Bernstein Polynomials and the Weierstrass Approximation Theorem

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Sergei Natanovich Bernstein, 1880 - 1968

Works Cited

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Bernstein Polynomials

A Bernstein Polynomial, is a polynomial in the Bernstein form, that is a linear combination of Bernstein basis polynomials.

Bernstein basis polynomials of degree n greater than or equal to 0 are defined as

$$b_{v,n} = \binom{n}{v}\cdot x^v(1-x)^{n-v}, \qquad v = 0,\dots,n \quad .(1)$$

Their linear combination

$$B_n(x) = \sum_{v=0}^{n} \beta_v b_{v,n}(x) \qquad (2)$$

is how we come to define the Bernstein Polynomial. The coefficients, β_v, are referred to as Bernstein or Bézier coefficients.

Properties of Bernstein Polynomials

Property 1. $b_{\nu,n} = 0$ if $\nu < 0$ or $\nu > n$

Proof. We proceed by cases. In case one, $\nu < 0$. We consider that it must be true that $n \geq 0$. By definition, $\binom{n}{\nu} = \frac{n!}{\nu!(n-\nu)!}$, and extending the definition of a factorial to the Gamma function, $\nu! = \infty$, while $n!$ and $(n-\nu)!$ are discrete, positive integers. Hence, $\binom{n}{\nu} = 0$, given the definition of the Bernstein basis polynomial, $b_{\nu,n} = 0$. In case two, $\nu > n$. Once again, $n \geq 0$, so $\nu \geq 0$; however, while $n!$ and $\nu!$ are both discrete, positive integers, $(n-\nu)! = \infty$. Thus, it is the case that $\binom{n}{\nu} = 0$, so once again, $b_{\nu,n} = 0$ given the definition of the Bernstein basis polynomial.

Property 2. $\sum_{\nu=0}^{n} b_{\nu,n} = 1$

Proof. By definition, $\sum_{\nu=0}^{n} b_{\nu,n} = \sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu}$.

We then take note of the equality $(a+b)^n = \sum_{\nu=0}^{n}\binom{n}{\nu}a^\nu b^{n-\nu}$.

Thus, $\sum_{\nu=0}^{n}\binom{n}{\nu}x^\nu(1-x)^{n-\nu} = (x+(1-x))^n = (1)^n = 1$.

Thus, $\sum_{\nu=0}^{n} b_{\nu,n} = 1$.

Property 3. $b'_{\nu,n}(x) = n(b_{\nu-1,n-1}(x) - b_{\nu,n-1}(x))$

Proof. Let $B_{\nu,n}(x) = \binom{n}{\nu}x^\nu(1-x)^n$. By algebra, $B_{\nu-1,n-1} = \binom{n-1}{\nu-1}x^{\nu-1}(1-x)^{(n-1)-(\nu-1)}$ and $B_{\nu,n-1} = \binom{n-1}{\nu}x^\nu(1-x)^{(n-1)-\nu}$. Thus,

$$B'_{\nu,n}(x) = \binom{n}{\nu}\nu x^{\nu-1} - \binom{n}{\nu}(n-\nu)x^\nu(1-x)^{n-\nu-1}$$

$$\binom{n}{\nu}\nu = \frac{n!}{\nu!(n-\nu)!}\cdot\nu = \frac{n(n-1)!}{(\nu-1)!(n-\nu)!}$$

$$= \frac{n(n-1)!}{(\nu-1)!(n-1-(\nu-1))!} = n\binom{n-1}{\nu-1}$$

$$B'_{\nu,n}(x) = n\binom{n-1}{\nu-1}x^{\nu-1}(1-x)^{(n-1)-(\nu-1)} - \binom{n}{\nu}(n-\nu)x^\nu(1-x)^{n-\nu-1}$$

$$\binom{n}{\nu}(n-v) = \frac{n!}{\nu!(n-\nu)!}\cdot(n-\nu) = \frac{n!}{\nu!((n-1)-\nu)!} = n\binom{n-1}{\nu}$$

$$B'_{\nu,n}(x) = n((\binom{n-1}{\nu-1}x^{\nu-1}(1-x)^{(n-1)-(\nu-1)} - \binom{n-1}{\nu}x^\nu(1-x)^{(n-1)-\nu}$$

$$B'_{\nu,n}(x) = n(B_{\nu-1,n-1} - B_{\nu,n-1})$$

Weierstrass Approximation Theorem

If f, g are continuous on the interval $[a,b]$, define

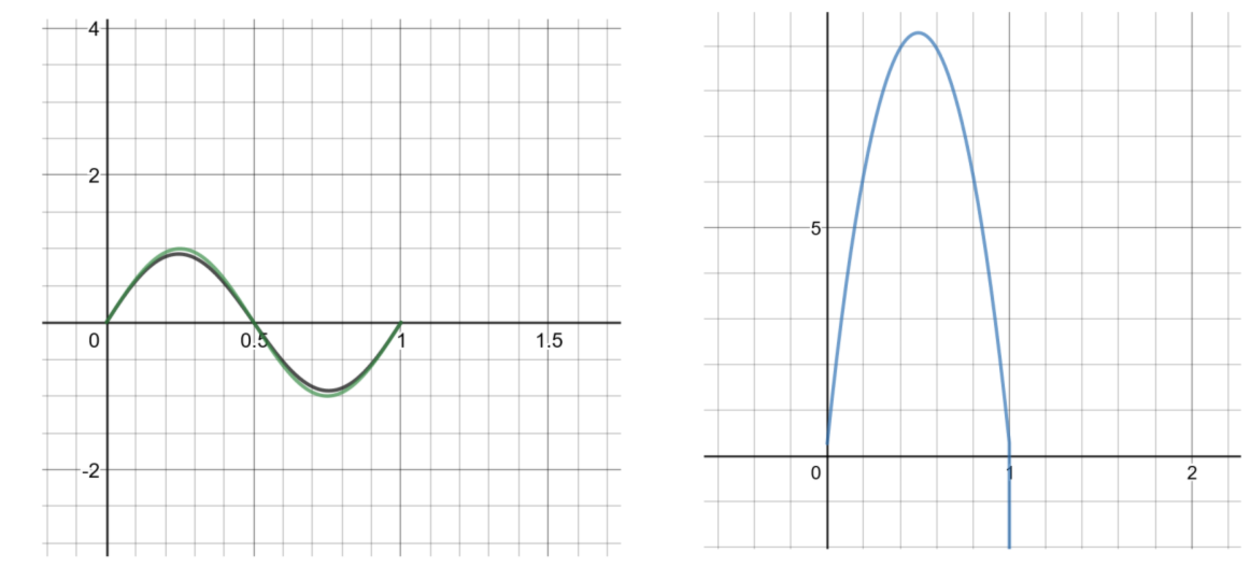
$$d(f,g) = \max_{x\in[a,b]} ||f(x) - g(x)||$$

Theorem(Weierstrass, 1885): If f is continuous on [a,b] and $\varepsilon > 0$ is given, there exists a polynomial $p(x)$ such that $d(p,f) \leq \varepsilon$.

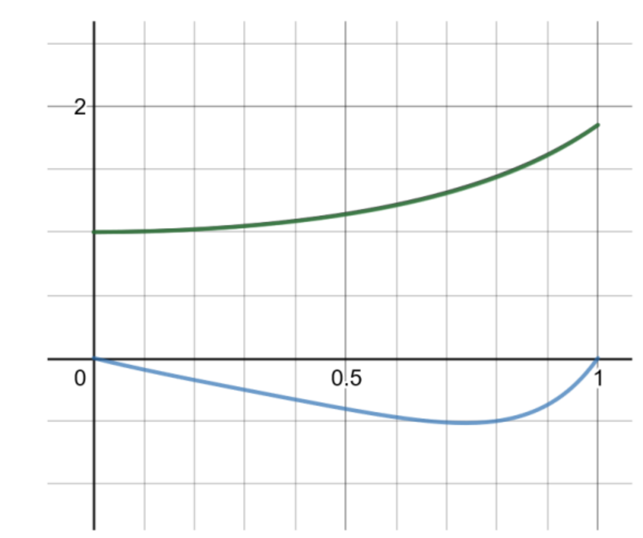
Bernstein Polynomials build a constructive proof of the Weierstrass approximation theorem, by creating a sum of polynomials that converge to a given function.

Bernstein Examples

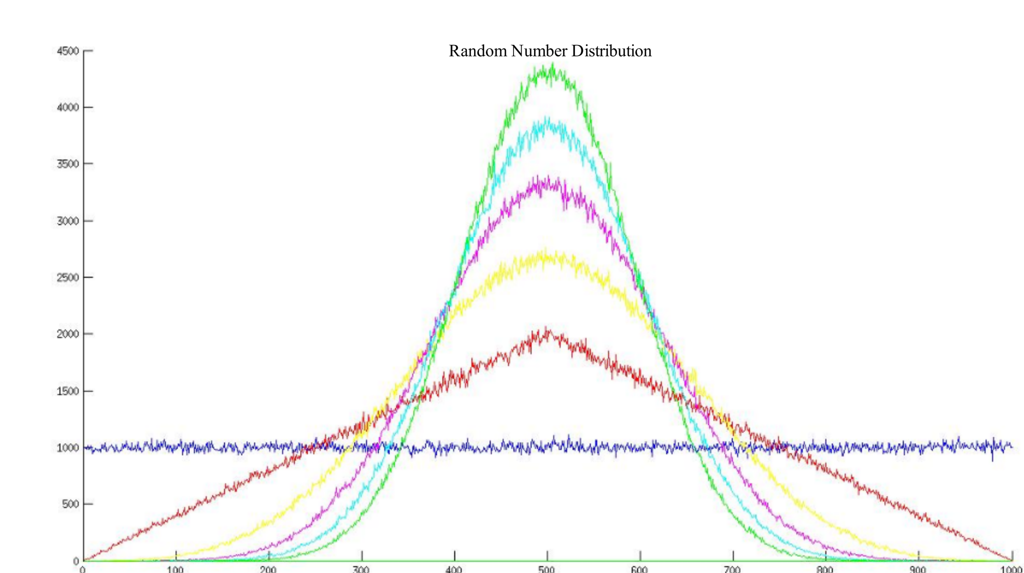
$\sin(2\pi x)$



$\frac{1}{\cos(x)}$



Using a value of n=50, we approximated the two functions above, the blue line shows the relative error, and the green line is the original function, with the Bernstein Polynomial barely visible in black due to the small error.



N.B.: The proof of the Weierstrass Approximation Theorem is omitted; however, one may deduce that Bernstein polynomials uniformly approximate conintuous functions given the Central Limit Theorem. The above image is a graphic representation of the Central Limit Theorem.