

# UR机械臂正逆运动学求解

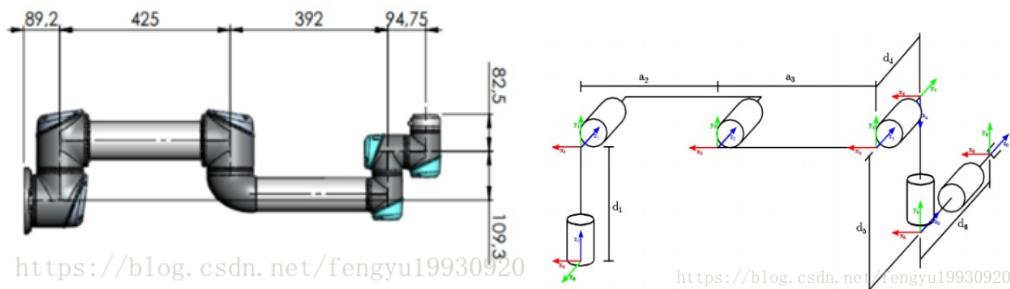
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文章标签: Positive and Inverse Kinematic

最近有个任务: 求解UR机械臂正逆运动学, 在网上参考了一下大家的求解办法, 众说纷纭, 其中有些朋友求解过程非常常规, 但是最后求解的8:4组可用。在这里我介绍一个可以求解8组解析解的方法, 供大家参考。

以UR5机械臂结构和尺寸参数为例进行正逆运动学求解, 下图分别是UR5结构图和标准DH系参数:



$i^\circ$	$d_i^\circ$	$a_i^\circ$	$\alpha_i^\circ$	$\theta_i^\circ$
1 <sup>o</sup>	$d_1 = 89.459^\circ$	$0^\circ$	$\pi / 2^\circ$	$\theta_1^\circ$
2 <sup>o</sup>	$0^\circ$	$a_2 = -425^\circ$	$0^\circ$	$\theta_2^\circ$
3 <sup>o</sup>	$0^\circ$	$a_3 = -392.25^\circ$	$0^\circ$	$\theta_3^\circ$
4 <sup>o</sup>	$d_4 = 109.15^\circ$	$0^\circ$	$\pi / 2^\circ$	$\theta_4^\circ$
5 <sup>o</sup>	$d_5 = 94.65^\circ$	$0^\circ$	$-\pi / 2^\circ$	$\theta_5^\circ$
6 <sup>o</sup>	$d_6 = 82.3^\circ$	$0^\circ$	$0^\circ$	$\theta_6^\circ$

## 1. 正运动学求解

正运动学是已知关节六个角度求变换矩阵T

其中:

$${}^{i-1}i T = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

整理得:

$${}^{i-1}i T = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i \cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i \sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

带入 DH参数 , 求解:

$${}^0_1 T = \begin{bmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^1_2 T = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2 \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2 \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^2_3 T = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3 \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3 \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4 T = \begin{bmatrix} \cos(\theta_4) & 0 & \sin(\theta_4) & 0 \\ \sin(\theta_4) & 0 & -\cos(\theta_4) & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^4_5 T = \begin{bmatrix} \cos(\theta_5) & 0 & -\sin(\theta_5) & 0 \\ \sin(\theta_5) & 0 & \cos(\theta_5) & 0 \\ 0 & -1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} {}^5_6 T = \begin{bmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

最终变换矩阵:

$$T = {}^0_1 T \cdot {}^1_2 T \cdot {}^2_3 T \cdot {}^3_4 T \cdot {}^4_5 T \cdot {}^5_6 T$$

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正运动学求解完毕。

## 2. 逆运动学求解

逆运动学是已知变换矩阵T, 求六个关节角度。逆运动学求解有解析法, 几何法, **迭代法**, 这里采用解析法求解。

### 2.1 两个简单的数学方法

#### 2.1.1 求角度

这个逆运动学算法求解的角度范围是

$$\theta \in [-\pi, +\pi]$$

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因为标准的反正切arctan的值域是

$$[-\pi/2, +\pi/2]$$

所以不能使用, 这里介绍一个改进的反正切求法 Atan2(y, x) (Matlab 里有这个函数), 它的值域可以满足要求。

#### 2.1.2 解方程

$$-\sin(\theta)p_x + \cos(\theta)p_y = d$$

首先进行三角恒等变换, 令

$$p_x = \rho \cos(\phi), p_y = \rho \sin(\phi)$$

其中:

$$\rho = \sqrt{p_x^2 + p_y^2}, \phi = \text{Atan2}(p_y, p_x)$$

然后带入原方程:

$$\cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) = d / \rho$$

$$\sin(\phi - \theta) = \frac{d}{\rho} \quad \text{则} \quad \cos(\phi - \theta) = \pm \sqrt{1 - \frac{d^2}{\rho^2}}$$

则

$$\phi - \theta = A \tan 2\left(\frac{d}{\rho}, \pm \sqrt{1 - \frac{d^2}{\rho^2}}\right)$$

$$\theta = \text{Atan2}(p_y, p_x) - A \tan 2(d, \pm \sqrt{p_x^2 + p_y^2 - d^2})$$

### 2.2 约定

为了简化书写, 约定:

$$c_{23} = \cos(\theta_2 + \theta_3)$$

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## 2.3 求解1, 5, 6关节角度

已知:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$T = {}^0T \cdot {}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T \cdot {}^5T \quad \text{https://blog.csdn.net/fengyu19930920}$$

则

$${}^0T^{-1} \cdot T \cdot {}^5T^{-1} = {}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T \cdot {}^5T \quad \text{https://blog.csdn.net/fengyu19930920}$$

其中:

$$\begin{aligned} {}^0T^{-1} &= \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ s_1 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \\ {}^5T^{-1} &= \begin{bmatrix} c_6 & s_6 & 0 & 0 \\ -s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & -d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned} \quad \text{https://blog.csdn.net/fengyu19930920}$$

等式左边:

$${}^0T^{-1} \cdot T \cdot {}^5T^{-1} = \begin{bmatrix} c_6(n_x c_1 + n_y s_1) - s_6(o_x c_1 + o_y s_1) & s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1) & a_x c_1 + a_y s_1 & p_x c_1 - d_6(a_x c_1 + a_y s_1) + p_y s_1 \\ n_x c_6 - o_x s_6 & o_x c_6 + n_y s_6 & a_z & p_z - d_1 - a_z d_6 \\ s_6(o_y c_1 - o_x s_1) - c_6(n_y c_1 - n_x s_1) & -s_6(n_y c_1 - n_x s_1) - c_6(o_y c_1 - o_x s_1) & a_x s_1 - a_y c_1 & -p_y c_1 + d_6(a_y c_1 - a_x s_1) + p_x s_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{https://blog.csdn.net/fengyu19930920}$$

等式右边:

$${}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T \cdot {}^5T = {}^1T = \begin{bmatrix} c_{234}c_5 & -s_{234} & -c_{234}s_5 & a_3c_{23} + a_2c_2 + d_5s_{234} \\ s_{234}c_5 & c_{234} & -s_{234}s_5 & a_3s_{23} + a_2s_2 - d_5c_{234} \\ s_5 & 0 & c_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{https://blog.csdn.net/fengyu19930920}$$

### 2.3.1 求关节角1

利用等式左右两边第3行, 第4列对应相等求关节角1。

$$-p_y c_1 + d_6(a_y c_1 - a_x s_1) + p_x s_1 = d_4 \quad \text{https://blog.csdn.net/fengyu19930920}$$

整理得:

$$(d_6 a_y - p_y) c_1 - (a_x d_6 + p_x) s_1 = d_4 \quad \text{https://blog.csdn.net/fengyu19930920}$$

设:

$$d_6 a_y - p_y = m \quad | a_x d_6 + p_x = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

则

$$m c_1 - n s_1 = d_4 \quad \text{https://blog.csdn.net/fengyu19930920}$$

根据前面介绍的解方程的方法:

$$\theta_1 = A \tan 2(m, n) - A \tan 2(d_4, \pm \sqrt{m^2 + n^2 - d_4^2}) \quad (\text{其中 } m^2 + n^2 - d_4^2 \geq 0) \quad \text{https://blog.csdn.net/fengyu19930920}$$

### 2.3.2 求关节角5

利用等式左右两边第3行，第3列对应相等求关节角5。

$$a_x s_1 - a_y c_1 = c_5 \quad \text{https://blog.csdn.net/fengyu19930920}$$

解得：

$$\theta_5 = \pm \arccos(a_x s_1 - a_y c_1) \quad (\text{其中 } a_x s_1 - a_y c_1 \leq 1) \quad \text{https://blog.csdn.net/fengyu19930920}$$

### 2.3.3 求关节角6

利用等式左右两边第3行，第1列对应相等求关节角6。

$$s_6(o_y c_1 - o_z s_1) - c_6(n_y c_1 - n_z s_1) = s_5 \quad \text{https://blog.csdn.net/fengyu19930920} \quad \text{设:}$$

$$n_y s_1 - n_z c_1 = m \quad o_z s_1 + o_y c_1 = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

则

$$m c_6 - n s_6 = s_5 \quad \text{https://blog.csdn.net/fengyu19930920} \quad \text{根据前面介绍的方}$$

$$\theta_6 = A \tan 2(m, n) - A \tan 2(s_5, \pm \sqrt{m^2 + n^2 + s_5^2}) \quad \text{https://blog.csdn.net/fengyu19930920}$$

其实可以通过化简得到式中

$$m^2 + n^2 - s_5^2 = 0 \quad \text{https://blog.csdn.net/fengyu19930920}$$

则

$$\theta_6 = A \tan 2(m, n) - A \tan 2(s_5, 0) = A \tan 2(m/s_5, n/s_5) \quad (\text{其中 } s_5 \neq 0) \quad \text{https://blog.csdn.net/fengyu19930920}$$

## 2.4 求解2, 3, 4关节角度

已知：

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$T = {}^0T \cdot {}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T \cdot {}^5T \quad \text{https://blog.csdn.net/fengyu19930920}$$

则

$${}^0T^{-1} \cdot T \cdot {}^5T^{-1} \cdot {}^4T^{-1} = {}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T \quad \text{https://blog.csdn.net/fengyu19930920}$$

其中：

$${}^4T^{-1} = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ -s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{https://blog.csdn.net/fengyu19930920}$$

等式左边等于

$${}^0T^{-1} \cdot T \cdot {}^5T^{-1} \cdot {}^4T^{-1} = \begin{bmatrix} -c_5(x_0(o_x c_1 + o_y s_1) - c_6(n_x c_1 + n_y s_1)) - s_5(x_0(o_x c_1 + o_y s_1) - c_6(n_x c_1 + n_y s_1)) & c_5(x_0(o_x c_1 + o_y s_1) - c_6(n_x c_1 + n_y s_1)) - s_5(x_0(o_x c_1 + o_y s_1) - c_6(n_x c_1 + n_y s_1)) & -s_5(n_x c_1 + n_y s_1) - c_5(o_x c_1 + o_y s_1) & d_5(x_0(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1)) - d_5(o_x c_1 + o_y s_1) + p_x c_1 + p_y s_1 \\ c_5(n_x c_6 - o_y s_6) - o_z s_5 & x_0(n_x c_6 - o_y s_6) + o_z s_5 & -o_z c_6 - n_y s_6 & p_z - d_5 - o_x d_6 + d_5(o_x c_6 + n_y s_6) \\ c_5(x_0(o_x c_1 - o_y s_1) - c_6(n_x c_1 - n_y s_1)) + s_5(x_0(o_x c_1 - o_y s_1) - c_6(n_x c_1 - n_y s_1)) & x_0(x_0(o_x c_1 - o_y s_1) - c_6(n_x c_1 - n_y s_1)) - c_5(o_x c_1 - o_y s_1) & s_5(n_x c_1 - n_y s_1) + c_5(o_x c_1 - o_y s_1) & d_5(o_x c_1 - o_y s_1) - d_5(x_0(n_x c_1 - n_y s_1) + c_6(o_x c_1 - o_y s_1)) - p_z c_1 + p_y s_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{https://blog.csdn.net/fengyu19930920}$$

等式右边等于

$$\frac{1}{2}T \cdot \frac{2}{3}T \cdot \frac{3}{4}T = \frac{1}{4}T = \begin{bmatrix} c_{234} & 0 & s_{234} & a_3c_{23} + a_2c_2 \\ s_{234} & 0 & -c_{234} & a_3s_{23} + a_2s_2 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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### 2.4.1 求解关节角3

利用等式左右两边第1行，第4列对应相等，第2行，第4列对应相等，求关节角3。

$$d_5(s_6(n_xc_1 + n_ys_1) + c_6(o_xc_1 + o_ys_1)) - d_6(a_xc_1 + a_ys_1) + p_xc_1 + p_ys_1 = a_3c_{23} + a_2c_2 \quad (1)$$

$$p_z - d_1 - a_zd_6 + d_5(o_zc_6 + n_zs_6) = a_3s_{23} + a_2s_2 \quad (2)$$

为了简化，设：

$$\begin{aligned} d_5(s_6(n_xc_1 + n_ys_1) + c_6(o_xc_1 + o_ys_1)) - d_6(a_xc_1 + a_ys_1) + p_xc_1 + p_ys_1 &= m \\ p_z - d_1 - a_zd_6 + d_5(o_zc_6 + n_zs_6) &= n \end{aligned}$$

将m,n带入上式得

$$m = a_3c_{23} + a_2c_2 \quad (3)$$

$$n = a_3s_{23} + a_2s_2 \quad (4)$$

式子③④平方和为

$$a_2^2 + a_3^2 + 2a_2a_3(s_{23}s_2 + c_{23}c_2) = m^2 + n^2$$

因为

$$s_{23}s_2 + c_{23}c_2 = \cos(\theta_3)$$

所以

$$\theta_3 = \pm \arccos\left(\frac{m^2 + n^2 - a_2^2 - a_3^2}{2a_2a_3}\right) \quad (\text{其中 } m^2 + n^2 \leq (a_2 + a_3)^2)$$

### 2.4.2 求解关节角2

将③④展开得：

$$(a_3c_3 + a_2)c_2 - a_3s_3s_2 = m \quad (5)$$

$$a_3s_3c_2 + (a_3c_3 + a_2)s_2 = n$$

将关节角3带入⑤⑥，求关节角2得

$$s_2 = \frac{(a_3c_3 + a_2)n - a_3s_3m}{a_2^2 + a_3^2 + 2a_2a_3c_3}$$

$$c_2 = \frac{m + a_3s_3s_2}{a_3c_3 + a_2}$$

则

$$\theta_2 = \arctan(2(s_2, c_2))$$

### 2.4.3 求解关节角4

用 $\frac{1}{2}T$ 的第2行第2列，第1行第2列求 $\theta_2 + \theta_3 + \theta_4$

$$s_{234} = -s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1)$$

$$c_{234} = o_x c_6 + n_x s_6$$

$$\theta_2 + \theta_3 + \theta_4 = A \tan 2(-s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1), o_x c_6 + n_x s_6) - \theta_2 - \theta_3$$

则

$$\theta_4 = A \tan 2(-s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1), o_x c_6 + n_x s_6) - \theta_2 - \theta_3$$

## 2.5 总结

### 2.5.1 求解公式

$$d_6 a_y - p_y = m \quad | a_x d_6 + p_x = n$$

$$\theta_1 = A \tan 2(m, n) - A \tan 2(d_4, \pm \sqrt{m^2 + n^2 - d_4^2}) \quad (\text{其中 } m^2 + n^2 - d_4^2 \geq 0)$$

$$\theta_3 = \pm \arccos(a_x s_1 - a_y c_1) \quad (\text{其中 } a_x s_1 - a_y c_1 \leq 1)$$

$$n_x s_1 - n_y c_1 = m \quad | o_x s_1 + o_y c_1 = n$$

$$\theta_6 = A \tan 2(m, n) - A \tan 2(s_5, 0) = A \tan 2(m / s_5, n / s_5) \quad (\text{其中 } s_5 \neq 0)$$

$$d_5(s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1)) - d_5(a_x c_1 + a_y s_1) + p_x c_1 + p_y s_1 = m$$

$$p_z - d_1 - a_z d_6 + d_5(o_x c_6 + n_x s_6) = n$$

$$\theta_3 = \pm \arccos\left(\frac{m^2 + n^2 - a_2^2 - a_3^2}{2a_2 a_3}\right) \quad (\text{其中 } m^2 + n^2 \leq (a_2 + a_3)^2)$$

$$s_2 = \frac{(a_3 c_3 + a_2) n - a_3 s_3 m}{a_2^2 + a_3^2 + 2a_2 a_3 c_3}$$

$$c_2 = \frac{m + a_3 s_3 s_2}{a_3 c_3 + a_2} \quad | \quad https://blog.csdn.net/fengyu19930920$$

$$\theta_2 = A \tan 2(s_2, c_2) \quad | \quad https://blog.csdn.net/fengyu19930920$$

$$\theta_4 = A \tan 2(-s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1), o_x c_6 + n_x s_6) - \theta_2 - \theta_3$$

### 2.5.2 奇异位置

#### 1.肩关节奇异位置

$$n_x s_1 - n_y c_1 = m \quad | o_x s_1 + o_y c_1 = n$$

$$m^2 + n^2 - d_4^2 = 0$$

此时末端执行器参考点O6位于轴线z1和z2构成的平面内，关节角1无法求解。

#### 2.肘关节奇异位置

$$d_5(s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1)) - d_5(a_x c_1 + a_y s_1) + p_x c_1 + p_y s_1 = m$$

$$p_z - d_1 - a_z d_6 + d_5(o_x c_6 + n_x s_6) = n$$

$$m^2 + n^2 - (a_2 + a_3)^2 = 0$$

此时关节角2无法求解。

#### 3.腕关节奇异位置

## 2.6 实例

利用Matlab机器人 库 ur5 DH参数：

```
alpha1 = pi/2;      a1=0;          d1=89.459;
alpha2 = 0;        a2=-425;       d2=0;
alpha3 = 0;        a3=-392.25;    d3=0;
alpha4 = pi/2;      a4=0;          d4=109.15;
alpha5 = -pi/2;     a5=0;          d5=94.65;
alpha6 = 0;        a6=0;          d6=82.3;
```

取 theta1 = 1; theta2 = 1; theta3 = 1; theta4 = 1; theta5 = 1; theta6 = 1; (不要纠结theta选这6个数值是否有实际意义，这里只验证算法的有效性)

- 将theta带入正运动学公式，求T:

```
T =
0.1623   -0.3938   0.9047  137.6508
-0.5888   0.6972   0.4091  -69.9381
-0.7919   -0.5991  -0.1187 -540.9083
0           0           0       1.0000
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```

- 将T带入逆运动学公式，反求theta

theta =

```
1.0000   1.0000   1.0000   1.0000   1.0000   1.0000
1.0000   1.9562  -1.0000   2.0438   1.0000   1.0000
1.0000   0.6743   1.7374  -2.5533  -1.0000  -2.1416
1.0000   2.3170  -1.7374  -0.7212  -1.0000  -2.1416
0.0954   1.1462   1.0086   0.8611   1.8976   0.8826
0.0954   2.1106  -1.0086   1.9139   1.8976   0.8826
0.0954   0.8754   1.7300  -2.7311  -1.8976  -2.2590
0.0954   2.5114  -1.7300  -0.9071  -1.8976  -2.2590
https://blog.csdn.net/fengyu19930920
```

- 再将8个theta带入正运动学公式，反求8个T:

8个T均等于

```
0.1623   -0.3938   0.9047  137.6508
-0.5888   0.6972   0.4091  -69.9381
-0.7919   -0.5991  -0.1187 -540.9083
0           0           0       1.0000
https://blog.csdn.net/fengyu19930920
```

验证了算法的有效性