1 Calculus 2019-1

- Quizzes: 2 quizzes at 5/Nov/2019, 7/Jan/2020
- Score: 1. Attendancy 10% and Practices：20%
- Two Tests：total 70%, each 35%

- **Reference**: Larson: Essential Calculus 3rd Ed., Cengage Learning, 2017.
- Office Time: 12:00~13:00, Tue and Wed,

1.1 Functions

A **function** from set A to set B is an assign rule for each x in A if and only if to one y in B.

In [38]:
```
from sympy import Symbol,symbols,solve
from sympy import pi
```

In [6]:
```
x,y = symbols('x y')
h=Symbol("h")
```

1.2 Example

Suppose that $f(x) = x^2 + 2x - 1$

In [*]:
```
def f(x):
    return x**2+2*x-1
```

In [4]:
```
f(-1),
```

Out[4]: -2

In [7]:
```
f(pi)
```

Out[7]: -1 + 2*pi + pi**2

In [8]:
```
f(x+h)
```

Out[8]: 2*h + 2*x + (h + x)**2 - 1

In [9]:
```
f(2*x)
```

Out[9]: 4*x**2 + 4*x - 1

1.3 Graph of a function

The graph of a function, $f(x)$, is a set of all points, (x, y) sunc that $y = f(x)$ where x lies in the domain of f.

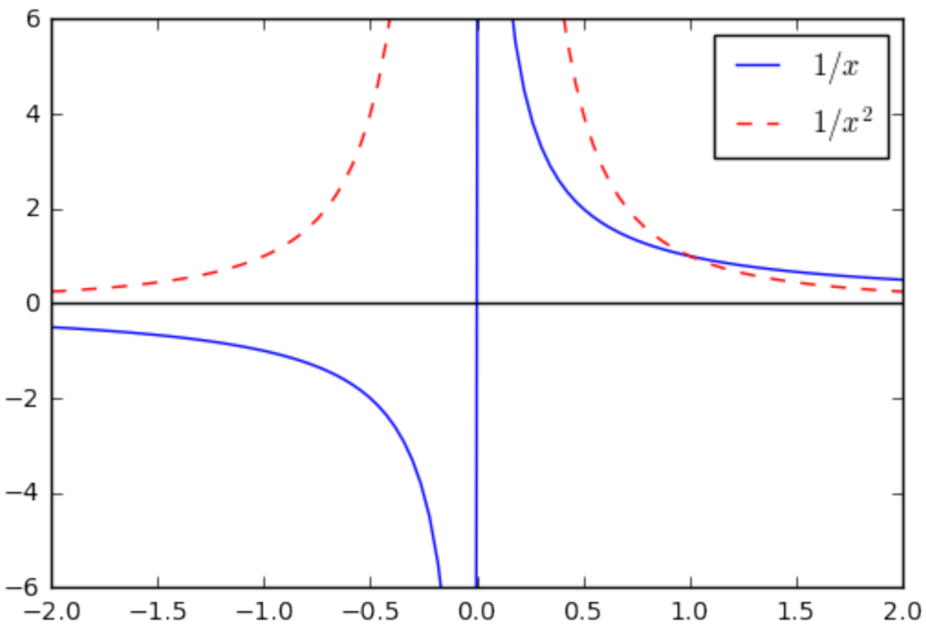
In [1]:
```
%matplotlib inline
import numpy as np
from numpy import pi,sin,cos
import matplotlib.pylab as plt
```

1.4 Example

Graphs of $1/x$ and $1/x^2$.

In [6]:
```
t=np.linspace(-2,2,100)
plt.plot(t,1/t,label='$1/x$')
plt.plot(t,1/t/t,'r--',label='$1/x^2$')
plt.plot(t,0*t,'k-')
plt.ylim([-6,6])
plt.legend()
```

Out[6]:
```
<matplotlib.legend.Legend at 0x1086964e0>
```

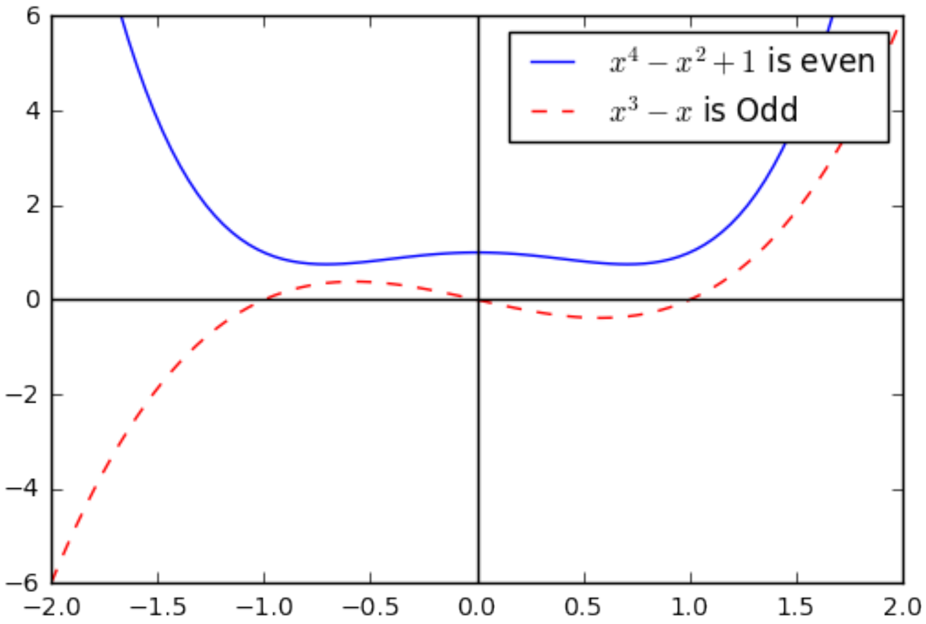


1.5 Even and Odd Funnctions

- Even function: $f(-x) = f(x)$,
- Odd function: $f(-x) = -f(x)$.

In [36]:
```
plt.plot(t,t**4-t**2+1,label='$x^4-x^2+1$ is even')
plt.plot(t,t**3-t,'r--',label='$x^3-x$ is Odd')
plt.plot(t,0*t,'k-')
plt.plot([0,0],[-6,6],'k')
plt.ylim([-6,6])
plt.legend()
```

Out[36]:
```
<matplotlib.legend.Legend at 0x10b754ac8>
```



1.6 Trigonometric Functions

sin, cos, tan, cot, sec, csc

1.7 Example

Determine whether the function,

$$\mathbf{f(x)} = \frac{\sin \mathbf{2x}}{\sqrt{\mathbf{1} + \cos^2 \mathbf{x}} + \mathbf{1}}$$

is even, odd, or neither.

Answer: ,

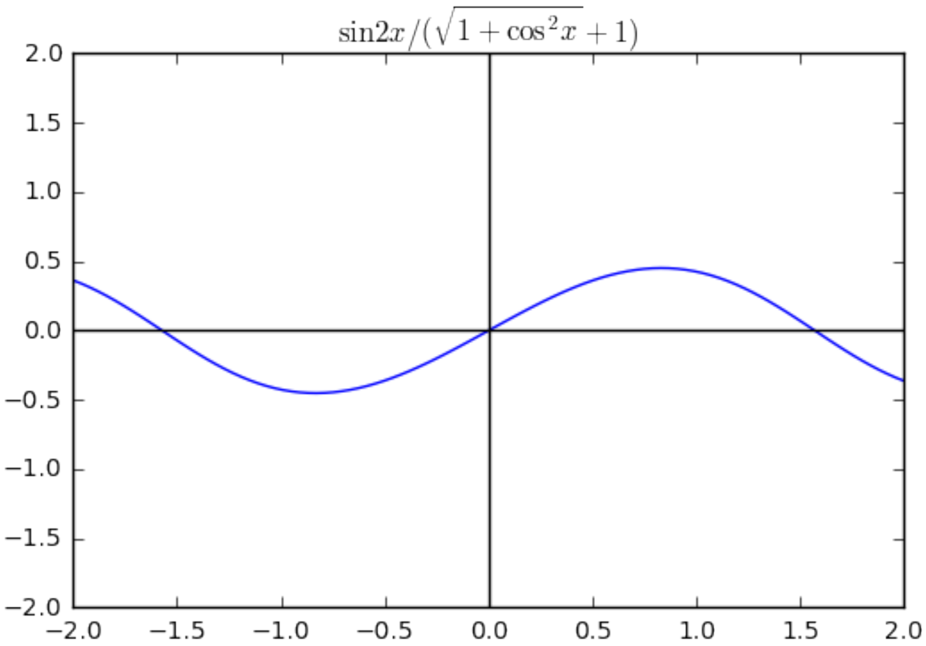
since

In [2]:
```
from numpy import sin,cos,sqrt
```

In [37]:
```
f=sin(2*t)/((sqrt(1+cos(t)*cos(t))+1))
plt.plot(t,f)
plt.title('$\sin 2x/(\sqrt{1+\cos^2x}+1)$')
plt.plot(t,0*t,'k-')
plt.plot([0,0],[-2,2],'k')

plt.ylim([-2,2])
```

Out[37]: (-2, 2)



Trigonalmetric functions are Transcendetal functiona since their calculation are much complicated than algebraic functiona.

1.8 Example

Solve $\cos x - 2\sin^2 x + 1 = 0$ where $0 \leq x \leq 2\pi$.

In [40]:
```
from sympy import sin,cos
solve(cos(x)-2*(sin(x))**2+1,x)
```

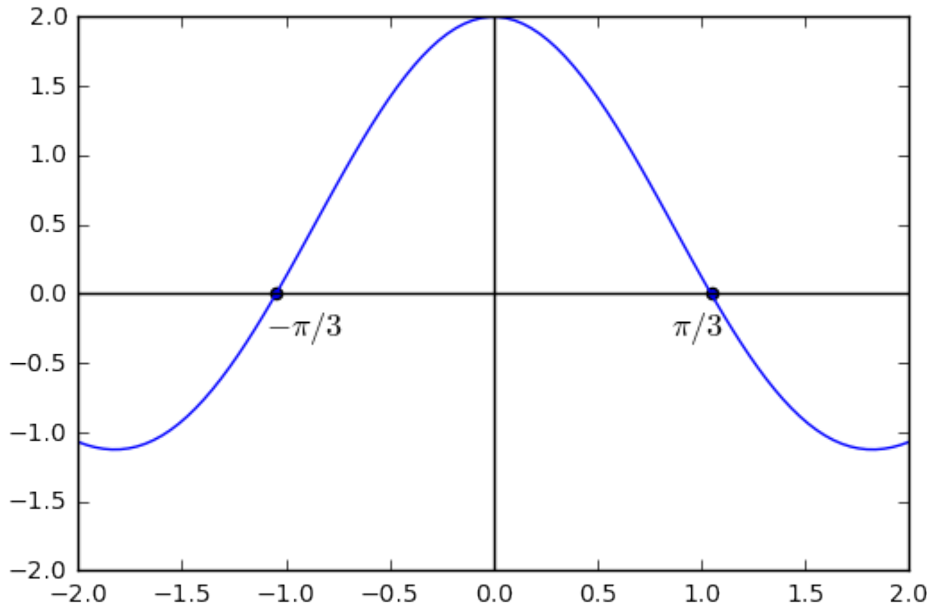
Out[40]: [-pi/3, pi/3]

In [63]:

```
from numpy import sin,cos,pi
g=cos(t)-2*sin(t)**2+1
plt.plot(t,g)
plt.plot(t,0*t,'k-')
plt.plot([0,0],[-2,2],'k')
plt.scatter([-np.pi/3,np.pi/3],[0,0])
plt.text(-1.1,-0.3,'$-\pi/3$',size=13)
plt.text(0.85,-0.3,'$\pi/3$',size=13)

plt.ylim([-2,2])
plt.xlim([-2,2])
```

Out[63]: (-2, 2)



1.9 Period

h is called period of f if $f(x+h) = f(x)$.

1.10 Example

$f(x) = 3\sin(x/2)$

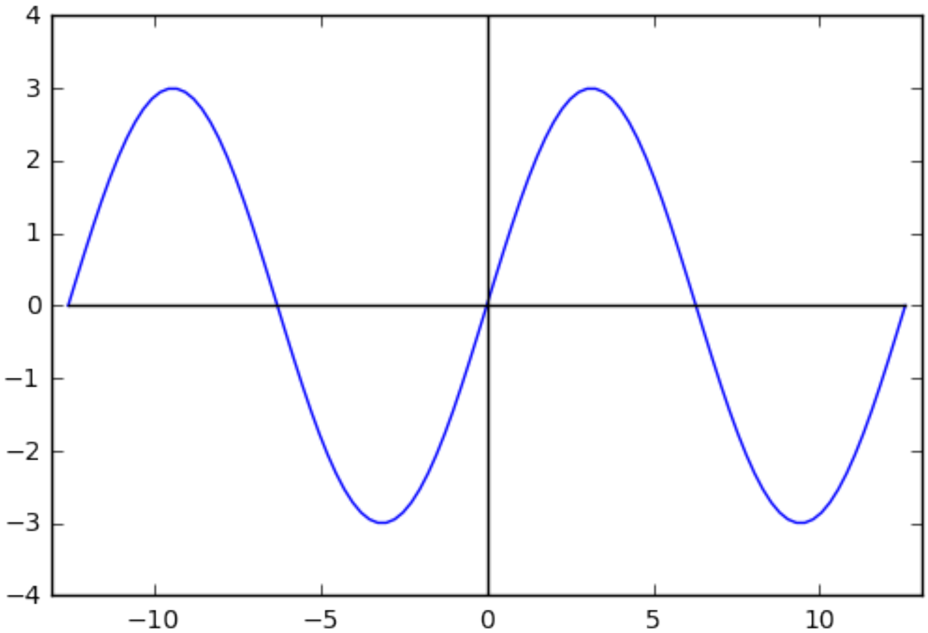
Period is (   ).

In [4]:

```
t=np.linspace(-4*pi,4*pi,101)
F=3*np.sin(t/2)
plt.plot(t,F)
plt.plot(t,0*t,'k-')
plt.plot([0,0],[-4.5,4.5],'k')

plt.ylim([-4,4])
plt.xlim([-4*pi-0.5,4*pi+0.5])
```

Out[4]: (-13.066370614359172, 13.066370614359172)



The limit as x goes to a of f is L if $f(x)$ can be made as close to L by taking x sufficiently close to a and expressed by

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f(x) = L}$$

If $f(x)$ is *continuous* at $x = a$, the $L = f(a)$.

1.11 One-sided Limits

- Right-hand Limit: The limit as x goes to a of f is L if $f(x)$ can be made as close to L by taking x sufficiently close to a but greater than a, and expressed by
$$\lim_{x\to a^+} f(x) = L$$
- Left-hand Limit: The limit as x goes to a of f is L if $f(x)$ can be made as close to L by taking x sufficiently close to a but less than a, and expressed by
$$\lim_{x\to a^-} f(x) = L$$

1.12 Theorem

Let f be a function defined on an interval containing a, possibly except at $x = a$. Then

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

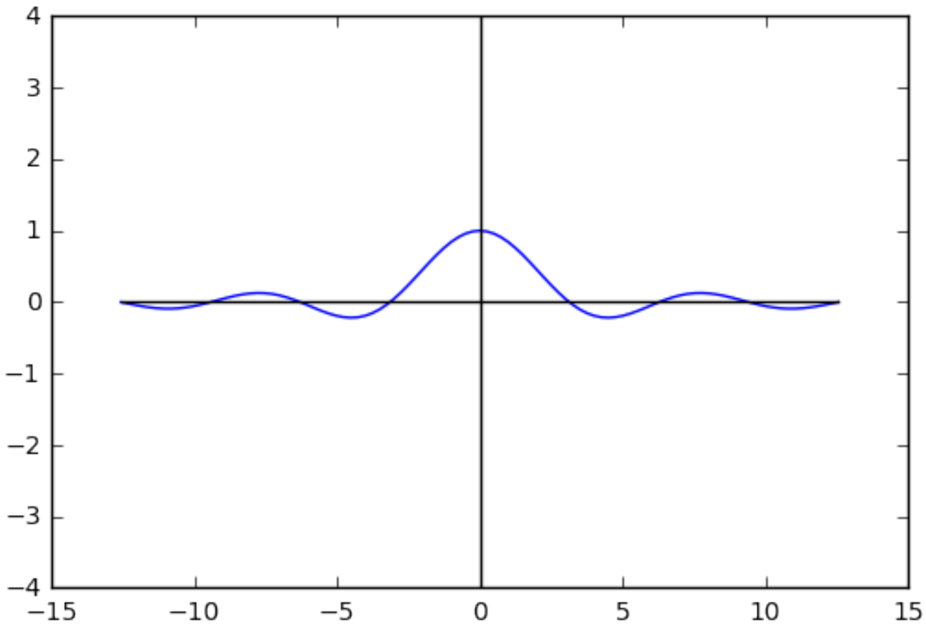
1.13 Example

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In [5]:
```
sinc=np.sin(t)/t
plt.plot(t,sinc)
plt.plot(t,0*t,'k-')
plt.plot([0,0],[-4,4],'k')

plt.ylim([-4,4])
```

Out[5]: (-4, 4)



In [3]:
```
from sympy import limit,sin,Symbol

x=Symbol("x")
limit(sin(x)/x,x,0)
```

Out[3]: 1

1.14 Example

$$\lim_{x\to 0} \sin\frac{1}{x} = ?$$

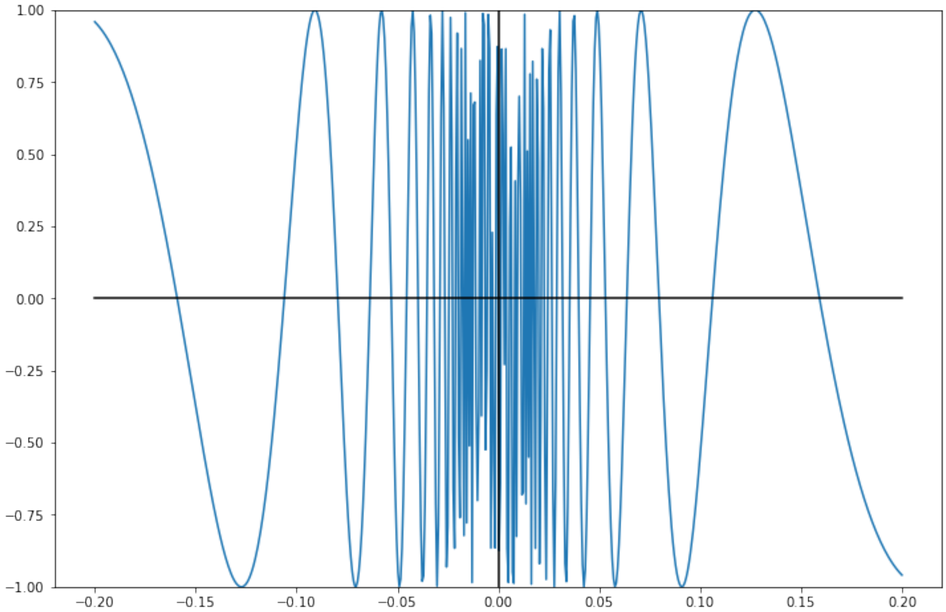
In [4]:
```
limit(sin(1/x),x,0)
```

Out[4]: <-1, 1>

In [3]:
```
t=np.linspace(-0.2,0.2,600)
plt.figure(figsize=(12,8))
f=np.sin(1/t)
plt.plot(t,f)
plt.plot(t,0*t,'k-')
plt.plot([0,0],[-4,4],'k')

plt.ylim([-1,1])
```

Out[3]: (-1, 1)



1.15 Rules of finding limit

In [86]:
```
limit(2*x**3-4*x**2+3,x,2)
```

Out[86]: 3

In [90]:
```
limit((3*x**2+2*x+1)**5,x,-1)
```

Out[90]: 32

In [91]:
```
from sympy import sqrt
limit((sqrt(1+x)-1)/x,x,0)
```

Out[91]: 1/2

In [94]:
```
from sympy import floor
limit(floor(x),x,0)
```

Out[94]: 0

In [97]:
```
limit(sin(x)/x,x,0)
```

Out[97]: 1

In [98]:
```
limit(sin(2*x)/(3*x),x,0)
```
Out[98]: 2/3

In [100]:
```
from sympy import cos
limit((1-cos(x))/x,x,0)
```
Out[100]: 0

In [101]:
```
limit((1-cos(x))/x**2,x,0)
```
Out[101]: 1/2

1.16 Squeeze Theorem

Suppose that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval around a, except possibly at a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \Rightarrow \lim_{x \to a} g(x) = L$$

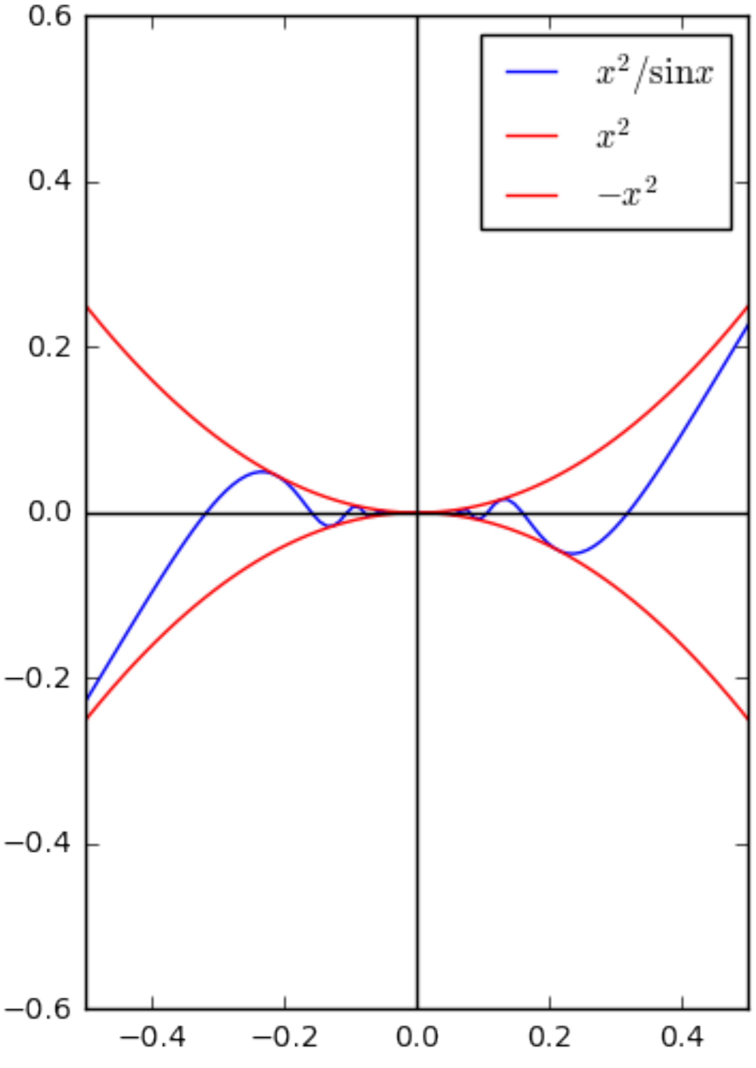
In [102]:
```
limit(x*x*(sin(1/x)),x,0)
```
Out[102]: 0

In [16]:

```
plt.figure(figsize=(4,6))
x=np.linspace(-0.5,.5,100)
plt.plot(x,x*x*sin(1/x),label='$x^2/\sin x$')
plt.plot(x,x*x,'r-',label='$x^2$')
plt.plot(x,-x*x,'r-',label='$-x^2$')
plt.plot(x,0*x,'k-')
plt.plot([0,0],[-0.6,0.6],'k')
plt.ylim([-.6,.6])
plt.xlim([-.5,.5])
plt.legend()
```

Out[16]:

```
<matplotlib.legend.Legend at 0x10a0e25c0>
```



1.17 Theorem

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

1.18 Example

1. $$\lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta} = \frac{2}{3}$$
2. $$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1$$
3. $$\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$$

1.19 Homework

p. 101:

1. (16)
$$\lim_{w \to 0} \frac{\sqrt{w+1} - \sqrt{w^2+4}}{(w+2)^2 - (w+1)^2} = (-\frac{1}{3})$$
2. (54)
$$\lim_{x \to 4} \frac{x-4}{\sqrt{x} - 2} = (4)$$
3. (63)
$$\lim_{x \to 2^-} (x - [x]) = (1)$$
4. (76)
$$\lim_{x \to 0^-} \sqrt{\frac{\tan x - \sin x}{x^2}} = (0)$$
5. (92) The function
$$f(x) = \begin{cases} 1 & \text{if } x \text{ rational,} \\ 0 & \text{if } x \text{ irrational,} \end{cases}$$
has no limit at any $x = a$.

1.20 1.3 Definition of Limit

1.21 Example

$$\lim_{x \to 2} (2x - 1) = 3$$

1.22 Example

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

Then $\lim_{x\to 0} f(x)$ does not exist.

1.23 Properties of Limit

Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

1. $\lim_{x\to a} L = L$ for any a and L in $\mathbb{R}$.
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} f(x) \pm g(x) = L \pm M$
4. $\lim_{x\to a} f(x)g(x) = LM$
5. $\lim_{x\to a} f(x)/g(x) = L/M$ if $M \neq 0$.
6. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, where n is odd and $x \in \mathbb{R}$ or n is even and $x > 0$.

1.24 Homework

p. 111:

1. (6) Find δ such that

$$\lim_{x\to -2} \frac{x^2-4}{x+2} = -4, \epsilon = .005$$

1.25 Answer:

For $0 \neq |x-(-2)| = |x+2| < \delta$,

$$\begin{aligned} \epsilon = 0.005 > & \left| \frac{x^2-4}{x+2} - (-4) \right| \\ &= |x-2+4| \\ &= |x+2| \\ \Rightarrow \quad & \delta = 0.005 \end{aligned}$$

- (26) Let

$$f(x) = \begin{cases} 0 & \text{if } x \text{ rational}, \\ -1 & \text{if } x \text{ irrational}, \end{cases}$$

  Then $\lim_{x\to 0} f(x)$ does not exist.

1.26 Proof

Suppose that the limit exists and is equal to L. Then for $\epsilon = |L|/2 > 0$, there is a $\delta > 0$ such that:

$$|x-0| = |x| < \delta \Rightarrow |f(x) - L| < \frac{|L|}{2}$$

And the last result implies

$-\dfrac{|L|}{2} < |f(x)| - |L| \Rightarrow \dfrac{|L|}{2} < |f(x)| \text{ or } f(x) < -\dfrac{|L|}{2}$ (i.e.$f(x)$ ≠
However, it is contradict to the fact that
$$f(x) = 0$$
at some rational, x, and $|x| < \delta$.

1.27 1.4 Continuous Function

Definition $f(x)$ is continuous at point a, if the following hold:

a) $f(a)$ is defined,

b) $\lim_{x\to a} f(x)$ exists,

c) $\lim_{x\to a} f(x) = f(a)$

$f(x)$ is continuous on set D if it is continuous for any $x \in D$.

1.28 Proberties of Continuity

Assume that $f(x)$ and $g(x)$ are continuous. Then

1. $f(x) \pm g(x)$ continuous;
2. $cf(x)$, $f(x) \cdot g(x)$ are continuous;
3. $f(x)/g(x)$ is also continuous if $g(x) \neq 0$.
4. If f is continuous on L and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(g(x)) = f(L)$.

1.29 $\varepsilon - \delta$ Definition

$f(x)$ is continuous at $x = a$ means that
$$\forall\varepsilon, \exists\delta \to |f(x) - f(a)| < \epsilon \text{ if } 0 < |x - a| < \delta$$

1.30 The Intermediate Theorem

If $f(x)$ is continuous on $[a, b]$ and $f(a)$ is not equal to $f(b)$. Then there exists at least one c in the interval, such that $f(c) = k$ for each k between $f(a)$ and $f(b)$.

1.31 Example

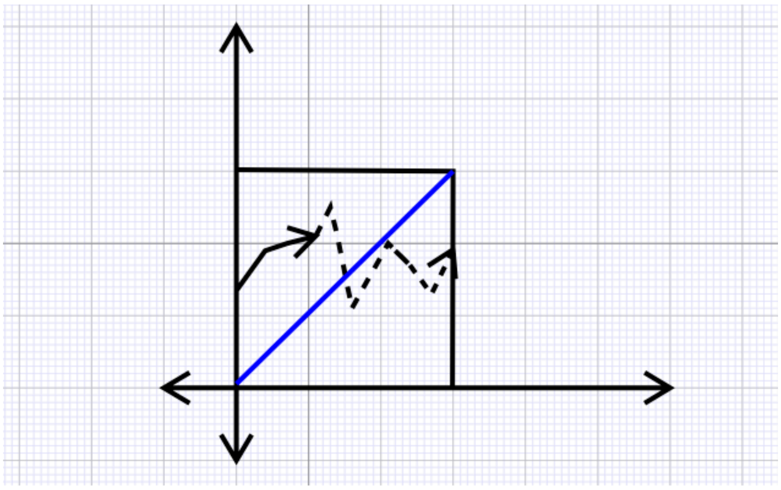
Suppose that
$$f(x) : [0, 1] \xrightarrow{\text{cont}} [0, 1]$$
Then there exists at least one c such that $f(c) = c$.

1.32 Proof

Let $F(x) = x - f(x)$,

$F(x)$ is continuous on $[0, 1]$.

$$F(0) = 0 - f(0) \leq 0$$
$$F(1) = 1 - f(1) \geq 0$$



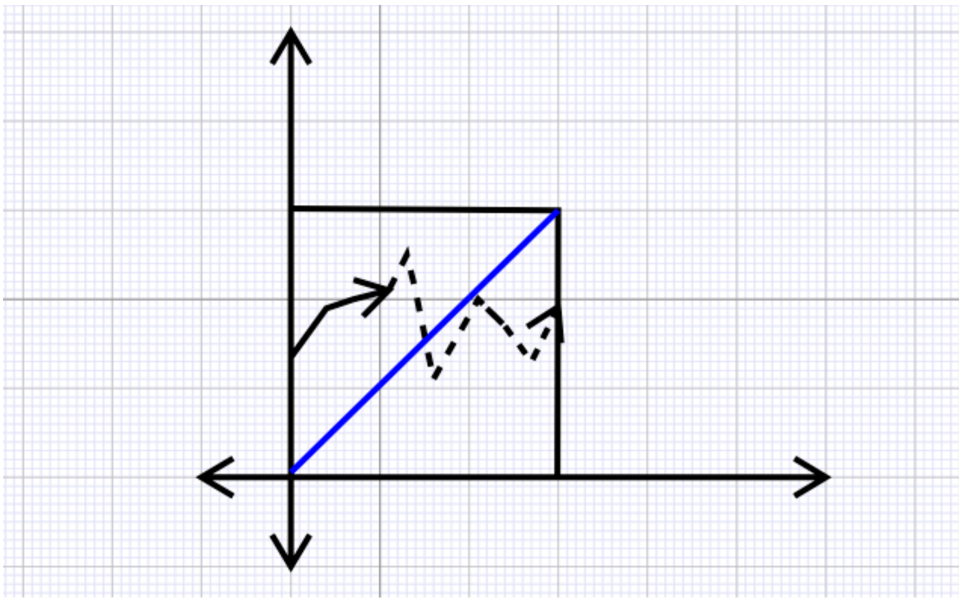
And

$$F(0) \leqslant 0 \leqslant F(1)$$

then $\exists c \in [0, 1] \rightarrow 0 = F(c) = c - f(c) \Rightarrow c = f(c)$.

In [2]:
```
from IPython.display import Image
Image("imgs/0/ch2p-13.png")
```

Out[2]:



In [ ]: