Exam

1 $U = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$

$\omega^A = (1, 0, 1, 0, \dots)$
$\omega^B = (0, 1, 0, 1, \dots)$

Consumer problem (generic form) [CP]

$$\max\ U(c^i) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t)) \mathbb{P}(s^t) \quad \text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

Here, $s_0 = 1$, $s_t = 0$ for t odd and $s_t = 1$ for t even; and $y_t^A(s^t) = s_t$, $y_t^B(s^t) = 1 - s_t$

Competitive equilibrium

A competitive equilibrium consists of a price system $\{q_t^0(s^t)\}$ and a consumption allocation $\{c_t^i(s^t)\}_{t=0}^{\infty}$ for $i = A, B$ such that

[1] Given $\{q_t^0(s^t)\}_{t=0}^{\infty}$, $\{c_t^i(s^t)\}_{t=0}^{\infty}$ solves [CP]

[2] Markets clear

$$\sum_i c_t^i(s^t) = \sum_i y_t^i(s^t) \quad \forall t, s^t$$

FINDING ALLOCATIONS AND PRICES

$$L = \sum_{t=0}^{\infty} \beta^t \ln(c_t^i) + \mu^i \left\{ \sum_{t=0}^{\infty} q_t^0 \left[y_t^i - c_t^i \right] \right\}$$

$$\forall t, i: \ \beta^t \frac{1}{c_t^i} = \mu^i q_t^0 \Rightarrow q_t^0 = \frac{1}{\mu^i} \frac{\beta^t}{c_t^i}$$

Across agents: $\frac{c_t^j}{c_t^i} = \frac{\mu^i}{\mu^j} \Rightarrow c_t^j = \frac{\mu^i}{\mu^j} c_t^i$

Market clearing: $\frac{\mu^i}{\mu^j} c_t^i + c_t^i = y_t^i + y_t^j = 1 \Rightarrow c_t^i = \bar{c}^i$ constant across time

$\Rightarrow c_t^j = \bar{c}^j$ as well

Budget constraints

$$\sum_{t=0}^{\infty} q_t^0 c_t^i = \sum_{t=0}^{\infty} q_t^0 y_t^i$$

$$\Rightarrow \sum_{t=0}^{\infty} \frac{1}{\mu^i} \frac{\beta^t}{\bar{c}^i} \left[\bar{c}^i - y_t^i \right] = 0 \Rightarrow \underbrace{\sum_{t=0}^{\infty} \beta^t \bar{c}^i}_{\frac{\bar{c}^i}{1-\beta}} = \sum_{t=0}^{\infty} \beta^t y_t^i$$

$$\Rightarrow \bar{c}^i = (1-\beta) \sum_{t=0}^{\infty} \beta^t y_t^i$$

$$\therefore \bar{c}^A = (1-\beta) \sum_{t=0}^{\infty} \beta^t s_t = (1-\beta) \left[1 + \beta^2 + \beta^4 + \cdots \right] = (1-\beta) \frac{1}{1-\beta^2} = \frac{1}{1+\beta}$$

$$\bar{c}^B = (1-\beta) \sum_{t=0}^{\infty} \beta^t (1-s_t) = (1-\beta) \left[\beta + \beta^3 + \beta^5 + \cdots \right] = (1-\beta) \frac{\beta}{1-\beta^2} = \frac{\beta}{1+\beta}$$

What about prices?

$$q_t^0 = \frac{1}{\mu^i} \frac{\beta^t}{\bar{c}^i}$$

▷ If we normalize $q_0^0 = 1$, then $1 = \frac{1}{\mu^i} \frac{1}{\bar{c}^i} \Rightarrow \mu^i = \frac{1}{\bar{c}^i}$

Therefore, $q_t^0 = \beta^t$ ▨

$\boxed{2}$ Pure exchange economy

$$U = \mathbb{E}_0\left(\sum_{t=0}^{\infty} \beta^t \ln(c_t^i)\right)$$

Consumer problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \ln(c_t^i)\right] \quad \text{s.t.} \quad c_t + a_{t+1} = (1+r)a_t + \omega_t$$

In recursive formulation:

$$V(a,\omega) = \max_{a'} \left\{ \ln\left((1+r)a + \omega - a'\right) + \beta \mathbb{E}_{\omega'}\left[V(a',\omega') | \omega\right] \right\}$$

▶ We need to define a borrowing limit: $a' \geq \underline{a}$

Stationary equilibrium

A stationary equilibrium is a policy function for assets $a' = g_a(a,\omega)$, a policy function for consumption $c = g_c(a,\omega)$, a price r and a stationary distribution $\lambda(a,\omega)$ such that:

[1] Given r, $g_a(a,\omega)$ and $g_c(a,\omega)$ solve the value function

[2] Markets clear

$$\int_S g_a(a,\omega)\, d\lambda = \sum_{a,\omega} g_a(a,\omega) = 0 \quad \text{[loan market]}$$

$$\int_S g_c(a,\omega)\, d\lambda = \int_S \omega\, d\lambda \quad \text{[consumption market]}$$

[3] $\lambda(a,\omega)$ is a stationary probability measure (i.e., distribution) induced by $(\mathbb{P}_\omega, \omega)$ and $g_a(a,\omega)$

$$\lambda(B) = \sum_{(\omega,a)\in B} \lambda(a,\omega)\mathbb{P}_\omega$$

ALGORITHM.

[1] Given a guess $r = \hat{r}$, solve the household's problem to find $g_a(a,\omega)$.

[2] Given $g_a(a,\omega)$, iterate on

$$\lambda_{t+1}(a',\omega') = \sum_{\omega} \sum_{\{a : a' = g_a(a,\omega)\}} \lambda_t(a,\omega)\,\mathbb{P}(\omega'|\omega)$$

[3] Using the converged distribution, compute $e = \sum_{\omega,a} \lambda(a,\omega)\, g_a(a, g_\omega)$

[4] If $e < \varepsilon$, update $r^{[j+1]} > r^{[j]}$ and if $e > \varepsilon$, update $r^{[j+1]} < r^{[j]}$ and return to step [1].

If $|e| < \varepsilon$, stop.

▶ Unconditional distribution of (a_t, ω_t) pairs: $\lambda_t(a,\omega) = \mathbb{P}(a_t = a, \omega_t = \omega)$.

The exogenous Markov transition matrix $\mathcal{P}$ on ω and the policy function $a' = g_a(a,\omega)$ induce a law of motion for the distribution λ_t

$$\lambda_{t+1}(a',\omega') = \mathbb{P}(a_{t+1} = a', \omega_{t+1} = \omega') = \sum_a \sum_s \mathbb{P}(a_{t+1} = a', \omega_{t+1} = \omega' | a_t = a, \omega_t = \omega)\,\mathbb{P}(a_t = a, \omega_t = \omega)$$

(Law of Total Probability: $\mathbb{P}(A) = \sum_n \mathbb{P}(A \cap B_n) = \sum_n \mathbb{P}(A|B_n)\mathbb{P}(B_n)$)

$$= \sum_a \sum_s \mathbb{P}(a_{t+1} = a' | a_t = a, \omega_t = \omega; \omega_{t+1} = \omega')\,\mathbb{P}(\omega_{t+1} = \omega' | \omega_t = \omega, a_t = a)\,\mathbb{P}(a_t = a, \omega_t = \omega)$$

$$= \sum_a \sum_s \mathbb{P}(a_{t+1} = a' | a_t = a; \omega_t = \omega)\,\mathbb{P}(\omega_{t+1} = \omega' | \omega_t = \omega)\,\mathbb{P}(\omega_t = \omega, a_t = a)$$

$$= \sum_a \sum_s \lambda_t(a,\omega)\,\mathbb{P}(\omega_{t+1} = \omega' | \omega_t = \omega)\,\mathbb{1}\{a' = g_a(a,\omega)\}$$

$$\therefore \lambda_{t+1}(a',\omega') = \sum_s \sum_{\{a : a' = g_a(a,\omega)\}} \lambda_t(a,\omega)\,\mathcal{P}(\omega,\omega')$$