# Macroeconomics III
# Lecture 10: Continuous Time Models

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# Road Map

See Ben Moll's website (many lectures, codes, and articles). My lecture is closer to his lecture labeled **Hamilton-Jacobi-Bellman Equations**

http://www.princeton.edu/~moll/ECO2149_2018/Lecture3_2149.pdf

# Introduction: Continuous Time Models

Advantages:

- It can give closed form solutions even when they do not exist for the discrete time counterpart.
- It can be very fast to solve.

Disadvantages:

- Intuition is a bit tricky.
- Contraction Mapping Theorems/Convergence results might go out the window. The latter can create issues for numerical computing.

# The Solow Growth Model

The Solow growth model is represented by the following equations.

$$Y_t = K_t^{\alpha}(A_t N_t)^{1-\alpha},\ \alpha \in (0,1),$$

$$K_{t+1} = I_t + (1-\delta)K_t,\ \delta \in (0,1),\ \text{given } K_0,$$

$$A_{t+1} = (1+g)A_t,\ N_{t+1} = (1+n)N_t,\ \text{given } A_0,\ \text{and } N_0,$$

$$S_t = sY_t,\ \text{and } S_t = I_t.$$

Along the Balanced Growth Path (BGP) all variables growth at Constant rate. It can be shown that in this case:

$$\frac{K_{t+1}}{K_t} = (1+g)(1+n) = \frac{Y_{t+1}}{Y_t}.$$

**Define:** $k_t = \frac{K_t}{A_t N_t}$. Clearly $k_t$ is stationary along the BGP.

Then:

$$K_{t+1} = sK_t^{\alpha}(A_t N_t)^{1-\alpha} + (1-\delta)K_t.$$

Divide both sides by $A_t N_t$:

$$\frac{K_{t+1}}{A_t N_t} = sk_t^{\alpha} + (1-\delta)k_t,$$

$$(1+g)(1+n)k_{t+1} = sk_t^{\alpha} + (1-\delta)k_t.$$

Along the BGP, we have that $k_{t+1} = k_t = k$:

$$k = \left(\frac{s}{g+n+gn+\delta}\right)^{\frac{1}{1-\alpha}}.$$

# From Discrete Time to Continuous Time I

- Continuous time is not a state in itself, but is the effect of a limit. A derivative is a limit, an integral is a limit, the sum to infinity is a limit, and so on.
- Continuous time is the name we use for the behaviour of an economy as intervals between time periods approaches zero.
- The right approach is therefore to derive this behaviour as a limit

# From Discrete Time to Continuous Time II

- Suppose that the length of each time period was one month. Now we want to rewrite the model on a biweekly frequency.
- It seems reasonable to assume that in two weeks we produce half as much as we do in one month:

$$0.5Y_t = 0.5K_t^{\alpha}(A_tN_t)^{1-\alpha}.$$

- It also seems reasonable to assume that capital depreciates at a slower rate, i.e. $0.5\delta$.

# From Discrete Time to Continuous Time III

- We still have $N_t$ worker and $K_t$ units of capital: Stocks are not affected by the length of time intervals (although the accumulation of them will).
- The propensity to save is the same, but with half of the income saving is halved too (and therefore investment).
- What happens to the exogenous processes for $A_t$ and $N_t$.

# From Discrete Time to Continuous Time IV

We had:

$$A_{t+1} = (1+g)A_t, \text{ and } N_{t+1} = (1+n)N_t.$$

Now, we have:

$$A_{t+0.5} = (1+0.5g)A_t, \text{ and } N_{t+0.5} = (1+0.5n)N_t.$$

Suppose that the time period is not one month but $\Delta\times$ one month. Then:

$$A_{t+\Delta} = (1+\Delta g)A_t, \text{ and } N_{t+\Delta} = (1+\Delta n)N_t.$$

Rearrange:

$$\frac{A_{t+\Delta} - A_t}{\Delta} = gA_t, \text{ and } \frac{N_{t+\Delta} - N_t}{\Delta} = gN_t.$$

Taking limit $\Delta \to 0$: (and defining $\frac{\partial X}{\partial t} = \dot{X}$)

$$\dot{A} = gA \text{ (or } A = e^{gt}A(0)) , \quad \text{and } \dot{N} = gN \text{ (or } N = e^{gt}N(0)) .$$

# The Solow Growth Model (Again)

Model in $\Delta$ of one month:

$$A_{t+\Delta} = (1+\Delta g)A_t, \; N_{t+\Delta} = (1+\Delta n)N_t,$$

$$K_{t+\Delta} = s\Delta K_t^{\alpha}(A_t N_t)^{1-\alpha} + (1-\Delta\delta)K_t.$$

Divide both sides of the last equation by $A_t N_t$.

$$\frac{K_{t+\Delta}}{A_t N_t} = s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t, \;\; \text{and}$$

$$(1+\Delta g)(1+\Delta n)k_{t+\Delta} = s\Delta k_t^{\alpha} + (1-\Delta\delta)k_t.$$

Then:

$$k_{t+\Delta} - k_t = \Delta[sk_t^{\alpha} - \delta k_t] - \Delta(g+n+\Delta gn)k_t.$$

$$k_{t+\Delta} - k_t = \Delta[sk_t^{\alpha} - \delta k_t] - \Delta(g + n + \Delta gn)k_t,$$

Dividing both sides by $\Delta$:

$$\frac{k_{t+\Delta} - k_t}{\Delta} = sk_t^{\alpha} - (\delta + g + n + \Delta gn)k_t.$$

Taking limits when $\Delta \to 0$:

$$\dot{k}_t = sk_t^{\alpha} - (\delta + g + n)k_t.$$

Along the BGP $\dot{k}_t = 0$. Therefore:

$$k = \left(\frac{s}{\delta + g + n}\right)^{\frac{1}{1-\alpha}}.$$

# The Solow Growth Model: Solution

$$\dot{k}_t = sk_t^\alpha - (\delta + g + n)k_t.$$

- The equation above is an ODE.
- Declare it as a function with respect to time, $t$, and capital, $k$, in Matlab as

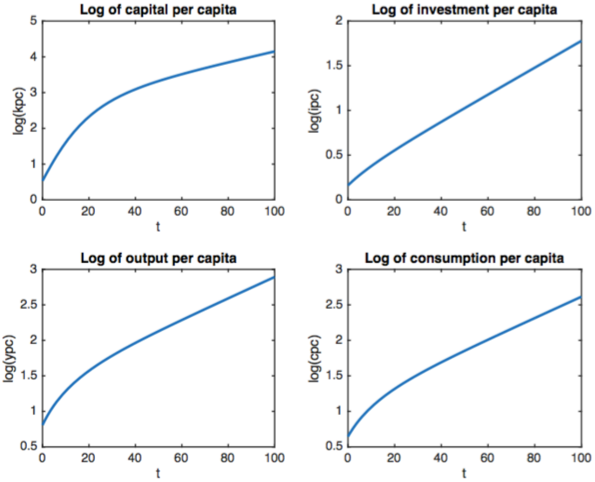
$$Solow = @(t,k)\ sk^\alpha - (\delta + g + n)k$$

- Then simulate it for, say 100 units of time, with initial condition $k_0$ as

$$[time, capital] = ode45(Solow, [0\ 100], k_0).$$

- The ODE function in Matlab uses the so-called Runge Kutta method to vary the step-size $\Delta$ in an optimal way.

# Transition Dynamics in the Solow Growth Model



# The Ramsey Growth Model

- Now consider the Ramsey growth model:

$$V(k_t) = \max_{c_t, k_{t+1}} \{u(c_t) + (1-\rho)V(k_{t+1})\},$$

subject to

$$c_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t.$$

- In $\Delta$ units of time:

$$V(k_t) = \max_{c_t, k_{t+\Delta}} \{\Delta u(c_t) + (1-\Delta\rho)V(k_{t+\Delta})\},$$

subject to

$$k_{t+\Delta} - k_t = \Delta[k_t^\alpha - \delta k_t - c_t].$$

# Some Comments

- Notice that all flows change when the length of the time period on which they are defined changes.
- Capital stock, $k_t$, is the same.
- The future is discounted at rate $(1 - \Delta\rho)$ instead of $(1 - \rho)$ (or with $e^{-\Delta\rho}$ instead of $e^{-\rho}$, but these are, in the limit, equivalent).

  Let $X_{t+1} = (1 - \rho)X_t$ then $X_{t+\Delta} = (1 - \Delta\rho)X_t$ and $X_{t+\Delta} - X_t = -\Delta\rho X_t$. Dividing by $\Delta$ and taking limits imply that $\dot{X} = -\rho X$, or $X(t) = e^{-\rho}X(0)$.

# The Ramsey Growth Model

- We have:

$$V(k_t) = \max_{c_t, k_{t+\Delta}} \{\Delta u(c_t) + (1 - \Delta\rho)V(k_{t+\Delta})\},$$

subject to

$$k_{t+\Delta} = \Delta[k_t^\alpha - \delta k_t - c_t] + k_t.$$

- Subtract $V(k_t)$ from both sides:

$$0 = \max_{c_t}\{\Delta u(c_t) + V(\Delta[k_t^\alpha - \delta k_t - c_t] + k_t) - V(k_t)$$

$$-\Delta\rho V(\Delta[k_t^\alpha - \delta k_t - c_t] + k_t)\}.$$

- Divide both sides by $\Delta$

$$0 = \max_{c_t}\{u(c_t) + \frac{V(\Delta[k_t^\alpha - \delta k_t - c_t] + k_t) - V(k_t)}{\Delta}$$

$$-\rho V(\Delta[k_t^\alpha - \delta k_t - c_t] + k_t)\}.$$

# The Ramsey Growth Model

- Taking limits for $\Delta \to 0$ and rearranging terms:

$$\rho V(k_t) = \max_{c_t}\{u(c_t) + V'(k_t)(k_t^\alpha - \delta k_t - c_t)\}.$$

  This is know as the **Hamilton-Jacobi-Bellman (HJB) Equation.**

- Dropping time notation we have:

$$\rho V(k) = \max_{c}\{u(c) + V'(k)(k^\alpha - \delta k - c)\}.$$

# Solution

- Equation

$$\rho V(k) = \max_c \{u(c) + V'(k)(k^\alpha - \delta k - c)\}$$

can be very fast to solve. First-order condition:

$$u'(c) = V'(k).$$

- So if we know $V'(k)$ we know optimal $c$ without searching for it!

# How Do We Find $V'(k)$?

- Suppose we have hypothetical values of $V(k)$ on a uniformly spaced grid of $k$, $\mathbf{K} = \{k_1, k_2 ..., k_N\}$ with stepsize $\Delta k$.
- We can then approximate $V'(k)$ at gridpoint $k_i$ for $i \neq 1, N$ as:

$$V'(k_i) = \frac{1}{2}\left(\frac{V(k_{i+1}) - V(k_i)}{\Delta k}\right) + \frac{1}{2}\left(\frac{V(k_i) - V(k_{i-1})}{\Delta k}\right).$$

$$V'(k_i) = \frac{V(k_{i+1}) - V(k_{i-1})}{2\Delta k}, \ \forall i \neq 1, N.$$

- For $k_1$ and $k_N$, we would have

$$V'(k_1) = \frac{V(k_2) - V(k_1)}{\Delta k} \ \text{ and } \ V'(k_N) = \frac{V(k_N) - V(k_{N-1})}{\Delta k},$$

respectively.

# How Do We Find $V'(k)$?

- There are many ways of doing this. If you have a vector of $V(k)$ values - call it $V$ - then $dV = gradient(V)/dK$.
- Construct a matrix $D$ as:

$$D = \begin{pmatrix} -1/dk & 1/dk & 0 & 0 & \cdots & 0 \\ -0.5/dk & 0 & 0.5/dk & 0 & \cdots & 0 \\ 0 & -0.5/dk & 0 & 0.5/dk & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1/dk & 1/dk \end{pmatrix}$$

- Then $V'(k) \approx D \times V(k)$.

# The Ramsey Growth Model: Solving

**Algorithm:**

1. Construct a grid for k.
2. For each point on the grid, guess a value of $V_0$.
3. Calculate the derivative as $dV_0 = D \times V_0$.
4. Find $V_1$ from

$$\rho V_1(k) = u(c) + dV_0(k)(k^\alpha - \delta k - c)$$

$$\text{with: } u'(c) = dV_0(k).$$

5. Check whether or not $V_1 \approx V_0$, if not go to step 3 until convergence.

# The Ramsey Growth Model: Solving

**Caveat:** The Contraction Mapping Theorem does not work, so convergence is an issue. **Solution:** Update slowly, i.e $V_1 = \gamma V_1 + (1-\gamma)V_0$, for a small value for $\gamma$.

$$\rho(\gamma V_1(k) + (1-\gamma)V_0(k)) = u(c) + dV_0(k)(k^\alpha - \delta k - c)$$

$$\text{with: } u'(c) = dV_0(k).$$

$$V_1(k) = \frac{1}{\rho\gamma}\left(u(c) + dV_0(k)(k^\alpha - \delta k - c)\right) - \frac{(1-\gamma)}{\gamma}V_0(k).$$

$$V_1(k) = \frac{1}{\rho\gamma}\left(u(c) + dV_0(k)(k^\alpha - \delta k - c) - \rho V_0(k)\right) + V_0(k).$$

$$\text{with: } u'(c) = dV_0(k).$$

# The Ramsey Growth Model: Euler Equation

- Let's go back to the Hamilton-Jacobi-Bellman (HJB) equation

$$\rho V(k) = u(c) + V'(k)(k^\alpha - \delta k - c)$$

$$\text{with } \; u'(c) = V'(k).$$

- Then:

$$\rho V'(k) = V''(k)(k^\alpha - \delta k - c) + V'(k)(\alpha k^{\alpha-1} - \delta), \; \text{ and}$$

$$V''(k) = u''(c)c'(k).$$

- Therefore:

$$\rho u'(c) = u''(c)c'(k)(k^\alpha - \delta k - c) + u'(c)(\alpha k^{\alpha-1} - \delta), \; \text{ or}$$

$$-u''(c)c'(k)(k^\alpha - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho).$$

# The Ramsey Growth Model: Euler Equation

$$-u''(c)c'(k)(k^\alpha - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho).$$

- Suppose CRRA utility, such that $\frac{u''(c)c}{u'(c)} = \sigma$. Then:

$$\sigma \frac{c'(k)}{c}\dot{k} = (\alpha k^{\alpha-1} - \delta - \rho).$$

- Observe that:

$$\dot{c} = \frac{\partial c}{\partial t} = \frac{\partial c}{\partial k}\frac{\partial k}{\partial t} = c'(k)\dot{k}.$$

- Then:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}(\alpha k^{\alpha-1} - \delta - \rho).$$

# The Ramsey Growth Model: Dynamics

- Two Equations:

$$
\begin{aligned}
\dot{c} &= \frac{c}{\sigma}(\alpha k^{\alpha-1} - \delta - \rho), \\
\dot{k} &= k^{\alpha} - \delta k - c.
\end{aligned}
$$

- Steady-state

$$
\begin{aligned}
0 &= \alpha k^{\alpha-1} - \delta - \rho, \\
0 &= k^{\alpha} - \delta k - c.
\end{aligned}
$$

- Two boundary conditions: $k_0$ and $\lim_{T\to\infty} e^{-\rho T} u'(c_T) k_T = 0$.

# Back to Euler Equation

$$-u''(c)c'(k)(k^{\alpha} - \delta k - c) = u'(c)(\alpha k^{\alpha-1} - \delta - \rho).$$

$$\frac{c'(k)}{c}(k^{\alpha} - \delta k - c) = \frac{1}{\sigma}(\alpha k^{\alpha-1} - \delta - \rho).$$

Solve for $c$:

$$c = \frac{c'(k)(k^{\alpha} - \delta k)}{\frac{1}{\sigma}(\alpha k^{\alpha-1} - \delta - \rho) + c'(k)}.$$

# The Ramsey Growth Model: Euler Equation Solution

**Algorithm:**

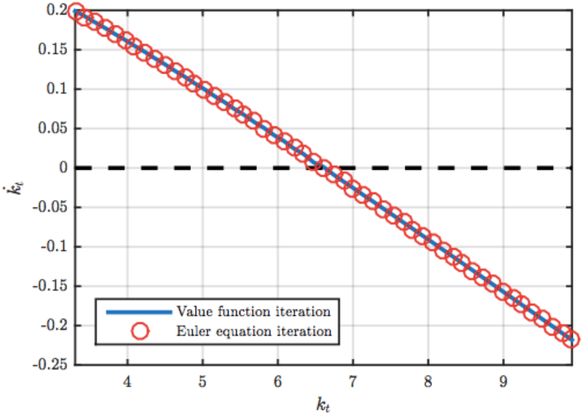
1. Construct a grid for $k$.
2. For each point on the grid, guess for a value of $c_0$.
3. Calculate the derivative as $dc_0 = D \times c_0$.
4. Find $c_1$ from

$$c_1 = \frac{dc_0(k^\alpha - \delta k)}{\frac{1}{\sigma}(\alpha k^{\alpha-1} - \delta - \rho) + dc_0}.$$

5. Check if $c_1 \approx c_0$.
6. If not, back to step 3 with $c_1$ replacing $c_0$. Repeat until convergence.

**Caveat:** No guaranteed convergence. Update slowly.

# The Ramsey Growth Model: Solution



# The Ramsey Growth Model: Solution