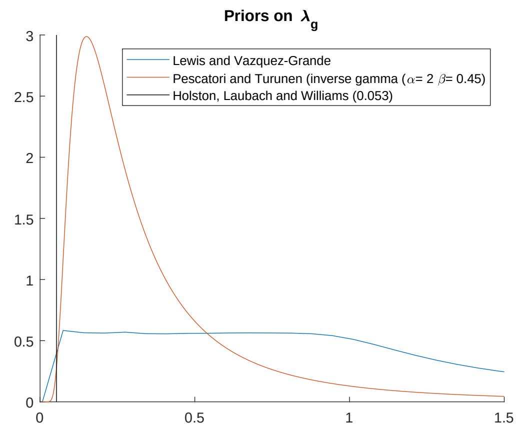

Figure A.5: Priors of λ_g

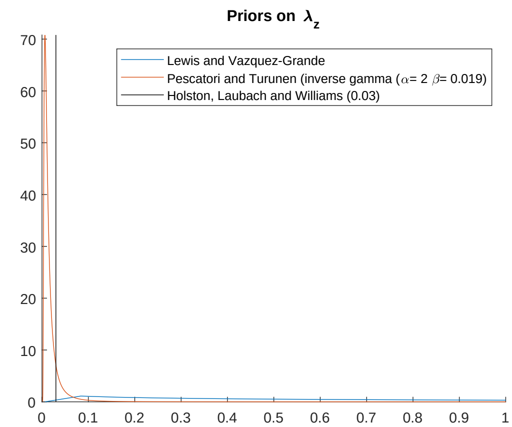

Figure A.6: Priors of λ_z

NOTES: These inverse gamma prior parameters are consistent with the moments reported on Pescatori and Turunen (2016).

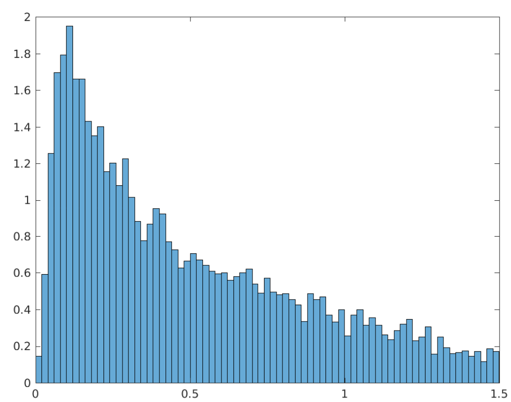

Figure A.7: Posterior Distribution of σ_3

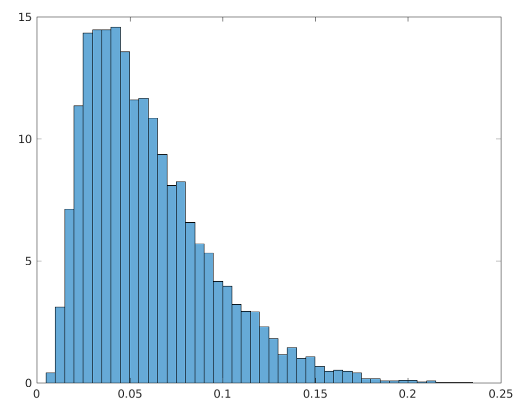

Figure A.8: Posterior Distribution of σ_5

NOTES: Posterior Distributions of unobserved volatilities where the "pileup problem" was a concern show no evidence of pileup.

A.6 The Long-run natural rate

An important difference between the baseline and alternative specifications is that in the baseline specification r^* is, by construction, a long-run object. Having introduced transitory shocks in the alternative specification, we will need to transform our new measure of r^* to align it better for a direct comparison. To do this, we extract the lower-frequency component of the new r^* measure. Following Del Negro, Giannone, Giannoni, and Tambalotti (2017), we using the medium term forecast (specifically the ten-year projection) of the rate as our long-run r^*:

$$r_t^{*\,LR} = E_t\left(r_{t+40}^*\right). \tag{A.8}$$

Figure A.9 shows the path of long-run r^* under the alternative specification along with the median path from the baseline specification. The baseline specification remains in a relatively tight area around the alternative specification for much of the sample, then plummets during the financial crisis. While the median path of the baseline model drops about three percentage points to around -1, the dip in the alternative specification, driven more significantly by the growth rate, is significantly less. Thus, a major factor in determining the level of long-run r^* in 2017 would appear to be the assumption that all the shocks during the financial crisis are permanent.

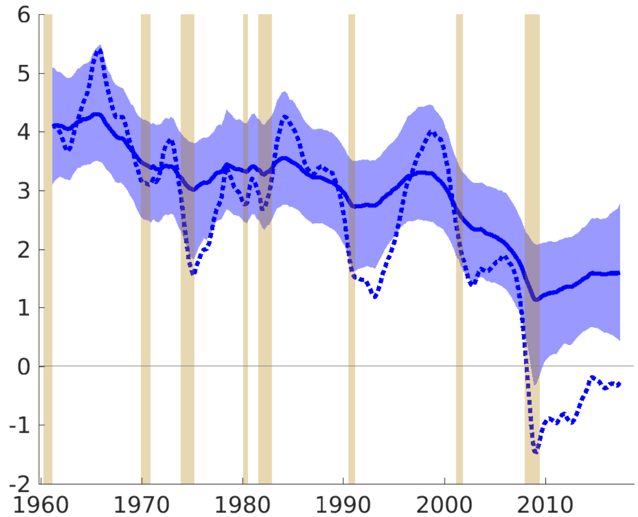

Figure A.9: r^* Path

NOTES: A comparison of the path of long-run r^* under the baseline and alternative models. The solid blue line shows the median path of the smoothed (two-sided) estimate of the alternative specification and the blue-shaded area is bounded by the 5th and 95th percentiles of this estimated path. The dotted blue line shows the median estimated path of the long-run r^* under the baseline specification. The vertical shaded bars represent NBER-dated recessions.

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