Macroeconomics III
Lecture 8: Incomplete Markets with Capital Accumulation

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Road Map

S. Rao Aiyagari (1994, QJE) model: **Uninsured Idiosyncratic Risk and Aggregate Saving.**

Introduction

- It was simultaneous and independent work of Hugget.
- The main difference is that in the Aiyagari-world, households both underwrite debt contracts to each other, but also rent out resources to firms.
- Of course, this means that there can be positive savings in the economy which determines the capital stock.
- In addition, wages are not simple endowments, but paid by firms in a competitive market.

Question studied by Aiyagari (1994): Are precautionary savings important for aggregate saving/capital accumulation?

Main conclusion: "*... the contribution of uninsured idiosyncratic risk to aggregate saving is quite modest, at least for moderate and empirically plausible values of risk aversion, variability, and persistence in earnings.*

The aggregate saving rate is higher by no more than three percentage points."

But: "*In contrast to representative agent models (see Cochrane [1989]), it turns out that access to asset markets is quite important in enabling consumers to smooth out earnings fluctuations.*

The model is also consistent, at least qualitatively, with certain features of income and wealth distributions. The distributions are positively skewed (median < mean), the wealth distribution is much more dispersed than the income distribution, and inequality as measured by the Gini coefficient is significantly higher for wealth than for income."

Aiyagari (1994) Model: Firms

- Firms can hire workers on a labor sport market at wage rate w, and rent capital at rental rate $\tilde{r}$.
- Technology: $Y = F(K, N) = K^{\alpha} N^{1-\alpha}$, so we can work with a representative firm (capital depreciates at rate δ).
- The problem of the representative firm is:

$$\max_{K,N} \{K^{\alpha} N^{1-\alpha} - wN - \tilde{r}K\}, \quad \text{then}$$

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha},$$

$$\tilde{r} = \alpha\left(\frac{K}{N}\right)^{\alpha-1}.$$

Threfore: $w = (1-\alpha)\left(\frac{\alpha}{\tilde{r}}\right)^{\frac{\alpha}{1-\alpha}}$

Aiyagari (1994) Model: Households

Consider the following problem:

$$\max_{c_t, a_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to:

$$c_t + a_{t+1} = (1 + r_t)a_t + w_t z_t, \; c \geq 0$$

$$z_t \sim \text{a Markov Process.}$$

- $V(a, z) = \max_{a'} \{u(((1 + r)a + wz - a') + \beta E[V(a', z')/z = \hat{z}]\}$.
  For each $i = 1, ..., n$:

$$V(a, z_i) = \max_{a'} \{u(((1 + r)a + wz_i - a') + \beta \sum_{j=1}^{n} P_{ij} V(a', z_j)\}.$$

- Agent wants: $a' = -\infty$.

- Natural borrowing limit (Aiyagari, 1994). Agent is able to repay in its worst state.

$$a' \geq -\frac{w\underline{z}}{r} = \underline{a}.$$

Derivation: $a_{t+1} + c_t \leq (1 + r_t)a_t + w_t z_t$. Let $z_t = \underline{z}$ and consider the stationary eq. with constant prices. If $c_t = 0$,

$$a_{t+1} \leq (1 + r)a_t + w\underline{z}, \Rightarrow a_{t+2} \leq (1 + r)a_{t+1} + w\underline{z}$$

$$a_{t+2} \leq (1+r)^2 a_t + \sum_{j=0}^{1}(1+r)^j w\underline{z}, \Rightarrow a_t \geq \frac{a_{t+2}}{(1+r)^2} - \sum_{i=1}^{2}\frac{w\underline{z}}{(1+r)^i}.$$

$$\text{Repeating T times: } a_t \geq \frac{a_{t+T}}{(1+r)^T} - \sum_{i=1}^{T}\frac{w\underline{z}}{(1+r)^i}, \ \forall t.$$

$$a_{t+1} \geq \frac{a_{t+1+T}}{(1+r)^T} - \sum_{i=1}^{T}\frac{w\underline{z}}{(1+r)^i}.$$

$$\text{Taking limits: } a' \geq -\frac{w\underline{z}}{r} = \underline{a}.$$

- Endogenous borrowing limit (Kehoe and Levine, 1993). Penalty for those who default on their obligations is the subsequent exclusion from capital and credit markets. Such that

$$\underline{V}(z) = u(\gamma w z) + \beta E[\underline{V}(z')/z = \hat{z}],$$

where $\gamma \in (0, 1]$ corresponds to a pecuniary loss due to the default.

$$V(a, z) \geq \underline{V}(z).$$

$V(a, z)$ is increasing in a whereas $\underline{V}(z)$ is independent of a, then the condition above defines $\underline{a}^{EB}$, such that

$$a' \geq \underline{a}^{EB}.$$

$$V(a, z_i) = \max_{\underline{a} \leq a' \leq \bar{a}} \{u(((1+r)a + wz_i - a') + \beta \sum_{j=1}^{n} P_{ij} V(a', z_j)\}.$$

- Arbitrage: $\tilde{r} = r + \delta$.
- Again, if $\beta(1+r) \leq 1$, we can find an $\bar{a} < \infty$ such that $a' \in [\underline{a}, \bar{a})$ and this problem is a **Contraction Mapping**.
- And for a given $\tilde{r}$ (and therefore w and r) the associated policy function is $a' = g(a, z)$ (and $c = g_c(a, z)$).

Aiyagari (1994, QJE): Equilibrium

DEFINITION: A recursive stationary equilibrium is an interest rate, r, wage rate, w, a policy function, $g(a,z)$, and a stationary distribution $\lambda(a,z)$, such that:

1. Given r and w, The policy function $g(a,z)$ solves $V(a,z)$;
2. Given r and w, the representative firm maximizes profits: $w = F_N(K,N)$ and $r + \delta = F_K(K,N)$.
3. Markets clear:

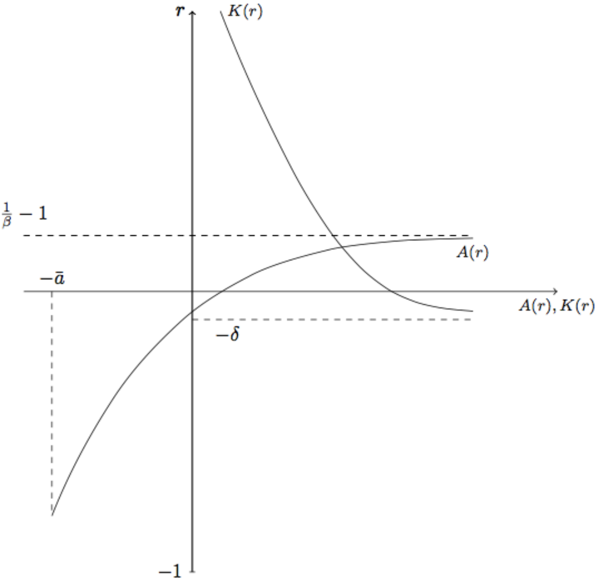
$$K' = \sum_{z,a} \lambda(a,z)a.$$

$$N = \pi' z.$$

4. The stationary distribution $\lambda(a,z)$ is induced by (P,z) and $g(a,z)$:

$$\lambda(B) = \sum_{X=[\underline{a},\bar{a}]\times Z\in B} Q(X,B).$$

Stationary Equilibrium



Aiyagari (1994, QJE): Algorithm

1. Guess $r \in \left(-\delta, \frac{1}{\beta} - 1\right)$, get $\tilde{r} = r + \delta$ and $w = (1-\alpha)\left(\frac{\alpha}{\tilde{r}}\right)^{\frac{\alpha}{1-\alpha}}$;
2. Given r, get $K = \left(\frac{\alpha}{r+\delta}\right)^{\frac{1}{1-\alpha}} \pi' z$;
3. Solve the households problem and get $g(a, z)$;
4. Calculate the associated stationary distribution: $\lambda(a, z)$;
5. Evaluate: $e_K = K - \sum_{z,a} \lambda(a, z) a$;
6. If $e_K > 0$, update (bisection) $r' > r$; if $e_K < 0$, update (bisection) $r' < r$. Go to step 1;
7. If $|e_K| < \epsilon$, stop.

Aiyagari (1994) with Endogenous Labor Supply

- Continuum of agents $[0, 1]$ with preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - l_t\right)$$

- Labor income at period t is

$$w_t l_t z_t,$$

  Shock z_t follows a Markov Chain (Z, Π)
  ($Z = \{z_1 < z_2 < \ldots < z_{n_z}\}$ and an element $\pi_{i,j}$ of Π is
  $Prob(z_{t+1} = z_j \mid z_t = z_i)$.

- There is a representative firm which has access to the a CRS technology:

$$Y_t = F\left(K_t, L_t\right)$$

  firms rents input in competitive markets and capital depreciates at rate δ.

Aiyagari (1994) with Endogenous Labor Supply

$$V\left(z_i, a\right) = \max_{c,l,a'} \left\{ u\left(c, l\right) + \beta \sum_{j=1}^{n_z} \pi_{i,j} V\left(z_j, a'\right) \right\}$$

subject to

$$\begin{aligned} a' + c &= a\left(1 + r\right) + wlz_i \\ l &\in [0, 1] \\ c &\geq 0 \\ a' &\geq \underline{a} \\ & r \text{ and } w \text{ are taken as given} \end{aligned}$$

Get the optimal decision rules $a'\left(z_i, a\right)$, $c\left(z_i, a\right)$ and $l\left(z_i, a\right)$.

Aiyagari (1994) with Endogenous Labor Supply

A recursive stationary equilibrium consists of:

- time invariant prices w and r;
- a value function $V(z_i, a)$;
- optimal decision rules $a'(z_i, a)$, $c(z_i, a)$ and $l(z_i, a)$;
- an invariant distribution function $\lambda(z_i, a)$
- a vector of aggregates: K, L;

such that

Aiyagari (1994) with Endogenous Labor Supply

1. **Consumer optimization:** given the prices r and w, $V(z_i, a)$ is a solution of the individual problem and $a'(z_i, a)$, $c(z_i, a)$ and $l(z_i, a)$ are the associated optimal decision rules;
2. **Firm's optimization**: prices r and w satisfy:

$$\begin{aligned} r &= F_K(K, L) - \delta \\ w &= F_L(K, L) \end{aligned}$$

3. **Consistency:** $\Phi(z_i, a)$ is a stationary distribution consistent with the optimal decision rule $a'(z_i, a)$ and the Markov chain (Z, Π) ;
4. **Aggregation:** The aggregate capital stock and labour supply are consistent with $F(s_i, a)$:

$$\begin{aligned} K &= \sum_{i=1}^{n_z} \int_A a d\lambda(z_i, a) \\ L &= \sum_{i=1}^{n_z} \int_A l(z_i, a) z_i d\lambda(z_i, a) \end{aligned}$$

Aiyagari (1994) with Endogenous Labor Supply

Households problem:

- If you choose VFI you need to implement VFI with 2 variable optimization.
- OR...assuming $u(c, 1-l) = \frac{\left(c^{\gamma}(1-l)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma}$
  - Intra-temporal condition and budget constraint:

$$\begin{aligned} wlz_i &= wz_i - \frac{1-\gamma}{\gamma}c, \\ a' + c &= a(1+r) + wlx_i, \end{aligned}$$

Imply that $c = \frac{1}{\gamma}[(1+r)a + wz_i - a']$ and $(1-l) = \frac{1-\gamma}{\gamma}\frac{c}{z_i w}$. Or

$$l = \widetilde{l}(z_i, a, a')$$

  - if we plug back this into the original problem we can find the optimal a' in the same way as always and obtain:

$$a'(z_i, a)$$

Aiyagari (1994) with Endogenous Labour Supply

The Value function becomes:

$$V(z_i, a) = \max_{a' \in [a_{\min}, \bar{a})} \left\{ u(a(1+r) + w\tilde{l}(z_i, a, a') z_i - a', 1 - \tilde{l}(z_i, a, a') \right.$$
$$\left. + \beta \sum_{j=1}^{n_z} \pi_{i,j} V(z_j, a') \right\}.$$

We can solve it again for a' since the labour-leisure choice is an instantaneous decision and does not affect the future value directly.

- We can obtain: $a'(z_i, a)$;
- then $l = \tilde{l}(z_i, a, a') = \tilde{l}(z_i, a, a'(z_i, a)) = l(z_i, a)$
- then you can get: $c = c(z_i, a)$
- Now you are ready to solve the problem

Aiyagari (1994) with Endogenous Labour Supply

1. Guess r
2. Compute the capital-labour ratio k from
$$r = F_K(k) - \delta$$
3. Compute w :
$$w = F_L(k)$$
4. Given r and w solve the consumer optimization problem finding $a'(s_i, a)$, $c(s_i, a)$ and $l(s_i, a)$.
5. Using $a'(s_i, a)$ and Π find the stationary distribution of the assets: $\lambda(s_i, a)$

Aiyagari (1994) with Endogenous Labor Supply

6. Compute the aggregate capital supply K and the aggregate labour supply L :

$$K = \sum_{i=1}^{n_s} \int_A a d\lambda\left(s_i, a\right)$$

$$L = \sum_{i=1}^{n_s} \int_A l\left(s_i, a\right) s_i d\lambda\left(s_i, a\right)$$

7. the excess demand function can be defines as:

$$D\left(r\right) = k - \frac{K}{L}$$

find a fix point in $D\left(r\right)$ such that $D\left(r\right) = 0$ using a bisection method with bounds:

0.1 lower bound:$r = -\delta$ (the return of capital after depreciation approaches $-\delta$ as k goes to infinity)

0.2 upper bound: $r < \frac{1}{\beta} - 1$ (for $r = \frac{1}{\beta} - 1$ the household accumulates an infinite amount of asset then capital supply goes to infinity)

Reading

- Aiyagari "Uninsured Idiosyncratic Risk and Aggregate Savings", QJE, 1994
- Heer and Maussner, Chapter 5;
- Hopenhayn, Prescott "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies", Econometrica 1992
- Huggett "The Risk Free Rate in Heterogeneous-Agents, Incomplete Insurance Economies", JEDC 1993
- Miao, Chapter 17
- Rios-Rull "Computation of Equilibria in Heterogeneous-Agents Model" in Marimon and Scott eds, Computational Methods for the Study of Economic Dynamics, Chapter 11