Exam
Macroeconomics III

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1. Consider a finite horizon pure exchange economy with a single consumption good in each period. There are two types of consumers, A and B, with equal measure. You can normalise it to one. There are two periods 0 and 1, and two states in period 1, $s \in \{s_1, s_2\}$. Event s_1 occurs with probability $\pi \in [0, 1]$, and event s_2 with probability $1 - \pi$. The endowments of agents A and B are:

$$\omega^A = \{\omega_0^A, \omega_1^A(s_1), \omega_1^A(s_2)\} = \{0.5, 0.25, 0.75\},$$
$$\omega^B = \{\omega_0^B, \omega_1^B(s_1), \omega_1^B(s_2)\} = \{0.5, 0.75, 0.25\},$$

respectively. Let preferences be represented by the following utility function:

$$U^i = \sum_{t=0}^{1} \sum_{s^t} \beta^t u(c_t^i(s^t)) Pr(s^t), \;\; u(c_t^i(s^t)) = \ln(c_t^i(s^t)), \;\; i \in \{A, B\}.$$

   (a) Define an Arrow-Debreu Competitive Equilibrium for this economy.

   (b) Characterise the competitive equilibrium. Comment.

   (c) Define the Social Planner's problem, and solve the allocations of this problem. Explain.

   (d) Suppose that agents can only buy a one period bond at discount and this bond is not state contingent. This is what we call an incomplete markets economy since agents cannot write state contingent contracts for all states. Write down the problem of the agents and the market clearing conditions.

   (e) Show that he/she cannot attain the same level of consumption as in the complete markets case. *Hint:* you might want to proceed along the lines of a proof by contradiction.

2. Consider the following growth model:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} \leq Af(k_t),\ c_t \geq 0,\ k_{t+1} \geq 0, A > 0,\ \text{and } k_0 > 0.$$

Assume that $u(\cdot)$ is strictly concave and increasing for $c_t \geq 0$, and $f(\cdot)$ is concave and increasing.

(a) Formulate this maximisation problem as a dynamic programming problem.

(b) Is the value function associated with this problem a Contraction Mapping? What are the implications of that?

(c) Assuming that the value function is differentiable and all orther functions are differentiable, characterise the Euler equation that determines the optimal path of consumption and capital accumulation.

(d) Is this Euler equation enough to determine the path of k and c? If not, what other conditions do we need to impose? Write down these conditions and explain them intuitively.

(e) Describe an algorithm to describe the solution of this recursive problem. Go as far as you can in describing the details of your code. Describe how you could simulate a permanent rise in productivity.

(f) Could you think about any tricky to increase the efficiency of your code?

Sketch of the Solution (NOT THE FULL SOLUTION):

1. Answer of question 1:

   (a) Define an Arrow-Debreu Competitive Equilibrium for this economy.

   Definition: A Competitive Equilibrium for this economy is a series of prices $\{q_t^0(s^t)\}_{t=0}^1$, a series of allocations for each household $i \in \{A, B\}$ $\{c_t^i(s^t)\}_{t=0}^1$, such that:

   - Given $\{q_t^0(s^t)\}_{t=0}^1$, $\{c_t^i(s^t)\}_{t=0}^1$ solves:

   $$\max \sum_{t=0}^{1} \sum_{s^t} \beta^t u(c_t^i(s^t)) Pr(s^t) \quad \text{subject to}$$

   $$\sum_{t=0}^{1} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) \leq \sum_{t=0}^{1} \sum_{s^t} q_t^0(s^t) \omega^i(s^t).$$

   - Market clearing:

   $$\sum_i c^i(s^t) = \sum_i \omega^i(s^t) = \bar{\omega}(s^t), \ \forall t \text{ and } \forall s^t.$$

   (b) Characterise the competitive equilibrium. Comment.

   $$L^i = \sum_{t=0}^{1} \sum_{s^t \in \{s_1, s_2\}} \beta^t u(c_t^i(s^t)) Pr(s^t) + \mu^i \left[\sum_{t=0}^{1} \sum_{s^t \in \{s_1, s_2\}} q_t^0(s^t)[y_t^i(s^t) - c_t^i(s^t)] \right].$$

   First order conditions: (recall that $s^0 = s_0$ is given, $Pr(s^0) = 1$ and $q_0^0(s^0) = 1$)

   $$c_0^i : \ u'(c_0^i) - \mu^i = 0,$$

   $$c_1^i(s^1) : \ \beta u'(c_1^i(s^1)) Pr(s^1) - q_1^0(s^1)\mu^i = 0, \forall s^1$$

   Arrow-Debrue state price:

   $$q_1^0(s_1) = \frac{\beta u'(c_1^i(s_1))\pi}{u'(c_0^i)}.$$

   $$q_1^0(s_2) = \frac{\beta u'(c_1^i(s_2))(1-\pi)}{u'(c_0^i)}.$$

   You can show that

   $$\frac{c_0^A}{c_1^A(s_1)} = \frac{c_0^B}{c_1^B(s_1)}, \ \frac{c_0^A}{c_1^A(s_2)} = \frac{c_0^B}{c_1^B(s_2)}.$$

   Market clearing conditions imply:

   $$c_0^A + c_0^B = 1, \ c_1^A(s_1) + c_1^B(s_1) = 1, \ c_1^A(s_2) + c_1^B(s_2) = 1.$$

Then $c_0^A = c_1^A(s_1) = c_1^A(s_2)$ and $c_0^B = c_1^B(s_1) = c_1^B(s_2)$, and

$$q_0^0 = 1,\ q_1^0(s_1) = \beta\pi,\ q_1^0(s_2) = \beta(1-\pi).$$

Using the budget constraint of each individual, we have that:

$$c_0^A = c_1^A(s_1) = c_1^A(s_2) = \frac{1}{1+\beta}\left(0.5 + 0.75\beta - 0.5\beta\pi\right),$$

$$c_0^B = c_1^B(s_1) = c_1^B(s_2) = \frac{1}{1+\beta}\left(0.5 + 0.25\beta + 0.5\beta\pi\right).$$

Therefore, we have characterised all prices and allocations. Agent's A consumption is decreasing with the probability of the first state in period 1 while agent's B consumption is increasing with this probability.

(c) Define the Social Planner's problem, and solve the allocations of this problem. Explain.

The Social Planner's problem would be to choose $\{c_t^i(s^t)\}_{t=0}^1$, $i \in \{A, B\}$ so that to

$$\max \sum_{t=0}^{1} \sum_{s^t \in \{s_1,s_2\}} \beta^t Pr(s^t)[\alpha u(c_t^A(s^t)) + (1-\alpha)u(c_t^B(s^t))], \text{ subject to}$$

$$c_0^A + c_0^B = 1,\ c_1^A(s_1) + c_1^B(s_1) = 1,\ c_1^A(s_2) + c_1^B(s_2) = 1.$$

Set the Lagrangen to this problem (there will be one Lagrange multiplier to each resource constraint), you can show that

$$\frac{1-\alpha}{\alpha} = \frac{u'(c_0^A)}{u'(c_0^B)} = \frac{u'(c_1^A(s_1))}{u'(c_1^B(s_1))} = \frac{u'(c_1^A(s_2))}{u'(c_1^B(s_2))},$$

and

$$\frac{u'(c_1^A(s_1))}{u'(c_0^A)} = \frac{u'(c_1^B(s_1))}{u'(c_0^B)},\ \frac{u'(c_1^A(s_2))}{u'(c_0^A)} = \frac{u'(c_1^B(s_2))}{u'(c_0^B)}.$$

So

$$\frac{c_0^A}{c_1^A(s_1)} = \frac{c_0^B}{c_1^B(s_1)},\ \frac{c_0^A}{c_1^A(s_2)} = \frac{c_0^B}{c_1^B(s_2)}.$$

Given the resource constraints, we have that $c_0^A = c_1^A(s_1) = c_1^A(s_2) = \alpha$ and $c_0^B = c_1^B(s_1) = c_1^B(s_2) = 1-\alpha$. Then you can see that the only α consistent (check the budget constraint of the agents) with the prices in the competitive equilibrium solution is

$$\alpha = \frac{1}{1+\beta}\left(0.5 + 0.75\beta - 0.5\beta\pi\right).$$

(d) Suppose that agents can only buy a one period bond at discount and this bond is not state contingent. This is what we call an incomplete markets economy since agents cannot write state contingent contracts for all states. Write down the problem of the agents and the market clearing conditions.

The problem of the agent will be to choose $\{c_0^i, c_1^i(s_1), c_1^i(s_2), a_1^i\}$ to maximise

$$\max \sum_{t=0}^{1} \sum_{s^t} \beta^t u(c_t^i(s^t)) Pr(s^t) \text{ subject to}$$

$$c_0^i + q_1 a_1^i = \omega_0^i, c_1^i(s_1) = \omega_1^i(s_1) + a_1^i, c_1^i(s_2) = \omega_1^i(s_2) + a_1^i.$$

The market clearing conditions are:

$$c_0^A + c_0^B = 1,\ c_1^A(s_1) + c_1^B(s_1) = 1,\ c_1^A(s_2) + c_1^B(s_2) = 1,$$

and

$$a_1^A + a_1^B = 0.$$

(e) Show that he/she cannot attain the same level of consumption as in the complete markets case. *Hint:* you might want to proceed along the lines of a proof by contradiction.

You can show that now the marginal condition of each agent implies that

$$\frac{\pi u'(c_1^A(s_1)) + (1-\pi)u'(c_1^A(s_2))}{u'(c_0^A)} = \frac{\pi u'(c_1^B(s_1)) + (1-\pi)u'(c_1^B(s_2))}{u'(c_0^B)},$$

and this does not necessarily imply that

$$\frac{u'(c_1^A(s_1))}{u'(c_0^A)} = \frac{u'(c_1^B(s_1))}{u'(c_0^B)}, \text{ and } \frac{u'(c_1^A(s_2))}{u'(c_0^A)} = \frac{u'(c_1^B(s_2))}{u'(c_0^B)},$$

as in the complete markets case.

2. Answer of question 2:

Consider the following growth model:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} \leq Af(k_t), \text{ and } k_0 > 0,\ A > 0.$$

Assume that $u(\cdot)$ is strictly concave and increasing for $c \geq 0$, and $f(\cdot)$ is concave and increasing.

(a) Formulate this maximisation problem as a dynamic programming problem.

$$V(k) = \max_{0 \leq k' \leq f(k)} \{u(Af(k) - k') + \beta V(k')\}.$$

(b) Show that this value function is a Contraction Mapping. What are the implications of that?

In the first part, we just need to use the Blackwell's sufficient conditions for a Contraction. Notice that, since $k < \infty$, then the operator $V(\cdot)$ maps continuous bounded functions into continuous bounded function. Monotonicity and discounting can be shown easily, as we did in the lectures. Therefore, the operator is a contraction. Concavity of $V(k)$ follows from the corollary of the contraction mapping theorem, when we apply the theorem twice. (students show show all steps)

(c) Assuming that $V(k)$ is differentiable and all other functions are differentiable, characterise the Euler equation that determines the optimal path of consumption and capital accumulation.

First order condition: $u'(c) = \beta V'(k)$.

Envelope theorem: $V'(k) = u'(c)Af'(k)$. Therefore:

$$u'(c) = \beta Af'(k')u'(c').$$

(d) Is this Euler equation enough to determine the path of k and c? If not, what other conditions do we need to impose? Write down these conditions and explain them intuitively.

Notice that, the Euler equation can be rewritten as:

$$u'(f(k_t) - k_{t+1}) = \beta Af'(k_{t+1})u'(Af(k_{t+1}) - k_{t+2}).$$

This is a second-order difference equation in k_t. We need two boundary conditions. First, we know that k_0 is given. Also, we can impose the transversality condition:

$$\lim_{T\to\infty} \beta^T u'(c_T)k_{T+1} = 0.$$

This condition is rather intuitive: At a terminal date, either the marginal utility of consumption is zero $\beta^T u'(c_T)$, or the capital stock has to be zero, $k_{T+1} = 0$. There is also the resource constraint:

$$c_t = Af(k_t) - k_{t+1}.$$

(e) Describe an algorithm to describe the solution of this recursive problem. Go as far as you can in describing the details of your code. Describe how you could simulate a permanent productivity shock in A.

We had describe in lectures in details the algorithm to solve this problem. Students should start with the parameterisation of the model; discretisation of the state space, consumption, utility; value function iteration (show matrices); objects that are solution of this problem (Value function and Policy function). They can also describe how to simulate the model after a permanent shock in A.

(f) Could you think about any tricky to increase the efficiency of your code?

Howard's improvement algorithm. Students should modify the code of the previous question to introduce this one.