Homework 1

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Problem 1

Unless specifically mentioned, the sum and product operators below all range within set $\{x_1, \cdots, x_8\}$.
(a) Likelihood function can be found as follow:

$$L = \prod P(X = x_i) = \prod e^{-\lambda} \frac{\lambda^{x_i}}{x_i!} = e^{-8\lambda} \cdot \frac{\lambda^{\sum x_i}}{\prod x_n!}$$

(b) The log likelihood can be found as follow:

$$\mathcal{L} = ln(L) = -8\lambda + ln(\lambda) \cdot \sum x_i - \sum ln(x_i!)$$

(c) Function $\mathcal{L}(\lambda) = c_0 + c_1 \cdot \lambda + c_2 \cdot ln(\lambda)$ is the sum of convex functions, hence $\mathcal{L}$ is convex. Therefore $\mathcal{L}$ reaches its maximum if and only if its derivative is 0. Solving this equation we have:

$$\frac{d}{d\lambda}\mathcal{L} = -8 + \frac{1}{\lambda} \cdot \sum x_i = 0$$
$$\lambda = \frac{1}{8} \sum x_i$$

Problem 2

(a) The estimation for λ is

```python
import numpy as np

counts = np.array([32,25,28,22,31,34,23,17])
l = sum(counts) / 8

print('The estimation for lambda is {}'.format(np.round(l,5)))
```

```
The estimation for lambda is 26.5
```

(b) Below is the function that calculates the negative log likelihood:

```python
from scipy.special import factorial

def nll(l, counts):
    '''
    Returns the negative log likelihood of given observation counts and rate
    parameter l.
    '''
    return 8*l - np.log(l)*np.sum(counts) + np.sum(np.log(factorial(counts)))
```

(c) We use the Scipy optimizer to find the minimum negative loss:

```python
from scipy.optimize import minimize

# Using initial guess 20 which is decently close to l
l_hat = minimize(nll, [20], args=counts).x[0]
# Does the analytic solution agree with the optimization?
assert np.isclose(l,l_hat)

print('The estimator found using scipy is {}'.format(np.round(l_hat, 5)))
print('The estimator calculated in Problem 1 is {}'.format(np.round(l, 5)))
```

```
The estimator found using scipy is 26.49999
The estimator calculated in Problem 1 is 26.5
```

The calculated estimator is reasonably close to the optimized estimator.

Problem 3

(a) Denote the value of the two dice d_1, d_2. The conditional probability can be calculated with:

$$P(d_1 = 4 | d_1 + d_2 = 7) = \frac{P(d_1 = 4 \cap d_1 + d_2 = 7)}{P(d_1 + d_2 = 7)} = \frac{(1/6)^2}{(1/6)^2 \cdot 6} = \frac{1}{6}$$

(b) Denote families owning dogs D and cats C. The conditional probability can be calculated with:

$$P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{1/3 \cdot 0.6}{0.4} = \frac{1}{2}$$