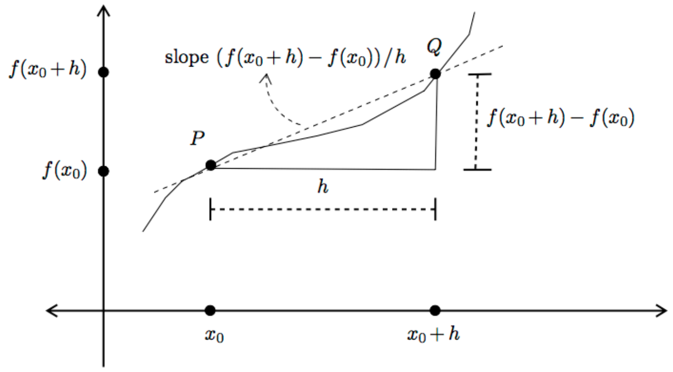
Differentiation

Suppose that $f(x)$ is continuous on $[a, b]$ and let $P = (x_0, f(x_0))$ and $Q = (x_0 + h, f(x_0 + h))$ on the graph of $f(x)$:



Then the slope of $\overleftrightarrow{PQ}$, secant line passing through P and Q, is

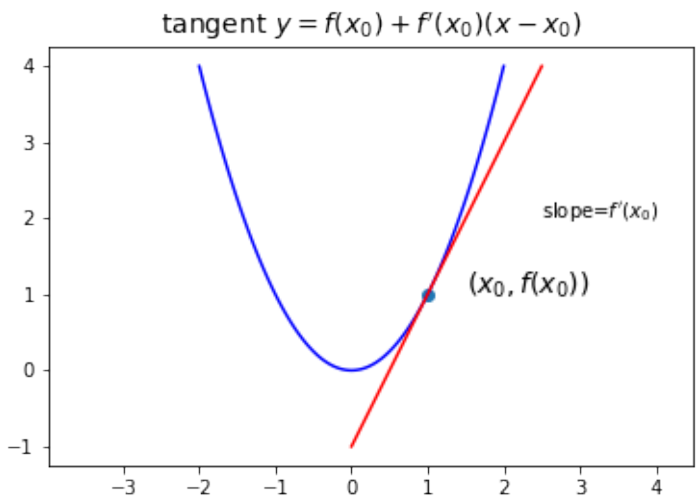
$$\mathbf{m} = \lim_{\mathbf{h}\to 0} \frac{\mathbf{f(x_0 + h) - f(x_0)}}{\mathbf{h}}$$

so called Newton quotient of $f(x)$ at $x = x_0$. If the limit of m as h approaches 0 exists, it is called the derivative of $f(x)$ at $x = x_0$, i.e. it is the slope of tangent line of $f(x)$ passing through $(x_0, f(x_0))$, and denoted as $f'(x_0)$. The process for finding derivative is called differentiation.

In [1]:
```
%matplotlib inline
import numpy as np
import matplotlib.pylab as plt
```

In [2]:
```
x=np.linspace(-2,2,101)
plt.plot(x,x**2,"-b")
plt.plot([0,2.5],[-1,4],'-r')
plt.scatter([1],[1])
plt.text(1.5,1,'$(x_0,f(x_0))$',size=14)
plt.text(2.5,2,'slope=$f\'(x_0)$')
```

```
plt.text(2.5,2,'slope $f\'(x_0)$')
plt.title('tangent $y=f(x_0)+f\'(x_0)(x-x_0)$',size=14)
plt.axis("equal");
```



Remarks

There are several symbols for derivatives as follows:

1. the derivatives value of $f(x)$ at $x = x_0$:

$$f'(x_0) = \frac{d}{dx} f(x)\Big|_{x=x_0} = \frac{d}{dx} f(x_0) = D_x f(x_0)$$

- the derivative of $f(x)$:

$$f'(x_0) = \frac{d}{dx} f(x) = D_x f(x)$$

a). $|x|$ is differentiable everywhere except at $x = 0$.
b). Neither $x^{1/3}$ nor $x^{2/3}$ are not differentiable at $x = 0$.

Theorem

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = 0$.

Proof

$$\begin{aligned}\lim_{x \to a} |f(x) - f(a)| &= \lim_{x \to a} \left| \frac{f(x) - f(a)}{x - a} \right| \cdot |x - a| \\ &= |f'(a)| \cdot 0\end{aligned}$$

$$= 0$$

This means $\lim_{x \to a} f(x) = f(a)$, i.e. $f(x)$ is continuous at $x = a$.

Example

Suppose that

$$f(x) = \begin{cases} x \sin(1/x), \text{ if } x \neq 0, \\ 0, , \text{ if } x = 0 \end{cases}$$

Then $f(x)$ is continuous for any real number by the squeeze value theorem. And the the derivative at $x = 0$, is as follows:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h \sin(1/h)}{h} \\ &= \lim_{h \to 0} \sin(1/h) \end{aligned}$$

This concludes that the limit fails to exist.

Example

The derivative of sine function is derived as follows:

$$\begin{aligned} (\sin x)' &= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h \to 0} \left(\sin x \frac{\cos h - 1}{h} + \cos x \frac{\sin h}{h} \right) \\ &= \cos x \end{aligned}$$

since $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h}$ =0.

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(61). Let

$$f(x) = \begin{cases} x^2 \sin(1/x), \text{ if } x \neq 0, \\ 0, , \text{ if } x = 0, \end{cases}$$

a). $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0}{h} \\ &= \lim_{h \to 0} h \sin(1/h) = 0 \end{aligned}$$

The last hols since

$$|h \sin(1/h)| \leq |h| \to 0 \text{ as } h \to 0$$

b). Its image for $(x, y) \in [-0.5, 0.5] \times [-0.1, 0.1]$. Reference the following picture.

True or False

(66). If $f(x)$ is differentiable at $x = a$, and $g(x)$ is not differentiable at $x = a$, then $fg(x)$ is not differentiable at $x = a$. (   ).

(67) If both $f(x)$ and $g(x)$ are not differentiable at $x = a$, then $fg(x)$ is not differentiable at $x = a$. (   ).

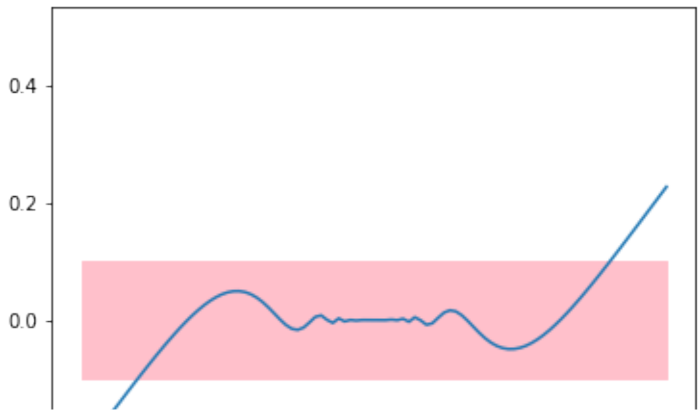
(68) If both $f(x)$ and $g(x)$ are not differentiable at $x = a$, then $f + g$ is not differentiable at $x = a$. (   ).

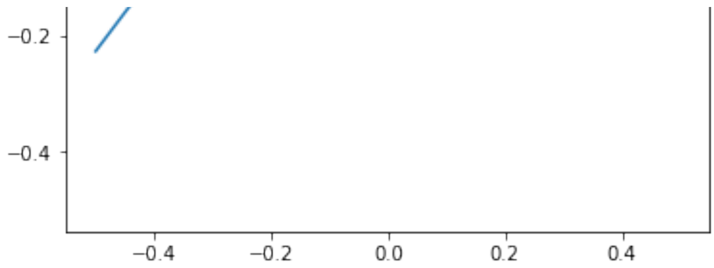
(69). The domain of $f'(x)$ is the same as $f(x)$. (   ).

(70). If $n \in \mathbb{N}$, the there exists a function f such that f is differentiable everywhere except at n number. (   ).

In [6]:
```
from numpy import sin
eps=1e-6
x=np.linspace(-0.5,0.5,101)

plt.figure(figsize=(6,6))
plt.plot(x,x*x*sin(1/(x+eps)))
plt.ylim(-0.5,0.5)
plt.xlim(-0.5,0.5)
plt.fill_between([-0.5,0.5,0.5,-0.5,-0.5],[-0.1,-0.1,0.1,0.1,-0.1],
color="pink")
plt.axis("equal");
```





2.2 Rules of Differentiation

1. $(c)' = 0$ for any constant c.

- $(x^n)' = nx^{n-1}$ for $x > 0$ and $n \in \mathbb{R}$. Here, a example for $n = -1$:

$$\begin{aligned}\left(\frac{1}{x}\right)' &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{xh(x+h)} \\ &= \frac{-1}{x^2}\end{aligned}$$

- $(f \pm g)' = f' \pm g'$,
- $(f \cdot g)' = f'g + g'f$,
- $(f/g)' = (f'g - g'f)/g^2$ suppose that $g(x) \neq 0$ for any x,
- (Power Rule) $(x^r)' = rx^{r-1}$ for $r > 0$,
- (Chain's Rule) $(f \circ g)' = (f(g(x)))' = f'(g(x))g'(x)$

$$(f(g(x)))' = \frac{df(g(x))}{d\cancel{g}(x)} \cdot \frac{d\cancel{g}(x)}{dx}$$

The following diagram says everything about chain's rule:

$$x \xrightarrow{g(x)} u = g(x) \xrightarrow{f(u)} f(g(x))$$

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
g'(x) & \times & f'(u) \\
& \downarrow & \\
& f'(g(x))g'(x) &
\end{array}
$$

- $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$, $(\cot x)' = -\csc^2 x$, $(\sec x)' = \sec x \tan x$, $(\csc x)' = -\csc x \cot x$;
- (Logarithmic Rule):
  - $(\ln x)' = (\log_e x)' = 1/x$, where $\ln(\bullet)$ means logarithm with base $e \cong 2.71428\cdots$ where e, Euler number, is an irrational number;
  - $(\log_a x)' = 1/(x \ln a)$ where $a > 0$;
- (Exponential Rules):
  - $(e^x)' = e^x$;
  - $(a^x)' = a^x \ln a$, where $a > 0$ and $a \neq 1$.

Example

$$
\begin{aligned}
\left(\frac{\sin x}{1-\cos x}\right)' &= \frac{(\sin x)'(1-\cos x) - \sin x(1-\cos x)'}{(1-\cos x)^2} \\
&= \frac{\cos x(1-\cos x) - \sin x \sin x}{(1-\cos x)^2} \\
&= \frac{1}{\cos x - 1}
\end{aligned}
$$

Example

$\left[(1+x^2)^{100}\right]' = 2x \cdot 100(1+x^2)^{99} = 200x(1+x^2)^{99}$

$$
\begin{array}{ccccc}
x & \xrightarrow{1+x^2} & 1+x^2 & \xrightarrow{(\cdot)^{100}} & (1+x^2)^{100} \\
& \downarrow & & \downarrow & \\
& 2x & \cdot & 100(\cdot)^{99} & \\
& & \downarrow & & \\
& & 2x \cdot 100(1+x^2)^{99} & &
\end{array}
$$

Example

$\left(\sin(x^2)\right)' = 2x \cos x^2$

$$
x \xrightarrow{x^2} x^2 \xrightarrow{\sin(\cdot)} \sin(x^2)
$$

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
2x & \cdot & \cos(\cdot) \\
 & \downarrow & \\
 & 2x \cdot \cos(x^2) &
\end{array}
$$

Example

$\left(\sin^2 x\right)' = 2 \sin x \cos x$

$$
\begin{array}{ccccc}
x & \xrightarrow{\sin x} & \sin x & \xrightarrow{(\cdot)^2} & \sin^2(x) \\
 & \downarrow & & \downarrow & \\
 & \cos x & \cdot & 2(\cdot) & \\
 & & \downarrow & & \\
 & & \cos x \cdot 2(\sin x) & &
\end{array}
$$

In [40]:
```
from sympy import Symbol, diff,sin,cos,sqrt,pprint,Function,tan
```

In [41]:
```
x=Symbol("x")
u=Symbol("u")
def ChainRule(f,g):
    u=Symbol("u")
    l1=diff(f,u)
    l2=diff(g,x)
    print("f'(g(x)) =",l1*l2,", where g(x) = u = ",g)
```

Examples

Differentiate a) $\sqrt{2x^2 - 1}$ b). $u = x^3 + 1, u^3 - u^2 + u + 1$ c). $3 \cos x^2$

In [27]:
```
ChainRule(sqrt(u),2*x*x-1)
```

```
f'(g(x)) = 2*x/sqrt(u) , where g(x) = u =  2*x**2 - 1
```

In [28]:
```
ChainRule(u*u*u-u*u+u+1,x*x*x+1)
```

```
f'(g(x)) = 3*x**2*(3*u**2 - 2*u + 1) , where g(x) = u =  x**3 + 1
```

In [29]:
```
ChainRule(3*cos(u),x*x)
```

```
f'(g(x)) = -6*x*sin(u) , where g(x) = u =  x**2
```

In [31]:
```
ChainRule(tan(u)*tan(u)*tan(u),3*x*x+1)
```

```
f'(g(x)) = 6*x*(3*tan(u)**2 + 3)*tan(u)**2 , where g(x) = u =  3*x
**2 + 1
```

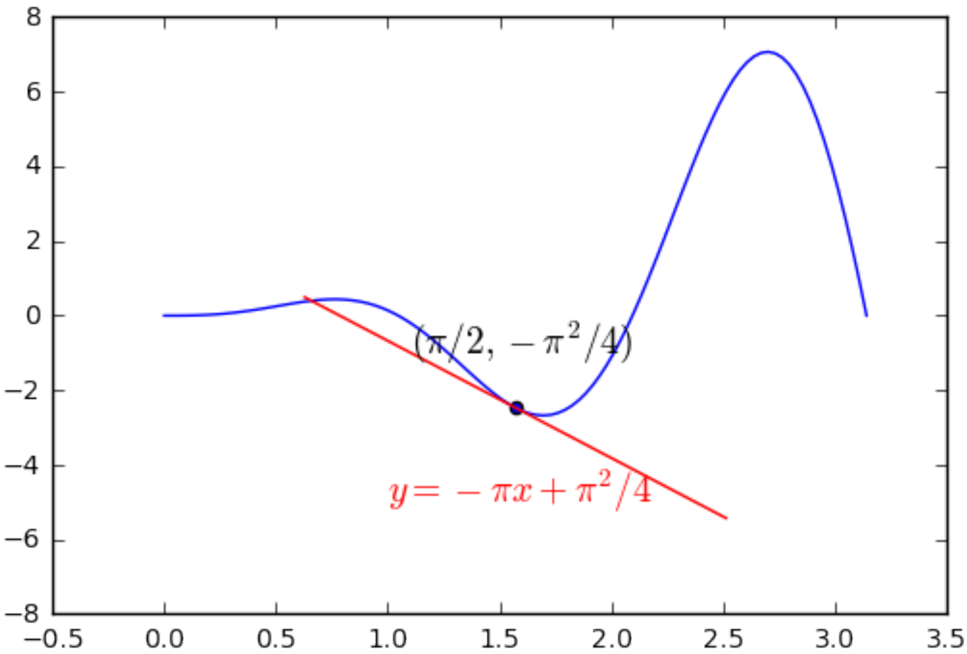
Example

Find the tangent line of $f(x) = x^2 \sin 3x$ at $x = \pi/2$

a). $(x^2 \sin 3x)' = 2x\sin(3x) + x^2 \cdot 3\cos 3x\Big|_{x=\pi/2} = -\pi$
b). $f(\pi/2) = -\pi^2/4$
c). tangent line: $(y - f(\pi/2)) = f'(\pi/2)(x - \pi/2)$, i.e. $y + \pi^2/4 = -\pi(x - \pi/2)$

In [46]:
```
from numpy import pi,sin
x=np.linspace(0,pi,101)
plt.plot(x,x*x*sin(3*x))
t=x[20:-20]
f=-pi*t+pi*pi/4
plt.plot(t,f,color='red')
plt.scatter([pi/2],[-pi*pi/4])
plt.text(1.1,-1,'$(\pi/2,-\pi^2/4)$',size=14)
plt.text(1,-5,'$y=-\pi x+\pi^2/4$',size=14,color='red')
```

Out[46]:
```
<matplotlib.text.Text at 0x10bbc6ef0>
```



Example

Differentiate the following function, x^x.

Right or Wrong

- If it is a power function, then $(x^x)' = xx^{x-1} = x^x$;
- If it is an exponential function, then $(x^x)' = x^x \ln x$.

But it is neither power function nor exponential function. Then we have to modify it to be one of them. As well known, $2 = 10^{\log_{10} 2}$, then

$$x^x = e^{x \log_e x} = e^{x \ln x}$$

Then

$$\begin{array}{ccccc}
 & x \ln x & & e^{(\cdot)} & \\
x & \longrightarrow & x \ln x & \longrightarrow & e^{x \ln x} \\
 & \downarrow & & \downarrow & \\
 & \ln x + 1 & \cdot & e^{(\cdot)} & \\
 & & (\ln x + 1) \cdot e^{x \ln x} & &
\end{array}$$

i.e. $(x^x)' = (1 + \ln x)x^x$.

Higher-order Derivatives

Suppose that $f(x)$ is smooth function; the second order derivative is defined as follows:

$$f''(x) = \frac{d^2}{dx^2} f(x) = D_x^2 f(x) = \frac{d}{dx} f'(x)$$

And the more higher order derivatives, (n-th)-order for instance, are defined by the recurive formula:

$$\begin{aligned}
f^{(3)}(x) &= \frac{d^3}{dx^3} f(x) = D_x^3 f(x) = \frac{d}{dx} f''(x) \\
&\vdots \\
f^{(n)}(x) &= \frac{d^n}{dx^n} f(x) = D_x^n f(x) = \frac{d}{dx} f^{(n-1)}(x)
\end{aligned}$$

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In [46]:

```
# 20
from sympy import Symbol
t=Symbol("t")
diff((t**5-3*t**3+2*t**2)/2/t**2,t)
```

Out[46]:

```
(5*t**4/2 - 9*t**2/2 + 2*t)/t**2 - 2*(t**5/2 - 3*t**3/2 + t**2)/t*
*3
```

In [ ]:
```
# 72
```

In [ ]:

Example

Investigate the derivatives of $f(x) = x^2 \sin x$.

Sol:

The first order of derivative is:

$$\begin{aligned} f'(x) &= (x^2 \sin x)' \\ &= (x^2)' \sin x + x^2(\sin x)' \\ &= 2x \sin x + x^2 \cos x \end{aligned}$$

And the second order derivative is:

$$\begin{aligned} f''(x) &= (x^2 \sin x)'' \\ &= (2x \sin x + x^2 \cos x)' \\ &= (2x)' \sin x + 2x(\sin x)' + (x^2)' \cos x + x^2(\cos x)' \\ &= 2 \sin x + 2x \cos x + 2x \cos x - x^2 \sin x \\ &= 2 \sin x + 4x \cos x - x^2 \sin x \end{aligned}$$

Exactly, the last result is equal to the following:

$$C(2,0)(x^2)'' \sin x + C(2,1)(x^2)'(\sin x)' + C(2,2)x^2(\sin x)''$$

where $C(n,k) = n!/k!(n-k)!$ and $n! = 1 \cdot 2 \cdot 3 \cdots n$, the factorial of n.

This relation can be extended to the more higher differentiation:

$$\begin{aligned} f^{(n)} &= (x^2 \sin x)^{(n)} \\ &= C(n,2)(x^2)''(\sin x)^{(n-2)} + C(n,1)(x^2)'(\sin x)^{(n-1)} + C(n,0)x^2(\sin x)^{(n)} \\ &= n(n-1)(\sin x)^{(n-2)} + 2nx(\sin x)^{(n-1)} + x^2(\sin x)^{(n)} \end{aligned}$$

By the last result, the derivative of 10-th order is

$$f^{(10)}(x) = 90 \sin x + 20x \cos x - x^2 \sin x$$

Differention with Sympy

In [1]:
```
from sympy import Symbol, diff,sin,cos,exp,pi,pprint,Abs
x=Symbol("x")
```

In [3]:
```
diff(x*x*sin(x),x,10)
```

Out[3]:
```
-x**2*sin(x) + 20*x*cos(x) + 90*sin(x)
```

In [4]:
```
for i in range(1,11):
    print("%d). the %d-order derivative is: " %(i,i))
    pprint(diff(x*x*sin(x),x,i))
    print("\n")
```

```
1).  the 1-order derivative is:
```

```
1). the 1-order derivative is:
 2
x ·cos(x) + 2·x·sin(x)


2). the 2-order derivative is:
   2
- x ·sin(x) + 4·x·cos(x) + 2·sin(x)


3). the 3-order derivative is:
   2
- x ·cos(x) - 6·x·sin(x) + 6·cos(x)


4). the 4-order derivative is:
 2
x ·sin(x) - 8·x·cos(x) - 12·sin(x)


5). the 5-order derivative is:
 2
x ·cos(x) + 10·x·sin(x) - 20·cos(x)


6). the 6-order derivative is:
   2
- x ·sin(x) + 12·x·cos(x) + 30·sin(x)


7). the 7-order derivative is:
   2
- x ·cos(x) - 14·x·sin(x) + 42·cos(x)


8). the 8-order derivative is:
 2
x ·sin(x) - 16·x·cos(x) - 56·sin(x)


9). the 9-order derivative is:
 2
x ·cos(x) + 18·x·sin(x) - 72·cos(x)


10). the 10-order derivative is:
   2
- x ·sin(x) + 20·x·cos(x) + 90·sin(x)
```

In [11]:

```python
u=Symbol("u")
def ChainRule(f,g):
    u=Symbol("u")
    l1=diff(f,u).subs({u:g})
    l2=diff(g,x)
    print("f'(g(x)) =",l1*l2,", where u = ",g)
```

In [24]:

```python
u=Symbol("u")
def ChainRule2(f,g):
    u=Symbol("u")
    l1=diff(f,u).subs({u:g})
```

```
    l2=diff(g,x)
    print("(f(g(x)))' =",l1*l2,", \nwhere f(u) = ",f, " and u = g =
",g)
```

In [2]:
```
u=Symbol("u")
def ChainRule3(f,g):
    u=Symbol("u")
    fu=diff(f,u)
    fx=diff(f,u).subs({u:g})
    gx=diff(g,x)
    fg=f.subs({u:g})
    print("    ",g,"\t",f,"\n", \
          "x","  ⟶   ",g,"    ⟶    ",fg, "\n",\
          "     ↓     ×       ↓", "\n",\
          "    ",gx,"\t",fu,"\n",\
          "           ↓    \n", \
          "     ", gx*fx
         )
    #print("(f(g(x)))' =",l1*l2,", \nwhere f(u) = ",f, " and u = g
= ",g)
```

In [ ]:
```
```

In [3]:
```
u=Symbol("u")
def ChainRule3(f,g):
    """
    Differentiate f(g(x))
    f=f(u)
    g=u=g(x)
    e.g. (sin(x**2+1))' uses
    ChainRule3(sin(u),x**2)
    """
    u=Symbol("u")
    fu=str(diff(f,u))
    fx=str(diff(f,u).subs({u:g}))
    gx=str(diff(g,x))
    fgx= str(diff(g,x)*diff(f,u).subs({u:g}))
    gfunc=str(g)
    ffunc0=str(f)
    ffunc=str(f.subs({u:g}))
    print('{:3}'.format(" "),'{:10.8}'.format(gfunc),'{:12}'.format
```

```
(" "),'{:8.8}'.format(ffunc0))
    print('{:3}'.format("x"),'{:4}'.format("⟶"),'u =','{:12}'.format(gfunc),
          '{:5}'.format("⟶"),'{:12}'.format(ffunc))
    print('{:3}'.format(" "),'{:20}'.format("↓"),'{:2}'.format(" "),"↓")
    print('{:3}'.format(" "),'{:10.8}'.format(gx),'{:8}'.format(" × "),'{:2}'.format(" "),'{:8}'.format(fu))
    print('{:15}'.format(" "),"↓")
    print('(',ffunc,")'=",'{:15}'.format(fgx))
```

In [24]:

```

```

In [53]:

```
ChainRule(u**2,sin(x))
```

```
f'(g(x)) = 2*sin(x)*cos(x) , where u =  sin(x)
```

In [54]:

```
ChainRule2(sin(u),x**2)
```

```
(f(g(x)))' = 2*x*cos(x**2) ,
where f(u) =  sin(u)  and u = g =  x**2
```

In [16]:

```
ChainRule3(sin(u),x**2)
```

```
    x**2                     sin(u)
x   ⟶      u = x**2          ⟶      sin(x**2)
    ↓                        ↓
    2*x          ×          cos(u)
                 ↓
( sin(x**2) )'= 2*x*cos(x**2)
```

In [17]:

```
ChainRule3(u**2,cos(x))
```

```
    cos(x)                   u**2
x   ⟶      u = cos(x)        ⟶      cos(x)**2
    ↓                        ↓
    -sin(x)      ×          2*u
                 ↓
( cos(x)**2 )'= -2*sin(x)*cos(x)
```

In [18]:

```
ChainRule3(sin(u),x**2+x)
```

```
    x**2 + x                 sin(u)
x   ⟶      u = x**2 + x      ⟶      sin(x**2 + x)
    ↓                        ↓
    2*x + 1      ×          cos(u)
                 ↓
( sin(x**2 + x) )'= (2*x + 1)*cos(x**2 + x)
```

In [20]:

```
ChainRule3(cos(pi-u),x**2+x)
```

```
    x**2 + x                 -cos(u)
x   ⟶      u = x**2 + x      ⟶      -cos(x**2 + x)
    ↓                        ↓
    2*x + 1      ×          sin(u)
```

```
                  ↓
( -cos(x**2 + x) )'= (2*x + 1)*sin(x**2 + x)
```

Exercise, p170

16.

$$\left(\frac{2t^2-3t^{3/2}}{5t^{1/2}}\right)'$$

In [5]:
```
#16
t=Symbol("t")
pprint(diff((2*t**2-3*t**(3/2))/5*t**(1/2),t))
```

```
            ⎛   2       1.5⎞                         
       -0.5 ⎜2⋅t     3⋅t   ⎟    0.5 ⎛         0.5   4⋅t⎞
0.5⋅t     ⋅⎜──── - ──────⎟ + t   ⋅⎜- 0.9⋅t     + ───⎟
            ⎝ 5        5   ⎠        ⎝                5 ⎠
```

36.

$$\left(\frac{x}{x^4-2x^2-1}\right)'|_{x=-1}$$

In [7]:
```
#36
df=diff((x)/(x**4-2*x**2-1),x)
pprint(df)
```

```
  ⎛     3      ⎞              
x⋅⎝- 4⋅x  + 4⋅x⎠         1      
──────────────── + ────────────
              2     4      2    
 ⎛ 4      2    ⎞   x  - 2⋅x  - 1
 ⎝x  - 2⋅x  - 1⎠                
```

In [15]:
```
df.subs({x:-1})
```

Out[15]: -1/2

Exercise, p190

(20).

$$\left(\frac{1-\tan x}{1+\cot x}\right)' = \frac{(1-\tan x)'(1+\cot x)-(1-\tan x)(1+\cot x)'}{(1+\cot x)^2}$$

$$=$$

(48).

' '

$$\lim_{h\to 0}\frac{\frac{1}{\sin(x+h)}-\frac{1}{\sin x}}{h}=\lim_{h\to 0}\frac{\csc(x+h)-\csc x}{h}$$
$$=$$

Exercise, p190

12.

$$\left(\frac{\cos\theta}{1-\sin\theta}\right)'$$

In [2]:
```
from sympy import diff, Symbol,symbols,sin,cos,pprint
```

In [3]:
```
x,t=symbols(" x t")
```

In [6]:
```
#12
pprint(diff(cos(x)/(1-sin(x)),x))
```

```
                         2
     sin(x)           cos (x)
- ──────────── + ───────────────
  -sin(x) + 1                  2
                 (-sin(x) + 1)
```

In [ ]:
```

```

22.

$$\left(\frac{a\sin\theta}{1+b\cos\theta}\right)'$$

In [9]:
```
#22
a,b =symbols("a b")
pprint(diff(a*sin(x)/(1+b*cos(x)),x))
```

```
           2
  a⋅b⋅sin (x)          a⋅cos(x)
─────────────────── + ────────────
                  2    b⋅cos(x) + 1
(b⋅cos(x) + 1)
```

42. Foe $n=0,1,2,$

$$(\cos\theta)^{(n)}=?$$

And find out the formula of devivatives for all the $n=1,2,3,4,\cdots$.

In [13]:
```
# 42
for n in range(8):
    pprint(diff(sin(x),x,n))
```

```
sin(x)
cos(x)
-sin(x)
-cos(x)
sin(x)
cos(x)
-sin(x)
-cos(x)
```

Whatever for $\sin x$ or $\cos x$, absolute value of its derivative is no more than 1.

Exercise, p201

12.

$$\left(\left(\frac{x^2+3}{x}\right)^{-2}\right)'$$

In [60]:
```
# 12
pprint(diff(((x*x+3)/x)**(-2),x))
```

```
          3                
       4⋅x           2⋅x   
  - ─────────── + ─────────
              3           2
     ⎛ 2     ⎞    ⎛ 2     ⎞
     ⎝x  + 3⎠    ⎝x  + 3⎠
```

14.

$$\left((2x-1)^2(x^2+1)^3\right)'$$

In [17]:
```
pprint(diff((2*x-1)**2*(x**2+1)**3,x))
```

```
                          2                        3
                 2 ⎛ 2    ⎞                ⎛ 2    ⎞ 
6⋅x⋅(2⋅x - 1) ⋅ ⎝x  + 1⎠  + (8⋅x - 4)⋅ ⎝x  + 1⎠ 
```

22. Evaluate $f'(x)$ where

$$\left(\frac{x+2}{x-3}\right)^{3/2}$$

In [18]:
```
##22
pprint(diff(((x+2)/(x-2))**(3/2),x))
```

```
        1.5
⎛x + 2⎞              ⎛  1.5     1.5⋅(x + 2)⎞
⎜─────⎟     ⋅(x - 2)⋅⎜───── - ───────────⎟
⎝x - 2⎠              ⎜x - 2              2 ⎟
```

```
\                (x - 2)  /
────────────────────────────
           x + 2
```

22. Evaluate $f'(x)$ where

$$\left(\frac{(t+1)^3}{(t^2+2t)^2}\right)$$

In [63]:
```
#24
pprint(diff((t+1)**3/(t**2+2*t)**2,t))
```

```
                    3               2
(-4⋅t - 4)⋅(t + 1)       3⋅(t + 1)
─────────────────────  + ───────────
             3                    2
   ⎛ 2      ⎞            ⎛ 2      ⎞
   ⎝t  + 2⋅t⎠            ⎝t  + 2⋅t⎠
```

In [65]:
```
#29
ChainRule(u**3,sin(x))
```

```
f'(g(x)) = 3*u**2*cos(x) , where u =  sin(x)
```

In [22]:
```
#29
ChainRule3(u**3,sin(x))
```

```
     sin(x)                      u**3
x     ⟶      u = sin(x)          ⟶       sin(x)**3
      ↓                          ↓
     cos(x)        ×           3*u**2
                   ↓
( sin(x)**3 )'= 3*sin(x)**2*cos(x)
```

32. y' if $y = \cos(x^2 - 3x + 1) + \tan(2/x)$

In [19]:
```
from sympy import tan
pprint(diff(cos(x*x-3*x)+1+tan(2/x),x))
```

```
                                        ⎛    2 ⎛2⎞    ⎞
                                      2⋅⎜tan ⎜─⎟ + 1⎟
                       ⎛ 2      ⎞       ⎝    ⎝x⎠    ⎠
- (2⋅x - 3)⋅sin⎝x  - 3⋅x⎠ - ─────────────────
                                         2
                                        x
```

36. z' if $z = (1 + \csc^2 x)^4$

In [4]:
```
from sympy import csc
z=(1+(csc(x))**2)**4
pprint(diff(z,x))
```

```
                     3
    ⎛   2        ⎞               2
-8⋅⎝csc (x) + 1⎠  ⋅cot(x)⋅csc (x)
```

42. y' if

$$y = \frac{x + \sin 2x}{2 + \cos 3x}$$

In [2]:
```
from sympy import sin,cos
y=(x+sin(x))/(2+cos(3*x))
pprint(diff(y,x))
```

```
3⋅(x + sin(x))⋅sin(3⋅x)     cos(x) + 1
─────────────────────── + ────────────
                    2     cos(3⋅x) + 2
     (cos(3⋅x) + 2)
```

52. y' if $y = x\tan^2(2x+3)$

In [6]:
```
from sympy import tan
y=x*tan(2*x+3)**2
pprint(diff(y,x))
```

```
   ⎛     2              ⎞                    2
x⋅ ⎝4⋅tan (2⋅x + 3) + 4⎠ ⋅tan(2⋅x + 3) + tan (2⋅x + 3)
```

60. y' if $y = x\sin 1/x$

In [7]:
```
y=x*sin(1/x)
pprint(diff(y,x))
```

```
             ⎛1⎞
          cos⎜─⎟
   ⎛1⎞       ⎝x⎠
sin⎜─⎟ - ──────
   ⎝x⎠     x
```

64. Find the tangent line of $h(t) = 2\cos^2 \pi t$ at $t = 1/4$.

In [5]:
```
from sympy import pi,cos,symbols,diff
x=symbols("x")
h=2*cos(pi*x)**2
t0=1/4
df=diff(h,x)
y0=h.subs({x:t0})
m=df.subs({x:t0})

print("The tangent line of f(x) at x = %s is (y-%s)=%s(x-%s)" %(t0,
y0,m,t0))
```

```
The tangent line of f(x) at x = 0.25 is (y-1)=-2*pi(x-0.25)
```

In [66]:
```
#36
ChainRule(cos(u),x**3)
```

```
f'(g(x)) = -3*x**2*sin(u) , where u =  x**3
```

In [19]:
```
#36
ChainRule3(cos(u),x**3)
```

```
     x**3                        cos(u)
x    ⟶      u = x**3              ⟶      cos(x**3)
     ↓                            ↓
     3*x**2       ×              -sin(u)
                  ↓
( cos(x**3) )'= -3*x**2*sin(x**3)
```

In [67]:
```
#44
pprint(diff(x*sin(1/x),x))
```

```
                  ⎛1⎞
              cos ⎜─⎟
     ⎛1⎞          ⎝x⎠
sin ⎜─⎟ -  ──────
     ⎝x⎠       x
```

In [68]:
```
#60
pprint(diff(x*sin(1/x),x,2))
```

```
      ⎛1⎞
-sin ⎜─⎟
      ⎝x⎠
────────
    3
   x
```

(77). Find the derivative of

$$\frac{|x|}{\sqrt{2-x^2}}$$

for $x \neq 0$ and make its picture (like bullet-nose).

Since

$$|x| = \begin{cases} x, & \text{if } x > 0, \\ -x, & \text{if } x < 0 \end{cases}$$

- $x > 0$

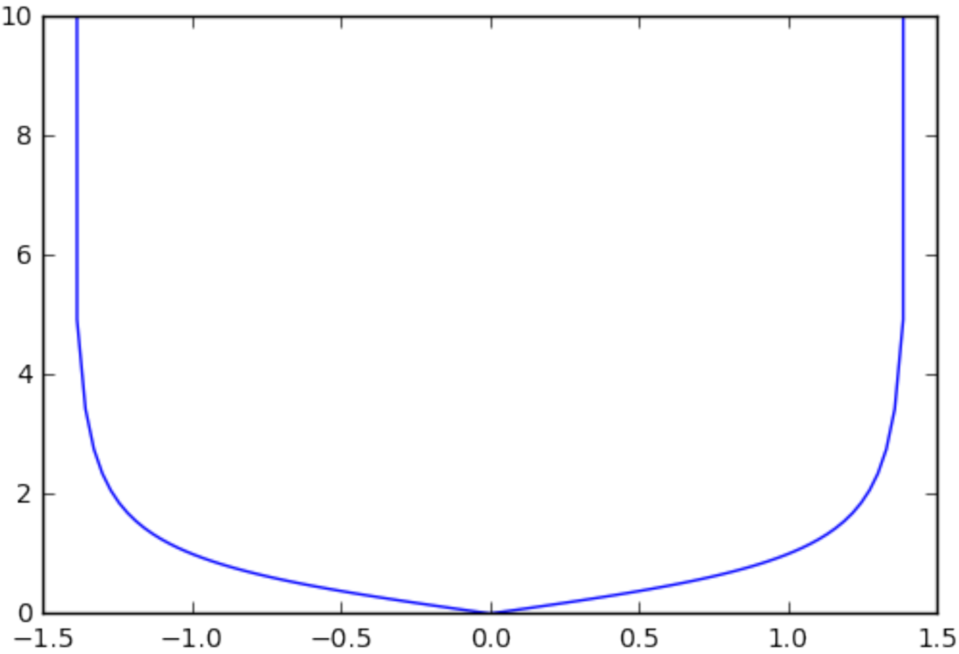
$$f'(x) = \left(\frac{x}{\sqrt{2-x^2}}\right)' = \frac{2}{(2-x^2)^{3/2}}$$

- $x < 0$

$$f'(x) = \left(\frac{-x}{\sqrt{2-x^2}}\right)' = \frac{-2}{(2-x^2)^{3/2}}$$

In [19]:
```
from numpy import abs,sqrt
epsilon=1e-6
x=np.linspace(-sqrt(2)+epsilon,sqrt(2)-epsilon,101)
plt.plot(x,abs(x)/sqrt(2-x*x))
plt.ylim(0,10)
```

Out[19]: (0, 10)



(99).

Suppose that

$$f(x) = \begin{cases} x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

a). For $x \neq 0$,

$$f'(x) = 2x\sin(1/x) + x^2 \cdot (-1/x^2)\cos(1/x) = 2x\sin(1/x) - \cos(1/x)$$

b). If $x = 0$, then

b). If $x = 0$, then

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin(1/x) - 0}{x} = 0$$

by squeeze value theorem. Thus

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

c). For $x \neq 0$,

$$f''(x) = (2x\sin(1/x) - \cos(1/x))' = 2\sin(1/x) - \frac{2\cos(1/x)}{x} - \frac{\sin(1/x)}{x^2}$$

d). but $f''(0)$ fails to exist since the limit

$$f''(0) = \lim_{x\to 0} \frac{2x\sin(1/x) - \cos(1/x)}{x}$$

fails to exist.

In [70]:
```
pprint(diff(x**2*sin(1/x),x))
```

```
         ⎛1⎞      ⎛1⎞
2⋅x⋅sin ⎜─⎟ - cos ⎜─⎟
         ⎝x⎠      ⎝x⎠
```

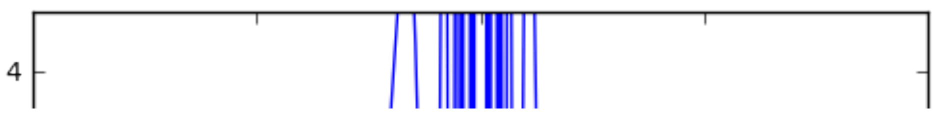
In [69]:
```
pprint(diff(x**2*sin(1/x),x,2))
```

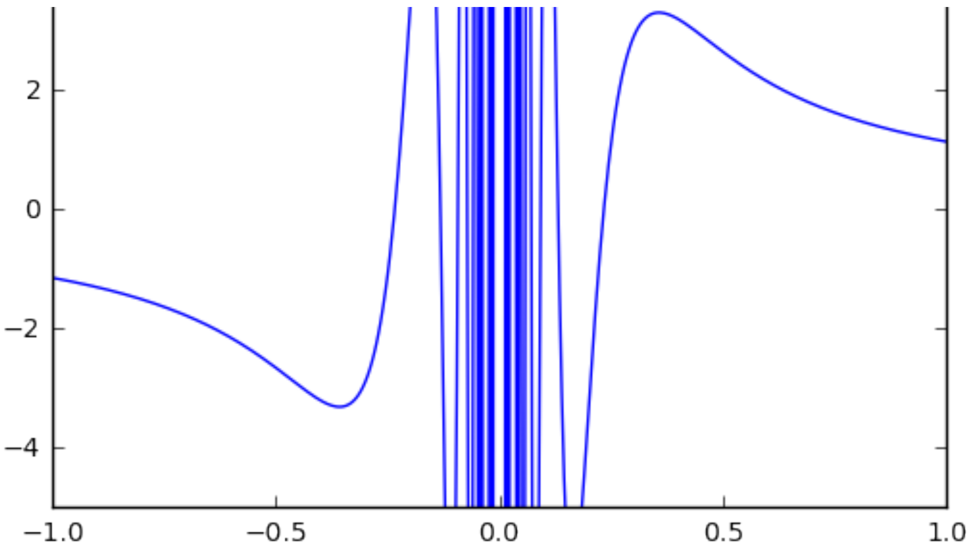
```
                 ⎛1⎞       ⎛1⎞
           2⋅cos ⎜─⎟   sin ⎜─⎟
      ⎛1⎞        ⎝x⎠       ⎝x⎠
2⋅sin ⎜─⎟ - ───────── - ───────
      ⎝x⎠       x           2
                           x
```

In [75]:
```
from numpy import sin,cos
epsilon=1e-6
x=np.linspace(-1,1,401)
plt.plot(x,(2*x*sin(1/x)-cos(1/x))/x)
plt.ylim(-5,5)
```

```
/Users/cch/anaconda/lib/python3.5/site-packages/ipykernel/__main__
.py:4: RuntimeWarning: divide by zero encountered in true_divide
/Users/cch/anaconda/lib/python3.5/site-packages/ipykernel/__main__
.py:4: RuntimeWarning: invalid value encountered in sin
/Users/cch/anaconda/lib/python3.5/site-packages/ipykernel/__main__
.py:4: RuntimeWarning: invalid value encountered in cos
```

Out[75]: (-5, 5)





In [ ]:

(100).

$$|u|' = (\sqrt{u^2})' = 2u \cdot u' \cdot \frac{1}{2\sqrt{u^2}} = \frac{u'u}{|u|}$$

Implicit Differentiation

Definition

$F(x, y) = 0$ is called equation of x and y of y is called an implicit function of x.

Example

The circle, $x^2 + y^2 = r^2$, is a famous known equation. And how do we find the tangent line at the point on the circle? Implicit differentiation says that y can be treated as function of x locally, i.e. $y = y(x)$. Then

$$\begin{aligned}
& x^2 + (y(x))^2 = r^2 \\
\Longrightarrow & [x^2 + (y(x))^2]' = [r^2]' \quad \text{by differentiating both sides} \\
\Longrightarrow & \; 2x + 2y(x)y'(x) = 0 \\
\Longrightarrow & \quad y'(x) = -\frac{x}{y(x)} \qquad \text{if } y(x) \neq 0
\end{aligned}$$

This means that $y'(x)$ depends on both x and y. For example,

- At $(x, y) = (1/\sqrt{2}, 1/\sqrt{2})$, $y'(x) = -\sqrt{2}/\sqrt{2} = -1$;
- at $(x, y) = (-3/5, 4/5)$, $y'(x) = -[(-3/5)/(4/5)] = 3/4$

Example

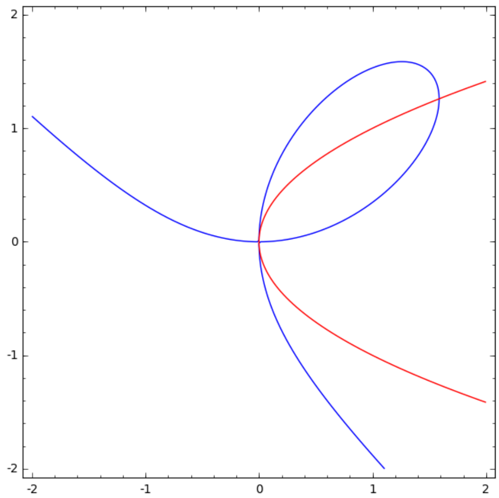
Find out $y'(x)$ if $x^3 + y^3 - 3xy = 0$.

By implicit differention:

$$\begin{aligned} & [x^3 + y^3 - 3xy]' && = 0 \\ \Longrightarrow & 3x^2 + 3y^2 y' - 3(y + xy') && = 0 \\ \Longrightarrow & y'(3y^2 - 3x) && = 3(y - x^2) \\ \Longrightarrow & y' && = \frac{x^2 - y}{x - y^2} \end{aligned}$$

if $y^2 - x \neq 0$ or $x \neq y^2$.

Why no Derivative at $(x, y) = (0, 0)$ and $\left(2^{2/3}, 2^{1/3}\right)$



It's very clear:

- the curve of $x^3 + y^3 - 3xy = 0$ crosses origin, $(0, 0)$, twice with different tangent; and this is why its tangent can't exist at origin even that the tangent lines exist.
- The latter case, it is obvious that there exists a vertical tangent line at the case, i.e. $f'|_{(2^{2/3}, 2^{1/3})} = \infty$, does not exist.

Note

Above image was created by another CAS, Sage (www.sagemath.org), which owns powerful plotting functionalities for math works.

In [22]:
```
from sympy import Function,solve,Derivative
y=Function("y")
y=y(x)
```

In [23]:
```
l=diff(x**3+y**3-3*x*y,x);
l
```

Out[23]:
```
3*x**2 - 3*x*Derivative(y(x), x) + 3*y(x)**2*Derivative(y(x), x) -
3*y(x)
```

In [9]:
```
solve(l,Derivative(y,x))
```

Out[9]:
```
[(x**2 - y(x))/(x - y(x)**2)]
```

In [24]:
```
def ImplicitDiff(express):
    l=diff(express,x);
    print("y'(x) =",solve(l,Derivative(y,x))[0])
```

In [1]:
```
from sympy.plotting import plot_implicit
from sympy import  symbols, Eq, solve
x, y = symbols('x y')

def ImplicitPlot(express):
    eq = Eq(express)
    plot_implicit(eq)
```

In [25]:
```
ImplicitDiff(x**3+y**3-3*x*y)
```

```
y'(x) = (x**2 - y(x))/(x - y(x)**2)
```

In [26]:
```
ImplicitDiff(x**2*y+y+cos(x)+1)
```

```
y'(x) = (-2*x*y(x) + sin(x))/(x**2 + 1)
```

In [27]:
```
ImplicitDiff(y**4-2*y**3+x**3*y**2-cos(x)-8)
```

```
y'(x) = -(3*x**2*y(x)**2 + sin(x))/(2*(x**3 + 2*y(x)**2 - 3*y(x))*
y(x))
```

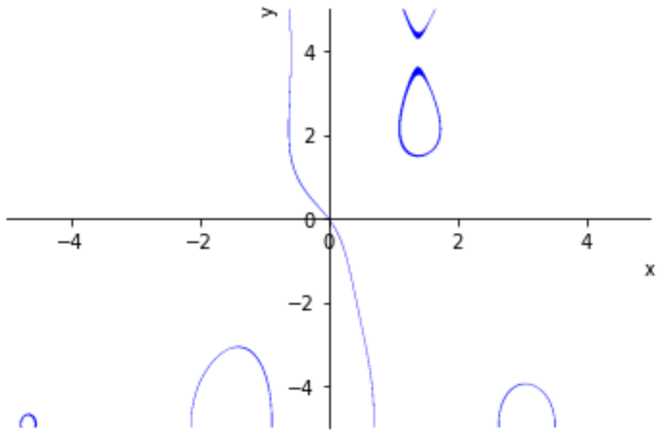
Example

Evaluate $y'(x)$ for $x \sin y - y \cos 2x = 2x$ at $(x, y) = (\pi/2, \pi)$

In [28]:
```
ImplicitDiff(x*sin(y)-y*cos(2*x)-2*x)
```

```
y'(x) = (-2*y(x)*sin(2*x) - sin(y(x)) + 2)/(x*cos(y(x)) - cos(2*x)
)
```

In [4]:
```
from sympy import sin,cos,exp
ImplicitPlot(x*sin(y)-y*cos(2*x)-2*x)
%matplotlib inline
```



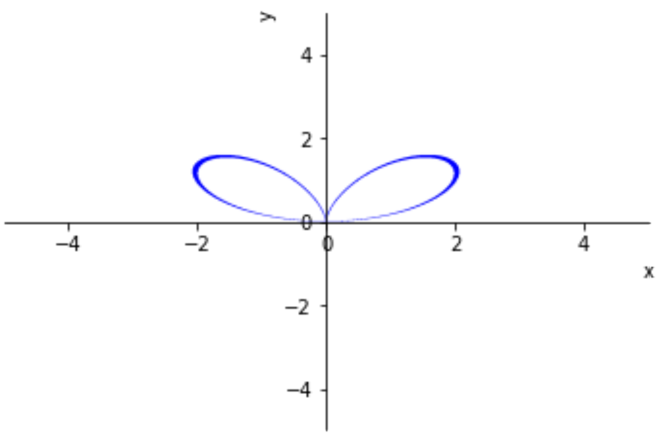
Thus

$$y'(x)\,|_{(x,y)=(\pi/2,\pi)} = \frac{-2\pi\sin\pi - \sin\pi + 2}{\pi/2\cos\pi - \cos\pi} = \frac{4}{2-\pi}$$

Example

Find an equation of tangent line of $4x^4 + 8x^2y^2 - 25x^2y + 4x^4 = 0$ at $(x, y) = (2, 1)$

In [38]:
```
ImplicitPlot(4*x**4+8*x**2*y**2-25*x**2*y+4*y**4)
```



In [29]:
```
ImplicitDiff(4*x**4+8*x**2*y**2-25*x**2*y+4*y**4)
```

```
y'(x) = 2*x*(-8*x**2 - 8*y(x)**2 + 25*y(x))/(16*x**2*y(x) - 25*x**
2 + 16*y(x)**3)
```

The slope is $y' = 2\times 2(-8\times 2^2 - 8 + 25)/(16\times 2^2 - 25\times 2^2 + 16) = 3$. Thus the tangent line is

$$(y-1) = 3(x-2)$$

P. 212

10. Find $y'(x)$ if $\frac{x+y}{x-y} = y^2 + 1$, #16. $x + y^2 = \cos xy$

36. tangent line of $2y^2 - x^3 - x^2 = 0$ at $(1, 1)$.

42. normal line of $x^5 - 2xy + y^5 = 0$ at $(1, 1)$.

44. tangent line of $x^{2/3} + y^{2/3} = 1$ at $(3\sqrt{3}, 1)$

In [16]:
```
#10.
ImplicitDiff((x+y)/(x-y-y**2-1))
```

```
y'(x) = (y(x) + 1)/(2*x + y(x) - 1)
```

In [18]:
```
#16.
ImplicitDiff(x+y**2-cos(x*y))
```

```
y'(x) = -(y(x)*sin(x*y(x)) + 1)/(x*sin(x*y(x)) + 2*y(x))
```

In [19]:
```
#36.
ImplicitDiff(2*y**2-x**3-x**2)
```

```
y'(x) = x*(3*x + 2)/(4*y(x))
```

since $m = 1$, tangent line is $(y - 1) = x - 1$.

In [20]:
```
#42.
ImplicitDiff(x**5-2*x*y+y**5)
```

```
y'(x) = (5*x**4 - 2*y(x))/(2*x - 5*y(x)**4)
```

slope of normal line is negative inverse of tangent line, $n = -1/m = 1$, tangent line is $(y - 1) = x - 1$.

In [21]:
```
#44.
ImplicitDiff(x**(2/3)+y**(2/3)-4)
```

```
y'(x) = -y(x)**(1/3)/x**(1/3)
```

since $m = -\sqrt{3}$, tangent line is $(y-1) = -\sqrt{3}(x - 3\sqrt{3})$.

Inverse of Function

Inverse Differentiation

Suppose that the inverse of $y = f(x)$ exists and is equal to $x = f^{-1}(y)$. Then
$$\frac{dx}{dy} = \frac{1}{dy/dx}$$
i.e.
$$\left(f^{-1}(y)\right)' = \frac{1}{f'(x)}$$

Example

The inverse of $\sin$ is defined as follows:
$$y = \mathrm{Sin}^{-1} x \text{ if } x = \sin(y), \text{ for } y \in (-\pi/2, \pi/2)$$
$$\begin{aligned}\frac{d\mathrm{Sin}^{-1} x}{dx} &= \frac{dy}{dx}\\ &= \frac{1}{dx/dy}\\ &= \frac{1}{d(\sin y)/dy}\\ &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Note

Other derivatives for inverse trigonometric functions are:

- $\left(\mathrm{Tan}^{-1}(x)\right)' = \frac{1}{1+x^2}$,
- $\left(\mathrm{Sec}^{-1}(x)\right)' = \frac{1}{|x|\sqrt{x^2-1}}$,

In [85]:
```
from sympy import asin,atan,asec,simplify
```

In [79]:
```
diff(asin(x),x)
```

Out[79]:
```
1/sqrt(-x**2 + 1)
```

In [88]:
```
pprint(diff(atan(x),x))
```

```
   1
──────
 2
x  + 1
```

In [ ]: