5. Techniques And Applications of Integration

Review

Review the definition of Definite Integral:

$$|S| = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

In another words, sum of infinite constants can be evaluated by the integration operation.

Example (Length of Roll Toilet Paper)

Consider a solid roll toilet paper with radius b. Then area of its side surface is about $b^2\pi$

The area can be approximated by the following sum:

$$\sum_{k=1}^{n} \pi((k\Delta x)^2 - \pi((k-1)\Delta x)^2)$$

where $\Delta x = b/n$. And this sum can be evaluated by the following:

$$\begin{aligned}
&\sum_{k=1}^{n} \pi((k\Delta x)^2 - \pi((k-1)\Delta x)^2) \\
=\ &\sum_{k=1}^{n} \pi(2k-1)(\Delta x)^2 \\
=\ &\sum_{k=1}^{n} \pi[(2k-1)\Delta x]\Delta x \\
=\ &\sum_{k=1}^{n} 2\pi \left[\frac{(2k-1)}{2}\Delta x\right] \Delta x \\
\xrightarrow{n\to\infty}\ &\int_0^b 2\pi x dx = \pi b^2
\end{aligned}$$

Here we use the fact:

$$\begin{array}{ccccccccccccc}
x_0 & & x_1 & & x_2 & \cdots & x_{k-1} & & x_k & \cdots & x_{n-1} & & x_n \\
0 & & \Delta x & & 2\Delta x & \cdots & (k-1)\Delta x & & k\Delta x & \cdots & (n-1)\Delta x & & n\Delta x \\
& \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow & \\
& \frac{\Delta x}{2} & & \frac{3\Delta x}{2} & & \cdots & & \frac{(2k-1)\Delta x}{2} & & & & \frac{(2n-1)\Delta x}{2} & \\
& x_1^* & & x_2^* & & & & x_k^* & & & & x_n^* &
\end{array}$$

where $\Delta x = b/n$ and $\{x_k^*\}$ is the middle points in each sub-interval, $[x_{k-1}, x_k]$.

After expanding the roll paper, the area is also equal to $\Delta x \cdot L$. This says: the length, L, is $b^2\pi/\Delta x$.

Integration by Parts

1154

$$
\begin{aligned}
(f(x)\cdot g(x))' &= f'(x)\cdot g(x) + f(x)\cdot g'(x) \\
&\Downarrow \\
f'(x)\cdot g(x) &= (f(x)\cdot g(x))' - f(x)\cdot g'(x) \\
&\Downarrow \text{ Integrating both sides} \\
\int f'(x)g(x)dx &= f(x)\cdot g(x) - \int f(x)g'(x)dx
\end{aligned}
$$

And

$$
\begin{aligned}
\int_a^b f'(x)g(x)dx &= f(x)\cdot g(x)|_a^b - \int_a^b f(x)g'(x)dx \\
&= f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx
\end{aligned}
$$

This technique integration is called **integration by parts**.

What kind of functions need to be integrated by this method? In general, if the integrand is the product of any two types of the following functions, it does need:

$$x^r, e^{ax}, \sin bx \text{ or } \cos bx, \ln x, \sin^{-1} x \text{ or } \tan^{-1} x \text{ etc.}$$

Example

$(g(x) = x^n$ and $f'(x) = e^{ax})$

$$\int xe^{-x}dx$$

$$g'(x) = x' = 1$$

$$\int e^{-x}dx = -e^{-x} + c$$

$$
\begin{aligned}
&= -xe^{-x} + \int 1\cdot e^{-x}dx \\
&= -xe^{-x} - e^{-x} + c
\end{aligned}
$$

Note

$$\int x^n e^{ax}dx = e^{ax}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) + C$$

Example

\begin{eqnarray*} \int x^{} e^{- x} d x & = & e^{- x} (a_1 x + a_0) + C\ & \Downarrow & \ x^{} e^{- x} & = & - e^{- x} (a_1 x + a_0) + a_1 e^{- x}\ & \Downarrow & \ \begin{array}{l}

```
- a_1 = 1\\
a_1 - a_0 = 0
```

\end{array} & \Rightarrow & a_1 = a_0 = - 1 \end{eqnarray*} Then

$$\int xe^{-x}dx \quad = -xe^{-x} - e^{-x} + c$$

Example

Suppose that $\Gamma(n)$ function is defined as follows:

$$\begin{aligned}\Gamma(n) &= \int_0^\infty x^{n-1}e^{-x}dx \\ &= (n-1)\Gamma(n-1) \\ \Gamma(1) &= 0! = 1 \\ \Gamma(n) &= (n-1)! \text{if} n = 1,2,3,\cdots\end{aligned}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

We need: 1.

$$\begin{aligned}\Gamma(1) &= \int_0^\infty x^{1-1}e^{-x}dx \\ &= \frac{-1}{e^x}\Big|_0^\infty \\ &= -\left(\frac{1}{e^\infty} - 1\right) = 1\end{aligned}$$

- For $n = 1, 2, 3, \cdots$:

$$\begin{aligned}\Gamma(n) &= \int_0^\infty x^{n-1}e^{-x}dx \\ &= x^{n-1}\cdot(-e^{-x})|_0^\infty - (n-1)\int_0^\infty x^{n-2}(-e^{-x})dx \\ &= 0 + (n-1)\int_0^\infty x^{n-2}e^{-x}dx \\ &= (n-1)\Gamma(n-1)\end{aligned}$$

- for $n = 1, 2, 3, \cdots$:

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) \\ &= (n-1)(n-2)\Gamma(n-2) \\ &= \cdots \\ &= (n-1)\cdots 3\cdot 2\cdot\Gamma(2) \\ &= (n-1)\cdots 3\cdot 2\cdot 1\cdot\Gamma(1) = (n-1)!\end{aligned}$$

Example

(x^r and $\ln x$)

$$\int x^{1/2}\ln x dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{3/2}\frac{1}{x}dx = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$$

Example

$$\int x \ln x dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

Example

($g(x) = x^n$ and $f'(x) = \sin bx$ or $\cos bx$)

$$\begin{aligned}\int x^2 \sin x dx &= -x^2 \cos x + 2\int x \cos x dx \\ &= -x^2 \cos x + \left(2x \sin x - \int 2 \cdot \sin x dx\right) \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + c\end{aligned}$$

Example

(e^{ax} and $\sin bx$ or $\cos bx$)

```
$$ \begin{array}{ll} I = \int e^x \color{brown}{\sin x} d x = e^x \color{brown}{\sin x}

      - \int e^x \color{brown}{\cos x} d x &
    \end{array} $$
```

And in the similar way, we have

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$$

From these facts, we conclude:

$$\begin{aligned} I &= e^x \sin x - \int e^x \cos x dx \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x dx (= I) \\ &\Downarrow \\ 2I &= e^x \sin x - e^x \cos x + C \\ &\Downarrow \\ I &= \frac{1}{2}(e^x \sin x - e^x \cos x) + C \end{aligned}$$

Note

For such type of integration, we also have the following result:

$$\int e^{ax} \cdot \begin{matrix} \sin bx \\ \cos bx \end{matrix} dx = e^{ax}(A \sin bx + B \cos bx) + C$$

Back to our example:

$$\int e^x \cos x dx = e^x (A \sin x + B \cos x) + C$$

Differentiating both sides and comparing the coefficients get the following relations:

$$\begin{aligned} e^x \cos x &= e^x (A \sin x + B \cos x) + e^x (A \cos x - B \sin x) \\ &\Downarrow \\ A - B &= 0 \\ A + B &= 1 \\ &\Downarrow \\ A = 1/2 \text{and } B &= 1/2 \end{aligned}$$

Example

($\sin^{-1}$ or $\tan^{-1}$)

$$\int \sin^{-1} x dx = \int 1 \cdot \sin^{-1} x dx$$
$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$
$$= x + \sin^{-1} x + \sqrt{1-x^2} + C$$

Example

Integrate the following integrals:

1. $\int xe^{2x} dx$
2. $\int x^2 \cos x dx$
3. $\int x^2 \ln x dx$
4. $\int e^x \cos 2x dx$
5. $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Sol:

1. Let $f(x) = x$ and $g'(x) = e^{2x}$. Then

$$\int xe^{2x} dx = \frac{1}{2} xe^{2x} - \frac{1}{2} \int 1 \cdot e^{2x} dx$$
$$= \frac{1}{2} xe^{2x} - \frac{1}{4} e^{2x} + C$$

2. Let $f(x) = x^2$ and $g'(x) = \cos x$. Then

$$\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx$$
$$= x^2 \sin x + 2x \cos x - 2 \int \cos x dx$$
$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

3. Since the elementary integration formula don't include logarithmic function, let $f(x) = \ln x$ and $g'(x) = x^2$. Then

$$\int x^2 \ln x dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^3 \cdot \frac{1}{x} dx$$
$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C$$

4. The integration about product of exponential functions and sine function is a special case. Any type can be chosen as $f(x)$. So let $f(x) = \cos 2x$ and $g'(x) = e^x$.

$$\int e^x \cos 2x dx = e^x \cos 2x + 2 \int e^x \sin 2x dx$$
$$= e^x \cos 2x + 2e^x \sin 2x - 4 \int e^x \cos 2x dx$$

Moving the integral in right side to the left side gets:

$$\int e^x \cos 2x dx = \frac{1}{5} e^x \cos 2x + \frac{2}{5} e^x \sin 2x + C$$

5. Let $f(x) = \sin^{-1} x$ and $g'(x) = x/\sqrt{1-x^2}$:

$$\int \frac{x}{\sqrt{1-x^2}} \cdot \sin^{-1} x dx = -\sqrt{1-x^2} \sin^{-1} x + \int \frac{\cancel{\sqrt{1-x^2}}}{\cancel{\sqrt{1-x^2}}} dx$$
$$= -\sqrt{1-x^2} \sin^{-1} x + x + C$$

In [4]:
```
Int(x*exp(2*x))
```

```
The indefinite integral of  ∫ x*exp(2*x) dx is
             2⋅x
(2⋅x - 1)⋅e
────────────
     4
```

In [5]:
```
Int(x**2*cos(x))
```

```
The indefinite integral of  ∫ x**2*cos(x) dx is
 2
x ⋅sin(x) + 2⋅x⋅cos(x) - 2⋅sin(x)
```

In [6]:
```
Int(x**2*log(x))
```

```
The indefinite integral of  ∫ x**2*log(x) dx is
 3          3
x ⋅log(x)   x
───────── - ──
    3       9
```

In [7]:
```
Int(exp(x)*cos(2*x))
```

```
The indefinite integral of  ∫ exp(x)*cos(2*x) dx is
   x              x
2⋅e ⋅sin(2⋅x)    e ⋅cos(2⋅x)
───────────── + ───────────
      5              5
```

```
In [8]: Int(x/sqrt(1-x**2)*asin(x))
```

```
The indefinite integral of  ∫ x*asin(x)/sqrt(-x**2 + 1) dx is
              ________
             ╱    2
x -  ╲╱    - x   + 1 ⋅asin(x)
```

Note

Summary from the above results: we have some experienced formula:

1. $\int x^n e^{ax} dx = (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) e^{ax} + C,$
2. $\int x^n \cos bx dx = a_n x^n \sin bx + a_{n-1} x^{n-1} \cos bx + \cdots + C,$
3. $\int e^{ax} \begin{cases} \sin bx \\ \cos bx \end{cases} dx \backslash = A e^{ax} \sin bx + B e^{ax} \cos bx + C.$
4. $\int x^k \ln x dx = A x^{k+1} \ln x + B x^{k+1} + C$

We can differentiate the above equations to find out the coefficients to calculate integration.

```
In [12]: a,b=symbols("a b")
```

```
In [13]: Int(exp(a*x)*sin(b*x))
```

```
The indefinite integral of  ∫ exp(a*x)*sin(b*x) dx is
⎧                               0
for a = 0 ∧ b
⎪
⎪    -ⅈ·b·x                  -ⅈ·b·x                  -ⅈ·b·x
⎪x·e        ·sin(b·x)   ⅈ·x·e       ·cos(b·x)   ⅈ·e        ·sin(b·x)
⎪────────────────────── - ──────────────────────── + ──────────────────────
for a = -ⅈ·
⎪         2                         2                        2·b
⎪
⎪     ⅈ·b·x                    ⅈ·b·x              ⅈ·b·x
⎨  x·e      ·sin(b·x)   ⅈ·x·e       ·cos(b·x)   e      ·cos(b·x)
⎪  ────────────────────── + ──────────────────────── - ─────────────────────
for a = ⅈ·
⎪           2                        2                      2·b
⎪
⎪                     a·x                  a·x
⎪                  a·e   ·sin(b·x)     b·e   ·cos(b·x)
⎪                  ──────────────── - ────────────────
otherwise
⎪                       2    2               2    2
⎩                      a  + b               a  + b

 = 0


b



b
```

(Example revisited) Since

$$\int xe^{2x}dx = Axe^{2x} + Be^{2x} + C,$$

differentiating both sides gets:

$$xe^{2x} = 2Axe^{2x} + (A + 2B)e^{2x}$$

This implies

$$2A = 1 \text{ and } A + 2B = 0 \Rightarrow A = \frac{1}{2} \text{ and } B = -\frac{1}{4}$$

Example

Evaluate the integral, $\int \sec^n x dx$, for $n = 1, 2, 3, \cdots$.

Sol:

First, consider the case $n = 1$:

$$\int \sec x dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + C$$

And $n = 2$ is a trivial case:

$$\int \sec^2 x dx = \tan x + C$$

For the higher power, $n \geq 3$, we have

$$\begin{aligned}\int \sec^n x dx &= \int \sec^{n-2} x d\tan x \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \cdot \tan^2 x dx \\ &= \sec^{n-2} x \tan x - (n-2)\int \sec^n x dx + (n-2)\int \sec^{n-2} x dx\end{aligned}$$

Then this results the following recursive formula:

$$\int \sec^n x dx = \frac{1}{n-1}\sec^{n-2} x \tan x - \frac{n-2}{n-1}\int \sec^{n-2} x dx$$

For the case $n = 3$, the integral is

$$\int \sec^3 x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Exercise, p613

In [1]:
```
from sympy import *
x=Symbol("x")
```

In [2]:
```
def DefInt(f,a,b):
    print("                                 ",b)
    print("The definite integral of  ∫ %s dt is " %f)
    print("                                 ",a)
    pprint(integrate(f,(x,a,b)))
```

In [3]:
```
def Int(f,*args):
    if(len(args)!=0):
       a=args[0]
       b=args[1]
       print("                          ",b)
       print("The definite integral of  ∫ %s dx is " %f)
       print("                          ",a)
       pprint(integrate(f,(x,a,b)))
    else:
       print("The indefinite integral of  ∫ %s dx is " %f)
       pprint(integrate(f,x))
```

In [48]:
```
#**18.**
integrate(x*atan(x),x)
```

Out[48]: `x**2*atan(x)/2 - x/2 + atan(x)/2`

In [49]:
```
integrate(x*atan(x),(x,0,1))
```

Out[49]: `-1/2 + pi/4`

In [50]:
```
DefInt(x*atan(x),0,1)
```

```
                          1
The definite integral of  ∫ x*atan(x) dt is
                          0
  1   π
- ─ + ─
  2   4
```

In [51]:
```
Int(x*atan(x),0,1)
```

```
                          1
The definite integral of  ∫ x*atan(x) dx is
                          0
  1   π
- ─ + ─
  2   4
```

In [52]:
```
Int(x*atan(x))
```

```
The indefinite integral of  ∫ x*atan(x) dx is
 2
x ⋅atan(x)   x   atan(x)
────────── - ─ + ───────
    2        2      2
```

In [53]:
```
#30
Int((x**2-1)*cos(x))
```

```
The indefinite integral of  ∫ (x**2 - 1)*cos(x) dx is
 2
x ⋅sin(x) + 2⋅x⋅cos(x) - 3⋅sin(x)
```

In [54]:
```
#34.
Int(log(x+1),0,2)
```

```
                                2
The definite integral of  ∫ log(x + 1) dx is
                                0
-2 + 3·log(3)
```

In [55]:
```
#36.
Int(x*sin(2*x),0,pi)
```

```
                                pi
The definite integral of  ∫ x*sin(2*x) dx is
                                0
```

$$\frac{-\pi}{2}$$

In [56]:
```
#42.
Int(sin(sqrt(x)),0,pi**2/4)
```

```
                                pi**2/4
The definite integral of  ∫ sin(sqrt(x)) dx is
                                0
2
```

In [57]:
```
Int(sin(sqrt(x)))
```

```
The indefinite integral of  ∫ sin(sqrt(x)) dx is
-2·√x·cos(√x) + 2·sin(√x)
```

In [46]:
```
#44.
Int(atan(sqrt(x)),0,1)
```

```
                                1
The definite integral of  ∫ atan(sqrt(x)) dx is
                                0
```

$$-1 + \frac{\pi}{2}$$

In [47]:
```
#44.
Int(atan(sqrt(x)))
```

```
The indefinite integral of  ∫ atan(sqrt(x)) dx is
-√x + x·atan(√x) + atan(√x)
```

Trigonometric Integration

1455

In this section, we will discuss the integration technique about trigonometric functions, trigonmetric integration.

Consider

$$\int \sin^m x \cos^n x dx$$

where m and n are integer.

1. one of m and n is odd: suppose that n is odd and let $n = 2k + 1, k \in \mathbb{R}$
$$\begin{aligned}\int \sin^m x \cos^n x dx &= \int \sin^m x \cos^{2k} x \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k d \sin x \\ &= \int u^m (1 - u^2)^k du\end{aligned}$$
2. m and n are even: Use the following formula to simplify the integral:
$$\sin^2 x = \frac{1 - \cos 2x}{2} \text{ and } \cos^2 x = \frac{1 + \cos 2x}{2}$$

Example

Evaluate $\int \sin x \cos^4 xdx$.

Sol:

$$\int \sin x \cos^4 x dx = -\int \cos^4 x d \cos x = \frac{-1}{5} \cos^5 x + C$$

In [4]:
```
# EXAMPLE
Int(sin(x)**5*cos(x)**2)
```

```
The indefinite integral of  ∫ sin(x)**5*cos(x)**2 dx is
     7            5         3
  cos (x)    2⋅cos (x)   cos (x)
- ─────── + ───────── - ───────
     7            5         3
```

In [ ]:
```
Int(sin(x)**3*sqrt(cos(x)),0,pi/2)
```

Example

$$\begin{aligned}\int_0^{\pi/2} \sin^3 x \sqrt{\cos x} dx &= \int_0^{\pi/2} (1 - \cos^2 x)\sqrt{\cos x} \sin x dx \\ &= -\left(\frac{2}{3}\cos^{3/2} x - \frac{2}{7}\cos^{7/2} x\right)\Big|_0^{\pi/2} = \frac{8}{21}\end{aligned}$$

Example

Evaluate the integral $\int \cos^4 x dx$.

Sol:

$$\begin{aligned}\int \cos^4 x dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \frac{1}{4}\int (1+2\cos 2x + \cos^2 2x) dx \\ &= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{\cos 4x}{2}\right) dx \\ &= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{\sin 4x}{32} + C\end{aligned}$$

Here, we use:

$$\int \cos ax dx = \frac{1}{a}\int \cos {\color{red}ax} d{\color{red}ax} = \frac{1}{a}\sin {\color{red}ax} + C$$

In [5]:
```
#Example
Int(sin(x)**4)
```

```
The indefinite integral of  ∫ sin(x)**4 dx is
             3
3⋅x    sin (x)⋅cos(x)    3⋅sin(x)⋅cos(x)
─── - ────────────── - ───────────────
 8          4                 8
```

The second case is the integral with the integrand being $\tan^m x \sec^n x\}$ then

1. If m is odd, say $m = 2k+1, k \in \mathbb{R}$
$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x d\sec x \\ &= \int (u^2-1)^k u^{n-1} du\end{aligned}$$
where $u = \sec x$.
2. If n is even, say $n = 2k, k \in \mathbb{R}$
$$\begin{aligned}\int \tan^m x \sec^n x dx &= \int \tan^m x \sec^{2k-2} x \sec^2 x dx \\ &= \int \tan^m x(\tan^2 x + 1)^{k-1} d\tan x \\ &= \int v^m (v^2+1)^{k-1} dv\end{aligned}$$
where $v = \tan x$.

In [ ]:
```
Int(tan(x)**3*sec(x)**7)
```

In [ ]:
```
Int(sqrt(tan(x))*sec(x)**6)
```

Example

$$\int_0^{\pi/4} \sec^6 x\sqrt{\tan x}dx = \int_0^{\pi/4} (1+\tan^2 x)^2\sqrt{\tan x}\sec^2 xdx$$
$$= \left(\frac{2}{3}\tan^{3/2} x + \frac{4}{7}\tan^{7/2} x + \frac{2}{11}\tan^{11/2} x\right)\Bigg|_0^{\pi/2} = \frac{328}{231}$$

In [4]:
```
Int(cot(x)**5*csc(x)**5)
```

```
The indefinite integral of  ∫ cot(x)**5*csc(x)**5 dx is
  ⎛        4             2       ⎞
- ⎝63⋅sin (x) - 90⋅sin (x) + 35⎠
─────────────────────────────────
                      9
             315⋅sin (x)
```

In [5]:
```
Int(tan(x)/sec(x)**2)
```

```
The indefinite integral of  ∫ tan(x)/sec(x)**2 dx is
   2
sin (x)
───────
   2
```

Example

Evaluate $I = \int \sec^2 x \tan^3 xdx$.

Sol:

- Let $\tan x$ be the variable, then
$$I = \int \tan^3 xd\tan x = \frac{1}{4}\tan^4 x + C$$
- Let $\sec x$ be the variable, then $$ I = \int \sec x (\sec^2 x - 1) d \sec x = \frac{1}{4} \sec^4 x +

```
\frac{1}{2} \sec^2 x + C $$
```

None of above satisfies, then by other techniques.

Example

Consider the integral $\int \tan^n x dx$ where $n = 1, 2, 3, \cdots$.

Sol:

When $n = 1$,

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

If $n = 2$, we have

$$\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

The more higher power, $n \geq 3$, integrals can be derived by the following recursive formula:

$$\begin{aligned}\int \tan^n x dx &= \int \tan^{n-2} x (\sec^2 x - 1) dx \\ &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx\end{aligned}$$

For $n = 3$, the integral is:

$$\int \tan^3 x dx = \frac{1}{2} \tan^2 x - \int \tan x dx = \frac{1}{2} \tan^2 x - \ln|\sec x| + C$$

There is still a exceptional case under these conditions: n being even and m being odd. Certainly, no absolute procedure may be used in such case. But it does not mean no way to solve this type.

Example

Solve $\int \frac{\tan^2 x}{\sec x} dx$.

Sol:

This example is just the exceptional case out of control. Consider that transform the $\tan x$ and $\sec x$ into $\sin x$ and $\cos x$:

$$\begin{aligned}\int \frac{\tan^2 x}{\sec x} dx &= \int \frac{\sec^2 x - 1}{\sec x} dx \\ &= \int \sec x dx - \int \cos x dx \\ &= \ln|\sec x + \tan x| - \sin x + C\end{aligned}$$

Before the end of this section, we introduce the type of integration of $\sin ax \cos bx$ where $a \neq b$. In such cases, the following rules are needed:

$$\sin ax \cos bx = \frac{1}{2}(\sin(a+b)x + \sin(a-b)x)$$
$$\cos ax \cos bx = \frac{1}{2}(\cos(a+b)x + \cos(a-b)x)$$
$$\sin ax \sin bx = \frac{1}{2}(\cos(a-b)x - \cos(a+b)x)$$

Example

Calculate $\int \sin 3x \sin 7x dx$.

Sol:

$$\begin{aligned} \int \sin 3x \sin 7x dx &= \frac{1}{2}\int (\cos(7-3)x - \cos(7+3)x) dx \\ &= \frac{1}{8}\sin 4x - \frac{1}{20}\sin 10x + C \end{aligned}$$

In [6]:
```
Int(sin(4*x)*cos(5*x))
```

```
The indefinite integral of  ∫ sin(4*x)*cos(5*x) dx is
5·sin(4·x)·sin(5·x)    4·cos(4·x)·cos(5·x)
─────────────────── + ───────────────────
         9                     9
```

Example

Calculate the following definite integrals, $a, b \in \mathbb{N}$:

1. $$\int_0^{2\pi} \cos ax \cos bx = \pi\delta(a-b),$$

   where $\delta(x) = 1$ if $x = 0$ and $\delta(x) = 0$ if $x \neq 0$.

2.

$$\int_0^{2\pi} \sin ax \sin bx = \pi\delta(a-b),$$

3.

$$\int_0^{2\pi} \sin ax \cos bx = 0$$

`Sol:`

1. Suppose that $a \neq b$, then

$$\begin{aligned}
\int_0^{2\pi} \cos ax \cos bx dx &= \int_0^{2\pi} \frac{1}{2}(\cos(a+b)x + \cos(a-b)x)dx \\
&= \frac{1}{2(a+b)}\sin(a+b)x + \frac{1}{2(a-b)}\sin(a-b)x\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

2. If $a = b$, then

$$\begin{aligned}
\int_0^{2\pi} \cos ax \cos bx dx &= \int_0^{2\pi} \cos^2 axdx \\
&= \int_0^{2\pi} \frac{1+\cos(2ax)}{2}dx \\
&= \frac{x}{2} + \frac{\sin(2ax)}{4a}\Big|_0^{2\pi} \\
&= \pi
\end{aligned}$$

and 3) are the same.

Consider the type of integral of power of sine function:

$$I_n = \int f^n(x)dx$$

Example

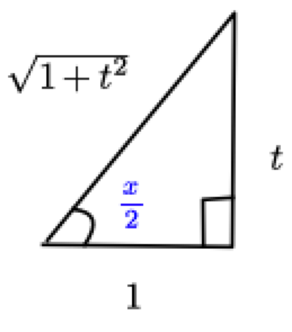
For $n \geq 2$, consider

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin^n x dx \\ &= \int_0^{\pi/2} (-\cos x)' \cdot \sin^{n-1} x dx \\ &= -\cos x \sin^{n-1} x\Big|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx \\ &= 0 + (n-1)\int_0^{\pi/2} \sin^{n-2} x(1-\sin^2 x)dx \\ &\Downarrow \\ I_n &= \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx = \frac{n-1}{n} I_{n-2} \end{aligned}$$

This implies

1. $\int_0^{\pi/2} \sin^{10} x dx = \frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2} \cdot \frac{\pi}{2}$;
2. $\int_0^{\pi/2} \sin^{11} x dx = \frac{10\cdot 8\cdot 6\cdot 4\cdot 2}{11\cdot 9\cdot 7\cdot 5\cdot 3} \cdot 1$.

Type of $F(a\sin x + b\cos x), a \neq b$



Scheme

1. take new variable, $\mathbf{t} = \tan\frac{\mathbf{x}}{\mathbf{2}}$.
2. $\sin x = 2\sin(x/2)\cos(x/2) = 2\tan(x/2)\cos^2(x/2) = \frac{\mathbf{2t}}{\mathbf{1+t^2}}$,
3. $\cos x = 2\cos^2(x/2) - 1 = \frac{\mathbf{1-t^2}}{\mathbf{1+t^2}}$,
4. $x = 2\tan^{-1} t \Longrightarrow \mathbf{dx} = \frac{\mathbf{2dt}}{\mathbf{1+t^2}}$.

Example

By the help of $t = \tan\frac{x}{2}$, we have:

$$\begin{aligned}
\int \frac{dx}{4\sin x + 3\cos x} &= \int \frac{1}{4\frac{2t}{1+t^2} + 3\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} \\
&= \int \frac{2dt}{3 + 8t - 3t^2} \\
&= \int \frac{1}{5}\left(\frac{3}{3t+1} - \frac{1}{t-3}\right) dt \\
&= \frac{1}{5}(\ln|3t+1| - \ln|t-3|) + C \\
&= \frac{1}{5}\ln\left|\frac{3t+1}{t-3}\right| + C \\
&= \frac{1}{5}\ln\left|\frac{3\tan\frac{x}{2} + 1}{\tan\frac{x}{2} - 3}\right| + C
\end{aligned}$$

In [58]:
```
Int(1/(4*sin(x)+3*cos(x)))
```

```
The indefinite integral of  ∫ 1/(4*sin(x) + 3*cos(x)) dx is
     /     /x\    \      /     /x\    1\
  log| tan |-| - 3|   log| tan |-| + -|
     \     \2/    /      \     \2/    3/
- ---------------- + ----------------
         5                  5
```

P.623

In [61]:
```
#6
Int(cos(x)**3)
```

```
The indefinite integral of  ∫ cos(x)**3 dx is
     3
  sin (x)
- ─────── + sin(x)
     3
```

In [62]:
```
# 10
Int(sin(2*x)**2*cos(2*x)**4)
```

```
The indefinite integral of  ∫ sin(2*x)**2*cos(2*x)**4 dx is
                    5                    3
x    sin(2⋅x)⋅cos (2⋅x)   sin(2⋅x)⋅cos (2⋅x)   sin(2⋅x)⋅cos(2⋅x)
── - ────────────────── + ────────────────── + ─────────────────
16           12                   48                   32
```

In [63]:
```
#18
Int(x*cos(x)**2)
```

```
The indefinite integral of  ∫ x*cos(x)**2 dx is
 2    2        2    2                              2
x ⋅sin (x)    x ⋅cos (x)    x⋅sin(x)⋅cos(x)    cos (x)
────────── + ────────── + ─────────────── + ───────
    4             4              2              4
```

In [64]:
```
#20
Int(tan(pi-x)**3)
```

```
The indefinite integral of  ∫ -tan(x)**3 dx is
     ⎛   2     ⎞
  log⎝sin (x) - 1⎠          1
- ──────────────── + ──────────────
         2                2
                     2⋅sin (x) - 2
```

In [65]:
```
#22
Int(tan(x)**5*sec(x)**3)
```

```
The indefinite integral of  ∫ tan(x)**5*sec(x)**3 dx is
      4            2
35⋅cos (x) - 42⋅cos (x) + 15
────────────────────────────
               7
        105⋅cos (x)
```

In [66]:
```
##24
Int(tan(x)**2*sec(x)**2,0,pi/4)
```

```
                                 pi/4
The definite integral of  ∫ tan(x)**2*sec(x)**2 dx is
                                 0
1/3
```

In [ ]:
```
##38
Int((1+cot(x))**2*csc(x))
```

In [72]:
```
Int((sin(x)+cos(x))/sin(x)**3)
```

The indefinite integral of ∫ (sin(x) + cos(x))/sin(x)**3 dx is

$$-\frac{\tan^{2}\left(\frac{x}{2}\right)}{8}+\frac{\tan\left(\frac{x}{2}\right)}{2}-\frac{1}{2\cdot\tan\left(\frac{x}{2}\right)}-\frac{1}{8\cdot\tan^{2}\left(\frac{x}{2}\right)}$$

In [69]:
```
#40
Int(sin(3*x)*sin(4*x))
```

The indefinite integral of ∫ sin(3*x)*sin(4*x) dx is

$$-\frac{4\cdot\sin(3\cdot x)\cdot\cos(4\cdot x)}{7}+\frac{3\cdot\sin(4\cdot x)\cdot\cos(3\cdot x)}{7}$$

In [70]:
```
#48, cos^2=(cos x+sin(x))(cos(x)-sin(x))
Int(cos(2*x)/(cos(x)+sin(x)))
```

The indefinite integral of ∫ cos(2*x)/(sin(x) + cos(x)) dx is

$$\int\frac{\cos(2\cdot x)}{\sin(x)+\cos(x)}\,dx$$

Trigonometric Substitution

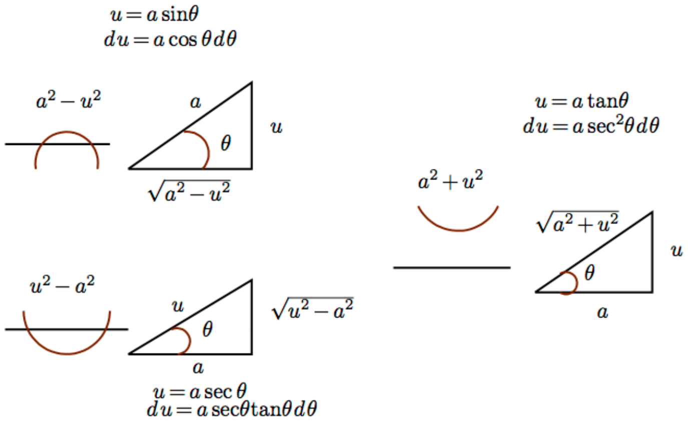
1793

Sometimes, the integrand of functions based on quadratic function can be transformed into function of trigonometric functions and integrated out by trigonometric integration technique introduced above.

With suitable trigonometric functions substitution, $f(ax^2 + bx + c)$ can be simplified into function concerned with trigonometric functions.

1. Type of $ax^2 + bx + c \rightarrow a^2 - u^2$: let $u = a\sin\theta$.
2. Type of $ax^2 + bx + c \rightarrow a^2 + u^2$: let $u = a\tan\theta$.
3. Type of $ax^2 + bx + c \rightarrow u^2 - a^2$: let $u = a\sec\theta$.

with the following transformation respectively:



Example

1. $a^2 - x^2 \Rightarrow x = a\sin\theta$ and $dx = a\cos\theta d\theta$

$$\begin{aligned}
\int \frac{dx}{(9-x^2)^{3/2}} &= \int \frac{3\cos\theta d\theta}{(9-(3\sin\theta)^2)^{3/2}} \\
&= \frac{1}{9}\int \frac{d\theta}{\cos^2\theta} \\
&= \frac{1}{9}\tan\theta + C \\
&= \frac{x}{9\sqrt{9-x^2}} + C
\end{aligned}$$

2. $a^2 + x^2 \Rightarrow x = a\tan\theta$ and $dx = a\sec^2\theta d\theta$

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \int \frac{\sec^2\theta d\theta}{\sec^2\theta} \\
&= \int 1 d\theta \\
&= \theta + C = \tan^{-1} x + C
\end{aligned}$$

3. $x^2 - a^2 \Rightarrow x = a\sec\theta$ and $dx = a\sec\theta\tan\theta d\theta$

$$\begin{aligned}
\int \frac{dx}{x^2-4} &= \int \frac{2\sec\theta\tan\theta d\theta}{4\tan^2\theta} \\
&= \int \frac{d\theta}{2\sin\theta} \\
&= -\frac{\ln|\csc\theta + \cot\theta|}{2} + C \\
&= -\frac{\ln|\frac{x}{\sqrt{x^2-4}} + \frac{2}{\sqrt{x^2-4}}|}{2} + C \\
&= -\frac{1}{2}\left(\ln|x-2| - \ln\sqrt{x^2-4}\right) + C \\
&= \frac{\ln|x-2|}{4} - \frac{\ln|x+2|}{4} + C
\end{aligned}$$

In [8]:
```
Int(x**2/sqrt(9-x**2))
```

```
The indefinite integral of  ∫ x**2/sqrt(-x**2 + 9) dx is
                                   ⎛x⎞
        ________            9⋅asin ⎜─⎟
       ╱    2                      ⎝3⎠
  x⋅╲╱   - x  + 9
- ─────────────── + ──────────
         2               2
```

In [7]:
```
Int(sqrt(1+x**2))
```

The indefinite integral of ∫ sqrt(x**2 + 1) dx is

$$\frac{x\cdot\sqrt{x^{2} + 1}}{2} + \frac{\operatorname{asinh}(x)}{2}$$

In [9]:
```
Int((sqrt(4+x**2))**3)
```

The indefinite integral of ∫ (x**2 + 4)**(3/2) dx is

$$\frac{x^{3}\cdot\sqrt{x^{2} + 4}}{4} + \frac{5\cdot x\cdot\sqrt{x^{2} + 4}}{2} + 6\cdot\operatorname{asinh}\left(\frac{x}{2}\right)$$

In [12]:
```
Int(sqrt(x**2-16)/x)
```

The indefinite integral of ∫ sqrt(x**2 - 16)/x dx is

$$\begin{cases} - \dfrac{i\cdot x}{\sqrt{-1 + \dfrac{16}{x^{2}}}} - 4\cdot i\cdot\operatorname{acosh}\left(\dfrac{4}{x}\right) + \dfrac{16\cdot i}{x\cdot\sqrt{-1 + \dfrac{16}{x^{2}}}} & \text{for } 16\cdot\left|\dfrac{1}{x^{2}}\right| > 1 \\ \dfrac{x}{\sqrt{1 - \dfrac{16}{x^{2}}}} + 4\cdot\operatorname{asin}\left(\dfrac{4}{x}\right) - \dfrac{16}{x\cdot\sqrt{1 - \dfrac{16}{x^{2}}}} & \text{otherwise} \end{cases}$$

Example

$$\int \frac{dx}{\sqrt{x^2 + 4x + 7}} = \int \frac{du}{\sqrt{u^2 + 3}} = \left|\ln \frac{\sqrt{u^2 + 3}}{\sqrt{3}} + \frac{u}{\sqrt{3}}\right| + C$$

In [15]:
```
Int(1/sqrt(x**2+3))
```

The indefinite integral of ∫ 1/sqrt(x**2 + 3) dx is

$$\operatorname{asinh}\left(\frac{\sqrt{3}\cdot x}{3}\right)$$

Example

Evaluate the integral:

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^2} &= \tan^{-1} x\Big|_0^\infty \\ &= \tan^{-1}\infty - \tan^{-1} 0 \\ &= \frac{\pi}{2}\end{aligned}$$

Example

Take $x = 2\sin t$:

$$\begin{aligned}\int \frac{dx}{2+\sqrt{4-x^2}} &= \int \frac{2\cos t dt}{2+\sqrt{4-4\sin^2 t}} \\ &= \int \frac{\cos t dt}{1+\cos t} \\ &= \int \left(1 - \frac{1}{1+\cos t}\right) dt \\ \left(\cos\frac{t}{2} = \sqrt{\frac{1+\cos t}{2}}\right) &= t - \int \frac{dt}{2\cos^2 \frac{t}{2}} \\ &= t - \frac{1}{2}\int \sec^2 \frac{t}{2} dt \\ &= t - \tan\frac{t}{2} + C\end{aligned}$$

Example

$(a^2 - x^2$ and $a^2 + x^2)$

\begin{eqnarray*} \int \frac{d x}{(4 - x^2) \sqrt{4 + x^2}} & = & \int \frac{d (2 \tan \theta)}{(4 - 4 \tan^2 \theta) \sqrt{4 + 4 \tan^2 \theta}} (\color{brown}{x = 2 \tan \theta})\ & = & \frac{1}{4} \int \frac{\sec^{} \theta}{1 - \tan^2 \theta} d \theta\ & = & \frac{1}{4} \int \frac{\cos \theta}{\cos^2 \theta - \sin^2 \theta} d \theta\ & = & \frac{1}{4} \int \frac{d t}{1 - 2 t^2} (\color{brown}{t = \sin \theta})\ & = & \frac{1}{8} \int \left(\frac{1 / \sqrt{2}}{t + \frac{1}{\sqrt{2}}}

```
- \frac{1 / \sqrt{2}}{t - \frac{1}{\sqrt{2}}} \right) d t\\
& = & \frac{1}{8 \sqrt{2}} \ln \frac{t + \frac{1}{\sqrt{2}}}{t -
\frac{1}{\sqrt{2}}} + C\\
& = & \frac{1}{8 \sqrt{2}} \ln \frac{\sin \theta +
\frac{1}{\sqrt{2}}}{\sin \theta - \frac{1}{\sqrt{2}}} + C\\
& = & \frac{1}{8 \sqrt{2}} \ln \frac{\frac{x}{\sqrt{4 + x^2}} +
\frac{1}{\sqrt{2}}}{\frac{x}{\sqrt{4 + x^2}} - \frac{1}{\sqrt{2}}} + C
```

\end{eqnarray*}

P.631, Exercises

In [17]:
```
#4
Int(sqrt(4-x**2)/x**2)
```

```
The indefinite integral of  ∫ sqrt(-x**2 + 4)/x**2 dx is

                 __________
                ╱    2
       ⎛x⎞    ╲╱  - x  + 4
- asin⎜─⎟ - ─────────────
       ⎝2⎠          x
```

In [18]:
```
#6
Int(x**3*sqrt(1+x**2))
```

```
The indefinite integral of  ∫ x**3*sqrt(x**2 + 1) dx is

     ________       ________        ________
 4  ╱  2         2  ╱  2           ╱  2
x ⋅╲╱  x  + 1   x ⋅╲╱  x  + 1   2⋅╲╱  x  + 1
────────────── + ────────────── - ────────────
      5               15              15
```

In [20]:
```python
#8
Int(1/x**3/sqrt(x**2-4))
```

The indefinite integral of ∫ 1/(x**3*sqrt(x**2 - 4)) dx is

$$\begin{cases} \dfrac{i\cdot\operatorname{acosh}\left(\frac{2}{x}\right)}{16} + \dfrac{i\cdot\sqrt{-1+\frac{4}{x^2}}}{8\cdot x} & \text{for } 4\cdot\left|\frac{1}{x^2}\right| > 1 \\ -\dfrac{\operatorname{asin}\left(\frac{2}{x}\right)}{16} + \dfrac{1}{8\cdot x\cdot\sqrt{1-\frac{4}{x^2}}} - \dfrac{1}{2\cdot x^3\cdot\sqrt{1-\frac{4}{x^2}}} & \text{otherwise} \end{cases}$$

Explicitly,

$$\begin{aligned} \int \frac{dx}{x^3\sqrt{x^2-4}} &= \int \frac{\sec t\tan t dt}{\sec^3 t\tan t} \\ &= \int \cos^2 t dt \\ &= \int \frac{1+\cos 2t}{2} dt = \frac{2t+\sin 2t}{4} + C \end{aligned}$$

In [22]:
```python
#24
Int((2*x+3)/sqrt(1-x**2))
```

The indefinite integral of ∫ (2*x + 3)/sqrt(-x**2 + 1) dx is

$$-2\cdot\sqrt{-x^2+1} + 3\cdot\operatorname{asin}(x)$$

In [24]:
```python
#26
Int(exp(x)*sqrt(exp(x)+1))
```

The indefinite integral of ∫ sqrt(exp(x) + 1)*exp(x) dx is

$$\frac{2\cdot\sqrt{e^x+1}\cdot e^x}{3} + \frac{2\cdot\sqrt{e^x+1}}{3}$$

In [25]:
```
#30
Int(x**2/sqrt(4*x-x**2))
```

```
The indefinite integral of  ∫ x**2/sqrt(-x**2 + 4*x) dx is
⌠
⎮        2
⎮       x
⎮ ────────────── dx
⎮   ____________
⎮ ╲╱ -x⋅(x - 4)
⌡
```

Sympy could not *solve* this integral directly. Let us use change the variable, $u = x - 2$, and re-run integration:

$$u = x - 2$$
$$du = dx$$
$$\frac{x^2}{\sqrt{4x - x^2}} = \frac{(u+2)^2}{\sqrt{4 - u^2}}$$

In [26]:
```
# x=t-2
Int((x+2)**2/sqrt(4-x**2))
```

```
The indefinite integral of  ∫ (x + 2)**2/sqrt(-x**2 + 4) dx is
        __________
       ╱    2
   x⋅╲╱  - x  + 4          _________
                          ╱    2               ⎛x⎞
 - ─────────────── - 4⋅╲╱  - x  + 4  + 6⋅asin⎜─⎟
          2                                    ⎝2⎠
```

7.4 Partial Fraction

The form of partial fraction functions is in form as follows:

$$\frac{P_n(x)}{Q_m(x)}$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of n, m respectively.

Since improper partial fraction can be represented as sum of polynomial and proper rational function, it is only to consider how to manipulate the integration on proper rational function if want to manipulate integation over partial fraction.

Rules

1. $$\mathbf{\int\frac{dx}{(x-a)^n}=\cases{\frac{(x-a)^{-n+1}}{-n+1}+C, &if $n\ne1$\

   `\ln |x-a|+C, &if $n=1$}}$$`

2. $ax^2+2bx+c=a(x+b/a)^2+c-b^2/a$ and use **trigonometric substitution** to take integration over it;
3.

$$\mathbf{\frac{1}{(x-a)^m(ax^2+2bx+c)^n}=\frac{A_1}{(x-a)}+\cdots+\frac{A_m}{(x-a)^m}+\frac{B_1x+C_1}{(ax^2+2bx+c)}+\cdots+\frac{B_nx+C}{(ax^2+2bx}}$$

In [28]:
```
f=(4*x**2-4*x+6)/(x**3-x**2-6*x)
print('f=',f,' = ',apart(f))
```

```
f= (4*x**2 - 4*x + 6)/(x**3 - x**2 - 6*x)  =  3/(x + 2) + 2/(x - 3
) - 1/x
```

In [29]:
```
Int(f)
```

```
The indefinite integral of  ∫ (4*x**2 - 4*x + 6)/(x**3 - x**2 - 6*
x) dx is
-log(x) + 2⋅log(x - 3) + 3⋅log(x + 2)
```

Example

$$\int \frac{\mathbf{4x^3 + x}}{\mathbf{2x^2 + x - 3}}\mathbf{dx}$$

1. $$\frac{\mathbf{4x^3 + x}}{\mathbf{2x^2 + x - 3}} = \mathbf{2x - 1} + \frac{\mathbf{8x - 3}}{\mathbf{2x^2 + x - 3}}$$

By polynomial quotient rule:

$$
\begin{array}{rrrrr}
 & 2x & -1 & & \\
\hline
2x^2+x-3 \;) & 4x^3 & +0\,x^2 & +x & \\
 & 4x^3 & +2\,x^2 & -6\,x & \\
\hline
 & & -2\,x^2 & 7\,x & \\
 & & -2\,x^2 & -x & 3 \\
\hline
 & & & 8x & -3
\end{array}
$$

2.

$$\frac{\mathbf{8x - 3}}{\mathbf{2x^2 + x - 3}} = \frac{\mathbf{8x - 3}}{\mathbf{(2x + 3)(x - 1)}} = \frac{\mathbf{6}}{\mathbf{2x + 3}} + \frac{\mathbf{1}}{\mathbf{x - 1}}$$

3.

$$
\begin{aligned}
\int \frac{\mathbf{4x^3 + x}}{\mathbf{2x^2 + x - 3}}\mathbf{dx} &= \int \left(2x - 1 + \frac{8x - 3}{2x^2 + x - 3}\right) dx \\
&= \int \left(2x - 1 + \frac{6}{2x + 3} + \frac{1}{x - 1}\right) dx \\
&= \mathbf{x^2 - x + \ln|x - 1| + 3\ln|2x + 3| + C}
\end{aligned}
$$

In [49]:
```
def FracInt(f,g):
    func="(%s)/(%s)" %(f,g)
    print("1. Integrand: (%s)/(%s) could be expressed as folllows:"
%(f,g))
    pf=apart(f/g)
    pprint(pf)
    print("2.")
    Int(f/g)
```

```
In [50]: FracInt(4*x**3+x,2*x**2+x-3)
```

```
1. Integrand: (4*x**3 + x)/(2*x**2 + x - 3) could be expressed as
folllows:
            6         1
2⋅x - 1 + ─────── + ─────
          2⋅x + 3   x - 1
2.
The indefinite integral of ∫ (4*x**3 + x)/(2*x**2 + x - 3) dx is
 2
x  - x + log(x - 1) + 3⋅log(x + 3/2)
```

```
In [32]: f=(4*x**3+x)/(2*x**2+x-3)
         print('f=',f,' = ',apart(f))
         Int(f)
```

```
f= (4*x**3 + x)/(2*x**2 + x - 3)  =  2*x - 1 + 6/(2*x + 3) + 1/(x
- 1)
The indefinite integral of ∫ (4*x**3 + x)/(2*x**2 + x - 3) dx is
 2
x  - x + log(x - 1) + 3⋅log(x + 3/2)
```

```
In [34]: f=(2*x**2+3*x+7)/(x**3+x**2-x-1)
         print('f=',f,' = ',apart(f))
         Int(f)
```

```
f= (2*x**2 + 3*x + 7)/(x**3 + x**2 - x - 1)  =  -1/(x + 1) - 3/(x
+ 1)**2 + 3/(x - 1)
The indefinite integral of ∫ (2*x**2 + 3*x + 7)/(x**3 + x**2 - x
- 1) dx is
                                3
3⋅log(x - 1) - log(x + 1) + ─────
                            x + 1
```

```
In [51]: FracInt(x**4+3*x**3+14*x**2+14*x+41,(x**2+4)*(x**2+2*x+5))
```

```
1. Integrand: (x**4 + 3*x**3 + 14*x**2 + 14*x + 41)/((x**2 + 4)*(x
**2 + 2*x + 5)) could be expressed as folllows:
   x + 4                1
──────────── + 1 + ──────
 2                   2
x  + 2⋅x + 5        x  + 4
2.
The indefinite integral of ∫ (x**4 + 3*x**3 + 14*x**2 + 14*x + 41
)/((x**2 + 4)*(x**2 + 2*x + 5)) dx is
                                   ⎛x⎞          ⎛x   1⎞
        ⎛ 2            ⎞      atan ⎜─⎟   3⋅atan ⎜─ + ─⎟
     log⎝x  + 2⋅x + 5⎠            ⎝2⎠          ⎝2   2⎠
x + ─────────────────── + ──────── + ──────────────
               2              2             2
```

Note

- For $a \neq 0$,

$$\int \frac{\mathbf{dx}}{\mathbf{x^2 + a^2}} = \frac{\mathbf{1}}{\mathbf{a}} \int \frac{\mathbf{d}\left(\frac{\mathbf{x}}{\mathbf{a}}\right)}{\left(\frac{\mathbf{x}}{\mathbf{a}}\right)^2 + \mathbf{1}} = \frac{\mathbf{1}}{\mathbf{a}} \tan^{-1} \frac{\mathbf{x}}{\mathbf{a}} + \mathbf{C}$$

- $$\int \frac{\mathbf{x+4}}{\mathbf{x^2+2x+5}}\mathbf{dx} = \int \frac{\mathbf{x+1}}{\mathbf{(x+1)^2+2^2}}\mathbf{dx} + \int \frac{\mathbf{3}}{\mathbf{2^2+(x+1)^2}}\mathbf{dx} = \frac{\mathbf{1}}{\mathbf{2}}\ln|\mathbf{(x+1)^2+4}| + \frac{\mathbf{3}}{\mathbf{2}}\mathrm{ta}$$

  the former comes *substitution method* and the latter comes from *above*.

Example

Evaluate

$$\int \frac{\mathbf{x^3 - 2x^2 + 3x + 2}}{\mathbf{x(1+x^2)^2}}\mathbf{dx}$$

In [52]:
```
FracInt(x**3-2*x**2+3*x+2,x*(x**2+1)**2)
```

```
1. Integrand: (x**3 - 2*x**2 + 3*x + 2)/(x*(x**2 + 1)**2) could be
expressed as folllows:
  2⋅x - 1     2⋅(2⋅x - 1)     2
- ───────── - ──────────── + ─
   2                    2     x
  x  + 1        ⎛ 2    ⎞
                ⎝x  + 1⎠
2.
The indefinite integral of  ∫ (x**3 - 2*x**2 + 3*x + 2)/(x*(x**2 +
1)**2) dx is
x + 2                         ⎛ 2    ⎞
────── + 2⋅log(x) - log ⎝x  + 1⎠  + 2⋅atan(x)
 2
x  + 1
```

Steps

1. $$\frac{\mathbf{x^3 - 2x^2 + 3x + 2}}{\mathbf{x(1+x^2)^2}} = \frac{\mathbf{a}}{\mathbf{x}} + \frac{\mathbf{bx+c}}{\mathbf{1+x^2}} + \frac{\mathbf{dx+e}}{\mathbf{(1+x^2)^2}}$$
2. get the values, a, b, c, d, e:

$$\begin{aligned}
x^3 - 2x^2 + 3x + 2 &= a((1+x^2)^2 + (bx+c)x(1+x^2) + (dx+e)x \\
x = 0 &\Rightarrow a = 2 \\
\text{coefficient of } x^4 &\Rightarrow (a+b) = 0 \Rightarrow b = -2 \\
\text{coefficient of } x^3 &\Rightarrow c = 1 \\
\text{coefficient of } x^2 &\Rightarrow 2a + b + d = -2 \Rightarrow d = -4 \\
\text{coefficient of } x &\Rightarrow c + e = 3 \Rightarrow e = 2
\end{aligned}$$

imply

$$\frac{x^3 - 2x^2 + 3x + 2}{x(1+x^2)^2} = \frac{2}{x} + \frac{-2x+1}{1+x^2} + \frac{-4x+2}{(1+x^2)^2}$$

3. integrate term by term:

-

$$\int \frac{2dx}{x} = 2\ln|x| + C$$

$$\int \frac{-2x+1}{1+x^2}dx = -\int \frac{2xdx}{1+x^2} + \int \frac{dx}{1+x^2} = -\ln|1+x^2| -$$

$$\begin{aligned}
\int \frac{-4x+2}{(1+x^2)^2}dx &= \int \frac{-2(2x)+2}{(1+x^2)^2}dx \\
&= -2\int \frac{(2x)dx}{(1+x^2)^2}dx + \int \frac{2}{(1+x^2)^2}dx \\
&= \frac{2}{1+x^2} + 2\int \frac{\sec^2 t dt}{(1+\tan^2 t)^2} \\
&= \frac{2}{1+x^2} + 2\int \cos^2 t dt = \frac{2}{1+x^2} + \int (1+\cos 2t)dt \\
&= \frac{2}{1+x^2} + t + \frac{\sin 2t}{2} + C = \frac{2}{1+x^2} + t + \sin t \cos t + C \\
&= \frac{2}{1+x^2} + \tan^{-1} x + \frac{1 \cdot x}{1+x^2} + C
\end{aligned}$$

Result

$$\int \frac{x^3 - 2x^2 + 3x + 2}{x(1+x^2)^2}dt = \frac{x+2}{1+x^2} + 2\ln|x| - \ln|1+x^2| + 2\tan^{-1} x + C$$

P.642 Exercise

```
In [53]: #10
FracInt(2*x-1,2*x**2-x)
```

```
1. Integrand: (2*x - 1)/(2*x**2 - x) could be expressed as folllow
s:
1
─
x
2.
The indefinite integral of  ∫ (2*x - 1)/(2*x**2 - x) dx is
log(x)
```

```
In [54]: #20
f=(x**4-3*x**2-3*x-2)/(x**3-x**2-2*x)
FracInt(x**4-3*x**2-3*x-2,x**3-x**2-2*x)
```

```
1. Integrand: (x**4 - 3*x**2 - 3*x - 2)/(x**3 - x**2 - 2*x) could
be expressed as folllows:
               1           2         1
x + 1 - ───────── - ───────── + ─
          3⋅(x + 1)   3⋅(x - 2)   x
2.
The indefinite integral of  ∫ (x**4 - 3*x**2 - 3*x - 2)/(x**3 - x*
*2 - 2*x) dx is
 2
x                    2⋅log(x - 2)   log(x + 1)
── + x + log(x) - ──────────── - ──────────
2                        3              3
```

```
In [55]: #28
f=x**2/(x**2+4*x+3)**2
FracInt(x**2,(x**2+4*x+3)**2)
```

```
1. Integrand: (x**2)/((x**2 + 4*x + 3)**2) could be expressed as f
olllows:
    3             9             3             1
───────── + ────────── - ───────── + ──────────
4⋅(x + 3)            2   4⋅(x + 1)            2
            4⋅(x + 3)                4⋅(x + 1)
2.
The indefinite integral of  ∫ x**2/(x**2 + 4*x + 3)**2 dx is
      5⋅x + 6        3⋅log(x + 1)   3⋅log(x + 3)
- ────────────── - ──────────── + ────────────
     2                  4              4
  2⋅x  + 8⋅x + 6
```

44.

$$\int \frac{\cos x}{\sin^2 x - \sin x - 6} dx = \int \frac{tdt}{t^2 - t - 6} = \frac{\ln|t-3| - \ln|t+2|}{5} + C$$

In [56]:
```
#44
f=cos(x)/(sin(x)**2-sin(x)-6)
#print('f=',f,":")
#pprint(apart(f))
Int(f)
```

```
The indefinite integral of  ∫ cos(x)/(sin(x)**2 - sin(x) - 6) dx i
s
log(sin(x) - 3)    log(sin(x) + 2)
───────────────  - ───────────────
       5                  5
```

46.

$$\int \frac{e^x dx}{e^{2x} + 2e^x - 8} = \int \frac{dt}{t^2 + 2t - 8} = \frac{\ln|t-2| - \ln|t+4|}{6} + C$$

In [57]:
```
#46
f=exp(x)/(exp(2*x)+2*exp(x)-8)
Int(f)
```

```
The indefinite integral of  ∫ exp(x)/(exp(2*x) + 2*exp(x) - 8) dx
is
    ⎛ x    ⎞       ⎛ x    ⎞
log ⎝e  - 2⎠   log ⎝e  + 4⎠
──────────── - ────────────
     6              6
```