

一、符号定义

1.1 MDH参数（常量）

符号	值 (m)	说明
d_1	0.342	基座高度
a_1	0.040	肩关节偏移
a_2	0.275	大臂长度
a_3	0.025	小臂偏移
d_4	0.280	腕部偏移
d_t	0.073	末端工具长度

1.2 各关节MDH参数表

关节 i	$\alpha(i-1)$	$a(i-1)$	d_i	θ_{offset}	说明
1	0	0	d_1	0	基座高度
2	-90°	a_1	0	-90°	肩关节偏移
3	0	a_2	0	0	大臂
4	-90°	a_3	d_4	0	小臂、腕部
5	90°	0	0	0	腕关节
6	-90°	0	d_t	0	末端工具

1.3 三角函数简写

简写	含义
c_i	$\cos(\theta_i)$
s_i	$\sin(\theta_i)$
c_{ij}	$\cos(\theta_i + \theta_j)$
s_{ij}	$\sin(\theta_i + \theta_j)$

1.4 末端位姿矩阵

目标位姿矩阵 0T_6 定义为：

$${}^0T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{n} & \vec{o} & \vec{a} & \vec{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

其中：

- $\vec{n} = (n_x, n_y, n_z)^T$ ：法向量 (Normal)
- $\vec{o} = (o_x, o_y, o_z)^T$ ：方向向量 (Orientation)
- $\vec{a} = (a_x, a_y, a_z)^T$ ：接近向量 (Approach)
- $\vec{p} = (p_x, p_y, p_z)^T$ ：位置向量 (Position)

二、正运动学

2.1 通用齐次变换矩阵 (MDH Convention)

根据Modified DH参数，相邻坐标系间的齐次变换矩阵为：

$${}^{i-1}T_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & a_{i-1} \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_{\alpha_{i-1}} & s_{\alpha_{i-1}} & 0 & 0 \\ -s_{\alpha_{i-1}} & c_{\alpha_{i-1}} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 各关节变换矩阵

矩阵 0T_1

$${}^0T_1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta_1 = q_1$$

矩阵 1T_2

$${}^1T_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta_2 = q_2 - \frac{\pi}{2}$$

矩阵 2T_3

$${}^2T_3 = \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta_3 = q_3$$

矩阵 3T_4

$${}^3T_4 = \begin{bmatrix} c_4 & -s_4 & 0 & a_3 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta_4 = q_4$$

矩阵 4T_5

$${}^4T_5 = \begin{bmatrix} c_5 & -s_5 & 0 & 0 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta_5 = q_5$$

矩阵 5T_6

$${}^5T_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_t \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \theta_6 = q_6$$

2.3 复合变换矩阵

$$\mathbf{\{T\}_3} = \mathbf{\{T\}_2} \cdot \mathbf{\{T\}_3} = \begin{bmatrix} c_{23} & -s_{23} & 0 & a_2 c_2 + a_1 \backslash 0 & 0 & 1 \\ 0 & -s_{23} & -c_{23} & 0 & -a_2 s_2 \backslash 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{aligned} \mathcal{T}_6 &= \mathcal{T}_5 \cdot \mathcal{T}_6 = \begin{pmatrix} c_5 c_6 & -c_5 s_6 & -s_5 & -d_t s_5 \\ s_5 c_6 & s_5 s_6 & c_5 & d_t c_5 \\ 0 & 0 & s_5 & 0 \\ 0 & 0 & c_5 & 1 \end{pmatrix} \end{aligned}$$
$$\begin{matrix} \{^3T_6 = \{^3T_4 \cdot \{^4T_6 = \begin{bmatrix} n_x^{(36)} & o_x^{(36)} & a_x^{(36)} & p_x^{(36)} & n_y^{(36)} & o_y^{(36)} & a_y^{(36)} & p_y^{(36)} & n_z^{(36)} & o_z^{(36)} & a_z^{(36)} & p_z^{(36)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

| 向量 | x 分量 | y 分量 | z 分量 | | :---: | :---: | :---: | :---: | | $\vec{n}^{(36)}$ | $c_4 c_5 c_6 - s_4 s_6$ | $s_5 c_6$ | $-s_4 c_5 c_6 - c_4 s_6$ | | $\vec{o}^{(36)}$ | $-c_4 c_5 s_6 - s_4 c_6$ | $-s_5 s_6$ | $s_4 c_5 s_6 - c_4 c_6$ | | $\vec{a}^{(36)}$ | $-c_4 s_5$ | c_5 | $s_4 s_5$ | | $\vec{p}^{(36)}$ | $a_3 - d_t c_4 s_5$ | $d_4 + d_t c_5$ | $d_t s_4 s_5$ |

$$\begin{matrix} \{T_6\} = \{T_3\} \cdot \{T_6\} = \begin{bmatrix} n_x^{(16)} & o_x^{(16)} & a_x^{(16)} & p_x^{(16)} & n_y^{(16)} & o_y^{(16)} & a_y^{(16)} & p_y^{(16)} & n_z^{(16)} & o_z^{(16)} & a_z^{(16)} & p_z^{(16)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$
$$T_6 = \begin{bmatrix} c_{23} n_x^{(36)} - s_{23} n_y^{(36)} & c_{23} o_x^{(36)} - s_{23} o_y^{(36)} & c_{23} a_x^{(36)} - s_{23} a_y^{(36)} & p_x^{(16)} \setminus n_z^{(36)} & o_z^{(36)} & a_z^{(36)} \\ & & & p_y^{(16)} \setminus -s_{23} n_x^{(36)} - c_{23} n_y^{(36)} & -s_{23} o_x^{(36)} - c_{23} o_y^{(36)} & -s_{23} a_x^{(36)} - c_{23} a_y^{(36)} \\ & & & & & p_z^{(16)} \setminus 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{aligned} p_x^{(16)} &= c_{23}(a_3 - d_t c_4 s_5) - s_{23}(d_4 + d_t c_5) + a_2 c_2 + a_1 \backslash \\ p_y^{(16)} &= d_t s_4 s_5 \backslash p_z^{(16)} &= -s_{23}(a_3 - d_t c_4 s_5) - c_{23}(d_4 + d_t c_5) - a_2 s_2 \\ \end{aligned}$$
$$\mathbf{\{^0T_6\}} = \mathbf{\{^0T_1\}} \cdot \mathbf{\{^1T_6\}} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x & r_{21} & r_{22} & r_{23} & p_y & r_{31} & r_{32} & r_{33} & p_z & 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{aligned} p_x &= c_1 \left[c_{23}(a_3 - d_t c_4 s_5) - s_{23}(d_4 + d_t c_5) + a_2 c_2 + a_1 \right] - s_1 \\ &\left[d_t s_4 s_5 \right] \quad p_y = s_1 \left[c_{23}(a_3 - d_t c_4 s_5) - s_{23}(d_4 + d_t c_5) + a_2 c_2 + a_1 \right] \\ &+ c_1 \left[d_t s_4 s_5 \right] \quad p_z = -s_{23}(a_3 - d_t c_4 s_5) - c_{23}(d_4 + d_t c_5) - a_2 s_2 + \\ &d_1 \end{aligned}$$

2.4.2 旋转矩阵

第一列（法向量 \vec{n} ）：

$$\begin{aligned} r_{11} &= c_1 \left[c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6 \right] - s_1 \left[-s_4 c_5 \right. \\ &c_6 - c_4 s_6 \left. \right] \quad r_{21} = s_1 \left[c_{23}(c_4 c_5 c_6 - s_4 s_6) - s_{23} s_5 c_6 \right] + c_1 \left[- \right. \\ &s_4 c_5 c_6 - c_4 s_6 \left. \right] \quad r_{31} = -s_{23}(c_4 c_5 c_6 - s_4 s_6) - c_{23} s_5 c_6 \end{aligned}$$

第二列（方向向量 \vec{o} ）：

$$\begin{aligned} r_{12} &= c_1 \left[c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_5 s_6 \right] - s_1 \left[s_4 c_5 \right. \\ &s_6 - c_4 c_6 \left. \right] \quad r_{22} = s_1 \left[c_{23}(-c_4 c_5 s_6 - s_4 c_6) + s_{23} s_5 s_6 \right] + c_1 \left[\right. \\ &s_4 c_5 s_6 - c_4 c_6 \left. \right] \quad r_{32} = -s_{23}(-c_4 c_5 s_6 - s_4 c_6) + c_{23} s_5 s_6 \end{aligned}$$

第三列（接近向量 \vec{a} ）：

$$\begin{aligned} r_{13} &= c_1 \left[-c_{23} c_4 s_5 - s_{23} c_5 \right] - s_1 \left[s_4 s_5 \right] \quad r_{23} \\ &= s_1 \left[-c_{23} c_4 s_5 - s_{23} c_5 \right] + c_1 \left[s_4 s_5 \right] \quad r_{33} = s_{23} c_4 s_5 - c_{23} \\ &c_5 \end{aligned}$$

注：旋转矩阵各元素可通过将 $\{{}^3T_6$ 中的 $n^{(36)}$ 、 $o^{(36)}$ 、 $a^{(36)}$ 代入 $\{{}^1T_6$ 的表达式，再与 $\{{}^0T_1$ 相乘得到。

令 $\theta_1 = 0^\circ$, $\theta_2 = -90^\circ$, $\theta_3 = 0^\circ$, $\theta_4 = 0^\circ$, $\theta_5 = 0^\circ$, $\theta_6 = 0^\circ$ ，最终得到 $\{{}^0T_6$ 为

$$\{{}^0T_6 = \begin{bmatrix} 0 & 0 & 1 & d_4 + d_t + a_1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & a_3 + a_2 + d_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

三、逆运动学

$$\begin{aligned} \{{}^0T_6 &= \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \{{}^0T_1 \} \cdot \{{}^1T_2 \} \cdot \{{}^2T_3 \} \cdot \{{}^3T_4 \} \cdot \{{}^4T_5 \} \\ &\cdot \{{}^5T_6 \} \end{aligned}$$

3.1 求解 θ_1

步骤1：构建方程

从矩阵方程 $\left(\{{}^0T_1\right)^{-1} \cdot \{{}^0T_6 = \{{}^1T_6$ 的 (2,4) 和 (2,3) 元素：

$$\left(\{{}^0T_1\right)^{-1} \cdot \{{}^0T_6 = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \end{bmatrix} \cdot \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \{{}^1T_6$$

$$-s_1 p_x + c_1 p_y = d_t(s_4 s_5) \quad \text{..(1)}$$

$$-s_1a_x + c_1a_y = s_4s_5 \quad \text{...(2)}$$

步骤2：消元

将方程(2)代入方程(1)消去 s_4s_5 ：

$$-s_1p_x + c_1p_y = d_t(-s_1a_x + c_1a_y)$$

步骤3：整理求解

$$s_1(d_{ta_x} - p_x) = c_1(d_{ta_y} - p_y)$$

◆ 公式1： θ_1 求解公式

$$\boxed{\theta_1 = \text{atan2}(p_y - d_{ta_y}, p_x - d_{ta_x})}$$

物理意义： $(p_x - d_{ta_x}, p_y - d_{ta_y})$ 是腕部中心点在基坐标系XY平面的投影