IID stochastic growth model

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \, \mathrm{u}\,(c_t)$$

$$c_t + k_{t+1} = (1-\delta)k_t + y_t$$

$$y_t = z_t \, \mathrm{F}(k_t)$$

IID shocks

$$z_t = \begin{cases} z_1 & \text{w.p. } \pi_1 \\ \vdots & \\ z_n & \text{w.p. } \pi_n \end{cases}, \text{ for all } t$$

Recursive formulation I

- We assume the solution to the previous problem is the same as the solution to the following functional equation problem

$$V(k, z) = \max_{\substack{k' \geq 0 \\ c \geq 0}} \left\{ \mathrm{u}\left(c\right) + \beta\, \mathrm{E}_{z'}\left[V\left(k', z'\right)\right] \right\}$$

$$\text{subject to}$$

$$c + k' = (1 - \delta)k + y$$

$$y = z\, \mathrm{F}(k)$$

Recursive formulation II

- Plugging in the constraints, we have

$$V(k,z) = \max_{\substack{k' \geq 0 \\ c \geq 0}} \left\{ \mathrm{u}\left((1-\delta)k + z\,\mathrm{F}(k) - k'\right) + \beta\,\mathrm{E}_{z'}\left[V\left(k', z'\right)\right] \right\}$$

Recursive formulation III

- We proceed as if we knew V was differentiable and obtain the FOC

$$-\,\mathrm{u}'(c) + \beta\,\mathrm{E}_{z'}\left[V_{k'}\left(k', z'\right)\right] = 0$$

- Problem: we do not know $V_{k'}\left(k', z'\right)$
- If we ensure the conditions for the Benveniste-Scheinkman Envelope Theorem are satisfied, then we ensure differentiability of the value function

$$V_k\left(k, z\right) = \mathrm{u}'(c)\left[(1-\delta) + z\,\mathrm{F}'(k)\right]$$

Recursive formulation IV

- Since this envelope condition holds for any t

$$V_{k'}\left(k', z'\right) = \mathrm{u}'(c')\left[(1-\delta) + z\,\mathrm{F}'(k')\right]$$

- We arrive at the Euler equation

$$\mathrm{u}'(c) = \beta\,\mathrm{E}_{z'}\left[V_{k'}\left(k', z'\right)\right] = \beta\,\mathrm{E}_{z'}\left[\mathrm{u}'(c')\left[(1-\delta) + z\,\mathrm{F}'(k')\right]\right]$$

Functional forms

$$\mathrm{u}(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

$$\mathrm{F}(k) = k^{\alpha}$$

Euler equation

$$c^{-\gamma} = \beta\,\mathrm{E}_{z'}\left[c^{-\gamma}\left[(1-\delta) + z'\alpha k'^{\alpha-1}\right]\right]$$

Steady-state

- We look at a deterministic steady-state where all shocks are equal to its mean ($z = z' = \bar{z}$)

$$k_{SS} = \left(\frac{\bar{z}\alpha}{\frac{1}{\beta} + \delta - 1} \right)^{\frac{1}{1-\alpha}}$$