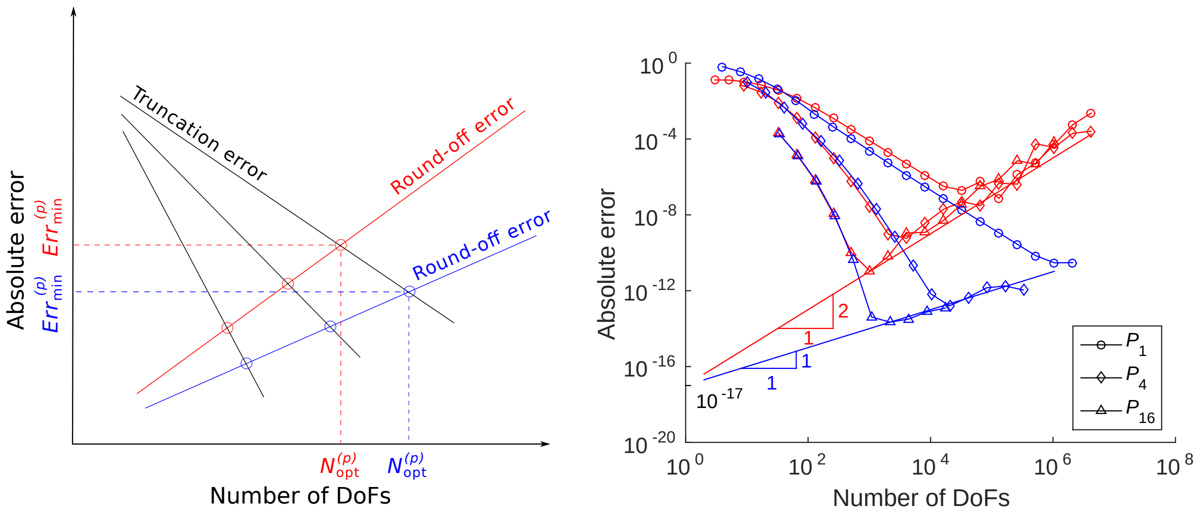
A practical a posteriori strategy to ascertain the optimal number of degrees of freedom for hp-refinement in finite element methods

Jie Liu, Matthias Möller, Henk M. Schuttelaars

Delft Institute of Applied Mathematics, Delft University of Technology,
Van Mourik Broekmanweg 6, 2628 XE Delft, The Netherlands

To improve the accuracy of solutions obtained with finite element methods, h-, p- and hp- refinements are widely used. They all aim at decreasing the truncation error by increasing the number of degrees of freedom ("DoFs"). However, when the number of DoFs becomes larger than a critical number $N_{\text{opt}}^{(p)}$, where p is the degree of the element, round-off errors accumulate and start to exceed the truncation error, and thus dominate the total error. Since further refinements will even result in less accurate solutions, the error $Err_{\text{min}}^{(p)}$ corresponding to $N_{\text{opt}}^{(p)}$ is the minimum attainable error for the element of order p. The conceptual sketch of the dependency of the truncation and round-off error for the solution on the number of DoFs is shown in Fig. 1(a) [1].



(a) Conceptual sketch. **(b)** Numerical results.

Fig. 1. Absolute errors using the standard and mixed FEM. The red color relates to the standard FEM, the blue color relates to the mixed FEM, and the black color relates to both of them.

It shows that $N_{\text{opt}}^{(p)}$ strongly depends on p, with $N_{\text{opt}}^{(p)}$ decreasing for increasing p, both for the standard and mixed FEM. The same trend is applied for derivative quantities. Thus, by taking higher-order elements and the optimal number of DoFs, the best $Err_{\text{min}}^{(p)}$ can be achieved, resulting in most accurate solutions. This leads us to develop a practical a posteriori hp-refinement strategy that adjusts both the mesh width $h^{(p)}$ (as a function of p) and p simultaneously so that for each p the optimal mesh width $h_{\text{opt}}^{(p)}$ correlates with $N_{\text{opt}}^{(p)}$.

To illustrate a systematic method to identify $N_{\text{opt}}^{(p)}$ and $Err_{\text{min}}^{(p)}$ a posteriori, we focus on the following model problem:

$$\nabla \cdot (D(\boldsymbol{x})\nabla u) + r(\boldsymbol{x})u(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega = [0,\, 1]^{\text{dim}},$$

with u denoting the unknown variable, $f \in L^2(\Omega)$ a prescribed right-hand side, D and r coefficient functions, and dim the space dimension. Both the standard and mixed FEM are implemented in deal.II [2].

Based on an extensive numerical analysis of the one-dimensional model problem [1], we validated the robustness of the trend in Fig. 1(a) and have identified the possible influence factors, such as the L_2 norm of the solution $||u||_2$ in the standard FEM and both $||u||_2$ and the L_2 norm of the first derivative $||u_x||_2$

in the mixed FEM, floating-point precision, computational mesh, type and implementation of boundary conditions and choice of solvers. In particular, to make the round-off error independent of $||u||_2$ and/or $||u_x||_2$, scaling schemes for the right-hand side are put forward for both the standard and mixed FEM.

The practical usability of the proposed *hp*-refinement strategy is demonstrated for the one-dimensional Helmholtz equation [1], see Fig. 1(b), where the value of $N_{\text{opt}}^{(p)}$ is determined based on the extrapolated truncation error from coarse grid computations. Last but not least, preliminary results show that the same trend in the one-dimensional model problem carries over to the two-dimensional one.

References

[1] J. Liu, M. Möller, H. M. Schuttelaars, A practical a posteriori strategy to ascertain the optimal number of degrees of freedom for *hp*-refinement in finite element methods, in preparation.

[2] G. Alzetta, D. Arndt, W. Bangerth, V. Boddu, B. Brands, D. Davydov, R. Gassmöller, T. Heister, L. Heltai, K. Kormann, et al., The deal. ii library, version 9.0, Journal of Numerical Mathematics 26 (4) (2018) 173–183.