7. 1 Multi-variable Calculus


```
from IPython.core.display import HTML
css_file = 'css/ngcmstyle.css'
HTML(open(css_file, "r").read())
```

In [1]:

```
%matplotlib inline

#rcParams['figure.figsize'] = (10,3) #wide graphs by
import scipy
import numpy as np
import time
from sympy import symbols,diff,pprint,sqrt,exp,sin,co


from mpl_toolkits.mplot3d import Axes3D
from IPython.display import clear_output,display,Math
import matplotlib.pylab as plt
```

1.1 Partial Differentiation

1.2 Definition

Suppose that (x_0, y_0) is in the domain of $z = f(x, y)$ **1.** the partial derivative with respect to x at (x_0, y_0) is the limit

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x_0}, \mathbf{y_0}) = \lim_{\mathbf{h} \to 0} \frac{\mathbf{f}(\mathbf{x_0} + \mathbf{h}, \mathbf{y_0}) - \mathbf{f}(\mathbf{x_0}, \mathbf{y_0})}{\mathbf{h}}$$

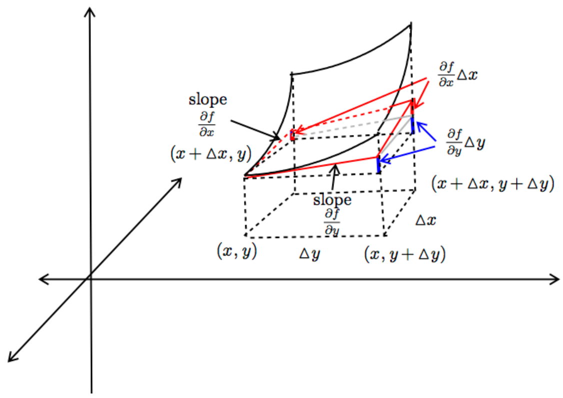
Geometrically, the value of this limit is the slope of the tangent line of $z = f(x, y)$ in the plane $y = y_0$. And this quantity is the rate of change of $f(x, y)$ at (x_0, y_0) along the x-direction.

2. the partial derivative with respect to y at (x_0, y_0) is the limit

$$\frac{\partial \mathbf{f}}{\partial \mathbf{y}}(\mathbf{x_0}, \mathbf{y_0}) = \lim_{\mathbf{k} \to 0} \frac{\mathbf{f}(\mathbf{x_0}, \mathbf{y_0} + \mathbf{k}) - \mathbf{f}(\mathbf{x_0}, \mathbf{y_0})}{\mathbf{k}}$$

Geometrically, the value of this limit is the slope of the tangent line of $z = f(x, y)$ in the plane $x = x_0$. And this quantity is the rate of change of $f(x, y)$ at (x_0, y_0) along the y-direction.

Here is a geometric meaning about partial derivative:



In [2]:
```
import plotly.graph_objs as go

import plotly
from plotly.offline import init_notebook_mode,iplot
init_notebook_mode()
```

In [4]:
```
from numpy import sqrt
X = np.arange(.2, 1, 0.02)
Y = np.arange(0.2, 1, 0.02)
t = np.arange(-0.2, 1.2, 0.02)
s = np.arange(0.4, 0.8, 0.01)
X,Y = np.meshgrid(X,Y)
f= sqrt(X*X + Y*Y)
z0=sqrt(0.4**2+0.4**2)
u=np.arange(0., z0, 0.01)

surface = go.Surface(x=X, y=Y, z=f)#,colorscale=0.5)
Xaxis = go.Scatter3d(x=t, y=0*t, z=0*t,
            mode = "lines",
```

```
                line = dict(
                            color='black',
                            width = 5
                )
            )
Yaxis = go.Scatter3d(x=0*t, y=t, z=0*t,
                mode = "lines",
                line = dict(
                            color='black',
                            width = 5
                )
            )
X0 = go.Scatter3d(x=s, y=0.4+0*s, z=0*s,
                mode = "lines",
                line = dict(
                            color='brown',
                            width = 5
                )
        )
X01= go.Scatter3d(x=0.4+0*u, y=0.4+0*u, z=u,
                mode = "lines",
                line = dict(
                            color='blue',
                            width = 5
                )
        )
Y0 = go.Scatter3d(y=s, x=0.4+0*s, z=0*s,
                mode = "lines",
                line = dict(
                            color='orange',
                            width = 5
                )
        )
#Line2 = go.Scatter3d(x=0*t, y=t, z=0*t)
#Line3 = go.Scatter3d(x=t, y=t, z=np.ones(len(t))/2)
#Line4 = go.Scatter3d(x=t, y=-t, z=-np.ones(len(t))/2
data = [surface,Xaxis,Yaxis,X0,Y0,X01]

fig = go.Figure(data=data)
iplot(fig)
```

The same definition can be also applied to the functions with more than two variables.

1.3 Definition

Suppose that $(x_0^i) = (x_0^1, x_0^2, \cdots, x_0^n)$ in the domain of $f(\mathbf{x}) = f(x^1, x^2, \cdots, x^n)$. The partial derivative with respect to x^i at (x_0^i) is defined as

$$\begin{aligned}\mathbf{f_i(x_0)} &= \frac{\partial \mathbf{f}}{\partial \mathbf{x^i}}(\mathbf{x})\bigg|_{\mathbf{x=x_0}} \\ &= \lim_{\mathbf{k}\to \mathbf{0}} \frac{\mathbf{f(x_0^1, \cdots, x_0^{i-1}, {\color{red}x_0^i + k}, x_0^{i+1}, \cdots, x_0^n) - f(x_0^1, x_0^2, \cdots, x_0^{i-1}}}{\mathbf{k}}\end{aligned}$$

1.4 Definition

${\color{red}(f_1, \cdots, f_n)}$ is called gradient of $f(\vec{x})$, denoted as ${\color{red}\nabla f}$.

In [ ]:

```

```

1.5 Example

Find the $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}(1,3), \frac{\partial f}{\partial y}(2,-4)$ if $f(x,y) = x^3 + 4x^2y^3 + y^2$.

In [34]:

```python
from sympy import symbols, diff,pprint,sqrt
x,y=symbols('x y')
f=x**3+4*x*x*y**3+y*y

grad = lambda func, vars :[diff(func,var) for var in

df=grad(f,[x,y])
pprint(df)
```

```
⎡   2          3        2  2       ⎤
```

```
⎣3⋅x  + 8⋅x⋅y , 12⋅x ⋅y  + 2⋅y⎦
```

In [4]:
```
def df_val(f,val):
    return [ff.subs({x:val[0],y:val[1]}) for ff in f]
df_val(df,[1,3])
```

Out[4]:
```
[-9*sin(9) + cos(9), -6*sin(9)]
```

In [3]:
```
df_val(df,[2,-4])
```

Out[3]:
```
[-1012, 760]
```

1.6 Example

Find the $\frac{\partial f}{\partial x}$ if $f(x, y) = x^3 + 4x^2y^3 + y^2$.

In [5]:
```
f=2*x**2*y**3-3*x*y**2+2*x**2+3*y*y*1

grad = lambda func, vars :[diff(func,var) for var in

df=grad(f,[x,y])
pprint(df)
```

```
⎡      3             2      2   2              ⎤
⎣4⋅x⋅y  + 4⋅x - 3⋅y , 6⋅x ⋅y  - 6⋅x⋅y + 6⋅y⎦
```

1.7 Example

Find the $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ if $f(x, y) = x \cos xy^2$.

In [2]:
```
from sympy import cos
x,y=symbols("x y")
f=x*cos(x*y**2)

grad = lambda func, vars :[diff(func,var) for var in

df=grad(f,[x,y])
pprint(df)
```

```
⎡      2    ⎛   2⎞         ⎛   2⎞          2       ⎛   2⎞
⎤
⎣- x⋅y ⋅sin ⎝x⋅y ⎠ + cos ⎝x⋅y ⎠, -2⋅x ⋅y⋅sin ⎝x⋅y ⎠⎦
```

Example

```
  Find all the first partial derivatives of Cobb-
Douglas function with $n$
  inputs,
  $$ f (x_1, \cdots, x_n) = A x_1^{\alpha_1} \cdots
x_n^{\alpha_n}
     \text{ where } A > 0, 0 < \alpha_1, \cdots
\alpha_n < 1 $$
  Sol:

\begin{eqnarray*}
      \frac{\partial f}{\partial x_i} & = & A
x_1^{\alpha_1} \cdots x_{i -
      1}^{\alpha_{i - 1}}  \color{brown}{\alpha_i
x_i^{\alpha_i - 1}} x_{i +
      1}^{\alpha_{i + 1}} \cdots x_n^{\alpha_n}\\
      & = & A \alpha_i x_1^{\alpha_1} \cdots x_{i -
1}^{\alpha_{i - 1}}
      x_i^{\alpha_i} x_{i + 1}^{\alpha_{i + 1}}
\cdots x_n^{\alpha_n} / x_i\\
      & = & \alpha_i \frac{f (x_1, \cdots, x_n)}{x_i}
\text{ for } i = 1,
      \cdots, n
    \end{eqnarray*}
```

1.9 Eexample

A factory produces two kinds of machine parts, says A and B. If the totally daily cost function of production of x hundred units of A and y hundred units of B is:

$$C(x, y) = 200 + 10x + 20y - \sqrt{x + y}$$

In [13]:
```
C = 200+10*x+20*y-sqrt(x+y)
Cxy=grad(C,[x,y])
df_val(Cxy,[5,6])
```

Out[13]:
```
[-sqrt(11)/22 + 10, -sqrt(11)/22 + 20]
```

$\frac{\partial C}{\partial x}(5, 6) = 10 - \frac{1}{22}\sqrt{11}$,i.e. an increase for x from 5 to 6 while y kept at 6 will result in an increase in daily cost function approximately 9.85. And $\frac{\partial C}{\partial y}(5, 6) = 20 - \sqrt{11}/22$, i.e. an increase for y from 6 to 7 while x kept at 5 will result in an increase in daily cost function approximately 19.85.

1.10 Example

If $f(x, y) = x^2 e^{y^3} + \sqrt{2x + 3y}$, $\frac{\partial f}{\partial x} = 2xe^{y^3} + (2x + 3y)^{-1/2}$ and $\frac{\partial f}{\partial y} = 3x^2 y^2 e^{y^3} + \frac{3}{2}(2x + 3y)^{-1/2}$

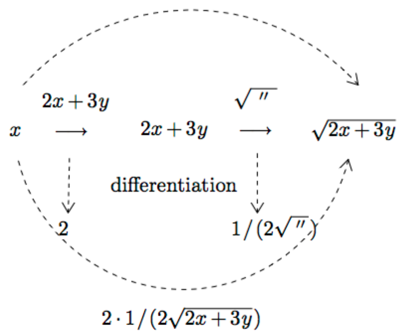
1.11 Solution

Since the partial differentiation only works for the defaulted variables, in other words, the left variables are treated as constants in such operation. Therefore

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(x^2 e^{y^3} + \sqrt{2x + 3y}\right) &= e^{y^3} \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial x} \sqrt{2x + 3y} \\
&= e^{y^3} \cdot 2x + 2 \cdot \frac{1}{2\sqrt{2x + 3y}}
\end{aligned}
$$

$$= 2xe^{y^3} + \frac{}{\sqrt{2x + 3y}}$$

Note that the last result comes from the {\tmstrong{Chain Rule}} as follows:



As the same reason, we also have the result for partial derivative with respect to y:

$$\begin{aligned} \frac{\partial}{\partial y}\left(x^2 e^{y^3} + \sqrt{2x+3y}\right) &= x^2 \frac{\partial}{\partial y} e^{y^3} + \frac{\partial}{\partial y}\sqrt{2x+3y} \\ &= 3x^2 y^2 e^{y^3} + \frac{3}{2\sqrt{2x+3y}} \end{aligned}$$

1.12 Example

Find out the first order derivatives for $f(x, y) = x\sqrt{y} - y\sqrt{x}$.

In [19]:
```
f=x*sqrt(y)-y*sqrt(x)
grad(f,[x,y])
```

Out[19]:
```
[sqrt(y) - y/(2*sqrt(x)), -sqrt(x) + x/(2*sqrt(y))]
```

1.13 Example

For Cobb-Douglas production function, $f(K, L) = 20K^{1/4}L^{3/4}$

1. **The marginal productivity of capital** when $K = 16$ and $L = 81$ is $\frac{\partial f}{\partial K}(16, 81) = \frac{135}{8}$, i.e. an increase in K from 16 to 17 will result in an increase of approximately $\frac{135}{8}$ units of productions.
2. **The marginal productivity of labor** when $K = 16$ and $L = 81$ is $\frac{\partial f}{\partial L}(16, 81) = 10$, i.e. an increase in L from 81 to 82 will result in an increase of approximately 10 units of productions.

Description

Note the partial derivatives are as follows:

$$\frac{\partial f}{\partial K} = 5\left(\frac{L}{K}\right)^{3/4}$$
$$\frac{\partial f}{\partial L} = 15\left(\frac{K}{L}\right)^{1/4}$$

1.14 Example

For Cobb-Douglas production function, $f(K, L) = 20K^{2/3}L^{1/3}$

1. **The marginal productivity of capital** when $K = 125$ and $L = 27$ is $\frac{\partial f}{\partial K}(125, 27) = 8$, i.e. an increase in K from 125 to 126 will result in an increase of approximately 8 units of productions.
2. **The marginal productivity of labor** when $K = 125$ and $L = 27$ is $\frac{\partial f}{\partial L}(125, 27) = 18\frac{14}{27}$, i.e. an increase in L from 27 to 28 will result in an increase of approximately $18\frac{14}{27}$ units of productions.

Description

Note the partial derivatives are as follows:

$$\frac{\partial f}{\partial K} = \frac{40}{3}\left(\frac{L}{K}\right)^{1/3}$$
$$\frac{\partial f}{\partial L} = \frac{20}{3}\left(\frac{K}{L}\right)^{2/3}$$

In [ ]:

```

```

Two products are said to be **competitive** with each other if an increase in demand for one results in a decrease in demand for the other. **Complementary** products have just the opposite relation to each other. Suppose that $f(p, q)$ and $g(p, q)$ are the demand for products, A and B, at respective price p and q. We have

1. $\frac{\partial f}{\partial p} < 0$ and $\frac{\partial g}{\partial q} < 0$ sine raising price always results in a decrease in demand.
2. If $\frac{\partial f}{\partial q} > 0$ and $\frac{\partial g}{\partial p} > 0$ Then A and B are in **competitive** case at price level (p, q).
3. If $\frac{\partial f}{\partial q} < 0$ and $\frac{\partial g}{\partial p} < 0$ Then A and B are in **complementary** case at price level (p, q).

1.15 Example

If $f(p, q) = 400 - 5p^2 + 16q$ and $g(p, q) = 600 + 12p - 4q^2$, then A and B are competitive since

$$\frac{\partial f}{\partial q} = 16 > 0$$

$$\frac{\partial g}{\partial p} = 12 > 0$$

1.16 Example

If $f(p,q) = \frac{30p}{2p+3q}$ and $g(p,q) = \frac{10q}{p+4q}$, then A and B are complementary since

$$\begin{aligned}
\frac{\partial f}{\partial q} &= \frac{\partial}{\partial q}\frac{30p}{2p+3q} \\
&= \frac{-90p}{(2p+3q)^2} < 0 \\
\frac{\partial g}{\partial p} &= \frac{\partial}{\partial p}\frac{10q}{p+4q} \\
&= \frac{-10q}{(p+4q)^2} < 0
\end{aligned}$$

1.17 Implicit Differentiation

Suppose that z is differentiable and defined implicitly as follows:

$$x^2 + y^3 - z + 2yz^2 = 5.$$

In [9]:
```
from sympy import Function,solve
x,y  = symbols('x y')
z = Function('z')(x,y)
```

In [12]:
```
eq= x**2+y**3-z+2*y*z**2-5
gradv=grad(eq,[x,y])
gradv
```

Out[12]:
```
[2*x + 4*y*z(x, y)*Derivative(z(x, y), x) - Derivativ
e(z(x, y), x),
 3*y**2 + 4*y*z(x, y)*Derivative(z(x, y), y) + 2*z(x,
y)**2 - Derivative(z(x, y), y)]
```

In [17]:
```
pprint("dz/dx = %s" %solve(gradv[0],diff(z, x))[0])
```

```
dz/dx = -2*x/(4*y*z(x, y) - 1)
```

In [22]:
```
pprint("dz/dy = %s" %solve(gradv[1],diff(z, y))[0])
```

```
dz/dy = -(3*y**2 + 2*z(x, y)**2)/(4*y*z(x, y) - 1)
```

1.18 Example

a). If $f(x,y,z) = x^2y + y^2z + zx$, then $f_x = 2xy + z$;

b). If $h(x,y,zw) = \frac{xw^2}{y+\sin zw}$, then $h_w =$

In [24]:
```python
from sympy import sin
x,y,z,w=symbols(" x y z w")
f=x*w**2/(y+sin(z*w))
pprint(diff(f,w))
```

```
    2
   w ·x·z·cos(w·z)        2·w·x
- ──────────────── + ──────────────
                 2   y + sin(w·z)
  (y + sin(w·z))
```

1.19 Note

Higher order partial derivative. As the functions of single variable, we can define what the higher order partial derivatives of functions with multiple variables as follows:

1. Two variables: Suppose that $f(x, y)$ is smooth enough,

Partial derivatives for $f(x, y)$

Order	Partial Derivatives
1st	$\mathbf{f_1} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}, \mathbf{f_2} = \frac{\partial \mathbf{f}}{\partial \mathbf{y}}$
2nd	$\mathbf{f_{ij}} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x^j} \partial \mathbf{x^i}}$
More	$\mathbf{f_{\dots i}} = \frac{\partial \mathbf{f_{\dots}}}{\partial \mathbf{x^i}}$

2. More than two variables: Suppose that $f(\vec{x}) = f(x^1, \cdots, x^n)$ (or denoted as $f(x^i)$) is smooth enough,

Order	partial derivatives
1st	$\mathbf{f_i} = \frac{\partial \mathbf{f}}{\partial \mathbf{x^i}}$, for $i = 1, \cdots, n$
2nd	$\mathbf{f_{ij}} = \frac{\partial^2 \mathbf{f}}{\partial \mathbf{x^i} \partial \mathbf{x^j}}$, for $1 \leqslant i, j \leq n$
More	$\mathbf{f_{\dots i}} = \frac{\partial}{\partial \mathbf{x^i}} \mathbf{f_{\dots}}$, for $i = 1, \cdots, n$

1.20 Example

Find the second order derivatives of $f(x, y) = x^2y^3 + e^{4x}\ln y$.

All the 1st order partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = 2x \cdot y^3 + 4e^{4x} \cdot \ln y \text{and } \frac{\partial f}{\partial y} = x^2 \cdot 3y^2 + e^{4x} \cdot \frac{1}{y}$$

And all the 2nd order of partial derivatives are as follows:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(2xy^3 + 4e^{4x}\ln y) \\
&= 2 \cdot y^3 + 16e^{4x} \cdot \ln y \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(3x^2y^2 + e^{4x}\frac{1}{y}\right) \\
&= 6x^2y - e^{4x}\frac{1}{y^2} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(2xy^3 + 4e^{4x}\ln y) \\
&= 2x \cdot 3y^2 + 4e^{4x} \cdot \frac{1}{y} \\
\frac{\partial^2 f}{\partial x \partial y} &= 6xy^2 + 4e^{4x}/y
\end{aligned}$$

In [15]:
```
from sympy import exp,log

f=x*x*y**3+exp(4*x)*log(y)
X=[x,y]
g=grad(f,X)
```

```
y
```

Out[15]: [2*x*y**3 + 4*exp(4*x)*log(y), 3*x**2*y**2 + exp(4*x)/y]

In [18]:
```
pprint([[diff(gg,var) for gg in g] for var in X])
```

```
⎡⎡                                          4⋅x⎤   ⎡
4⋅x              4⋅x⎤⎤
⎢⎢   3          4⋅x                 2   4⋅e    ⎥   ⎢      2
4⋅e          2       e    ⎥⎥
⎢⎢2⋅y  + 16⋅e    ⋅log(y), 6⋅x⋅y  + ──────⎥, ⎢6⋅x⋅y  +
──────, 6⋅x ⋅y - ─────⎥⎥
⎢⎣                                       y   ⎦   ⎢
y                  2  ⎥⎥
⎣                                                ⎣
y  ⎦⎦
```

1.21 Example

Find the first three order derivatives of $f(x, y) = 4x^2 - 6xy^3$.

All the 1st order partial derivatives are as follows:

$$\frac{\partial f}{\partial x} = 8x - 6y^3 \text{and} \ \frac{\partial f}{\partial y} = 0 - 6x \cdot 3y^2$$

And the 2nd order of partial derivatives are as follows:

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(8x - 6y^3) \\
&= 8 \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}(-18xy^2) \\
&= -36xy \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(8x - 6y^3) \\
&= 0 - 12y^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= -12xy^2
\end{aligned}$$

And 3rd order of partial derivatives are as follows:

$$\begin{aligned}
f_{111} &= \frac{\partial}{\partial x}(8) = 0 \\
f_{222} &= \frac{\partial}{\partial y}(-36xy) = -36x \\
f_{112} = f_{121} = f_{211} &= \frac{\partial}{\partial y}(8 - 6y^3) = -18y^2 \\
f_{122} = f_{212} = f_{221} &= \frac{\partial}{\partial x}(-36xy) = -36x
\end{aligned}$$

In [26]:
```
f=4*x**2-6*x*y**3
diff(f,x,y)==diff(f,y,x)
```
Out[26]:
```
True
```

1.22 Example

Let $f(x, y, z) = xe^{yz}$, then

1. $f_{xzy} = (1 + yz)e^{yz}$,
2. $f_{yzx} = (1 + yz)e^{yz}$,

They are equal to with respectively.

1.23 Theorem

If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ are all continuous near (x_0, y_0), then $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$.

1.24 Example

For $f(x, y, z) = xe^{yz}$,

In [28]:
```
from sympy import exp
f=x*exp(y*z)
pprint("fxzy= %s" %diff(f,x,z,y) )
```
```
fxzy= (y*z + 1)*exp(y*z)
```

In [29]:
```
from sympy import exp
f=x*exp(y*z)
pprint("fyxz= %s" %diff(f,y,x,z) )
```
```
fyxz= (y*z + 1)*exp(y*z)
```

1.25 Example

Suppose that

$$f(x, y) = \begin{cases} \frac{x^3y - y^3x}{x^2+y^2} & \text{if } (x, y) \neq (0.0) \\ 0 & \text{if } (x, y) = (0.0) \end{cases}$$

$f(x, y)$ is continuous at $(0, 0)$.

1. first partial derivatives of $f(x. y)$ where $(x, y) \neq (0, 0)$:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(3x^2y + y^3)(x^2 + y^2) - 2x(x^3y + y^3x)}{(x^2 + y^2)^2} \\ &= \frac{-y^5 + 4x^2y^3 + x^4y}{(x^2 + y^2)^2} \\ \frac{\partial f}{\partial y} &= \frac{x^5 - 4y^2x^3 - y^4x}{(x^2 + y^2)^2} \end{aligned}$$

2. first partial derivatives of $f(x. y)$ at $(x, y) = (0, 0)$:

$$\begin{aligned} \frac{\partial f}{\partial x}(0, 0) &= \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= 0 \\ \frac{\partial f}{\partial x}(0, 0) &= 0 \end{aligned}$$

3. Second partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ of $f(x. y)$ where $(x, y) = (0, 0)$:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(0, 0) &= \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h^5 + 40^2h^3 + 0^4h}{(h^2+0^2)^2} - 0}{h} = -1 \\ \frac{\partial^2 f}{\partial y \partial x}(0, 0) &= \lim_{k\to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \\ &= \lim \frac{\frac{k^5 - 40^2k^3 - 0^4k}{(0^2+k^2)^2} - 0}{} = 1 \end{aligned}$$

$$\lim_{k\to 0} \qquad k$$

The last result shows $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

1.26 Definition

A function, $u(x, y)$,is called harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1.27 Example

$u = e^x \cos y$ is harmonic,

In [64]:
```
x,y=symbols("x y")
u=exp(x)*cos(y)
diff(u,x,x)+diff(u,y,y)
```

Out[64]: 0

1.28 Exercise p. 1069

In [5]:
```
# 20 log(exp(x)+exp(y))
from sympy import log,exp,cos,sin,diff,integrate,symb
z=log(exp(x)+exp(y))
pprint(grad(z,[x,y]))
```

```
⎡     x          y    ⎤
⎢    e          e     ⎥
⎢───────,  ───────────⎥
⎢ x     y   x     y   ⎥
⎣e  + e    e  + e     ⎦
```

In [9]:
```
# 22 partial derivative of int_x^y cos t dt

t=symbols("t")
f=integrate(cos(t),[t,x,y])
pprint(grad(f,[x,y]))
```

```
[-cos(x), cos(y)]
```

In [40]:
```
#32 2cos(x+2y)+sin yz -1=0
z=Function("z")(x,y)
eq= 2*cos(x+2*y)+sin(y*z)-1

gradv=grad(eq,[x,y])

pprint("dz/dx = %s" %solve(gradv[0],diff(z, x))[0])
pprint("dz/dy = %s" %solve(gradv[1],diff(z, y))[0])
```

```
dz/dx = 2*sin(x + 2*y)/(y*cos(y*z(x, y)))
dz/dy = (-z(x, y) + 4*sin(x + 2*y)/cos(y*z(x, y)))/y
```

In [41]:

```
#38 Second derivate of \sqrt(x^2+y^2)
from sympy import sqrt
f= sqrt(x**2+y**2)
pprint("fxx = %s" %diff(f, x,x))
pprint("fxy = fyx = %s" %diff(f, x,y))
pprint("fyy = %s" %diff(f, y,y))
```

```
fxx = (-x**2/(x**2 + y**2) + 1)/sqrt(x**2 + y**2)
fxy = fyx = -x*y/(x**2 + y**2)**(3/2)
fyy = (-y**2/(x**2 + y**2) + 1)/sqrt(x**2 + y**2)
```

In [10]:

```
#46 f=exp(-2x)cos(3y), fxy=fyx
f=exp(-2*x)*sin(3*y)
```

In [6]:

```
def highdiff(f,xy):
    fpart=f
    for x in xy:
        fpart=diff(fpart,x)
    return fpart
```

In [7]:

```
fxy = highdiff(f,[x,y])
fyx = highdiff(f,[y,x])
if (fxy == fyx):
    print("fxy = fyx = %s" %fxy)
else:
    print("fxy ≠ fyx and  fxy= %s, fyx= %s" %(fxy,fyx
```

```
fxy = fyx = -2*x**2*y**3*cos(x*y**2) - 4*x*y*sin(x*y*
*2)
```

#90.

Suppose that

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0.0) \\ 0 & \text{if } (x,y) = (0.0) \end{cases}$$

$f(x, y)$ is continuous at $(0, 0)$.

a. first partial derivatives of $f(x. y)$ where $(x, y) \neq (0, 0)$:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{(3x^2y + y^3)(x^2 + y^2) - 2x(x^3y + y^3x)}{(x^2 + y^2)^2} \\ &= \frac{-y^5 + 4x^2y^3 + x^4y}{(x^2 + y^2)^2} \\ \frac{\partial f}{\partial y} &= \frac{x^5 - 4y^2x^3 - y^4x}{(x^2 + y^2)^2} \end{aligned}$$

b. first partial derivatives of $f(x. y)$ at $(x, y) = (0, 0)$:

$$\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= 0 \\ \frac{\partial f}{\partial x}(0,0) &= 0 \end{aligned}$$

c. Second partial derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ of $f(x. y)$ where $(x, y) = (0, 0)$:

$$\begin{aligned} \frac{\partial^2 f}{\partial x \partial y}(0,0) &= \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h^5+40^2h^3+0^4h}{(h^2+0^2)^2} - 0}{h} = -1 \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) &= \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \\ &= \lim_{k\to 0} \frac{\frac{k^5-40^2k^3-0^4k}{(0^2+k^2)^2} - 0}{k} = 1 \end{aligned}$$

The last result shows $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$. This does not contradict **Clairaut's Theorem** since f_{xy}, f_{yx} are not continuous at $(0, 0)$ since

$$\lim_{x=0, y\to 0} f_{x,y}(x,y) \neq f_{x,y}(0,0)$$

1.29 Differentials

Let $f(x, y)$, and let $\triangle x, \triangle y$ be the increments of x and y respectively. Then the (total) differential dz is

$$\mathbf{dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = f_x dx + f_y dy}$$

Example Let $f(x, y) = 2x^2 - xy$

In [8]:
```
def differential(func,xy):
    df=""
    for x in xy:
        fx=diff(func,x)
        if df!="":
            df="%s + (%s) d%s" %(df,fx,x)
        else:
            df="(%s) d%s" %(fx,x)
    return df
```

In [9]:
```
f=2*x**2-x*y
df=differential(f,[x,y])
print(df)
```

```
(4*x - y) dx + (-x) dy
```

While (x, y) changes from (1,1) to (0.98,1.03):

1. $dx = 0.98 - 1 = 0.02, dy = 1.03 - 1),$
2. $dz = (4 \times 1 - 1)dx - 1dy = -0.09,$
3. $\triangle z = z(0.98, 1.03) - z(1, 1) \approx -0.0886 \sim dz$

1.30 Example, body mass index (BMI)

The body mass index (BMI) or Quetelet index is a value derived from the mass (weight) and height of an individual. The BMI is defined as the body mass divided by the square of the body height, and is universally expressed in units of kg/m2,

$$\text{BMI} = \frac{\text{weight}}{\text{height}^2}$$

resulting from mass in kilograms and height in metres.

What's the increase of BMI if one's weight increases from 68 kg to 70 kg and height increases from 169cm to 170cm?

Sol. As problem stated, assume

$$\text{BMI}(w, h) = \frac{w}{h^2}$$

where w, h represent one's weight (in kg) and height (in m).

In [7]:
```
w,h=symbols("w h")
BMI= w/h/h
dBMI=grad(BMI,[w,h])
#df_val(BMI,[2,0.01])
dBMI
```

Out[7]:
```
[h**(-2), -2*w/h**3]
```

In [9]:
```
differential(BMI,[w,h])
```

Out[9]:
```
'(h**(-2)) dw + (-2*w/h**3) dh'
```

In [31]:
```
h=1.69
dh=0.01
w=68
dw=2
whh0=w/h/h
whh1=(w+dw)/(h+dh)/(h+dh)
exact=(whh1-whh0)/whh0
dBMIvalpercent=(dw/h/h-2*dh*w/h/h/h)/whh0
print("BMI increases from %5.3f to %5.3f, approximate
      %(whh0,whh1,dBMIvalpercent,exact))
```

```
BMI increases from 23.809 to 24.221, approximately 0.
018 (exactly 0.017)
```

p. 1082

In [10]:
```
#16
```

```
z=symbols("z")
w=sqrt(x*x+x*y+z**2)
wxyz=differential(w,[x,y,z])
pprint(wxyz)
```

```
((x + y/2)/sqrt(x**2 + x*y + z**2)) dx + (x/(2*sqrt(x
**2 + x*y + z**2))) dy +
(z/sqrt(x**2 + x*y + z**2)) dz
```

1.31 Chain Rule

As result in one-variable function:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

we also have the similar result for multivariate functions:

$$\left(\frac{\partial z}{\partial t^i}\right) = \left(\frac{\partial z}{\partial x^j}\right)\left(\frac{\partial x^j}{\partial t^i}\right)$$

where

$$\left(\frac{\partial z}{\partial t^i}\right) = \left(\frac{\partial z}{\partial t^1}, \frac{\partial z}{\partial t^2}, \cdots, \frac{\partial z}{\partial t^n}\right)$$

$$\left(\frac{\partial x^j}{\partial t^i}\right) = \begin{pmatrix} \frac{\partial x^1}{\partial t^1} & \cdots & \frac{\partial x^1}{\partial t^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x^m}{\partial t^1} & \cdots & \frac{\partial x^m}{\partial t^n} \end{pmatrix}$$

In [48]:
```
def ChainRule(func, x,t,xt,output=False):
    z=Matrix([func])
    X=Matrix(x)
    n=len(x)
    Xt=Matrix(x).subs({x[n]:xt[n] for n in range(len(
    T=Matrix(t)
    dzdt=z.jacobian(X).subs({x[n]:xt[n] for n in rang
    if len(t)!=1:
       print("∂ %s /∂ %s\n" %(func,t))
       pprint(dzdt)
    else:
       print("d ( %s) /d %s\n" %(func,t))
       pprint(dzdt)
    if output==True:
       return dzdt
```

1.32 Example, ($\mathbb{R}(t) \rightarrow \mathbb{R}^n(\mathbf{x}_{1\times n}) \rightarrow \mathbb{R}(f(\mathbf{x}(t)))$

$$\frac{dW}{dt} = \frac{\partial W}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} = \begin{bmatrix} \frac{\partial W}{\partial x^1} & \frac{\partial W}{\partial x^2} & \cdots & \frac{\partial W}{\partial x^n} \end{bmatrix}_{1\times n} \begin{bmatrix} \frac{dx^1}{dt} \\ \frac{dx^2}{dt} \\ \vdots \\ \frac{dx^n}{dt} \end{bmatrix}_{n\times 1}$$

e.g. $n = 2$

$$\frac{dW}{dt} = \frac{\partial W}{\partial (x,y)} \frac{d(x,y)}{dt} = \begin{bmatrix} \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt}$$

Let $w = x^2y - xy^3$,$(x, y) = (\cos t, e^t)$. Find dw/dt and its value at $t = 0$

In [44]:

```
x,y,s,t=symbols("x y s t")
w=Matrix([x**2*y-x*y**3])
X=Matrix([cos(t),exp(t)])

pprint(w.jacobian(Matrix([x,y]))*X.jacobian(Matrix([t
```

```
⎡⎛ 2          2⎞  t    ⎛            3⎞        ⎤
⎣⎝x  - 3⋅x⋅y  ⎠⋅e  - ⎝2⋅x⋅y - y ⎠⋅sin(t)⎦
```

In [49]:

```
ChainRule(x**2*y-x*y**3, [x,y],[t],[cos(t),exp(t)])
```

```
d ( x**2*y - x*y**3) /d [t]

⎡⎛      2⋅t                  2    ⎞  t    ⎛    3⋅t        t
⎞         ⎤
⎣⎝- 3⋅e    ⋅cos(t) + cos (t)⎠⋅e  - ⎝- e     + 2⋅e ⋅co
s(t)⎠ ⋅sin(t)⎦
```

$$\left.\frac{dw}{dt}\right|_{t=0} = (-3 + 1) \cdot 1 - (-1 + 2) \cdot 0 = -2$$

Example

$$\mathbb{R}^m(\mathbf{u}_{1\times m}) \to \mathbb{R}^n(\mathbf{x}_{1\times n}) \to \mathbb{R}(f(\mathbf{x}(\mathbf{u})))$$

$$\frac{\partial W}{\partial \mathbf{u}} = \frac{\partial W}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{u}}$$

$$\begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & \cdots & \frac{\partial x^1}{\partial u^m} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial W}{\partial x^1} & \frac{\partial W}{\partial x^2} & \cdots & \frac{\partial W}{\partial x^n} \end{bmatrix}_{1\times n} \begin{bmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^1}{\partial u^2} & & \frac{\partial x^1}{\partial u^m} \\ \frac{\partial x^2}{\partial u^1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial x^n}{\partial u^1} & \frac{\partial x^n}{\partial u^2} & & \frac{\partial x^n}{\partial u^m} \end{bmatrix}_{n\times m}$$

1.33 e.g. $m, n = 2, 2$

$$\frac{\partial W}{\partial(u,v)} = \frac{\partial W}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial W}{\partial x} & \frac{\partial W}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Let $w = 2x^2y$,$(x, y) = (u^2 + v^2, u^2 - v^2)$. Find $\partial w/\partial u$ and $\partial w/\partial u$.

In [5]:
```
x,y,u,v=symbols("x y u v")
z=[2*x*x*y]
X=Matrix([x,y])
Xt=Matrix([x,y]).subs({x:u*u+v*v,y:u*u-v*v})
T=Matrix([u,v])
pprint(Matrix(z).jacobian(X)*Xt.jacobian(T))
```

```
⎡       2                   2         ⎤
⎣4⋅u⋅x  + 8⋅u⋅x⋅y  - 4⋅v⋅x  + 8⋅v⋅x⋅y⎦
```

In [11]:
```
ChainRule(z, [x,y],[u,v],[u*u+v*v,u*u-v*v])
```

```
∂ [2*x**2*y] /∂ [u, v]

⎡                                                   2
⎢     ⎛ 2    2⎞ ⎛ 2    2⎞            ⎛ 2    2⎞
⎛ 2    2⎞ ⎛ 2    2⎞           ⎛ 2
⎣8⋅u⋅⎝u  - v ⎠⋅⎝u  + v ⎠  + 4⋅u⋅⎝u  + v ⎠     8⋅v⋅⎝u
- v ⎠⋅⎝u  + v ⎠  - 4⋅v⋅⎝u  +

   2⎤
  2⎞ ⎥
 v ⎠ ⎦
```

1.34 Example

Suppose that

$$z = f(x, y) = \sin(x + y^2)$$
$$(x, y) = (st, s^2 + t^2)$$

Then

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(\cos(x+y^2) \quad 2y\cos(x+y^2)\right)$$

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} t & s \\ 2t & 2s \end{pmatrix}$$

$$\left(\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}\right) = \left(\cos(x+y^2) \quad 2y\cos(x+y^2)\right)\begin{pmatrix} t & s \\ 2t & 2s \end{pmatrix}$$

$$= \left(t\cos(x+y^2) + 4ty\cos(x+y^2) \quad s\cos(x+y^2) + \right.$$

$$= \cos((t^2+s^2)^2 + st) \cdot ((4s(t^2+s^2)+t), (4t(t^2+s^2)$$

In [64]:

```python
x,y,s,t=symbols("x y s t")
f=sin(x+y*y)
Xt=[s*t,s*s+t*t]
ChainRule(f, [x,y],[s,t],Xt)
```

```
∂ sin(x + y**2) /∂ [s, t]

⎡                          ⎛                        2⎞              ⎛
2⎞            ⎛
⎢        ⎛ 2     2⎞        ⎜             ⎛ 2     2⎞  ⎟              ⎜
⎜ 2     2⎞   ⎟             ⎜             ⎜
⎣4⋅s⋅ ⎝s  + t ⎠ ⋅cos ⎝s⋅t +  ⎝s  + t ⎠  ⎠  + t⋅cos ⎝s⋅t
+  ⎝s  + t ⎠  ⎠   s⋅cos ⎝s⋅t +  ⎝s

          2⎞                          ⎛                      2⎞ ⎤
2      2⎞   ⎟            ⎛ 2     2⎞       ⎜          ⎛ 2     2⎞   ⎟
⎟
  + t ⎠  ⎠  + 4⋅t⋅ ⎝s  + t ⎠ ⋅cos ⎝s⋅t +  ⎝s  + t ⎠  ⎠ ⎦
```

1.35 Example ($\mathbb{R}^2(r,s) \to \mathbb{R}^3(x,y,z) \to \mathbb{R}(f(x,y,z))$)

Suppose that

$$w(x,y,z) = x^2y + y^2z^3$$
$$(x,y,z) = (r\cos s, r\sin s, re^s)$$

Find $\partial w/\partial s$ at $(r,s) = (1,0)$

In [59]:

```python
x,y,z,r,s=symbols("x y z r s")
w=x*x*y+y*y*z*z*z
Xt=[r*cos(s),r*sin(s),r*exp(s)]
wxyz=ChainRule(w, [x,y,z],[r,s],Xt,output=1)
```

```
∂ x**2*y + y**2*z**3 /∂ [r, s]

⎡    4   3⋅s      2            2                2          ⎛    4   3⋅
s              2      2     ⎞
⎣3⋅r ⋅e     ⋅sin (s) + 2⋅r ⋅sin(s)⋅cos (s) +  ⎝2⋅r ⋅e     ⋅
sin(s) + r ⋅cos (s)⎠ ⋅sin

            5   3⋅s      2              3      2                        ⎛
4   3⋅s                  2      2
```

```
(s)   3·r  ·e     ·sin (s) - 2·r  ·sin (s)·cos(s) + r·⎝2·r
·e    ·sin(s) + r ·cos (s

 ⎞         ⎤
)⎠ ·cos(s)⎦
```

In [62]:
```
wxyz.subs({r:1,s:0})[1]
```

Out[62]: 1

And

$$\left.\frac{\partial w}{\partial s}\right|_{(r,s)=(1,0)} = 3\cdot 1\cdot 0 - 2\cdot 1\cdot 0 + 1\cdot(2\cdot 0 + 1\cdot 1) = 1$$

1.36 Example

Suppose that

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2}$$
$$(x,y,z) = (r\cos t, r\sin t, r)$$

Then

$$\begin{pmatrix} f_x & f_y & f_z \end{pmatrix} = \frac{-2}{(x^2+y^2+z^2)^2}\begin{pmatrix} x & y & z \end{pmatrix}$$
$$\frac{\partial(x,y,z)}{\partial(r,t)} = \begin{pmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \\ 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} f_r & f_t \end{pmatrix} = \frac{-2}{(x^2+y^2+z^2)^2}\begin{pmatrix} x & y & z \end{pmatrix}\begin{pmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \\ 1 & 0 \end{pmatrix}$$
$$= \frac{-2}{(x^2+y^2+z^2)^2}\begin{pmatrix} x\cos t + y\sin t + z & ry\cos t \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{2r\sin(t)^2+2r\cos(t)^2+2r}{(r^2\sin(t)^2+r^2\cos(t)^2+r^2)^2} & 0 \end{pmatrix}$$

In [65]:
```
x,y,z,r,t=symbols("x y z r t")
f=1/(x*x+y*y+z*z)
Xt=[r*cos(t),r*sin(t),r]
ChainRule(f, [x,y,z],[r,t],Xt)
```

```
∂ 1/(x**2 + y**2 + z**2) /∂ [r, t]

⎡                         2
2
⎢              2·r·sin (t)                                  2·r·co
s (t)
⎢- ──────────────────────────────────── - ─────────────────
─────────────────── - ────────
⎢                                 2
2
⎢    ⎛ 2      2           2      2        2⎞       ⎛ 2      2
2      2         2⎞     ⎛ 2
```

```
⎣   ⎝r ·sin (t) + r ·cos (t) + r ⎠      ⎝r ·sin (t) +
r ·cos (t) + r ⎠      ⎝r ·sin

                                                    ⎤
         2·r                                        ⎥
                                                0 ⎥
                                           2        ⎥
2          2     2          2⎞                      ⎥
 (t) + r ·cos (t) + r ⎠                             ⎦
```

1.37 Exercise

1. Suppose that $f(x, y) = x^2 + 3xy + y^2$ and $(x, y) = (st, s^2 t)$. Find all the first-order partial derivatives of f with respect to (x, y) and (s, t).
2. Suppose that $f(x^1, x^2, \cdots, x^n) = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$ and $x^i = ((t)^i)$, i.e. the i-th coordinate, x^i, is equal to power i of t. Find all the first-order partial derivatives of f with respect to (x, y) and t.

1.38 Answer

1.

$$\left(\frac{\partial f}{\partial x^i}\right) = \begin{pmatrix} 2x + 3y & 3x + 2y \end{pmatrix}$$

$$\left(\frac{\partial x^i}{\partial(st)}\right) = \begin{pmatrix} t & s \\ 2st & s^2 \end{pmatrix}$$

$$\left(\frac{\partial f}{\partial(st)}\right) = \begin{pmatrix} 2x+3y & 3x+2y \end{pmatrix} \begin{pmatrix} t & s \\ 2st & s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2st+3s^2t & 3st+2s^2t \end{pmatrix} \begin{pmatrix} t & s \\ 2st & s^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2st^2+9s^2t^2+4s^3t^2 & 3s^2t+6s^3t+2s^4t \end{pmatrix}$$

2.

$$(\partial f/\partial x^i) = \left(\frac{x^i}{\sqrt{(x^1)^2+\cdots+(x^n)^2}}\right)$$

$$(\partial f/\partial t) = \left(\frac{x^i}{\sqrt{(x^1)^2+\cdots+(x^n)^2}}\right)(\partial x^i/\partial t)$$

$$= \sum_{i=1}^{n} \frac{i x^i (t)^{i-1}}{\sqrt{(x^1)^2+\cdots+(x^n)^2}}$$

$$= \sum_{i=1}^{n} \frac{i (t)^{2i-1}}{\sqrt{(t)^2+\cdots+(t)^{2n}}}$$

1.39 p. 1093

4. $w = \ln(x+y^2), (x,y) = (\tan t, \sec t)$

In [23]:

```
from sympy import tan,sec
x,y,t=symbols("x y  t")
w=log(x+y**2)
Xt=[tan(t),sec(t)]

ChainRule(w, [x,y],[t],Xt)
```

```
d ( log(x + y**2)) /d [t]

⎡       2                          2       ⎤
⎢  tan (t) + 1             2⋅tan(t)⋅sec (t)⎥
⎢─────────────────── + ────────────────────⎥
⎢                2                      2  ⎥
⎣tan(t) + sec (t)       tan(t) + sec (t)   ⎦
```

8. $w = x\sqrt{y^2+z^2}, (x,y,z) = (1/t, e^{-t}\cos t, e^{-t}\sin t)$

In [24]:

```
x,y,z,t=symbols("x y z t")
w=[x*sqrt(y*y+z*z)]
Xt=[1/t,exp(-t)*cos(t),exp(-t)*sin(t)]

ChainRule(w, [x,y,z],[t],Xt)
```

```
d ( [x*sqrt(y**2 + z**2)]) /d [t]

⎡
⎢ ⎛   -t              -t          ⎞   -t              ⎛   -t
-t         ⎞   -t
⎢ ⎝- e   ⋅sin(t) - e   ⋅cos(t)⎠ ⋅e   ⋅cos(t)    ⎝- e   ⋅si
n(t) + e   ⋅cos(t)⎠ ⋅e   ⋅sin(
⎢──────────────────────────────────────── + ──────────
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        ────────────────────────────────       ────
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⎢      ╱  -2⋅t    2         -2⋅t    2              ╱  -
2⋅t    2     -2⋅t    2
⎣ t⋅╲╱   e     ⋅sin (t) + e     ⋅cos (t)      t⋅╲╱   e
⋅sin (t) + e     ⋅cos (t)

                                          ⎤
         ╱  -2⋅t    2         -2⋅t    2   ⎥
t)     ╲╱   e     ⋅sin (t) + e     ⋅cos (t)  ⎥
── - ─────────────────────────────────────⎥
                        2                 ⎥
_                      t                  ⎥
                                          ⎦
```

10. $w = \sin xy, (x, y) = \left((u+v)^3, \sqrt{v}\right)$

In [25]:

```python
x,y,u,v=symbols("x y u v")
w=sin(x*y)
Xt=[(u+v)**3,sqrt(v)]

ChainRule(w, [x,y],[u,v],Xt)
```

```
∂ sin(x*y) /∂ [u, v]

⎡
3
⎢                 2       ⎛            3⎞                  2
⎛           3⎞     (u + v) ⋅cos
⎢ 3⋅√v⋅(u + v) ⋅cos ⎝√v⋅(u + v) ⎠   3⋅√v⋅(u + v) ⋅cos ⎝√
v⋅(u + v) ⎠  + ────────────
⎣
2

⎛           3⎞ ⎤
⎝√v⋅(u + v) ⎠ ⎥
──────────────⎥
⋅√v           ⎦
```

20. Let $w = x\sqrt{y} + \sqrt{x}, (x, y) = (2s + t, s^2 - 7t)$; evaluate $\partial w / \partial t$ at $(s, t) = (4, 1)$.

In [50]:
```python
x,y,u,v=symbols("x y u v")
w=x*sqrt(y)+sqrt(x)
Xt=[2*s+t,s*2-7*t]

wst=ChainRule(w, [x,y],[s,t],Xt,output=1)
```

```
∂ sqrt(x) + x*sqrt(y) /∂ [s, t]

⎡     _____________              1              2⋅s + t          __
 _____________        1
⎢2⋅╲╱  2⋅s - 7⋅t   +  ──────────── + ─────────────   ╲╱
2⋅s - 7⋅t   +  ──────────────
⎢                          _________       _____________
 _____________
⎣                      ╲╱  2⋅s + t   ╲╱  2⋅s - 7⋅t
2⋅╲╱  2⋅s + t

      7⋅(2⋅s + t)    ⎤
  - ─────────────────⎥
          ___________⎥
     2⋅╲╱  2⋅s - 7⋅t ⎦
```

In [57]:
```python
wst.subs({s:4,t:1})[1]
```

Out[57]: -91/3

$$\left.\frac{\partial \mathbf{w}}{\partial \mathbf{t}}\right|_{(\mathbf{s},\mathbf{t})=(4,1)} = 1 + 1/6 - 63/2 = -91/3$$

28. Given $x = (u^2 - v^2)/2$, $y = uv$, find $\partial(x, y)/\partial(u, v)$, $\partial(u, v)/\partial(x, y)$

In [41]:
```python
x,y,u,v = symbols("x y u v")
fxy=Matrix([(u*u-v*v)/2, u*v])
print("∂(x,y)/∂(u,v) = %s" %fxy.jacobian([u,v]))
```

```
∂(x,y)/∂(u,v) = Matrix([[u, -v], [v, u]])
```

From the fact,

$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^{-1} = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}^{-1}$$
$$= \frac{1}{u^2+v^2}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

In [ ]:
```python

```

Type *Markdown* and LaTeX: α^2

1.40 Tangent Plane

1. Let $P(a, b, c)$ on the surface S, at which satisfies $F(x, y, z) = 0$. Then the normal vector $\vec{n}$ at P is parallel to $\nabla F(a, b, c)$, i.e.

$$\frac{x-a}{F_x(a,b,c)} = \frac{y-b}{F_y(a,b,c)} = \frac{z-c}{F_z(a,b,c)}$$

2. Partial derivative represents the ratio of changes in the respective direction. In the case of $\mathbb{R}^2$, there could exist a tangent plane for $z = f(x, y)$ at certain point (x_0, y_0) which is perpendicular with $(\nabla f, -1)$ vector.

Suppose that the surface in $\mathbb{R}^3$ satisfies:

$$\begin{aligned} z &= f(x,y) \\ &\Downarrow \\ 0 &= F(x,y,z) \\ &= f(x,y) - z \end{aligned}$$

And suppose that all the partial derivatives of $f(x, y)$ are continuous. Any curve on the surface can be represented as follows:

$$(x(t), y(t), z(t))$$

Thus we have:

$$\begin{aligned} 0 &= F(x(t), y(t), z(t)) \\ &\Downarrow \\ 0 &= \frac{dF(t)}{dt} \\ &= \frac{\partial F}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial F}{\partial z}\cdot\frac{dz}{dt} \\ (1^\circ) &= \nabla F \cdot \frac{d(x,y,z)}{dt} \\ (2^\circ) &= \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + (-1)\cdot\frac{dz}{dt} \\ &= (\nabla f, -1)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \end{aligned}$$

Given that $(x_0, y_0, f(x_0, y_0))$ lies on the surface, and so in the tangent, then for any other point (x, y, z) in the tangent plane, the vector $(x - x_0, y - y_0, z - f(x_0, y_0))$ must lie in the tangent plane, and so must be normal to the normal to the curve and this tangent plane is always in such form as follows:

$$\begin{aligned} 0 &= (\nabla f, -1)\cdot(x - x_0, y - y_0, z - f(x_0, y_0)) \\ &\Downarrow \\ f(x,y) - f(x_0,y_0) &= \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) \\ &\Downarrow \\ f(x,y) &= f(x_0,y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) \end{aligned}$$

1.41 Example

The gradient of $f(x,y)=\sqrt{x}+\sqrt{y}$ at $(x,y)=(1,1)$ is:

$$\begin{aligned}\nabla f(1,1) &= \left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right|_{(1,1)}\\ &= \left.\left(\frac{1}{2\sqrt{x}},\frac{1}{2\sqrt{y}}\right)\right|_{(1,1)}\\ &= \left(\frac{1}{2},\frac{1}{2}\right)\end{aligned}$$

Then the normal vector of the tangent plane passing throught $(1,1,2)$ is:

$$0=\left(\frac{1}{2},\frac{1}{2},-1\right)\cdot(x-1,y-1,z-2)$$

$$\Downarrow$$

$$2z=x+y+2$$

1.42 Example

The normal line and tangent plane of $4x^2+y^2+4z^2=16$ at $(1,2,\sqrt{2})$ are

$$\frac{x-1}{8}=\frac{y-2}{4}=\frac{z-\sqrt{2}}{8\sqrt{2}}$$

$$8(x-1)+4(y-2)+8\sqrt{2}(z-\sqrt{2})=0$$

1.43 Example

The normal line and tangent plane of $f(x,y)=4x^2+y^2+2$ at $(x,y)=(1,1)$ are

$$\frac{x-1}{-8}=\frac{y-1}{-2}=\frac{z-7}{1}$$

$$-8(x-1)-2(y-1)+(z-7)=0$$

In [4]:
```
from sympy import log,exp,cos,sin,diff,integrate,symb

x,y,z=symbols("x y z")
```

In [37]:
```
grad = lambda func, vars :[diff(func,var) for var in

def df_val(f,val):
    return [ff.subs({x:val[0],y:val[1],z:val[2]}) for

def tangentplane(f,X,A):
    if len(A)==2:
        A=[A[0],A[1],0]
```

```
    A[2]= f.subs({X[0]:A[0],X[1]:A[1]})
    if len(X)==2:
        f=f-z
        X=[X[0],X[1],z]
    df=grad(f,X)
    df0=df_val(df,A)

    print(df0[0]*(X[0]-A[0])+df0[1]*(X[1]-A[1])+df0[2
```

In [10]:
```
f=4*x**2+y**2+4*z**2-16
tangentplane(f,[x,y,z],[1,2,sqrt(2)])
```

```
8*x + 4*y + 8*sqrt(2)*(z - sqrt(2)) - 16 = 0
```

1.44 p. 1115 Exercise

20. tangent plane of $xyz = -4$ at $(P = (2, -1, 2)$ is

In [7]:
```
f=x*y*z+4
tangentplane(f,[x,y,z],[2,-1,2])
```

```
-2*x + 4*y - 2*z + 12 = 0
```

26. tangent plane of $z = \exp(x)\sin(\pi y)$ at $(P = (0, 1, 0))$ is

In [36]:
```
f=exp(x)*sin(pi*y)-z
tangentplane(f,[x,y,z],[0,1,0])
```

```
-z - pi*(y - 1) = 0
```

In [35]:
```
f=exp(x)*sin(pi*y)
tangentplane(f,[x,y],[0,1])
```

```
-z - pi*(y - 1) = 0
```

The change of f in the other directions different to $x, y, \cdots$, can be evaluated by the following:

1.45 Definition

The directional derivative in the unitary direction, $\vec{e} = (e^1, \cdots, e^n)$ is:

$$D_{\vec{e}} f = \nabla f \cdot \vec{e}$$

where $\cdot$ means inner product.

In [ ]:
```

```

1.46 Example

The directional derivative of $f(x, y) = \sqrt{x} + \sqrt{y}$ at $(x, y) = (1, 1)$ in the $(3, 4)$ direction is calculated as:

$$
\begin{aligned}
(3,4) &\rightarrow \frac{1}{5}(3,4) \\
D_{\vec{e}} f(1,1) &= \nabla f(1,1) \cdot \vec{e} \\
&= \frac{1}{2}(1,1) \cdot \frac{1}{5}(3,4) \\
&= \frac{7}{10}
\end{aligned}
$$

In which direction does the directional derivative attain its maximum?

Since the inner product of two vectors, $\vec{a}$ and $\vec{b}$ is:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

where θ is the intersection angle between $\vec{a}$ and $\vec{b}$, the directional derivative will attainn its maximum if ∇f and $\vec{e}$ are parallel.

1.47 Example

The directional derivative of $f(x, y) = \exp^x \cos y$ at $(x, y) = \left(1, \frac{\pi}{4}\right)$ in the $(2, 3)$ direction is calculated as:

$$
\begin{aligned}
(2,3) &\rightarrow \frac{1}{\sqrt{13}}(2,3) \\
D_{\vec{e}} f(0, \pi/4) &= \nabla f(0, \pi/4) \cdot \vec{e} \\
&= (0, -2) \cdot \frac{1}{\sqrt{13}}(2,3) \\
&= \frac{-6}{\sqrt{13}}
\end{aligned}
$$

1.48 Theorem

Directional derivative will attain its maximum (minimum) if

$$\vec{e} = \nabla f / ||\nabla f|| (-\nabla f / \nabla f||)$$

1.49 Example

The maximum of directional derivative of $f(x, y) = \sqrt{x} + \sqrt{y}$ at $(x, y) = (1, 1)$ will occur at the direction:

$$\vec{e} = \nabla f / ||\nabla f|| = (1/2, 1/2) / \sqrt{(1/2)^2 + (1/2)^2} = (1/\sqrt{2}, 1/\sqrt{2})$$

and is equal to:

$$\text{maximum of } D_{\vec{e}} f(1,1) = (1/2, 1/2) \cdot (1/\sqrt{2}, 1/\sqrt{2}) = 1/\sqrt{2}$$

In [15]:
```
from numpy import sqrt
#X = np.arange(.2, 1, 0.02)
#Y = np.arange(0.2, 1, 0.02)
x = np.arange(0, 1.4, 0.1)
y = np.arange(0, 1.4, 0.1)
```

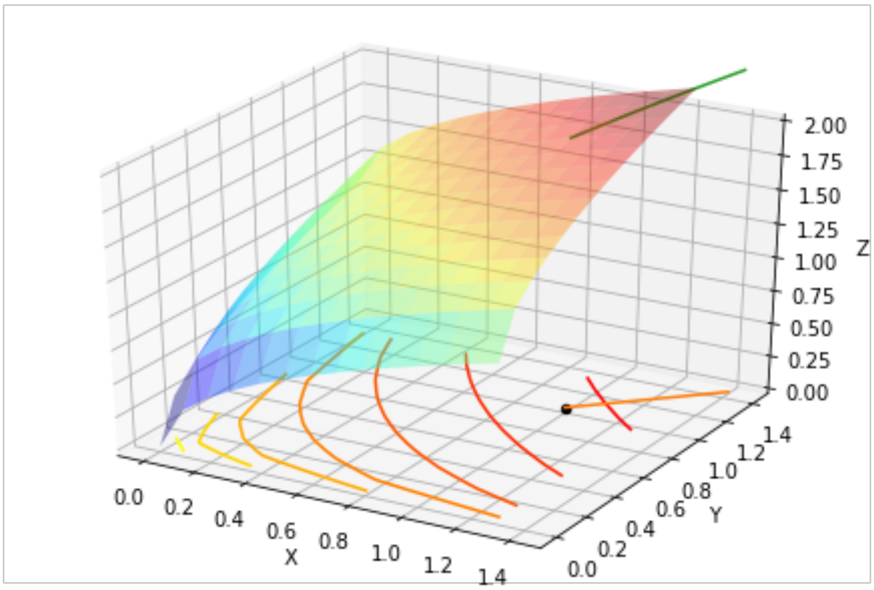
```
t = np.arange(-0.2, 1.2, 0.02)
s = np.arange(0.4, 0.8, 0.01)
X,Y = np.meshgrid(x,y)
f= sqrt(X) + sqrt(Y)
z0=sqrt(0.4**2+0.4**2)
u=np.arange(0., z0, 0.01)
Pf=(X-1)/2+(Y-1)/2+2
surface = go.Surface(x=X, y=Y, z=f,opacity=0.95)
P = go.Surface(x=X, y=Y, z=Pf,opacity=1)
Xaxis = go.Scatter3d(x=t, y=0*t, z=0*t,
            mode = "lines",
            line = dict(
                    color='black',
                    width = 5
            )
        )
Yaxis = go.Scatter3d(x=0*t, y=t, z=0*t,
            mode = "lines",
            line = dict(
                    color='black',
                    width = 5
            )
        )
X0 = go.Scatter3d(x=[1,1], y=[1,1], z=[0,2],
            mode = "lines",
            line = dict(
                    color='black',
                    width = 5
            )
    )
XY = go.Scatter3d(x=[1,1+1/2.], y=[1,1+1/2.], z=[0,0]
            mode = "lines",
            line = dict(
                    color='blue',
                    width = 3
            )
    )
N = go.Scatter3d(x=[1,1+1/2.], y=[1,1+1/2.], z=[2,2-1
            mode = "lines",
            line = dict(
                    color='blue',
                    width = 3
            )
    )
Y0 = go.Scatter3d(y=s, x=0.4+0*s, z=0*s,
            mode = "lines",
            line = dict(
                    color='orange',
                    width = 5
            )
    )
#Line2 = go.Scatter3d(x=0*t, y=t, z=0*t)
#Line3 = go.Scatter3d(x=t, y=t, z=np.ones(len(t))/2)
#Line4 = go.Scatter3d(x=t, y=-t, z=-np.ones(len(t))/2
data = [surface,Xaxis,Yaxis,X0,XY,N,P]

fig = go.Figure(data=data)
iplot(fig)
```

In [4]:

```
def plot3d(x,y,z):
    fig = plt.figure()
    ax = Axes3D(fig)
    ax.plot_surface(x, y, z, rstride=1, cstride=1, cm
    ax.contour(x, y, z,  lw=3, cmap="autumn_r", lines
    ax.set_xlabel('X')
    ax.set_ylabel('Y')
    ax.set_zlabel('Z')
    ax.set_zlim(0, 2)
    ax.scatter3D([1],[1],[0],color=(0,0,0));
    ax.arrow(x=1,y=1,dx=0.1,dy=0.1)
    xt=np.linspace(1,1.414,100)
    yt=np.linspace(1,1.414,100)
    zt=np.zeros(100)
    ax.plot3D(xt,yt,zt)
    ax.plot3D(xt,yt,np.sqrt(xt)+np.sqrt(yt))
```

In [5]:

```
x = np.arange(0, 1.4, 0.1)
y = np.arange(0, 1.4, 0.1)
x,y=np.meshgrid(x,y)
f= np.sqrt(x)+np.sqrt(y)
plot3d(x,y,f)
```



From above picture, the value of $f(x, y)$ inscreases fastest along the (positive) gradient direction, which projection on the $X - Y$ plane is orthogonal to the level curves.

1.50 Example

Find directional derivative of $f(x, y) = x^2 - 2xy$ at $(x, y) = (1, -2)$ from $(-1, 2)$ to $(2, 3)$, i.e. the direction:

$$(2 - (-1), 3 - 2) = (3, 1) \to \frac{1}{\sqrt{10}}(3, 1)$$

$$\begin{aligned} D_{\vec{e}} f(1, -2) &= \nabla f(1, -2) \cdot \vec{e} \\ &= (6, -2) \cdot \frac{1}{\sqrt{10}}(3, 1) \\ &= \frac{16}{\sqrt{10}} \end{aligned}$$

1.51 Example

Suppose that $f(x) = x^2 \sin(\pi y/6)$. **1.** The gradient of $f(x)$ at $(x, y) = (1, 1)$ is:

$$\nabla f(1,1) = \left(2x\sin(\pi y/6), \pi x^2 \cos(\pi y/6)/6\right|_{(x,y)=(1,1)}$$
$$= \left(1, \frac{\sqrt{3}\pi}{12}\right)$$

2. The directional derivative at the direction, $\vec{u} = (1, 0)$, is:

$$\nabla_{\vec{u}} f(1,1) = \left(1, \frac{\sqrt{3}\pi}{12}\right) \cdot (1,0) = 1$$

3. The directional derivative at the direction, $\vec{v} = (1, 1)$, is:

$$\vec{v} \Rightarrow (1,1)/\sqrt{1^2+1^2} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$\Downarrow$$
$$\nabla_{\vec{v}} f(1,1) = \left(1, \frac{\sqrt{3}\pi}{12}\right) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{\sqrt{6}\pi}{24}$$

4. The maximum of the directional derivative is:

$$\|\nabla f(1,1)\| = \sqrt{1^2 + \left(\frac{\sqrt{3}\pi}{12}\right)^2}$$

and in the direction:

$$\vec{e} = \nabla f(1,1)/\|\nabla f(1,1)\|$$

1.52 Example

Suppose that $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$. Find the directional derivative of $f(x, y, z)$ at $P = (1, 2, 3)$ in the directions $\overrightarrow{e}_1 = (2, 1, -2)$, b). Find the direction at which the directional derivative increases fastest and what is the maximal rate of increase.

- gradient at P

$$\nabla f(P) = \left(\frac{-x/\sqrt{x^2+y^2}}{(\sqrt{x^2+y^2}+z^2)^2}, \frac{-y/\sqrt{x^2+y^2}}{(\sqrt{x^2+y^2}+z^2)^2}, \frac{-2z}{(\sqrt{x^2+y^2}} \right.$$

$$= \left(\frac{-1/\sqrt{5}}{(\sqrt{5}+9)^2}, \frac{-2/\sqrt{5}}{(\sqrt{5}+9)^2}, \frac{-6}{(\sqrt{5}+9)^2} \right)$$

- unit direction:

$$\vec{v} \Rightarrow (2, 1, 2)/\sqrt{2^2+1^2+2^2} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)$$

- directional derivative:

$$\nabla f(P) \cdot \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right) = \frac{-12}{(\sqrt{5}+9)^2}$$

- At the direction, $\nabla f(P)$, the directional derivative increases rapidly with rate $||\nabla f||(P)$.

1.53 Exercise

Suppose that $f(x, y) = 3x^2 + 4xy + 5y^2$. Find the directional derivative of $f(x, y)$ at $(1, 1)$ in the directions a) $\overrightarrow{e}_1 = (3, -4)$, b) $\overrightarrow{e}_2 = (1, 1)$. Find the direction at which the directional derivative attains its maximum.

In [20]:
```
f=3*x*x+4*x*y+5*y*y
df=grad(f,[x,y])
dfv=df_val(df,[1,1])
def df_dir(f,val):
    l=f[0]*val[0]+f[1]*val[1]
    return l/(sqrt(val[0]**2+val[1]**2))
df_dir(dfv,[3,-4])
```

Out[20]: -26/5

In [21]:
```
df_dir(dfv,[1,1])
```

Out[21]: 12*sqrt(2)

Exercise p. 1105

8. Find the gradient of $f(x, y, z) = (x + y)/(x + z)$ at $(1, 2, 3)$.

In [2]:
```
x,y,z=symbols("x y z")
```

In [9]:
```
grad = lambda func, vars :[diff(func,var) for var in

def df_valX(f,X,P):
    """
    input
    f: function
    X: [x,y,...], variables
    P: position
    output
    gradient vector at P
    """
    df=grad(f,X)
    return [ff.subs({X[i]:P[i] for i in range(len(X))

def norm(v):
    """norm of v"""
    d=0
    for i in range(len(v)):
        d+=v[i]**2
    return sqrt(d)
def df_dir(f,X,P,vec):
    """
    Input
    f: function
    X: [x,y,...], variables
    P: position
    vec: direction
    output
    directional derivative of f at P in direction vec
    """
    dotsum=0
    dfv=df_valX(f,X,P)
    for i in range(len(dfv)):
        dotsum+=dfv[i]*vec[i]
    return dotsum/norm(vec)
```

In [10]:
```
f=(x+y)/(x+z)
df_valX(f,[x,y,z],[1,2,3])
```

Out[10]:
```
[1/16, 1/4, -3/16]
```

In [5]:
```
grad(f,[x,y,z])
```

Out[5]:
```
[-(x + y)/(x + z)**2 + 1/(x + z), 1/(x + z), -(x + y)
/(x + z)**2]
```

12. Find the gradient of $f(x, y, z) = x^3 - y^3$ at $(2, 1)$ in the directional

$(1, 1)$.

In [11]:
```
f=x**3-y**3
X=[x,y]
P=[2,1]
v=[1,1]

pprint(df_dir(f,X,P,v))
```

```
9·√2
────
 2
```

20. Find the gradient of $f(x, y, z) = x^2 + 2xy^2 + 2yz^3$ at $(2, 1, -1)$ in the directional $(1, 2, 2)$.

In [12]:
```
f=x*x+2*x*y*y+2*y*z**3
X=[x,y,z]
P=[2,1,-1]
v=[1,2,2]

df_dir(f,X,P,v)
```

Out[12]:
```
10
```

38. Find the direction at which the directional derivative of $f(x) = xe^{-y^2}$ at $(1, 0)$ increases rapidly.

In [8]:
```
f=x*exp(-y**2)

df_valX(f,[x,y],[1,0])
```

Out[8]:
```
[1, 0]
```

This concludes that directional derivative increases rapidly at the the direction $(1, 0)$.

In [1]:
```
!jupyter nbconvert --to html 6*Differ*-2.ipynb
```

```
[NbConvertApp] Converting notebook 6 Multi-variable C
alculus-Differentiation-2.ipynb to html
[NbConvertApp] Writing 2506000 bytes to 6 Multi-varia
ble Calculus-Differentiation-2.html
```

In [ ]:
```

```

In [ ]:
```

```