Derivations of a working example based on Brunnermeier and Sannikov (2014)

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This document presents a first working example for the Matlab toolbox from d'Avernas et al. (2020)[1]. We derive the model and relate the different expressions to the equations needed to solve the model numerically using the toolbox.
In this first example, we present a general extension of Brunnermeier and Sannikov (2014) where two agents have Epstein and Zin (1989) utility functions and aggregate volatility is time-varying. The framework can easily be modified to any other general equilibrium framework with n-agents and two state variables.

Preferences There are two types of agents: households $h \in H$ and intermediaries $i \in I$. Both agents have stochastic differential utility, as developed by Duffie and Epstein (1992). The utility of agent j over his consumption process c_t^j is defined as

$$U_t^j = \mathbb{E}_t \left(\int_t^\infty f\left(c_s^j, U_s^j\right) ds \right).$$

The function $f_j(c,u)$ is a normalized aggregator of consumption and continuation value in each period defined as

$$f(c,U) = \frac{1-\gamma}{1-1/\zeta} U \left[\left(\frac{c}{((1-\gamma)U)^{1/(1-\gamma)}} \right)^{1-1/\zeta} - \rho \right]$$

where ρ is the rate of time preference, γ is the coefficient of relative risk aversion, and ζ determines the elasticity of intertemporal substitution. Each agent chooses its optimal consumption c_t^j, investment risk σ_t^r, and portfolio weight w_t^j on capital

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[1] Available for download under https://github.com/adavernas/toolbox

holdings in order to maximize discounted infinite life time expected utilities U_t^j. At any time, the following budget constraint has to be satisfied:

$$\frac{dn_t^j}{n_t^j} = \big((1-w_t^j)r_t + w_t^j \mu_t^{r,j} - \mathfrak{c}_t^j\big)dt + w_t^j \sigma_t^{q,\sigma} dZ_t^\sigma + w_t^j\big(\sigma_t + \sigma_t^{q,k}\big)dZ^k,$$

where n_t^j is the wealth of agent j, $\mathfrak{c}_t^j = c_t^j/n_t^j$ his consumption rate, and the portfolio weight w_t^j are choice variables. Z_t^σ and Z_t^k are two standard Brownian motions that hit aggregate volatility and capital growth respectively.

Technology The production technology in the economy is given by:

$$y_t^j = \big(a^j - \iota_t^j\big)k_t^j$$

and

$$\frac{dk_t}{k_t} = \Phi\left(\iota_t^j\right)dt + \sigma_t dZ_t^k,$$

where $\Phi(\cdot)$ is a concave investment function. In this model, we work with the following functional form:

$$\Phi(\iota_t^j) = log(1+\kappa_p \iota_t^j)/\kappa_p + \delta^j$$

The price of a unit of capital is q_t. The volatility of capital returns follows a diffusion:

$$\frac{d\sigma_t}{\sigma_t} = \kappa\big(\sigma_t - \bar{\sigma}\big)dt + \varsigma dZ_t^\sigma.$$

The stochastic law of motion of q_t follows:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^{q,\sigma} dZ_t^\sigma + \sigma_t^{q,k} dZ_t^k.$$

The variables μ_t^q, $\sigma_t^{q,k}$, and $\sigma_t^{q,\sigma}$ are to be determined endogenously. We can use

Ito's lemma to write the process of the value of capital:

$$\frac{d(q_t k_t^j)}{q_t k_t^j} = \left(\Phi_t + \mu_t^q + \sigma_t \sigma_t^{q,k}\right)dt + \sigma_t^{q,\sigma} dZ_t^\sigma + \left(\sigma_t + \sigma_t^{q,k}\right)dZ^k.$$

Hence, the return on physical asset is:

$$dr_t^j = \underbrace{\left(\frac{a^j - \iota_t}{q_t} + \Phi_t + \mu_t^q + \sigma_t \sigma_t^{q,k}\right)}_{\mu_t^{r,j}} dt + \sigma_t^{q,\sigma} dZ_t^\sigma + \left(\sigma_t + \sigma_t^{q,k}\right)dZ^k.$$

Solving the HJB We will guess and verify that the homotheticity of preferences allows us to write the value function for agents of type j as:

$$U\left(n_t^j, \xi_t^j\right) = \frac{\left(n_t^j\right)^{1-\gamma} \xi_t^j}{1-\gamma},$$

where variable ξ_t^j follows

$$\frac{d\xi_t^j}{\xi_t^j} = \mu_t^{\xi,j} dt + \sigma_t^{\xi,\sigma,j} dZ_t^\sigma + \sigma_t^{\xi,k,j} dZ_t^k.$$

We can write the HJB equation corresponding to the problem of agent j as

$$\begin{aligned}
0 = &\max_{\mathfrak{c}_t^j, \iota_t^j, w_t^j} f\left(\mathfrak{c}_t^j n_t^j, U_t^j\right) \\
&+ \left(\left(1 - w_t^j\right) r_t + w_t^j \mu_t^{r,j} - \mathfrak{c}_t^j\right) n_t^j U_n(n_t^j, \xi_t^j) + \mu_t^{\xi,j} \xi_t^j U_\xi(n_t^j, \xi_t^j) \\
&+ \frac{1}{2}\left[\left(w_t^j \sigma_t^{q,\sigma} n_t^j\right)^2 + \left(w_t^j\left(\sigma_t + \sigma_t^{q,k}\right) n_t^j\right)^2\right] U_{nn}(n_t^j, \xi_t^j) \\
&+ \frac{1}{2}\left[\left(\sigma^{\xi,\sigma,j} \xi_t^j\right)^2 + \left(\sigma^{\xi,k,j} \xi_t^j\right)^2\right] U_{\xi\xi}(n_t^j, \xi_t^j) \\
&+ \left[w_t^j \sigma_t^{q,\sigma} n_t^j \sigma_t^{\xi,\sigma,j} \xi_t + w_t^j \left(\sigma_t + \sigma_t^{q,k}\right) n_t^j \sigma_t^{\xi,k,j} \xi_t^j\right] U_{n\xi}(n_t^j, \xi_t^j).
\end{aligned}$$

Substituting the guess from equation (**??**), the HJB becomes

$$0 = \max_{\mathfrak{c}_t^j, \iota_t^j, w_t^j} \frac{1}{1-1/\zeta} \left[\frac{\left(\mathfrak{c}_t^j\right)^{1-1/\zeta}}{\left(\xi_t^j\right)^{\frac{1-1/\zeta}{1-\gamma}}} - \rho \right] + (1-w_t^j) r_t + w_t^j \mu_t^{r,j} - \mathfrak{c}_t^j + \frac{\mu^{\xi,j}}{1-\gamma}$$
$$-\frac{\gamma}{2}\left(w_t^j \sigma_t^{q,\sigma}\right)^2 - \frac{\gamma}{2}\left(w_t^j \sigma_t + w_t^j \sigma_t^{q,k}\right)^2 + w_t^j \sigma_t^{q,\sigma} \sigma_t^{\xi,\sigma,j} + w_t^j \left(\sigma_t + \sigma_t^{q,k}\right) \sigma_t^{\xi,k,j}.$$

Optimality Conditions The first order conditions with respect to $\mathfrak{c}_t^j$, ι_t^j, and w_t^j are given by

$$\left(\mathfrak{c}_t^j\right)^{-1/\zeta} = \left(\xi_t^j\right)^{\frac{1-1/\zeta}{1-\gamma}},$$

$$1/q_t = \Phi_\iota(\iota_t),$$

$$\mu_t^{r,j} - r_t - \gamma w_t^j \left(\sigma_t^{q,\sigma}\right)^2 - \gamma w_t^j \left(\sigma_t + \sigma_t^{q,k}\right)^2 + \sigma_t^{q,\sigma} \sigma_t^{\xi,\sigma,j} + \left(\sigma_t + \sigma_t^{q,k}\right) \sigma_t^{\xi,k,j} = 0.$$

Plugging in the optimality conditions in the HJB gives:

$$0 = \frac{1}{1-1/\zeta}\left(\mathfrak{c}_t^j - \rho\right) + r_t - \mathfrak{c}_t^j + \frac{\gamma}{2}\left(w_t^j \sigma_t^{q,\sigma}\right)^2 + \frac{\gamma}{2}\left(w_t^j \sigma_t + w_t^j \sigma_t^{q,k}\right)^2 + \frac{\mu^{\xi,j}}{1-\gamma}.$$

Market Clearing Conditions We start by providing the definition of such an equilibrium in the state variables $\{\eta_t, \sigma_t\}$, where η_t is defined as the share of wealth in the hands of the intermediaries:

$$\eta_t = \frac{n_t^i}{n_t^h + n_t^i} = \frac{n_t^i}{q_t k_t}.$$

Then, we can use the market clearing condition for consumption to find q_t. Market clearing for consumption dictates that consumption from both types of agents equals the surplus from the production technology:

$$
\begin{aligned}
&\mathfrak{c}_t^i n_t^i + \mathfrak{c}_t^h n_t^h = (a^i - \iota_t^i)k_t^i + (a^h - \iota_t^h)k_t^h \\
&\frac{\mathfrak{c}_t^i n_t^i}{q_t k_t} + \frac{\mathfrak{c}_t^h n_t^h}{q_t k_t} = \frac{(a^i - \iota_t)k_t^i}{q_t k_t} + \frac{(a^h - \iota_t)k_t^h}{q_t k_t} \\
&\mathfrak{c}_t^i \eta_t + \mathfrak{c}_t^h (1-\eta_t) = \frac{(a^i - \iota_t)n_t^i k_t^i}{n_t^i q_t k_t} + \frac{(a^h - \iota_t)n_t^h k_t^h}{n_t^h q_t k_t} \\
&\mathfrak{c}_t^i \eta_t + \mathfrak{c}_t^h (1-\eta_t) = \frac{(a^i - \iota_t)\eta_t k_t^i}{n_t^i} + \frac{(a^h - \iota_t)(1-\eta_t)k_t^h}{n_t^h} \\
&\mathfrak{c}_t^i \eta_t + \mathfrak{c}_t^h (1-\eta_t) = (a^i - \iota_t^i)\eta_t w_t^i / q_t + (a^h - \iota_t^h)(1-\eta_t) w_t^h / q_t
\end{aligned}
$$

Now let

$$
\psi_t \equiv \frac{w_t^i n_t^i}{w_t^i n_t^i + w_t^h n_t^h} = w_t^i \eta_t,
$$

So finally, we have

$$
\left(\mathfrak{c}_t^i \eta_t + \mathfrak{c}_t^h (1-\eta_t)\right) q_t = \psi_t (a^i - \iota_t) + (1-\psi_t)(a^h - \iota_t)
$$

The market clearing condition for capital allows us to identify r_t:

$$
\begin{aligned}
&k_t^i + k_t^h = k_t \\
&\frac{k_t^i}{k_t} + \frac{k_t^h}{k_t} = 1 \\
&\frac{k_t^i n_t^i q_t}{n_t^i q_t k_t} + \frac{k_t^h n_t^h q_t}{n_t^h q_t k_t} = 1 \\
&w_t^i \eta_t + w_t^h (1-\eta_t) = 1.
\end{aligned}
$$

Further, using our definition of η_t and Ito's lemma, we can derive the law of motion of η_t as:

$$
\begin{aligned}
\frac{d\eta_t}{\eta_t} =& \Bigg(r_t + w_t^i(\mu_t^{r,j} - r_t) - \mathfrak{c}_t^i - \Phi_t - \mu_t^q - \sigma_t \sigma_t^{q,k} \\
& - w_t^i \left(\sigma_t^{q,\sigma}\right)^2 + \left(\sigma_t^{q,\sigma}\right)^2 + \left(\sigma_t + \sigma_t^{q,k}\right)^2 - w_t^i \left(\sigma_t + \sigma_t^{q,k}\right)^2 \Bigg) dt \\
& + \left(w_t^i - 1\right)\sigma_t^{q,\sigma} dZ_t^\sigma + \left(w_t^i - 1\right)\left(\sigma_t + \sigma_t^{q,k}\right) dZ^k
\end{aligned}
$$

By applying Ito's lemma, we can find $\sigma_t^{q,\sigma}$, $\sigma_t^{q,k}$, $\sigma_t^{\xi,\sigma,j}$, $\sigma_t^{\xi,k,j}$, and μ_t^q from:

$$
\begin{aligned}
q(\sigma_t, \eta_t)\sigma_t^{q,\sigma} &= q_\sigma(\sigma_t, \eta_t)\varsigma\sigma_t + q_\eta(\sigma_t, \eta_t)\left(w_t^i - 1\right)\sigma_t^{q,\sigma}\eta_t, \\
q(\sigma_t, \eta_t)\sigma_t^{q,k} &= q_\eta(\sigma_t, \eta_t)\left(w_t^i - 1\right)\left(\sigma_t + \sigma_t^{q,k}\right)\eta_t, \\
\xi^j(\sigma_t, \eta_t)\sigma_t^{\xi,\sigma,j} &= \xi_\sigma^j(\sigma_t, \eta_t)\varsigma\sigma_t + \xi_\eta^j(\sigma_t, \eta_t)\left(w_t^i - 1\right)\sigma_t^{q,\sigma}\eta_t, \\
\xi^j(\sigma_t, \eta_t)\sigma_t^{\xi,k,j} &= \xi_\eta^j(\sigma_t, \eta_t)\left(w_t^i - 1\right)\left(\sigma_t + \sigma_t^{q,k}\right)\eta_t, \\
q(\sigma_t, \eta_t)\mu_t^q &= q_\sigma(\sigma_t, \eta_t)\mu_t^\sigma\sigma_t + q_\eta(\sigma_t, \eta_t)\mu_t^\eta\eta_t + \frac{1}{2}q_{\sigma\sigma}(\sigma_t, \eta_t)\left(\varsigma\sigma_t\right)^2 \\
&\quad + \frac{1}{2}q_{\eta\eta}(\sigma_t, \eta_t)\left[\left(\left(w_t^i - 1\right)\sigma_t^{q,\sigma}\eta_t\right)^2 + \left(\left(w_t^i - 1\right)\left(\sigma_t + \sigma_t^{q,k}\right)\eta_t\right)^2\right] \\
&\quad + q_{\sigma\eta}(\sigma_t, \eta_t)\varsigma\sigma_t\left(w_t^i - 1\right)\sigma_t^{q,\sigma}\eta_t.
\end{aligned}
$$

Linking the model to the code We start by collecting parameters and variables. The model parameters are reported in table 1. The parameters have to be specified (as well as given a value) in the section *Parameters* in the file `model.m` of the toolbox. Next, we focus on the variables. Model specific variables are shown in table **??**. Endogenous variables are specified in the array `vars` in section *Variables* in the file `model.m` of the toolbox, while secondary variables are listed in `vars2`. The secondary variables are defined as follows. The two state variables are

$$\eta_t = e \tag{1}$$

$$\sigma_t = z \tag{2}$$

Table 1: Model parameters

Parameter	Definition
γ^i	relative risk aversion
γ^h	relative risk aversion
ϖ	intertemporal elasticity of substitution
ρ	discount rate
a^i	productivity of agent i
a^h	productivity of agent h
κ_p	investment costs
κ	drift of volatility
$\bar{\sigma}$	average volatility
ς	loading of volatility process

Table 2: Variables

Variables	Definition
Endogenous	q_t, ψ_t, μ_t^η, σ_t^{qk}, σ_t^{qs}
Secondary	w_t^i, w_t^h, $\mathfrak{c}_t^i$, $\mathfrak{c}_t^h$, ι_t^i, ι_t^h, μ_t^{ri}, μ_t^{rh}, μ_t^k, μ_t^q, μ_t^{ni}, μ_t^{nh}, μ_t^z, σ_t, $\sigma_t^{\eta s}$, $\sigma_t^{\eta k}$, σ_t^{nik}, σ_t^{nhk}, σ_t^{nis}, σ_t^{nhs}, $\sigma_t^{\xi ik}$, $\sigma_t^{\xi hk}$, $\sigma_t^{\xi is}$, $\sigma_t^{\xi hk}$

In the code, the wealth multipliers are

$$\xi_t^i = vi \tag{3}$$
$$\xi_t^h = vh \tag{4}$$

Leverage was defined as

$$w_t^i = \frac{\psi_t}{\eta_t} \tag{5}$$
$$w_t^h = \frac{1-\psi_t}{1-\eta_t} \tag{6}$$

Consumption-to-wealth ratio is given by the first order condition:

$$c_t^i = (\xi_t^i)^{\frac{1-\zeta^i}{1-\gamma^i}} \tag{7}$$
$$c_t^h = (\xi_t^h)^{\frac{1-\zeta^h}{1-\gamma^h}} \tag{8}$$

The investment ratio is also given by its first order condition:

$$\iota_t^i = \frac{q_t - 1}{\kappa_p} \tag{9}$$
$$\iota_t^h = \frac{q_t - 1}{\kappa_p} \tag{10}$$

The functional form for Φ^j was assumed to be:

$$\Phi^i = log(1 + \kappa_p \iota_t^i)/\kappa_p - \delta^i \tag{11}$$
$$\Phi^h = log(1 + \kappa_p \iota_t^h)/\kappa_p - \delta^h \tag{12}$$

The drift of the state variable σ_t was assumed to be

$$\mu^\sigma = \kappa(\sigma_t - \bar{\sigma}) \tag{13}$$

Using Ito's lemma and the process for k_t^i and k_t^h, we get the drift of aggregate capital:

$$\mu^k = \psi\Phi^i + (1-\psi)\Phi^h \tag{14}$$

From the law of motion of wealth, we have the following loadings:

$$\sigma_t^{nis} = w_t^i \sigma_q^{qs} \tag{15}$$
$$\sigma_t^{nhs} = w_t^h \sigma_q^{qs} \tag{16}$$
$$\sigma_t^{nik} = w_t^i (\sigma_t + \sigma_q^{qk}) \tag{17}$$
$$\sigma_t^{nhk} = w_t^h (\sigma_t + \sigma_q^{qk}) \tag{18}$$

Similarly for η_t:

$$\sigma_t^{\eta s} = \sigma_t^{nis} - \sigma_t^{qs} \tag{19}$$
$$\sigma_t^{\eta k} = \sigma_t^{nik} - (\sigma_t + \sigma_t^{qk}) \tag{20}$$

The law of motion for ξ_t^j was derived using Ito's lemma and yielded

$$\sigma_t^{\xi ik} = \frac{\xi_\eta^i}{\xi_t^i}\sigma_t^{\eta k}\eta_t \tag{21}$$
$$\sigma_t^{\xi hk} = \frac{\xi_\eta^h}{\xi_t^h}\sigma_t^{\eta k}\eta_t \tag{22}$$
$$\sigma_t^{\xi is} = \frac{\xi_\eta^i}{\xi_t^i}\sigma_t^{\eta s}\eta_t + \frac{\xi_\sigma^i}{\xi_t^i}\sigma_t^s\sigma_t \tag{23}$$
$$\sigma_t^{\xi hs} = \frac{\xi_\eta^h}{\xi_t^h}\sigma_t^{\eta s}\eta_t + \frac{\xi_\sigma^h}{\xi_t^h}\sigma_t^s\sigma_t \tag{24}$$

The same was done for

$$\begin{aligned}\mu_t^q =& \frac{q_\sigma}{q_t}\mu_t^\sigma \sigma_t + \frac{q_\eta}{q_t}\mu_t^\eta \eta_t + \frac{1}{2}\frac{q_{\sigma\sigma}}{q_t}\big(\varsigma\sigma_t\big)^2 \\ &+ \frac{1}{2}\frac{q_{\eta\eta}}{q_t}\Big[\big(\big(w_t^i - 1\big)\sigma_t^{q,\sigma}\eta_t\big)^2 + \big(\big(w_t^i - 1\big)\big(\sigma_t + \sigma_t^{q,k}\big)\eta_t\big)^2\Big] \\ &+ \frac{q_{\sigma\eta}}{q_t}\varsigma\sigma_t\big(w_t^i - 1\big)\sigma_t^{q,\sigma}\eta_t\end{aligned} \tag{25}$$

Finally, the drift of r_t and n_t^j using Ito's lemma:

$$\mu_t^{ri} = (a^i - \iota_t^i)/q_t + \Phi^i + \mu_t^q + \sigma_t\sigma_t^{qk} \tag{26}$$

$$\mu_t^{rh} = (a^h - \iota_t^h)/q_t + \Phi^h + \mu_t^q + \sigma_t\sigma_t^{qk} \tag{27}$$

$$r_t = \mu_t^{ri} - \gamma^i w_t^i((\sigma_t^{qs})^2 + (\sigma_t + \sigma_t^{qk})^2) + \sigma_t^{qs}\sigma_t^{\xi s} + (\sigma_t + \sigma_t^{qk})\sigma_t^{\xi k} \tag{28}$$

$$\mu_t^{ni} = r_t + w_t^i(\mu_t^{ri} - r_t) - \mathfrak{c}_t^i \tag{29}$$

$$\mu_t^{nh} = r_t + w_t^i(\mu_t^{rh} - r_t) - \mathfrak{c}_t^h \tag{30}$$

Important here is that a secondary variable cannot be used before it is specified. Once the secondary variables are defined, the endogenous variables can be specified. The first endogenous variable we define in the code is the drift of the state variable η_t. It was derived using Ito's lemma. The equation is written down as:

$$eqmue = (\mu_t^{ni} - \mu_t^q - mu_t^k - \sigma_t\sigma_t^{qk} + (\sigma_t^{qk} + \sigma)^2 + \sigma_t^{qs} - w^i(\sigma_t^{qs})^2 - w_t^i(\sigma_t^{qk} + \sigma_t)^2) - \mu_t^\eta \tag{31}$$

The price of capital q_t comes from the market clearing condition of consumption:

$$eqq = \qquad (\mathfrak{c}_t^i\eta_t + \mathfrak{c}_t^h(1 - \eta_t))q_t = (a^i - \iota_t^i)\eta_t w_t^i + (a^h - \iota_t^h)(1 - \eta_t)w_t^h \tag{32}$$

ψ_t solves the difference between the first order condition for w_t^i and w_t^h:

$$\begin{aligned} eqpsii =& \mu_t^{ri} - \mu_t^{rh} \\ &+ \gamma^h w_t^h((\sigma_t^{qs})^2 + (\sigma_t + \sigma_t^{qk})^2) \\ &- \gamma^i w_t^i((\sigma_t^{qs})^2 + (\sigma_t + \sigma_t^{qk})^2) \\ &+ \sigma_t^{qs}\sigma_t^{\xi is} + (\sigma_t + \sigma_t^{qk})\sigma_t^{\xi ik} \\ &- \sigma_t^{qs}\sigma_t^{\xi hs} + (\sigma_t + \sigma_t^{qk})\sigma_t^{\xi hk} \end{aligned} \tag{33}$$

Finally, using Ito's lemma, we have:

$$eqsigqs = (\varsigma q_\sigma \sigma_t + \sigma_t^{\eta s} q_\eta \eta) - \sigma_t^{qs} q_t \tag{34}$$

$$eqsigqk = \sigma_t^{\eta k} q_\eta \eta - \sigma_t^{qk} q_t \tag{35}$$

And finally, using all the optimality conditions and solved for μ_t^ξ, we update the HJB in the file `HJB.m`

$$\mu^{\xi,j} = -\left(1-\gamma^j\right)\left(\frac{1}{1-1/\zeta^j}\left(\mathfrak{c}_t^j - \rho\right) + r_t - \mathfrak{c}_t^j + \frac{\gamma^j}{2}\left(w_t^j \sigma_t^{q,\sigma}\right)^2 + \frac{\gamma^j}{2}\left(w_t^j \sigma_t + w_t^j \sigma_t^{q,k}\right)^2\right) \tag{36}$$