1 Multi-variable Calculus


```
from IPython.core.display import HTML
css_file = 'css/ngcmstyle.css'
HTML(open(css_file, "r").read())
```

1.1 Relative Maxima and Minima

From now on, let $\vec{\mathbf{x}} = (x^1, \cdots, x^n) \in \mathbb{R}^n$ and $\overrightarrow{\mathbf{x}_0} = (x_0^1, \cdots, x_0^n) \in \mathbb{R}^n$. Also let $f_i(\vec{x}) = \frac{\partial f}{\partial x^i}$ be the i-th partial derivative.

1.2 Definition

$z_0 = f(\overrightarrow{\mathbf{x}}_0)$ is called a relative maximum if there is $r > 0$ such that we have $f(\overrightarrow{\mathbf{x}}) \le f(\vec{\mathbf{x}}_0)$ for $|\vec{\mathbf{x}} - \vec{\mathbf{x}}_0| < r$. $z_0 = f(\vec{\mathbf{x}}_0)$ is called a relative minimum if there is $r > 0$ such that we have $f(\vec{\mathbf{x}}) \ge f(\vec{\mathbf{x}}_0)$ for $|\vec{\mathbf{x}} - \vec{\mathbf{x}}_0| < r$.

1.3 Theorem

If $z_0 = f(\vec{\mathbf{x}}_0)$ is a relative extremum then $f_i(\vec{\mathbf{x}}_0) = 0$ for all $i = 1, \cdots, n$ if they exist. All such points are called **critical points** or **stationary points**.

Although, the theorem can not state at which function attains its extrema, but we can still use this theorem to find out all the places at which extrema of functions attain.

1.4 Example

Find all the critical points of the following functions:

1. $f(x, y) = x^2 + 4y^2$. (reference the following picture at left)

$f_x = 2x = 0$, $f_y = 8y = 0$ implies $(x, y) = (0, 0)$ is the only critical point of $f(x, y)$. Since $f(x, y) \geq 0 = f(0, 0)$, $f(x, y)$ attains its minimum at $(0, 0)$.

2. For $f(x, y) = x^2 - 4y^2$ (reference the following picture at right) ,
$f_x = 2x = 0$, $f_y = -8y = 0$ implies $(x, y) = (0, 0)$ which is also the only one critical point of $f(x, y)$. But $f(0, 0)$ can not be any extremum since

$$f(\delta, 0) \geqslant f(0, 0) \geq f(0, \delta) \text{ for any } \delta > 0$$

3. From the condition,

$$f_x = 4x - 1 = 0, f_y = 2y - 2 = 0, f_z = 8z = 0$$

the critical point is $(\frac{1}{4}, 1, 0)$ and $f(x, y, z)$ attainsits relative minimum at this critical point.

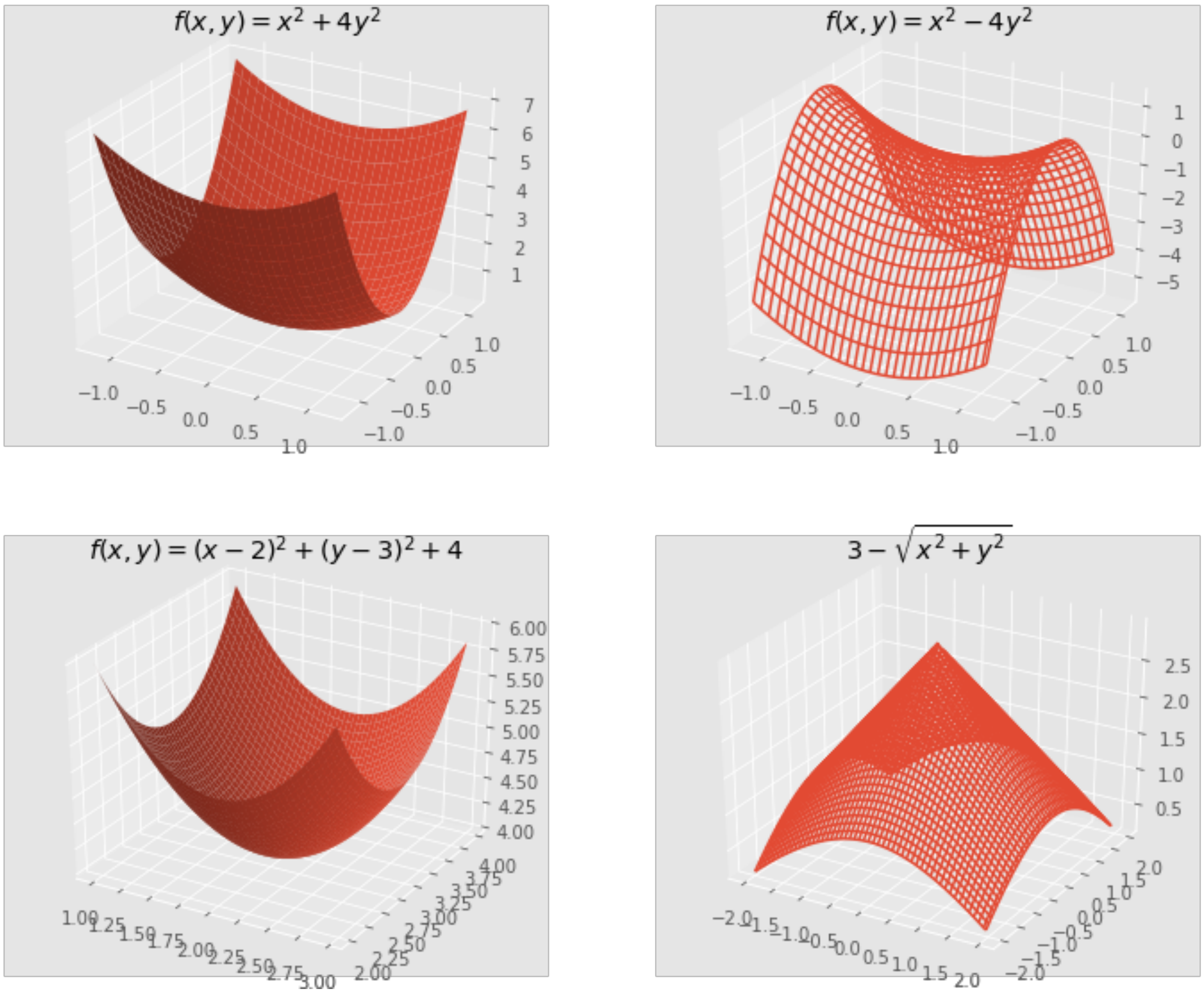
4. $f(x, y) = x^2 + y^2 - 4x - 6y + 17 = (x - 2)^2 + (y - 3)^2 + 4 \geq 4$: this implies only critical point at $(x, y) = (2, 3)$ which attains its minimuum, 4.

5. $f(x, y) = 3 - \sqrt{x^2 + y^2} \leq 3$: critical point $(0, 0)$ and maximum is 3:

$$\nabla f = \left[\frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}} \right] = (0, 0)$$

Note: ∇f fails to exit at $(0, 0)$.

```
Text(0.5, 0.92, '$3-\\sqrt{x^2+y^2}$')
```




```
{x: 1/4, y: 1, z: 0}
```

1.5 Example

Suppose that $x^i, i = 1, 2, \cdots, n$ satisfies

$$f_{\mu,\sigma}(x^i) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x^i-\mu)^2}{2\sigma^2}\right)$$

Define $L(\mu, \sigma^2)$ as follows:

$$\begin{aligned} L(\mu, \sigma^2) &= \prod_{i=1}^{n} f_{\mu,\sigma}(x^i) \\ &= \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x^i-\mu)^2}{2\sigma^2}\right)\right] \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\sum_{i=1}^{n}\frac{(x^i-\mu)^2}{2\sigma^2}\right) \end{aligned}$$

What values of (μ, σ^2) will makes $L(\mu, \sigma^2)$ attains its maximum? Before answering this question, note that

- assume $(\hat{\mu}, \hat{\sigma}^2)$ is the value such that maximizes $L(\mu, \sigma^2)$;
- Suppose that the value of $(\hat{\mu}, \hat{\sigma}^2)$ maximizes $L(\mu, \sigma^2)$. This it also maximizes $\ln L(\mu, \sigma^2)$. This means that we have to find the value of $(\hat{\mu}, \hat{\sigma}^2)$ that maximizes the following:

$$\ln L(\mu, \sigma^2) = \ln\left[\frac{1}{(2\pi\sigma^2)^{n/2}}\exp\left(-\sum_{i=1}^{n}\frac{(x^i-\mu)^2}{2\sigma^2}\right)\right]$$
$$= -\frac{n}{2}(\ln 2\pi + \ln\sigma^2) - \left(\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x^i-\mu)^2\right)$$

Then the critical value $(\hat{\mu}, \hat{\sigma}^2)$ satisfies:

$$0 = \frac{\partial}{\partial\mu}\ln L(\mu, \sigma^2)$$
$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x^i-\mu)$$
$$0 = \frac{\partial}{\partial\sigma^2}\ln L(\mu, \sigma)$$
$$= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x^i-\mu)^2$$

From the first result, we can get the value, $\hat{\mu}$, as:

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}(x^i-\mu) = 0$$
$$\Longrightarrow \quad \sum_{i=1}^{n}(x^i-\mu) = 0$$
$$\Longrightarrow \quad n\mu = \sum_{i=1}^{n}x^i$$
$$\Longrightarrow \quad \hat{\mu} = \sum_{i=1}^{n}x^i/n = \bar{x}$$

i.e. $\hat{\mu}$ is the mean of sum of x^i, $i = 1, 2, \cdots, n$, called **sample mean**.

$$-\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x^i-\mu)^2 = 0$$

we have:

$$\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(x^i-\hat{\mu})^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}(x^i-\bar{x})^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}(x^i)^2 - \frac{2\bar{x}}{n}\sum_{i=1}^{n}x^i + \frac{1}{n}\sum_{i=1}^{n}(\bar{x})^2$$
$$= \frac{1}{n}\sum_{i=1}^{n}(x^i)^2 - 2(\bar{x})^2 + (\bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x^i)^2 - (\bar{x})^2$$

so called the **sample variance**.

Note that the term in last result, $-mx_0 - mx_1 + m^2$, is equal to $-m^2$ since $\sum x^i = n \cdot \bar{x}$

1.6 Note

However, $\widehat{\sigma^2}$ is so-called biased estimator, since

$$
\begin{aligned}
E\,\widehat{\sigma^2} &= E\left(\frac{1}{n}\sum_{i=1}^{n}(x^i-\bar{x})^2\right) \\
&= \frac{1}{n}E\sum_{i=1}^{n}(x^i-\mu-(\bar{x}-\mu))^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}E\left(\frac{n-1}{n}(x^i-\mu)-\frac{1}{n}\sum_{j=1 j\neq i}^{n}(x^j-\mu)\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{n-1}{n}\right)^2\sigma^2+\frac{n-1}{n^2}\sigma^2\right] \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{n-1}{n}\sigma^2 \\
&= \frac{n-1}{n}\sigma^2 \neq \sigma^2
\end{aligned}
$$

And

$$
n\,\widehat{\sigma^2}/(n-1) = \frac{1}{n-1}\sum_{i=1}^{n}(x^i-\bar{x})^2
$$

is called the **unbiased estimator** for σ^2 since

$$
E\left(\frac{1}{n-1}\sum_{i=1}^{n}(x^i-\bar{x})^2\right) = \sigma^2.
$$

```
---------------------------------------------------------------------------
AttributeError                            Traceback (most recent call last)
<ipython-input-6-0c198d241708> in <module>
      8 M=Matrix(1,n,lambda i,j: mu )
      9 XX=(Xn-M)*(Xn-M).transpose()
---> 10 f=1/sqrt(2*pi*sigma)**n*exp(-XX[0]/(2*sigma))
     11 L=log(f)
     12 dL=grad(L,[mu,sigma])

AttributeError: 'Mul' object has no attribute 'sqrt'
```

1.7 Example

(Multi-product Monopoly) A company produces twokinds of goods, A and B, with relative demand functions, p_1 and p_2. Suppose the functions satisfy:

$$A = 100 - 2p_1 + p_2$$
$$B = 120 + 3p_1 - 5p_2$$

Assume the cost function for producing A and B has been estimated as
$C = 50 + 10A + 20B$
Then the profit function is

$$\begin{aligned} P(p_1, p_2) &= p_1 A + p_2 B - C \\ &= p_1(100 - 2p_1 + p_2) + p_2(120 + 3p_1 - 5p_2) - (50 + 10(100 - 2p_1 + p_2) \\ &= 1350 + 60p_1 + 210p_2 - 2p_1^2 - 5p_2^2 + 4p_1p_2 \end{aligned}$$

The critical point of profit function can be obtained by

$$P_{p_1} = P_{p_2} = 0$$
$$\Longrightarrow \quad p_1 = 60 \qquad p_2 = 45$$

And these imply $A = 25$ and $B = 75$. Here, it is difficulty to check whether profit is an extremum at this critical point. Later, we will give some conditions to confirm whether the function value is an extremum at critical points.

1.8 Example

Certain company sells its goods in two different brands, A and B. Suppose that the relations between price function, p_i, and demand function, $x_{i,}$ $i = 1, 2$ in different sub-markets are:

$$p_1 = 100 - x_1$$
$$p_2 = 120 - 2x_2$$

with the total cost function:
$C = 20(x_1 + x_2)$
Then profit function can be obtained by:

$$\begin{aligned} P(x_1, x_2) &= p_1x_1 + p_2x_2 - C \\ &= 80x_1 - x_1^2 + 100x_2 - 2x_2^2 \end{aligned}$$

It is trivial that the maximum exists. From the first partial derivatives properties, the relative extrema must occurs at

$$P_{x_1} = 80 - 2x_1 = 0$$
$$P_{x_2} = 100 - 4x_2 = 0$$

And this implies that there exists only one stationary point, $(x_1, x_2) = (40, 25)$, and at which the maximum attains.

1.9 Example

Two firms produce identical products with same price:

$$p = 150 - x_1 - x_2$$

and with zero cost. For each firms, their profits are in the following manners respectively:

$$P_1 = px_1 = (150 - x_1 - x_2)x_1$$
$$P_2 = px_2 = (150 - x_1 - x_2)x_2$$

Then if they want to get maximum profits, the first partial derivatives have to be held:

$$\frac{\partial P_1}{\partial x_1} = 0$$
$$\frac{\partial P_2}{\partial x_2} = 0$$

And these imply

$$\widehat{x_1} = \frac{150 - x_2}{2}$$
$$\widehat{x_2} = \frac{150 - x_1}{2}$$

From above results, the maximum occurs dependent on other stationary point. Therefore, two results have to be held simultaneously. In other words, we have

$$x_1 = x_2 = \frac{150}{3} = 50$$

From the graphs, the demand functions are intersected at this point, $\left(\frac{100}{3}, \frac{100}{3}\right)$:

This point is called **Cournot equilibrium**.

1.10 Example

Suppose that there are n little firms produce identical products now. The price function is

$p = 150 - \sum_{i=1}^{n} x_i$

with zero cost for simplicity. Then at stationary points, the following relation holds for any $k = 1, \cdots, n$:

$$
\begin{aligned}
P_k &= px_k = x_k \left(150 - \sum_{i=1}^{n} x_i\right) \\
\Longrightarrow 0 = \frac{\partial P_k}{\partial x_k} &= \left(150 - \sum_{i=1}^{n} x_i\right) - x_k
\end{aligned}
$$

Since the equation still remains unchanged even by interchanging indexes with any two different variables, x_i and x_j, it is obvious that all the x_i's are equal. Thus replace all x_k's with $\widehat{x}$, then:

$$
\begin{aligned}
0 &= \left(150 - \sum_{i=1}^{n} \widehat{x}\right) - \widehat{x} \\
\Longrightarrow \qquad \widehat{x} &= \frac{150}{n+1} \qquad\qquad \text{for } k = 1, \cdots, n
\end{aligned}
$$

The last example is just in the case, $n = 2$.

In above examples, the first derivative properties do not assure at which critical points are extrema. Some strongly conditions are needed to confirm whether there are extrema at critical point.

1.11 Definition

Suppose that $M = (m_{ij})_{n \times n}$ is a $n \times n$ square matrix and $\vec{h} = (h_1, h_2, \cdots, h_n)_{1 \times n}$ is any $1 \times n$ vector. Then M is called **positive definite** if $\vec{h}^t M \vec{h} = \sum_{i,j=1}^{n} h_i m_{ij} h_j > 0$ and called **negative definite** if $\vec{h}^t M \vec{h} < 0$.

Define the Hessian matrix of function $f(\vec{x})$ with its minor matrices as:

$$
\begin{aligned}
H &= (f_{ij})_{n \times n} \\
H_1 &= (f_{11}) \cdots \ H_k = (f_{ij})_{k \times k}, k = 1, \cdots, n-1
\end{aligned}
$$

and let $|H_i|$ be the determinant of H_i. For multi-variable functions, we have the following theorem:

The definite properties of Hessian matrices can be determined by calculating determinants of Hessian matrices and their minor matrices.

1.12 Theorem

Let

$A = \dfrac{\partial^2 f}{\partial x^2}(x_0, y_0), B = \dfrac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \dfrac{\partial^2 f}{\partial x \partial y}(x_0, y_0), C = \dfrac{\partial^2 f}{\partial y^2}(x_0, y_0)$ and $\dfrac{\partial f}{\partial x}(x_0,$

where (x_0, y_0) is the critical point of $f(x, y)$, $D = AC - B^2$, then

1. if $D > 0$ and $A < 0$, $f(x_0, y_0)$ is a relative maximum,
2. if $D > 0$ and $A > 0$, $f(x_0, y_0)$ is a relative minimum,
3. if $D < 0$, $(x_0, y_0, f(x_0, y_0))$ is a saddle point,
4. if $D = 0$, no conclusion.

Here the **Hessian matrix** for function $f(x, y)$ is defined as follows:

$$H = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y \partial x} \\ \dfrac{\partial^2 f}{\partial x \partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

and with determinant $D = |H|$. For higher dimension case, $n \geq 2$, then:

1. if $|H_k| > 0$, for $k = 1, \cdots, n$, $f(\vec{x}_0)$ is a relative minimum,
2. if $|H_1| < 0, |H_2| > 0, \cdots$, $f(\vec{x}_0)$ is a relative maximum,

   where H_k is the submatrix of H and defineds as follows:

$$\begin{array}{c} \begin{array}{cccccc} H_1 & H_2 & H_3 & & H_{n-1} & H_n \end{array} \\ \begin{pmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1,n-1} & f_{1n} \\ f_{21} & f_{22} & f_{23} & & & \\ f_{31} & f_{32} & f_{33} & & & \vdots \\ \vdots & & & \ddots & & \\ f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & & f_{n-1,n-1} & f_{n-1,n} \\ f_{n1} & f_{n2} & f_{n3} & & f_{n,n-1} & f_{nn} \end{pmatrix} \end{array}$$

1.13 Example

Suppose that $f(x, y) = x^3 + y^2 - 2xy + 7x - 8y + 2$. Then the critical value comes from the following relations:

$$
\begin{aligned}
& f_x = 3x^2 - 2y + 7 = 0 \text{ and } f_y = 2y - 2x - 8 = 0 \\
\implies & \qquad\qquad (x, y) = (3, 1)
\end{aligned}
$$

i.e. there are two critical values, $(x, y) = P_1 = (1, 5), (x, y) = P_2 = (-1/3, 11/3)$. By the way, we also have

$$
\begin{aligned}
H_2 &= \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \\
&= 12x - 4
\end{aligned}
$$

1. P_1, $f_{xx}(P_1) = 6 > 0, H(P_1) = 8 > 0$ imply $f(P_1)$ is local minimum;
2. P_2, $H(P_2) = -8 < 0$ imply $f(P_2)$ is saddle point;

1.14 Definition

$f(\mathbf{x_0})$ is called absolute maximum of $f(\mathbf{x})$ if $f(\mathbf{x_0}) \geq f(\mathbf{x})$ for any $\mathbf{x}$ in its domain; and is called absolute minimum if $f(\mathbf{x_0}) \leq f(\mathbf{x})$.

1.15 Theorem

Continuous function $f(\mathbf{x})$ should attain its both absolute maximum and minimum in closed domain. This means that the absolute extrema of $f(\mathbf{x})$ could be found from the following steps:

1. find out all critical values.
2. evaluate the function values at the points of (1) and the boundary of D,
3. absolute maximum is the largest of (2) and the absolute minimum is smallest of (2).

1.16 Note

Extrema on the boundary could be evaluated by the **Lagrange's method** or called **mountain-pass** theorem, introduced later.

1.17 Example

Find the both absolute extrema values of $f(x, y) = 2x^2 + y^2 - 4x - 2y + 3$ on the rectangle:
$D = \{0 \le x \le 3, 0 \le y \le 2\}$

1. critical value:
$$\begin{aligned} & f_x = 4x - 4 = 0 \text{ and } f_y = 2y - 2 = 0 \\ \Longrightarrow \quad & (x, y) = (1, 1) \end{aligned}$$
i.e. there are only one critical value, $(x, y) = (1, 1)$ with $f(x, y)|_{(x,y)=(1,1)} = 0$.
2. Let the boundary of D be:
$L_1 \cup L_2 \cup L3 \cup L_4 = \{y = 0\} \cup \{x = 3\} \cup \{y = 2\} \cup \{x = 0\}$

   - $L_1 = \{y = 0\}$:
$$\begin{aligned} f_{L_1} &= f(x, 0) \text{ and } 0 \le x \le 3 \\ &= 2x^2 - 4x \\ f'_{L_1} &= 0 \Rightarrow 4x - 4 = 0 \rightarrow x = 1 \text{ and } x = 3 \\ &\rightarrow f(1, 0) = 1, f(3, 0) = 9 \end{aligned}$$
   - $L_2 = \{x = 3\}$:
$$\begin{aligned} f_{L_2} &= f(y, 3) \text{ and } 0 \le y \le 2 \\ &= y^2 - 2y + 9 \\ f'_{L_2} &= 0 \Rightarrow 2y - 2 = 0 \rightarrow y = 1 \text{ and } y = 0, 2 \\ &\rightarrow f(3, 1) = 8, f(3, 0) = f(3, 2) = 9 \end{aligned}$$
   - $L_3 = \{y = 2\}$:
$$\begin{aligned} f_{L_3} &= f(x, 0) \text{ and } 0 \le x \le 3 \\ &= 2x^2 - 4x + 3 \\ f'_{L_3} &= 0 \Rightarrow 4x - 4 = 0 \rightarrow x = 1 \text{ and } x = 3 \\ &\rightarrow f(1, 2) = 1, f(3, 2) = 9 \end{aligned}$$
   - $L_4 = \{x = 0\}$:
$$\begin{aligned} f_{L_4} &= f(y, 0) \text{ and } 0 \le y \le 2 \\ &= y^2 - 2y + 3 \\ f'_{L_4} &= 0 \Rightarrow 2y - 2 = 0 \rightarrow y = 1 \text{ and } y = 0, 2 \\ &\rightarrow f(0, 1) = 2, f(0, 0) = f(0, 2) = 3 \end{aligned}$$

   Thus the absolute maximum is 9 and mabsolute minimum is 0.

1.18 Example

Suppose that $f(x, y) = 4x - 2y - x^2 - 2y^2 + 2xy - 10$. Then the critical value comes from the following relations:

$$\begin{aligned} & f_1 = 4 - 2x + 2y = 0 \text{ and } f_2 = -2 - 4y + 2x = 0 \\ \Longrightarrow \quad & (x, y) = (3, 1) \end{aligned}$$

i.e. only one critical value $(x, y) = (3, 1)$. By the way, we also have

$$H_2 = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -4 \end{pmatrix}$$

$$\Longrightarrow f_{11} = -2 < 0 \text{ and } D = f_{11}f_{22} - (f_{12})^2 = 4 > 0$$

Therefore $f(3, 1) = -5$ is relative maximum.

1.19 Example

$f(x, y) = x^4 + y^4 - 4xy$,Then the critical values come from the following relations:

$$\begin{aligned} & f_1 = 4x^3 - 4y = 0 \text{ and } f_2 = 4y^3 - 4x = 0 \\ \Longrightarrow \quad & x = y^3 \text{ and } y = x^3 (\text{i.e. } x = x^9) \\ \Longrightarrow \quad & (x, y) = (0, 0) \text{ or } (\pm 1, \pm 1) \end{aligned}$$

And the Hessian matrix is:

$$H = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$$

1. at $(x, y) = (0, 0)$,

$$D = \left| \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \right| = -16 < 0$$

saddle point.

2. at $(x, y) = (\pm 1, \pm 1)$:

$$D = \left| \begin{pmatrix} 12 & -4 \\ -4 & 12 \end{pmatrix} \right| = 128 > 0$$

with $f_{11}(\pm 1, \pm 1) = 12 > 0$. Then $f(-1, -1) = -2$ is a relative minimum and $f(1, 1) = -128$ is also a relative minimum.

1.20 Example

Revisit the previous example $f(x, y, z) = 2x^2 + y^2 + 4z^2 - x - 2y$. We have:

$$\begin{aligned} f_1 &= 4x - 1 \\ f_2 &= 2y - 2 \\ f_3 &= 8z \\ H &= \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} \\ &= \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \end{aligned}$$

From the last result, we have:

$$\begin{aligned} |H| &= 64 > 0 \\ |H_1| &= 4 > 0 \\ |H_2| &= 4 \cdot 2 = 8 > 0 \\ |H_3| &= |H| > 0 \end{aligned}$$

Therefore, $f(x, y, z)$ actually attains its relative minimum at the critical point, $(1/4, 1, 0)$.

1.21 Example

If a company produces two products, A and B, with prices 100 and 300 respectively. The total cost in producing x units of A and y units of B is

$$C(x, y) = 2000 + 50x + 80y + x^2 + 2y^2$$

Revenue $R = 100x + 300y$ and

$$\begin{aligned} P(x, y) &= R(x, y) - C(x, y) \\ &= -2000 + 50x + 220y - x^2 - 2y^2 \end{aligned}$$

with $o \leqslant x, y$. First we want to find the critical point:

$$\begin{aligned} \nabla P = \vec{0} &\implies \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y} \right) = \vec{0} \\ &\implies 50 - 2x = 0 \text{and} 220 - 4y = 0 \end{aligned}$$

i.e. only one critical point $(x, y) = (25, 55)$. Also the Hessian matrix is as follows:

$$H = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

$P_1 = -2 < 0$ but $|H| = 8 > 0$. This means that $P(x, y)$ attains its relative maximum at $(25, 55)$. At this point, $P(x, y)$ attains its maximum too.

1.22 Exercise

Find the relative extrema of $f(x, y) = x^3 + y^3 - 3xy$ if any.

1. Find the critical point(s) as follows:

$$
\begin{aligned}
\nabla f = \vec{0} &\Longrightarrow (3x^2 - 3y, 3y^2 - 3x) = (0, 0) \\
&\Longrightarrow y = x^2 \text{ and } x = y^2 (i.e.\, x = x^4) \\
&\Longrightarrow (x, y) = (0, 0) \text{ or } (1, 1)
\end{aligned}
$$

There exist two critical points. **2.** Find Hessian matrix:

$$
H = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}
$$

- $(x, y) = (0, 0)$: $|H| = -9 < 0 \Rightarrow f(0, 0)$ is a saddle point.
- $(x, y) = (1, 1)$: $|H| = 36 - 9 > 0$ and $f_{11} = 6 > 0 \Rightarrow f(1, 1)$ is a relative minimum.

1.23 Exercise

Find the relative extrema of $f(x, y) = \exp(-x^2 - y^2)$ if any.

1. Find the critical point(s) as follows:

$$
\begin{aligned}
\nabla f = \vec{0} &\Longrightarrow (-2xe^{-x^2-y^2}, -2ye^{-x^2-y^2}) = (0, 0) \\
&\Longrightarrow (x, y) = (0, 0)
\end{aligned}
$$

There exists one critical point.

2. Find Hessian matrix:

$$
H = \begin{pmatrix} (4x^2 - 2)e^{-x^2-y^2} & 4xye^{-x^2-y^2} \\ 4xye^{-x^2-y^2} & (4y^2 - 2)e^{-x^2-y^2} \end{pmatrix}
$$

$(x, y) = (0, 0)$: $|H| = 4 > 0$ and $f_{11}(0, 0) = -2 < 0 \Longrightarrow f(0, 0)$ is a relative maximum.

In general, it is not difficulty to find extrema for functions without any restriction. The following theorem is an extension in the case of all variables defined within intervals, i.e. $x^i \in [a^i, b^i]$, for all $1 \leq i \leq n$:

1.24 Theorem

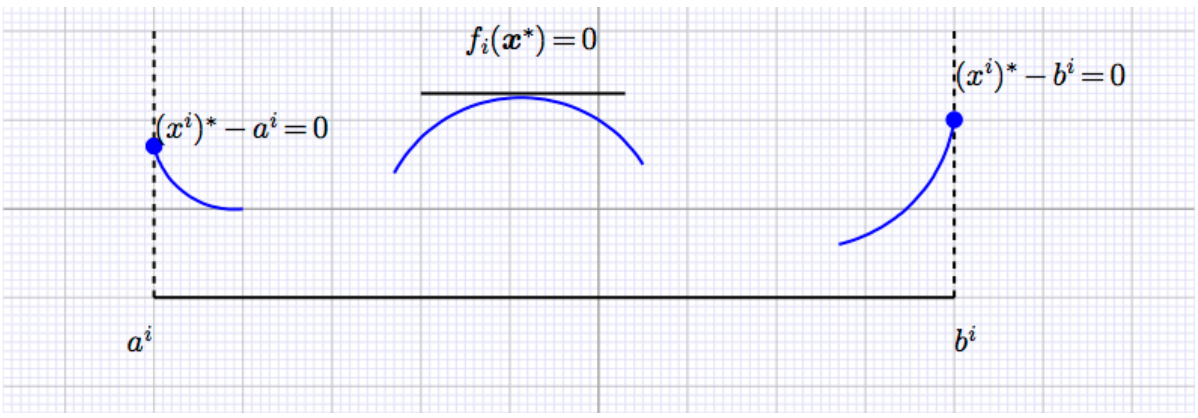
Suppose that

$$\text{Domain}(f) = \left\{\mathbf{x} | x^i \in [a^i, b^i],\ \text{for all } 1 \leqslant i \leqslant n\right\}$$

and $f(\mathbf{x})$ is smooth with all the first order derivatives in domain. Suppose that $f(\mathbf{x})$ attains it global maximum at $\mathbf{x} = \mathbf{x}^*$, then one or both following condition(s) must hold:

1. $f_i(\mathbf{x}^*) \leq 0$ and $((x^i)^* - a^i)f_i(\mathbf{x}^*) = 0$;
2. $f_i(\mathbf{x}^*) \geq 0$ and $(b^i - (x^i)^*)f_i(\mathbf{x}^*) = 0$. for all $1 \leqslant i \leqslant n$.

1.25 Proof



Since the domain is in the rectangular form, there are only three possibilities in each direction:

1. $\mathbf{x}^*$ is an interior point of domain, then $f_i(\mathbf{x}^*) = 0$ for any i since it is smooth.
2. It is at boundary and suppose that it is increasing, then $(x^i)^* = b^i$ and $f_i(\mathbf{x}^*) > 0$.
3. It is at boundary and suppose that it is decreasing, then $(x^i)^* = a^i$ and $f_i(\mathbf{x}^*) < 0$.

And these prove this theorem.

Similarly, we have the following theorem to describe the behavior as minimum of function within bounded region:

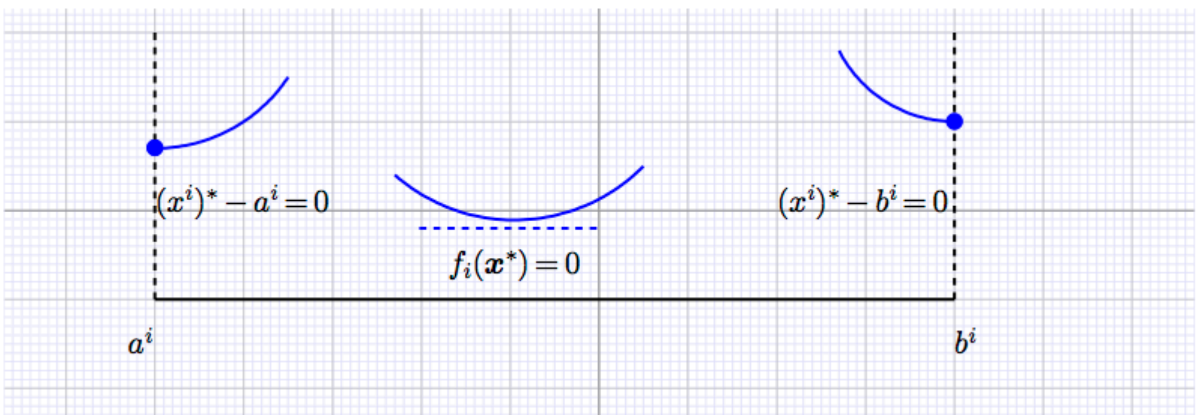
1.26 Theorem

Suppose that
$\text{Domain}(f) = \left\{\vec{x}|x_i \in [a_i, b_i], \text{ for all } 1 \leqslant i \leqslant n:\right\}$
and $f(\mathbf{x})$ is smooth with all its first order derivatives. Suppose that $f(\mathbf{x})$ attains it global minimum at $\mathbf{x} = \mathbf{x}^*$, then one or both following condition(s) must hold:

1. $f_i(\mathbf{x}^*) \geqslant 0$ and $((x^i)^* - a^i)f_i(\mathbf{x}^*) = 0$;
2. $f_i(\mathbf{x}^*) \leqslant 0$ and $(b^i - (x^i)^*)f_i(\mathbf{x}^*) = 0$

   for all $1 \leqslant i \leqslant n$.

Proof



There are only three possibilities in each direction:

1. $\mathbf{x}^*$ is an interior point of domain, then $f_i(\mathbf{x}^*) = 0$ for any i since it is smooth.
2. It is at boundary and suppose that it is decreasing, then $(x^i)^* = b^i$ and $f_i(\mathbf{x}^*) < 0$.
3. It is at boundary and suppose that it is increasing, then $(x^i)^* = a^i$ and $f_i(\mathbf{x}^*) > 0$.

   And these prove this theorem.

1.27 Example

Suppose that one factory inputs its goods from two different supply plants, A and B, with different costs, 4 and 6 each respective. And suppose the price function in the market is the same and is decided as $p(x, y) = 100 - x - y$ where x and y are the demand functions. Discuss the the maxima problem due to the following situations:

1. $0 \leqslant x, y$
2. $0 \leqslant x \leqslant 30, 0 \leqslant y \leqslant 30$

Solve

1. The profit function is

$$\begin{aligned} P(x, y) &= p(x, y)(x + y) - 4x - 6y \\ &= 100(x + y) - (x + y)^2 - 4x - 6y \end{aligned}$$
with all its first derivative:

$$P_x(x, y) = 100 - 2(x + y) - 4$$
$$P_y(x, y) = 100 - 2(x + y) - 6$$
Obviously, it is impossible to get the equations simultaneously:

$$P_x(x, y) = 0 \Longrightarrow 100 - 2(x + y) - 4 = 0$$
$$P_y(x, y) = 0 \Longrightarrow 100 - 2(x + y) - 6 = 0$$

since it is absolutely inconsistent. Does it imply no maximum for $P(x, y)$? Certainly not. Think about it: if you can input more cheaper goods, 4 each, why not do so? The first derivative test can only be used to confirm that the maximum does not occur at the point, i.e. satisfying $P_x = P_y = 0$.

In this case, we always choose to input from A with cost 4 each but not from B. This just implies:

$$y = 0 \text{ and } \; P(x, y) = 100x - x^2 - 4x$$

Then $P(x, y)$ attains its maximum at $(x, y) = (48, 0)$. Note that we have

$$\begin{aligned} \vec{x}^* = (x^*, y^*) = (48, 0) &\Longrightarrow (x^* - 0)P_x(\vec{x}^*) = 0 \text{ and } P(\vec{x}^*) = 0 \\ &\Longrightarrow P_y(x^*, y^*) = -2 - 2y^* < 0 \text{ and } (y^* - 0)P_x(\vec{x}^*) = 0 \end{aligned}$$

2. Since the it is beneficial for us to input from A, certainly x is equal to 30, the maximal output from A. This concludes

$$P(x, y) = P(30, y) = 100(30 + y) - (30 + y)^2 - 10 - 6y$$
and

$$P_x(x, y) = 0 \Longrightarrow 100 - 2(x + y) - 4 = 0$$
$$P_y(x, y) = 0 \Longrightarrow 100 - 2(x + y) - 6 = 0$$

By the first derivative test, $P(x, y)$ will attains its maxima at $x = 30$ (boundary point) and $y = 17$, coming from the fact, $P_y(30, y) = 0$. In this case,

$$\begin{aligned} \vec{x}^* = (x^*, y^*) = (30, 17) &\Longrightarrow (30 - x^*)P_x(\vec{x}^*) = 0 \text{ and } P_x(\vec{x}^*) < 0 \\ &\Longrightarrow P_y(\vec{x}^*) = 0 \end{aligned}$$

1.28 Example

Two kinds of eggs, white and brown, are sold. The daily sales for white eggs will be $W(x, y) = 30 - 15x + 3y$ and daily sales for brown eggs will be $B(x, y) = 20 - 12y + 2x$ where x, y are the sale prices for white and brown eggs respectively. Then the revenue is

$$\begin{aligned} R(x, y) &= xW(x, y) + yB(x, y) \\ &= x(30 - 15x + 3y) + y(20 - 12y + 2x) \\ &= 30x + 20y - 15x^2 + 5xy - 12y^2 \end{aligned}$$

The critical point is calculated as follows:

$$\begin{aligned} \nabla R = \vec{0} &\Longrightarrow (30 - 30x + 5y, 20 + 5x - 24y) = (0, 0) \\ &\Longrightarrow (x, y) = \left(\frac{164}{139}, \frac{150}{139} \right) \end{aligned}$$

Also the Hessian matrix is as follows:

$$H = \begin{pmatrix} -30 & 5 \\ 5 & -24 \end{pmatrix}$$

$R_1 = -30 < 0$ and $|H| > 0$. Then at $(x, y) = \left(\frac{164}{139}, \frac{150}{139}\right)$, $R(x, y)$ attains its relative maximum and maximum too.

1.29 Example

Suppose that a firm produces one kind of output, y, by two inputs, K, called capital, and L, called labor. Each unit of capital costs u and each unit of labor costs v. And the production function follows the Cobb-Douglas relation:

$$P(K, L) = AK^{\alpha}L^{\beta} \text{ where } A, \alpha \text{ and } \beta > 0$$

If the price function is constant p, find the condition at which the profit function attains its extremum and determine the condition at which profit function actually attains its maximum at the critical point.

Sol

The profit function, by definition, is

$$\begin{aligned} P(K, L) &= R(K, L) - C(K, L) \\ &= pAK^{\alpha}L^{\beta} - uK - vL \end{aligned}$$

By the first order condition, the extremum occurs at which

$$\begin{aligned} 0 &= \frac{\partial P}{\partial K} = p\alpha AK^{\alpha-1}L^{\beta} - u \\ 0 &= \frac{\partial P}{\partial L} = p\beta AK^{\alpha}L^{\beta-1} - v \end{aligned}$$

Then

$$\begin{aligned} & \frac{L}{K} && = \frac{u\beta}{v\alpha} \\ \Longrightarrow & p\alpha AK^{\alpha-1}\left(\frac{u\beta}{v\alpha}K\right)^{\beta} && = u \\ \Longrightarrow & \hat{K} && = \left(\frac{v^{\beta}\alpha^{\beta-1}}{pAu^{\beta-1}\beta^{\beta}}\right)^{\frac{1}{\alpha+\beta-1}} \\ \Longrightarrow & \hat{L} && = \left(\frac{u^{\alpha}\beta^{\alpha-1}}{pAv^{\alpha-1}\alpha^{\alpha}}\right)^{\frac{1}{\alpha+\beta-1}} \end{aligned}$$

If $P(K, L)$ attains its maxima at $(\hat{K}, \hat{L})$, then

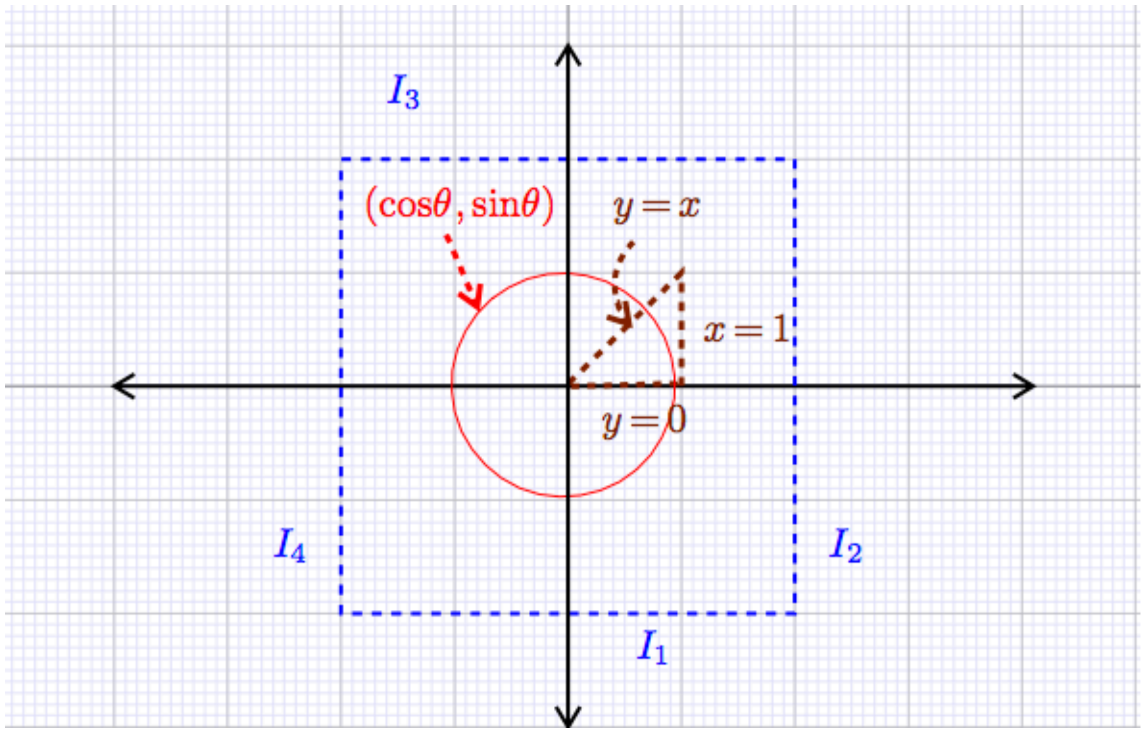
$$\begin{aligned} & |H| > 0 \text{ and } |H_1| < 0 \\ \Longrightarrow & \frac{\partial^2 P}{\partial L^2} = p\alpha(\alpha-1)AK^{\alpha-2}L^{\beta} < 0 \\ & |H| = (pA)^2\alpha(\alpha-1)\beta(\beta-1)K^{\alpha+\beta-2}L^{\alpha+\beta-2} \\ & \quad - (pA\alpha\beta)^2K^{\alpha+\beta-2}L^{\alpha+\beta-2} < 0 \\ \\ \Longrightarrow & \alpha < 1 \text{ and } \alpha + \beta < 1 \end{aligned}$$

Sometimes, what the range of domain does influences the extrema.

1.30 Example

Find the extrema of $f(x, y) = x^2 - xy + y^2 - x + y - 6$ for (x, y) within the following region Ω respectively:

1. $\Omega = \{(x, y) | x^2 + y^2 \leq 1\}$;
2. $\Omega = \{(x, y) | -2 \leq x, y \leq 2\}$;
3. $\Omega = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leqslant x\}$.



Sol: First we want to find the critical point(s) as follows:

$$\begin{aligned} \vec{0} &= (f_1, f_2) \\ &= (2x - y - 1, 2y - x + 1) \\ \Longrightarrow \quad & x = 1/3 \text{ and } y = -1/3 \end{aligned}$$

1. Critical point is within this region and $f(1/3, -1/3) = -6\frac{1}{3}$. And the possible positions for $f(x, y)$ attaining its extrema is at the boundary, $\partial\Omega = \{(x, y) | x^2 + y^2 = 1\}$, i.e. $x = \cos\theta$ and $y = \sin\theta$, $0 \leq \theta \leq 2\pi$. On the boundary,

$$\begin{aligned} f(x, y) &= f(\cos\theta, \sin\theta) \\ &= \cos^2\theta - \sin\theta\cos\theta + \sin^2\theta - \cos\theta + \sin\theta - 6 \\ &= -\sin\theta\cos\theta - \cos\theta + \sin\theta - 5 \end{aligned}$$

$$\frac{df}{d\theta} = 0 \Longrightarrow \sin^2\theta - \cos^2\theta + \sin\theta + \cos\theta = 0$$

$$\Longrightarrow (\sin\theta + \cos\theta)(\sin\theta - \cos\theta + 1) = 0$$

a. $\sin\theta + \cos\theta = 0$: $\theta = 3\pi/4$ and $7\pi/4$, this implies

$$f\left(\cos\frac{3\pi}{4}, \sin\frac{3\pi}{4}\right) = -\sin\frac{3\pi}{4}\cos\frac{3\pi}{4} - \cos\frac{3\pi}{4} + \sin\frac{3\pi}{4} - 5 = -4\frac{1}{2}$$

$$f\left(\cos\frac{7\pi}{4}, \sin\frac{7\pi}{4}\right) = -4\frac{1}{2}$$

b. $\sin\theta - \cos\theta + 1 = 0$: $\theta = 0$ and $3\pi/2$, this implies

$$f(\cos 0, \sin 0) = -6$$
$$f\left(\cos\frac{3\pi}{2}, \sin\frac{3\pi}{2}\right) = -6$$

These conclude: maximum is $-4\frac{1}{2}$ and minimum is $-6\frac{1}{3}$ at $(x, y) = (1/3, -1/3)$.

2. In this square case, $\partial\Omega$ contains four components:

a. $I_1 = \{(x, y)| - 2 \leqslant x \leqslant 2, y = -2\}$;
b. $I_2 = \{(x, y)| - 2 \leqslant y \leqslant 2, x = 2\}$;
c. $I_3 = \{(x, y)| - 2 \leqslant x \leqslant 2, y = 2\}$;
d. $I_4 = \{(x, y)| - 2 \leqslant y \leqslant 2, x = -2\}$;

- For $f(x, y)|_{I_1} = f(x, -2) = x^2 + x - 4, -2 \leqslant x \leqslant 2$:
  - maximum: 2 at $(x, y) = (2, -2)$;
  - minimum: $-4\frac{1}{4}$ at $(x, y) = (-1/2, -2)$.
- $f(x, y)|_{I_2} = f(2, y) = y^2 - y - 4, -2 \leqslant y \leqslant 2$:
  - maximum: 2 at $(x, y) = (2, -2)$;
  - minimum: $-4\frac{1}{4}$ at $(x, y) = (2, 1/2)$.
- $f(x, y)|_{I_3} = f(x, 2) = x^2 - 3x, -2 \leqslant x \leqslant 2$:
  - maximum: 10 at $(x, y) = (-2, 2)$;
  - minimum: $-9/4$ at $(x, y) = (3/2, 2)$.
- $f(x, y)|_{I_4} = f(-2, y) = y^2 + 3y, -2 \leqslant y \leqslant 2$:
  - maximum: 10 at $(x, y) = (-2, 2)$;
  - minimum: $-9/4$ at $(x, y) = (-2, -3/2)$.

These conclude: maximum is 10 at $(-2, 2)$ and minimum is $-6\frac{1}{3}$ at $(x, y) = (1/3, -1/3)$.

3. the last case, we have these conclusions: maximum is 10 at $(x, y) = (-2, 2)$ and minimum is $-6\frac{1}{3}$ at $(x, y) = (1/3, -1/3)$. Since critical point is not included. Then the extrema might attain at the boundary $\partial\Omega$ as follows:

- $I_1 = \{(x, y)|0 \leqslant x \leqslant 1, y = 0\}$;
- $I_2 = \{(x, y)|0 \leqslant y \leqslant 1, x = 1\}$;
- $I_3 = \{(x, y)|0 \leqslant x \leqslant 1, y = x\}$;

and

- $f(x, y)|_{I_1} = f(x, 0) = x^2 - x - 6, 0 \leqslant x \leqslant 1$:
  - maximum : - 6 at $(x, y) = (0, 0)$;
  - minimum : $-6\frac{1}{4}$ at $(x, y) = (1/2, 0)$.
- $f(x, y)|_{I_2} = f(1, y) = y^2 - 6, 0 \leqslant y \leqslant 1$:
  - maximum: - 5 at $(x, y) = (1, 1)$;
  - minimum: - 6 at $(x, y) = (0, 1)$.
- $f(x, y)|_{I_3} = f(x, x) = x^2 - 6, 0 \leqslant x \leqslant 1$:
  - maximum: - 5 at $(x, y) = (1, 1)$;
  - minimum: - 6 at $\}(x, y) = (0, 0)$.

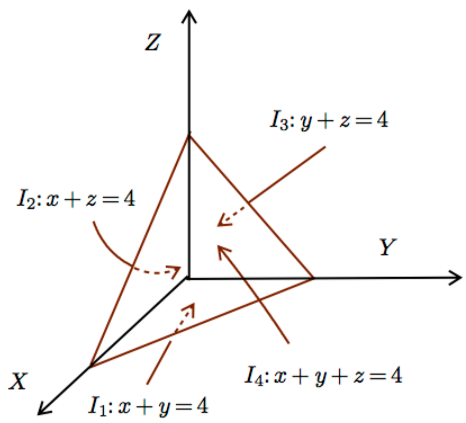
These conclude: maximum is -5 at $(x, y) = (1, 1)$ and minimum is $-6\frac{1}{4}$ at $(x, y) = (1/2, 0)$.

1.31 Example

Find the extrema of $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ for $x + y + z \leq 4$ and $0 \leqslant x, y, z$.

The critical point is $(1, 1, 1)$ (with $f(1, 1, 1) = 0$) since
$\nabla f = 2(x-1, y-1, z-1) = \vec{0}$
and it is within the domain. Since $f(x, y, z) \geqslant 0$, then $f(x, y, z)$ attain its minimum at $(1, 1, 1)$. Another positions at which $f(x, y, z)$ possibly attains its extrema (or maximum) are on the boundary:

- $I_1 : \{(x, y, 0) | x + y \leqslant 4, x, y \geqslant 0\}$;
- $I_2 : \{(x, 0, z) | x + z \leqslant 4, x, z \geqslant 0\}$;
- $I_3 : \{(0, y, z) | y + z \leqslant 4, y, z \geqslant 0\}$;
- $I_4 : \{(x, y, z) | x + y + z = 4, x, y, z \geqslant 0\}$



By the symmetry, the maximum in I_1, I_2, I_3 are the same. Therefore only I_1 and I_4 are necessarily to be considered:

1. On I_1: $g(x, y) = f(x, y, 0) = (x-1)^2 + (y-1)^2 + 1$
   - critical point is $(1, 1)$ (with $g(1, 1) = 1$) since
     $\nabla g = 2(x-1, y-1) = \vec{0}$
   - boundary points:
     - $0 \leqslant x \leqslant 4, y = 0$: $g(x, y) = (x-1)^2 + 2$ implies that maximum is 11 at $x = 4$.
     - $0 \leqslant y \leqslant 4, x = 0$: $g(x, y) = (y-1)^2 + 2$ implies that maximum is 11 at $y = 4$.
     - $x + y = 4, x, y \geqslant 0$: $g(x, y) = 2x^2 - 8x + 11, 0 \leqslant x \leqslant 4$ implies that maximum is 11 at $x = 0$ or 4.
2. On I_4: $f(x, y, z) = (x-1)^2 + (y-1)^2 + (z-1)^2$ with $x + y + z = 4, x, y, z \geqslant 0$. This can be solved the following technique, Lagrange multiplier, introduced in the following section:

Let

$$\begin{aligned} L(x, y, z, \lambda) &= f(x, y, z) + \lambda(4 - x - y - z) \\ &= (x-1)^2 + (y-1)^2 + (z-1)^2 + \lambda(4 - x - y - z) \end{aligned}$$

and

$$\nabla L = \vec{0} \Longrightarrow 2(x-1) = 2(y-1) = 2(z-1) = \lambda$$
By symmetry, $x = y = z$, then

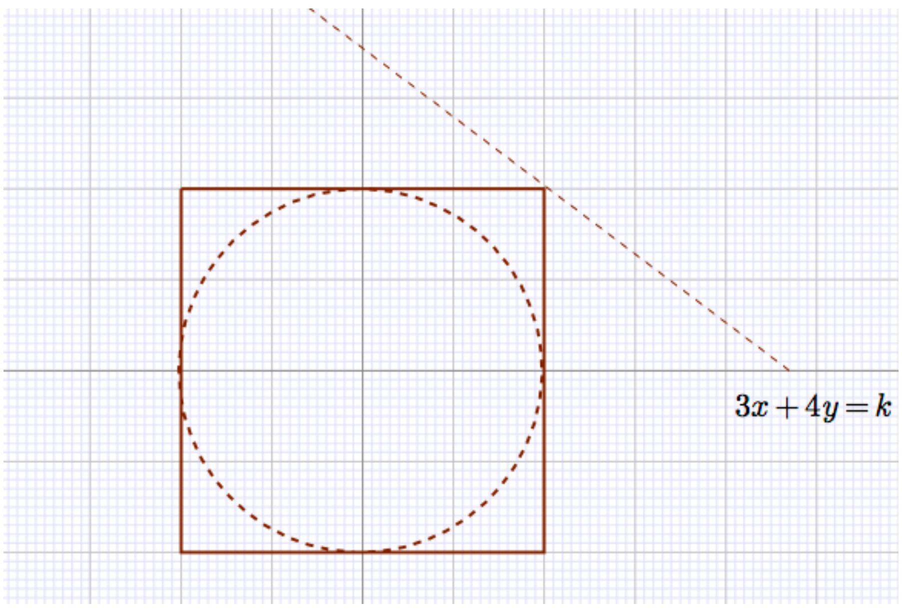
$$x + y + z = 4 \Rightarrow x = y = z = 4/3$$
This implies $f(x, y, z)$ attains its minimum $1/3$ (why?) at this point $(4/3, 4/3, 4/3)$, but not maximum.

Therefore the maximum is 11 and minimum is 0.

1.32 Exercise

Suppose that $f(x, y) = 3x + 4y$. Find the extrema of $f(x, y)$ in the following regions Ω respectively:

1. $\Omega = \{(x, y) | -2 \leqslant x, y \leqslant 2\}$;
2. $\Omega = \{(x, y) | x^2 + y^2 \leqslant 2^2\}$;



1.
$$\begin{aligned}\partial\Omega &= \{y = -2\} \cup \{x = 2\} \cup \{y = 2\} \cup \{x = -2\} \\ &= I_1 \cup I_2 \cup I_3 \cup I_4\end{aligned}$$
And the extrema appear at the ends, i.e. the four corners of the region, ($-2, -2$), $(2, -2)$, $(2, 2)$ and (-2,2). And the functions values at these points are $-14, -2, 14$ and 2. Therefore, the maximum is 14 and minimum is -14

- Since $\partial\Omega = \{x^2 + y^2 = 4\}$, consider the Lagrangian function:
  $L(x, y, \lambda) = f(x, y) + \lambda(4 - x^2 - y^2)$
  Then

$$\begin{aligned}\nabla L = \vec{0} &\Longrightarrow (3 - 2\lambda x, 4 - 2\lambda y, 4 - x^2 - y^2) = \vec{0} \\ &\Longrightarrow x = \frac{3}{2\lambda}, y = \frac{2}{\lambda} \\ &\Longrightarrow y = \frac{4}{3}x \\ &\Longrightarrow (x, y) = \left(\pm\frac{6}{5}, \pm\frac{8}{5}\right)\end{aligned}$$
Then the maximum is $f(6/5, 8/5) = 10$ and minimum is $f(-6/5, -8/5) = -10$.

1.33 Exercise

Suppose that $f(x,y)=x^2+y^2$. Find the extrema of $f(x,y)$ in the following regions Ω respectively:

1. $\Omega=\{(x,y)|-2\leqslant x,y\leqslant 2\}$;
2. $\Omega=\{(x,y)|3x+4y\leqslant 5, 0\leqslant x,y\}$;

1.

$$\begin{aligned}\partial\Omega &= \{y=-2\}\cup\{x=2\}\cup\{y=2\}\cup\{x=-2\}\\ &= I_1\cup I_2\cup I_3\cup I_4\end{aligned}$$

And the extrema appear at the ends, i.e. the four corners of the region, ($-2,-2), (2,-2), (2,2)$ and (-2,2). The functions values at these points are all 8 and $0\leqslant x^2+y^2$. Therefore, the maximum is 8 and minimum is 0.

2. Since

$$\begin{aligned}\partial\Omega &= \{y=0, 0\leqslant x\leqslant 5/3\}\cup\{x=0, 0\leqslant y\leqslant 5/4\}\cup\{3x+4y=5\}\\ &= I_1\cup I_2\cup I_3\end{aligned}$$

Consider the Lagrangian function:
$L(x,y,\lambda)=f(x,y)+\lambda(5-3x-4y)$
Then

1. $(x,y)\in I_1$: $f(x,y)=x^2$ and $0\leqslant x\leqslant 5/3$. Then maximum is $25/9$ and minimum is 0.
2. $(x,y)\in I_2$: $f(x,y)=y^2$ and $0\leqslant y\leqslant 5/4$. Then maximum is $25/16$ and minimum is 0.
3. $(x,y)\in I_3$

$$\begin{aligned}\nabla L=\vec{0} &\Longrightarrow (2x-3\lambda x, 2y-4\lambda y, 5-3x-4y)=\vec{0}\\ &\Longrightarrow x=\frac{2}{3\lambda}, y=\frac{1}{2\lambda}\\ &\Longrightarrow y=\frac{3}{4}x\\ &\Longrightarrow (x,y)=\left(\frac{5}{6},\frac{5}{8}\right)\end{aligned}$$

Then the value of $f(x,y)$ is $625/576$ and is the minimum on I_3 (why?). These conclude that minimum is 0 and maximum is 25/9.

1.34 P.1125

Classify the types of extrema or saddle point of $f(x,y)$:
*14. * $f(x,y)=xy((3-x-y)$

1. $\nabla f = (y(3-x-y) - xy, x(3-x-y) - xy) = (y(3-2x-y), x(3-x-2y))$
2. Hessian matrix
$$H = \begin{pmatrix} -2y & 3-2x-2y \\ 3-2x-2y & -2x \end{pmatrix}$$
3. critical values:
   a. $y = 0$ implies $x = 0$ or $3 - x = 0$, i.e. $(0,0), (3,0)$; $(H < 0)$
   b. $3 - 2x - y = 0$ implies

   - $x = 0$ and $y = 3$, $(H < 0)$
   - $3 - x - 2y = 0$ implies $(x, y) = (1, 1)$ $(H > 0, a < 0)$

16. * $f(x, y) = 4y/(x^2 + y^2 + 1)$

1. $\nabla f = (-8xy/(x^2 + y^2 + 1)^2, (4x^2 - 4y^2 + 4)/(x^2 + y^2 + 1)^2) = (0, 0)$
2. Hessian matrix
$$H = \frac{8}{(3x^2 + y^2 + 1)^3} \begin{pmatrix} y(x^2 - 1 - y^2) & x(3y^2 - 1 - x^2) \\ x(3y^2 - x^2 - 1) & y(x^2 - 3y^2 + 1) \end{pmatrix}$$
3. critical values: $x = 0$ implies $y = \pm 1$, i.e. $(0, 1), (0, -1)$; $(H > 0)$
   a. $(x, y) = (0, 1)$: $H > 0, a < 0$ b. $(x, y) = (0, -1)$: $H > 0, a > 0$

36. * Find the absolute extrema of $f(x, y) = 3x^2 + 2xy + y^2$ on the triangle with vertices $(-2, -1), (1, -1), (1, 2)$.

These conclude: maximum is 17, munimum is 0.

40. * Find the absolute extrema of $f(x, y) = 4x^2 + 2x + y^2 - y$ on the elipse $4x^2 + y^2 \leq 1$.

These conclude: maximum is $1 + \sqrt{2}$, munimum is -0.5.

44. Find the point on the surface $xy^2z = 4$ that are closest to the origin and what is the shortest distance between them?

62. Let $f(x, y) = x^2 - y^2 + 2xy + 2$.

1. no extrema since $f(x, y) \to \infty$, if $x \to \infty, y = 0$, and $f(x, y) \to -\infty$, if $y \to \infty, x = 0$.
2. Find extrema on $D = \{x^2 + 4y^2 \leq 4\}$

1.35 Note

For $(x, y) \in \partial D$,

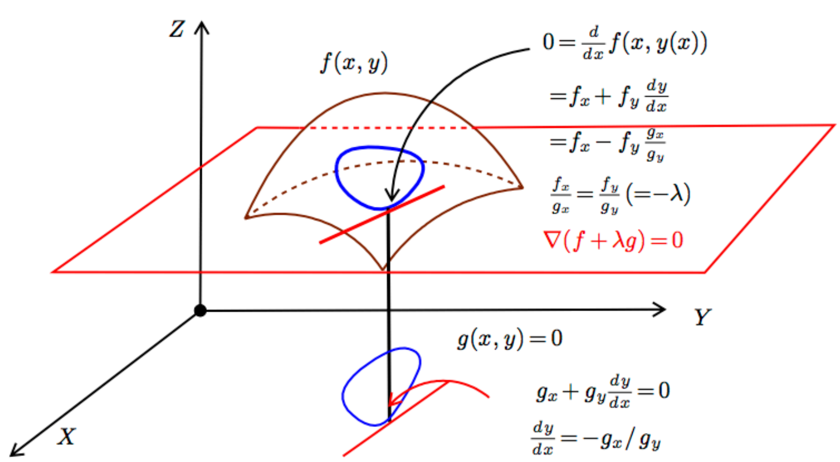
$$
\begin{aligned}
f(x, y) &= f(2\cos t, \sin t) \\
&= 4\cos^2 t - \sin^2 t + 4\sin t\cos t + 2 \\
&= 2(1+\cos 2t) - \frac{1}{2}(1-\cos 2t) + 2\sin 2t + 2 \\
&= \frac{5}{2}\cos 2t + 2\sin 2t + \frac{3}{2} \\
\Rightarrow \quad & -\sqrt{(5/2)^2 + 2^2} + \frac{3}{2} \le f \le \sqrt{(5/2)^2 + 2^2} + \frac{3}{2}
\end{aligned}
$$

1.36 Optimization Problem with Constraints

1.37 Theorem

If a relative extrema of $f(x, y)$ and $g(x, y) = 0$ occurs at (x, y), then there exists a λ for which (x, y, λ) is the critical point, (a, b), of $L = f(x, y) + \lambda g(x, y)$, i.e.:
$\nabla f(a, b) + \lambda \nabla g(a, b) = \mathbf{0}$

This function is called **Lagrangian** of $f(x)$ and $g(x)$.



In real world, Lagrangian is more realistic than other kinds of optimization problem: finding extrema under resources limited.

For general case, $\mathbf{x} \in \mathbb{R}^n$, the extrema of $f(\mathbf{x}$ with constraint $g(\mathbf{x} = 0$ at $\mathbf{x}_0$ should satisfy the following:
$\nabla f(\mathbf{x}_0) + \lambda \nabla g(\mathbf{x}_0) = \mathbf{0}$
where $\lambda \neq 0$ and $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$.

1.38 Example

Suppose that the Cobb-Douglas function is
$P(x, y) = 100x^{1/4}y^{3/4}$
where x is the unit of labor and y is the unit of capital. Each unit of labor costs 200 and each of capital costs 100. If the total of 800 worth of labor and capital is to be used, Find the maximum of $P(x, y)$.

First note:

the critical point at which $P(x, y)$ attains it maximum is also the critical point for $\ln P(x, y)$ that attains its maximum.

Since the constraint is

$200x + 100y = 800$
consider the Lagrangian function:

$$\begin{aligned} L(x, y, \lambda) &= \ln P(x, y) + \lambda(800 - 200x - 100y) \\ &= \ln 100 + \frac{1}{4}\ln x + \frac{3}{4}\ln y + \lambda(800 - 200x - 100y) \end{aligned}$$

Note: The critical value of $P(x, y)$ is also critical value of $\ln P(x, y)$!}

Now the critical point(s) is as follows:

$$\begin{aligned} \vec{0} &= \nabla L(x, y, \lambda) \\ &= \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial \lambda}\right) \\ &= \left(\frac{1}{4x} - 200\lambda, \frac{3}{4y} - 100\lambda, 800 - 200x - 100y\right) \\ \Longrightarrow \quad & x = \frac{1}{800\lambda}, y = \frac{3}{400\lambda}(\text{ i.e. } y = 6x) \\ \Longrightarrow \quad & x = 1 \text{ and } y = 6 \end{aligned}$$

$(x, y) = (1, 6)$ is the only one critical point. Since $0 \leqslant x, y \leq 8$, $P(x, y)$ has to be a maximum in such closed region. Therefore, $P(x, y)$ attains it maximum at $(1, 6)$.

1.39 Implement Python Lagrange's Multiplier solver

1.40 Example

Suppose you wishes to allocate your available time of 16 hours in this week between quizzes, English and Calculus, held in the next week. What would you do in such way to maximize your grade average?

Solution

Suppose that

1. $f(t_1) = 20 + 20\sqrt{t_1}$: the time will be spent for English course with t_1 hour per week,
2. $g(t_2) = 50 + 3t$: the time will be spent for Calculus course with t_2 hour per week.

Then the problem is turned to be:

$$\text{Maximize } S(t_1, t_2) = \frac{f(t_1) + g(t_2)}{2}$$
$$\text{subject to} \quad t_1 + t_2 = 16$$

Consider the Lagrangian:
$L(t_1, t_2) = 20 + 20\sqrt{t_1} + 50 + 3t_2 + \lambda(16 - t_1 - t_2)$
The extremum occurs at the place that satisfies:

$$\frac{\partial L}{\partial t_1} = 0, \frac{\partial L}{\partial t_2} = 0, \frac{\partial L}{\partial \lambda} = 0.$$

And these imply:

$$\frac{10}{\sqrt{t_1}} - \lambda = 0, \ 3 - \lambda = 0$$
$$\Longrightarrow t_1 = \left(\frac{10}{3}\right)^2 \quad \text{and } t_2 = 16 - \left(\frac{10}{3}\right)^2$$

How do I know at which the maximum occurs? Since $S(t_1, t_2)$ is continuous for both t_1 and t_2 are in bounded intervals, the function obtains its maximum and minimum. Comparing with the function values at boundary, $S(t_1, t_2)$ will obtains its maximum at $(t_1, t_2) = (1, 15)$ with the constraint $t_1 + t_2 = 16$.

1.41 Theorem (Heron's formulae)

Suppose that The lengths of sides of triangle are x, y and z respectively and $x + y + z = l$. Then the area of this triangle is $\sqrt{s(s-x)(s-y)(s-z)}$ where $s = l/2$.

1.42 Example

Suppose that the perimeter is 24. Find the dimension of this triangle such that it owns maximum area.

Suppose x, y, z are the lengths of sides of this triangle and let A be its area. Then

$$\text{Maximize } A = \sqrt{12(12-x)(12-y)(12-z)} \text{ with } x + y + z = 24$$
$$\Longrightarrow \text{Maximize } 12(12-x)(12-y)(12-z) \text{ with } x + y + z = 24$$
$$\Longrightarrow \text{Maximize } L(x, y, z, \lambda) = \ln(12(12-x)(12-y)(12-z)) + \lambda(24 - x - y - z)$$

Since
$L(x, y, z, \lambda) = \ln 12 + \ln(12 - x) + \ln(12 - y) + \ln(12 - z) + \lambda(24 - x - y - z)$
then the critical value of $L(x, y, z, \lambda)$ can be found by the following steps:

$$
\begin{aligned}
\vec{0} &= \nabla L(x, y, z, \lambda) \\
&= \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial z}, \frac{\partial L}{\partial \lambda}\right) \\
&= \left(-\frac{1}{12-x} - 1, -\frac{1}{12-y} - 1, -\frac{1}{12-z} - 1, 24 - x - y - z\right)
\end{aligned}
$$

This implies that $x = y = z = 8$, i.e. this triangle is equilateral.

1.43 Exercise

Assume that $P(x, y) = 100x^{1/4}y^{3/4}$. Then $\frac{\partial P}{\partial x} = \frac{25y^{\frac{3}{4}}}{x^{\frac{3}{4}}}$ and $\frac{\partial P}{\partial y} = \frac{75x^{\frac{1}{4}}}{y^{\frac{1}{4}}}$.

1. $\frac{\partial P}{\partial x}(1, 6) = 25\sqrt[4]{6^3}$
2. $\frac{\partial P}{\partial y}(1, 6) = 75/\sqrt[4]{6}$
3. $\frac{\partial P}{\partial x}(1, 6)/\frac{\partial P}{\partial y}(1, 6) = 25\sqrt[4]{6^3}/(75/\sqrt[4]{6}) = 6/3 = 2$
4. Let $F(x, y) = P(x, y) + \lambda(80 - 20x - 10y)$. Then

$$
0 = \frac{\partial F}{\partial x} = \frac{25y^{\frac{3}{4}}}{x^{\frac{3}{4}}} - 20\lambda \text{ and } 0 = \frac{\partial F}{\partial y} = \frac{75x^{\frac{1}{4}}}{y^{\frac{1}{4}}} - 10\lambda
$$

These imply $y = 6x$. Therefore $F(x, y)$ attains its maximum under the constraint $20x + 10y = 80$ as $x = 10$ and $y = 60$.

1.44 Example

Find the extrema of $f(x, y) = x^2 - 2y$ subject to $x^2 + y^2 = 9$.

1. Let $L = f + \lambda(9 - x^2 - y^2) = x^2 - 2y + \lambda(9 - x^2 - y^2)$.

2.

$$
\begin{aligned}
\vec{0} &= \nabla L(x, y, \lambda) \\
&= \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial \lambda}\right) \\
&= \left(2x - 2\lambda x, -2 - 2\lambda y, 9 - x^2 - y^2\right)
\end{aligned}
$$

3.

- $x = 0 \to 0^2 + y^2 = 9 \to y = \pm 3$
- $\lambda = 1 \to y = -1 \to x = \pm 2\sqrt{2}$
- $f(0, -3) = 6, f(0, 3) = -6(\text{min}), f(\pm 2\sqrt{2}, -1) = 10(\text{max})$

1.45 Example

Find the extrema of $f(x, y) = 2x^2 + y^2 - 2y + 1$ subject to $x^2 + y^2 \leq 4$.

0. critical point:
$\nabla f = (0, 0) \rightarrow (x, y) = (0, 1)$
and $f(0, 1) = 0$.

1. Let $L = f + \lambda(4 - x^2 - y^2) = 2x^2 + y^2 - 2y + 1 + \lambda(4 - x^2 - y^2)$.
2.

$$\begin{aligned}
\vec{0} &= \nabla L(x, y, \lambda) \\
&= \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial \lambda}\right) \\
&= \left(4x - 2\lambda x, 2y - 2 - 2\lambda y, 4 - x^2 - y^2\right)
\end{aligned}$$

3.

- $x = 0 \rightarrow 0^2 + y^2 = 4 \rightarrow y = \pm 2$
- $\lambda = 2 \rightarrow y = -1 \rightarrow x = \pm\sqrt{3}$
- $f(0, 1) = 0$ (minimum), $f(0, -2) = 9$, $f(0, 2) = 1$, $f(\pm\sqrt{3}, -1) = 10$ (maximum).

1.46 Example (With two constraints)

Find the extrema of $f(x, y, z) = 3x + 2y + 4z$ subject to $x - y + 2z = 1$ and $x^2 + y^2 = 4$.

1. Let
$L = f + \lambda(1 - x + y - 2z) + \mu(4 - x^2 - y^2) = 3x + 2y + 4z + \lambda(1 - x + y - 2z)$
2. Find critical point(s):

$$\begin{aligned}
\vec{0} &= \nabla L(x, y, z, \lambda, \mu) \\
&= \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial z}, \frac{\partial L}{\partial \lambda}, \frac{\partial L}{\partial \mu}\right) \\
&= (3 - \lambda - 2\mu x, 2 + \lambda - 2\mu y, 4 - 2\lambda, \ldots, \ldots)
\end{aligned}$$

3.

- $\lambda = 2 \rightarrow x = 1/(2\mu)$ and $y = 2/\mu$
- $x^2 + y^2 = 4 \rightarrow \mu = \pm\sqrt{17}/4 \rightarrow (x, y, z) = \left(\pm 2/\sqrt{17}, \pm 8/\sqrt{17}, \frac{1}{2}\left(1 \pm \frac{6}{\sqrt{17}}\right)\right)$
- Maximum $f\left(2/\sqrt{17}, 8/\sqrt{17}, \frac{1}{2}\left(1 + \frac{6}{\sqrt{17}}\right)\right) = 2(1 + \sqrt{17})$,
- Minimum $f\left(-2/\sqrt{17}, -8/\sqrt{17}, \frac{1}{2}\left(1 - \frac{6}{\sqrt{17}}\right)\right) = 2(1 - \sqrt{17})$.

1.47 Exercise

1. Find the extrema of $f(x, y) = xy^2$ with $2x^2 + y^2 = 12$.
2. Find the extrema of $f(x, y, z) = 10x + 2y + 6z$ with $x^2 + y^2 + z^2 = 35$.

1.48 Exercise p1138

10. Find the extrema of $f(x, y) = x^2 + y^2$ with $x^4 + y^4 = 1$.

14. Find the extrema of $f(x, y) = x^2 + y^2 + z^2$ with $y - x = 1$.

Since $x^2 + y^2 + z^2 \geq 0$, only minimum attains and at $(x, y, z) = (-1/2, 1/2, 0)$, i.e. min= $1/2$.

18. Find the extrema of $f(x, y) = x + y + z$ with $x^2 + y^2 = 1$ with $x + z = 2$.

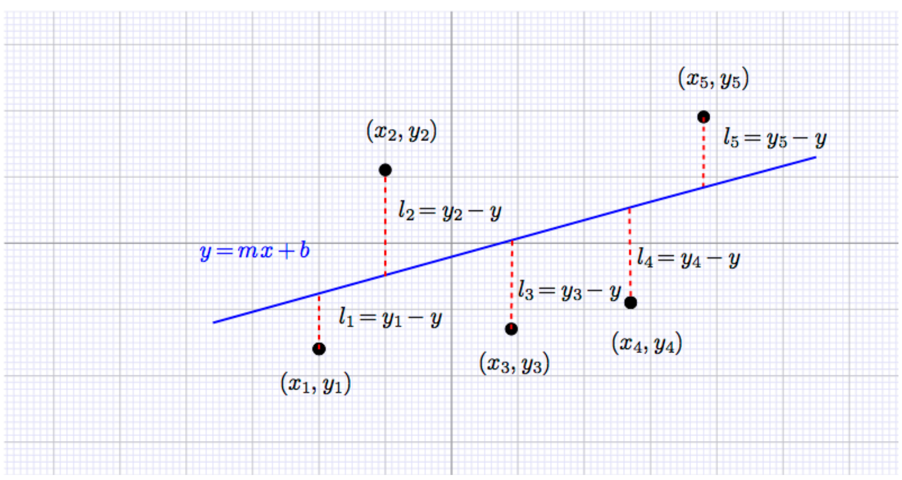
22. Find the extrema of $f(x, y) = x^2y$ with $4x^2 + y^2 \leq 4$.

Actually, set of critical points is $(x, y) = \{(0, y), y \in (-2, 2)\}$, infinite points included.

After all, Maximum is $4\sqrt{3}/9$, and Minimum is $-4\sqrt{3}/9$.

1.49 The Method of Least Squares

Suppose that there are n paired data, $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, that were observed in one experiment. Can we find the approximate relation between $X = x_i$ and $Y = y_i$?



The real relation between X and Y can not be found out even by rigours theory. Thus the approximation will be a good replaced solution. There are still some problems that have to be considered:

1. Which relation do we want? The simple linear relation, i.e. $Y = mX + b$, is a good suggestion;
2. Since the relation is an approximation, the error can not be ignored. As shown in the last picture:
   $l_1, l_2, \cdots, l_5$ errors

The linear approximation will be a good candidate if the sum of errors

$E = l_1 + \cdots + l_5$
is minimum! But different signs of errors will reduce the sum of errors. This means that the sum of errors is very small but the differences between exact values and observed data are very large. Therefore we can consider to minimize the sum of the square of errors:

$$E' = l_1^2 + \cdots + l_5^2$$

The following famous theorem given by Gauss describes how to predict the relation from the data come from the real world:

1.50 Theorem

The line $l = mx + b$ that best fits the data points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ is the line for which sum of the sum of square errors
$E_1 + E_2 + \cdots + E_n, \text{ where } E_i = (y_i - mx_i - b)^2$
is a minimum and

$$m = \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2}, b = \frac{\overline{x^2}\,\bar{y} - \bar{x}\,\overline{xy}}{\overline{x^2} - \bar{x}^2}$$
Here, we use the notation of "average of variable", $\bar{\cdot}$, to represent the sum of relative variables, for example,

$$\overline{xy} = \frac{1}{n}\sum_{k=1}^{n} x_k y_k$$

Proof

Since the sum of total error, E_i , is sum of (positive) square of errors, it must attain its minimum, i.e.

$$E = \sum_{k=1}^{n} E_i = \sum_{k=1}^{n} (y_i - mx_i - b)^2$$
$\min E$ exists since the last sum is sum of positive squares. Then the critical value satisfies

$$0 = \frac{\partial E}{\partial m} = -2\sum_{k=1}^{n} x_i(y_i - mx_i - b)$$

$$0 = \frac{\partial E}{\partial b} = -2\sum_{k=1}^{n}(y_i - mx_i - b)$$

And these can be reduced into the following linear system equations:

$$m\sum_{k=1}^{n} x_k^2 + b\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k x_k \longrightarrow {\color{red}m}\overline{x^2} + {\color{red}b}\bar{x} = \overline{xy}$$

$$m\sum_{k=1}^{n} x_k + nb = \sum_{k=1}^{n} y_k \longrightarrow {\color{red}m}\overline{x} + {\color{red}b} = \overline{y}$$

And by the general procedure of solving linear system of equations:

$$\begin{aligned} m &= \frac{\begin{vmatrix} \overline{xy} & \bar{x} \\ \overline{y} & 1 \end{vmatrix}}{\begin{vmatrix} \overline{x^2} & \bar{x} \\ \overline{x} & 1 \end{vmatrix}} \\ &= \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2} \\ b &= \frac{\begin{vmatrix} \overline{x^2} & \overline{xy} \\ \overline{x} & \overline{y} \end{vmatrix}}{\begin{vmatrix} \overline{x^2} & \bar{x} \\ \overline{x} & 1 \end{vmatrix}} \\ &= \frac{\overline{x^2}\,\bar{y} - \bar{x}\,\overline{xy}}{\overline{x^2} - \bar{x}^2} \end{aligned}$$

we have the result.

1.51 Example

The discount rate (in $\%$) for 5 months beginning in June of a given year in U.S.A. is as following table:

June	July	August	September	October
14	13	10	10	9

1. Find the linear approximation for discount rate with respect to time (month), i.e. the least square regression.
2. Predict the discount rates in November and December.

Sol: Let x_i be the number of i-months after June with y_i be the discount rate during this month, then we have

$$\sum_{i=1}^{5} x_i = 10; \sum_{i=1}^{5} y_i = 56; \sum_{i=1}^{5} x_i^2 = 30; \sum_{i=1}^{5} x_i y_i = 99.$$

*1. *

$$y = mx + b = \frac{5 \cdot 99 - 10 \cdot 56}{5 \cdot 30 - 10^2} x + \frac{30 \cdot 56 - 10 \cdot 99}{5 \cdot 30 - 10^2} = -\frac{13}{10} x + \frac{69}{5}$$

2. Since Nov. and Dec. x' s are 5 and 6 respectively, we have

$$y_{\text{Nov}} = -\frac{13}{10}5 + \frac{69}{5} = 7.3\%;$$
$$y_{\text{Dec}} = -\frac{13}{10}6 + \frac{69}{5} = 6\%$$

1.52 Example

Suppose the last 7 records of average week stock price for certain a company (PRIME VIEW INTERNATIONAL CO., LTD.) in Taiwan is listed as follows:

Week	$3/26 \sim 3/30$	$4/2 \sim 4/6$	$4/9 \sim 4/13$	$4/16 \sim 4/20$	$4/23 \sim 4/27$	$4/30$
Price	17.8	19.6	21.9	20.85	23.0	23.6

1. Find the linear approximation of the relation between time (in week) and stock price.
2. Predict the stock price that will be in the next week (i.e. $5/14 \sim 5/18$). (The real value is 25.85)

Sol:

Suppose that (x_i, y_i) be the paired (week,price) record for $i = 1, 2, 3, 4, 5, 6, 7$. Then
$$\sum_{i=1}^{7} x_i = 28, \sum_{i=1}^{7} y_i = 151.15, \sum_{i=1}^{7} x_i^2 = 140, \sum_{i=1}^{7} x_i y_i = 642.5$$

1.

$$\begin{aligned} m &= \frac{\overline{xy} - \bar{x}\bar{y}}{\overline{x^2} - \bar{x}^2} \\ &\sim 1.032 \\ b &= \frac{\overline{x^2}\bar{y} - \bar{x}\overline{xy}}{\overline{x^2} - \bar{x}^2} \\ &\sim 17.46 \end{aligned}$$

This means
$y = 1.032x + 17.46$

2. The price in the next week will be approximately estimated as:

$$1.032 \cdot 8 + 17.46 = 25.72$$

Comparing with the real value (25.85), the error is within 1%!

1.53 Exercise

Find the least square approximations for the following data:

1. $(x_{i,} y_i) = (0, 20), (2, 24), (4, 25), (6, 32), (8, 34)$ for i=1,2,3,4,5
2. $(x_{i,} y_i) = (0, 160), (1, 164), (2, 168), (3, 171), (4, 175)$ for i=1,2,3,4,5

Answer

1.
$$\sum_{i=1}^{5} x_i = 20; \sum_{i=1}^{5} y_i = 135; \sum_{i=1}^{5} x_i^2 = 120; \sum_{i=1}^{5} x_i y_i = 612.$$
Therefore
$$y = mx + b = \frac{9}{5}x + \frac{99}{5} \text{ and } y_{1994} = \frac{189}{5} y_{2000} = \frac{243}{5}$$
2. $\sum_{i=1}^{5} x_i = 10; \sum_{i=1}^{5} y_i = 838; \sum_{i=1}^{5} x_i^2 = 30; \sum_{i=1}^{5} x_i y_i = 1713.$ Therefore
$y = mx + b = \frac{37}{10}x + \frac{801}{5}$ and $y_{1995} = y|_{x=5} = 178.7$ $y_{1998} = y|_{x=5.6} = 180.92$