Calculus

7.4 Other Convergent tests pp.759-761

11°). Take $a_n = \frac{(-1)^{n+1}\sqrt{n}}{x^{3/2}}$. Since
a°). By alternating n-term test, it is convergent since

$$\frac{\sqrt{n}}{x^{3/2}} = \frac{1}{n} \to 0$$

b°). But it is divergent by p-series, $p = 1$:

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$$

It is conditional convergent.

12°). Take

$$a_n = \frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = (-1)^{n+1} \frac{2^n (n!)^2}{2n!}$$

Since,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2^{n+1}((n+1)!)^2/(2n+2)!}{2^n (n!)^2/2n!} = \frac{2(n+1)^2}{(2n+1)(2n+2)} \to$$

It is absolutely convergent by ratio test.

28°). Take

$$\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{1+n}$$

It is conditional convergent since it is alternative harmonic series.

64°). By ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{3(n+1)!} \Big/ \frac{((n)!)^2}{3n!} = \frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)}$$

It is absolutely convergent.

74°). Take

$$a_n = \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

Since,

$$(a_n)^{1/n} = \frac{1}{n} - \frac{1}{n^2} = \frac{n-1}{n^2} \to 0$$

It is absolutely convergent by root test.

93°). The recursive sequences defined as follows:

$$a_1 = 1/3, a_{n+1} = \left(1 + \frac{1}{n}\right) a_n = \frac{n+1}{n} a_n$$

By mathematical induction law, we have:

$$a_{n+1} = \frac{n+1}{n} a_n = \frac{n+1}{\cancel{n}} \frac{\cancel{n}}{n-1} \cdot a_{n-1} = \cdots = \frac{n+1}{3}$$

It is trivial,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n+1}{3}$$

is divergent.

99°). Find the value of x in the following such that the series is convergent:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n (x+1)^n}{n}$$

By the ratio test, it would be convrgent if

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \frac{n|x+1|}{n+1} = |x+1| \end{aligned}$$

for any $|x+1| < 1$, i.e. $x \in (-2, 0)$; and is divergent for $x > 0$ or $x < -2$.
For the left cases, $x = 0$ and $x = -2$:

- $x = 0$: convergent by alternating n-term test:
$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$
- $x = -2$: divergent by p-series, $p = 1$:
$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

100°). Find the value of x in the following such that the series is convergent:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} n! \left(\frac{x}{2}\right)^n$$

By the ratio test, it would be convrgent if

$$\begin{aligned} 1 &> \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} (n+1)\frac{|x|}{2} \end{aligned}$$

But this limit is smaller than 1 only if $x = 0$, otherwise, it is divergent.

101°). Find the value of x in the following such that the series is convergent:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(x+1)^n}{3n!}$$

By the ratio test, it would be convrgent if

$$\begin{aligned} 1 > & \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ = & \lim_{n\to\infty} \frac{|x+1|}{n+1} = 0 \end{aligned}$$

for any $x \in \mathbb{R}$; the series is convergent for any $x \in \mathbb{R}$.