STATISTISCHE METHODEN DER DATENANALYSE

Excersise Sheets

Physik

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1 Sheet

1.1 Maximum likelihood Estimators for the multivariate Normal distribution

The density of the multivariate normal distribution is

$$f = \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{|V|}} \exp\left(-\frac{1}{2}(x-\mu)^T V^{-1}(x-\mu)\right) \tag{1.1}$$

The likelihood function is

$$\begin{aligned} g(\mu, V|x_1, \ldots, x_n) &= \prod_{i=1}^{n} \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{|V|}} \exp\left(-\frac{1}{2}(x_i-\mu)^T V^{-1}(x_i-\mu)\right) && (1.2)\\ &= (2\pi)^{-\frac{nd}{2}} |V|^{-\frac{n}{2}} \prod_{i=1}^{n} \exp\left(-\frac{1}{2}(x_i-\mu)^T V^{-1}(x_i-\mu)\right) && (1.3)\end{aligned}$$

Calculating the log likelihood:

$$l = \ln g(\mu, V|x_1, \ldots, x_n) = -\frac{nd}{2}\ln(2\pi) - \frac{n}{2}\ln(|V|) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^T V^{-1}(x_i-\mu) \tag{1.4}$$

1.1.1 Calculating $\hat{\mu}$

$$\begin{aligned} \nabla_\mu \ln g &= -\frac{1}{2}\sum_{i=1}^{n} \nabla_\mu \left[(x_i-\mu)^T V^{-1}(x_i-\mu)\right] && (1.5)\\ &= -\frac{1}{2}\sum_{i=1}^{n} \nabla_\mu \left[x_i^T V^{-1} x_i - x_i^T V^{-1}\mu - \mu^T V^{-1} x_i + \mu^T V^{-1}\mu\right] && (1.6)\\ &= -\frac{1}{2}\sum_{i=1}^{n} \nabla_\mu \left[-x_i^T V^{-1}\mu - \mu^T V^{-1} x_i + \mu^T V^{-1}\mu\right] && (1.7)\\ &= -\frac{1}{2}\sum_{i=1}^{n} \left[-x_i^T V^{-1} - v^{-1} x_i + 2V^{-1}\mu\right] && (1.8)\\ &= -\frac{1}{2}\sum_{i=1}^{n} 2V^{-1}(\mu - x_i) && (1.9)\\ &= V^{-1}\sum_{i=1}^{n}(x_i-\mu) && (1.10)\end{aligned}$$

Setting this to zero yields

$$\sum_{i=1}^{n}(x_i-\mu)\overset{!}{=}0 \tag{1.11}$$

$$\Leftrightarrow \hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}x_i \tag{1.12}$$

since V is invertible.

1.1.2 Calculating $\hat{V}$

Use

$$|V^{-1}|=\frac{1}{|V|}\Rightarrow -\ln|V|=\ln\left|V^{-1}\right| \tag{1.13}$$

$$\nabla_{V^{-1}}\ln g=\frac{n}{2}\nabla_{V^{-1}}\left[\ln|V|\right]-\frac{1}{2}\sum_{i=1}^{n}\nabla_{V^{-1}}\left[(x_i-\mu)^T V^{-1}(x_i-\mu)\right] \tag{1.14}$$

Now we have the terms

$$\frac{n}{2}\nabla_{V^{-1}}\left[\ln|V|\right]=\frac{n}{2}V \tag{1.15}$$

and

$$\begin{aligned}\nabla_{V^{-1}}\left[(x_i-\mu)^T V^{-1}(x_i-\mu)\right] &= \nabla_{V^{-1}}\left[\mathrm{tr}\left((x_i-\mu)^T V^{-1}(x_i-\mu)\right)\right] &(1.16)\\ &= \nabla_{V^{-1}}\left[\mathrm{tr}\left((x_i-\mu)^T (x_i-\mu)V^{-1}\right)\right] &(1.17)\\ &= \left((x_i-\mu)^T(x_i-\mu)\right)^T &(1.18)\\ &= (x_i-\mu)(x_i-\mu)^T &(1.19)\end{aligned}$$

The trace can be added, because the term is a scalar. Also we used

$$\nabla_B\,\mathrm{tr}\{AB\}=A^T \tag{1.20}$$

Together

$$\nabla_{V^{-1}}\ln g=\frac{n}{2}V-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)^T\overset{!}{=}0 \tag{1.21}$$

which means

$$\hat{V}=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)(x_i-\mu)^T \tag{1.22}$$

1.2 Radioactive source

We are testing $H_0 : \lambda \geq \frac{5}{s}$. Our sample size is $n = 240$. The test statistik is $T = 1182$.

$$P(T) = \sum_{k=0}^{T} \frac{(n\lambda_0)^k e^{-n\lambda_0}}{k!} \tag{1.23}$$

$$= \sum_{k=0}^{T} \frac{1200^k e^{-1200}}{k!} \approx 0.308 \overset{!}{<} \alpha = 0.05 \tag{1.24}$$

This can also be thought of as chopping the 240s into 1s bits and summing to get the test statistik.

$$Z = \frac{T - n\lambda_0}{\sqrt{n\lambda_0}} \tag{1.25}$$

$$= \frac{1182 - 1200}{\sqrt{1200}} \tag{1.26}$$

$$\approx -0.52 \tag{1.27}$$

$$\approx z_{.30} \tag{1.28}$$

1.3 Drunk intervention

Use the t-Test for paired samples. $W_i = Y_i - X_i$. Assume W_i are normally distributed. We assume Alcohol may only increase reaction times and use a one sided t-Test.

$H_0 : \mu_w = \mu_{w0}$

The test Statistic is

$$T = \frac{\sqrt{n}\left(\overline{W} - \mu_{w0}\right)}{S} \approx 3.397 \overset{!}{>} t_{1-\alpha;n-1} \approx 1.9 \tag{1.29}$$

H_0 is rejected. Alcohol has an influence on reaction times.