Macroeconomics III
Lecture 12: Continuous Time Models (cont.)

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Road Map

See Ben Moll's website (many lectures, codes, and articles). My lecture is closer to his lectures labeled **University of Chicago/Penn/UCLA/Bonn/Rochester Mini-Course 'Heterogeneous Agent Models in Continuous Time"**

http://www.princeton.edu/~moll/Lecture1_Rochester.pdf

http://www.princeton.edu/ moll/Lecture2_Rochester.pdf

Aiyagari Model

$$r(t) = F_K(K(t), 1), \;\; w(t) = F_L(K(t), 1) \tag{P}$$

$$K(t) = \int a g_1(a,t) da + \int a g_2(a,t) da \tag{K}$$

$$\rho V_j(a,t) = u(c_j(a,t)) + \partial_a V_j(a,t)(w(t)z_j + r(t)a - c_j(a,t)) \tag{HJB}$$

$$+\lambda_j(V_{-j}(a,t) - V_j(a,t)) + \partial_t V_j(a,t)$$

$$\partial_t g_j(a,t) = -\partial_a[s_j(a,t)g(a,t)] - \lambda_j g_j(a,t) + \lambda_{-j} g_{-j}(a,t) \tag{KF}$$

$$s_j(a,t) = w(t)z_j + r(t)a - c_j(a,t), \; c_j(a,t) = (u')^{-1}(\partial_a V_j(a,t))$$

Transition Dynamics

- Discretised equations for the stationary solution

$$\rho \mathbf{V} = \mathbf{u}(\mathbf{V}) + \mathbf{P}\mathbf{V} \tag{HJB}$$

where $\mathbf{u}(\mathbf{V}) = \mathbf{u}^{-\mathbf{1}}(\mathbf{V}')$

$$\mathbf{0} = \mathbf{P}'\mathbf{g} \tag{KF}$$

- Finite difference method transforms the solution into a solution of a system of sparse matrix equations

- For the transition dynamics:
  - Besides discretising a, discretise also t^n, $n = 1, ..., N$ (step of length Δt) (n is a superscript).
  - Denote $V_{i,j}^n = V_j(a_i, t^n)$ and stack into $\mathbf{V^n}$
  - Denote $g_{i,j}^n = g_j(a_i, t^n)$ and stack into $\mathbf{g^n}$

Transition Dynamics

- Solution of the transition dynamics is similar:
  1. Guess the entire path for prices $r(t)$ and get $w(t)$ from FOCs of the firm.
  2. Use non-stationary discretised conditions:

$$\rho \mathbf{V}^n = \mathbf{u}(\mathbf{V}^{n+1}) + \mathbf{P^{n+1}}\mathbf{V}^n + \frac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta t} \tag{HJB}$$

     where $\mathbf{u}(\mathbf{V}^{n+1}) = \mathbf{u^{-1}}(\mathbf{V'^{n+1}})$

$$\frac{\mathbf{g}^{n+1} - \mathbf{g}^n}{\Delta t} = (\mathbf{P}^n)'\mathbf{g}^{n+1} \tag{KF}$$

  3. Terminal conditions:

$$\mathbf{V}^N = \mathbf{V},\ \mathbf{g}^1 = \mathbf{g}^0$$

Transition Dynamics: Algorithm

- (HJB) looks forward, runs backwards in time
- (KF) looks backward, runs forward in time
- Algorithm: Guess $K(t^n)$ and then for $n = 0, 1, 2, \ldots$
  1. find prices $r(t^n)$ and $w(t^n)$
  2. solve (HJB) backwards in time given $V^N = V$
  3. solve (KF) forward in time given given $g^1 = g^0$
  4. compute $S(t^n) = \int a g_1(a, t^n) da + \int a g_2(a, t^n) da$
  5. update $K(t^{n+1}) = (1 - \xi) K(t^n) + \xi S(t^n)$ with $\xi \in (0, 1)$
- Ben Moll provides the codes with full explanation: Aiyagari_poisson_MITshock.m

More General Income Processes

- In our model so far $y = wz$ takes only two values.
- Brownian Motion: a standard Brownian motion is a stochastic process W

$$W(t + \Delta t) - W(t) = \epsilon_t \sqrt{\Delta t}, \ \epsilon_t \sim N(0, 1), \ W(0) = 0$$

- Then: $W(t) \sim N(0, t)$
- Continuous time analogue of a discrete time random walk

$$W_{t+1} = W_t + \epsilon_t, \ \epsilon_t \sim N(0, 1).$$

Brownian Motion

- Can be generalised

$$x(t) = x(0) + \mu t + \sigma W(t)$$

- Since $E(W(t)) = 0$ and $Var(W(t)) = t$, the:

$$E(x(t) - x(0)) = \mu, \; Var(x(t) - x(0)) = \sigma^2 t$$

- This is called a Brownian motion with drift μ and variance σ^2
- Standard to write this as

$$dx(t) = \mu dt + \sigma dW(t),$$

  which is a stochastic differential equation.
- Analogue of stochastic difference equation:

$$x_{t+1} = \mu + x(t) + \sigma \epsilon_t, \; \epsilon_t \sim N(0, 1).$$

Diffusion Process

- Can be generalised further (suppressing dependence of x, W on t)

$$dx = \mu(x)dt + \sigma(x)dW.$$

- This is called a "diffusion process" or the continuous time analogous of a Markov Process

- For practical reasons in our case $y \in [\underline{y}, \bar{y}]$ and

$$dy = \mu(y)dt + \sigma(y)dW.$$

Stationary System

$$\rho V(a,y) = \max_c u(c) + \partial_a V(a,y)(y + ra - c) + \partial_y V(a,y)\mu(y) \quad \textbf{(HJB)}$$

$$+\frac{1}{2}\partial_{yy} V(a,y)\sigma^2(y)$$

$$0 = -\partial_a[s(a,y)g(a,y)] - \partial_y[\mu(y)g(a,y)] + \frac{1}{2}\partial_{yy}[\sigma^2 g(a,y)] \quad \textbf{(KF)}$$

where $s(a,y) = y + ra - c(a,y)$ and $c(a,y) = (u')^{-1}(\partial_a V(a,y))$.
At the borrowing limit:

$$\partial_a V(\underline{a}, y) \geq u'(y + r\underline{a}), \ \forall y$$

Smooth pasting conditions:

$$\partial_y V(a, \underline{y}) = 0, \partial_y V(a, \bar{y}) = 0, \ \forall a.$$

Stationary System

- Also, market clearing:

$$K = \int_{\underline{y}}^{\overline{y}} \int_{\underline{a}} a g(a, y) da dy.$$

$$\int_{\underline{y}}^{\overline{y}} \int_{\underline{a}} g(a, y) da dy = 1$$

Point: The algorithm is similar to outlined in Lecture 10!

Ito's Lemma

- Let x be a scalar diffusion

$$dx = \mu(x)dt + \sigma(x)dW$$

Lemma Let $y(t) = f(x(t))$ be any twice differentiable function. It follows that

$$df(x) = \left(\mu(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)\right)dt + \sigma(x)f'(x)dW$$

- it says that any (twice differentiable) function of a diffusion is also a diffusion

Heuristical Derivation of Stochastic HJB Equation

In Δt units of time, value function is:

$$V(a_t, y_t) = \max_{c_t} \Delta t u(c_t) + (1 - \Delta t \rho) E[V(a_{t+\Delta t}, y_{t+\Delta t})].$$

Subtract both sides by $V(a_t, y_t)$ and divide by Δt

$$\rho V(a_t, y_t) = \max_{c_t} u(c_t) + E[\frac{1}{\Delta t}(V(a_{t+\Delta t}, y_{t+\Delta t}) - V(a_t, y_t)].$$

Taking the limit for $\Delta t \to 0$,

$$\rho V(a, y) = \max_{c} u(c) + E[\partial_t V(a, y)].$$

Using Ito's Lemma and get

$$\rho V(a, y) = \max_{c} u(c) + \partial_a V(a, y)(y + ra - c) + \partial_y V(a, y)\mu(y) \quad \textbf{(HJB)}$$

$$+ \frac{1}{2}\partial_{yy} V(a, y)\sigma^2(y)$$

Using Ito's Lemma

$$dV(a,y) =$$

$$\left(\partial_a V(a,y)(y+ra-c) + \partial_y V(a,y)\mu(y) + \frac{1}{2}\partial_{yy}V(a,y)\sigma^2(y)\right) dt$$

$$\left(\partial_a V(a,y)(y+ra-c) + \partial_y V(a,y)\right)\sigma(y)dW$$

Given that $E(dW) = 0$, then substituting $E\left(\partial_t V(a,y)\right)$ into

$$\rho V(a,y) = \max_c u(c) + E[\partial_t V(a,y)] \text{ yields}$$

$$\rho V(a,y) = \max_c u(c) + \partial_a V(a,y)(y+ra-c) + \partial_y V(a,y)\mu(y) \text{ (HJB)}$$

$$+\frac{1}{2}\partial_{yy}V(a,y)\sigma^2(y)$$

Solution

1. Guess r and get w from FOCs of the firm.
2. Use stationary discretised conditions:

$$\rho \mathbf{V} = \mathbf{u}(\mathbf{V}) + \mathbf{A}\mathbf{V} \tag{HJB}$$

   where $\mathbf{u}(\mathbf{V}) = \mathbf{u}^{-\mathbf{1}}(\partial_{\mathbf{a}}\mathbf{V})$ and A describes the dynamics of the states (a, y)

$$0 = (\mathbf{A})'\mathbf{g} \tag{KF}$$

3. Aggregate and compare demands with supply of capital and iterate on r until convergence

Summary

- Similar finite difference method can be applied!
- See the file HJB_diffusion_implicit.m at Moll's page.
- The paper Achdou et al (2017) contains the details of the model and extensions
  - Derive analytical results
  - Describes the conditions for the numerical solution to converge
  - Other examples including with housing and non-convex choices
- This is indeed a promising methodology to apply to macro problems
- Superb work by Moll to provide all codes and explanation and details of the codes.