Consensus Problems in Networks of Agents with Switching Topology and Time-Delays

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Abstract

此为原文 Paper 的总结部分

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1   Introduction

2   Consensus Problems

$$N_J := \cup_{i \in J} N_i = \{j \in \mathcal{I} : i \in J, ij \in \mathcal{E}\} \tag{1}$$

$$\dot{x}_i = f(x_i, u_i), i \in \mathcal{I} \tag{2}$$

$$u_i = k_i(x_{j_1}, \ldots, x_{j_{m_i}}) \tag{3}$$

3   Consensus Protocols

$$\dot{x}_i(t) = u_i(t) \tag{4}$$

$$x_i(k+1) = x_i(k) + \epsilon u_i(k) \tag{5}$$

$$\dot{x}(t) = -Lx(t) \tag{6}$$

$$l_{ij} = \begin{cases} \sum_{k=1, k \neq i}^{n} a_{ik}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \tag{7}$$

$$\dot{x}(t) = -L_k x(t), \ k = s(t) \tag{8}$$

$$x(k+1) = P_\epsilon x(k) \tag{9}$$

$$P_\epsilon = I - \epsilon L \tag{10}$$

$$u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \tag{A1}$$

$$u_i(t) = \sum_{j \in N_i} a_{ij}[x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})] \tag{A2}$$

4 Algebraic Graph Theory: Properties of Laplacians

$$\deg_{in}(v_i) = \sum_{j=1}^{n} a_{ji}, \quad \deg_{out}(v_i) = \sum_{j=1}^{n} a_{ij} \tag{11}$$

$$L = \mathcal{L}(G) = \Delta - \mathcal{A} \tag{12}$$

Theorem 1. 定义 $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ 是关于拉普拉斯矩阵 L 的加权有向图。那么，当且仅当 $\text{rank}(L) = n - 1$ 时，G 就是强连接的。

$$\Phi_G(x) = x^T L x = \frac{1}{2} \sum_{ij \in \mathcal{E}} (x_j - x_i)^2 \tag{13}$$

$$\min_{\substack{x \neq 0 \\ \mathbf{1}^T x = 0}} \frac{x^T L x}{||x||^2} = \lambda_2(L) \tag{14}$$

Theorem 2. (spectral localization) 定义 $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ 是关于拉普拉斯矩阵 L 的加权有向图。记图 G 的节点的最大化出度为 $d_{max}(G) = \max_i \deg_{out}(v_i)$。那么，所有的 $L = \mathcal{L}(G)$ 的特征值都位于下面的圆盘中

$$D(G) = \{z \in \mathbb{C} : |z - d_{max}(G)| \leq d_{max}(G)\} \tag{15}$$

$$D_i = \{z \in \mathbb{C} : |z - l_{ii}| \leq \sum_{j \in \mathcal{I}, j \neq i} |l_{ij}|\} \tag{16}$$

5 A Counterexample for Average-Consensus

$$
\begin{aligned}
x(t) &= exp(-Lt)x(0) \\
x(t) &= e^{-Lt}x(0) \\
u(t) &= -Lx(t)
\end{aligned}
\tag{17}
$$

Theorem 3. 假设 G 是一个强连通图，拉普拉斯矩阵 L 满足 $Lw_r = 0$，$w_l^T L = 0$，$w_l^T w_r = 1$。那么

$$R = \lim_{t \to +\infty} exp(-Lt) = w_r w_l^T \in M_n \tag{18}$$

6 Networks with Fixed or Switching Topology

6.1 Balanced Graphs and Average-Consensus on Digraphs

Theorem 4. 考虑一个含有有向信息流 $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ 的积分器网络，网络是强连通的。那么，当且仅当图 G 是平衡图时，协议 (A1) 全局渐进的解决了平均一致性问题。

Theorem 5. 考虑一个积分器智能体网络图 $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ 是强连通的。那么，当且仅当 $\mathbf{1}^T L = 0$ 时，协议 (A1) 全局渐进的解决了平均一致性问题。

$$\alpha = \frac{\sum_i \gamma_i x_i(0)}{\sum_i \gamma_i} \tag{19}$$

$$\gamma_i \dot{x}_i = u_i, \quad \gamma_i > 0, \forall i \in \mathcal{I} \tag{20}$$

$$\alpha = \frac{\sum_i \gamma_i x_i(0)}{\sum_i \gamma_i} \tag{21}$$

Theorem 6. 定义一个具有邻接矩阵 $\mathcal{A} = [a_{ij}]$ 的图 $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$。那么，下列所有的条件都是等价的：

   i) 图 G 是平衡的；

   ii) $w_l = \mathbf{1}$ 是图 G 拉普拉斯矩阵相对于 0 特征值的左特征向量，即 $\mathbf{1}^T L = 0$。

   iii) 对 $\forall x \in \mathbb{R}^n$ 有 $\sum_{i=1}^n u_i = 0$，且 $u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$。

6.2 Performance of Group Agreement and Mirror Graphs

$$x = \alpha \mathbf{1} + \delta \tag{22}$$

$$\dot{\delta} = -L\delta \tag{23}$$

$$\Phi_G(x) = x^T L x \tag{24}$$

$$\hat{a}_{ij} = \hat{a}_{ji} = \frac{a_{ij} + a_{ji}}{2} \geq 0 \tag{25}$$

Theorem 7. 定义 G 是一个具有邻接矩阵 $\mathcal{A} = adj(G)$ 和拉普拉斯矩阵 $L = \mathcal{L}(G)$ 的图。那么当且仅当 G 是平衡图时，$L_s = Sym(L) = (L + L^T)/2$ 是一个关于 $\hat{G} = \mathcal{M}(G)$ 的可用拉普拉斯矩阵，即有下述转换形式

$$G \xrightarrow{adj} \qquad \mathcal{A} \xrightarrow{\mathcal{L}} L$$

$$\mathcal{M} \downarrow \qquad Sym \quad \downarrow Sym \downarrow \tag{26}$$

$$\hat{G} \xrightarrow[adj]{} \qquad \hat{\mathcal{A}} \xrightarrow[\mathcal{L}]{} \hat{L}$$

Theorem 8. （一致性的性能）考虑带有有向信息流 G 是平衡和强连通的积分器网络。那么，给出协议 (A1)，下述状态满足：

i）群组的非一致向量 δ 是非一致变化在公式 (23) 全局渐进消失的结果，消失速度等于 $\kappa = \lambda_2(\hat{G})$（或图 G 镜像的费德勒特征值），即

$$||\delta(t)|| \le ||\delta(0)|| exp(-kt) \tag{27}$$

ii）下述的流畅的，正定的，适当的函数

$$V(\delta) = \frac{1}{2}||\delta||^2 \tag{28}$$

是一个关于非一致动态变化的有效李亚普诺夫函数。

$$\dot{V} = -\delta^T L \delta = -\delta^T L_s \delta = -\delta^T \hat{L} \delta \le -\lambda_2(\hat{G})||\delta||^2 = -2\kappa V(\delta) < 0, \forall \delta \ne 0 \tag{29}$$

$$\delta^T \hat{L} \delta \ge \lambda_2(\hat{G})||\delta||^2, \quad \forall \delta : \mathbf{1}^T \delta = 0 \tag{30}$$

6.3 Consensus in Networks with Switching Topology

$$\Gamma_n = \{G = (\mathcal{V}, \mathcal{E}, \mathcal{A}) : \text{rank}(\mathcal{L}(G)) = n - 1, \mathbf{1}^T \mathcal{L}(G) = 0\} \tag{31}$$

$$\dot{x}(t) = -\mathcal{L}(G_k)x(t), \quad k = s(t), G_k \in \Gamma_n \tag{32}$$

Theorem 9. 对于任意的切换信号 $s(\cdot)$，切换系统 (32) 的解决方法是全局渐进收敛到 $Ave(x(0))$（即达成平均一致性）。此外，下述的平滑，正定，合适函数

$$V(\delta) = \frac{1}{2}||\delta||^2 \tag{33}$$

$$\dot{\delta}(t) = -\mathcal{L}(G_k)\delta(t), \quad k = s(t), G_k \in \Gamma_n. \tag{34}$$

$$\kappa^* = \min_{G \in \Gamma_n} \lambda^2(\mathcal{L}(\hat{G})). \tag{35}$$

$$\dot{V} = \delta^T \mathcal{L}(G_k)\delta = -\delta^T \mathcal{L}(\hat{G}_k)\delta \leq -\lambda_2(\mathcal{L}(\hat{G}_k))||\delta||^2 \leq -\kappa^*||\delta||^2 = -2\kappa^* V(\delta) < 0, \forall \delta \neq 0 \quad (36)$$

7 Networks with Communication Time-Delays

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij}[x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})]. \quad (37)$$

$$sX_i(s) - x_i(0) = \sum_{j \in N_i} a_{ij} h_{ij}(s)(X_j(s) - X_i(s)) \quad (38)$$

$$X(s) = (s + L(s))^{-1}x(0) \quad (39)$$

Theorem 10. 考虑一个具有相等通信时滞 $\tau > 0$ 的积分器网络。假设网络信息流 G 是无向且连通的。那么，当且仅当以下两个等价情况任何一个满足时，带有 $\tau_{ij} = \tau$ 的协议 (A2) 能全局渐进解决平均一致性问题：

i) $\tau \in (0, \tau^*)$ with $\tau^* = \frac{\pi}{2\lambda_n}, \lambda_n = \lambda_{max}(L)$.

ii) 关于 $\Gamma(s) = e^{-\tau s}/s$ 的奈奎斯特图 (Nyquist plot) 有在 $-1/\lambda_k, \forall k > 1$ 附近的零包围。

$$\tau \leq \frac{\pi}{4d_{max}(G)} \quad (40)$$

8 Max-Consensus and Leader Determination

$$x_i(k+1) = max(x_i(k), u_i(k)) \quad (41)$$

$$x_i(k+1) = \frac{1}{2}(x_i(k) + u_i(k) + |x_i(k) - u_i(k)|) \quad (42)$$

$$u_i(k) = \max_{j \in N_i} x_j \quad (A4)$$

$$f_i(k+1) = K(f_i(k), x_i(k), u_i(k)) := \begin{cases} f_i(k) & x_i(k+1) = x_i(k) \\ \bar{f}_i(k) & x_i(k+1) > x_i(k) \end{cases} \quad (43)$$

Theorem 11. 考虑一个最大智能体网络具有如下动态变化：

$$\begin{cases} x_i(k+1) = \max(x_i(k), u_i(k)) \\ f_i(k+1) = K(f_i(k), x_i(k), u_i(k)) \end{cases} \quad (44)$$

9 Simulation Results

10 Conclusions

```matlab
clear;
clc;
% 此为 Paper 中 Scetion5 代码
% 拉普拉斯矩阵如下
L = [2, -1, -1;
     0,  1, -1;
    -1,  0,  1;];

% 根据所给出的拉普拉斯矩阵推断可知:
% 节点顺序为 3-1
% 度矩阵为
D = [2, 0, 0;
     0, 1, 0;
     1, 0, 1;];
% 邻接矩阵为
A = [0, 1, 1;
     0, 0, 1;
     1, 0, 0;];

% 求其特征值和特征向量
% 依次返回（右）特征向量，特征值，左特征向量
[eigenVector, eigenValue, lefteigenVector] = eig(L);

% Equation 17
syms x1 x2 x3 real;
X0 = [x1 x2 x3]';
Xt = -L * X0;
sum(Xt);

% Equation 18
syms t;
R = expm(-L * t);
R1 = limit(R, t, inf);
R2 = eigenVector * lefteigenVector';
```