1 4 Antiderivatives and the Definite Integrals

1.1 4.1 Antiderivatives

1.2 Definition

The function $F(x)$ is called an antiderivative for the function $f(x)$ if $F(x)$ is differentiable and if

$$F'(x) = f(x)$$

for all x in the domain of $F(x)$.

1.3 Example

- $F(x) = x^2 + x$ is the antiderivative of $f(x) = 2x + 1$.
- $F(x) = \frac{1}{2}e^{2x} + \ln x + 5$ is the antiderivative of $f(x) = e^{2x} + \frac{1}{x}$. Note that $\frac{1}{2}e^{2x} + \ln x + e$ is also an antiderivative of $f(x)$.

From the last example, we find that $f(x)$ has at least two antiderivatives. The following theorem will show that there are infinite antiderivatives for $f(x)$ if there is one antiderivative.

1.4 Lemma

If $F'(x) = 0$ then $F(x) = C$ for some C.

1.5 Proof

Suppose that $F(a) \neq F(b)$ for any distinct $a, b \in \text{Domain}(F)$. Then by **Mean value theorem**,

$$0 = F'(c) = \frac{F(b) - F(a)}{b - a} \neq 0$$

this is a contraction.

1.6 Theorem

Let $F(x)$ and $G(x)$ be the antiderivatives for $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C$$

1.7 proof

Consider new function $F(x) - G(x)$. And

$$(F(x) - G(x))' = f(x) - f(x) = 0$$

This means

$$F(x) - G(x) = C \Longrightarrow F(x) = G(x) + C$$

for some $C \in \mathrm{R}$.

In other words, $F(x) + C$ is also antiderivative of $f(x)$ if $F(x)$ is antiderivative of $f(x)$. Then the antiderivative is denoted by the following

$$\int f(x)dx$$

where $\int$ is called the integral sign, $f(x)$ is called integrand and dx is called differential. And it is called indefinite integral of $f(x)$.

1.8 Example

Confirm the following indefinite integrals:

1) $\int \frac{1}{2\sqrt{x}}dx = \sqrt{x} + C,$

2) $\int \frac{1}{x}dx = \ln|x| + C,$

3) $\int e^{ax}dx = e^{ax}/a + C,$

4) $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1.$

In 1), it is obvious that

$$(\sqrt{x} + C)' = \frac{1}{2\sqrt{x}}$$

i.e. equal to integrand. As the same way, we can prove the others.

1.9 Proposition

For any continuous function, $f(x)$, $g(x)$,

1. $\int cf(x)dx = c\int f(x)dx$
2. $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

1.10 Practice

Find the indefinte integrals of the forllowing functions:

1. $2x^2 - 3x - 9$,
2. $|x|$,
3. $\sin(ax + c)$,
4. $(x + 3)^9$,
5. $\frac{1}{9+x^2}$,
6. $x^{3/2}$,
7. 2^x.

In [1]:
```
from sympy import Symbol,symbols,Abs,sin,exp,ir
```

In [2]:
```
x,a,c=symbols("x a c")
```

In [3]:
```
pprint(integrate(2*x**2-3*x-9,x))
```

```
   3      2
2⋅x    3⋅x
──── - ──── - 9⋅x
 3      2
```

In [4]:
```
pprint(integrate(Abs(x),x))
```

```
⌠
⎮ │x│ dx
⌡
```

In [5]:
```
pprint(integrate(sqrt(x**2),x))
```

```
 2
x
──
2
```

In [6]:
```
pprint(integrate(sin(a*x+c),x))
```

```
⎧   x⋅sin(c)      for a = 0
⎪
⎨-cos(a⋅x + c)
⎪──────────────   otherwise
⎩      a
```

In [7]:
```
pprint(integrate((x+3)**9,x))
```

```
 10              8                            5
x         9    81⋅x          7          6    30618⋅x
4           3    59049
─── + 3⋅x   + ────── + 324⋅x   + 1701⋅x   + ──────── +
15309⋅x   + 26244⋅x   + ─────
 10              2                          5
2

  2
⋅x
─── + 19683⋅x
```

In [8]:
```
pprint(integrate(1/(9+x**2),x))
```

```
    ⎛x⎞
atan⎜─⎟
    ⎝3⎠
───────
   3
```

In [9]:
```
pprint(integrate(x**(3/2),x))
```

```
     2.5
0.4⋅x
```

In [10]:
```
pprint(integrate(2**x,x))
```

```
   x
  2
──────
log(2)
```

For the case $|x|$, Sympy got the wrong answer.

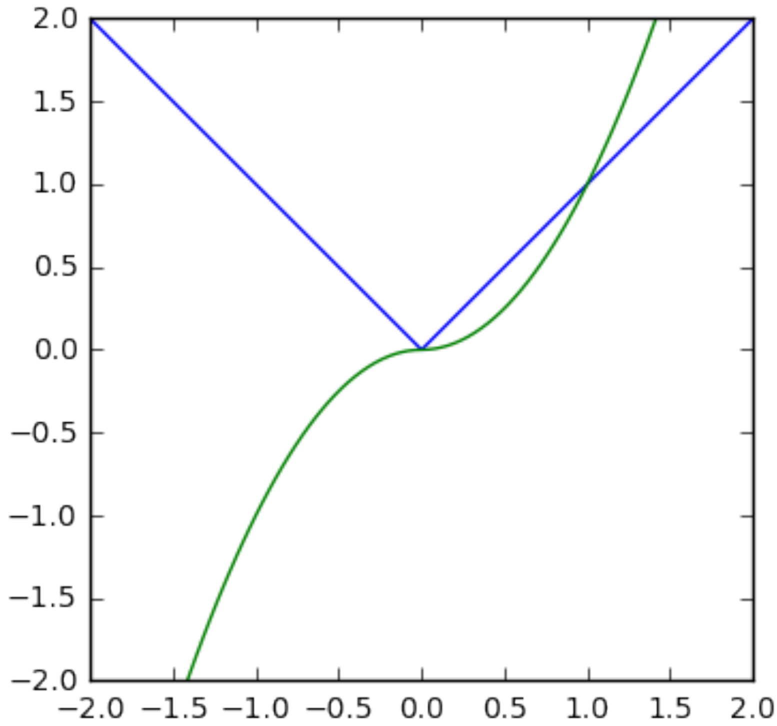
In [11]:
```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

In [12]:
```
def antiabs(x):
    result=x*x
    result[x<0]=-x[x<0]**2
    return result
```

In [15]:

```
x=np.linspace(-2,2,101)
plt.figure(figsize=(4,4))
plt.plot(x,np.abs(x))
plt.plot(x,antiabs(x))
plt.axis("equal")
plt.xlim([-2,2])
plt.ylim([-2,2])
```

Out[15]: (-2, 2)



1.11 Formulas of Antiderivatives

1.12 Table of Integrals

1. $\int a dx = ax + C$
2. $\int x^r dx = \frac{1}{1+r}x^{r+1} + c$ if $r \neq -1$ and $\ln|x| + C$ if $x = -1$
3. $\int \sin x dx = -\cos x + C$, $\int \cos x dx = \sin x + C$, $\int \tan x \sec x dx = \sec x + C$, $\int \cot x \csc x dx = -\csc x + c$,$\int \sec^2 x dx = \tan x + C$, $\int \csc^2 x dx = -\cot x + c$
4. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$
5. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
6. $\int e^x dx = e^x + C$

1.13 Example

Evaluate the following integrals:

- $\int(x^3-3x^2+1)dx$
- $\int(x+3)/x^2dx$
- $\int(x+3\cos x)dx$
- $\int \sec x(\tan x+\sec x)dx$

1.14 Sol

- $$\begin{aligned}\int(x^3-3x^2+1)dx &= \int x^3dx-3\int x^2dx+\int 1dx\\ &= \frac{1}{4}x^4-x^3+x+C\end{aligned}$$

- $$\begin{aligned}\int(x+3)/x^2dx &= \int\frac{1}{x}dx+\int\frac{3}{x^2}dx\\ &= \ln|x|-\frac{3}{x}+C\end{aligned}$$

- $$\begin{aligned}\int(x+3\cos x)dx &= \int xdx+3\int\cos xdx\\ &= \frac{x^2}{2}+3\sin x+C\end{aligned}$$

- $$\begin{aligned}\int\sec x(\tan x+\sec x)dx &= \int\sec x\tan xdx+\int\sec^2xdx\\ &= \sec x+\tan x+C\end{aligned}$$

1.15 Compute by Sympy

Evaluate the following integrals:

a) $\int(x^2-x+3x^{-2})dx$
b) $\int(2\sin x-3\cos x)dx$
c) $\int(4\sin x-x^{-2})dx$
d) $\int\frac{1+x^2}{x}dx$

In [56]:
```
pprint(integrate(x**2-x+3*x**(-2),x))
```

```
 3    2
x    x    3
── - ── - ─
3    2    x
```

In [59]:
```
pprint(integrate(2*sin(x)-3*cos(x),x))
```

```
-3⋅sin(x) - 2⋅cos(x)
```

In [60]:
```
pprint(integrate(4*sin(x)-x**(-2),x))
```

```
            1
-4⋅cos(x) + ─
            x
```

In [62]:
```
pprint(integrate((1+x**2)/x,x))
```

```
 2
x
── + log(x)
2
```

1.16 Example

Evaluate the following integrals:

1. $\int 2x^3 dx$
2. $\int (2x + \sin x) dx$
3. $\int (2x^2 - 1)/x^2 dx$
4. $\int (3x^5 - 2x^3 + 2 - x^{-1/3}) dx$
5. $\int \sin x / \cos^2 x dx$

1.17 Sol

1. $2\int x^3 dx = 2 \cdot x^4/4 + C = x^4/2 + C$
2. $2\int x dx + \int \sin x dx = x^2 - \cos x + C$
3. $\int 2dx - \int x^{-2} dx = 2x + 1/x + C$
4. $3\int x^5 dx - 2\int x^3 + \int 2dx - \int x^{-1/3} dx = x^6/2 - x^4/2 + 2x - 9x/2 + C$
5. $\int (\sin x/\cos x) \cdot (1/\cos x) dx = \int \tan x \sec x dx = \sec x + C$

1.18 Exercise, p. 358

12. $\int (x^2 + 1)/x^2 dx = \int (1 + x^{-2}) dx = x - 1/x + c$

16. $\int (\pi^2 + \pi + 1) dx = (\pi^2 + \pi + 1)x + c$

23. $\int \cos x/(1 - \cos^2 x) dx = \int \cos x/\sin x \sin x dx = \int \cot x \csc x dx = -\csc x + c$

24. $\int \sin 2x/\cos x dx = \int 2\sin x dx = -2\cos x + C$

26.
$\int \cos^2 x/(\cos x - \sin x)dx = \int (\cos^2 - \sin^2 x)/(\cos x - \sin x)dx = \int (\cos x + \sin x)dx =$

28. $\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + C$

81-86, True or False. Assume that $F'(x) = f(x)$, $G'(x) = g(x)$.

81. $\int f'(x)dx = f(x) + C$,

.

82. $\int f(x)g(x)dx = F(x)G(x) + C$,

.

83. $\int xf(x)dx = x\int f(x)dx = xF(x) + C$,

.

84. $\int (2f(x) - 3g(x))dx = 2F(x) - 3G(x) + C$,

.

85. $\int (f(x)/g(x))dx = F(x)/(G(x)) + C$,

.

86. $\int (\int f(x)dx)dx = \mathrm{F}(x) + C_1 x + C_2$, where $\mathrm{F}'(x) = F(x)$,

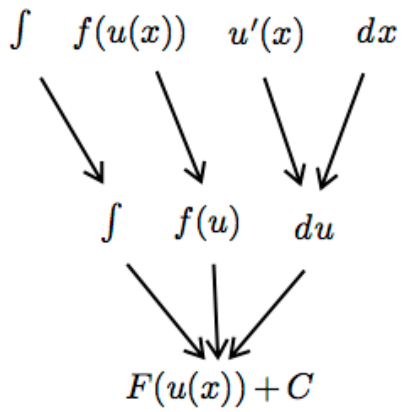
.

1.19 4.2 Integration by Substitution

1.20 Theorem

Suppose that $F'(x) = f(x)$ then

$$\int f(u(x)) \cdot u'(x)dx = F(u(x)) + C$$



1.21 Example

1. Let $u = x^2 + 3$, $du = 2xdx$,

$$\begin{aligned}\int 2x\sqrt{x^2+3}dx &= \int \sqrt{u}du \\ &= \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{3}(x^2+3)^{3/2} + C\end{aligned}$$

- Let $u = 2x - 4$, $du = 2dx$,

$$\begin{aligned}\int \frac{1}{(2x-4)^3}dx &= \frac{1}{2}\int \frac{du}{u^{-3}} \\ &= -\frac{1}{4}u^{-2} + C \\ &= -\frac{1}{4(2x-4)^2} + C\end{aligned}$$

- Let $u = 2x - 1$, then $du = 2dx$,

$$
\begin{aligned}
\int (x+1)\sqrt{2x-1}dx &= \int (u/2+3/2)u^{1/2}\frac{du}{2} \\
&= \int \left(\frac{u^{3/2}}{4} + \frac{\sqrt{u}}{4}\right)du \\
&= \frac{u^{5/2}}{10} + \frac{u^{3/2}}{2} + C \\
&= \frac{(2x-1)^{5/2}}{10} + \frac{(2x-1)^{3/2}}{2} + C
\end{aligned}
$$

- Let $u = 5x$, then $du = 5dx$,

$$
\begin{aligned}
\int \sin 5x dx &= \int \sin u \frac{du}{5} \\
&= -\frac{\cos u}{5} + C \\
&= -\frac{\cos 5x}{5} + C
\end{aligned}
$$

- Let $u = \sqrt{x}$, then $du = \frac{dx}{2\sqrt{x}}$,

$$
\begin{aligned}
\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx &= \int 2\cos u du \\
&= 2\sin u + C \\
&= 2\sin\sqrt{x} + C
\end{aligned}
$$

- Let $u = \sin x$, then $du = \cos x dx$,

$$
\begin{aligned}
\int \cos x \sin^3 x dx &= \int u^3 du \\
&= \frac{u^4}{4} + C \\
&= \frac{\sin^4 x}{4} + \ + C
\end{aligned}
$$

1.22 Example

1. Let $u = x^2$, $du = 2xdx$,

$$\begin{aligned}\int 2x\sin x^2 dx &= \int \sin u du \\ &= -\cos u + C \\ &= -\cos x^2 + C\end{aligned}$$

2. Let $u = x^3$, $du = 3x^2 dx$,

$$\begin{aligned}\int \frac{x^2}{1+x^3}dx &= \frac{1}{3}\int \frac{du}{u} \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{\ln|1+x^3|}{3} + C\end{aligned}$$

3. Let $u = 9 + x^2$, then $du = 2xdx$,

$$\begin{aligned}\int 3x\sqrt{9+x^2}dx &= 3/2\int u^{1/2}du \\ &= u^{3/2} + C \\ &= \sqrt{(9+x^2)^3} + C\end{aligned}$$

4. Let $u = 5x^{6/5} - 3x$, then $du = 3(2x^{1/5} - 1)dx$,

$$\begin{aligned}\int (5x^{6/5} - 3x)^{3/2}(6x^{1/5} - 3)dx &= \int u^{3/2}du \\ &= 2/5u^{5/2} + C \\ &= 2/5\sqrt{(5x^{6/5} - 3x)^5} + C\end{aligned}$$

1.23 Computer practicea

Evaluate the following indefinite integrals:

a) $\int (x^3 + x^2 + x + 1)(3x^2 + 2x + 1)dx$
b) $\int x/\sqrt{1+x^2}dx$
c) $\int x\sqrt[3]{1+x}dx$
d) $\int \frac{xdx}{1+x^2}$
e) $\int \tan^2 x \sec^2 x dx$
f) $\int xe^{-x^2/2}dx$
g) $\int (x + \frac{1}{x})(1 - \frac{1}{x^2})dx$
h) $\int (1 + 1/x)^2/x^2 dx$

In [63]:
```
pprint(integrate((x**3+x**2+x+1)*(3*x**2+2*x+1)
```

```
 6           4           2
x     5   3⋅x       3   3⋅x
── + x  + ──── + 2⋅x  + ──── + x
2          2             2
```

In [64]:
```
pprint(integrate((x)/sqrt(x**2+1),x))
```

```
   ________
  ╱  2
╲╱  x  + 1
```

Note

$$x(1+x)^{1/3} = (t-1)t^{1/3} = t^{4/3} - t^{1/3}$$

where $t = 1 + x$.

In [71]:
```
t=Symbol("t")
pprint(integrate((t)**(4/3)-t**(1/3),t))
```

```
        1.33333333333333                         2.333
33333333333
- 0.75⋅t                 + 0.428571428571429⋅t
```

In [72]:
```
pprint(integrate((x)/(x**2+1),x))
```

```
   ⎛ 2    ⎞
log⎝x  + 1⎠
───────────
     2
```

In [74]:
```
from sympy import sec,tan
pprint(integrate(sec(x)**2+tan(x)**2,x))
```

```
     2⋅sin(x)
-x + ────────
      cos(x)
```

In [75]:
```
pprint(integrate(x*exp(-x**2/2),x))
```

```
   2
 -x
 ───
  2
-e
```

In [76]:
```
pprint(integrate((x+1/x)*(1-1/x**2),x))
```

```
 2
x      1
── + ────
2       2
     2⋅x
```

In [77]:
```
pprint(integrate((1+1/x)**2/x**2,x))
```

```
 ⎛   2          ⎞
-⎝3⋅x  + 3⋅x + 1⎠
──────────────────
          3
       3⋅x
```

1.24 Another Application for Integration

Solve the initial value problem, (**IVP**):

$$f'(x) = x^3(x^2+1)^{1/2}, f(0) = 0$$

1. this is equivalent to evaluate the integration:

Let $u = x^2 + 1$, $du = 2xdx$,

$$\begin{aligned}\int x^3\sqrt{x^2+1}dx &= \int (u-1)\sqrt{u}\frac{du}{2}\\ &= \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3} + C\\ &= \frac{(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} + C\end{aligned}$$

2. since $f(0) = 0$,

$$0 = f(0) = \frac{(0^2+1)^{5/2}}{5} - \frac{(0^2+1)^{3/2}}{3} + C \rightarrow C = \frac{2}{15}$$

Thus $f(x) = \frac{(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} + \frac{2}{15}$.

1.25 Exercise, p367

In [1]:
```
from sympy import Symbol,integrate,sin,cos,sqrt
x=Symbol("x")

```

In [2]:

```python
def Int(func,var):
    #print("the integral of ")
    Integral(func,x)
    pprint(func)
    print("is:")
    pprint(integrate(func,var))
```

In [29]:

```python
# 8. u=2x**3-1
pprint(integrate(x**2*(2*x**3-1)**4,x))
```

$$\frac{16 \cdot x^{15}}{15} - \frac{8 \cdot x^{12}}{3} + \frac{8 \cdot x^{9}}{3} - \frac{4 \cdot x^{6}}{3} + \frac{x^{3}}{3}$$

In [17]:

```python
func1=x**2*(2*x**3-1)**4
Int(func1,x)
```

the integral of

$$x^{2} \cdot \left(2 \cdot x^{3} - 1\right)^{4}$$

is:

$$\frac{16 \cdot x^{15}}{15} - \frac{8 \cdot x^{12}}{3} + \frac{8 \cdot x^{9}}{3} - \frac{4 \cdot x^{6}}{3} + \frac{x^{3}}{3}$$

In [3]:

```python
# 16 2x(1-4x**2)*(1/3)
func2=2*x*(1-4*x**2)**Rational(1,3)
Int(func2,x)
```

$$2 \cdot x \cdot \sqrt[3]{- 4 \cdot x^{2} + 1}$$

is:

$$\frac{3 \cdot x^{2} \cdot \sqrt[3]{- 4 \cdot x^{2} + 1}}{4} - \frac{3 \cdot \sqrt[3]{- 4 \cdot x^{2} + 1}}{16}$$

In [ ]:

```python
# 26 (2x+3)(x-1)**(1/2)
func3=(2*x+3)*(x-1)**(1/2)
Int(func3,x)
```

In [33]:

```
# 36 sin x/(1+cos x)**3
func4=sin(x)/(1+cos(x))**3
Int(func4,x)
```

```
    sin(x)
────────────
           3
(cos(x) + 1)
is:
           1
────────────────────────
     2
2⋅cos (x) + 4⋅cos(x) + 2
```

In [4]:

```
#49 1/(x**(1/2)+(1+x)**(1/2))
func5=1/(sqrt(x)+sqrt(1+x))
Int(func5,x)
```

```
      1
──────────────
         _______
√x + ╲╱ x + 1
is:
             _______
  2⋅√x⋅╲╱ x + 1                 4⋅x                      2
────────────────── + ────────────────── + ─────────
─────────
                  _______               _______
      _______
3⋅√x + 3⋅╲╱ x + 1    3⋅√x + 3⋅╲╱ x + 1    3⋅√x + 3⋅
╲╱ x + 1
```

In [36]:

```
from sympy import assume
a=Symbol("a")

```

In [38]:

```python
# 50 (a**2-x**2)/x**4, (x=1/t)

func6= (sqrt(a**2-x**2)/x**4)
Int(func6,x)
```

$$\frac{\sqrt{a^{2} - x^{2}}}{x^{4}}$$

is:

$$\begin{cases} - \frac{\sqrt{\frac{a^{2}}{x^{2}} - 1}}{3 \cdot x^{2}} + \frac{\sqrt{\frac{a^{2}}{x^{2}} - 1}}{3 \cdot a^{2}} & \text{for } \left|\frac{a^{2}}{x^{2}}\right| > 1 \\ - \frac{i \cdot \sqrt{- \frac{a^{2}}{x^{2}} + 1}}{3 \cdot x^{2}} + \frac{i \cdot \sqrt{- \frac{a^{2}}{x^{2}} + 1}}{3 \cdot a^{2}} & \text{otherwise} \end{cases}$$

In [41]:

```python
# slope of tangent line of function is x/(2x**2
Int(x/(2*x**2+1)**(Rational(3,2)),x)
```

$$\frac{x}{\left(2 \cdot x^{2} + 1\right)^{3/2}}$$

is:

$$\frac{-1}{2 \cdot \sqrt{2 \cdot x^{2} + 1}}$$

Since $f(2) = -1/6, f(x) = -1/\left(2\sqrt{2x^2+1}\right)$.

1.26 True or False

61. If $f(x)$ is continuous, then $\int xf(x^2)dx = \frac{1}{2}\int f(u)du$, where $u = x^2$. (T, by substitution mathod)

62. If $f(x)$ is continuous, then $\int f(ax+b)dx = \int f(x)dx$. (F, $f(x) = x$)

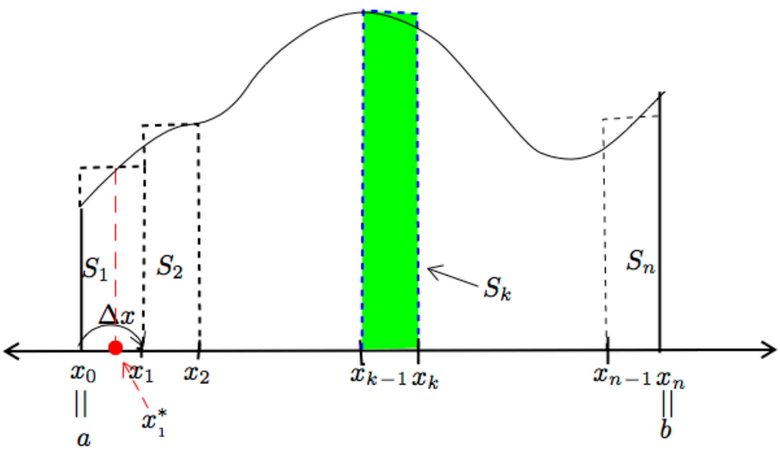
1.27 4.3&4.4 Area and Definite Integral

Assume that $f(x)$ is continuous and nonnegative (i.e. $f(x) \geqslant 0$) on $[a, b]$. Riemann gave a clear procedure to find out the area of the region under the graph of $f(x)$ and above X-axis. First partition $[a, b]$ into n non-overlapping and equally-like subintervals with partition points:

$$a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{n-1} \leqslant x_n = b$$

$$\Delta x = \frac{b-a}{n}$$

$$x_k = a + k\Delta x \text{ for } k = 0, 1, \cdots, n$$



where

1. S: the region under the graph of $f(x)$ and above X-axis from $x = a$ to $x = b$,
2. $|S|$: area of S,
3. S_k: the region under the graph of $f(x)$ and above X-axis from $x = x_{k-1}$ to $x = x_k$, for $k = 1, \cdots, n$.
4. $|S_k|$: area of S_k.

Consider the region, S_k, take any point, says x_k^*, in $[x_{k-1}, x_k]$. Then we have:

$$|S_k| \backsim f(x_k^*)\Delta x$$

i.e. the area of S_k can be approximated by area of rectangle with height, $f(x_k^*)$. Then we have the approximation of area of all the region, S:

$$|S| \backsim R_n(f) = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

where $R_n(f)$ is called the Riemann sum of f. Since $f(x)$ is continuous on $[a, b]$, and also on subintervals $[x_{k-1}, x_k]$ for $k = 1, \cdots, n$. From the existence of extrema, there exist the maximum, M_k, and minimum, m_k, of $f(x)$ on $[x_{k-1}, x_k]$ for each k. Then

$$\sum_{k=1}^{n} m_k \Delta x \leqslant |S| \leqslant \sum_{k=1}^{n} M_k \Delta x$$

Consider the difference between the sum of maxima and sum of minima:

$$\begin{aligned} \left|\sum_{k=1}^{n} (M_k \Delta x - m_k \Delta x)\right| \quad &\leqslant \quad \sum_{k=1}^{n} \left|M_k - m_k\right| \Delta x \\ &\leqslant \quad \max_{k=1}^{n} \left|M_k - m_k\right| \sum_{k=1}^{n} \Delta x \\ &\leqslant \quad (b-a) \cdot \max_{k=1}^{n} \left|M_k - m_k\right| \\ &\longrightarrow 0 \end{aligned}$$

as $n \to \infty$. This means that

$$\lim_{n\to\infty} \sum_{k=1}^{n} m_k \Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} M_k \Delta x$$

Therefore by squeeze theorem, $|S|$ exists and is equal to the above limit.

1.28 Example

The region under the graph of $f(x) = x^2$ from $x = 0$ to $x = 1$ is $1/3$.

1. Partition $[0, 1]$ into $[0, 1/n, 2/n, \cdots, n/n = 1]$ and $\triangle x = 1/n$.
2. $f(x_i^*) = (i-1)/n$ where x_i^* is the left end in sub-interval, $[x_{i-1}, x_i]$.
3. Area of S_k, S:

$$
\begin{aligned}
S_k &= f(x_k^*) \triangle x = \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n} \\
S &= \sum_{k=1}^{n} S_k \\
&= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \cdot \frac{1}{n} \\
&= \sum_{k=1}^{n} \frac{(k+\ -1)^2}{n^3} \\
&= \frac{(n-1)(n)(2n-1)}{6n^3} \to 1/3
\end{aligned}
$$

while $n \to \infty$.

1.29 Theorem

Suppose that $f(x)$ is continuous and nonnegative on $[a, b]$. Then the area of the region under the graph of $f(x)$ and above X-axis exists and is equal to:

$$|S| = \lim_{n \to \infty} R_n$$

where R_n is the Riemann sum defined as above. $f(x)$ is called integrable in such case. Also

the area is denoted as the following symbol, called definite integral of $f(x)$ on $[a, b]$:

$$|S| = \int_a^b f(x)dx$$

where a, b are called the *lower* limit and *upper* limit, $f(x)$ is called *integrand* and dx is the *differential* of x. Furthermore, the condition of nonnegative property of $f(x)$ can be removed. In such condition, the definite integral is equal to the sum of area of region above X-axis minus the sum of area.

1.30 Properties of Definite Integral

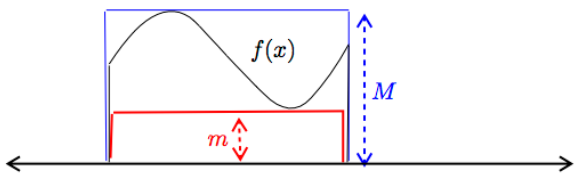
Properties of Definite Integral

Assume that $f(x)$ and $g(x)$ are continuous on $[a, b]$, $c \in (a, b)$, $k \in \mathrm{R}$,

1. $\int_a^a f(x)dx = 0$
2. $\int_b^a f(x)dx = -\int_a^b f(x)dx$
3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. If $f \geqslant 0$, then $\int_a^b f(x)dx \geqslant 0$
6. If $f(x) \geqslant g(x)$, then $\int_a^b f(x)dx \geqslant \int_a^b g(x)dx$
    - Since $f(x) - g(x) \geqslant 0$, then the definite integral of this difference function is also nonnegative. Then the property is proved.
7. $|\int_a^b f(x)dx| \leqslant \int_a^b |f(x)|dx$
    - Note that $-|f(x)| \leqslant f(x) \leqslant |f(x)|$. This can be proved by last property.
8. Suppose that M, m are the max and min of $f(x)$ on $[a, b]$, then

$$m \leqslant I_f = \frac{1}{b-a}\int_a^b f(x)dx \leqslant M$$

   where I_f is called the average of $f(x)$ over $[a, b]$.



   - $$(b-a)m \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$

     since the related region is bounded by the rectangles with the same base and the heights which are the maximum and minimum of $f(x)$ on $[a, b]$.
9. (Mean Value Theorem for Integration) If $f(x)$ is \ continuous on $[a, b]$, there exists at least one c in (a, b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

   And it is called the average of $f(x)$ on $[a, b]$.

   - Since I_f, the average of $f(x)$ over $[a, b]$, is between the maximum and minimum of $f(x)$ on $[a, b]$, there exists at least one point c in (a, b) such that $f(c)$ is equal to I_f by intermediate value theorem.
10. Suppose that $f(x)$ is continuous on $[-a, a]$ and is odd where $a > 0$. Then

$$\int_{-a}^{a} f(x)dx = 0$$

by symmetry. Suppose that $f(x)$ is continuous on $[-a, a]$ and is even where $a > 0$. Then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

by symmetry.

1. $\int_{-2}^{2}(x^2 - 2x + 4)dx = 2\int_{0}^{2}(x^2 + 4)dx = 21\frac{1}{3}$.
2. $\int_{2}^{2}(x^2 - 2x + 4)dx = 0$.
3. Estimate $\int_{1}^{3}\sqrt{3 + x^2}dx$.

$$4 = \int_{1}^{3}\sqrt{3 + 1^2}dx \leq \int_{1}^{3}\sqrt{3 + x^2}dx \leq \int_{1}^{3}\sqrt{3 + 3^2}dx = 4\sqrt{3}$$

4. The region under the graph of $f(x) = 4 - x^2$ from $x = -1$ to $x = 3$ is $1/3$.

- Partition $[-1, 3]$ into $[-1, -1 + 4/n, -1 + 2 \cdot 4/n, \cdots, -1 + n \cdot 4/n = 3]$ and $\triangle x = (3 - (-1))/n = 4/n$.
- $f(x_i^*) = -1 + i \cdot 4/n$ where x_i^* is the right end in sub-interval, $[x_{i-1}, x_i]$.
- Area of S_k, S:

$$
\begin{aligned}
S_k &= f(x_k^*) \triangle x = \left(4 - \left(-1 + \frac{4k}{n}\right)^2\right) \cdot \frac{4}{n} \\
&= \left(12 + \frac{32k}{n} - \frac{64k^2}{n^2}\right) \cdot \frac{1}{n} \\
S &= \sum_{k=1}^{n} S_k \\
&= \sum_{k=1} \left[\frac{12}{n} + \frac{32k}{n^2} - \frac{64k^2}{n^3}\right] \\
&= \frac{12n}{n} + \frac{32n(n+1)}{2n^2} - \frac{64n(n+1)(2n+1)}{6n^3} \\
&\Longrightarrow 12 + 16 - \frac{64}{3} = 6\frac{2}{3}
\end{aligned}
$$

while $n \to \infty$. This means

$$\int_{-1}^{3}(4 - x^2)dx = 6\frac{2}{3} \text{ and } \int_{3}^{-1}(4 - x^2)dx = -\int_{-1}^{3}(4 - x^2)dx = -6\frac{2}{3}$$

5.

a.) $\int_0^1(x^2 - 4)dx = \int_0^1 x^2 dx - \int_0^1 4dx = 1/3 - 4;$
b). $\int_0^1(5x^2)dx = 5\int_0^1 x^2 dx = 5/3.$

6. The mean value of $f(x) = 4 - 2x$ on $[0, 2]$ is evaluated by the following:

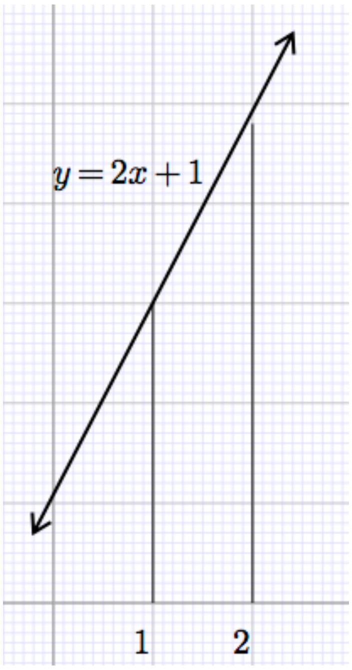
$$
\begin{aligned}
4 - 2c = f(c) = &\frac{1}{2-0}\int_0^2 f(x)dx = 4 \\
\rightarrow \quad & c = 1
\end{aligned}
$$

7. Find the

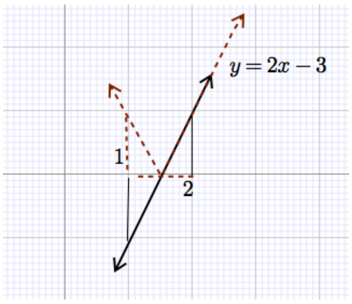
1.31 Example

Find out the definite integrals:

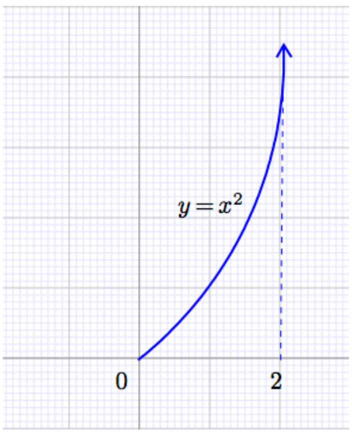
1. $\int_1^2 (2x+1)dx = 4$



2. $\int_1^2 (2x-3)dx = 0$

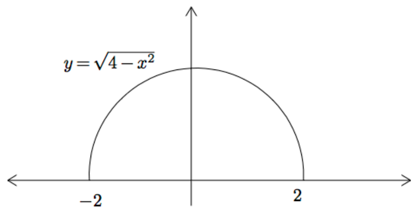


3. $\int_1^2 |2x-3| dx = 1/2$
4. $\int_0^2 x^2 dx$ can not be calculated by the area method, but can be done by the Riemann method:



$$\int_0^2 x^2 dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2k}{n}\right)^2\cdot\frac{2}{n}$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{8}{n^3}\cdot k^2$$
$$= \lim_{n\to\infty}\frac{8}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1)$$
$$= \frac{8}{3}$$

5. $\int_0^2\sqrt{4-x^2}dx = \frac{4\pi}{4} = \pi$, since the region under the graph $\sqrt{4-x^2}$ on $[0,2]$ is one quarter of circle with radius 2.



6. Find the average value of $f(x) = \sqrt{4-x^2}$ over $[0,2]$.

   - By the definition of average value of function, we have

$$I_f = \frac{\pi}{2-0} = \frac{\pi}{2}$$

   and by the formula, it occurs at:

$$\sqrt{4-x_0^2} = \frac{\pi}{2} \implies 4-x_0^2 = \left(\frac{\pi}{2}\right)^2$$
$$\implies x_0 = \sqrt{4-\left(\frac{\pi}{2}\right)^2}$$

7.

- Since $\sin x$ is odd, $\int_{-a}^{a}\sin x dx = 0$ for any $a \in \mathrm{R}$.
- $\int_{-1}^{1}|x|dx = \int_0^1 x dx = 1$.

1.32 Example

1. Area of region under the graph of $f(x) = 4-x^2$ on $[-1,3]$, $\int_{-1}^{3}(4-x^2)dx = 6\frac{2}{3}$.
2. $\int_{-4}^{4}\sqrt{16-x^2}dx = 8\pi$

1.33 Computer practice

Calculate the definite integrals:

a). $\int_{-2}^{2}(x+1)dx$

b). $\int_{0}^{5}|2x-4|dx$

c). $\int_{-3}^{3}\sqrt{9-x^2}dx$

d). $\int_{-2}^{2}x^3dx$

e). $\int_{-2}^{2}(x^3+1)dx$

In [80]:
```
?integrate
```

In [81]:
```
integrate(x+1,(x,-2,2))
```

Out[81]: 4

In [83]:
```
integrate(-(2*x-4),(x,0,2))+integrate((2*x-4),(
```

Out[83]: 13

In [85]:
```
integrate(sqrt(9-x**2),(x,-3,3))
```

Out[85]: 9*pi/2

In [86]:
```
integrate(x**3,(x,-2,2))
```

Out[86]: 0

In [87]:
```
integrate(x**3+1,(x,-2,2))
```

Out[87]: 4

1.34 Computer Practice

Find the average of $f(x)=\sqrt{x+1}$ over $[1,4]$.

1. integrate $f(x)=\sqrt{x+1}$ over $[1,4]$:

2. divided by length of inteval:

In [3]:
```
from sympy import solve,symbols,integrate,diff
```

In [91]:
```
avg=integrate(sqrt(x+1),(x,1,4))/(4-1)
avg
```

Out[91]: -4*sqrt(2)/9 + 10*sqrt(5)/9

In [99]:
```
pprint(solve(sqrt(x+1)-avg,x)[0])
```

$$-\frac{80\cdot\sqrt{10}}{81} + \frac{451}{81}$$

In [100]:
```
def favg(func,a=0,b=1):
    avg=integrate(func,(x,a,b))/(b-a)
    pprint(solve(func-avg,x)[0])
```

In [101]:
```
favg(sqrt(x+1),a=1,b=4)
```

$$-\frac{80\cdot\sqrt{10}}{81} + \frac{451}{81}$$

1.35 Exercises, p401

14. $\lim_{n\to\infty} \sum_{k=1}^{n} 2c_k(1-c_k)^2 \triangle x = \int$ $\mathbf{dx}$

26. $\int_0^2 \sqrt{-x^2+2x}dx = \int_{\mathbf{0}}^{\mathbf{2}} \sqrt{} dx =$

36. prove $\int_0^1 x^2 dx \le \int_0^1 \sqrt{x}dx$; since $\mathbf{x^2} \le \sqrt{\mathbf{x}}$ for $x \in [0,1]$

44. Estimate $\int_{\pi/4}^{\pi/2} x\sin x dx$:

$$\frac{\pi^2}{16\sqrt{2}} = \int_{\pi/4}^{\pi/2} \frac{\pi}{4}\sin\frac{\pi}{4}dx \le \int_{\pi/4}^{\pi/2} x\sin x dx \le \int_{\pi/4}^{\pi/2} \frac{\pi}{2}\sin\frac{\pi}{2}dx = \frac{\pi^2}{8}$$

59. Determine whether the Dirichelet function:

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{otherwise} \end{cases}$$

- $x_k^* \in \mathrm{Q}$: $R_n(f) = \sum_{k=1}^{n} f(x_k^*)\frac{1}{n} = \sum_{k=1}^{n} 1 \cdot \frac{1}{n} = 1 \to 1;$
- $x_k^* \in \mathrm{Q} \setminus \mathrm{R}$: $R_n(f) = \sum_{k=1}^{n} f(x_k^*)\frac{1}{n} = \sum_{k=1}^{n} 0 \cdot \frac{1}{n} = 0 \to 0;$

  These limits, taking sample points being rationals or irrationals respectively, are different; this means that the limit of Riemann sum **is** not convergent!

64. (T or F) If $f(x)$ is continuous on $[a,b]$. and $\int_a^b f(x)dx > 0$. Then $f(x) > 0$ on $[a,b]$.

66. (T or F) If $f(x)$ is nonnegative and continuous on $[a,b]$. and $a < c < d < b$. Then $\int_c^d f(x)dx < \int_a^b f(x)dx$.

1.36 4.5 Fundamental Theorem of Calculus

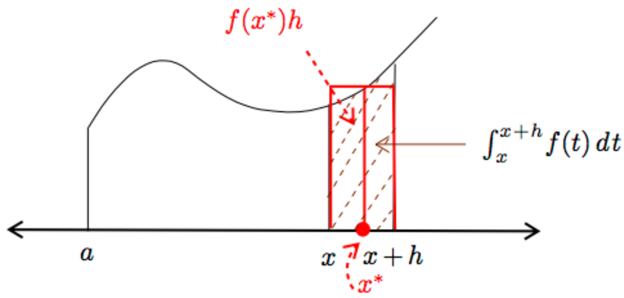
1.37 Fundamental Theorem of Calculus, FTC

(**Part I, Differentiation under Integral sign**)If $f(x)$ is \ continuous on $[a, b]$, then $\int_a^x f(t)dt$ is differentiable for $x \in (a, b)$,and

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

(**Part II, Evaluate Definite Integral via Indefinite Integral**) Assume that $F(x)$ is an indefinite integral of $f(x)$, then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$



From the above picture, let

$$F(x) = \int_a^x f(t)dt$$

Then

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} &= \frac{1}{h}\int_x^{x+h} f(t)dt \\ \Longrightarrow \quad \frac{1}{h}\int_x^{x+h} f(t)dt &= f(x^*) && \text{by MVT} \\ \text{and } f(x^*) &\longrightarrow f(x) && \text{as } h \to 0 \\ \overset{h\to 0}{\Longrightarrow} \quad F'(x) &= f(x) \end{aligned}$$

The last one holds since $f(x)$ is continuous. And the second part of FTC can be proved by the defintion of indefinite integral:

Obviously, $\int_a^x f(t)dt$ is also an indefinite integral of $f(x)$ by the first part of FTC. Then

$$\int_a^x f(t)dt = F(x) + C \text{ for some } C \in \mathrm{R}$$

Taking $x = a$ gets $C = -F(a)$. Taking $x = b$ proves the FTC.

1.38 Example

1. $\left(\int_{-1}^{x}\frac{dt}{1+t^2}\right)' = \frac{1}{1+x^2}$.
2. $\left(\int_{x}^{3}\sqrt{1+t^2}dt\right)' = \sqrt{1+x^2}$.
3. $\left(\int_{0}^{x^3}\cos t^2 dt\right)' = 3x^2\cos x^2$.

1.39 Example

Differentiate the following:

a) $\int_0^x te^{-t/2}dt$
b) $\int_2^x te^{-t/2}dt$
c) $\int_0^{2x} te^{-t/2}dt$
d) $\int_x^1 te^{-t/2}dt$
e) $\int_x^{2x} te^{-t/2}dt$

1.40 Solution

a), b) Let $F(x) = \int_c^x te^{-t/2}dt$ for any $c \in \mathbb{R}$. Then by FTC, we have

$$F'(x) = xe^{-x/2}$$

c) Since $\int_0^{2x} te^{-t/2}dt$ can be rewritten as $F(2x)$, by chain rule, its derivative is:

$$\frac{d}{dx}F(2x) = (2x)'F'(2x) = 2 \cdot 2xe^{-x}$$

d) Since

$$\int_x^1 te^{-t/2}dt = \int_0^1 te^{-t/2} - \int_0^x te^{-t/2}$$

differentiating both sites gets

$$\frac{d}{dx}\int_x^1 te^{-t/2}dt = -xe^{-x/2}$$

e) Note this integral is equal to $F(2x) - F(x)$. Theirfore the derivative of this integral is equal to

$$\frac{d}{dx}\int_x^{2x} te^{-t/2}dt = 4xe^{-x} - xe^{-x/2}$$

1.41 Computer practice

Let $F(x) = \int_0^x e^{-t^2} dt$. Evaluate the following:

a) $\frac{d}{dx}\int_0^{x^2} e^{-t^2} dt$

b) $\frac{d}{dx}\int_0^{1-x} e^{-t^2} dt$

c) $\frac{d}{dx}\int_{1-x}^{x^2} e^{-t^2} dt$

In [5]:
```python
from sympy import exp,pprint,symbols
t,x = symbols("t x")
```

In [7]:
```python
diff(integrate(exp(-t**2/2),(t,0,x)),x)
```

Out[7]: exp(-x**2/2)

In [1]:
```python
from sympy import diff

def FTCI(f,a,b):
    print("                             ",b)
    print("The derivative of  ∫ %s dt is " %f)
    print("                             ",a)
    pprint(diff(integrate(f,(t,a,b)),x))
```

In [11]:
```python
FTCI(exp(-t**2/2),0,x)
```

```
                    x
The derivative of  ∫ exp(-t**2/2) dt is
                    0
   2
 -x
 ───
  2
e
```

In [12]:
```python
FTCI(exp(-t**2/2),0,x**2)
```

```
                    x**2
The derivative of  ∫ exp(-t**2/2) dt is
                    0
       4
     -x
     ───
      2
2⋅x⋅e
```

In [13]:
```
FTCI(exp(-t**2/2),0,1-x)
```

```
                    -x + 1
The derivative of  ∫ exp(-t**2/2) dt is
                    0
          2
 -(-x + 1)
 ──────────
      2
-e
```

In [14]:
```
FTCI(exp(-t**2/2),1-x,x**2)
```

```
                    x**2
The derivative of  ∫ exp(-t**2/2) dt is
                    -x + 1
     4              2
  -x       -(-x + 1)
  ───      ──────────
   2           2
2⋅x⋅e     + e
```

1.42 Natural Logarithmic function

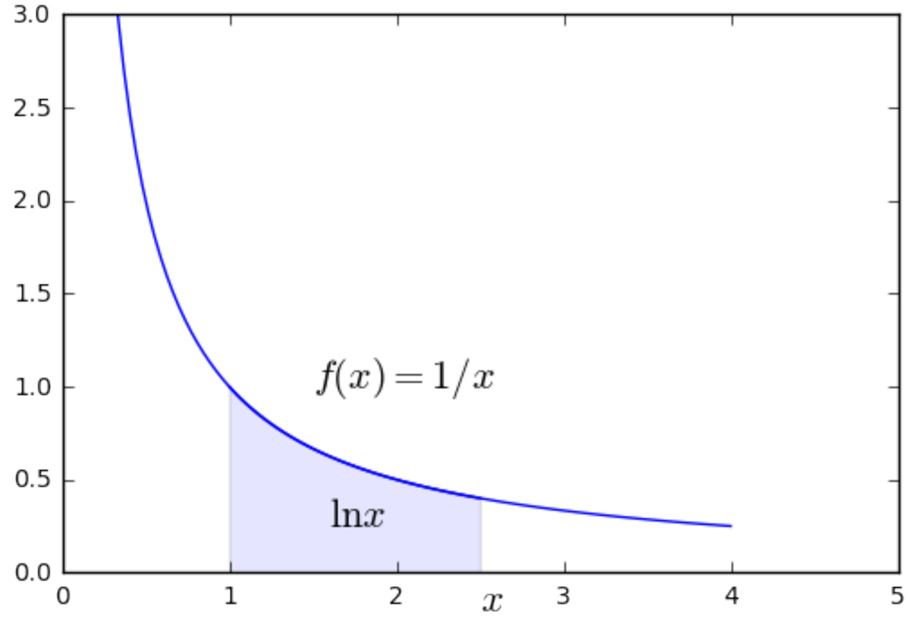
The natural logarithmic function, $\ln x$, is defined as the area of region of $f(x) = 1/x$ from 1 to x:

$$\ln x = \int_1^x \frac{dt}{t}$$

In [14]:

```
x=np.linspace(1,2.5,101)
y=np.linspace(1/3,4,101)
plt.plot(x,1/x,'blue')
plt.plot(y,1/y,'blue')
plt.fill_between(x,1/x,alpha=0.1)
plt.text(1.5,1, "$f(x)=1/x$",size=16)
plt.text(1.6,0.25, "$\ln x$",size=16)
plt.text(2.5,-0.2, "$x$",size=16)
plt.xlim(0,5)
plt.ylim(0,3)
```

Out[14]: (0, 3)



Derivative of $\ln x$ is

$$(\ln x)' = \frac{d}{dx}\int_1^x \frac{dt}{t} = \frac{1}{x}$$

by FTC.

1.43 Example

1. $(f(x))' = (\sqrt{x^2+1})' = \frac{x}{1+x^2}$.
2. (Logaritmic Differentiation) Assume $f(x) = \frac{(2x-1)^3}{\sqrt{3x+1}}$. Then

   - $(\ln f)' = f'/f$;
   - but $\ln(f(x)) = 3\ln(2x-1) - (3x+1)/2$
   - and

$$(f(x))' = f \cdot (\ln f)' = f \cdot \left(\frac{6}{2x-1} - \frac{3}{2(3x+1)}\right)$$

1.44 p.529

78. $\int \frac{\sqrt{1+\ln x}}{x} dx = 2u^{3/2}/3 + C = 2(1+\ln x)^{3/2}/3 + C$ where $u = 1 + \ln x$.

92.

$$\frac{d}{dx}\int_{2/x}^{x^2} \frac{dt}{t} = \frac{2x}{x^2} - \frac{-2x^{-2}}{2/x} = \frac{3}{x}$$

.

1.45 Example

Since an antiderivative for $f(x) = x$ is $F(x) = \frac{1}{2}x^2$. Thus, we have the following result:

$$\begin{aligned} \int_0^2 x dx &= \left.\frac{x^2}{2}\right|_0^2 \\ &= 8 \end{aligned}$$

1.46 Example

Calculate the following definite integral:

$$\begin{aligned} \int_{-1}^{1} (4x^3 - 3x^2 + 9) dx &= \left. x^4 - x^3 + 9x \right|_{-1}^{1} \\ &= 16 \end{aligned}$$

1.47 Natural Exponential Functions

$\exp x = y$ if and only if $\ln y = x$.

1.48 Derivative of Exponentials

$\frac{\ln y}{dx} = \frac{dx}{dx} = 1$ implies $y' = y = \exp x$.

1.49 Example

1. $(\exp x)'\Big|_{x=0} = \exp 0 = 1;$

2. $$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$

1.50 p.549

38. $(e^{-x}\tan e^x)' = -e^{-x}\tan e^x + \sec^2 e^x.$

92. $\int \frac{e^{-1/x}}{x^2}dx = e^u + c = e^{-1/x} + c$ where $u = -1/x.$

Since by basci property of integration, (10), the above example can be derived as:

$$\begin{aligned}\int_{-1}^{1}(4x^3 - 3x^2 + 9)dx &= 2\int_0^1(-3x^2 + 9)dx \\ &= 16\end{aligned}$$

1.51 Eexample

Evaluate the following differentiation:

$$\frac{d}{dx}\int_0^x t^{1/2}e^{-t}dt = x^{1/2}e^{-x}$$

Note that different lower limit does not inflence the result!

1.52 Example

Evaluate the following differentiation:

1. $$\begin{aligned}\frac{d}{dx}\int_x^1 t^{1/2}e^{-t}dt &= -\frac{d}{dx}\int_0^x t^{1/2}e^{-t}dt \\ &= -x^{1/2}e^{-x}\end{aligned}$$

2. $$\begin{aligned}\frac{d}{dx}\int_0^{x^2} t^{1/2}e^{-t}dt &= (x^2)' \cdot t^{1/2}e^{-t}\Big|_{t=x^2} \\ &= 2x^2e^{-x^2}\end{aligned}$$

3. $$\frac{d}{dx}\int_x^{x^2} t^{1/2}e^{-t}dt = 2x^2e^{-x^2} - x^{1/2}e^{-x}$$

The technique of differentiating under integral sign is usually used to find the the probability density function with known distribution function in Probability and Statistics.

1.53 Exercise

Evaluate $\int_0^2 \frac{x-1}{x+2}dx$ and $\int_0^{\ln 2} \frac{e^x}{1+e^x}dx$.

1.54 Solution

- Since $(x-1)/(x+2) = 1 - 3$ and $(x+2)$,we have

$$\int_0^2 \frac{x-1}{x+2}dx = \int_0^2 (1 - \frac{3}{x+2})dx = (x - 3\ln|x+2|)\Big|_0^2 = 2 - 3\ln 2$$

- Let $u = 1 + e^x$, then $du = e^x dx$ and

$$\int \frac{e^x}{1+e^x}dx = \ln|1+e^x| + C$$

Then

$$\int_0^{\ln 2} \frac{e^x}{1+e^x}dx = \ln|1+e^x|\Big|_0^{\ln 2} = \ln 3 - \ln 2$$

1. $\int_1^2 (x^3 - 2x^2 + 1)dx = \left(x^4/4 - 2x^3/3 + x\right)\Big|_1^2 = 1/12$
2. $\int_0^4 \sqrt{x}dx = 2x^{3/2}/3\Big|_0^4 = 32/3,$
3. $\int_0^{\pi/2} \cos x dx = \sin x|_0^{\pi/2} = 1.$

1.55 Example

$$f(x) = \begin{cases} -x^2 + 1 & \text{if } x < 0, \\ x^3 + 1 & \text{if } x \geq 0 \end{cases}$$

$$\begin{aligned} \int_{-2}^2 f(x)dx &= \int_{-2}^0 (-x^2 + 1)dx + \int_0^2 (x^3 + 1)dx \\ &= 16/3 \end{aligned}$$

1.56 Example

$$\int_0^2 x\sqrt{x^2+4}dx = \left.\frac{(x^2+4)^{3/2}}{3}\right|_0^2 = \frac{8}{3}(2\sqrt{2}-1)$$

1.57 Example

$$\int_0^{\pi/4} \cos^3 2x \sin 2x dx = \left.\frac{1}{8}\cos^4 2x\right|_0^{\pi/4} = \frac{1}{8}$$

1.58 Finding Areas by Integration

Assume that $f(x)$, $g(x)$ are continuous on $[a, b]$, the area of region bounded by the graphs of

$f(x)$, $g(x)$ from $x = a$ to $x = b$ is:

$$\int_a^b |f(x) - g(x)|dx$$

1.59 Example

- Calulate the area of the region R bounded by the graph of $f(x) = \sqrt{x} + 2$ and X-axis between $x = 1$ and $x = 4$.

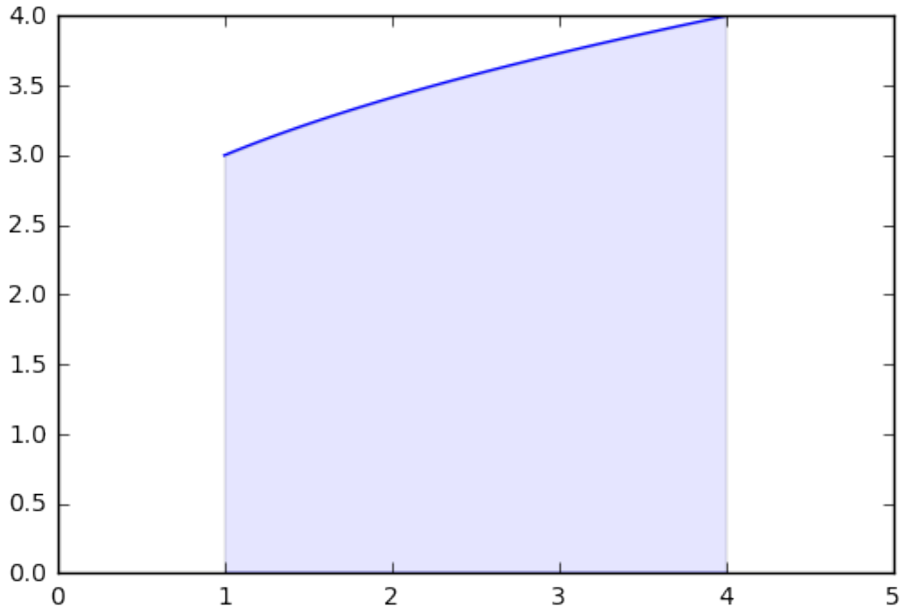
In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

%matplotlib inline
```

In [21]:
```
x=np.linspace(1,4,101)
plt.plot(x,np.sqrt(x)+2,x,0*x,'blue')
plt.fill_between(x,np.sqrt(x)+2,0*x,alpha=0.1)
plt.xlim(0,5)
```

Out[21]: (0, 5)



This means that $\int_1^4 |\sqrt{x}+2|\,dx = 32/3$

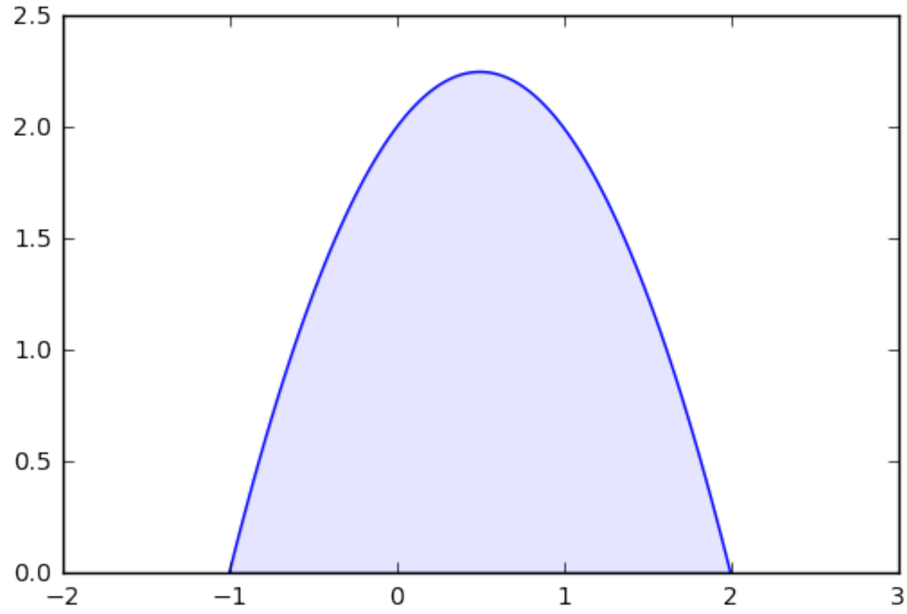
1.60 Example

The area of the region R bounded by the graph of $f(x) = 2 + x - x^2$ and X-axis is

$$\int_{-1}^{2} |2 + x - x^2 - 0|\,dx$$

where $2, -1$ are the x-intercept points of $f(x)$.

In [20]:
```
x=np.linspace(-1,2,101)
plt.plot(x,2+x-x**2,x,0*x,'b')
plt.fill_between(x,2+x-x**2,0*x,alpha=0.1)
plt.xlim(-2,3)
```
Out[20]: (-2, 3)



Thus, the area is $\int_{-1}^{2}(2+x-x^2)dx = 9/2$.

In [176]:
```
x=Symbol("x")
integrate(2+x-x**2,(x,-1,2))
```
Out[176]: 9/2

1.61 Example

- The area of the region R bounded above by the graph of $f(x) = |x+3|$ and below by \ X-axis between $x = 1$ and $x = 6$ is

$$\int_1^6 |x+2|\, dx = 13/2$$

.

- The area of the region R bounded above by the graph of $f(x) = e^x$ and below by \ $g(x) = 1/x$ for $1 \leqslant x \leqslant 2$ is

$$\int_1^2 (e^x - \frac{1}{x})dx = e^2 - e - \ln 2$$

.

- The area of the region R bounded by the graphs of $f(x) = x^3 - 4x$ and \ $g(x) = 5x$ \ is

$$\int_{-3}^3 |x^3 - 5x|\, dx = 81/2$$

.

1.62 Example

Evaluate these definite integrals: a) $\int_0^1 xe^{1-x^2} dx$ b) | $\int_0^4 \frac{x}{\sqrt{4+x^2}} dx$ c) $\int_1^e \frac{\sqrt{1+\ln x}}{x} dx$ d) $\int_0^1 \frac{x}{1+x^2} dx$

Also these can be evaluated by:

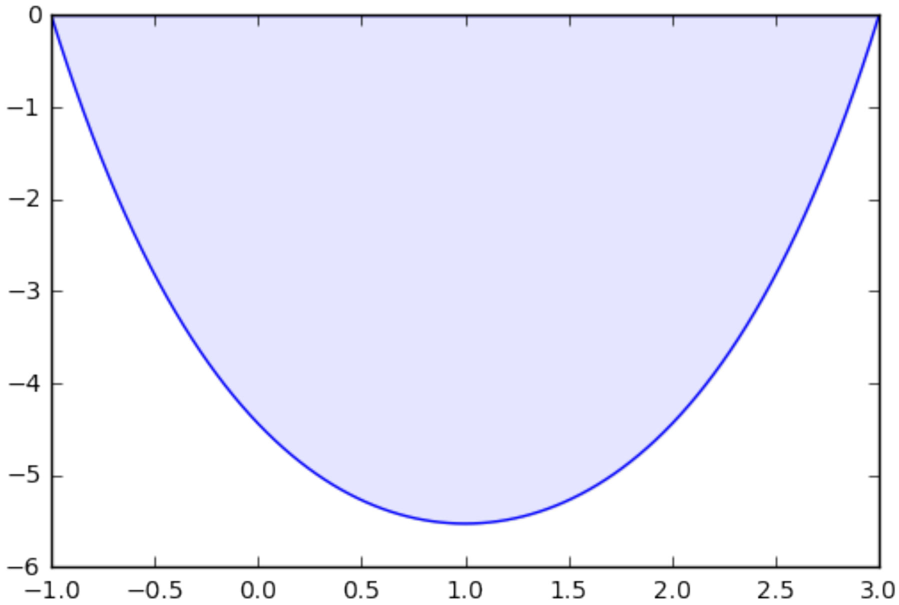
- $\int_0^1 xe^{1-x^2} dx = .\ \frac{1}{2} e^{1-x^2} \Big|_0^1 = (e-1)/2;$
- $\int_0^4 \frac{x}{\sqrt{4+x^2}} dx = \sqrt{4+x^2} \Big|_0^4 = \sqrt{20} - 2;$
- $\int_1^e \frac{\sqrt{1+\ln x}}{x} dx = 2\sqrt{(1+\ln x)^3}/3 \Big|_1^e = 2(\sqrt{8} - 1)/3$
- $\int_0^1 \frac{x}{1+x^2} dx = \ln(1+x^2)/2 \Big|_0^1 = \ln 2/2$

1.63 Computer practice

Find the area of the region R bounded by the graph of $e^{x-1} + e^{-x+1} - e^2 - e^{-2}$ and X-axis. Also find the area of the region R bounded by the graph of $x\sqrt{16 - x^2}$ and X-axis.

In [18]:
```
from numpy import exp
x=np.linspace(-1,3,101)
plt.plot(x,exp(x-1)+exp(-x+1)-exp(2)-exp(-2),x,
plt.fill_between(x,exp(x-1)+exp(-x+1)-exp(2)-ex
```

Out[18]: <matplotlib.collections.PolyCollection at 0x10a35c9
78>



In [197]:
```
from sympy import integrate,exp
x=Symbol("x")
integrate(exp(2)+exp(-2)-exp(x-1)-exp(-x+1),(x,
```

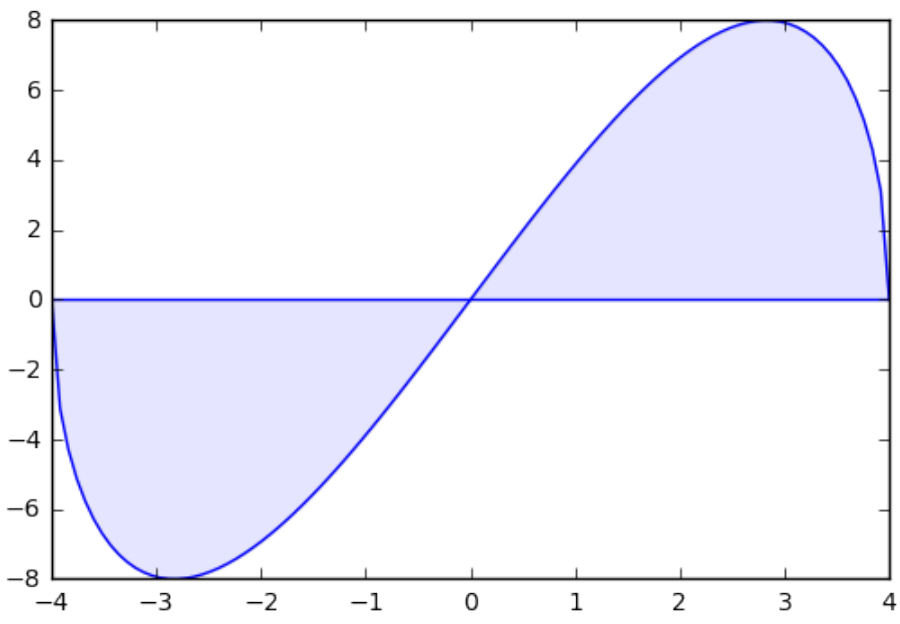
Out[197]: -2*exp(2) + 2*exp(-2) + 4*(1 + exp(4))*exp(-2)

$\int_{-1}^{3} |e^{x-1} + e^{-x+1} - e^2 - e^{-2}|\,dx = 2e^2 + 6e^{-2}$

In [19]:
```
from numpy import sqrt
x=np.linspace(-4,4,101)
plt.plot(x,x*sqrt(16-x**2),x,0*x,'b')
plt.fill_between(x,x*sqrt(16-x**2),0*x,alpha=0.

```

Out[19]: <matplotlib.collections.PolyCollection at 0x10a08ac c0>



In [194]:
```
from sympy import integrate,sqrt
x=Symbol("x")
2*integrate(x*sqrt(16-x**2),(x,0,4))
```

Out[194]: 128/3

$\int_0^4 |x\sqrt{16-x^2}|\,dx = 128/3$

1.64 Exercise p416

16. $\left(\int_{\sqrt{x}}^{5} \frac{\sin t^2}{t} dt\right)' = -\frac{\sin x}{2x}$

In [8]:
```
from sympy import sin,sqrt,pi,integrate
from sympy import *
FTCI(sin(t**2)/t,sqrt(x),5)
```

```
                     5
The derivative of  ∫ sin(t**2)/t dt is
                     sqrt(x)
-sin(x)
───────
  2⋅x
```

28. $\int_0^{\pi} |\cos x|\, dx = \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{\pi} \cos x dx = 2$

48. $\int_{-\pi/4}^{\pi/4} \frac{\tan^3 x}{1+x^2} dx = 0$ sine the integrand is odd.

56. The area of the region of $f(x) = |\sin x|$ over $[-\pi/2, \pi]$ is

$$\int_{-\pi/2}^{\pi} |\sin x|\, dx = -\int_{-\pi/4}^{0} \sin x dx + \int_0^{\pi} \sin x dx = 2 + 1/\sqrt{2}$$

62.

$$\lim_{n\to\infty} \frac{2\pi}{n} \sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n}\right) = \int_0^{\pi/2} \cos x dx = 1$$

since $\triangle x = \frac{\pi/2 - 0}{n}$, i.e.$a = 0, b = \pi/2, \cos\left(\frac{k\pi}{2n}\right) = \cos(k \triangle x)$, i.e. x_k^* being right end point in each subinterval.

In [ ]:

```
1
```