MATH 2500

# Forced Vibrations

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# Contents

## Introduction

The displacement of a mass $m$ attached to a spring acted on by a driving force $f(t)$ can be written as follows

$$my'' + by' + ky = f(t) \tag{1}$$

where $m$ is given by the mass of the object, $b$ is the damping factor, and $k$ is the spring constant. Assuming a damping factor $b = 0$, we may perform some manipulations on the equation. Given that $f(t) = F_0 cos(\omega t)$, we have that

$$my'' + ky = F_0 cos(\omega t) \tag{2}$$

dividing by $m$ yields

$$y'' + \frac{k}{m} y = \frac{F_0}{m} cos(\omega t) \tag{3}$$

Let $\omega_0$ be expressed as $\sqrt{\frac{k}{m}}$, equation 3 becomes

$$y'' + \omega_0^2 y = \frac{F_0}{m} cos(\omega t) \tag{4}$$

We seek to derive a general solution function from this, solving for $y(t)$ in terms of $m, F_0, w_0,$ and $t$. To achieve this, we will find $y_c(t)$ such that by equation 4,

$$y'' + \omega_0^2 y = 0 \tag{5}$$

Additionally, we will find a particular solution $y_p(t)$ such that equation 4 holds for $y(t) = y_p(t)$. Finally, by combining $y_c(t)$ and $y_p(t)$ through addition, we will arrive at our general solution $y(t)$.

# 1 Problem 1

## 1.1

### 1a

We begin with finding $y_c(t)$. We solve equation 5 in terms of y to find our eigenvalues $r_1, r_2$.

$$y'' + \omega_0^2 y = 0 \implies \frac{-0 \pm \sqrt{-4\omega_0^2}}{2} \implies r = \pm\omega_0 i$$

By the formula for complex eigenvalues, where r is of the form $\alpha \pm \beta i$, where $\alpha, \beta \in \mathrm{R}$, there exists a general solution of the form

$$y(t) = C_1 e^{\alpha} cos(\beta t) + C_2 e^{\alpha} sin(\beta t)$$

Using our known values $\alpha = 0$ and $\beta = \omega_0$, this yields

$$y_c(t) = C_1 cos(\omega_0 t) + C_2 sin(\omega_0 t) \tag{6}$$

We must now find the accompanying particular solution, $y_p(t)$. Recalling that the right side of equation 4 is of the form $F_0 cos(\omega t)$ for some constant $F_0$, and that the left side contains no odd orders of the $y(t)$ function, we know that our solution will take the form $Acos(\omega t)$ for some constant $A$. We will solve for $A$ in terms of known constants by finding our solution function's derivatives and substituting them into equation 4.

| | |
|---|---|
| $f(t)$ | $Acos(\omega t)$ |
| $f'(t)$ | $-\omega Asin(\omega t)$ |
| $f''(t)$ | $-\omega^2 Acos(\omega t)$ |

Thus

$$-\omega^2 Acos(\omega t) + \omega_0^2 Acos(\omega t) = \frac{F_0}{m} cos(\omega t)$$

By trivial algebraic manipulation, we can find $A$ in terms of known constants as follows

$$-\omega^2 A + \omega_0^2 A = \frac{F_0}{m} \implies (\omega_0^2 - \omega^2)A = \frac{F_0}{m} \implies A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

Substituting our newly found $A$ back into our $y_p(t) = Acos(\omega t)$ gives us our particular solution

$$y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} cos(\omega t) \tag{7}$$

Finally, by combining our two solutions given by equations 6 and 7, we get our general solution

$$y(t) = y_c(t) + y_p(t) = \boxed{C_1 cos(\omega_0 t) + C_2 sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} cos(\omega t)} \tag{8}$$

## 2 Problem 2

## 2.1

### 2a

Let us assume that the mass in equation 4 lacks both initial velocity, and initial displacement. That is, to say $y(0) = y'(0) = 0$. We will use this information along with equation 8 to derive the specific solution, and eliminate the unknown constants $C_1$, and $C_2$. With our values at $t = 0$, equation 8 becomes

$$y(0) = C_1 cos(0) + C_2 sin(0) + \frac{F_0}{m(\omega_0^2 - \omega^2)} cos(0) = C_1 + \frac{F_0}{m(\omega_0^2 - \omega^2)} = 0$$

This gives us $C_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}$

Deriving $y(t)$ gives us the second piece of information, $y'(t)$, from which we can use our known $y'(0)$ to find $C_2$.

$$y'(t) = -\frac{F_0\omega_0}{m(\omega_0^2 - \omega^2)} sin(\omega_0 t) + C_2\omega_0 cos(\omega_0 t) - \frac{F_0\omega}{m(\omega_0^2 - \omega^2)} sin(\omega t)$$

By substitution

$$y'(0) = -\frac{F_0\omega_0}{m(\omega_0^2 - \omega^2)} sin(0) + C_2\omega_0 cos(0) - \frac{F_0\omega}{m(\omega_0^2 - \omega^2)} sin(0) = 0$$

which simplifies to

$$C_2\omega_0 = 0$$

Because $\omega_0$ is a constant known not to be 0, (otherwise our mass m would be zero) we now know $C_2$ to be 0. Equation 8 now becomes the specific solution

$$y(t) = -\frac{F_0}{m(\omega_0^2 - \omega^2)} cos(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} cos(\omega t) \tag{9}$$

Equation 9 can be simplified to

$$y(t) = \boxed{\frac{F_0}{m(\omega_0^2 - \omega^2)}\big(cos(\omega t) - cos(\omega_0 t)\big)} \tag{10}$$

## 2.2

### 2b

Recalling from equation 10, we have our constant value

$$\frac{F_0}{m(\omega_0^2 - \omega^2)}$$

and a variable function

$$cos(\omega t) - cos(\omega_0 t)$$

By using the Product-Sum trigonometric identity, given by

$$cos(\beta) - cos(\alpha) = 2sin(\frac{\alpha + \beta}{2})sin(\frac{\alpha - \beta}{2})$$

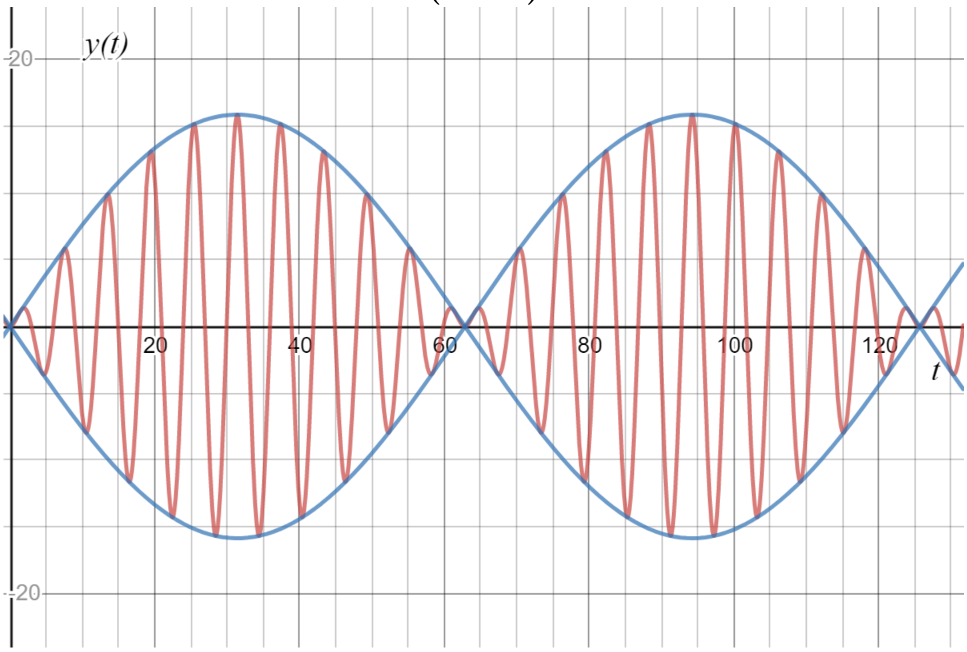
we may alter our variable function as shown below

$$cos(\omega t) - cos(\omega_0 t) = 2sin(\frac{\omega_0 - \omega}{2})sin(\frac{\omega_0 + \omega}{2})$$

which turns equation 10 into

$$\boxed{y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)}sin(\frac{\omega_0 - \omega}{2})sin(\frac{\omega_0 + \omega}{2})} \tag{11}$$

**Graph of equation 11 (Red) and its bounding sine functions (Blue)**




### 2.3

#### 2c

Looking at the graph above, we can see that the function found in equation 11 is an exponential sine function, meaning it oscillates like the sine function, but is additionally scaled down by another sine function. It is seen that the minimum and maximum of the function are given by

$$y(t) = \pm \frac{2F_0}{m(\omega_0^2 - \omega^2)} sin(\frac{(\omega_0 - \omega)t}{2})$$

Given our values $\omega = 1$, $\omega_0 = 1.1$, $F_0 = 0.5$, and $m = 0.3$, we can calculate the value of the constant part of these bounds:

$$\frac{2F_0}{m(\omega_0^2 - \omega^2)} = \frac{2(0.5)}{(0.3)((1.1)^2 - (1)^2} = \frac{1}{0.063} \approx 15.87$$

Because $-1 \leq sin(\frac{1}{20}t) \leq 1$, the function will never be greater than 15.87, or less than -15.87. Additionally, because $sin(n\pi) = 0 \; \forall n \in \mathrm{Z}$, $sin(\frac{1}{20}t)$ will be

0 when $t$ is a multiple of $20\pi$ or 0, thus, equation 11 will be 0 if $t$ satisfies either of these conditions.

# 3 Problem 3

## 3.1

### 3a

Given the differential equation

$$y'' + \omega_0^2 y = \frac{F_0}{m} cos(\omega_0 t) \tag{12}$$

we seek to derive its general solution. As in problem 1, we do this by finding the complimentary solution, and the particular solution, and then combining them. We begin with the complimentary solution, in other words, we start by solving

$$y'' + \omega_0^2 y = 0 \tag{13}$$

The equation we use to find the eigenvalues is given by

$$r^2 + \omega_0^2 = 0$$

and we solve for r. The quadratic formula gives us that

$$r = \pm \omega_0 i$$

Thus, by our formula for complex roots, $\alpha = 0,$ and $\beta = \omega_0.$ Plugging these values into the formula, we get that

$$y = C_1 cos(\omega_0 t) + C_2 sin(\omega_0 t) \tag{14}$$

Now to find the particular solution, we have that $f(t) = \frac{F_0}{m} cos(\omega_0 t)$, and thus, the particular solution $y_p(t)$ will take the form

$$[Acos(\omega_0 t) + Bsin(\omega_0 t)]t$$

We now have that

| | |
|---|---|
| $y_p(t)$ | $Atcos(\omega_0 t) + Btsin(\omega_0 t)$ |
| $y'_p(t)$ | $Acos(\omega_0 t) - Atsin(\omega_0 t) + Bsin(\omega_0 t) + Bt\omega_0 cos(\omega_0 t)$ |
| $y''_p(t)$ | $-2A\omega_0 sin(\omega_0 t) + 2B\omega_0 cos(\omega_0 t) - At\omega_0^2 cos(\omega_0 t) - Bt\omega_0^2 sin(\omega_0 t)$ |

Plugging $y_p(t)$ and $y''_p(t)$ back into equation 12 gives us

$$-2A\omega_0 sin(\omega_0 t) + 2B\omega_0 cos(\omega_0 t) = \frac{F_0}{m} cos(\omega_0 t)$$

which implies that

$$-2A\omega_0 = 0$$

and

$$2B\omega_0 = \frac{F_0}{m}$$

thus we have that

$$A = 0, \text{ and } B = \frac{F_0}{2\omega_0 m}$$

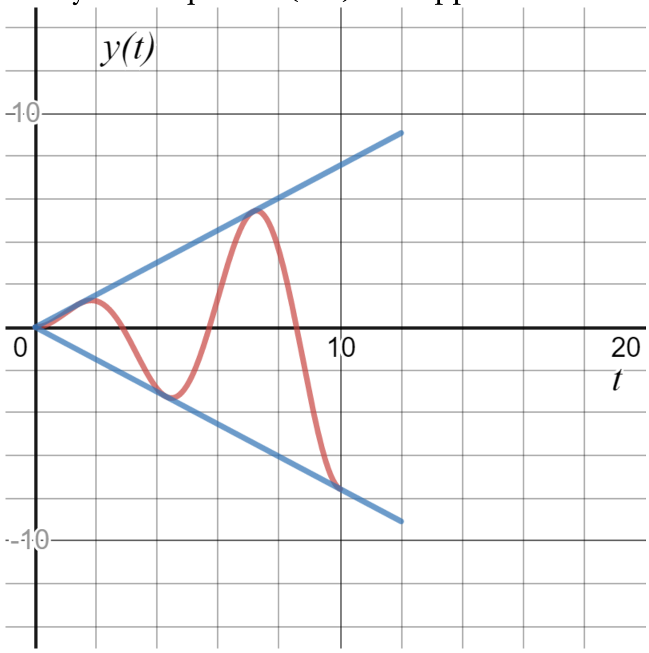
This gives us a general equation of

$$\boxed{y(t) = C_1 cos(\omega_0 t) + C_2 sin(\omega_0 t) + \frac{F_0}{2\omega_0 m} tsin(\omega_0 t)} \tag{15}$$

### 3.2

#### 3b

Graph of steady state equation (red) and upper/lower bounds (blue)




# 4 Problem 4

### 4.1

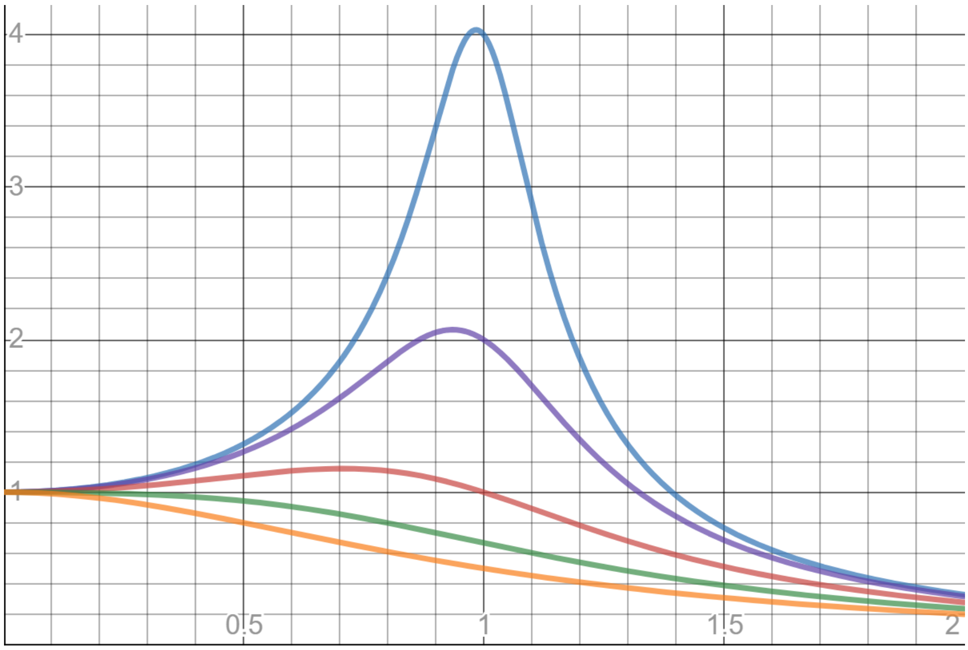
#### 4a

We define an **amplification factor** notated $M(\omega)$ with the following equation

$$M(\omega) = \frac{1}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} \tag{16}$$

When $\omega$ is optimized to give $M(\omega)$ its maximum value, that is called **practical resonance**, and this produces the largest possible steady state amplitude.

Graph of $M(\omega)$ with m $= \omega_0 = 1$




b $= \frac{1}{4}$(blue),$\frac{1}{2}$(purple),1(red),$\frac{3}{2}$(green),2(yellow)

## 4.2

### 4b

Let $\frac{dM(\omega)}{d\omega} = 0$. We seek to maximize this value, so we set the derivative to zero. Going back to equation 16, we find the derivative with respect to $\omega$, remembering that $b, m$, and $\omega_0$ are constants.

$$\frac{dM(\omega)}{d\omega} = \frac{d}{d\omega}\frac{1}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}$$

$$\frac{dM(\omega)}{d\omega} = \frac{-4\omega m^2(\omega_0^2-\omega^2)+2b^2\omega}{\sqrt{\left[m^2(\omega_0^2-\omega^2)^2+b^2\omega^2\right]^3}}$$

$$m^2(\omega_0^2-\omega^2)(-2)+b^2=0 \implies \omega^2 = -\frac{b^2}{2m^2}+\omega_0^2$$

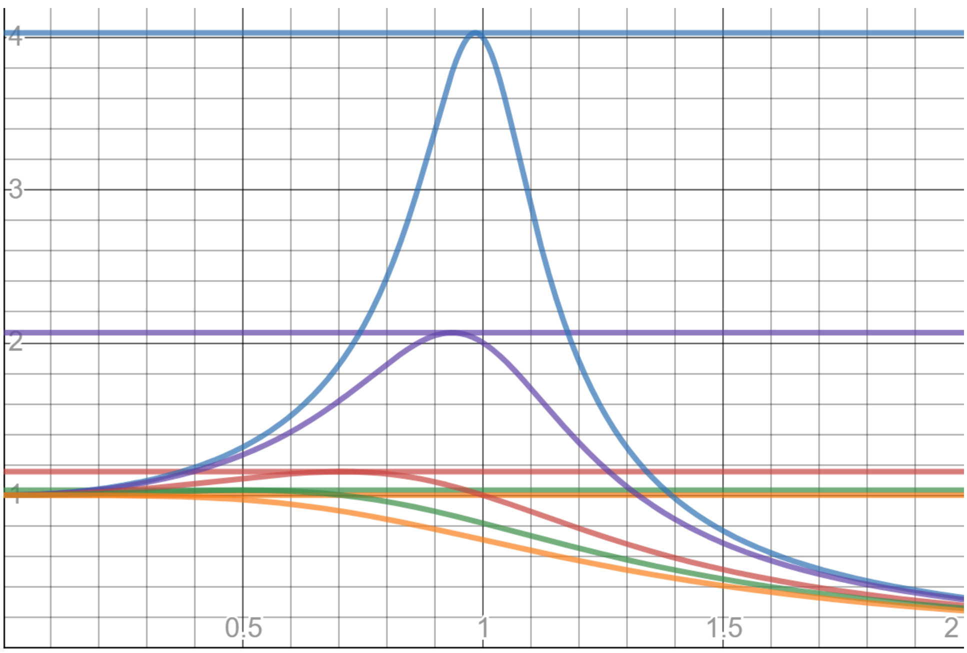
Recalling that $\omega_0 = \sqrt{\frac{k}{m}}$,

$$\boxed{\omega^2 = \frac{k}{m} - \frac{b^2}{2m^2} \implies \omega = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}}$$

### 4.3

#### 4c

Graph of $(\omega_b, M(\omega_b))$




Vertical lines represent maximum values of $M(\omega)$

## 5 Problem 5

### 5.1

#### 5a

Beats happen naturally when waves of different frequencies go into your ear, and the constructive and destructive interference makes all sound concurrently quiet and loud. There are many practical applications of beats in modern technology. For instance, Sonar technology in military vessels detects the existence of transient sound waves coming from enemy ships that exist because of beats. Pure resonance is achieved when there is no difference in frequency between these waves.[1]

# References

"LibGuides: Waves and Sounds - Honors Physics: A Word on Resonance and Beats." A Word on Resonance and Beats - Waves and Sounds - Honors Physics - LibGuides at Randolph School, https://library.randolphschool.net/c.php?g=237954p=1581843.