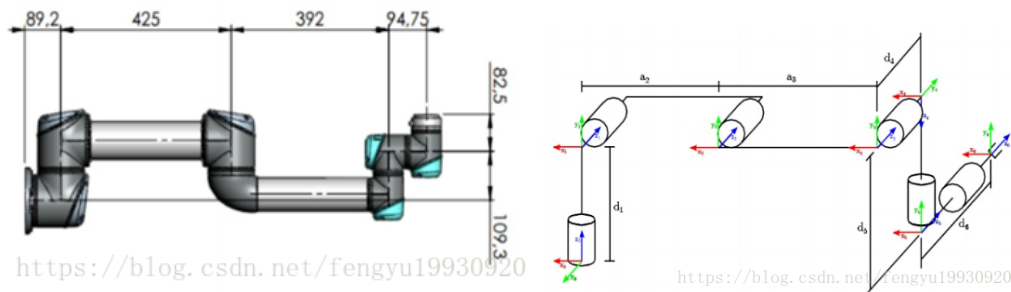


UR机械臂正逆运动学求解

最近有个任务：求解UR机械臂正逆运动学，在网上参考了一下大家的求解办法，众说纷纭，其中有些朋友求解过程非常常规，但是最后求解的8组可用。在这里我介绍一个可以求解8组解析解的方法，供大家参考。

以UR5机械臂结构和尺寸参数为例进行正逆运动学求解，下图分别是UR5结构图和标准DH系参数：



i	d_i	a_i	α_i	θ_i
1	$d_1=89.459$	0	$\pi / 2$	θ_1
2	0	$a_2=-425$	0	θ_2
3	0	$a_3=-392.25$	0	θ_3
4	$d_4=109.15$	0	$\pi / 2$	θ_4
5	$d_5=94.65$	0	$-\pi / 2$	θ_5
6	$d_6=82.3$	0	0	θ_6

1. 正运动学求解

正运动学是已知关节六个角度求变换矩阵T

其中：

$${}^{i-1}T_i = \begin{pmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

整理得：

$${}^{i-1}T_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos(\alpha_i) & \sin(\theta_i)\sin(\alpha_i) & a_i\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos(\alpha_i) & -\cos(\theta_i)\sin(\alpha_i) & a_i\sin(\theta_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

带入 DH参数 ， 求解：

$${}^0_1T = \begin{bmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & a_2 \cos(\theta_2) \\ \sin(\theta_2) & \cos(\theta_2) & 0 & a_2 \sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_3T = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & a_3 \cos(\theta_3) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & a_3 \sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^3_4T = \begin{bmatrix} \cos(\theta_4) & 0 & \sin(\theta_4) & 0 \\ \sin(\theta_4) & 0 & -\cos(\theta_4) & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^4_5T = \begin{bmatrix} \cos(\theta_5) & 0 & -\sin(\theta_5) & 0 \\ \sin(\theta_5) & 0 & \cos(\theta_5) & 0 \\ 0 & -1 & 0 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^5_6T = \begin{bmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

最终变换矩阵：

$$T = {}^0_1T \cdot {}^1_2T \cdot {}^2_3T \cdot {}^3_4T \cdot {}^4_5T \cdot {}^5_6T \quad \text{https://blog.csdn.net/fengyu19930920}$$

正运动学求解完毕。

2. 逆运动学求解

逆运动学是已知变换矩阵T，求六个关节角度。逆运动学求解有解析法，几何法，**迭代法**，这里采用解析法求解。

2.1 两个简单的数学方法

2.1.1 求角度

这个逆运动学算法求解的角度范围是

$$\theta \in [-\pi, +\pi] \quad \text{https://blog.csdn.net/fengyu19930920}$$

因为标准的反正切arctan的值域是

$$[-\pi/2, +\pi/2] \quad \text{https://blog.csdn.net/fengyu19930920}$$

所以不能使用，这里介绍一个改进的反正切求法 Atan2(y, x) (Matlab 里有这个函数)，它的值域可以满足要求。

2.1.2 解方程

$$-\sin(\theta)p_x + \cos(\theta)p_y = d \quad \text{https://blog.csdn.net/fengyu19930920}$$

首先进行三角恒等变换，令

$$p_x = \rho \cos(\phi), p_y = \rho \sin(\phi) \quad \text{https://blog.csdn.net/fengyu19930920}$$

其中：

$$\rho = \sqrt{p_x^2 + p_y^2}, \phi = \text{Atan2}(p_y, p_x) \quad \text{https://blog.csdn.net/fengyu19930920}$$

然后带入原方程：

$$\begin{aligned} \cos(\theta)\sin(\phi) - \sin(\theta)\cos(\phi) &= d / \rho \\ \sin(\phi - \theta) &= \frac{d}{\rho} \quad \text{则} \quad \cos(\phi - \theta) = \pm \sqrt{1 - \frac{d^2}{\rho^2}} \quad \text{https://blog.csdn.net/fengyu19930920} \end{aligned}$$

则

$$\phi - \theta = \text{Atan2}\left(\frac{d}{\rho}, \pm \sqrt{1 - \frac{d^2}{\rho^2}}\right)$$

$$\theta = \text{Atan2}(p_y, p_x) - \text{Atan2}\left(d, \pm \sqrt{p_x^2 + p_y^2 - d^2}\right) \quad \text{前提} (p_x^2 + p_y^2 - d^2 \geq 0) \quad \text{https://blog.csdn.net/fengyu19930920}$$

2.2 约定

为了简化书写，约定：

$$c_{23} = \cos(\theta_2 + \theta_3)$$

$$s_{23} = \sin(\theta_2 + \theta_3)$$

2.3 求解1, 5, 6关节角度

已知:

$$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = {}^0T_1 \cdot {}^1T_2 \cdot {}^2T_3 \cdot {}^3T_4 \cdot {}^4T_5 \cdot {}^5T_6$$

则

$${}^0T_1^{-1} \cdot T \cdot {}^5T_6^{-1} = {}^1T_2 \cdot {}^2T_3 \cdot {}^3T_4 \cdot {}^4T_5$$

其中:

$${}^0T_1^{-1} = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ s_1 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^5T_6^{-1} = \begin{bmatrix} c_6 & s_6 & 0 & 0 \\ -s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & -d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

等式左边:

$${}^0T_1^{-1} \cdot T \cdot {}^5T_6^{-1} = \begin{bmatrix} c_6(n_x c_1 + n_y s_1) - s_6(o_x c_1 + o_y s_1) & s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1) & a_x c_1 + a_y s_1 & p_x c_1 - d_6(a_x c_1 + a_y s_1) + p_y s_1 \\ n_x c_6 - o_x s_6 & o_x c_6 + n_x s_6 & a_z & p_z - d_1 - a_z d_6 \\ s_6(o_y c_1 - o_x s_1) - c_6(n_y c_1 - n_x s_1) & -s_6(n_y c_1 - n_x s_1) - c_6(o_y c_1 - o_x s_1) & a_x s_1 - a_y c_1 & -p_y c_1 + d_6(a_y c_1 - a_x s_1) + p_x s_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

等式右边:

$${}^1T_2 \cdot {}^2T_3 \cdot {}^3T_4 \cdot {}^4T_5 = {}^1T_2 = \begin{bmatrix} c_{234}c_5 & -s_{234} & -c_{234}s_5 & a_3c_{23} + a_2c_2 + d_3s_{234} \\ s_{234}c_5 & c_{234} & -s_{234}s_5 & a_3s_{23} + a_2s_2 - d_3c_{234} \\ s_5 & 0 & c_5 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3.1 求关节角1

利用等式左右两边第3行, 第4列对应相等求关节角1。

$$-p_y c_1 + d_6(a_y c_1 - a_x s_1) + p_x s_1 = d_4$$

整理得:

$$(d_6 a_y - p_y) c_1 - (a_x d_6 + p_x) s_1 = d_4$$

设:

$$d_6 a_y - p_y = m \quad a_x d_6 + p_x = n$$

则

$$m c_1 - n s_1 = d_4$$

根据前面介绍的解方程的方法:

$$\theta_1 = A \tan 2(m, n) - A \tan 2(d_4, \pm \sqrt{m^2 + n^2 - d_4^2}) \quad (\text{其中 } m^2 + n^2 - d_4^2 \geq 0)$$

2.3.2 求关节角5

利用等式左右两边第3行，第3列对应相等求关节角5。

$a_x s_1 - a_y c_1 = c_5$ <https://blog.csdn.net/fengyu19930920>

解得：

$\theta_5 = \pm \arccos(a_x s_1 - a_y c_1)$ (其中 $a_x s_1 - a_y c_1 \leq 1$) <https://blog.csdn.net/fengyu19930920>

2.3.3 求关节角6

利用等式左右两边第3行，第1列对应相等求关节角6。

$s_6(a_y c_1 - a_x s_1) - c_6(n_y c_1 - n_x s_1) = s_5$ <https://blog.csdn.net/fengyu19930920> 设：

$n_x s_1 - n_y c_1 = m \quad a_x s_1 - a_y c_1 = n$ <https://blog.csdn.net/fengyu19930920>

则

$m c_6 - n s_6 = s_5$ <https://blog.csdn.net/fengyu19930920> 根据前面介绍的方

$\theta_6 = A \tan 2(m, n) - A \tan 2(s_5, \pm \sqrt{m^2 + n^2 - s_5^2})$ <https://blog.csdn.net/fengyu19930920>

其实可以通过化简得到式中

$m^2 + n^2 - s_5^2 = 0$ <https://blog.csdn.net/fengyu19930920>

则

$\theta_6 = A \tan 2(m, n) - A \tan 2(s_5, 0) = A \tan 2(m / s_5, n / s_5)$ (其中 $s_5 \neq 0$) <https://blog.csdn.net/fengyu19930920>

2.4 求解2， 3， 4关节角度

已知：

$T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$ <https://blog.csdn.net/fengyu19930920>

$T = {}^0T \cdot {}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T \cdot {}^5T$ <https://blog.csdn.net/fengyu19930920>

则

${}^0T^{-1} \cdot T \cdot {}^5T^{-1} \cdot {}^4T^{-1} = {}^1T \cdot {}^2T \cdot {}^3T \cdot {}^4T$ <https://blog.csdn.net/fengyu19930920>

其中：

${}^4T^{-1} = \begin{bmatrix} c_5 & s_5 & 0 & 0 \\ 0 & 0 & -1 & d_5 \\ -s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ <https://blog.csdn.net/fengyu19930920>

等式左边等于

${}^0T^{-1} \cdot T \cdot {}^5T^{-1} \cdot {}^4T^{-1} = \begin{bmatrix} -c_5(s_6(o_x c_1 + o_y s_1) - c_6(n_x c_1 + n_y s_1)) - s_5(a_x c_1 + a_y s_1) & c_5(a_x c_1 + a_y s_1) - s_5(s_6(o_x c_1 + o_y s_1) - c_6(n_x c_1 + n_y s_1)) & -s_5(n_x c_1 + n_y s_1) - c_5(o_x c_1 + o_y s_1) & d_5(s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1)) - d_5(a_x c_1 + a_y s_1) + p_x c_1 + p_y s_1 \\ c_5(n_x c_1 - o_x s_1) - a_x s_5 & s_5(n_x c_1 - o_x s_1) + a_x s_5 & -o_x c_1 - n_x s_1 & p_x - d_5 - a_x d_5 + d_5(a_x c_1 + n_x s_1) \\ c_5(s_6(o_x c_1 - o_y s_1) - c_6(n_x c_1 - n_y s_1)) + s_5(a_x c_1 - a_y s_1) & s_5(s_6(o_x c_1 - o_y s_1) - c_6(n_x c_1 - n_y s_1)) - c_5(a_x c_1 - a_y s_1) & s_5(n_x c_1 - n_y s_1) + c_5(o_x c_1 - o_y s_1) & d_5(a_x c_1 - a_y s_1) - d_5(s_6(n_x c_1 - n_y s_1) + c_6(o_x c_1 - o_y s_1)) - p_x c_1 + p_y s_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ <https://blog.csdn.net/fengyu19930920>

等式右边等于

$${}^1_2T \cdot {}^2_3T \cdot {}^3_4T = {}^1_4T = \begin{bmatrix} c_{234} & 0 & s_{234} & a_3c_{23} + a_2c_2 \\ s_{234} & 0 & -c_{234} & a_3s_{23} + a_2s_2 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.4.1 求解关节角3

利用等式左右两边第1行，第4列对应相等，第2行，第4列对应相等，求关节角3。

$$d_5(s_6(n_xc_1 + n_ys_1) + c_6(o_xc_1 + o_ys_1)) - d_6(a_xc_1 + a_ys_1) + p_xc_1 + p_ys_1 = a_3c_{23} + a_2c_2 \quad (1)$$

$$p_z - d_1 - a_2d_6 + d_5(o_zc_6 + n_zs_6) = a_3s_{23} + a_2s_2 \quad (2)$$

为了简化，设：

$$d_5(s_6(n_xc_1 + n_ys_1) + c_6(o_xc_1 + o_ys_1)) - d_6(a_xc_1 + a_ys_1) + p_xc_1 + p_ys_1 = m$$

$$p_z - d_1 - a_2d_6 + d_5(o_zc_6 + n_zs_6) = n$$

将m,n带入上式得

$$m = a_3c_{23} + a_2c_2 \quad (3)$$

$$n = a_3s_{23} + a_2s_2 \quad (4)$$

式子③④平方和为

$$a_2^2 + a_3^2 + 2a_2a_3(s_{23}s_2 + c_{23}c_2) = m^2 + n^2$$

因为

$$s_{23}s_2 + c_{23}c_2 = c_{2+3}$$

所以

$$\theta_3 = \pm \arccos\left(\frac{m^2 + n^2 - a_2^2 - a_3^2}{2a_2a_3}\right) \quad (\text{其中 } m^2 + n^2 \leq (a_2 + a_3)^2)$$

2.4.2 求解关节角2

将③④展开得：

$$(a_3c_3 + a_2)c_2 - a_3s_3s_2 = m \quad (5)$$

$$a_3s_3c_2 + (a_3c_3 + a_2)s_2 = n$$

将关节角3带入⑤⑥，求关节角2得

$$s_2 = \frac{(a_3c_3 + a_2)n - a_3s_3m}{a_2^2 + a_3^2 + 2a_2a_3c_3}$$

$$c_2 = \frac{m + a_3s_3s_2}{a_3c_3 + a_2}$$

则

$$\theta_3 = A \tan 2(s_2, c_2)$$

2.4.3 求解关节角4

用 1_2T 的第2行第2列，第1行第2列求 $\theta_2 + \theta_3 + \theta_4$

$$s_{234} = -s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1) \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$c_{234} = o_z c_6 + n_z s_6 \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_2 + \theta_3 + \theta_4 = A \tan 2(-s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1), o_z c_6 + n_z s_6) \quad \text{https://blog.csdn.net/fengyu19930920}$$

则

$$\theta_4 = A \tan 2(-s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1), o_z c_6 + n_z s_6) - \theta_2 - \theta_3 \quad \text{https://blog.csdn.net/fengyu19930920}$$

2.5 总结

2.5.1 求解公式

$$d_6 a_y - p_y = m \quad | a_x d_6 - p_x = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_1 = A \tan 2(m, n) - A \tan 2(d_4, \pm \sqrt{m^2 + n^2 - d_4^2}) \quad (\text{其中 } m^2 + n^2 - d_4^2 \geq 0) \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_3 = \pm \arccos(a_x s_1 - a_y c_1) \quad (\text{其中 } a_x s_1 - a_y c_1 \leq 1) \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$n_x s_1 - n_y c_1 = m \quad o_x s_1 - o_y c_1 = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_5 = A \tan 2(m, n) - A \tan 2(s_3, 0) = A \tan 2(m / s_3, n / s_3) \quad (\text{其中 } s_3 \neq 0) \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$d_3(s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1)) - d_5(a_x c_1 + a_y s_1) + p_x c_1 + p_y s_1 = m \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$p_z - d_1 - a_z d_6 + d_5(o_z c_6 + n_z s_6) = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_3 = \pm \arccos\left(\frac{m^2 + n^2 - a_2^2 - a_3^2}{2a_2 a_3}\right) \quad (\text{其中 } m^2 + n^2 \leq (a_2 + a_3)^2) \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$s_2 = \frac{(a_3 c_3 + a_2)n - a_3 s_3 m}{a_2^2 + a_3^2 + 2a_2 a_3 c_3} \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$c_2 = \frac{m + a_3 s_3 s_2}{a_3 c_3 + a_2} \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_2 = A \tan 2(s_2, c_2) \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$\theta_4 = A \tan 2(-s_6(n_x c_1 + n_y s_1) - c_6(o_x c_1 + o_y s_1), o_z c_6 + n_z s_6) - \theta_2 - \theta_3 \quad \text{https://blog.csdn.net/fengyu19930920}$$

2.5.2 奇异位置

1. 肩关节奇异位置

$$n_x s_1 - n_y c_1 = m \quad o_x s_1 - o_y c_1 = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$m^2 + n^2 - d_4^2 = 0 \quad \text{https://blog.csdn.net/fengyu19930920}$$

此时末端执行器参考点O6位于轴线z1和z2构成的平面内，关节角1无法求解。

2. 肘关节奇异位置

$$d_3(s_6(n_x c_1 + n_y s_1) + c_6(o_x c_1 + o_y s_1)) - d_5(a_x c_1 + a_y s_1) + p_x c_1 + p_y s_1 = m \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$p_z - d_1 - a_z d_6 + d_5(o_z c_6 + n_z s_6) = n \quad \text{https://blog.csdn.net/fengyu19930920}$$

$$m^2 + n^2 - (a_2 + a_3)^2 = 0 \quad \text{https://blog.csdn.net/fengyu19930920}$$

此时关节角2无法求解。

3. 腕关节奇异位置

解。

2.6 实例

利用Matlab机器人 库 ur5 DH参数：

```
alpha1 = pi/2;    a1=0;        d1=89.459;
alpha2 = 0;        a2=-425;     d2=0;
alpha3 = 0;        a3=-392.25;  d3=0;
alpha4 = pi/2;     a4=0;        d4=109.15;
alpha5 = -pi/2;    a5=0;        d5=94.65;
alpha6 = 0;        a6=0;        d6=82.3;
```

取 $\theta_1 = 1$; $\theta_2 = 1$; $\theta_3 = 1$; $\theta_4 = 1$; $\theta_5 = 1$; $\theta_6 = 1$; (不要纠结 θ 选这6个数值是否有实际意义，这里只验证算法的有效性)

- 将 θ 带入正运动学公式，求T:

```
T =
    0.1623   -0.3938    0.9047   137.6508
   -0.5888    0.6972    0.4091   -69.9381
   -0.7919   -0.5991   -0.1187  -540.9083
         0         0         0         1.0000
```

- 将T带入逆运动学公式，反求 θ

```
theta =
    1.0000    1.0000    1.0000    1.0000    1.0000    1.0000
    1.0000    1.9562   -1.0000    2.0438    1.0000    1.0000
    1.0000    0.6743    1.7374   -2.5533   -1.0000   -2.1416
    1.0000    2.3170   -1.7374   -0.7212   -1.0000   -2.1416
    0.0954    1.1462    1.0086    0.8611    1.8976    0.8826
    0.0954    2.1106   -1.0086    1.9139    1.8976    0.8826
    0.0954    0.8754    1.7300   -2.7311   -1.8976   -2.2590
    0.0954    2.5114   -1.7300   -0.9071   -1.8976   -2.2590
```

- 再将8个 θ 带入正运动学公式，反求8个T:

8个T均等于

```
    0.1623   -0.3938    0.9047   137.6508
   -0.5888    0.6972    0.4091   -69.9381
   -0.7919   -0.5991   -0.1187  -540.9083
         0         0         0         1.0000
```

验证了算法的有效性