Macroeconomics III
Lecture 9: Local linear methods for solving macro models

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Road Map

- Local approximation methods
- Log-linearisation and linearisation
- Solving systems of (stochastic) difference equations with
  - [Blanchard and Kahn,1980] method
  - [Uhlig, 1999] method
  - [Christiano, 2002] method

A generic DSGE model

- Model

$$\begin{aligned}
U_0 &= \max_{x_t, y_t} E_0 \sum_{t=0}^{\infty} \beta^t u(x_t, y_t) \\
s.t.\ \ x_t &= g(x_{t-1}, y_t, z_t) \text{ (or } x_{t+1} = g\left(x_t, y_t, z_t\right)\text{)} \\
z_t &= f(z_{t-1}, \varepsilon_t),\ \varepsilon_t \sim iid(0, \Sigma) \\
& \quad x_{-1} \text{ (or } x_0\text{) and } z_0 \text{ given}
\end{aligned}$$

- Vector of endogenous state variables, $m \times 1$: x_t
- Vector of exogenous state variables (e.g. stochastic shocks): z_t
- Vector of all other endogenous variables (including control variables), $n \times 1$: y_t

Deriving equilibrium and 'solving' the model

- Write Lagrangian
- Take first-order conditions
- Collect all other equations that come from market clearing, policy rules, etc.
- Equilibrium conditions: a system of ν (first-order) difference equations of ν variables
- **Typically highly non-linear and stochastic**
- In general, DGSE models have large number of variables *and* large number of *state* variables $\Longrightarrow$ very complicated to solve with Global Methods.

First-Order Approximation: Log-linearisation

- Determine all conditions/equations that characterise the equilibrium in the economic environment.

- Find (deterministic) steady states for all variables – *not always trivial*

- Let x_t be a variable. Then define

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right) \approx \text{deviation of } x_t \text{ from its steady state } = \frac{x_t - \bar{x}}{\bar{x}}$$

$$\Rightarrow \boxed{x_t = \bar{x}\exp\hat{x}_t}$$

- Collect all equilibrium conditions.

- For each equation, replace variable x_t with $\bar{x}\exp\hat{x}_t$.

First-Order Approximation: Log-linearisation

- Replace $\exp \hat{x}_t$ with the approximation $\exp \hat{x}_t \approx \hat{x}_t + 1$ (First-order Taylor appr. $f(\hat{x}_t) = e^{\hat{x}_t}$)

- Collect all constant terms (use the steady-state relationships)

- Collect all variables, so that you finally have an equation **linear** and **homogeneous** in all variables

- **Collect all difference equations into a linear system of (stochastic) difference equations.** Analyse this system to determine whether:
  (a) There exist solutions or not, and how many
  (b) If unique solution exists, find matrices such that:

$$
\begin{aligned}
\hat{x}_t &= P\hat{x}_{t-1} + Q\hat{z}_t \text{ or } \hat{x}_{t+1} = P\hat{x}_t + Q\hat{z}_t \\
\hat{y}_t &= R\hat{x}_{t-1} + S\hat{z}_t \text{ or } \hat{y}_t = R\hat{x}_t + S\hat{z}_t \\
\hat{z}_t &= N\hat{z}_{t-1} + \varepsilon_t
\end{aligned}
$$

Example: Stochastic growth model

- The model:

$$\max_{k_t, c_t} E_0 \sum_{t=0}^{\infty} \beta^t \ln c_t$$

$$k_t = z_t k_{t-1}^{\alpha} - c_t$$

$$k_t, c_t \geq 0 \text{ and } k_0 \ \text{ given}$$

$$\log(z_{t+1}) = (1-\rho)\log(\bar{z}) + \rho \log(z_t) + \eta_t, \quad \eta_t \sim \ \text{iid} N(0, \sigma^2).$$

Example: Stochastic growth model

- State variables are: k_{t-1} (endogenous) and z_t (exogenous)
- Equilibrium conditions

$$\lambda_t = \frac{1}{c_t}, \quad \lambda_t = \alpha\beta E_t\left[\lambda_{t+1} z_{t+1} k_t^{\alpha-1}\right]$$

$$k_t = y_t - c_t, \; y_t = z_t k_{t-1}^{\alpha}, \; \log(z_{t+1}) = (1-\rho)\log(\bar{z}) + \rho\log(z_t) + \eta_t$$

- Log-linearise around the deterministic steady-state, taking $\bar{z} = 1$

$$\begin{aligned}
\lambda_t &= \frac{1}{c_t} \Longrightarrow \hat{\lambda}_t \approx -\hat{c}_t \\
\lambda_t &= \alpha\beta E_t\left(\lambda_{t+1} z_{t+1} k_t^{\alpha-1}\right) \Longrightarrow \hat{\lambda}_t \approx E_t\hat{\lambda}_{t+1} + \rho\hat{z}_t - (1-\alpha)\hat{k}_t \\
k_t &= y_t - c_t \Longrightarrow \bar{k}\left(\hat{k}_t + 1\right) \approx \bar{y}\left(\hat{y}_t + 1\right) - \bar{c}\left(\hat{c}_t + 1\right) \Longrightarrow \\
\hat{k}_t &\approx \frac{\bar{y}}{\bar{k}}\hat{y}_t - \frac{\bar{c}}{\bar{k}}\hat{c}_t \\
y_t &= z_t k_{t-1}^{\alpha} \Longrightarrow \hat{y}_t \approx \alpha\hat{k}_{t-1} + \hat{z}_t \\
\hat{z}_{t+1} &= \rho\hat{z}_t + \eta_t
\end{aligned}$$

First-Order Approximations: Linearisation

- Repeat same steps as for log-linearisation but instead use: $\hat{x}_t = x_t - \bar{x}$

- For every bit of every equation do a first-order Taylor approximation:

$$
\begin{aligned}
f(x_t) &= f(\bar{x}) + f'(\bar{x})\,\hat{x}_t \\
g(x_{1t}, x_{2t}) &= g(\bar{x}_1, \bar{x}_2) + g_1(\bar{x}_1, \bar{x}_2)\,\hat{x}_{1t} + g_2(\bar{x}_1, \bar{x}_2)\,\hat{x}_{2t}
\end{aligned}
$$

- *Linearisation sometimes gives less elegant expressions, but is appropriate when some variables may have zero steady-state, e.g. inflation*

- *Dynamic properties of approximate models with linearisation or log-linearisation should be similar*

Example: Stochastic growth model

- Solving by hand...

- Undetermined coefficients: Eliminate all variables apart from state vars

$$\left(1+\alpha^2\beta\right)\hat{k}_t = \alpha\beta E_t\hat{k}_{t+1} + \alpha\hat{k}_{t-1} + \left(1-\rho\alpha\beta\right)\hat{z}_t$$

- **Postulate:**

$$\begin{aligned} \hat{k}_t &= \phi_1\hat{k}_{t-1} + \phi_2\hat{z}_t \\ E_t\hat{k}_{t+1} &= \phi_1\hat{k}_t + \phi_2\rho\hat{z}_t \end{aligned}$$

- Then, collect terms:

$$\hat{k}_t = \frac{\alpha}{1+\alpha^2\beta-\alpha\beta\phi_1}\hat{k}_{t-1} + \frac{\alpha\beta\rho\phi_2+1-\alpha\beta\rho}{1+\alpha^2\beta-\alpha\beta\phi_1}\hat{z}_t$$

Example: Stochastic growth model

- Unique solution implies

$$\phi_1 = \frac{\alpha}{1 + \alpha^2\beta - \alpha\beta\phi_1} \text{ and } \phi_2 = \frac{\alpha\beta\rho\phi_2 + 1 - \alpha\beta\rho}{1 + \alpha^2\beta - \alpha\beta\phi_1}$$

which gives

$$\begin{aligned} \phi_1 &= \alpha \text{ or } \phi_1 = \frac{1}{\alpha\beta} \\ \phi_2 &= \frac{1 - \alpha\beta\rho}{\alpha^2\beta - \alpha\beta\rho - \alpha\beta\phi_1 + 1} = \begin{cases} 1 & \text{if } \phi_1 = \alpha \\ \frac{1-\alpha\beta\rho}{\alpha^2\beta - \alpha\beta\rho} & \text{if } \phi_1 = \frac{1}{\alpha\beta} \end{cases} \end{aligned}$$

- Similarly can retrieve law of motion for the remaining variables once we have chosen ϕ_1 and ϕ_2

Example: Stochastic growth model

- Note: since $\alpha, \beta \in (0, 1)$ the stable solution (satisfying TVC) is given by $\phi_1 = \alpha$, i.e.:

$$\hat{k}_t = \alpha \hat{k}_{t-1} + \hat{z}_t$$

  Then

$$\begin{aligned} \hat{y}_t &= \alpha \hat{k}_{t-1} + \hat{z}_t \\ \hat{c}_t &= \alpha \hat{k}_{t-1} + \hat{z}_t \\ \hat{z}_{t+1} &= \rho \hat{z}_t + \eta_t \end{aligned}$$

- Given the states (z_t, k_t) and realisation of the iid shock, η_t, we can construct endogenous series for k_t, y_t and c_t.

Comments

- Next we solve systems of (stochastic) difference equations with
  - [Blanchard and Kahn,1980] method
  - [Uhlig, 1999] method
  - [Christiano, 2002] method
- All these three solution methods are similar and should generate **similar results for the same model and parametrisation**

Example

Suppose:

$$y_t = \rho y_{t-1}$$

- infinite number of solutions, independent of the value of ρ.
- If y_0 is given, then unique solution, independent of the value of ρ.
- **Blanchard-Kahn** conditions apply to models that add as a requirement that the series do not explode:
  - $\rho > 1$: unique solution, namely $y_t = 0$ for all t;
  - $\rho < 1$: Many solutions; y_0 might determine the unique path;
  - $\rho = 1$: Many solutions; y_0 might determine the unique path (be careful with $\rho = 1$, uncertainty matters.

Blanchard-Kahn Method

- Write the system of linear difference equations as

$$A \begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + C \hat{z}_t$$

- Assuming that A is invertible, we transform to

$$\begin{bmatrix} \hat{x}_t \\ E_t \hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G \hat{z}_t$$

where $F = A^{-1}B$ is $(n+m) \times (n+m)$ and $G = A^{-1}C$

- Get the Jordan or spectrum decomposition of F

$$F = HJH^{-1} = \begin{bmatrix} v_1 & \dots & v_{n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+m} \end{bmatrix} \begin{bmatrix} v_1 & \dots & v_{n+m} \end{bmatrix}^{-1}$$

where the eigenvalues are ordered such that $|\lambda_1| < |\lambda_2| < \dots < |\lambda_{n+m}|$ and v_i are their corresponding eigenvectors.

Blanchard-Kahn Method

- Let h be the number of eigenvalues in J that are outside the unit circle, i.e. $|\lambda_i| > 1$.

- $\boxed{\textbf{Proposition}}$: [Blanchard-Kahn,1980].
  (a) If $h = n$, the system of stochastic difference equations has **a unique solution**.
  (b) If $h > n$, the system of linear stochastic difference equations has **no solutions.**
  (c) If $h < n$, the system of linear stochastic difference equations has **infinite solutions.**

- **Uniqueness**: For every free variable, you need one eigenvalue that is larger than one (saddle path stability).

- **Multiplicity**: Not enough eigenvalues larger than one (indeterminacy).

Blanchard-Kahn Method

- Suppose solution is **unique.** (hopefully the desirable outcome)

- To solve the system of difference equations

$$\begin{bmatrix} \hat{x}_t \\ E_t\hat{y}_{t+1} \end{bmatrix} = F \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} + G\hat{z}_t$$

  we do a change of variables according to

$$\begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} \text{ and } \begin{bmatrix} \tilde{x}_t \\ E_t\tilde{y}_{t+1} \end{bmatrix} = H^{-1} \begin{bmatrix} \hat{x}_t \\ E_t\hat{y}_{t+1} \end{bmatrix}$$

- Which implies

$$\begin{bmatrix} \tilde{x}_t \\ E_t\tilde{y}_{t+1} \end{bmatrix} = J \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + V\hat{z}_t$$

  where

$$V = H^{-1}G$$

Blanchard-Kahn Method

- The matrix J is diagonal, with the eigenvalues in the diagonal, ordered from smallest to largest.

- Partitioning:

$$\begin{bmatrix} \tilde{x}_t \\ E_t\tilde{y}_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \hat{z}_t$$

- So that

$$\tilde{x}_t = J_1\tilde{x}_{t-1} + V_1\hat{z}_t$$

  is a system of uncoupled difference equations which is **stable**, since all eigenvalues in J_1 are inside the unit circle.

- And

$$E_t\tilde{y}_{t+1} = J_2\tilde{y}_t + V_2\hat{z}_t \iff \tilde{y}_t = J_2^{-1}E_t\tilde{y}_{t+1} - J_2^{-1}V_2\hat{z}_t$$

Blanchard-Kahn Method

- Iterate the second expression forward and take expectations:

$$\tilde{y}_{t+1} = J_2^{-1} E_{t+1}\tilde{y}_{t+2} - J_2^{-1} V_2 \hat{z}_{t+2}$$

$$E_t\tilde{y}_{t+1} = J_2^{-1} E_t E_{t+1}\tilde{y}_{t+2} - J_2^{-1} V_2 \underbrace{E_t\hat{z}_{t+2}}_{=0} = J_2^{-1} E_t\tilde{y}_{t+2}$$

  and the above becomes after many iterations

$$\begin{aligned} \tilde{y}_t &= \left(J_2^{-1}\right)^2 E_t\tilde{y}_{t+2} - J_2^{-1} V_2 \hat{z}_t \\ &\quad ... \\ \tilde{y}_t &= -J_2^{-1} V_2 \hat{z}_t \end{aligned}$$

- So we have

$$\begin{aligned} \tilde{x}_t &= J_1 \tilde{x}_{t-1} + V_1 \hat{z}_t \\ \tilde{y}_t &= -J_2^{-1} V_2 \hat{z}_t \end{aligned}$$

Blanchard-Kahn Method

- This system can be solved with known methods for uncoupled difference equations, and we recover the original variables from using the above solutions and the change of variables

$$\begin{bmatrix} \hat{x}_{t-1} \\ \hat{y}_t \end{bmatrix} = H \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{y}_t \end{bmatrix}$$

Simple example

- Example (1st order)

$$
\begin{aligned}
x_{t-1} &= \phi x_t + E_t[z_{t+1}], \\
z_t &= 0.9 z_{t-1} + \epsilon_t.
\end{aligned}
$$

  - $|\phi| > 1$: Unique stable fixed point;
  - $|\phi| < 1$: No stable solutions; too many eigenvalues > 1.

Simple example

- Corresponding state space system:

$$\begin{bmatrix} \phi & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_t \\ z_{t+1} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0.9 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ z_t \end{bmatrix} = \begin{bmatrix} 0 \\ \epsilon_{t+1} \end{bmatrix}$$

$$\begin{bmatrix} 1/\phi & 0 \\ 0 & 0.9 \end{bmatrix}$$

- So we just need stability $\Rightarrow |\phi| > 1$.

Interest Rule in a New Keynesian Model

Consider a simple log-linearised New Keynesian model:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}, \text{ (Short-run Agg. Supply)}$$

$$x_t = E_t x_{t+1} - \frac{1}{\gamma}(i_t - E_t \pi_{t+1}) + \epsilon_t^x, \text{ (IS curve)}$$

$$i_t = \rho_r i_{t-1} + \epsilon_t, \text{ (Central bank's policy rule).}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\gamma} & 1 & \frac{1}{\gamma} \\ 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} i_t \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\kappa & 1 \end{bmatrix} \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \epsilon_t \\ -\epsilon_t^x \\ 0 \end{bmatrix}$$

Interest Rule in a New Keynesian Model

Pre-multiplying both sides by the inverse of the matrix on the left:

$$\begin{bmatrix} i_t \\ E_t x_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = W \begin{bmatrix} i_{t-1} \\ x_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{\gamma} & 1 & \frac{1}{\gamma} \\ 0 & 0 & \beta \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_t \\ -\epsilon_t^x \\ 0 \end{bmatrix}$$

where:

$$W = \begin{bmatrix} \rho_r & 0 & 0 \\ \frac{\rho_r}{\gamma} & 1 + \frac{\kappa}{\beta\gamma} & -\frac{1}{\beta\gamma} \\ 0 & -\frac{\kappa}{\beta} & \frac{1}{\beta} \end{bmatrix}$$

This system is indeterminate. There is only one eigenvalue outside the unit circle and two non-predetermined variables (x and π).

Uhlig's Method

- Essentially solving a linear system of difference equations using 'undetermined coefficients'.
- (Log)-Linearise all relevant equations
- Let x_t be an $m \times 1$ vector of endogenous state variables
- Let z_t be a $k \times 1$ vector of exogenous state variables
- Let y_t be a $n \times 1$ vector of other endogenous (control) variables
- Summarise system of linear (stochastic) difference equations as

$$
\begin{aligned}
0 &= E_t \left[F x_{t+1} + G x_t + H x_{t-1} + J y_{t+1} + K y_t + L z_{t+1} + M z_t \right] && (1) \\
z_{t+1} &= N z_t + \varepsilon_{t+1} && (2) \\
0 &= A x_t + B x_{t-1} + C y_t + D z_t && (3)
\end{aligned}
$$

where N has only stable eigenvalues

Uhlig's Method

- C is $l \times n$ and $rank(C) = n$ (with $l \geq n$, i.e. the number of equations in (3) is larger or equal than the number of the non-state endogenous variables)
- F is $(m + n - l) \times m$ (i.e. the number of expectational equations is at most equal to the number of endogenous state variables)
- The rest of the matrices conform with the above dimensions
- The total number of equations in (1) and (3) is equal to the total number of endogenous variables
- Given these, we are looking for a solution of the form

$$x_t = Px_{t-1} + Qz_t \quad (4)$$

$$y_t = Rx_{t-1} + Sz_t \quad (5)$$

Uhlig's Method

- **Special case** $n = l$ (i.e. C is square and full rank $\Longleftrightarrow$ invertible)

- Substitute postulated expressions

$$\begin{aligned} x_t &= Px_{t-1} + Qz_t \\ y_t &= Rx_{t-1} + Sz_t \end{aligned}$$

  into system of equations (1) - (3)

- Using that $E_t\left(\varepsilon_{t+1}\right) = 0$, we get

$$\begin{aligned} 0 &= \left(AP + B + CR\right) x_{t-1} + \left(AQ + CS + D\right) z_t \\ 0 &= \left(FP^2 + GP + H + JRP + KR\right) x_{t-1} \\ &\quad + \left(FPQ + FQN + GQ + JRQ + JSN + KS + LN + M\right) z_t \end{aligned}$$

Uhlig's Method

- Both need to be true for all x_{t-1} and z_t, i.e. need to solve the following for P, Q, R and S

$$\begin{aligned} AP + B + CR &= 0 & (6) \\ AQ + CS + D &= 0 & (7) \\ FP^2 + GP + H + JRP + KR &= 0 & (8) \\ FPQ + FQN + GQ + JRQ + JSN + KS + LN + M &= 0 & (9) \end{aligned}$$

- The first and third of these expressions pin down P and R, by substituting we get

$$\Psi P^2 = \Gamma P + \Theta \qquad (10)$$

where

$$\Psi = F - JC^{-1}A, \;\; \Gamma = \left(JC^{-1}B - G + KC^{-1}A\right), \;\; \Theta = KC^{-1}B - H$$

Uhlig's Method

- Quadratic equation in matrices (10) is a non-trivial problem, but can be done easily with Matlab functions that calculate the generalised eigenvalue problem (QZ decomposition)
- Basic idea: Expression (10) corresponds to solving a 2nd order system of linear difference equations,

$$\Psi x_{t+1} = \Gamma x_t + \Theta x_{t-1}$$

Then, rewrite this as

$$\begin{bmatrix} \Psi & 0_m \\ 0_m & I_m \end{bmatrix} \begin{bmatrix} x_{t+1} \\ x_t \end{bmatrix} = \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_m \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} \Longleftrightarrow \Delta w_{t+1} = \Xi w_t$$

Uhlig's Method

- The generalised eigenvalue problem for matrices Ξ and Δ is to find generalised eigenvalues λ_i and eigenvectors υ_i such that

$$\begin{aligned}
\Xi \upsilon_i &= \lambda_i \Delta \upsilon_i \Longleftrightarrow \begin{bmatrix} \Gamma & \Theta \\ I_m & 0_m \end{bmatrix} \begin{bmatrix} \upsilon_{1,i} \\ \upsilon_{2,i} \end{bmatrix} = \lambda_i \begin{bmatrix} \Psi & 0_m \\ 0_m & I_m \end{bmatrix} \begin{bmatrix} \upsilon_{1,i} \\ \upsilon_{2,i} \end{bmatrix} \\
\lambda_i \Psi \upsilon_{1,i} &= \Gamma \upsilon_{1,i} + \Theta \upsilon_{2,i} \text{ and } \upsilon_{1,i} = \lambda_i \upsilon_{2,i} \quad \Longleftrightarrow \quad \lambda_i^2 \Psi \upsilon_{2,i} = \lambda_i \Gamma \upsilon_{2,i} + \Theta \upsilon_{2,i} \\
\Psi V \Lambda^2 &= \Gamma V \Lambda + \Theta V
\end{aligned}$$

- Where Λ contains m eigenvalues and V contains m corresponding eigenvectors. Which ones?

- Select m eigenvalues that have m linearly independent corresponding eigenvectors (so that V^{-1} exists), then

$$\Psi V \Lambda^2 V^{-1} = \Gamma V \Lambda V^{-1} + \Theta$$

and therefore solution for P

$$P = V \Lambda V^{-1}$$

Uhlig's Method

- I.e. to solve (10), we just need to solve the generalised eigenvalue-eigenvector problem (matlab functions do that) and we get P.
- Then: All other matrices Q, R and S can be retrieved easily from (6)-(9).
- Is solution $x_t = Px_{t-1} + Qz_t$ stable?
- $\boxed{\textbf{Proposition}}$: If all the eigenvalues of P (i.e. the diagonal elements of Λ) are inside the unit circle, i.e. $|\lambda_i| < 1$, then the solution is stable.

Uhlig's Method

- Special case $n < l$ (i.e. C is not square anymore)
- Use the concept of pseudo inverses
- Everything is the same as before, apart from substituting C^{-1} with the pseudo-inverse of C (denoted with C^{+}). Matlab has an built-in function for finding pseudo-inverses, namely `pinv`
- You can download the toolkit here:https://www.wiwi.hu-berlin.de/de/professuren/vwl/wipo/research/MATLAB_Toolkit/version%2041

Example: Stochastic Growth Model with Ulhig's Method

- Order variables
  - End. State: k
  - Non-state: $\lambda,\ c,\ y$
  - Ex. State: z
- Rewrite the log-linear equations as

$$0 = E_t\left[0\hat{k}_{t+1} - (1-\alpha)\hat{k}_t + 0\hat{k}_{t-1} + \hat{\lambda}_{t+1} + 0\hat{c}_{t+1} + 0\hat{y}_{t+1} - \hat{\lambda}_t + 0\hat{c}_t + 0\hat{y}_t + \hat{z}_{t+1} + 0\hat{z}_t\right]$$

(Euler Equation)

$$0 = 0\hat{k}_t + 0\hat{k}_{t-1} + \hat{\lambda}_t + \hat{c}_t + 0\hat{y}_t + 0\hat{z}_t$$

(First-order condition for consumption)

$$0 = \hat{k}_t + 0\hat{k}_{t-1} + 0\hat{\lambda}_t + \frac{1-\alpha\beta}{\alpha\beta}\hat{c}_t - \frac{1}{\alpha\beta}\hat{y}_t + 0\hat{z}_t$$

(Resource constraint)

$$0 = 0\hat{k}_t - \alpha\hat{k}_{t-1} + 0\hat{\lambda}_t + 0\hat{c}_t + \hat{y}_t - \hat{z}_t$$

(Production function)

Example: Stochastic Growth Model with Ulhig's Method

- Then

$$
\begin{aligned}
F &= [0], \;\; G = [-(1-\alpha)], \;\; H = [0] \\
J &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \;\; K = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \\
L &= [1], \;\; M = [0] \\
A &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ -\alpha \end{bmatrix}, \\
C &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & \frac{1-\alpha\beta}{\alpha\beta} & -\frac{1}{\alpha\beta} \\ 0 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}
\end{aligned}
$$

Christiano's Method

- Also a method of undetermined coefficients, but more general, allowing for more lags and for different information sets (basic solution approach is the same)
- In Christiano's notation: system of linear/log-linear stochastic difference equations

$$\mathcal{E}_t \left[\sum_{i=0}^{r} \alpha_i z_{t+r-1-i} + \sum_{i=0}^{r-1} \beta_i s_{t+r-1-i} \right] = 0 \qquad (11)$$

- The notation $\mathcal{E}_t$ allows for various information sets (e.g. you can have E_t, E_{t-1}, etc.)
- **All endogenous variables** (states and controls) are included in z
- **All exogenous shocks** are included in s

Christiano's Method

- For any ARMA(p, q) shock, we can reset the exogenous variables to look like a VAR(1): $\theta_t = \rho\theta_{t-1} + \eta_t$
  - If all information sets are the same, then

$$s_t = \theta_t, \;\; P = \rho, \;\; \varepsilon_t = \eta_t$$

  - If any one of the information sets does not contain all of θ_t, then

$$s_t = (\theta_t \;\; \theta_{t-1})', \;\; P = \begin{pmatrix} \rho & 0 \\ I & 0 \end{pmatrix}, \;\; \varepsilon_t = \begin{pmatrix} \eta_t \\ 0 \end{pmatrix}$$

Christiano's method

	dims	explanation
z_{1t}	$n_1 \times 1$	(all) endogenous variables determined within period t
z_{2t}	$qn_1 \times 1$	$q \geq 0$ lagged elements of z_{1t}
z_t	$n_1(1+q) \times 1$	z_{1t} : all variables
		z_{2t} : vars relevant for determining z_{1t+1} at time $t+1$
s_t	$m \times 1$	vector of exogenous shocks, $s_t = Ps_{t-1} + \varepsilon_t$ including lags
θ_t	$m_\theta \times 1$	vector of shocks, written as VAR(1)
α_i	$n_1 \times n$	coefficient matrix for endogenous variables
β_i	$n_1 \times m$	coefficient matrix for shocks
$\mathcal{E}_t$		expectations operator that allows for various information sets
r	$r > q$	if $t+k$ is the largest lead, then $r = k+1$
α_0		coefficient matrix for the largest lead, $rank(\alpha_0) \neq 0$
τ	$m_\theta \times n_1$	$\tau_{ij} = \begin{matrix} 0, \text{if i-th element of } \theta \text{ is not in the info set of eq. } j \\ 1, \text{if i-th element of } \theta \text{ is in the info set of eq. } j \end{matrix}$

Christiano's Method

- We look for solutions of the form

$$\begin{pmatrix} z_{1t} \\ z_{2t} \end{pmatrix} = \underbrace{\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}}_{A} \begin{pmatrix} z_{1t-1} \\ z_{2t-1} \end{pmatrix} + \underbrace{\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}}_{B} \begin{pmatrix} \theta_t \\ \theta_{t-1} \end{pmatrix}$$

  with the initial condition z_{-1}.

- Functions `solvea.m` and `solveb.m`, give the solution matrices A and B
  (http://faculty.wcas.northwestern.edu/~lchrist/research/Solve/main.htm)

A Compact Summary of Christiano's Solution Method

- **STEP 1**: Solve for A
  - "Drop" expectations
  - Transform the original system into a first-order difference equation $aw_{t+1} + bw_t = 0$, by stacking zs (as with Uhlig's method), getting $aw_{t+1} = -bw_t$, where a and b are big matrices
  - Solve the above using generalised eigenvalue-eigenvector problem as before (QZ decomposition by Sims (2000))
  - Recover block of the solution matrix that yields A and condition for stability of system
- **STEP 2**: Solve for B
  - For a given A can be recovered easily (like other undetermined coefficient methods)

Example: Growth Model with Christiano's Method

- (*shock now denoted by* ω)
- Linearised equations

$$
\begin{aligned}
0 &= E_t\left[-(1-\alpha)\hat{k}_t + \hat{\lambda}_{t+1} - \hat{\lambda}_t + \hat{\omega}_{t+1}\right] \\
0 &= \hat{\lambda}_t + \hat{c}_t \\
0 &= \hat{k}_t + \frac{1-\alpha\beta}{\alpha\beta}\hat{c}_t - \frac{1}{\alpha\beta}\hat{y}_t \\
0 &= -\alpha\hat{k}_{t-1} + \hat{y}_t - \hat{\omega}_t
\end{aligned}
$$

- Variables determined at time t: $z_{1t} = z_t = \begin{bmatrix} \hat{k}_t & \hat{\lambda}_t & \hat{c}_t & \hat{y}_t \end{bmatrix}$
- $q = 0$
- $s_t = \hat{\omega}_t = \psi\hat{\omega}_{t-1} + \varepsilon_t$
- $\mathcal{E}_t = E_t$, $r = 2$
- $\tau = [1 \;\; 1 \;\; 1 \;\; 1]$
- System: $E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0$

Example: Growth Model with Christiano's Method

- Matrices

$$\alpha_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} -(1-\alpha) & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & \frac{1-\alpha\beta}{\alpha\beta} & -\frac{1}{\alpha\beta} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{pmatrix}, \quad \beta_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

Example 2: New Keynesian model with lagged inflation

- New Keynesian model with lagged inflation

$$\begin{aligned} y_t &= E_t\left(y_{t+1}\right) - \frac{1}{\sigma}\left(i_t - E_t\left(p_{t+1} - p_t\right)\right) \\ m_t &= \sigma y_t - \beta i_t + p_t \end{aligned}$$

$$p_t - p_{t-1} = \beta E_t\left(p_{t+1} - p_t\right) - \beta\gamma\left(p_t - p_{t-1}\right) + \gamma\left(p_{t-1} - p_{t-2}\right) + \kappa y_t$$

- Variables

$$\begin{aligned} z_{1t} &= \begin{bmatrix} y_t & i_t & p_t \end{bmatrix}' \\ z_{2t} &= \begin{bmatrix} y_{t-1} & i_{t-1} & p_{t-1} \end{bmatrix}' \end{aligned}$$

So that

$$z_t = \begin{bmatrix} y_t & i_t & p_t & y_{t-1} & i_{t-1} & p_{t-1} \end{bmatrix}'$$

Example 2: New Keynesian model with lagged inflation

- and

$$0 = E_t\left[y_t - y_{t+1} + \frac{1}{\sigma}i_t - \frac{1}{\sigma}p_{t+1} + \frac{1}{\sigma}p_t\right]$$

$$0 = E_t\left[m_t - \sigma y_t + \beta i_t - p_t\right]$$

$$0 = E_t\left[-\beta p_{t+1} + (1+\beta\gamma+\beta)\,p_t - (1+\gamma+\beta\gamma)\,p_{t-1} - \kappa y_t + \gamma p_{t-2}\right]$$

- Shock

$$s_t = m_t = \psi m_{t-1} + \eta_t$$

- $r = 2,\ q = 1$

- $\tau = [1\ \ 1\ \ 1]$

- Representation

$$E_t\left[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t\right] = 0$$

Example 2: New Keynesian model with lagged inflation

$$
\begin{aligned}
\alpha_0 &= \begin{pmatrix} -1 & 0 & -\frac{1}{\sigma} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta & 0 & 0 & 0 \end{pmatrix}, \\
\alpha_1 &= \begin{pmatrix} 1 & \frac{1}{\sigma} & \frac{1}{\sigma} & 0 & 0 & 0 \\ -\sigma & \beta & -1 & 0 & 0 & 0 \\ -\kappa & 0 & 1+\beta\gamma+\beta & 0 & 0 & 0 \end{pmatrix}, \\
\alpha_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\left(1+\gamma+\beta\gamma\right) & 0 & 0 & \gamma \end{pmatrix}, \\
\beta_0 &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

Example 2: New Keynesian model with lagged inflation

- Try these numbers to practice

$$\begin{aligned}\sigma &= 1,\ \beta = 0.99,\ \omega = 3/4 \\ \kappa &= \frac{(1-\omega)(1-\omega\beta)}{\omega},\ \psi = 0.7,\ \gamma = 0.66\end{aligned}$$

```
Solving for the feedback part (A matrix) of the solution...

Solving for the feedforward part (B matrix) of the solution...


The matrix A is:
         0         0   -1.1443         0         0    0.3923
         0         0    0.0000         0         0   -0.0000
         0         0    1.1443         0         0   -0.3923
    1.0000         0         0         0         0         0
         0    1.0000         0         0         0         0
         0         0    1.0000         0         0         0


The matrix B is:
    0.6372
   -0.2313
    0.1338
         0
         0
         0
```

Example 2: New Keynesian model with lagged inflation

- We have that

$$z_t = Az_{t-1} + B\theta_t$$

where

$$z_t = [y_t \quad i_t \quad p_t \quad y_{t-1} \quad i_{t-1} \quad p_{t-1}]'$$

- So that

$$\begin{aligned} y_t &= -1.1443 p_{t-1} + 0.3923 p_{t-2} + 0.6372 m_t \\ i_t &= -0.2313 m_t \\ p_t &= 1.1443 p_{t-1} - 0.3923 p_{t-2} + 0.1338 m_t \end{aligned}$$

Generating impulse response functions

- Generate a sequence of exogenous θ_t by setting

$$\eta_t = \begin{cases} \sigma_\eta & \text{for } t = q+2 \\ 0 & \text{otherwise} \end{cases}$$

- Can use $\eta_{q+2} = 1$ instead of σ_η if we don't care about absolute sizes
- Then

$$s_t = \begin{cases} \theta_t & \text{if all info sets same} \\ (\theta_t \;\; \theta_{t-1})' & \text{otherwise} \end{cases}$$

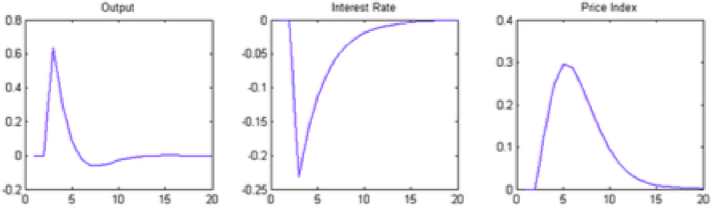
- Set $z_1 = 0$
- For $t = 1, ..., T$, generate

$$z_t = Az_{t-1} + Bs_t$$

- Plot the series z_{1t} (i.e. all endogenous variables) extracted easily from z_t

Impulse Response Functions

- Impulse responses to positive money shock in NK model with lagged inflation



Comparisons and comments

- Uhlig's approach and toolkit is more user friendly and practical to use. But:
  - Need to figure out the state/predetermined variables
  - Allows for one lead and one lag only
- Christiano's method is more general: it has the following features that Uhlig's toolkit does not do
  - Allows for unlimited number of leads and lags
  - Allows for different information sets, i.e. expectations dated at different points in time

Other toolboxes/methods

- **DYNARE** (developed by various researchers):
  - a pre-processor and a collection of GAUSS or MATLAB routines which solve non-linear models with forward looking variables
  - very convenient and easy to use DSGE solver
  - black box

Readings/References

- Blanchard and Kahn (1980). The Solution of Linear Difference Models under Rational Expectations. Econometrica.
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