Macroeconomics III
Lecture 7: Incomplete Markets

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Road Map

Huggett (1993, JEDC) model.

Introduction

- Dynamic Stochastic General Equilibrium models span a very wide spectrum of ideas:
- But two quite different strands can be detected:

1. Representative agent models without occasionally binding constraints, usually solved by linearization (or perturbation) methods, and can straigtforwardly be estimated (with, for instance, Dynare).
2. Incomplete market models, or Heterogenous Agents models, which commonly have occasionally binding constraints, "must" be solved by nonlinear methods, and are tricky to estimate (but not impossible).

A Familiar Problem

- Consider a measure one of agents and the following two-period problem:

$$\max_{\{c_0, c_1(\theta), a(\theta)\}_{\theta\in\Omega}} \{u(c_0) + \beta \sum_{\theta\in\Omega} u(c_1(\theta)) Pr(\theta)\},$$

subject to

$$c_0 + \sum_{\theta\in\Omega} q(\theta)a(\theta) = \omega_0,$$

$$c_1(\theta) = \omega_1(\theta) + a(\theta).$$

- θ is idiosyncratic: $\sum_{\theta\in\Omega} \omega_1(\theta) Pr(\theta) = \omega_1$.

A Familiar Problem

- First-order conditions:

$$u'(c_0)q(\theta) = \beta u'(c_1(\theta))Pr(\theta),\ \forall \theta \in \Omega.$$

$$u'(c_0) = \frac{\beta u'(c_1(\theta))Pr(\theta)}{q(\theta)} = \frac{\beta u'(c_1(\theta'))Pr(\theta')}{q(\theta')},\ \forall \theta, \theta \in \Omega.$$

- In equilibrium:

$$\int_0^1 c_0 = \omega_0,\ \int_0^1 c_1(\theta)Pr(\theta) = \omega_1,$$

$$q(\theta) = \frac{\beta u'(\omega_1)}{u'(\omega_0)}Pr(\theta).$$

- Such that $c_0 = \omega_0$ and $c_1(\theta) = \omega_1$.

Another Familiar Problem

- Consider the following two-period problem

$$\max_{c_0, c_1, a} \{u(c_0) + \beta u(c_1)\},$$

subject to

$$c_0 + qa = \omega_0,$$

$$c_1 = \omega_1 + a.$$

Another Familiar Problem

- First-order conditions:

$$u'(c_0)q = \beta u'(c_1).$$

- Then there exist equilibrium asset prices

$$q = \frac{\beta u'(\omega_1)}{u'(\omega_0)}.$$

- Such that $c_0 = \omega_0$ and $c_1 = \omega_1$.

What does this mean?

- The first problem: Loads of agents and loads of outcomes.
- Lots of trade in assets.
- The second problem: One agent, one outcome.
- No trade in assets.
- But same aggregate outcome!

What does this mean?

- A representative agent is not one agent.
- A representative agent does not exclude trade - it just occurs under the hood.
- Another word for representative agent models: Complete markets models.

What does this mean?

- But some time we wish to depart from representative agent:
  - Distributions and Aggregation matter.
  - Precautionary risk/savings from incomplete markets.
  - Could lead to interesting dynamics.
  - There is no full-insurance in the world!

The Huggett (1993) Model

- **The model of Hugget** basically takes a standard bond economy as the main starting point.
- In similarity to representative agent models, the continuum of agents are still ex-ante identical, but, importantly, they are now ex-post heterogenous.
- In particular, the agents are hit by only *partially insurable, idiosyncratic shocks*.

The Huggett (1993) Model

- One-period obligation contracts is the only source of insurance (*bonds*).
- That is, there are many more goods (states), than markets (incomplete markets models).
- There are no aggregate shocks to the economy, and therefore no aggregate risk (something that is relaxed in Krusell and Smith's (1998) model).

Huggett (1993, JEDC) I

Consider the following problem:

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to:

$$c_t + a_{t+1} = (1 + r_t)a_t + w_t z_t, \; c \geq 0$$

$$z \sim \text{a Markov Process.}$$

- $V(a, z) = \max_a' \{u(((1 + r)a + wz - a') + \beta E[V(a', z')/z = \hat{z}]\}$. For each $i = 1, ..., n$:

$$V(a, z_i) = \max_{a'} \{u(((1 + r)a + wz_i - a') + \beta \sum_{j=1}^{n} P_{ij} V(a', z_j)\}.$$

- Agent wants: $a' = -\infty$.

Huggett (1993, JEDC) II

- Ad-Hoc borrowing limit:

$$a' \geq \underline{a} > -\infty.$$

$$V(a, z_i) = \max_{\underline{a} \leq a' \leq \bar{a}} \{u(((1+r)a + wz_i - a') + \beta \sum_{j=1}^{n} P_{ij} V(a', z_j)\}.$$

- Euller equation: $u'(c_t) = \beta(1+r)E[u'(c_{t+1})]$. In the deterministic case:
  - If $\beta(1+r) = 1$, $c_t = c_{t+1} = c$;
  - If $\beta(1+r) < 1$, $c_t > c_{t+1}$;
  - If $\beta(1+r) > 1$, $c_t < c_{t+1}$.
- If $\beta(1+r) \leq 1$, we can find an $\bar{a} < \infty$ such that $a' \in (\underline{a}, \bar{a})$. But if $\beta(1+r) > 1$, there does not exist $\bar{a} < \infty$, such that $a' \in (\underline{a}, \bar{a})$.

Huggett (1993, JEDC) III

- This problem

$$V(a, z_i) = \max_{\underline{a} \leq a' \leq \bar{a}} \{u(((1+r)a + wz_i - a') + \beta \sum_{j=1}^{n} P_{ij} V(a', z_j)\}.$$

- You can easily show that this is a **Contraction Mapping**.
- $\Rightarrow$ Standard DP algorithm gives optimal policy function: $a' = g(a, z)$ and $c = g_c(a, z)$.

Huggett (1993, JEDC) IV

- Ok, so suppose that we have found $g(a,z)$ conditional on some r, now what?
- As there are idiosyncratic risk, each individual will be exposed to different shocks in different periods.
- An individual is identified through his history of shocks.
- Clearly, as the history of shocks affect an individual's wealth, and individuals have experienced different types of histories, there will be a cross-sectional distribution of wealth-holdings.
- We will denote the distribution of (a,z) at t as $\lambda_t(a,z)$.

Huggett (1993, JEDC) V

Law of Motion of the wealth-shock distribution:

- Unconditional distribution of (a_t, z_t) is $\lambda_t(a_t, z_t)$

$$\lambda_t(a_t, z_t) = Pr(a_t, z_t)$$

- Example: Suppose $a_t \in [a_1 < a_2]$ and $z_t \in [z_1 < z_2]$.

$$\begin{aligned}
&Pr(a_{t+1} = a_1, z_{t+1} = z_1) = \\
&Pr(a_{t+1} = a_1/a_t = a_1, z_t = z_1)Pr(z_{t+1} = z_1/z_t = z_1)Pr(a_t = a_1, z_t = z_1) + \\
&Pr(a_{t+1} = a_1/a_t = a_1, z_t = z_2)Pr(z_{t+1} = z_1/z_t = z_2)Pr(a_t = a_1, z_t = z_2) + \\
&Pr(a_{t+1} = a_1/a_t = a_2, z_t = z_1)Pr(z_{t+1} = z_1/z_t = z_1)Pr(a_t = a_2, z_t = z_1) + \\
&Pr(a_{t+1} = a_1/a_t = a_2, z_t = z_2)Pr(z_{t+1} = z_1/z_t = z_2)Pr(a_t = a_2, z_t = z_2).
\end{aligned}$$

Therefore:

$$Pr(a_{t+1} = a_1, z_{t+1} = z_1) = \sum_i \sum_j Pr(a_{t+1} = a_1/a_t = a_i, z_t = z_j)Pr(z_{t+1} = z_1/z_t = z_j)Pr(a_t = a_i, z_t = z_j).$$

Huggett (1993, JEDC) VI

Law of Motion of the wealth-shock distribution:

$Pr(a_{t+1} = a_1, z_{t+1} = z_1) = \sum_{a_t} \sum_{z_t} Pr(a_{t+1} = a_1/a_t = a_i, z_t = z_j)Pr(z_{t+1} = z_1/z_t = z_j)Pr(a_t = a_i, z_t = z_j)$.

$$\lambda_{t+1}(a', z') = \sum_{a} \sum_{z} \lambda_t(a', z')Pr(z', z)I(a', a, z),$$

Where: $I(a', a, z) = 1$ if $a' = g(a, z)$ and $I(a', a, z) = 0$ otherwise.

$$\lambda_{t+1}(a', z') = \sum_{z} \sum_{\{a : a' = g(a,z)\}} \lambda_t(a', z')Pr(z', z).$$

A time invariant distribution is one such that $\lambda_{t+1} = \lambda_t = \lambda$.

$$\lambda(a', z') = \sum_{z} \sum_{\{a : a' = g(a,z)\}} \lambda(a', z')Pr(z', z).$$

Huggett (1993, JEDC) VII

- One can show that under quite weak assumptions, λ_t (and for any λ_0) converges to a unique stationary distribution λ such that,

$$\lambda(a', z') = \sum_{z} \sum_{\{a:a'=g(a,z)\}} \lambda(a', z') Pr(z', z).$$

- This stationary distribution is very important to us.
- Given a constant interest-rate, the optimal household decision yields a stationary distribution with a constant excess-demand for bonds.
- Moreover, even if aggregates are constant (aggr. wealth, consumption, endowments etc.) individual specific variables are not: Agents jump frequently around in the distribution, but aggregates never change.

Huggett (1993, JEDC) VIII

The problem induces an endogenous Markov Chain:

$Pr(a_{t+1} = a', z_{t+1} = z'/a_t = a, z_t = z) = Pr(a_{t+1} = a'/a_t = a, z_t = z)Pr(z_{t+1} = z'/z_t = z) = I(a', a, z)P(z', z) = Q.$

This formula defines an $N \times N$ matrix, where:
N= Number of states of z$\times$Number of grid points of a.

DEFINITION: A stationary equilibrium is an interest rate, r, a policy function, $g(a, z)$, and a stationary distribution $\lambda(a, z)$, such that:

1. The policy function $g(a, z)$ solves $V(a, z)$;
2. The loan market clears:

$$\sum_{z,a} \lambda(a, z) g(a, z) = 0, \; (\sum_{z,a} \lambda(a, z) g_c(a, z) = w).$$

3. The stationary distribution $\lambda(a, z)$ is induced by (P, z) and $g(a, z)$:

$$\lambda(B) = \sum_{X=[\underline{a},\bar{a}]\times Z \in B} Q(X, B).$$

Huggett (1993, JEDC) IX

Solution algorithm:

1. Guess $r = r_j$
2. Solve household's problem using dynamic programming to find $g_j(a, z)$ and find $\lambda_j(a, z)$
3. Compute
$$e = \sum_{z,a} \lambda_j(a, z) g_j(a, z).$$
4. If $e > \epsilon$, update $r_{j+1} < r_j$ (if $e < \epsilon$, update $r_{j+1} > r_j$) and go back to step (1). If $|e| < \epsilon$ stop.

Computing issues

- Sounds easy, right? Intuitively it is indeed, easy. But,...
- Not necessarily! Calculating the cross-sectional distribution can be a pain.
- We will discuss two ways:
  - **Discretisation:** approximate the distribution function on a discrete number of grid points over the assets;
  - **Montecarlo simulation:** we take a sample of households and we track them over time.
- Let's digress a little and enter the wonderful world of calculating cross-sectional distributions.

Digression on Computing Distributions I

- When solving heterogenous models, we will often encounter stochastic variables with law of motion described by a probability density function

$$\lambda(\theta_{t+1}, \theta_t).$$

- **Interpretation:** The probability of θ_{t+1} occurring tomorrow given θ_t today.

- Quite general: θ can be a vector containing lagged values. θ can contain exogenous and endogenous variables.

Digression on Computing Distributions II

- What is the density of θ_{t+2} given θ_t?

$$\lambda(\theta_{t+2}, \theta_t) = \int_{\theta_{t+1}} \psi(\theta_{t+2}, \theta_{t+1})\lambda(\theta_{t+1}, \theta_t).$$

- What is the density of θ_{t+3} given θ_t?

$$\lambda(\theta_{t+3}, \theta_t) = \int_{\theta_{t+2}} \psi(\theta_{t+3}, \theta_{t+2})\lambda(\theta_{t+2}, \theta_t).$$

- In general

$$\lambda(\theta_{t+n}, \theta_t) = \int_{\theta_{t+n-1}} \psi(\theta_{t+n}, \theta_{t+n-1})\lambda(\theta_{t+n-1}, \theta_t).$$

Digression on Computing Distributions II

- Many times we are interested in the unconditional, or long-run, density

$$\lambda(\theta) = \lim_{n\to\infty} \lambda(\theta_{t+n}, \theta_t).$$

- This density must satisfy:

$$\lambda(\theta') = \int_{\theta'} \psi(\theta', \theta)\lambda(\theta).$$

Transition Matrices

- Transition matrix:

$$T := \begin{bmatrix} \psi(\theta_1, \theta_1) & \cdots & \psi(\theta_N, \theta_1) \\ \vdots & \ddots & \vdots \\ \psi(\theta_1, \theta_N) & \cdots & \psi(\theta_N, \theta_N) \end{bmatrix}$$

- **Each row must sum to one!**
- What is the distribution of θ_{t+1} given $\theta_t = \theta_j$?
- Its of course given by row j of T.

Distributions

- Let v_0 be a $1 \times N$ vector, with zeros everywhere apart from element j, where its one. Then given $\theta_t = \theta_j$:

$$\psi(\theta_{t+1}) = v_0 T = v_1.$$

- This makes things really really simple:

$$\psi(\theta_{t+n}) = v_{n-1} T = v_n.$$

- And the long-run unconditional distribution must solve:

$$vT = v \;\Rightarrow\; (T - I)v = 0.$$

- That is, v is the eigenvector associated with a unit eigenvalue (normalised to sum to one).

- Two ways of calculating it:
  - Solve

$$(T - I)v = 0.$$

  - Or iterate

$$v_{n-1}T = v_n$$

    and stop when v_n is close to v_{n-1}.

Transition Matrix: Example

- The job finding probability, f, in the United States is around 0.4 per month.
- How do I know?
  - The unemployment duration is around 2.5 months.
- Unemployment duration and job finding probability:

$$
\begin{aligned}
Duration &= 1 \times f + 2 \times (1-f)f + 3 \times (1-f)^2 f + \ldots \\
&= f[1 + 2 \times (1-f) + 3 \times (1-f)^2 + \ldots] \\
&= f \frac{\partial}{\partial(1-f)}[(1-f) + (1-f)^2 + (1-f)^3 + \ldots] \\
&= f \frac{\partial}{\partial(1-f)} \left(\frac{(1-f)}{1-(1-f)} \right) \\
&= \frac{1}{f}.
\end{aligned}
$$

- The separation rate in the United States is 3.4% (data)
- Transition matrix (employed (1) and unemployed (2)):

$$T := \begin{bmatrix} 0.966 & 0.034 \\ 0.4 & 0.6 \end{bmatrix}.$$

- Long run distribution:

$$v = [0.9217, 0.0783].$$

Back to the Huggett Model: Discretisation I

- Suppose 5 states for a and policy functions as

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}$$

- Can be written as transition matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Back to the Huggett Model: Discretisation II

- But in the Huggett model (and many others) we normally have two (one for each state) policy functions:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \underset{\text{if good state}}{\rightarrow} \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \\ 4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \underset{\text{if bad state}}{\rightarrow} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}$$

- With some transition matrix for good and bad states:

$$T = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

Back to the Huggett Model: Discretisation III

- Two transition matrices

$$M_g = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad M_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Back to the Huggett Model: Discretisation IV

- Full transition matrix is given by:

$$\begin{bmatrix} T(1,1)M_g & T(1,2)M_g \\ T(2,1)M_b & T(2,2)M_b \end{bmatrix}$$

- Endogenous transition matrix:

$$M = \begin{bmatrix} 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0.7 & 0 \end{bmatrix}$$

Back to the Huggett Model: Montecarlo Simulation

- Choose a sample size of Q individuals ($Q \simeq 1000$);
- Initialize each individual i with an initial asset holding a_i^0 and a productivity shock z_i
- Compute $a_i' = g(a_i, z_i)\ \forall i = 1, .., Q$;
- Generate the next period productivity shock $z_i'\ \forall i = 1, .., Q$;
- Calculate a set of statistics of the distribution of z and a (average and standard deviation);
- Iterate until convergence on the statistics.

Recall the Solution Algorithm

Solution algorithm:

1. Guess $r = r_j$
2. Solve household's problem using dynamic programming to find $g_j(a, z)$ and find $\lambda_j(a, z)$
3. Compute
$$e = \sum_{z,a} \lambda_j(a, z) g_j(a, z).$$
4. If $e > \epsilon$, update $r_{j+1} < r_j$ (if $e < \epsilon$, update $r_{j+1} > r_j$) and go back to step (1). If $|e| < \epsilon$ stop.

- How to adjust the Interest rate?
- Asset demand:

$$e = \sum_{z,a} \lambda_j(a,z) g_j(a,z).$$

  - e is an increasing function of r: $e(r)$.
  - We also know that $e(\frac{1}{\beta} - 1) \geq 0$, and, hopefully $e(0) < 0$.
  - Luckily, $e(r)$ also happens to be continuous.
  - An ideal numerical procedure to find $e(r) = 0$ is the bisection method.