Calculus

7.5~7.6 Taylor Series and Power Series pp.479

10°). Consider $\sum_{n=0}^{\infty} a_n$ where $a_n = \frac{(2n)!x^{2n}}{n!}$. It is convergent if

$$
\begin{aligned}
1 &> \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(2n+2)!x^{2n+2}}{(n+1)!}\bigg/\frac{(2n)!x^{2n}}{n!}\right| \\
&= \lim_{n\to\infty}\left|2(2n+1)x^2\right|
\end{aligned}
$$

This holds only when $x = 0$ otherwise the limit is divergent. Thus the convergent radius is 0.

12°). Consider $\sum_{n=1}^{\infty} a_n$ where $a_n = (-1)^n \frac{x^n}{n}$. It is convergent if

$$\begin{aligned} 1 > \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \\ = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)} \Big/ \frac{x^n}{n}\right| \\ = \lim_{n\to\infty} \left|\frac{n}{n+1} x\right| = |x| \end{aligned}$$

- $|x| < 1$, convergent; $|x| > 1$, divergent.
- In the case of $|x| = 1$, i.e. $x = 1$ or $x = -1$:
  - $x = 1$: Series is convergent (alternating harmonic series is convergent):
  $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
  - $x = -1$: Series is divergent (harmonic series is divergent):
  $$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

This series is convergent if $x \in (-1, 1]$ and radius of convergence is 1.

21°). Consider $f(x) = \sum_{n=1}^{\infty} (n+1)x^n$.

- Integrate $f(t)$ from $t=0$ to x:

$$\begin{aligned}
\int_0^x f(t)dt &= \sum_{n=1}^{\infty} \int_0^x (n+1)t^n dt \\
&= \sum_{n=1}^{\infty} \cancel{(n+1)} \frac{t^{n+1}}{\cancel{n+1}} \Big|_0^x \\
&= \sum_{n=1}^{\infty} x^{n+1} = \frac{1}{1-x} - 1 - x
\end{aligned}$$

  convergent for $|x| < 1$.
- derivative of $f(x)$ is convergent for $|x| < 1$:

$$\left(\sum_{n=1}^{\infty} (n+1)x^n \right)' = \sum_{n=1}^{\infty} n(n+1)x^{n-1} = \sum_{n=0}^{\infty} (n+2)(n+1)x^n$$

- As same reason, second derivative of $f(x)$ is convergent for $|x| < 1$:

$$f''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)x^n$$

24°). Consider $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(n!)^k x^n}{(kn)!}$. It is convergent if

$$
\begin{aligned}
1 > \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\
= \lim_{n\to\infty} \left| \frac{((n+1)!)^k x^n}{(kn+k)!} \Big/ \frac{(n!)^k x^n}{(kn)!} \right| \\
= \lim_{n\to\infty} \left| \frac{(n+1)^k x}{(kn+1)\cdot(kn+2)\cdots(kn+k)} \right| = \frac{|x|}{k^k}
\end{aligned}
$$

This implies that the series is convergent if

$$\frac{|x|}{k^k} < 1 \Longrightarrow |x| < k^k$$

Thus, radius of convergence is k^k, where $k = 1, 2, \cdots$.

33°). Consider $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(-1)^{n+1}(x-2)^n}{n}$. It is convergent if

$$
\begin{aligned}
1 &> \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{n+1}\Big/\frac{(x-2)^n}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n(x-2)}{n+1}\right| = |x-2|
\end{aligned}
$$

- $|x-2| < 1$, convergent; $|x-2| > 1$, divergent;
- When, $|x-2| = 1$, i.e. $x = 3$ or $x = 1$:
  - $x = 3$, convergent since alternating harmonic series is convergent:
  $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(3-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$
  - $x = 1$, convergent since harmonic series is divergent:
  $$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(1-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{2n+1}}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$$

Radius of convergence is 1 and convergent interval is $(1, 3]$.

50°).

$$
\begin{aligned}
& 1 + 2x + x^2 + 2x^3 + x^4 + 2x^4 + \cdots \\
&= (1 + x^2 + x^4 + \cdots) + (2x + 2x^3 + 2x^5 + \cdots) \\
&= \frac{1}{1 - x^2} + \frac{2x}{1 - x^2} = \frac{1 + 2x}{1 - x^2}
\end{aligned}
$$

for $|x^2| < 1$, (i.e. $|x| < 1$).

7.7~7.8 Representation of Functions of Power Series, Taylor and Maclaurin's Series pp.486

10°). Taylor Series of $\frac{4}{3x+2}$ **at** $x = 3$:

$$\begin{aligned}
\frac{4}{2+3x} &= \frac{4}{\boxed{11} + 3(x-3)} \\
&= \frac{4}{\boxed{11}} \cdot \frac{1}{1 + \left(\frac{3(x-3)}{\boxed{11}}\right)} \\
&= \frac{4}{\boxed{11}} \cdot \sum_{n=0}^{\infty} \boxed{-1}^{n} \left(\frac{\boxed{3(x-3)}}{\boxed{11}}\right)^{n} \\
&= \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n \cdot 4 \cdot 3^n}{11^{n+1}}} \cdot (x-3)^n
\end{aligned}$$

**25°). ** Differentiating the both sides :

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

gets

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$
$$= \sum_{n=0}^{\infty} (n+1)x^n$$

and convergent for $|x| < 1$.

**26°). ** Continue from above:

$$\frac{x}{(1-x)^2} = x \cdot \sum_{n=1}^{\infty} nx^{n-1}$$
$$= \sum_{n=1}^{\infty} nx^n$$

and also convergent for $|x| < 1$.

30°). Continue from 26°) above:

$$\frac{1}{(1-x)^2} = \frac{1}{x}\sum_{n=1}^{\infty} nx^n = 2$$

a°). at $x = 2/3$:

$$\frac{1}{3}\cdot\sum_{n=1}^{\infty} n(2/3)^n = \frac{1}{3}\cdot\frac{2/3}{(1-2/3)^2} = 2$$

b°). at $x = 9/10$:

$$\frac{1}{10}\cdot\sum_{n=1}^{\infty} n(9/10)^n = \frac{1}{10}\cdot\frac{9/10}{(1-9/10)^2} = 9$$

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15°). (Binomial series) If $r \neq 0, 1, 2, 3, \cdots$, and $r \in R$, then

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}(-x)^n$$

where $\binom{r}{n} = \frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}$. Here, $r = -1/2$ and replace x by $-x^2$:

$$(1-x^2)^{-1/2}=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(x^2)^n=\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}x^{2n}$$

Also the cofficient could be rewritten as follows:

$$\begin{aligned}
\binom{-\frac{1}{2}}{n} &= \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)\cdots(-\frac{1}{2}-n+1)}{n!}\\
&= \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}\\
&= (-1)^n\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}\\
&= (-1)^n\frac{1\cdot 3\cdot\cdots\cdot(2n-1)}{2^n n!}\cdot\frac{2\cdot 4\cdot\cdots\cdot(2n)}{2\cdot 4\cdot\cdots\cdot(2n)}\\
&= (-1)^n\frac{(2n)!}{2^n n!}\cdot\frac{1}{2^n\cdot 1\cdot 2\cdot\cdots\cdot(n-1)\cdot n}\\
&= (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}
\end{aligned}$$

Conclusion:

$$(1-x^2)^{-1/2}=\sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}x^{2n}$$

is convergent for $|x|<1$.

24°).

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \Longrightarrow \cos x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{4n}$$

is convergent for $x \in \mathbb{R}$.

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