1 Multi-variable Calculus

13.1 Functions of Several Variables
13.2 Limits and Continuuity
13.3 Partial Differentiation (6%20Multi-variable%20Calculus-Differentiation-2.ipynb#Partial-Differentiation)
13.4 Chain Rule (6%20Multi-variable%20Calculus-Differentiation-2.ipynb#Chain-Rule)
13.5 Tangent Plane (6%20Multi-variable%20Calculus-Differentiation-2.ipynb#Tangent-Plane)
13.6 Relative Extrema (6%20Multi-variable%20Calculus-Differentiation-3.ipynb#Relative-Maxima-and-Minima)
13.7 Lagrange Multiplier (6%20Multi-variable%20Calculus-Differentiation-3.ipynb#Optimization-Problem-with-Constraints)
13.8 Method of Least Squares (6%20Multi-variable%20Calculus-Differentiation-3.ipynb#The-Method-of-Least-Squares)

1.1 Appendix

Vissualization

- Mathbox, Javascript
- JSAnimation, IPython notebook UI enhanced
- plotly, cross-platform

2 Functions of Several Variables

Functions can be specified with more than two independent variables. For example, the Cobb-Douglas production function,
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is a model used to study the relationship between levels of labor, L, and capital goods, K.

2.1 Definition

1. Let $\mathbf{x} = (x^1, x^2, \cdots, x^n) = (x^i) \in \mathbb{R}^n$ be n-dimensional vector, where $x^i \in \mathbb{R}$ for $i = 1, 2, \cdots, n$;
2. Let $D = \{\mathbf{x} \in \mathbb{R}^n\}$ be certain a subset of $\mathbb{R}^n$.
3. $f(\mathbf{x}), (x) \in D$ ia a function of multiple-variable function over D if there is a rule that assigns a unique real value for each $(x) \in D$.
4. D is called domain of f, denoted as $D(F)$, and $f(D)$ is called range of f.
5. The graph of $f(\mathbf{x})$ is the set of $\{(\mathbf{x}, z) | z = f(\mathbf{x}), \mathbf{x} \in D(f)\} \in \mathbb{R}^{n+1}$.
6. The level curve (surface} of $f(\mathbf{x})$ at height k is the set
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Especially, $f(x, y)$ is called a function of two variables and Graphs of Functions of Two variables, $z = f(x, y)$, is the set
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which can be represented as the three-dimension Cartesian coordinate system or the **level curve** on the plane.

To Describe the property for simplicity, we also observe the following way:

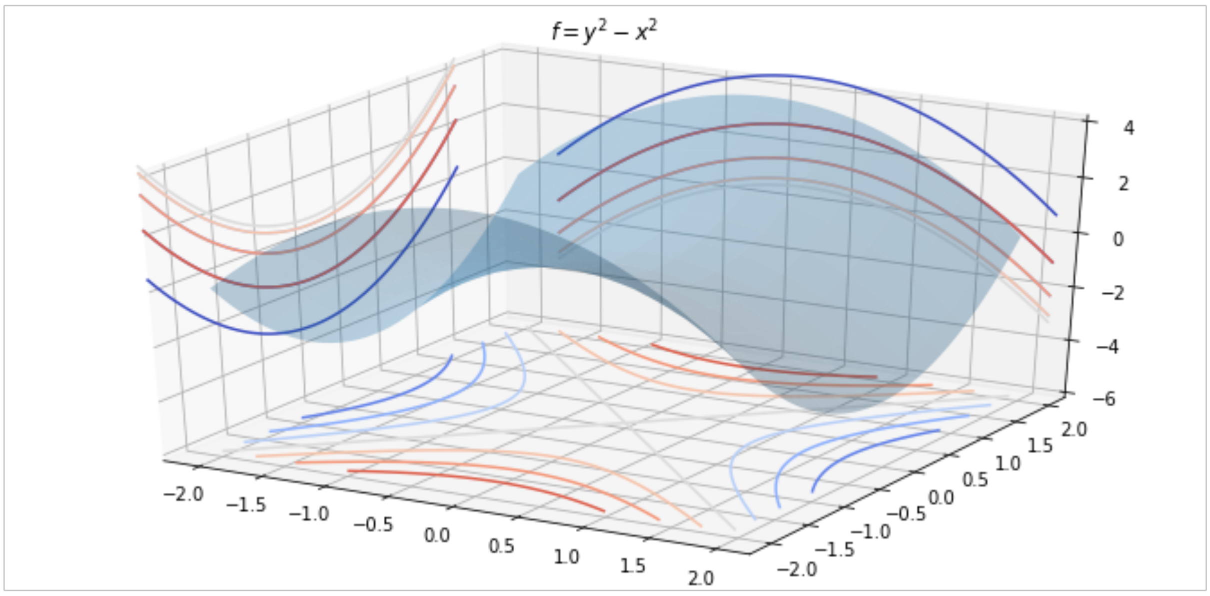
2.2 Definition

The level curves of a function of two variables are the curves on $X - Y$ plane with $f(x, y) = k$, where $k \in$ Range(f).

2.3 Example

Sketch grapg and contour of $z = f(x, y) = y^2 - x^2$.

```
Text(0.5, 0.92, '$f=y^2-x^2$')
```



2.4 Example

$z = f(x, y) = \sqrt{16 - x^2 - y^2}$. Here domain is $\{(x, y) | x^2 + y^2 \leq 16\}$ and the range is $[0, 4]$.

2.5 Example

Consider the function $z = \frac{6}{2-x-y}$.

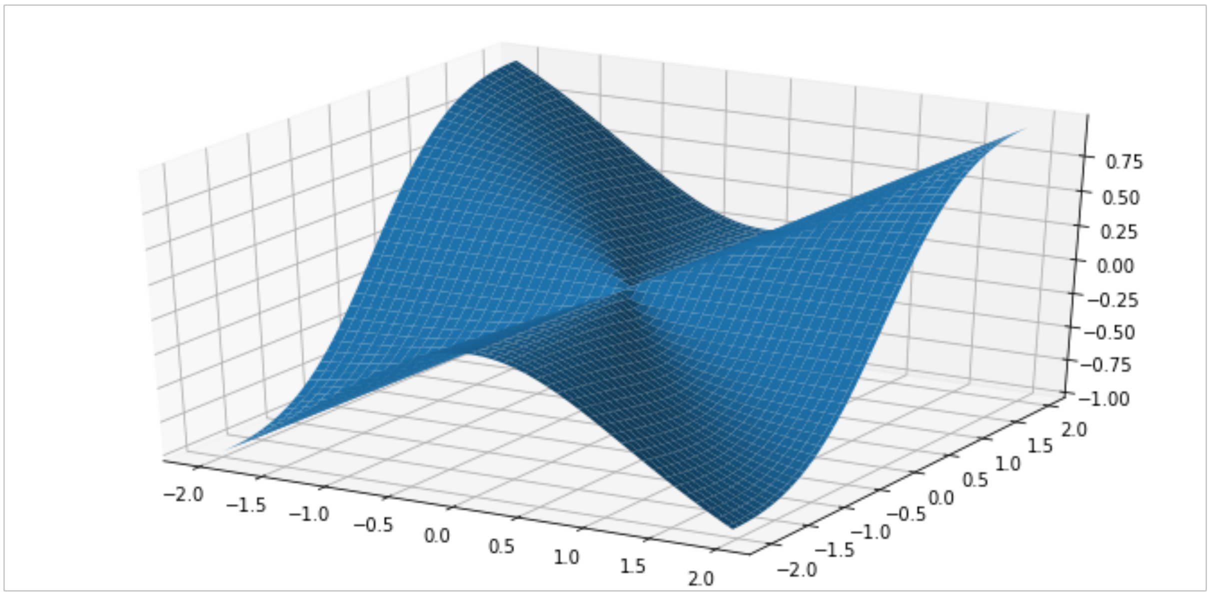
Domain=$\{(x, y) \in \mathbb{R}^2 | 2 - x - y \neq 0\}$ and Range= $\{z \in \mathbb{R} | z \neq 0\}$ since $2 - x - y$ can approach $\pm\infty$ and $2^{\pm}$.

2.6 Example

Since the domain for $\ln \cdot$ is $\{x > 0\}$, then the domain for $\ln xy$ is

$$\{xy > 0\} = \{(x, y) | x, y > 0 \text{ or } x, y < 0\}$$

It is obvious that the graphs of functions can help us to recognize the behavior of function, for example: the domain, the range and the extrema at which they occur. With the help of matplotlib, we can easily generate the graphs of functions not only 2D but 3D graphs with many useful features.



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Suppose that N is a region in $\mathbb{R}^2$ containing (a, b), called neighborhooh of (a, b).

2.8 Definition

$f(x, y)$ is called to have a limit, L, at (a, b) if the value of $f(x, y)$ can approach L arbitrarily while (x, y) is near (a, b) enough. This means:

$\forall \varepsilon > 0, \exists \delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ if $\|(x, y) - (a, b)\| < \delta$

where $\|(x, y) - (a, b)\| = \sqrt{(x-a)^2 + (y-b)^2}$ i.e. the distance between (x, y) and (a, b).

2.9 Example

Evaluate the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$$

1. Approach $(0, 0)$ along X-axis, we have

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{y=0, x\to 0} \frac{x-y}{x+y} = 1$$

2. Approach $(0, 0)$ along Y-axis, we have

$$\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x=0, y\to 0} \frac{x-y}{x+y} = -1$$

The limit fails to exist since the limits in (a) and (b) are not equal.

2.10 Example

Evaluate the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

1. Approach $(0, 0)$ along X-axis, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{y=0,x\to 0} \frac{x^2 - y^2}{x^2 + y^2} = 1$$

2. Approach $(0, 0)$ along Y-axis, we have

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x=0,y\to 0} \frac{x^2 - y^2}{x^2 + y^2} = -1$$

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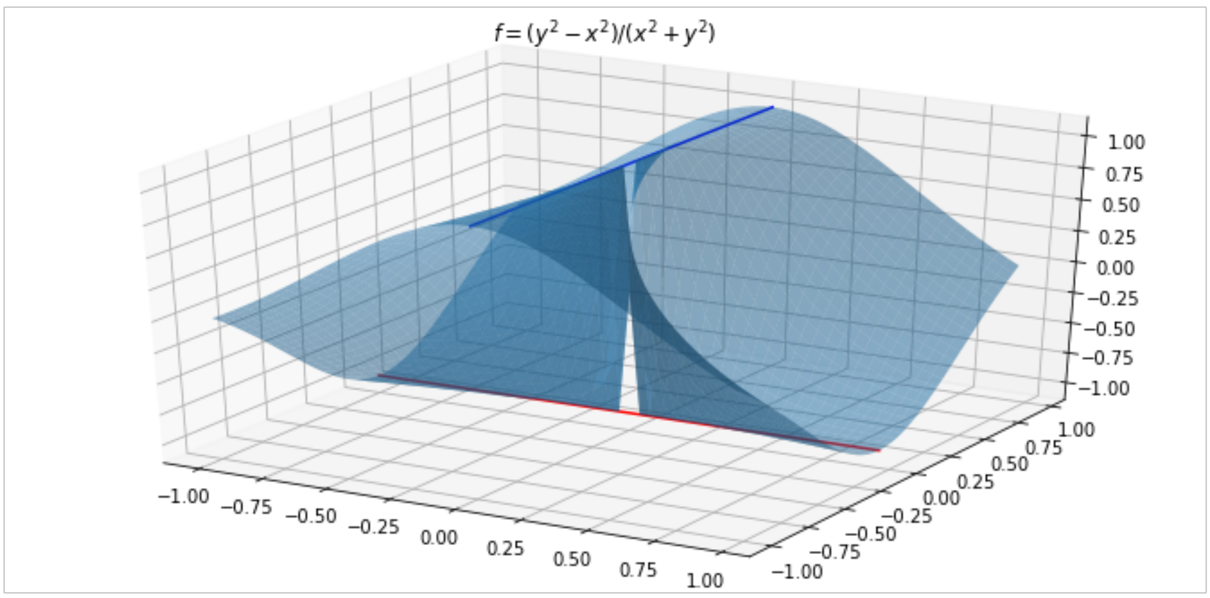


2.11 Example

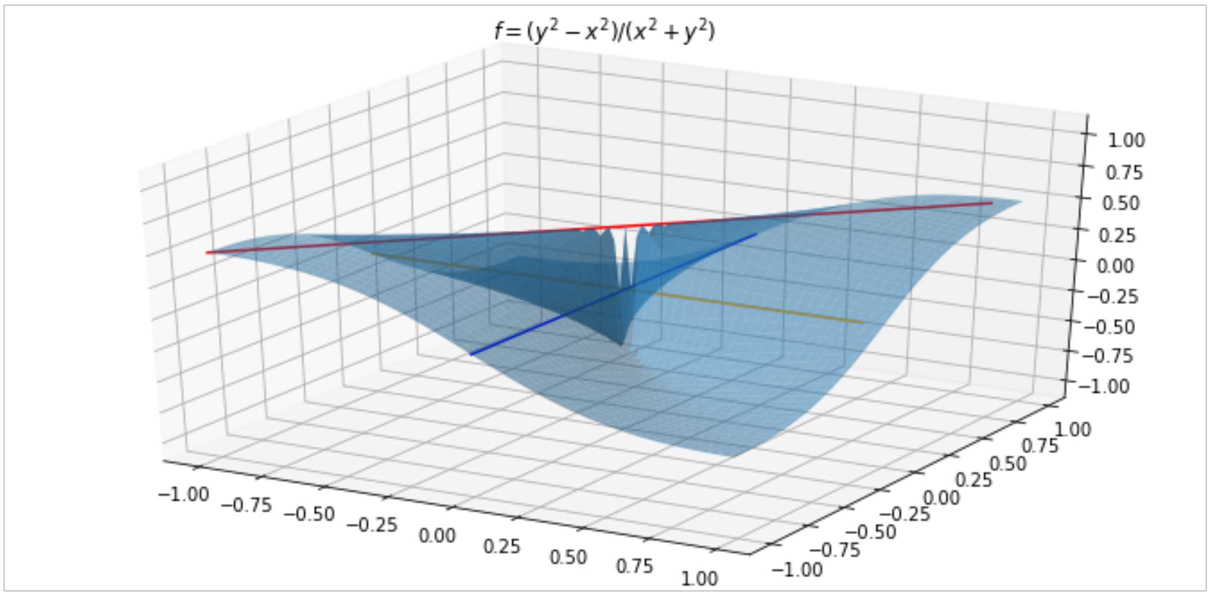
Evaluate the following limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

1. Along X-axis, $x \to 0$, the limit is 0.
2. Along Y-axis, $y \to 0$, the limit is 0.
3. Along $y = x$ and $x \to 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{x=y\to 0} \frac{xy}{x^2 + y^2}$$
$$= \lim_{x\to 0} \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

The limit fails to exist.



2.12 Definition

$f(x, y)$ is called continuous at (a, b) if
$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

2.13 Example

Suppose that
$$f(x, y) = \begin{cases} \frac{2x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0.0) \\ 0 & \text{if } (x, y) = (0.0) \end{cases}$$

Since the limit of $f(x, y)$ at $(0, 0)$ is equal to 0 (= $f(0, 0)$):
$$|f(x, y)| \leq \frac{(x^2 + y^2)^{3/2}}{x^2 + y^2} = 2\sqrt{x^2 + y^2} \to 0$$

$f(x, y)$ is continuous at $(0, 0)$.

2.14 Properties of Continuous Functions

Suppose that f, g continuous on $D \subset \mathbb{R}^n$.

1. $f \pm g$ are continuous,
2. $f \cdot g$ are continuous,
3. f/g are continuous if $g \neq 0$,
4. $f \circ g$ are continuous,

2.15 Example

- $$\lim_{(x,y,z)\to\left(\frac{\pi}{2},0,1\right)} \frac{e^{2y}(\sin x + \cos y)}{1 + y^2 + z^2} = 1$$
- $f(x, y, z) = \frac{\ln z}{\sqrt{1-x^2-y^2-z^2}}$ is continuous for $x^2 + y^2 + z^2 < 1$ and $z > 0$.

2.16 Exercise

Evaluate the following limit:

1. $\lim\limits_{(x,y)\to(0,0)} \frac{2x-3y}{x^2+y^2}$
2. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$
3. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y}$
4. $\lim\limits_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

2.17 Answer

1. As $y=0, x\to 0$, limit fails to exist.
2. Fail to exist since
   - Along X-axis approaching to origin point, $(0,0)$, limit is 0;
   - Along parabola, $(x,y)=(y^2,y)$, approaching to origin point, $(0,0)$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2} \neq 0$$

3. Consider

$$\begin{aligned} g : (x,y) &\to x+y \\ f : t &\to \frac{\sin(t)}{t} \end{aligned}$$

4. Since the limits of $g(x,y)$ and $f(t)$ exists, the composition of $f(g(x,y))$ also has limit and is equal to 1.

$$\begin{aligned} x &= r(\theta)\cos(\theta) \\ y &= r(\theta)\sin(\theta) \\ &\Downarrow \\ \frac{\sin(x^2+y^2)}{x^2+y^2} &= \frac{\sin r^2}{r^2} \\ &\xrightarrow{r\to 0} 1 \end{aligned}$$

2.18 Exercises, p.1058

4. $\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x^2+y^2}$ fails to exist since

- $\lim_{x\to 0, y=0} \cdot = 0,$
- $\lim_{x=y\to 0} \cdot = 1/2,$

28.

$$\lim_{(x,y)\to(0,0)} \frac{\sin 2x^2 + 2y^2}{x^2 + y^2} = \lim_{r\to 0} \frac{\sin 2r^2}{r^2} = 2$$

40. $F(x, y, z) = x \tan \frac{y}{z}$ is continuous if $y/z \neq n\pi \pm \pi/2, n \in \mathbb{N}$.

42. Let

$$f(x, y) = \begin{cases} \frac{x}{\sin x} + y & \text{if } x \neq 0 \\ 1 + y & \text{if } x = 0 \end{cases}$$

a). $f(x)$ is continuous if $x \neq n\pi, n = \pm 1, \pm 2, \cdots$

b). From the following picture, the surface would increase to infinte $x \to \pi$:

51.

$$\lim{(x, y) \to (0, 0)} \frac{3xy^3}{x^2 + y^2} = 0$$

since

a). for any $\varepsilon > 0$, there exists a $\delta = \sqrt{\varepsilon/3}$,

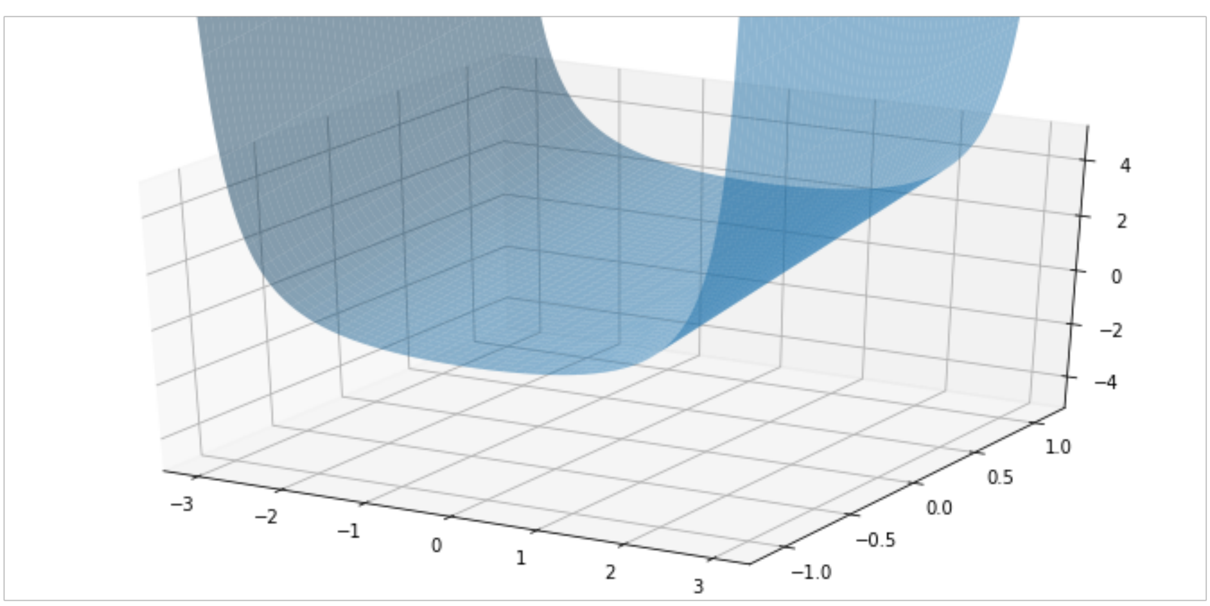
b). $\sqrt{x^2 + y^2} < \delta = \sqrt{\varepsilon/3}$, we have

$$\left|\frac{\mathbf{3xy^3}}{\mathbf{x^2 + y^2}}\right| \leq \frac{\mathbf{3\sqrt{x^2 + y^2}(x^2 + y^2)^{3/2}}}{\mathbf{x^2 + y^2}} = \mathbf{3(x^2 + y^2)} < \varepsilon$$

with the fact:

$$|x|, |y| \leq \sqrt{x^2 + y^2}.$$

```
(-5, 5)
```

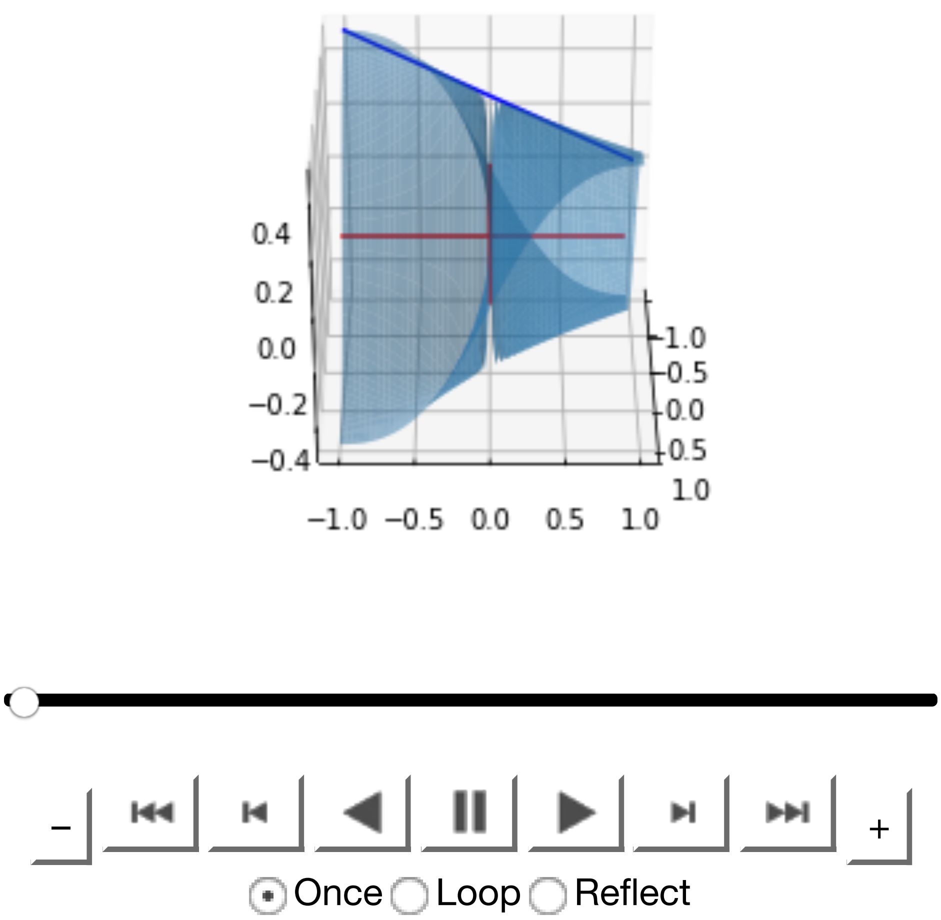


2.19 Note: Mathbox Show

A little bothering...

2.20 Note: JSAnimation Show

Another HTML binding display, installed by native Python installation way.



2.21 Plotly, offline mode

Installation

```
> conda install plotly
or
> pip install plotly
```

```
[NbConvertApp] Converting notebook 6 Multi-variable Ca
lculus-Differentiation-1.ipynb to html
[NbConvertApp] Writing 5022742 bytes to 6 Multi-variab
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```

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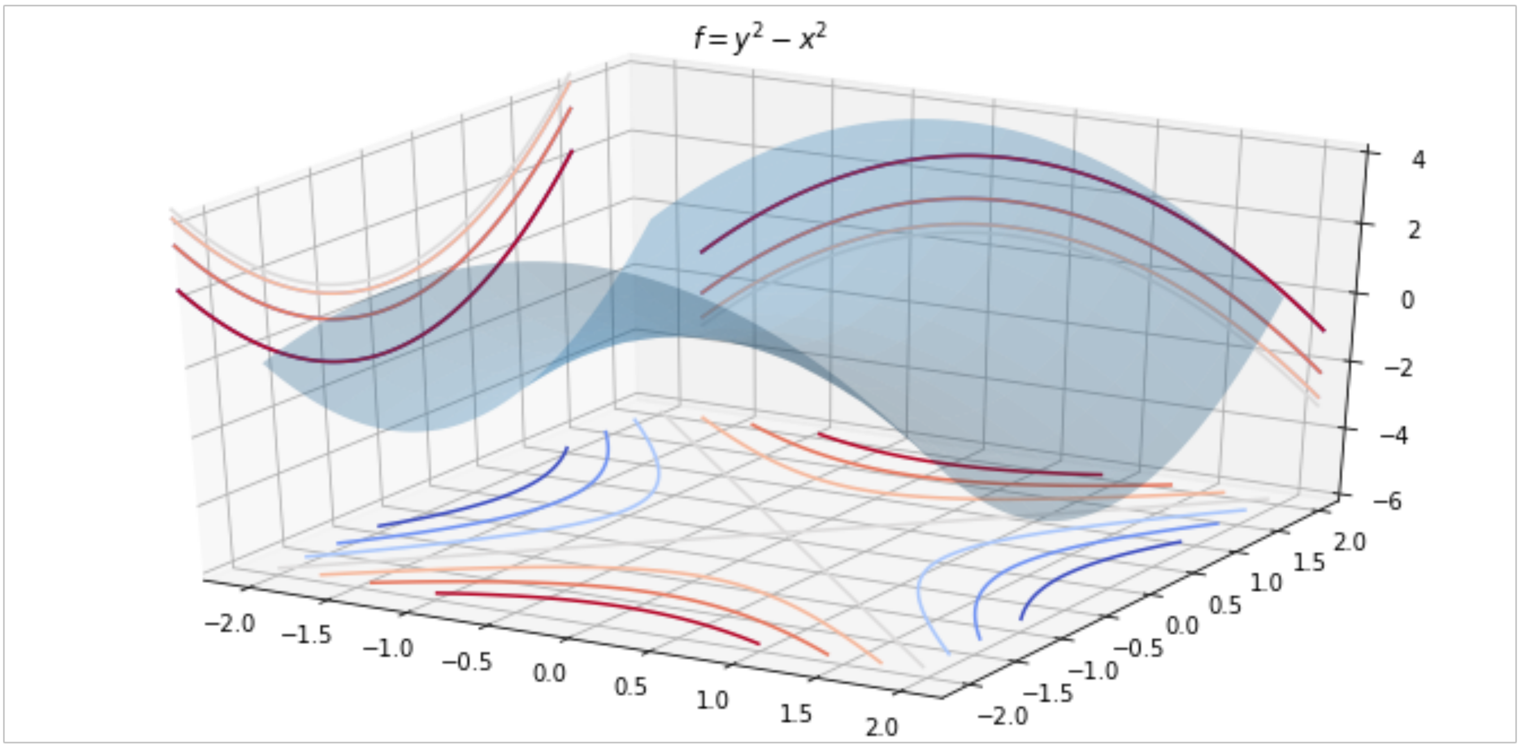
```
<matplotlib.text.Text at 0x1147cbe80>
```



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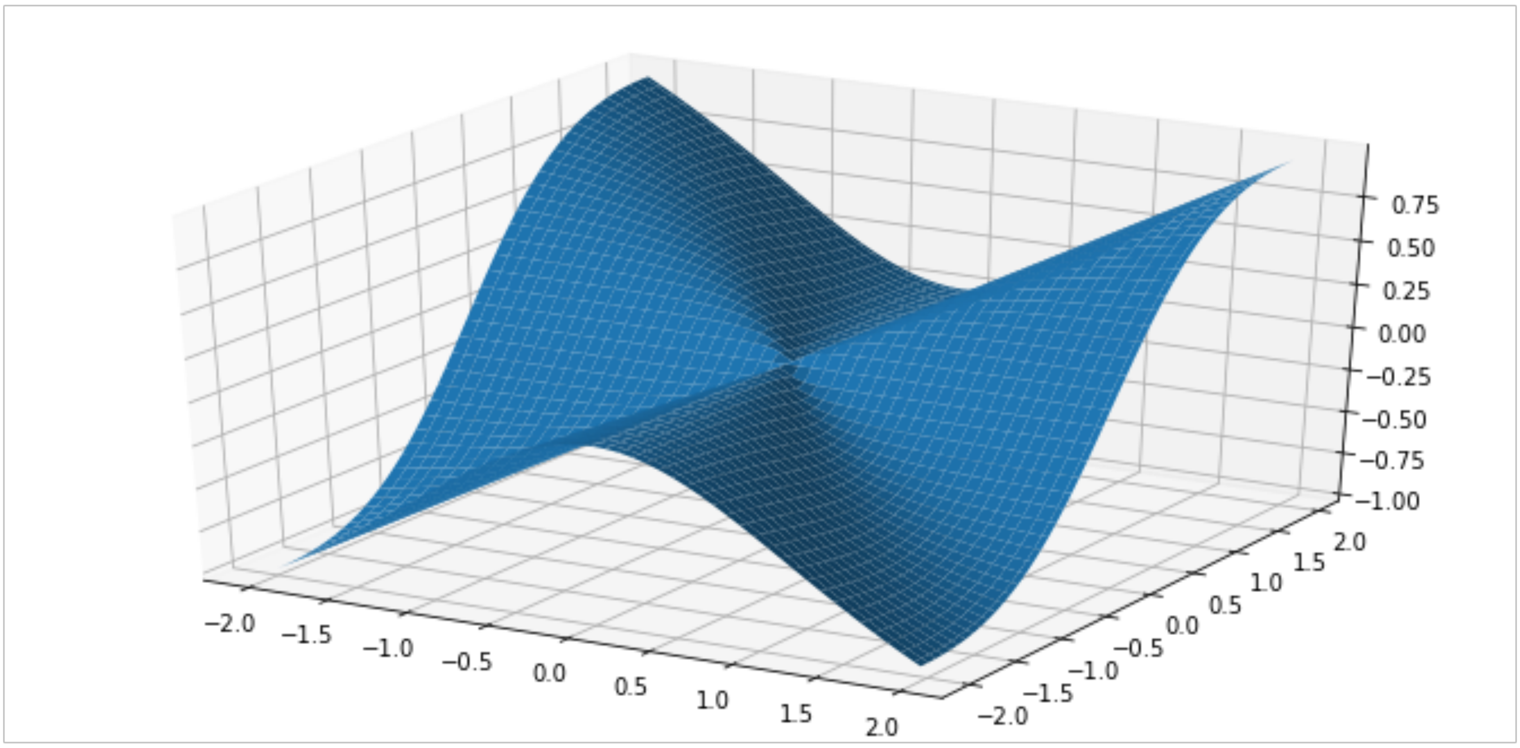

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Example

Evaluate the following limit: $\lim_{(x, y) \rightarrow (0, 0)} \frac{x - y}{x + y}$

1. Approach $(0, 0)$ along X-axis, we have

$$\begin{eqnarray*} \lim_{(x, y) \rightarrow (0, 0)} \frac{x - y}{x + y} & = & \lim_{y = 0, x \rightarrow 0} \frac{x - y}{x + y} = 1 \end{eqnarray*}$$

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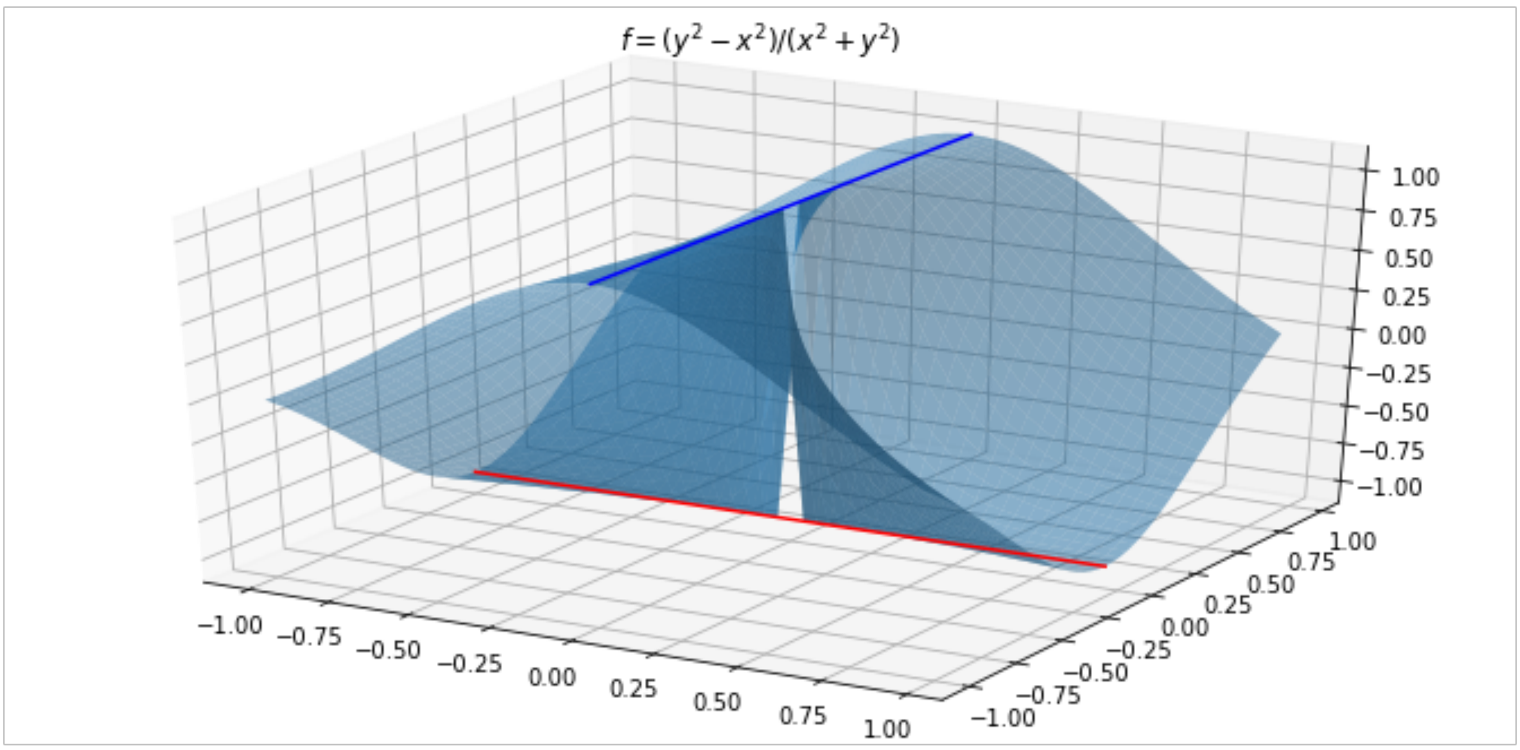

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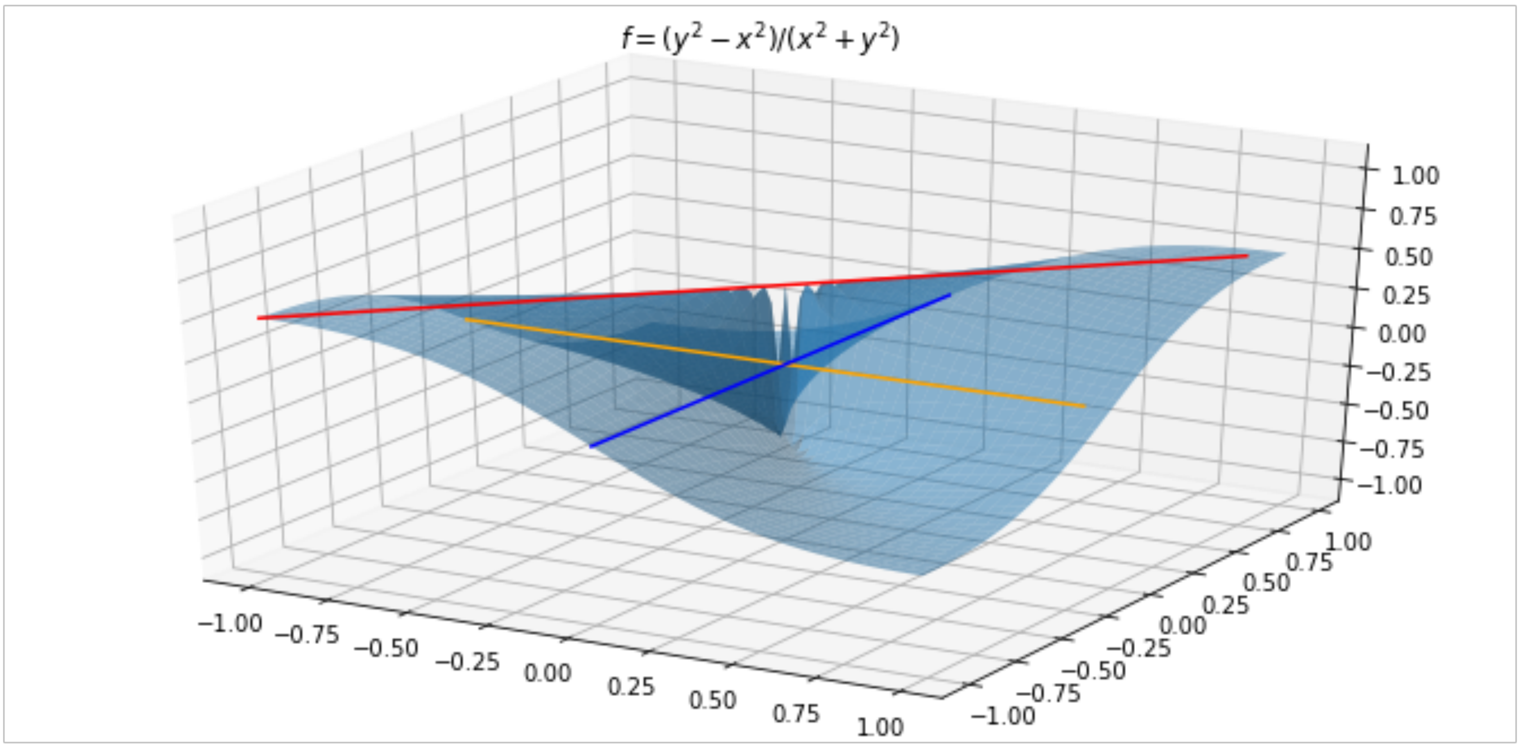

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Suppose that $f (x, y) = \left\{ \begin{array}{ll} \frac{2x^2 y}{x^2 + y^2} & \text{ if } (x, y) \neq (0.0) \\ 0 & \text{ if } (x, y) = (0.0) \end{array} \right.$

Since the limit of $f (x, y)$ at $(0, 0)$ is equal to 0 (= $f (0, 0)$): $|f(x,y)|\le\frac{(x^2 + y^2)^{3/2}}{x^2 + y^2}=2\sqrt{x^2 + y^2}\to 0$

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Properties of Continuous Functions

Suppose that f,g continuous on $D\subset \mathbb{R}^n$.

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Example

- $\lim_{(x,y,z)\to \left(\frac{\pi}{2},0,1\right)}\frac{e^{2y}(\sin x+\cos y)}{1+y^2+z^2}=1$
- $f(x,y,z)=\frac{\ln z}{\sqrt{1-x^2-y^2-z^2}}$ is continuous for $x^2+y^2+z^2<1$ and $z>0$.

Exercise

Evaluate the following limit:

1. $\lim_{(x, y) \rightarrow (0, 0)} \frac{2 x - 3 y}{x^2 + y^2}$
2. $\lim_{(x, y) \rightarrow (0, 0)} \frac{x y^2}{x^2 + y^4}$
3. $\lim_{(x, y) \rightarrow (0, 0)} \frac{\sin (x + y)}{x + y}$
4. $\lim_{(x, y) \rightarrow (0, 0)} \frac{\sin (x^2 + y^2)}{x^2 + y^2}$

Answer

1. As $y = 0, x \rightarrow 0$, limit fails to exist.
2. Fail to exist since
    - Along X-axis approaching to origin point, $(0, 0)$, limit is 0;
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4. Since the limits of $g (x, y)$ and $f (t)$ exists, the composition of $f (g(x, y))$ also has limit and is equal to 1.

$$\begin{eqnarray*} x & = & r (\theta) \cos (\theta)\\ y & = & r (\theta) \sin (\theta)\\ & \Downarrow & \\ \frac{\sin (x^2 + y^2)}{x^2 + y^2} & = & \frac{\sin r^2}{r^2}\\ & \overset{r \rightarrow 0}{\longrightarrow} & 1 \end{eqnarray*}$$

Exercises, p.1058

4. $\lim\limits_{(x,y)\to(0,0)}\frac{\sin x y}{x^2 + y^2}$ fails to exist since

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- $\lim\limits_{x =y\to 0}\cdot=1/2$,

28. $\lim\limits_{(x,y)\to(0,0)}\frac{\sin 2x^2+2 y^2}{x^2 + y^2}=\lim\limits_{r\to0}\frac{\sin 2r^2}{r^2 }=2$

40. $F(x,y,z)=x\tan\frac{y}{z}$ is continuous if $y/z\ne n\pi\pm\pi/2,n\in \mathbb N$.

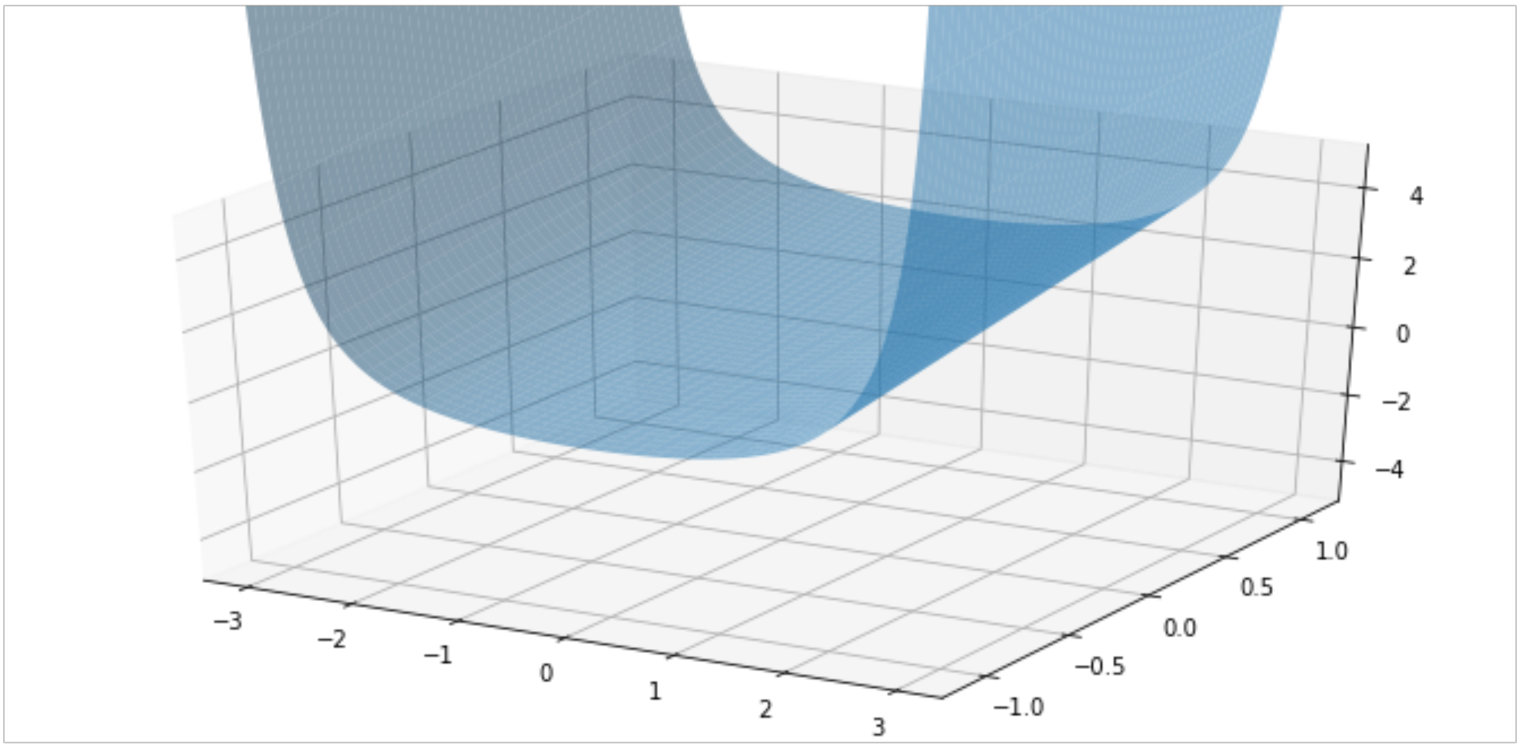
42. Let

$f(x,y) =\begin{cases} \frac{x}{\sin x}+y & \text{ if } x\ne0\\ 1+y & \text{ if } x=0 \end{cases}$ a). $f(x)$ is continuous if $x\ne n\pi, n=\pm1,\pm2,\cdots$
b). From the following picture, the surface would increase to infinte $x\to \pi$:

51. $\lim{(x,y)\to(0,0)}\frac{3xy^3}{x^2 + y^2}=0$ since

a). for any $\varepsilon>0$, there exists a $\delta=\sqrt{\varepsilon/3}$,
b). $\sqrt{x^2+y^2}<\delta=\sqrt{\varepsilon/3}$, we have $\mathbf{\left| \frac{3xy^3}{x^2 + y^2} \right|\le\frac{3\sqrt{x^2+y^2}(x^2+y^2)^{3/2}}{x^2 + y^2}=3(x^2+y^2)<\varepsilon}$ with the fact: $|x|,|y|\le\sqrt{x^2+y^2}$.

```
(-5, 5)
```



Note: Mathbox Show

A little bothering...