Details: Problem Set 1

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Question 4

A We first define a competitive equilibtrium.

Definition 1. *A list* $\{w_t, r_t, c_t, h_t, I_t, G_t, K_{t+1}, Y_t\}_{t=0}^{\infty}$ *is a competitive equilibrium if:*

1. *Agent's actions are optimal, in the sense that:*

$$
\begin{aligned}
\{c_t, h_t, I_t, K_{t+1}\}_{t=0}^{\infty} \in \quad & \underset{\{c_t, K_{t+1}, I_t, l_t, h_t\}_{t=0}^{\infty}}{\mathit{argmax}} \sum_{t=0}^{\infty} \beta^t N_t[\ln(c_t) + \theta \ln(l_t)] \\
s.t. \quad & (1+\tau^c)N_t c_t + I_t = (1-\tau^k) r_t K_t + (1-\tau^h) w_t N_t h_t \\
& K_{t+1} = (1-\delta)K_t + I_t \\
& l_t + h_t = 1 \\
& c_t, K_{t+1}, l_t, h_t \geq 0
\end{aligned}
$$

2. *Firms maximise their profits, i.e.,* $\forall t \geq 0$*:*

$$
(H_t, K_t^d) \in \underset{H_t, K_t}{\mathit{argmax}}\ K_t^{\alpha}(A_t H_t)^{(1-\alpha)} - w_t H_t - r_t K_t
$$

3. *The government's budget constraint is satisfied in every period, i.e.,* $\forall t \geq 0$*:*

$$
G_t = \tau^c N_t c_t + \tau^h w_t N_t h_t + \tau^k r_t K_t
$$

4. *Markets clear, i.e.,* $\forall t \geq 0$*:*

$$
\begin{aligned}
H_t &= N_t h_t \\
K_t^d &= K_t \\
Y_t &= N_t c_t + I_t + G_t
\end{aligned}
$$

Solving both the consumer and firm's problem, and plugging in the market-clearing conditions, we are left with the following equalities, which must hold in every period:

$$\frac{c_{t+1}}{c_t} = \beta(1-\delta+(1-\tau^k)r_{t+1}) \tag{1}$$

$$c_t = \frac{(1-h_t)(1-\tau^h)w_t}{\theta(1+\tau_c)} \tag{2}$$

$$(1+\tau^c)N_t c_t + K_{t+1} = [1-\delta+(1-\tau^k)r_t]K_t + (1-\tau^h)w_t N_t h_t \tag{3}$$

$$\lim_{t\to\infty} \beta^t \frac{K_t}{c_t} = 0 \tag{4}$$

$$w_t = K_t^{\alpha}(1-\alpha)(A_t N_t h_t)^{-\alpha} A_t \tag{5}$$

$$r_t = \alpha K_t^{\alpha-1}(A_t N_t h_t)^{1-\alpha} \tag{6}$$

B Does a balanced growth path exist? Let g_x denote the growth rate of endogenous variable x in a balanced growth path. If a balanced growth path exists, it follows from (1) that:

$$(1+g_c) = \beta(1-\delta+(1-\tau^k)r_{t+1}) \Rightarrow r_{t+1} = \bar{r}$$

In a balanced growth path, interest rates are constant. But then it follows from (6) that:

$$(1+g_k) = (1+\gamma)(1+\eta)(1+g_h)$$

Combine this with (5) and you get:

$$(1+g_w) = (1+\gamma)$$

And from (2):

$$(1+g_c) = (1+g_l)(1+g_w) = (1+g_l)(1+\gamma)$$

Where g_l is the growth rate of leisure. We claim that, in a balanced growth path, g_h (and consequently g_l) should be zero. Indeed, if $g_h > 0$, then, for t large enough, we would have $h = 1 \Rightarrow l = 0$, which is not optimal given the consumer's utility function (which satisfies the Inada conditions). Conversely, if $g_h < 0$, we have $g_l > 0$, and then, for t large enough, we would have $l = 1 \Rightarrow h = 0$. At that moment, there would be no production in the economy, which is not optimal given the choice of preferences. We thus conclude that, in a balanced growth path, it must be that $g_h = g_l = 0$.

Therefore, we have that, in a balanced growth path, aggregates grow at the following rates:

$$(1+g_c) = (1+\gamma)$$

$$(1+g_K) = (1+\gamma)(1+\eta)$$

$$g_h = g_l = 0$$

Given these rates, we can easily compute the growth rates of other endogenous variables, e.g. output grows at rate $(1+g_Y) = Y_{t+1}/Y_t = (K_{t+1}/K_t)^{\alpha}(A_{t+1}N_{t+1}h_{t+1}/A_tN_th_t)^{1-\alpha} = (1+\gamma)(1+\eta)$.

C Given the results above, a natural normalisation is to write variables in units of effective labour (A_tN_t). In particular, we write $\hat{c}_t = \frac{c_t}{A_t}$ and $\hat{K}_t = \frac{K_t}{A_tN_t}$. If we apply the normalisation to (1), (2), (3), (5) and (6), and then plug (5) and (6) into (1), (2) and (3), we are left with:

$$\hat{c}_t = \frac{(1-h_t)(1-\tau^h)(1-\alpha)\hat{K}_t^\alpha h_t^{-\alpha}}{\theta(1+\tau^c)} \tag{7}$$

$$\frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\beta}{(1+\gamma)}[1-\delta+(1-\tau^k)\alpha\hat{K}_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}] \tag{8}$$

$$(1+\tau_c)\hat{c}^t + (1+\gamma)(1+\eta)\hat{K}_{t+1} = [1-\delta+(1-\tau^k)\alpha\hat{K}_t^{\alpha-1}h_t^{1-\alpha}]\hat{K}_t + (1-\tau^h)(1-\alpha)\hat{K}_t^\alpha h_t^{1-\alpha} \tag{9}$$

We may further simplify our system by plugging (7) into (8) and (9):

$$\frac{1-h_{t+1}}{1-h_t}\left(\frac{\hat{K}_{t+1}}{\hat{K}_t}\right)^\alpha\left(\frac{h_{t+1}}{h_t}\right)^{-\alpha} = \frac{\beta}{(1+\gamma)}[1-\delta+(1-\tau^k)\alpha\hat{K}_{t+1}^{\alpha-1}h_{t+1}^{1-\alpha}] \tag{10}$$

$$\begin{aligned}\frac{(1-h_t)(1-\tau^h)(1-\alpha)\hat{K}_t^\alpha h_t^{-\alpha}}{\theta} + (1+\gamma)(1+\eta)\hat{K}_{t+1} = [1-\delta+(1-\tau^k)\alpha\hat{K}_t^{\alpha-1}h_t^{1-\alpha}]\hat{K}_t +& \\ +(1-\tau^h)(1-\alpha)\hat{K}_t^\alpha h_t^{1-\alpha}&\end{aligned} \tag{11}$$

Equations (10) and (11) are the ones used to solve the model in the computer. We also need to compute the steady state values of our variables (especially $\hat{K}^*$, which, along with the starting value of capital $\hat{K}_0$, allows us to find a solution to the system). This can be readily done. Denoting by an asterisk variables in steady state, note that evaluating (8) at steady state yields:

$$\frac{h^*}{\hat{K}^*} = \left[\frac{\frac{1+\gamma}{\beta}+\delta-1}{(1-\tau^k)\alpha}\right]^{\frac{1}{1-\alpha}} \tag{12}$$

Knowledge of this ratio allows us to compute h^*. Indeed, it follows from (11) that:

$$h^* = \left\{1+\frac{\theta}{(1-\tau_h)(1-\alpha)}\left[[1-\delta-(1+\gamma)(1+\eta)]\left(\frac{h^*}{\hat{K}^*}\right)^{\alpha-1}+\alpha(1-\tau_k)+(1-\alpha)(1-\tau_h)\right]\right\}^{-1} \tag{13}$$

Once these quantities are computed, one can easily calculate the steady-state value of the remaining variables (e.g. use (7) to find $\hat{c}^*$).

Problem 5 We solve parts of the model analytically. The solutions here are used as inputs to solve the model numerically.

Closed economy The closed-economy is described below. The consumer's problem is:

$$
\begin{aligned}
\max_{\{c_t, K_{t+1}, I_t\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t N_t \ln(c_t) \\
\text{s.t.} \quad & N_t c_t + I_t = r_t K_t + w_t N_t \\
& K_{t+1} = (1-\delta) K_t + I_t \\
& c_t, K_{t+1} \geq 0
\end{aligned}
$$

Firms' problem is given by:

$$
\max_{L_t, K_t} K_t^{\alpha} (A_t L_t)^{(1-\alpha)} - w_t L_t - r_t K_t
$$

Solving the above and using market clearing (in particular $N_t = L_t$), we are left with the following conditions, which characterise our economy:

$$
\frac{c_{t+1}}{c_t} = \beta(1 - \delta + r_{t+1}) \tag{14}
$$

$$
N_t c_t + K_{t+1} = [1 - \delta + r_t] K_t + w_t N_t \tag{15}
$$

$$
w_t = K_t^{\alpha} (1-\alpha)(A_t N_t)^{-\alpha} A_t \tag{16}
$$

$$
r_t = \alpha K_t^{\alpha - 1} (A_t N_t)^{1-\alpha} \tag{17}
$$

Using similar arguments as the ones in exercise 5, one may easily show that, in a balanced growth path, capital grows at rate gn, where $g := A_{t+1}/A_t$ and $n := N_{t+1}/N_t$ (I'm using the notation of the author). Furthermore, we have that, in a balanced growth path, consumption grows at rate g. One may then define $\tilde{k}_t = \frac{K_t}{A_t N_t}$ and $\tilde{c}_t = \frac{c_t}{A_t}$ and rewrite our system as:

$$
\frac{\tilde{c}_{t+1}}{\tilde{c}_t} = \frac{\beta}{g}(1 - \delta + r_{t+1}) \tag{18}
$$

$$
\tilde{c}_t + gn\tilde{k}_{t+1} = [1 - \delta + \alpha \tilde{k}_t^{\alpha-1}] \tilde{k}_t + (1-\alpha)\tilde{k}_t^{\alpha} \tag{19}
$$

$$
w_t = (1-\alpha)\tilde{k}_t^{\alpha} A_t \tag{20}
$$

$$
r_t = \alpha \tilde{k}_t^{\alpha-1} \tag{21}
$$

Where we already plugged the values of w_t and r_t into (19). Evaluating (18) at a steady-state, one then gets that:

$$
r^{ss} = \frac{g}{\beta} + \delta - 1 \tag{22}
$$

And combining it with (21) yields:

$$
\tilde{k}^{ss} = \left[\frac{\alpha}{\frac{g}{\beta} + \delta - 1} \right]^{\frac{1}{1-\alpha}} \tag{23}
$$

Finally, if we plug (19) and (21) into (18), we get a movement equation for $\tilde{k}_t$, which, along with $\tilde{k}_0$ and $\tilde{k}^{ss}$, allows us to solve the model numerically. Combining these, we get that:

$$\frac{-gn\tilde{k}_{t+2} + [1-\delta+\alpha\tilde{k}_{t+1}^{\alpha-1}]\tilde{k}_{t+1} + (1-\alpha)\tilde{k}_{t+1}^{\alpha}}{-gn\tilde{k}_{t+1} + [1-\delta+\alpha\tilde{k}_{t}^{\alpha-1}]\tilde{k}_{t} + (1-\alpha)\tilde{k}_{t}^{\alpha}} = \frac{\beta}{g}[1-\delta+\alpha\tilde{k}_{t+1}^{\alpha-1}] \tag{24}$$

Open economy Suppose now that we open the economy at $t = 0$. The economy is small and the rest of the world is in steady state. From $t = 0$ on, consumers may borrow or lend from the outside and there is free flow of capitals. Interest rates are thus equalised and it follows that the domestic rate is now equal to r^{ss} at all times. We then get from (21) that domestic capital immediately jumps so that $\tilde{k}_t = \tilde{k}^{ss}$ at all periods.

What is the consumer's problem now? Note that from $t = 0$ on, an agent may either invest in domestic capital or lend (borrow) from the outside. Since interest rates are equalised, the return of both activities is the same. We thus write B_t for the household overall assets and may then describe the consumer's problem as:

$$\begin{aligned}
\max_{\{c_t, B_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t N_t \ln(c_t) \\
\text{s.t.} \quad & \\
t = 0 \quad & N_0 c_0 + B_1 = (1-\delta+r^{ss})K_0 + w_0 N_0 \\
t > 0 \quad & N_t c_t + B_{t+1} = (1-\delta+r^{ss})B_t + w_t N_t
\end{aligned}$$

where we explicitly note that at $t = 0$ the consumer has no debt and only her capital stock accumulated. Note that this problem will lead to the same FOCs as before. In particular, we will have that the Euler equation is:

$$\frac{c_{t+1}}{c_t} = \beta(1-\delta+r^{ss}) \Rightarrow \frac{c_{t+1}}{c_t} = g \tag{25}$$

Thus, once we open up the economy, consumption at $t = 0$ (possibly) jumps, but from then on starts growing at rate g. Therefore, the only problem is to find c_0 (or $\tilde{c}_0$). We do so by iterating the agent's budget constraint. Note that we may rewrite the agent's constraint as:

$$\begin{aligned}
t = 0 \quad & \tilde{c}_0 + ng\tilde{b}_1 = (1-\delta+r^{ss})\tilde{k}_0 + (1-\alpha)(\tilde{k}^{ss})^{\alpha} \\
t > 0 \quad & \tilde{c}_t + ng\tilde{b}_{t+1} = (1-\delta+r^{ss})\tilde{b}_t + (1-\alpha)(\tilde{k}^{ss})^{\alpha}
\end{aligned}$$

where we already plug the value of w_t in the open economy and $\tilde{b}_t = \frac{B_t}{A_t N_t}$. Iterating forward and using the fact that $\tilde{c}_0 = \tilde{c}_t \ \forall t$ leads to:

$$\tilde{c}_0 \left[\sum_{t=0}^{\infty}\left(\frac{ng}{1-\delta+r^{ss}}\right)^t\right] = (1-\delta+r^{ss})\tilde{k}_0 + (1-\alpha)(\tilde{k}^{ss})^{\alpha}\left[\sum_{t=0}^{\infty}\left(\frac{ng}{1-\delta+r^{ss}}\right)^t\right] + \lim_{t\to\infty}\frac{(ng)^{t+1}\tilde{b}_{t+1}}{(1-\delta+r^{ss})^t}$$

Note that $\lim_{t\to\infty} \frac{(ng)^{t+1}\tilde{b}_{t+1}}{(1-\delta+r^{ss})^t} = (A_0 N_0)^{-1} \lim_{t\to\infty} \frac{B_{t+1}}{(1-\delta+r^{ss})^t} = 0$ by the no-Ponzi scheme condition. Do also note that, if $\beta n < 1$, then $\sum_{t=0}^{\infty} \left(\frac{ng}{1-\delta+r^{ss}}\right)^t = \sum_{t=0}^{\infty}(\beta n)^t = (1-\beta n)^{-1}$. We thus get that:

$$\tilde{c}_0 = \left(\frac{g}{\beta} - gn\right)\tilde{k}_0 + (1-\alpha)(\tilde{k}^{ss})^{\alpha} \tag{26}$$

Computing welfare differences Using the above results, we may solve the model and compute the Hicksian equivalent variation using the formula in the paper. To do this, we first need to compute the differences in utility between the open and closed economies, which are given by:

$$\Delta = U_{\text{open}} - U_{\text{closed}} = \sum_{t=0}^{\infty} \beta^t N_t \ln(c_t^{\text{o}}) - \sum_{t=0}^{\infty} \beta^t N_t \ln(c_t^{\text{c}}) \tag{27}$$

This simplifies to:

$$\begin{aligned} \Delta = \sum_{t=0}^{\infty} \beta^t N_t \ln(\tilde{c}_0^{\text{o}} A_t) - \sum_{t=0}^{\infty} \beta^t N_t \ln(\tilde{c}_t^{\text{c}} A_t) = \sum_{t=0}^{\infty} \beta^t N_t \ln(\tilde{c}_0^{\text{o}}) - \sum_{t=0}^{\infty} \beta^t N_t \ln(\tilde{c}_t^{\text{c}}) = \\ = \frac{N_0 \ln(\tilde{c}_0^{\text{o}})}{1-\beta n} - N_0 \sum_{t=0}^{\infty} \beta^t n^t \ln(\tilde{c}_t^{\text{c}}) \end{aligned}$$

Note that, in order to compute Δ, we would need to know N_0. We set it equal to 1 in our numerical exercise. Moreover, in order to compute the utility in the closed economy, one needs to assume the steady state from some T on. We set $T = 100$, which is the number of periods used to approximate the steady state in the numerical solution: after 100 periods, the closed economy is assumed to be in the steady state.