Calculus

7.4 Other Convergent tests pp.759-761

11° °). Take $a_n = \frac{(-1)^{n+1}\sqrt{n}}{x^{3/2}}$ $a_n = \frac{(-1)^{n+1}\sqrt{n}}{x^{3/2}}$.

Since

a° °). By alternating $n$$n$-term test, it is convergent since

$$\frac{\sqrt{n}}{x^{3/2}} = \frac{1}{n} \to 0$$

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b° °). But it is divergent by $p$$p$-series, $p = 1$$p = 1$:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$

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It is conditional convergent.

12° °). Take

$$a_n = \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = (-1)^{n+1} \frac{2^n (n!)^2}{2n!}$$

$$a_n = \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = (-1)^{n+1} \frac{2^n (n!)^2}{2n!}$$

Since,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}((n+1)!)^2/(2n+2)!}{2^n (n!)^2/2n!} = \frac{2(n -}{(2n+1)}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}((n+1)!)^2/(2n+2)!}{2^n (n!)^2/2n!} = \frac{2(n+1)^2}{(2n+1)(2n+2}$$

It is absolutely convergent by ratio test.

28° °). Take

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{1+n}$$

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It is conditional convergent since it is alternative harmonic series.

64° °). By ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{3(n+1)!} \Bigg/ \frac{((n)!)^2}{3n!} = \frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)} \to 0$$

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It is absolutely convergent.

74° °). Take

$$a_n = \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

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Since,

$$(a_n)^{1/n} = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} \to 0$$

$$(a_n)^{1/n} = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} \to 0$$

It is absolutely convergent by root test.

93° °). The recursive sequences defined as follows:

$$a_1 = 1/3, a_{n+1} = \left(1 + \frac{1}{n}\right) a_n = \frac{n+1}{n} a_n$$

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By mathematical induction law, we have:

$$a_{n+1} = \frac{n+1}{n} a_n = \frac{n+1}{\cancel{n}} \frac{\cancel{n}}{n-1} \cdot a_{n-1} = \cdots =$$

$$a_{n+1} = \frac{n+1}{n} a_n = \frac{n+1}{n/} \frac{n/}{n-1} \cdot a_{n-1} = \cdots = \frac{n+1}{3}$$

It is trivial,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+1}{3}$$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+1}{3}$$

is divergent.

99° °). Find the value of xx in the following such that the series is convergent:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$$

By the ratio test, it would be convrgent if

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \frac{n|x+1|}{n+1} = |x+1| \end{aligned}$$

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for any $|x+1| < 1$ $|x+1| < 1$, i.e. $x \in (-2, 0)$ $x \in (-2, 0)$; and is divergent for $x > 0$ $x > 0$ or $x < -2$ $x < -2$.

For the left cases, $x = 0$ $x = 0$ and $x = -2$ $x = -2$:

- $x = 0$ $x = 0$: convergent by alternating n n-term test:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

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- $x = -2$ $x = -2$: divergent by p p-series,

$p = 1$$p = 1$:

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100° °). Find the value of $x x$ in the following such that the series is convergent:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} n!\left(\frac{x}{2}\right)^n$$

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By the ratio test, it would be convrgent if

$$\begin{aligned} 1 > & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ = & \lim_{n\to\infty}(n+1)\frac{|x|}{2} \end{aligned}$$

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But this limit is smaller than 1 only if $x = 0x = 0$, otherwise, it is divergent.

101° °). Find the value of xx in the following such that the series is convergent:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(x+1)^n}{3n!}$$

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for any $x \in \mathbb{R}x \in \mathrm{R}$; the series is convergent for any $x \in \mathbb{R}x \in \mathrm{R}$.

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