Macroeconomics III
Lecture 11: Continuous Time Models (Cont.)

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Road Map

See Ben Moll's website (many lectures, codes, and articles). My lecture is closer to his lectures labeled **University of Chicago/Penn/UCLA/Bonn/Rochester Mini-Course 'Heterogeneous Agent Models in Continuous Time"**

http://www.princeton.edu/~moll/Lecture1_Rochester.pdf

http://www.princeton.edu/ moll/Lecture2_Rochester.pdf

Why is the Contraction Property Lost?

- Consider the deterministic Ramsey Growth model in discrete time:

$$V(k) = \max_{c}\{u(c) + (1-\rho)V(f(k) + (1-\delta)k - c)\}.$$

- We iterate

$$V_{n+1}(k) = \max_{c}\{u(c) + (1-\rho)V_n(f(k) + (1-\delta)k - c)\}.$$

- Under standard assumptions, this is a Contraction Mapping.
- We know that $V_n \to V$ from any V_0 that is bounded and continuous.

Why is the Contraction Property Lost?

- Let's convert this into continuous time

$$V_{n+1}(k) = \max_c \{\Delta u(c) + (1 - \Delta\rho)V_n(k + \Delta(f(k) - \delta k - c))\}.$$

$$0 = \max_c \{u(c) + \frac{V_n(k + \Delta(f(k) - \delta k - c)) - V_{n+1}(k)}{\Delta}$$
$$-\rho V_n(k + \Delta(f(k) - \delta k - c))\}.$$

- Taking limits and rearranging:

$$\rho V_n(k) = \max_c \{u(c) + \lim_{\Delta \to 0} \frac{V_n(k + \Delta(f(k) - \delta k - c)) - V_{n+1}(k)}{\Delta}\}.$$

Why is the Contraction Property Lost?

- Problem:

$$\lim_{\Delta \to 0} \frac{V_n(k + \Delta(f(k) - \delta k - c)) - V_{n+1}(k)}{\Delta} \neq V_n'(k)(f(k) - \delta k - c).$$

- The right hand side of the **HJB** equation contains V_{n+1} and that's a major issue.

How to Get It Back (Heuristically)

- Back to discrete time:

$$V_{n+1}(k) = \max_c \{u(c) + (1-\rho)V_n(f(k) + (1-\delta)k - c)\}.$$

- Call the optimal choice c_n (it's really a function of k).
- Howard's Improvement Algorithm says that we can then iterate on

$$V_{n+1}^{h+1}(k) = u(c_n) + (1-\rho)V_{n+1}^{h}(f(k) + (1-\delta)k - c_n),$$

with $V_{n+1}^0 = V_n$
- Iterate until $V_{n+1}^{h+1} \approx V_{n+1}^h$. This can speed things up considerably, and it can preserve the Contraction Property.

How to Get It Back (Heuristically)

- Suppose that it holds exactly, such that $V^{h+1}_{n+1} = V^{h}_{n+1}$, and let's just call this function V_{n+1}. Then:

$$V_{n+1}(k) = u(c_n) + (1-\rho)V_{n+1}(f(k) + (1-\delta)k - c_n).$$

- In Δ units of time:

$$V_{n+1}(k) = \Delta u(c_n) + (1-\Delta\rho)V_{n+1}(k + \Delta(f(k) + \delta k - c_n)).$$

How to Get It Back (Heuristically)

- Rearrange

$$0 = u(c_n) + \frac{V_{n+1}(k + \Delta(f(k) + \delta k - c_n)) - V_{n+1}(k)}{\Delta}$$
$$-\rho V_{n+1}(k + \Delta(f(k) + \delta k - c_n)),$$

- and take limits for $\Delta \to 0$:

$$\rho V_{n+1}(k) = u(c_n) + V'_{n+1}(k)(f(k) - \delta k - c_n).$$

- Now the awkward discrepancy between V_{n+1} and V_n is gone!

The Implicit Method

1. Start with a grid for capital $\mathbf{K} = \{k_1, k_2, ..., k_N\}$.
2. For each grid point for capital, guess $V_0(k_i)$.
3. So you have a vecor of N values of V_0. Call this $\mathbf{V}_0$.
4. You should also define a difference operator (an $N \times N$ matrix) $\mathbf{D}$, such that:

$$V'(K) \approx \mathbf{DV}, \ \forall k_i \in \mathbf{K}.$$

5. Optimal consumption choice given by FOC: $u'(\mathbf{c}_0) = \mathbf{DV}_0$. It is reasonable to call this $c(\mathbf{V_0})$ - an $N \times 1$ vector.

The Implicit Method

6. This implies another $N \times 1$ vector of savings

$$\mathbf{s_0} = (f(\mathbf{k}) - \delta\mathbf{k} - c(\mathbf{V_0})).$$

7. Create the $N \times N$ matrix $S_0 = diag(\mathbf{s}_0)$:

$$S = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_N \end{pmatrix}$$

8. Then, our **HJB** equation can be written as:

$$\rho\mathbf{V}_1 = u(c(\mathbf{V_0})) + \mathbf{S_0 D V_1}.$$

The Implicit Method

9. Manipulate:

$$(\rho \mathbf{I} - \mathbf{S_0 D})\mathbf{V}_1 = u(c(\mathbf{V_0})).$$

$$\mathbf{V}_1 = (\rho \mathbf{I} - \mathbf{S_0 D})^{-1} u(c(\mathbf{V_0})).$$

10. Generally:

$$\mathbf{V}_{n+1} = (\rho \mathbf{I} - \mathbf{S_n D})^{-1} u(c(\mathbf{V_n})).$$

Or even more generally

$$\mathbf{V}_{n+1} = ((\rho + 1/\Gamma)\mathbf{I} - \mathbf{S_n D})^{-1} [u(c(\mathbf{V_n})) + \mathbf{V_n}/\Gamma].$$

Low Γ implies slower update.

The Implicit Method: Improvement Tricks

- Recall our matrix $\mathbf{D}$ of central differences:

$$\mathbf{D} = \begin{pmatrix} -1/dk & 1/dk & 0 & 0 & \cdots & 0 \\ -0.5/dk & 0 & 0.5/dk & 0 & \cdots & 0 \\ 0 & -0.5/dk & 0 & 0.5/dk & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -1/dk & 1/dk \end{pmatrix}$$

- We can do better. In particular, $\mathbf{s_n}$ tells us where the economy is drifting for each $k_i \in \mathbf{K}$.

- So trick one is to use forward differences for all

$$\{k_i \in \mathbf{k} : s_i > 0\},$$

- and backward differences for all

$$\{k_i \in \mathbf{k} : s_i < 0\}.$$

The Implicit Method: Improvement Tricks

- This leads to

$$\mathbf{V}_{n+1} = ((\rho + 1/\Gamma)\mathbf{I} - \mathbf{S_n D_n})^{-1}[u(c(\mathbf{V_n})) + \mathbf{V_n}/\Gamma].$$

- Inspect matrix $((\rho + 1/\Gamma)\mathbf{I} - \mathbf{S_n D_n})$ and notice that all matrix are super sparse.

The Aiyagari Model in Continuous Time

Households:

- Two states: **High (employed)** and **Low (unemployed)**.
- High state: $(1 - \tau)w$.
- Low state: μw.
- An employed individual becomes unemployed with probability λ_e.
- An unemployed individual becomes employed with probability λ_u.

- Dynamics of aggregate employment and unemployment:

$$e_{t+1} = (1 - \lambda_e)e_t + \lambda_u u_t,$$

$$u_{t+1} = \lambda_e e_t + (1 - \lambda_u)u_t.$$

- In Δ units of time

$$e_{t+\Delta} = (1 - \Delta\lambda_e)e_t + \Delta\lambda_u u_t,$$

$$u_{t+\Delta} = \Delta\lambda_e e_t + (1 - \Delta\lambda_u)u_t.$$

- Rearranging and taking limits

$$\dot{e}_t = -\lambda_e e_t + \lambda_u u_t,$$

$$\dot{u}_t = \lambda_e e_t - \lambda_u u_t.$$

- System:

$$\dot{\mathbf{s}}_t = \mathbf{T}\mathbf{s}_\mathbf{t}.$$

  with

$$\mathbf{T} = \begin{pmatrix} -\lambda_e & \lambda_u \\ \lambda_e & -\lambda_u \end{pmatrix}$$

- Stationary equilibrium $\dot{\mathbf{s}}_t = 0$ and

$$\mathbf{0} = \mathbf{T}\mathbf{s},$$

  and $\mathbf{s}$ is an eigenvector associated with a zero eigenvalue, with the eigenvector normalised to sum to one.

Some Comments

- Can be solved as a regular eigenvalue problem.
- But we can use the following trick:
  1. Create a vector
  $$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and a matrix } \hat{\mathbf{T}} = \begin{pmatrix} 1 & 0 \\ \lambda_e & -\lambda_u \end{pmatrix}.$$
  2. Find $\hat{\mathbf{s}} = \hat{\mathbf{T}}^{-1} b$.
  3. Normalise $\hat{\mathbf{s}}$ to sum to one to find $\mathbf{s}$.
- The first element of $\mathbf{s}$ is then the stationary employment rate, and the second the stationary unemployment rate.

Government

- Government runs a balanced budget, so not deficits.
- The tax rate solves: $e\tau w = u\mu w$. Or:

$$\tau = \frac{u}{e}\mu.$$

Households

- Bellman equation for an employed agent:

$$V(a_t, e_t) = \max_{c_t}\{u(c_t)+(1-\rho)[(1-\lambda_e)V(w_t(1-\tau_t)+(1+r_t)a_t-c_t, e_t)$$

$$+\lambda_e V(w_t(1-\tau_t)+(1+r_t)a_t-c_t, u_t)]\},$$

subject to $a_t \geq 0$.

- In Δ units of time:

$$V(a_t, e_t) = \max_{c_t}\{\Delta u(c_t)+$$

$$(1-\Delta\rho)[(1-\Delta\lambda_e)V(a_t+\Delta(w_t(1-\tau_t)+r_t a_t-c_t), e_t)+$$

$$\Delta\lambda_e V(a_t+\Delta(w_t(1-\tau_t)+r_t a_t-c_t), u_t)]\}.$$

Households

- Rearrange and divide by Δ

$$0 = \max_{c_t}\{u(c_t) + \frac{V(a_t + \Delta(w_t(1-\tau_t) + r_t a_t - c_t), e_t) - V(a_t, e_t)}{\Delta}$$

$$-(\rho + \lambda_e + \Delta\rho\lambda_e V(a_t + \Delta(w_t(1-\tau_t) + r_t a_t - c_t), e_t)$$

$$+\lambda_e V(a_t + \Delta(w_t(1-\tau_t) + r_t a_t - c_t), u_t)]\}.$$

- Take limits for $\Delta \to 0$ and rearrange:

$$\rho V(a,e) = \max_c\{u(c) + V_a(a,e)(w(1-\tau) + ra - c)$$

$$-\lambda_e(V(a,e) - V(a,u))\}$$

Households

- So households' problem is given by the two **HJB** equations

$$\rho V(a,e) = \max_c \{u(c) + V_a(a,e)(w(1-\tau) + ra - c)$$

$$-\lambda_e(V(a,e) - V(a,u))\},$$

and

$$\rho V(a,u) = \max_c \{u(c) + V_a(a,u)(\mu w + ra - c)$$

$$-\lambda_u(V(a,u) - V(a,e))\},$$

Solution

1. Start with a linearly spaced grid for assets $\mathbf{A} = \{a_1, a_2, ..., a_N\}$, $da = a_{n+1} - a_n$.

2. For each point in the grid for assets guess a $V_0(a_i, s)$, $\forall a_i \in \mathbf{A}$, and $s \in \{e, u\}$. This gives us $\mathbf{V_{0,e}}$ and $\mathbf{V_{0,u}}$.

3. Call the stacked $2N \times 1$ vector $(\mathbf{V_{0,e}}, \mathbf{V_{0,u}})'$ as $\mathbf{V_0}$.

Solution

4. Creat two $N \times N$ difference operators:

$$D_f = \begin{pmatrix} -1/da & 1/da & 0 & 0 & \cdots & 0 \\ 0 & -1/da & 1/da & 0 & \cdots & 0 \\ \vdots & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -1/da & 1/da \\ 0 & \cdots & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$D_b = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ -1/da & 1/da & 0 & 0 & \cdots & 0 \\ \vdots & -1/da & 1/da & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & -1/da & 1/da \end{pmatrix}$$

Solution

5. Creat one $2N \times 2N$ matrix as:

$$B = \begin{pmatrix} -\lambda_e & 0 & \cdots & 0 & \lambda_u & 0 & \cdots & 0 \\ 0 & -\lambda_e & \cdots & 0 & 0 & \lambda_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\lambda_e & 0 & 0 & \cdots & \lambda_u \\ \lambda_e & 0 & \cdots & 0 & -\lambda_u & 0 & \cdots & 0 \\ 0 & \lambda_e & \cdots & 0 & 0 & -\lambda_u & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_e & 0 & 0 & \cdots & -\lambda_u \end{pmatrix}$$

Solution

6. Calculate the derivative of the value functions using both forward and backward differences.

$$\mathbf{V_{a,f}}(a,e) = \mathbf{D_f V_{0,e}}, \ \mathbf{V_{a,b}}(a,e) = \mathbf{D_b V_{0,e}}.$$

$$\mathbf{V_{a,f}}(a,u) = \mathbf{D_f V_{0,u}}, \ \mathbf{V_{a,b}}(a,u) = \mathbf{D_b V_{0,u}}.$$

7. Set the first element of $\mathbf{V_{a,b}}(a,e) = u'(w(1-\tau) + r\phi)$ and $\mathbf{V_{a,b}}(a,u) = u'(\mu w + r\phi)$, and the last elements of $\mathbf{V_{a,f}}(a,e) = u'(w(1-\tau) + ra_N)$ and $\mathbf{V_{a,f}}(a,u) = u'(\mu w + ra_N)$.

8. Find optimal consumption through:

$$u'(\mathbf{c_{e,f}}) = \mathbf{D_f V_{0,e}}, \ u'(\mathbf{c_{e,b}}) = \mathbf{D_b V_{0,e}}.$$

$$u'(\mathbf{c_{u,f}}) = \mathbf{D_f V_{0,u}}, \ u'(\mathbf{c_{u,b}}) = \mathbf{D_b V_{0,e}}.$$

Solution

9. Find optimal savings through:

$$\mathbf{s_{e,f}} = w(1-\tau) + ra - \mathbf{c_{e,f}},\ \mathbf{s_{e,b}} = w(1-\tau) + ra - \mathbf{c_{e,b}}.$$

$$\mathbf{s_{u,f}} = \mu w + ra - \mathbf{c_{u,f}},\ \mathbf{s_{u,b}} = \mu w + ra - \mathbf{c_{u,b}}.$$

10. Create indicator vectors:

$$\mathbf{I_{e,f}} = (I_{1,e,f}, I_{2,e,f}, ..., I_{N,e,f})',\ I_{e,b} = (I_{1,e,b}, I_{2,e,b}, ..., I_{N,e,b})',$$

$$\mathbf{I_{u,f}} = (I_{1,u,f}, I_{2,u,f}, ..., I_{N,u,f})',\ I_{u,b} = (I_{1,u,b}, I_{2,u,b}, ..., I_{N,u,b})',$$

where $I_{i,s,f} = 1$ if $s_{i,s,f} > 0$ and $I_{i,s,b} = 1$ if $s_{i,s,b} < 0$, for $i = 1, ..., N$ and $s \in \{e, u\}$.

Solution

11. Find consumption:

$$\mathbf{c_e} = \mathbf{I_{e,f}}\mathbf{c_{e,f}} + \mathbf{I_{e,b}}\mathbf{c_{e,b}}.$$

$$\mathbf{c_u} = \mathbf{I_{u,f}}\mathbf{c_{u,f}} + \mathbf{I_{u,b}}\mathbf{c_{u,b}}.$$

12. Find savings:

$$\mathbf{s_e} = \mathbf{I_{e,f}}\mathbf{s_{e,f}} + \mathbf{I_{e,b}}\mathbf{s_{e,b}}.$$

$$\mathbf{s_u} = \mathbf{I_{u,f}}\mathbf{s_{u,f}} + \mathbf{I_{u,b}}\mathbf{s_{u,b}}.$$

13. And matrices

$$\mathbf{S_e}\mathbf{D_e} = diag(\mathbf{I_{e,f}}\mathbf{s_{e,f}})\mathbf{D_f} + diag(\mathbf{I_{e,b}}\mathbf{s_{e,b}})\mathbf{D_b}.$$

$$\mathbf{S_u}\mathbf{D_u} = diag(\mathbf{I_{u,f}}\mathbf{s_{u,f}})\mathbf{D_f} + diag(\mathbf{I_{u,b}}\mathbf{s_{u,b}})\mathbf{D_b}.$$

Solution

14. Lastly, find the $2N \times 2N$ matrix $\mathbf{S_0 D_0}$ as:

$$\mathbf{S_0 D_0} = \begin{pmatrix} \mathbf{S_e D_e} & \mathbf{0} \\ \mathbf{0} & \mathbf{S_u D_u} \end{pmatrix}.$$

15. And the matrix $\mathbf{P_0}$ as

$$\mathbf{P_0} = \mathbf{S_0 D_0} + B$$

Back to the HJB Equations

- Using the implicit method the households' problem is given by the two **HJB** equations

$$\rho V_{n+1}(a,e) = u(c_n) + V_{a,n+1}(a,e)(w(1-\tau) + ra - c_n)$$

$$-\lambda_e(V_{n+1}(a,e) - V_{n+1}(a,u)),$$

and

$$\rho V_{n+1}(a,u) = u(c_n) + V_{a,n+1}(a,u)(\mu w + ra - c_n)$$

$$-\lambda_u(V_{n+1}(a,u) - V_{n+1}(a,e)).$$

Back to the HJB Equations

- These can be written as:

$$\rho \mathbf{V_{n+1}} = u(\mathbf{c_n}) + \mathbf{P_n} \mathbf{V_{n+1}}$$

with $\mathbf{c_n} = (\mathbf{c_{n,e}}, \mathbf{c_{n,u}})$.

- So we iterate on

$$\mathbf{V_{n+1}} = [(\rho + 1/\Gamma)\mathbf{I} - \mathbf{P_n}]^{-1}[u(\mathbf{c_n}) + \mathbf{V_n}/\Gamma]$$

until convergence.

Firms

- Firms solve a standard static optimisation problem:

$$\Pi_t = \max_{K_t, N_t} = \{K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - (r_t + \delta) K_t\}.$$

First-order conditions:

$$r_t + \delta = \alpha \left(\frac{K_t}{N_t}\right)^{\alpha - 1}, \; w_t = (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha}.$$

- In a stationary equilibrium:

$$r + \delta = \alpha \left(\frac{K}{1-u}\right)^{\alpha - 1}, \; w = (1-\alpha)\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

Stationary Distribution

What is the evolution of the endogenous stationary distribution of wealth and employment status?

- Denote the CDF as $G_{t+1}(a, e)$. This must satisfy

$$G_{t+1}(a, e) = (1 - \lambda_e)G_t(a^e_{-1}, e) + \lambda_u G_t(a^u_{-1}, u),$$

  where as a^s_{-1} denotes "where you came from".

- In Δ units of time approximate this as $a^e_{-1} = a - \Delta s_e$ and $a^u_{-1} = a - \Delta s_u$. Thus:

$$G_{t+\Delta}(a, e) = (1 - \Delta\lambda_e)G_t(a - \Delta s_e, e) + \Delta\lambda_u G_t(a - \Delta s_u, u).$$

Stationary Distribution

$$G_{t+\Delta}(a,e) = (1-\Delta\lambda_e)G_t(a-\Delta s_e, e) + \Delta\lambda_u G_t(a-\Delta s_u, u).$$

- Subtract $G_t(a,e)$ from both sides and divide by Δ:

$$\frac{G_{t+\Delta}(a,e) - G_t(a,e)}{\Delta} = \frac{G_t(a-\Delta s_e, e) - G_t(a,e)}{\Delta}$$

$$-\lambda_e G_t(a-\Delta s_e, e) + \lambda_u G_t(a-\Delta s_u, u).$$

- Taking limits for $\Delta \Rightarrow 0$:

$$\dot{G}_t(a,e) = -g_t(a,e)s_e(a) - \lambda_e G_t(a,e) + \lambda_u G_t(a,u).$$

Stationary Distribution/Kolmogorov Forward Equation

$$\dot{G}_t(a,e) = -g_t(a,e)s_e(a) - \lambda_e G_t(a,e) + \lambda_u G_t(a,u).$$

- Differentiate the above equation with respect to a:

$$\dot{g}_t(a,e) = -\frac{\partial[g_t(a,e)s_e(a)]}{\partial a} - \lambda_e g_t(a,e) + \lambda_u g_t(a,u).$$

- Thus the law of motion for the endogenous distribution is:

$$\dot{g}_t(a,e) = -\frac{\partial[g_t(a,e)s_e(a)]}{\partial a} - \lambda_e g_t(a,e) + \lambda_u g_t(a,u),$$

$$\dot{g}_t(a,u) = -\frac{\partial[g_t(a,u)s_u(a)]}{\partial a} + \lambda_e g_t(a,e) - \lambda_u g_t(a,u).$$

Stationary Distribution/Kolmogorov Forward Equation

- Remember the matrix:

$$\mathbf{P_n} = \mathbf{S_n D_n} + \mathbf{B}.$$

- When converged:

$$\mathbf{P} = \mathbf{SD} + \mathbf{B}.$$

- Turns out that

$$\dot{\mathbf{g}}_\mathbf{t} = P'\mathbf{g_t},$$

where $\mathbf{g_t}$ is the stacked vector $(\mathbf{g_t}(a,e), \mathbf{g_t}(a,u))'$.

Solving the Aiyagari Model: Summary

1. Guess for an interest rate r_n. Find w_n as

$$w_n = (1-\alpha)\left(\frac{\alpha}{r_n+\delta}\right)^{\frac{\alpha}{1-\alpha}}.$$

2. Find $\mathbf{V}$ as:

$$\mathbf{V} = [(\rho + 1/\Gamma)\mathbf{I} - \mathbf{P}]^{-1}[u(\mathbf{c}) + \mathbf{V}/\Gamma].$$

3. Find $\mathbf{g}$ by solving:

$$\mathbf{0} = \mathbf{P}'\mathbf{g},$$

and normalise to sum to one (remember how we found $\mathbf{s}$ above)

Solving the Aiyagari Model: Summary

1. Find K_n as
$$K_n = \mathbf{g}' \begin{pmatrix} \mathbf{a} \\ \mathbf{a} \end{pmatrix}.$$

2. Find $\hat{r}$ as:
$$\hat{r} = \alpha \left(\frac{K_n}{1-u} \right)^{\alpha-1} - \delta.$$

3. If $\hat{r} > r_n$ set $r_{n+1} > r_n$, else set $r_{n+1} < r_n$.

4. Repeat until convergence.

Model Presented in Continuous Time

- Households solve:

$$\max_{c_t \geq 0} E_0 \int_0^\infty e^{-\rho t} u(c_t) dt, \quad \text{subject to}$$

$$\dot{a}_t = \omega_t + r_t a_t - c_t, \quad \text{and} \quad a_t \geq -\phi,$$

$$\omega_t \in \{(1 - \tau_t) w_t, \mu w_t\}.$$

Poisson with intensities λ_e and λ_u

Summary

Solving Heterogenous Agents Model = Solving PDE

- A system of two PDE:
  1. **HJB** equation for individual outcomes
  2. **Kolmogorov Forward** equation for evolution of the wealth distribution

Solution

As Moll writes, the discretised solution for the stationary equilibrium of the Aiyagari model is described by the following equations:

$$\rho \mathbf{V} = \mathbf{u}(\mathbf{V}) + \mathbf{P}\mathbf{V} \tag{HJB}$$

where

$$\mathbf{u}(\mathbf{V}) = \mathbf{u}^{-1}(\mathbf{V}')$$

$$\mathbf{0} = \mathbf{P}'\mathbf{g} \tag{KF}$$

$$K = \mathbf{g}' \begin{pmatrix} \mathbf{a} \\ \mathbf{a} \end{pmatrix} \tag{K}$$

$$r = F_K(K, 1), \; w = F_L(K, 1) \tag{P}$$

Normalising labour productivity to 1 at the aggregate!