Macroeconomics III
Lecture 4: Discretization of State Spaces and Value Function Iterations

Tiago Cavalcanti

FGV/EESP
São Paulo

Road Map

1. Discretization of the state space.
2. Approximation of AR(1) processes into a Markov Chain.
3. Value Function Iterations.

RBC Model

$$
\begin{aligned}
\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \quad & E_0\left(\sum_{t=0}^{\infty} \beta^t u(c_t)\right) \\
\text{subject to} & \\
k_{t+1} + c_t &= z_t f(k_t) + (1-\delta)k_t, \\
\ln(z_{t+1}) &= \rho_z \ln(z_t) + \nu_{t+1},\ \nu_{t+1} \sim iidN(0, \sigma_\nu^2) \\
c_t &\geq 0
\end{aligned}
$$

Recursive Representation

- Remeber our RBC model:

$$V(k,z) = \max_{c,k'}\{u(c) + \beta E_z[V(k',z')]\}$$

subject to

$$c + k' = zf(k) + (1-\delta)k,$$

$$z' \sim \Gamma(z).$$

Iteration on Value Function

- Basic Idea:
  - Start with an initial guess for the value function, $V^0(k, z)$.
  - Maximize the right hand side and compute $V^1(k, z)$.
  - Repeate procedure until $||V^j(k, z) - V^{j-1}(k, z)|| < \epsilon$, where ϵ is a small number.
- What is the simplest way to solve this?
  - Discretize everything!
  - Create grids for the states and for the controls

With Discretized States and Controls

$$V(k,z) = \max_{c,k'}\{u(c) + \beta \sum_{z'} \pi(z'/z)V(k',z')\}$$

subject to

$$c + k' = zf(k) + (1-\delta)k,$$

for all $k \in \mathbf{K}$ and $z \in \mathbf{Z}$.

Questions:

- What are the relevant k and z? $\mathbf{K} \times \mathbf{Z}$.
- How can V be represented on a computer? As an array.
- How does one compute $E[z]$? Markov chain.

What is an appropriate/optimal discretization of $\mathbf{K}$ and $\mathbf{Z}$?

Discretizing the Grid for k

Questions:

- What are appropriate lower and upper bounds?
- How many points should there be and where?

Bounds for the k Grid

Upper and lower bounds $[\underline{k}, \bar{k}]$ for $\mathbf{K}$

- Many models have an explicit lower bound. There may also be an endogenous lower bound.
- Most models do not have an explicit upper bound instead relying on an endogenous upper bound.

Quantity and Location of the k Grid Points

Where should the points be?

- Linearly spaced (most common): often good starting point. In matlab, two options:
  - $k = 1 : 0.1 : 10$; generates a grid for k, such that $k \in \{1, 1.1, 1.2, ..., 10\}$ (size of grid will be 91);
  - $k = linspace(1, 10, 91)$ produces the same grid.

How many points should there be?

- As many as you can handle? Maybe too much.
- Ideally, adding points would not change your results.

Discretizing an AR(1) into a Markov Chain

- Consider the following AR(1) process:

$$z_t = (1 - \rho)\mu + \rho z_{t-1} + \eta_t,$$

- z_t is a continuous variable and η_t is a white-noise.

- Task: Approximate the continuous AR(1) process with a discrete first-order Markov chain.

- Aim: Discretize z so that the resulting process resembles the continuous AR(1) process.

Tauchen's Method

- Determine the range of the grid and the location and the number of the grid points;
- Choice of grid size depends on the persistence and the variance of the underlying continuous process;
- Higher variance $\sigma^2 \implies$ a larger range of the state space needs to be covered by the grid.
- The same for persistence, ρ.

Specifying the Grid

- Consider the following AR(1) process:

$$z_t = \mu(1-\rho) + \rho z_{t-1} + \eta_t$$

where $\eta_t \sim N(0, \sigma^2)$. The range of z_t is the real line and

$$\mu_z = \mu, \;\; \sigma_z = \frac{\sigma}{\sqrt{1-\rho^2}}.$$

- **Objective**: Discretise the range (or state space) of z_t into points $z_i, \; i = 1, ..., N$ and give each point z_j an approximate probability of occuring π_{ij}, given that the previous period state was z_i. The matrix $\Pi = [\pi_{ij}]$ consitutes the transition matrix.

Specifying the Grid

- Define the two bounds z_1 and z_N by:

$$z_1 = \mu - r\sqrt{\frac{\sigma^2}{1-\rho^2}}, \;\; z_N = \mu + r\sqrt{\frac{\sigma^2}{1-\rho^2}}$$

- Then $z_2, ..., z_{N-1}$ are defined by an **equispaced grid** of $[z_1, z_N]$:

$$d = \frac{z_N - z_1}{N-1} = \frac{2r\sigma_z}{N-1}$$

$$z_i = z_1 + (i-1)\, d = z_1 + (i-1)\,\frac{2r\sigma_z}{N-1}$$

- r is a scaling parameter.

Computing the Transition Probabilities

- Create the borders of each interval $[z_i, z_{i+1}]$

$$m_i = \frac{z_{i+1} + z_i}{2} = z_1 + (2i - 1)\frac{d}{2} = z_i + \frac{d}{2}$$

and

$$z_i \in \begin{cases} (-\infty, m_1] & \text{if } i = 1 \\ (m_{i-1}, m_i] & \text{if } 1 < i < N \\ (m_{N-1}, \infty) & \text{if } i = N \end{cases}$$

- If $j = 2, ..., N - 1$

$$\begin{aligned} \pi_{ij} &= \Pr(z_{t+1} = z_j | z_t = z_i) = \Pr\left(\mu(1-\rho) + \rho z_i + \eta_{t+1} = z_j\right) \\ &\approx \Pr\left(m_{j-1} \leq \mu(1-\rho) + \rho z_i + \eta_{t+1} \leq m_j\right) \\ &= \Phi\left(\frac{m_j - \rho z_i - \mu(1-\rho)}{\sigma}\right) - \Phi\left(\frac{m_{j-1} - \rho z_i - \mu(1-\rho)}{\sigma}\right) \end{aligned}$$

Computing the Transition Probabilities

- If $j = 1$

$$
\begin{aligned}
\pi_{i1} &= \Pr(z_{t+1} = z_1 | z_t = z_i) \\
&= \Pr\left(\mu(1-\rho) + \rho z_t + \eta_{t+1} = z_1 | z_t = z_i\right) \\
&= \Pr\left(\mu(1-\rho) + \rho z_i + \eta_{t+1} = z_1\right) \\
&\approx \Pr\left(\mu(1-\rho) + \rho z_i + \eta_{t+1} \leq m_1\right) \\
&= \Phi\left(\frac{m_1 - \rho z_i - \mu(1-\rho)}{\sigma}\right)
\end{aligned}
$$

- If $j = N$

$$
\begin{aligned}
\pi_{iN} &= \Pr(z_{t+1} = z_N | z_t = z_i) \\
&= 1 - \Phi\left(\frac{m_{N-1} - \rho z_i - \mu(1-\rho)}{\sigma}\right)
\end{aligned}
$$

Approximating an AR(1) Process with a Markov Chain

- The result of this discretization is a grid $\{z_i\}_{i=1}^N$ and the transition probability matrix Π.

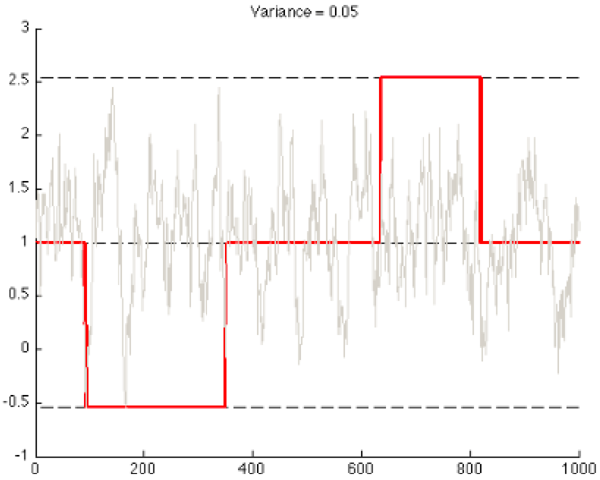
- **Example**: $N = r = 3$, $\mu = 1$, $\rho = 0.9$, $\sigma^2 = 0.05$. Then:

  $z_i \in \{-2.54, 1, 2.54\}$

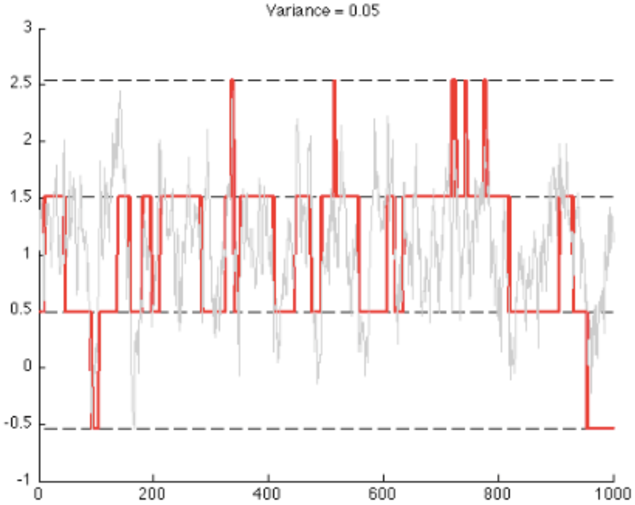
$$\Pi = \begin{pmatrix} 0.997 & 0.003 & 0 \\ 0.0003 & 0.9994 & 0.0003 \\ 0 & 0.003 & 0.997 \end{pmatrix}$$

- Let's look at the time path of the simulated Markov chain (for $N = 3, 4, 5, 10$).
- Tauchen makes the case (numerically) that the approximation is adequate for most purposes when $N = 9$ and $r = 3$.

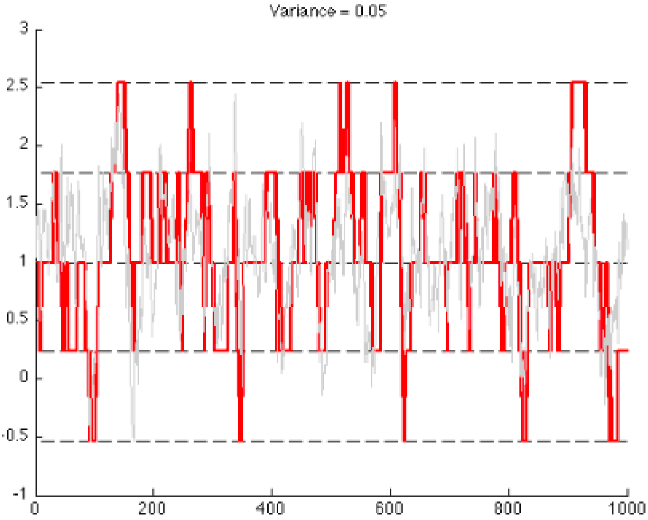
N=3



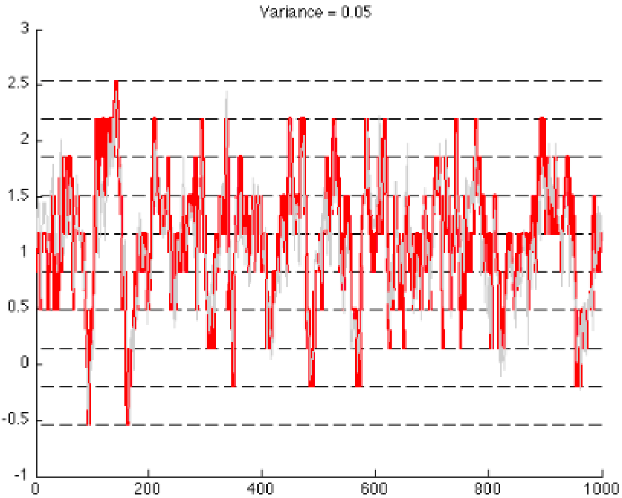
N=4



N=5



N=10



Comments

- We need to choose both N (number of grid points) and r. Tauchen (1986) shows (numerically) that $r = 3$ and $n = 9$ do a good job for AR(1) processes.

Simulating a Markov Chain

- Suppose that $\{z_t\}$ is a Markov Chain with the state space $\mathbf{Z} = \{z^1, z^2, ..., z^N\}$ and transition matrix Π_{ij}.
- Goal: Simulate this Markov Chain to generate $\{z_t\}_{t=0}^{T}$.
- **Step 1**: Compute the cumulative distribution of the Markov Chain, Π^c:

$$\Pi_{ij}^c = \sum_{k=1}^{j} \pi_{i,k}.$$

  This is the probability that the Markov Chain is in a state lower or equal to j given, that it was in state i at $t-1$.
- **Step 2**: Set the initial state, and simulate T random numbers from an Uniform Distribution over $[0,1]$: $\{p_t\}_{t=1}^{T}$.
- **Step 3**: Assume that the Markov Chain was in state i in $t-1$, and find the index j such that

$$\Pi_{i(j-1)}^c < p_t < \Pi_{ij}^c, \; j \geq 2.$$

  Then z^j is the state in period t. If $p_t \leq \pi_{i,1}$ then z^1 is the state in period t.

Simulating a Markov Chain

- Two m-files:
  - markovappr, which gives Tauchen's approximation of an AR(1). The inputs are the persistence of the process, standard deviation, scaling parameter and size of the grid.
  - markov which creates T realizations of a Markov Chain with transition matrix $PROB$.

Idea Behind Dynamic Programming I

Define *value function* $V(k_0)$ = optimal value of problem for a given k_0:

$$V(k_0) = \max_{\{c_t,k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \tag{1}$$

$$\text{s.t. } k_{t+1} = f(k_t) - c_t. \tag{2}$$

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1})$$

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} [u(f(k_0) - k_1) + \beta u(f(k_1) - k_2) + \beta^2 u(f(k_2) - k_3)\ldots]$$

Observe that:

$$V(k_0) = \max_{\{0\le k_{t+1}\le f(k_t)\}_{t=0}^{\infty}} \{u(f(k_0) - k_1) + \sum_{t=1}^{\infty} \beta^t u(f(k_t) - k_{t+1})\}.$$

Idea Behind Dynamic Programming II

- Since k_t for $t \geq 2$ does not appear in $u(f(k_0) - k_1)$, we can rewrite the problem as

$$V(k_0) = \max_{k_1} \{u(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} [\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})]\}.$$

- Notice that:

$$\begin{aligned} \max_{\{k_{t+1}\}_{t=1}^{\infty}} [\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})]\} &= \\ \max_{\{k_{t+1}\}_{t=1}^{\infty}} [u(f(k_1) - k_2) + \beta u(f(k_2) - k_3) + ...] &= V(k_1). \end{aligned}$$

- Therefore:

$$V(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \{u(f(k_0) - k_1) + \beta V(k_1)\}$$

Idea Behind Dynamic Programming II

- Since k_t for $t \geq 2$ does not appear in $u(f(k_0) - k_1)$, we can rewrite the problem as

$$V(k_0) = \max_{k_1}\{u(f(k_0) - k_1) + \beta \max_{\{k_{t+1}\}_{t=1}^{\infty}} [\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})]\}.$$

- Notice that:

$$\begin{aligned} \max_{\{k_{t+1}\}_{t=1}^{\infty}} [\sum_{t=1}^{\infty} \beta^{t-1} u(f(k_t) - k_{t+1})]\} &= \\ \max_{\{k_{t+1}\}_{t=1}^{\infty}} [u(f(k_1) - k_2) + \beta u(f(k_2) - k_3) + ...] &= V(k_1). \end{aligned}$$

- Therefore:

$$V(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \{u(f(k_0) - k_1) + \beta V(k_1)\}$$

Idea Behind Dynamic Programming III

- Observe that since the time horizon is infinite, the problem is the same in every period.

$$V(k) = \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V(k')\}, \qquad \text{(BE)}$$

where $(')$ stands for future variables.

- If function $V(k)$ were known, we could use (BE) to define *policy function* $c = g(k)$ [or $k' = h(k)$] that attains maximum in (BE). Given the capital stock today, agents decide about capital stock tomorrow.
- **Bellman's principle:** an individual following an optimal strategy from $t+1$ onwards, can do no better than optimize at time t taking future optimal plans as given.

Tools I

Complete Metric Space: A metric space (S, ρ) is complete if every Cauchy Sequence in S converges to an element in S (Banach Space).

Example: $C(X) = \{f : X \to \Re,$ Cont. & bounded functions$\}$ with $\|f\| = sup_{x \in X}|f(x)|$.

Contraction Mapping: let (S, ρ) be a metric space and $T : S \to S$ be a function. T is a *Contraction Mapping* with modulus β if for some $\beta \in (0, 1)$:

$$\rho(Tx, Ty) \leq \beta\rho(x, y), \; \forall \, x, y \in S.$$

Tools II

Contraction Mapping Theorem: If (S, ρ) is a complete metric space and $T : S \to S$ is a Contraction Mapping with modulus β, then:

i. T has exactly one fixed point V, such that $V = TV$.

ii. For any $V_0 \in S$, $\rho(T^n V_0, V) \leq \beta^n \rho(V_0, V)$, $n = 0, 1, ...$

The Contractions

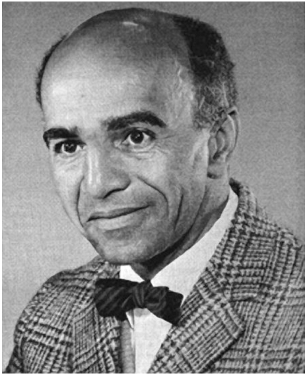
Tools III

A useful Corollary:

Let (S, ρ) be a complete metric space, and $T : S \to S$ is a Contraction with $V = TV$. Then, if S' is a closed subset of S, and $T(S') \subseteq S'$, then $V \in S'$.

- Useful when we want to apply the *CMT* twice: on a large space to establish uniqueness, and again on a smaller space to characterize the fixed point more precisely.
- The *CMT* can be applied to any metric space. Applying it to the *DP* will establish existence of a unique V. We have to show that the (BE) is a contraction mapping.

David Blackwell



Tools IV

Blackwell's sufficient conditions for a Contraction:

Let $X \subseteq \Re^l$, and let $B(X)$ be a space of bounded functions $f : X \rightarrow \Re$, with the *supnorm* : $||f|| = sup_x|f(x)|$. Let $T : B(X) \rightarrow B(X)$ be an operator satisfying:

- *Monotonicity*: $f, g \in B(X)$ and $f(x) \leq g(x), \; \forall \, x \in X$

$$T[f(x)] \leq T[g(x)],$$

- *Discounting*: $\exists \; \beta \in (0, 1)$:

$$T[f(x) + a] \leq T[f(x)] + \beta a, \; \forall f \in B(X), \; a \geq 0, \; x \in X.$$

- Then, T is a Contraction with modulus β.

Tools V

Theorem of the Maximum:

Problem: $h(x) = \max_{y \in \Gamma(x)} f(x, y)$.

Solution: $G(x) = \{y \in \Gamma(x) : h(x) = f(x, y)\}$.

a. Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^M$. Let $f : X \times Y \to \mathbb{R}$ be a continuous function; let $\Gamma : X \to Y$ be a continuous and compact-valued correspondence. Then the function $h : X \to \mathbb{R}$ is continuous and the correspondence $G : X \to Y$ is non-empty, compact-valued and upper-hemi continuous.

b. If in addition $\Gamma : X \to Y$ is convex-valued and the function $f : X \times Y \to \mathbb{R}$ is strictly concave in Y. Then $G(x)$ is single-valued and continuous.

Growth problem I

Is the Bellman equation

$$V_j(k) = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta V_{j-1}(k')\}, \qquad \text{(BE)}$$

a Contraction?

- If V_{j-1} is bounded, then V_j will also be bounded, because the set $[0, f(k)]$ is compact, given $k < \infty$.
- *Monotonicity:* Assume that $V(k) \le W(k)\ \forall k \in [0, f(k)]$. Then:

$$T[V(k)] = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta V(k')\}$$

$$T[W(k)] = \max_{0 \le k' \le f(k)} \{u(f(k) - k') + \beta W(k')\}$$

When $V(k) \le W(k)$, the objective function for which TW is the maximized value is uniformly higher than that for TV, thus $T[W(k)] \ge T[V(k)]$

Growth problem II

- *Discounting:*

$$
\begin{aligned}
T[V(k)+a] &= \max_{0\le k'\le f(k)} \{u(f(k)-k') + \beta[V(k')+a]\}, \\
T[V(k)+a] &= \max_{0\le k'\le f(k)} \{u(f(k)-k') + \beta V(k') + \beta a\}, \\
T[V(k)+a] &= \max_{0\le k'\le f(k)} \{u(f(k)-k') + \beta V(k')\} + \beta a, \\
T[V(k)+a] &= T[V(k)] + \beta a. \square
\end{aligned}
$$

Algorithm from Contraction Mapping Theorem

Algorithm to find V:

- Guess V_0; then use operator

$$V_1 = T[V_0] = \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V_0(k')\}$$

  to find V_1.
- Check if $V_0 \approx V_1$;
- if not, find $V_2 = T[V_1]$ from

$$V_2 = T[V_1] = \max_{0 \leq k' \leq f(k)} \{u(f(k) - k') + \beta V_1(k')\};$$

- continue that until $V_n \approx V_{n-1}$.

Example: Solution with pencil and paper

Brock and Mirman model:

$$
\begin{aligned}
\max_{\{c_t, k_{t+1}\}} \quad & \sum_{t=0}^{\infty} \beta^t \ln(c_t) \\
\text{s.t. } k_{t+1} + c_t &= A k_t^{\alpha}, \\
A &> 0; \alpha, \beta \in (0,1) \\
c_t &\geq 0.
\end{aligned}
$$

Value function:

$$V(k) = \max_{0 \leq k' \leq Ak^{\alpha}} \{ln(Ak^{\alpha} - k') + \beta V(k')\}.$$

Solution: Value function iteration I

- Start with $V_0(k) = 0$. Solve one period problem

$$V_1(k) = \max_{k'} \ln(Ak^\alpha - k')$$

Solution, $k' = 0 \Rightarrow c = Ak^\alpha$ and

$$V_1(k) = \ln A + \alpha \ln k.$$

- $V_2(k) = \max_{k'}\{\ln(Ak^\alpha - k') + \beta(\ln A + \alpha \ln k')\}$. Solution: $k' = \frac{\beta\alpha}{1+\beta\alpha}Ak^\alpha$, and

$$V_2(k) = (1+\beta)\ln A + \beta\alpha \ln \beta\alpha A - (1+\beta\alpha)\ln(1+\beta\alpha) + \alpha(1+\beta\alpha)\ln k$$

- $V_3(k) = \max_{k'}\{\ln(Ak^\alpha - k') + \beta(\alpha(1+\beta\alpha)\ln k' + ...)\}$. Solution: $k' = \frac{\beta\alpha(1+\beta\alpha)}{1+\beta\alpha(1+\beta\alpha)}Ak^\alpha$, ...
- Keep iterating, and in the limit: $k' = \alpha\beta Ak^\alpha$

$$V(k) = \frac{1}{1-\beta}\left\{\ln\left[A(1-\alpha\beta)\right] + \frac{\alpha\beta}{1-\alpha\beta}\ln(A\alpha\beta)\right\} + \frac{\alpha}{1-\alpha\beta}\ln k.$$

Euler Equation I

- Suppose that policy function $k' = h(k)$ solves (BE):

$$V(k) = u[f(k) - h(k)] + \beta V[h(k)]$$

  Notice that this is a *functional equation* to be solved for two unknown functions $V(.)$ and $h(.)$.

- If we knew $V(.)$ was *differentiable* and max value $k' = h(k)$ was interior, we have *first order condition*:

$$u'(c) = \beta V'(k')$$

- *Envelope condition*:

$$V'(k) = u'(c)f'(k)$$

- Intuition: marginal value of capital equals increment in production allocated to consumption.

Euler Equation II

We have

$$u'(c) = \beta V'(k'), \tag{FOC}$$

and

$$V'(k) = u'(c)f'(k), \tag{Envelope}$$

Since (Envelope) holds for any t:

$$u'(c) = \beta u'(c')f'(k') \Rightarrow \frac{u'(c)}{\beta u'(c')} = f'(k'). \tag{EE}$$

Optimal Growth Problem with Uncertainty I

- Suppose consumer maximizes

$$
\begin{aligned}
V(k_0, z_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} &= E_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right],\ 0 < \beta < 1 \quad (3) \\
\text{s.t. } k_{t+1} + c_t &\leq z_t f(k_t) \\
z_{t+1} = \rho z_t + \epsilon_{t+1} & \quad ,\ \rho \in (0,1),\ \epsilon \sim F(0, \sigma_\epsilon^2) \\
k_0 &> 0 \text{ given, and } c_t \geq 0.
\end{aligned}
$$

- Note: $u(.)$ and $f(.)$ satisfy standard properties.
- At period t agents know the realization of the shock z_t.

Optimal Growth Problem with Uncertainty II

- We can show that:

$$V(k_0, z_0) \quad = \quad \max_{0 \leq k_1 \leq z_0 f(k_0)} \{u(z_0 f(k_0) - k_1) + \beta E_0[V(k_1, z_1)]\}$$

- since the problem repeats itself in every period, we have that:

$$V(k, z) = \max_{0 \leq k' \leq z f(k)} \{u(z f(k) - k') + \beta E[V(k', z')]\}, \qquad \text{(BE)}$$

where $(')$ stands for future variables.

- Policy function: $c = g(k, z)$ [or $k' = h(k, z)$]

Markov Process

Feller Property: If $g(x,z)$ and $V(\cdot)$ are continuous and bounded functions, then

$$E[V(g(x',z'))/x,z] = \int V(g(x',z'))dF(z'/x,z)$$

is continuous and bounded.

Markov Process: A stochastic variable z_t follows a first-order Markov chains if for all $k \geq 1$ for each $i = 1, ..., n$:

$$Prob[z_t = \hat{z}/z_{t-1}, z_{t-2}, ..., z_{t-k}] = Prob[z_t = \hat{z}/z_{t-1}].$$

We can characterize a $n - dimensional$ Markov Process by the state space $z \in \mathbb{Z} = \{z^1, ..., z^n\}$ and by the nxn transition matrix, P, where:

$$P_{ij} = Prob[z_{t+1} = z^j/z_t = z^i], \text{ notice that } \sum_{j=1}^{n} P_{ij} = 1.$$