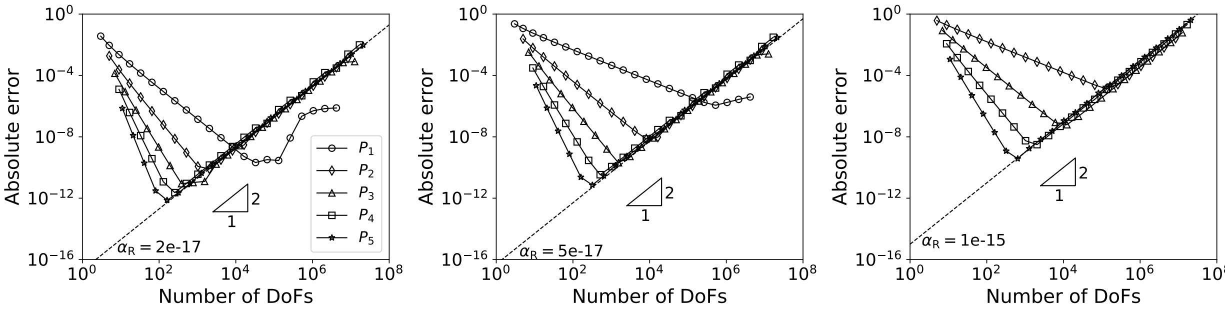
1. Numerical results for the benchmark Poisson, diffusion and Helmholtz equations

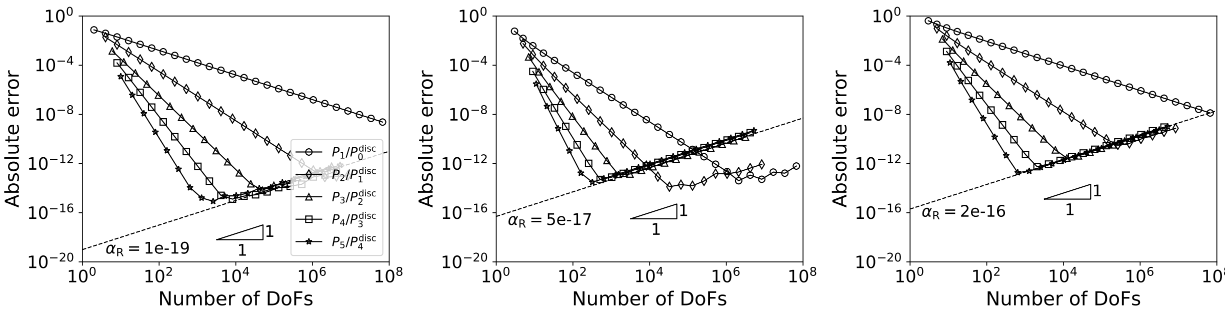
1.1. *The Poisson equation*

1.1.1. *p variant*



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 1. Absolute errors for the benchmark Poisson equation using the standard FEM.



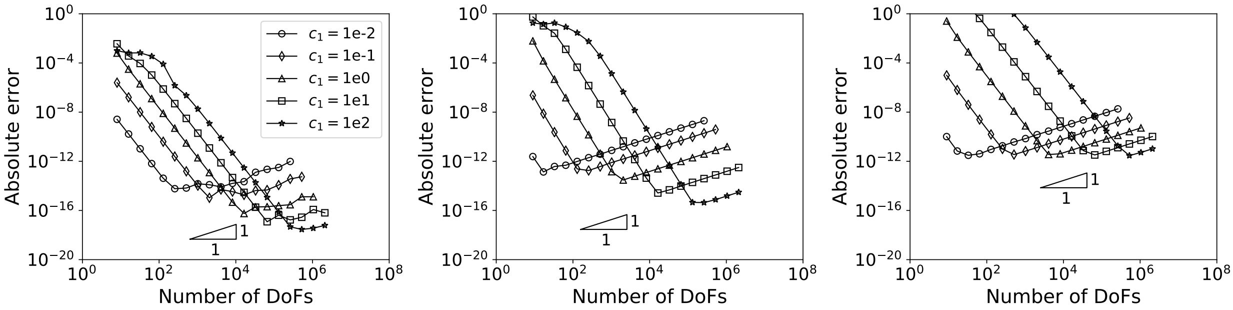
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 2. Absolute errors for the benchmark Poisson equation using the mixed FEM.

1.2. *The diffusion equation for* $p = (2\pi c_1)^{-2}\sin(2\pi c_1 x),\ d = 10$

The standard FEM.

The mixed FEM.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 3. Absolute errors for the diffusion equation with $p = (2\pi c_1)^{-2}\sin(2\pi c_1 x)$, $d = 10$ using the mixed FEM, no scaling.

1.3. The diffusion equation for p=sin$(2\pi x)$*,* d=1+0.5sin(cx)

To investigate the influence of the oscillation of the diffusion coefficient on the error, we consider $d = 1 + \frac{1}{2}\sin(cx)$ for the benchmark Poisson equation, resulting the first diffusion equation. For c ranging from 1 to 10^4, $\|d\|_2$ is of order 1, see Fig. 4. The oscillation of d magnifies when c increases. Using both the standard FEM and the mixed FEM, the errors are shown below. For the standard FEM, P_2 elements are used, and for the mixed FEM, P_4/P_3^{disc} elements are used.

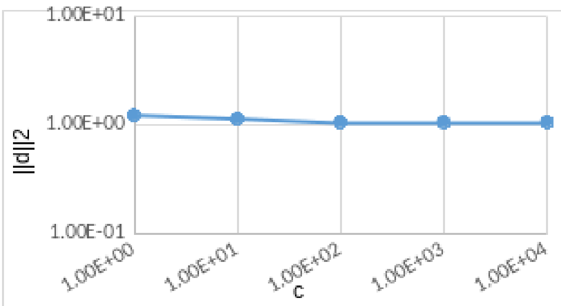
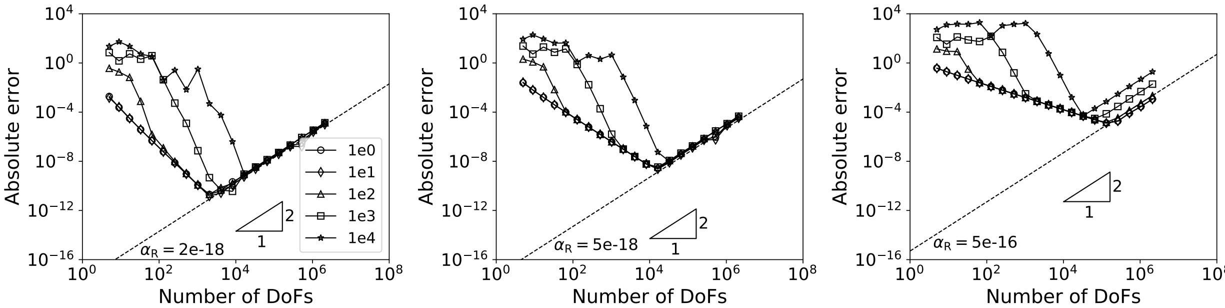


Fig. 4. Change of $\|d\|_2$ with the coefficient c for the first diffusion equation.

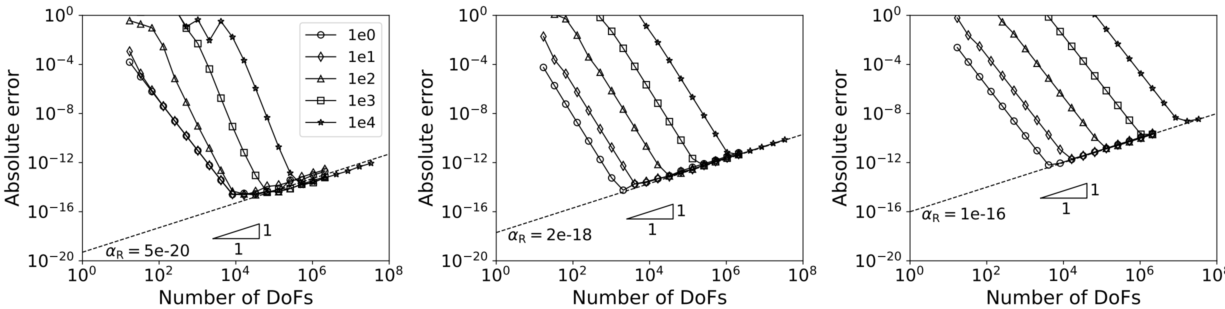
The standard FEM. The truncation error increases when the d oscillates relatively large, i.e. when c is larger than 10. The lines approximating the round-off error for the solution and first derivative are not affected by the oscillation of d, but that for the second derivative moves up a bit when c is larger than 10.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 5. Absolute errors for the first diffusion equation using the standard FEM.

The mixed FEM. The truncation error increases when d oscillates relatively large. The lines approximating the round-off error are not affected.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 6. Absolute errors for the first diffusion equation using the mixed FEM.

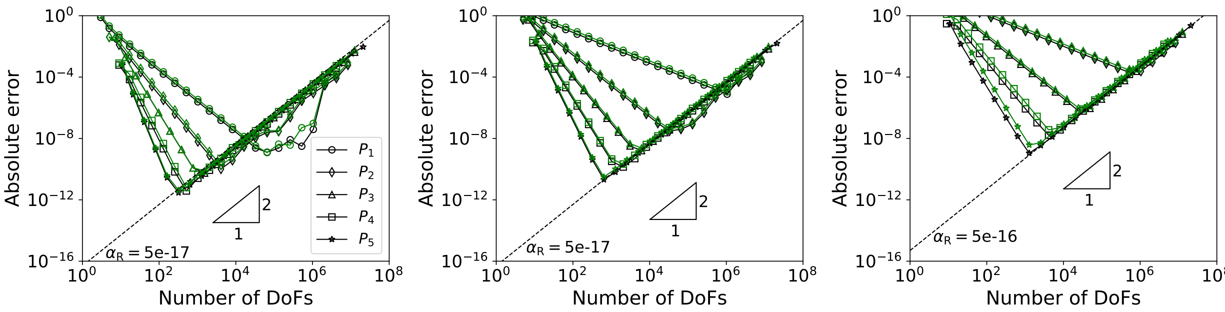
Last but not least, the offsets α_R tend to be smaller than that of the Poisson equation.

1.4. The diffusion equation for p=$\sin(2\pi x)$*, d=1+cx*

We consider the diffusion equation shown in Table 1 in [1]. We first find the offsets α_{R} for the diffusion equation using recommended scaling schemes for the standard and mixed FEMs, in which the element degree p ranges from 1 to 5. Next, we investigate α_{R} for different coefficients, i.e. $d(x) = 1 + cx$ with c ranging from 10^{-4} to 10^{4}, in which P_2 elements are used for the standard FEM, and P_4/P_3^{disc} elements are used for the mixed FEM.

1.4.1. p variant

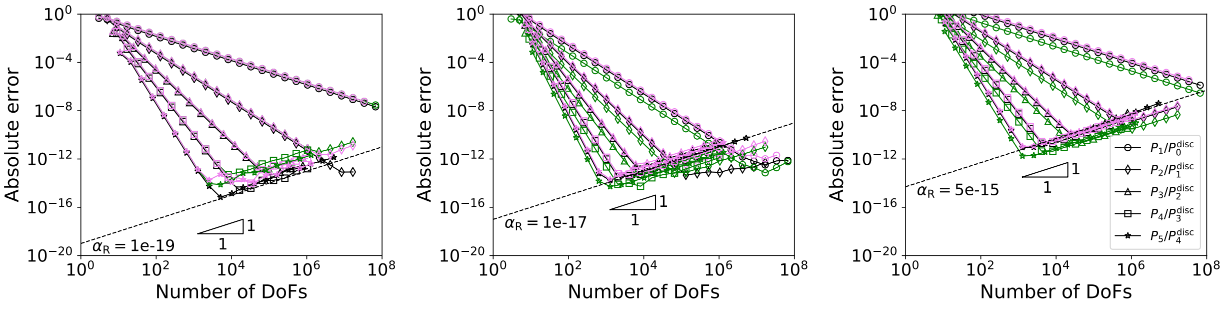
The standard FEM. Not using scaling and using scheme S, the errors are shown in Fig. 7.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 7. Absolute errors for the benchmark diffusion equation using the standard FEM. The black color denotes results without scaling, and the green color denotes results using scheme S.

The mixed FEM. Not using scaling and using schemes M_1 and M_2, the errors are shown in Fig. 8.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 8. Absolute errors for the benchmark diffusion equation using the mixed FEM. The black color denotes results without scaling, the green color denotes results using scheme M_1 and the violet color denotes results using scheme M_2.

The offsets α_{R} for different scaling schemes are summarized in Fig. 9. For the standard FEM, α_{R} are basically the same using scheme S or not. For the mixed FEM, α_{R} changes a bit for different scaling schemes.

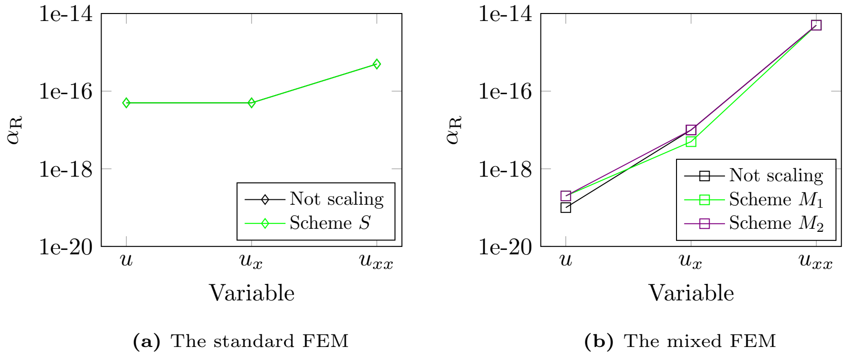


Fig. 9. α_{R} for the benchmark diffusion equation for different scaling schemes.

1.4.2. c variant

The values of $\|d\|_2$ for different c are given in Fig. 10. It shows that when $c < 1$, $\|d\|_2$ are basically the same; when $c > 1$, $\|d\|_2$ increases quickly.

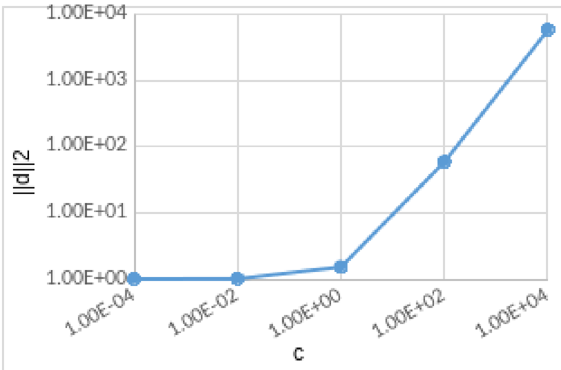
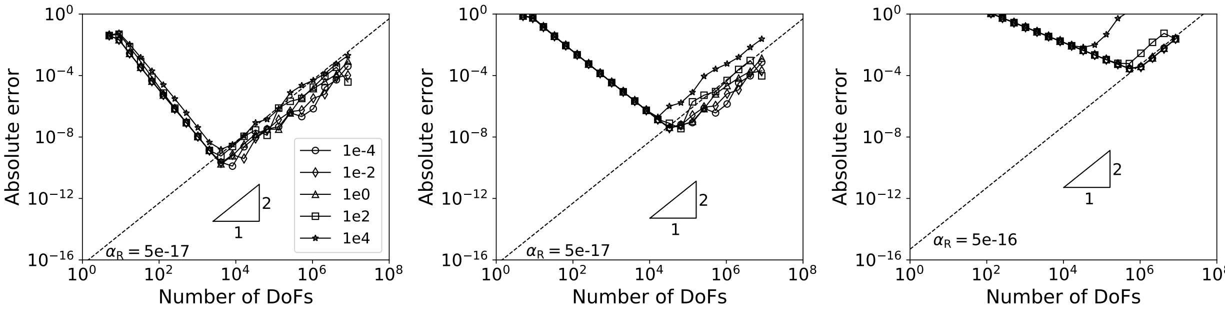


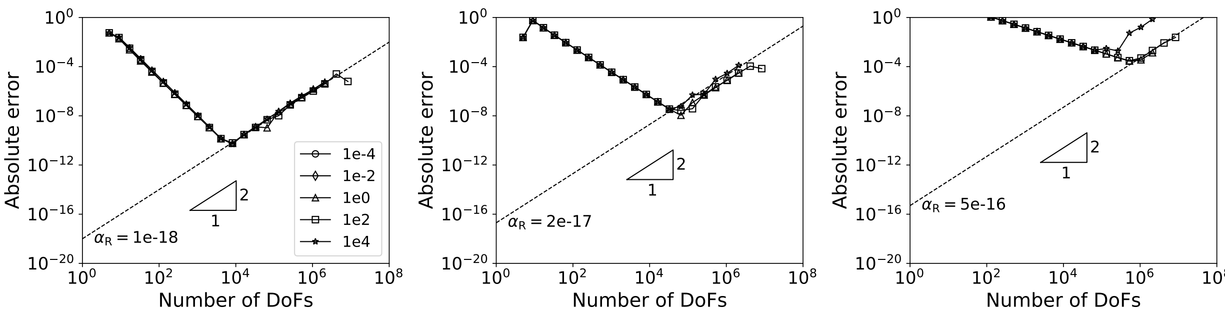
Fig. 10. Change of $\|d\|_2$ with the coefficient c for the diffusion equation.

The standard FEM. Using scheme S, the errors are shown in Fig. 11. When $c < 1$, like $\|d\|_2$, α_{R} for different c are basically the same; when $c > 1$, α_{R} increases with c, and the magnitude of the increase is larger for higher-order derivatives.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 11. Absolute errors for the benchmark diffusion equation using the standard FEM with scheme S, c variant.



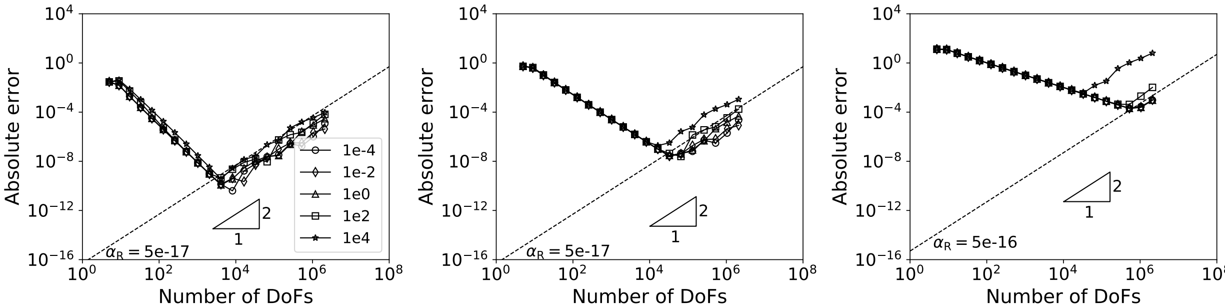
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 12. Absolute errors for the benchmark diffusion equation but only imposed by Dirichlet boundary conditions using the standard FEM with scheme S, c variant.

To clarify if the increase of α_{R} for higher-order derivatives is caused by the magnitude of $\|d\|_2$, we divide the equation by $\|d\|_2$, which results in

$$-(d/\|d\|_2 u_x)_x = f/\|d\|_2. \tag{1}$$

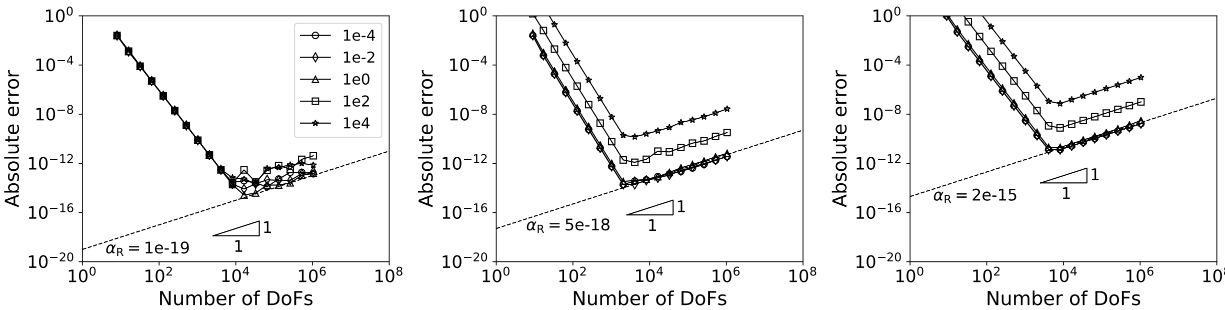
For the above equation, the errors of the solution, first and second derivatives are shown in Fig. 13. It shows that α_{R} also increases when c is relatively large.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 13. Absolute errors for Eq. (1) using the standard FEM, no scaling, c variant.

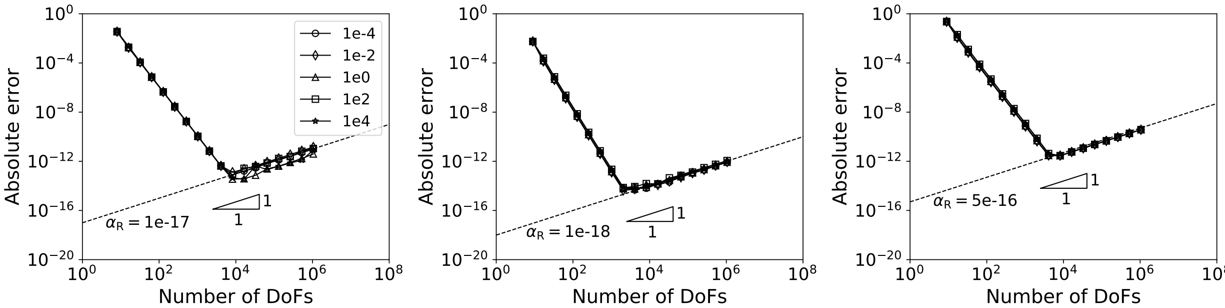
The mixed FEM. Not using scaling, the errors are shown in Fig. 14. It shows that when c is relatively large, α_{R} for the first and second derivatives increases. This is because the magnitude of the first derivative, which is $\|du_x\|_2$, increases with c when $c > 1$.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 14. Absolute errors for the benchmark diffusion equation using the mixed FEM with P_4/P_3^{disc} elements, $v = -du_x$, coefficient variant, no scaling.

If we use scheme M_1 in [1], the errors are shown in Fig. 15, where the convergence behavior of α_{R} is observed. Note that, $\|v\|_2$ is $\|du_x\|_2$ for the diffusion equation.



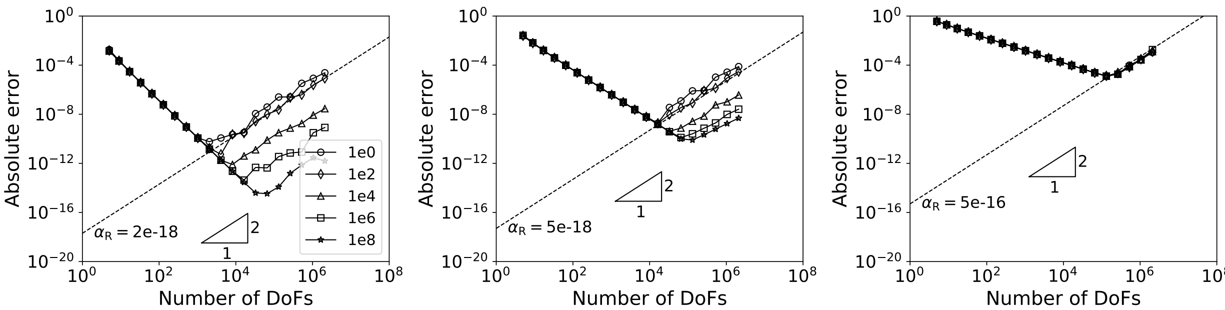
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 15. Absolute errors for the benchmark diffusion equation using the mixed FEM with P_4/P_3^{disc} elements, $v = -du_x$, coefficient variant, scheme M_1.

1.5. The Helmholtz equation

For $d(x) = 1 + 0.5\sin(x)$, we consider the influence of the magnitude of $r(x)$ on the error. We consider $r(x)$ as a constant, which ranges from 1 to 10^8.

The standard FEM. We use P_2 elements. The errors are shown in Fig. 16.

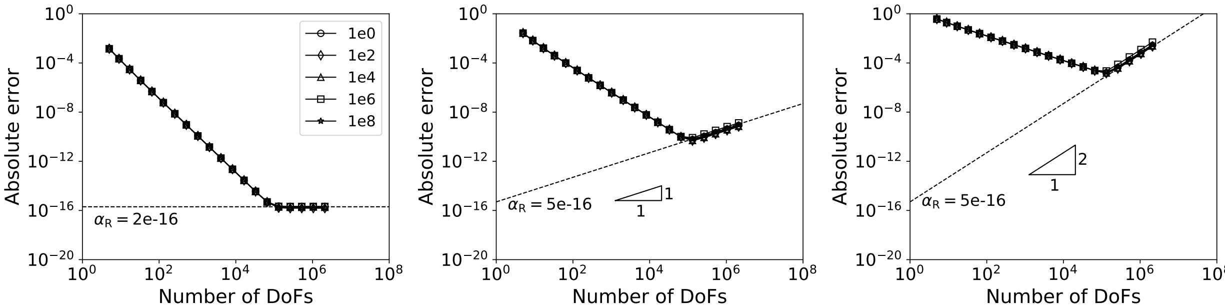


(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 16. Absolute errors for the Helmholtz equation using the standard FEM.

If we omit the diffusion part, i.e. concerning $d(x) = 0$, using the same set of $r(x)$, the errors are shown in Fig. 17.

Comparing Fig. 16 and Fig. 17, the slope 2 for the round-off error is led by the second-order derivative part in the differential equation.

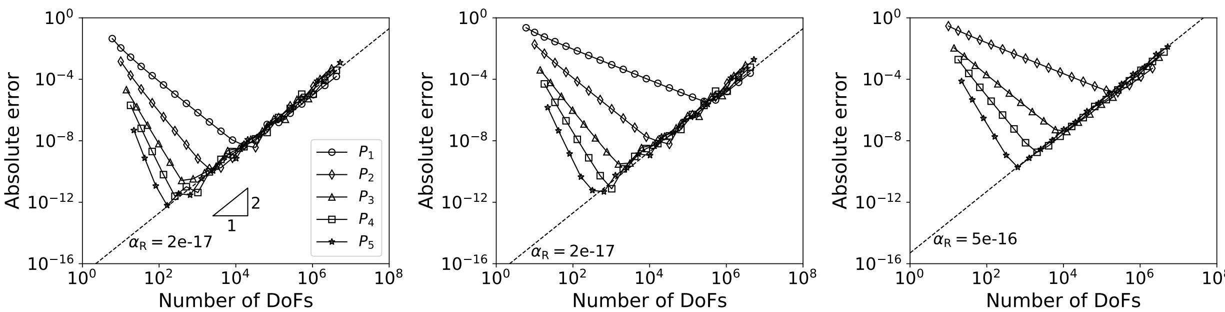


(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 17. Absolute errors for the Helmholtz equation, omitting the diffusion part, using the standard FEM.

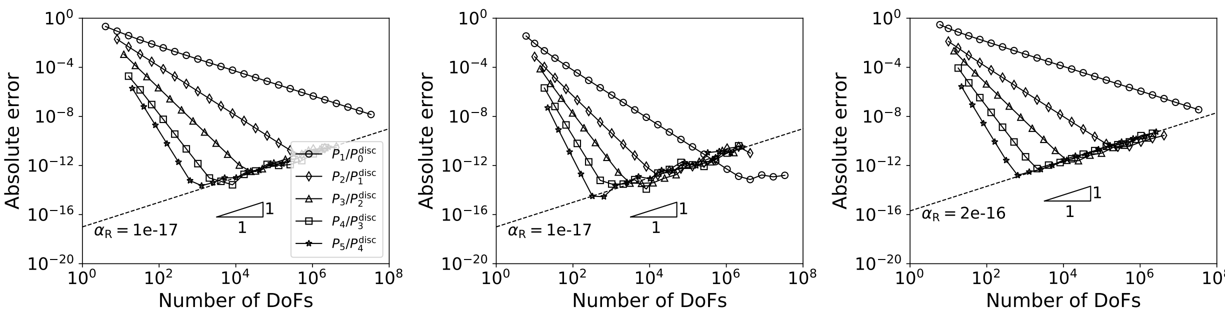
1.6. *The complex Helmholtz equation*

For the benchmark Helmholtz equation, using both the standard FEM and the mixed FEM, the absolute errors for all *three* variables are shown in Fig. 18 and Fig. 19, respectively.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 18. Absolute errors for the benchmark Helmholtz equation using the standard FEM.



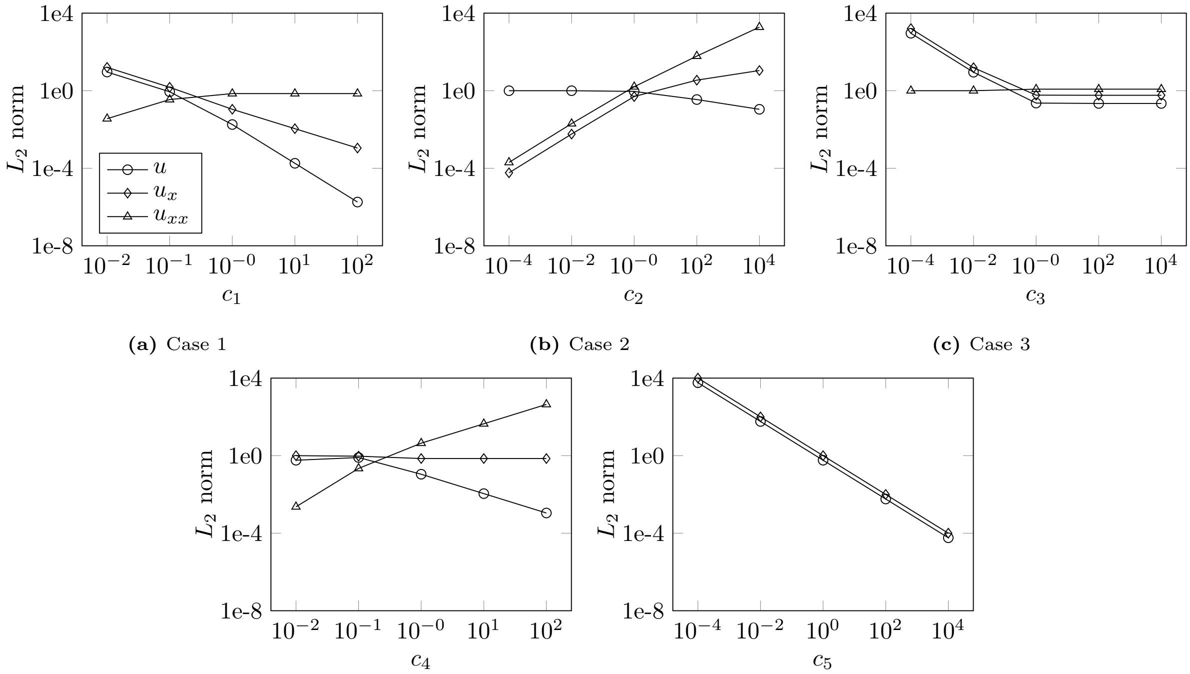
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 19. Absolute errors for the benchmark Helmholtz equation using the mixed FEM.

2. L_2 norms and absolute errors for different cases

2.1. L_2 norms

The L_2 norms of u, u_x and/or $u_{xx}(f)$ for different cases are shown in Fig. 20.



(a) Case 1 **(b)** Case 2 **(c)** Case 3

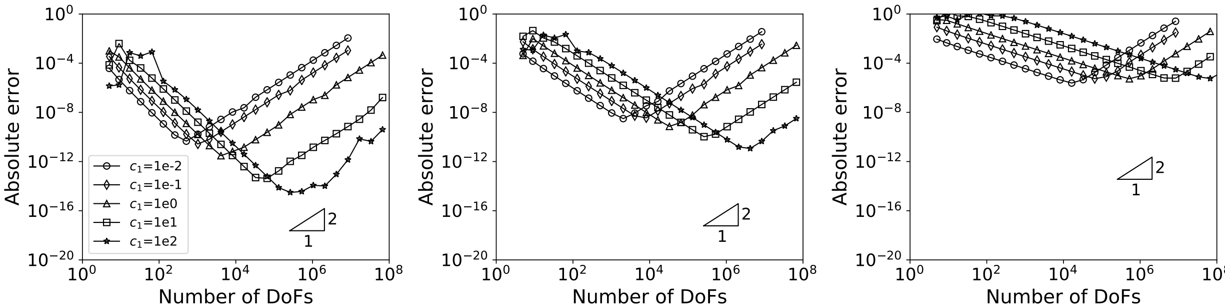
(d) Case 4 **(e)** Case 5

Fig. 20. L_2 norms of u, u_x and u_{xx} for different cases.

2.2. *Absolute errors*

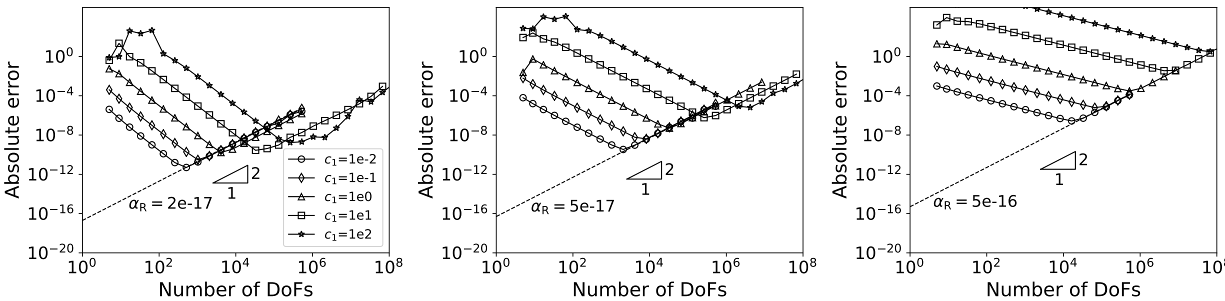
2.2.1. *The standard FEM*

Case 1. For Case 1, using the standard FEM without scaling the right-hand side and scheme S, the absolute errors are shown in Figs. 21–22.



(a) Solution **(b)** First derivative **(c)** Second derivative

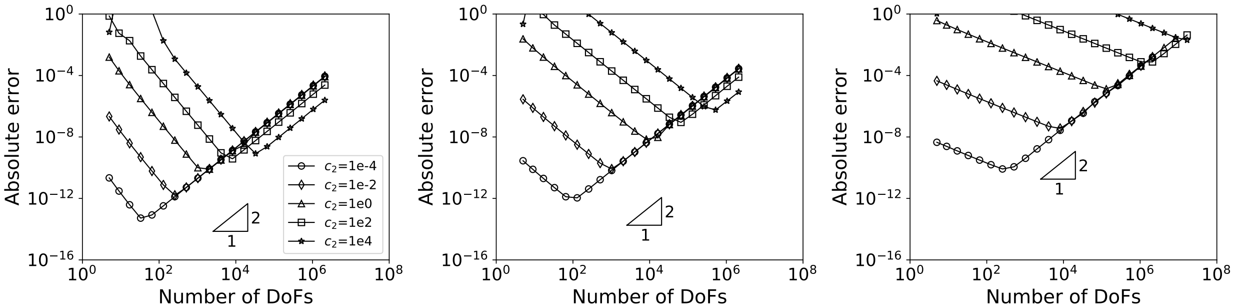
Fig. 21. Absolute errors for different c_1 using the standard FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

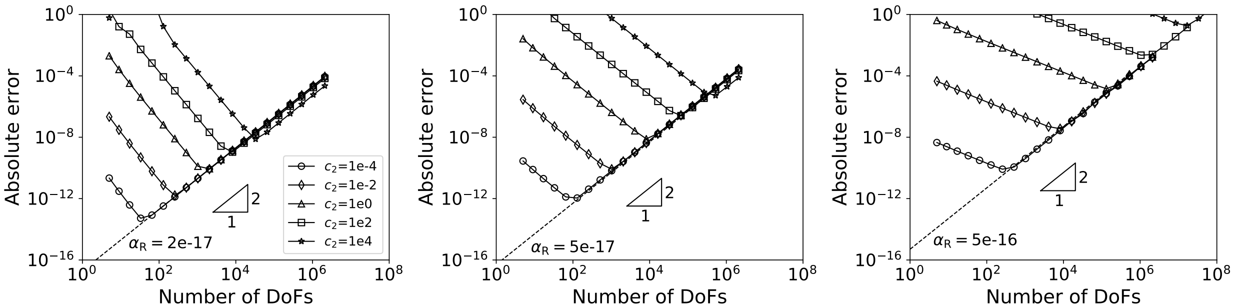
Fig. 22. Absolute errors for different c_1 using the standard FEM with scheme S.

Case 2. For Case 2, using the standard FEM without scaling the right-hand side and scheme S, the absolute errors are shown in Figs. 23–24.



(a) Solution **(b)** First derivative **(c)** Second derivative

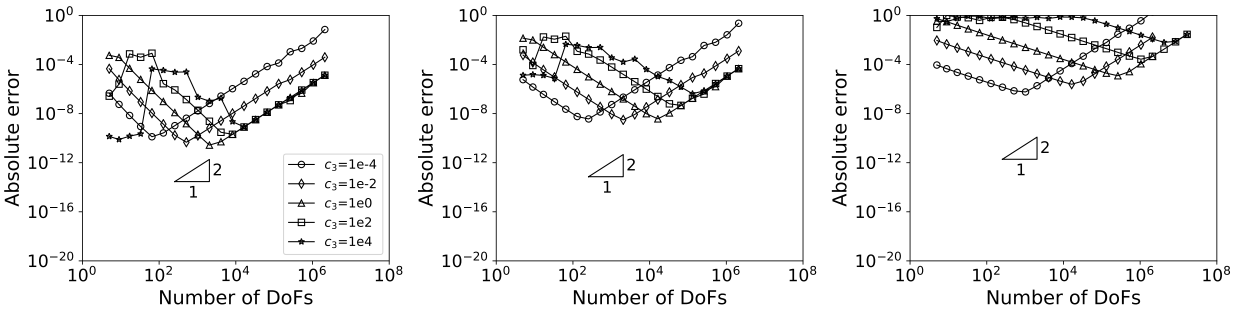
Fig. 23. Absolute errors for Case 2 using the standard FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

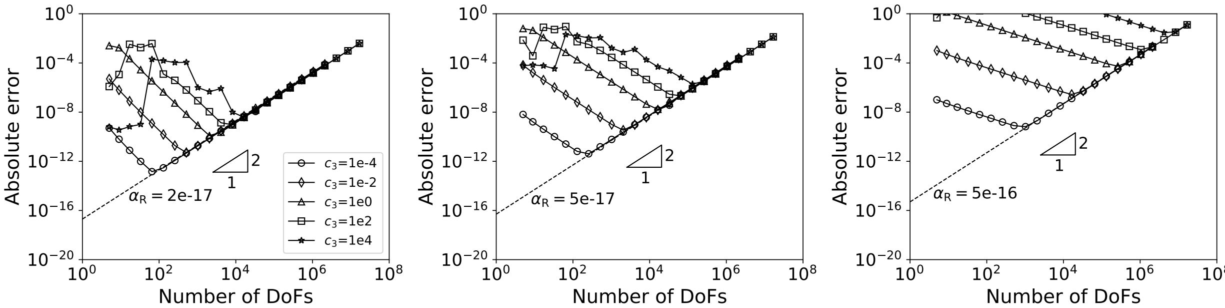
Fig. 24. Absolute errors for Case 2 using scheme S.

Case 3. For Case 3, using the standard FEM without scaling the right-hand side and scheme S, the absolute errors are shown in Figs. 25–26.



(a) Solution **(b)** First derivative **(c)** Second derivative

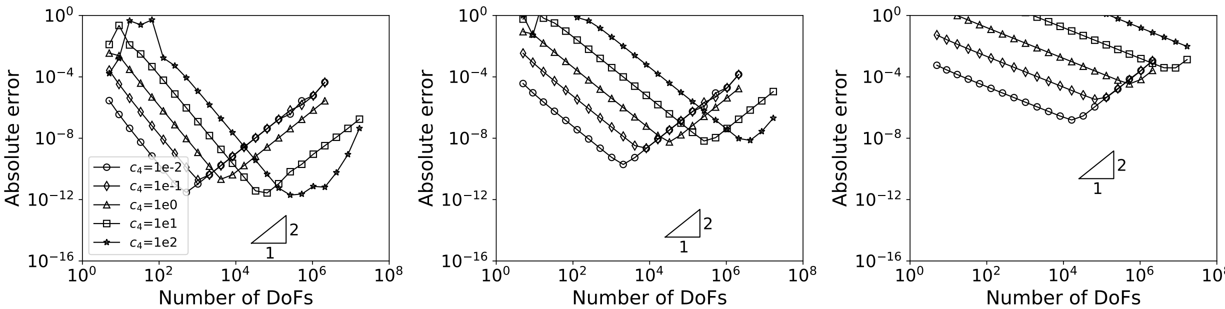
Fig. 25. Absolute errors for Case 3 using the standard FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

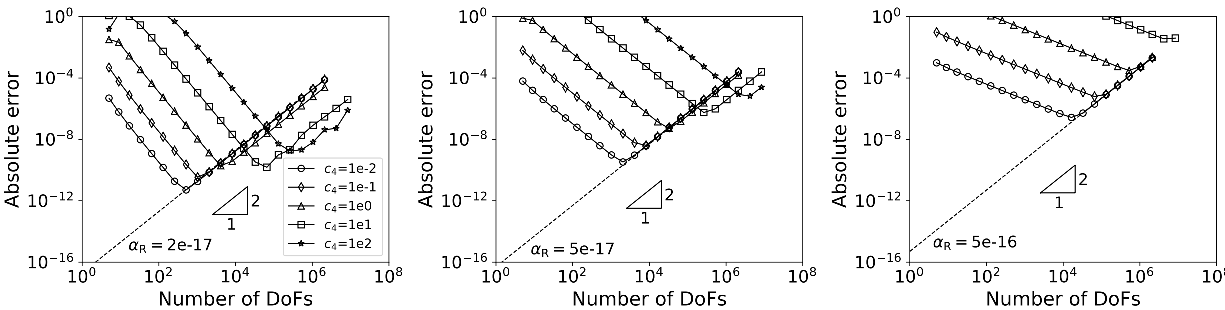
Fig. 26. Absolute errors of Case 3 using scheme S.

Case 4. For Case 4, using the standard FEM without scaling the right-hand side and scheme S, the absolute errors are shown in Figs. 27–28.



(a) Solution **(b)** First derivative **(c)** Second derivative

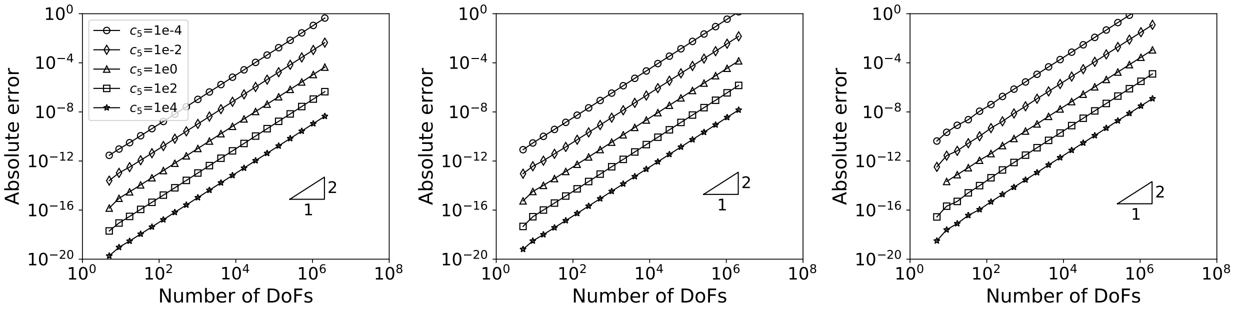
Fig. 27. Absolute errors for Case 4 using the standard FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

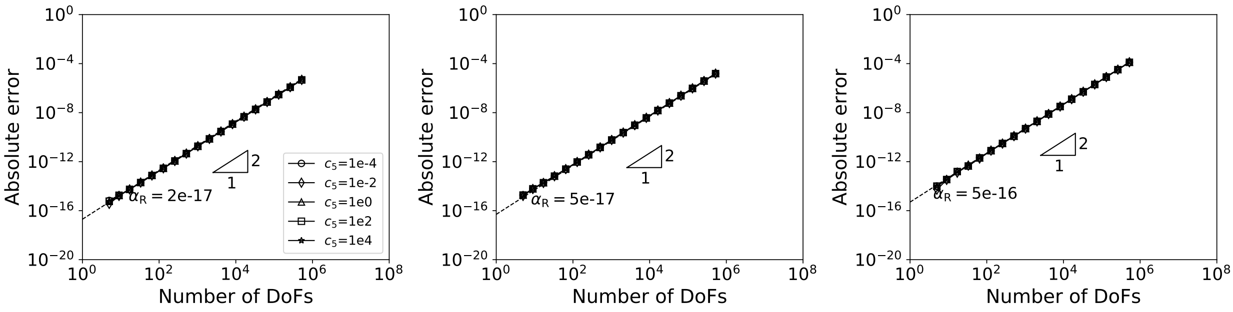
Fig. 28. Absolute errors of Case 4 using scheme S.

Case 5. For Case 5, using the standard FEM without scaling the right-hand side and scheme S, the absolute errors are shown in Figs. 29–30.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 29. Absolute errors for Case 5 using the standard FEM without scaling the right-hand side.

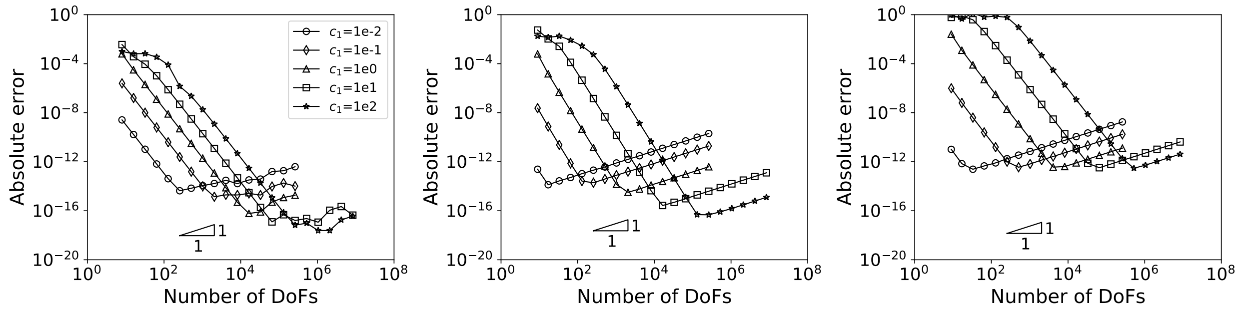


(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 30. Absolute errors of Case 5 using scheme S.

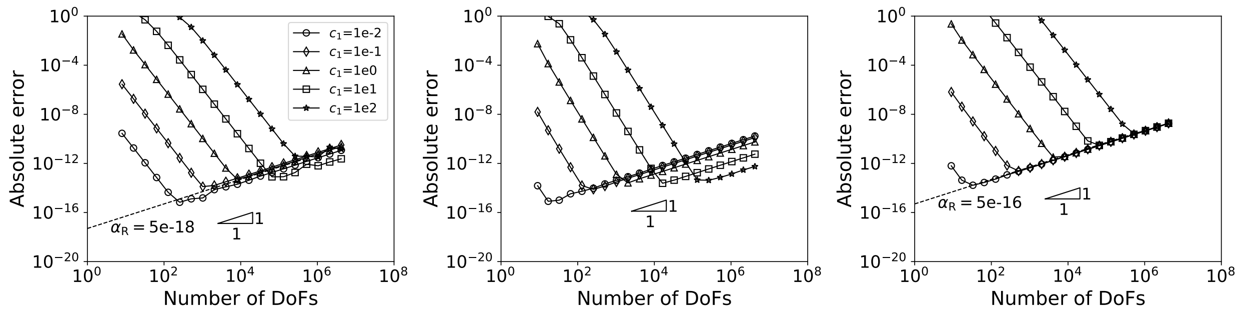
2.2.2. *The mixed FEM*

Case 1. For Case 1, using the mixed FEM without scaling the right-hand side and schemes M_1 and M_2, the absolute errors are shown in Figs. 31–33.



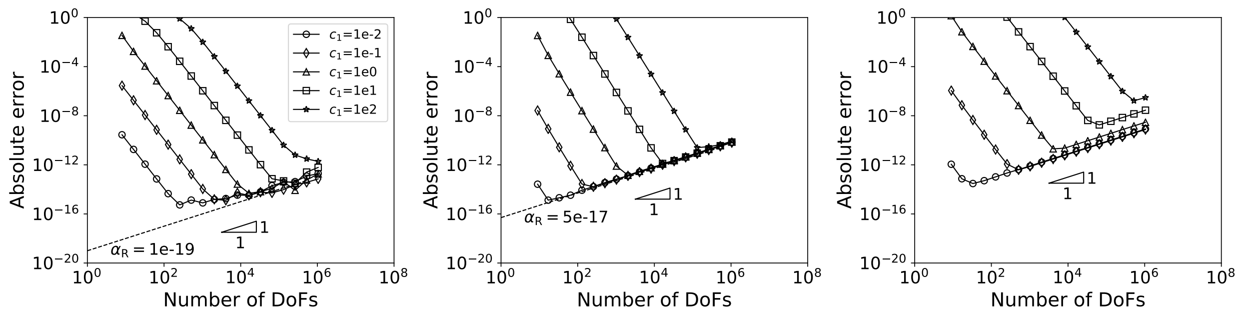
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 31. Absolute errors for different c_1 using the mixed FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

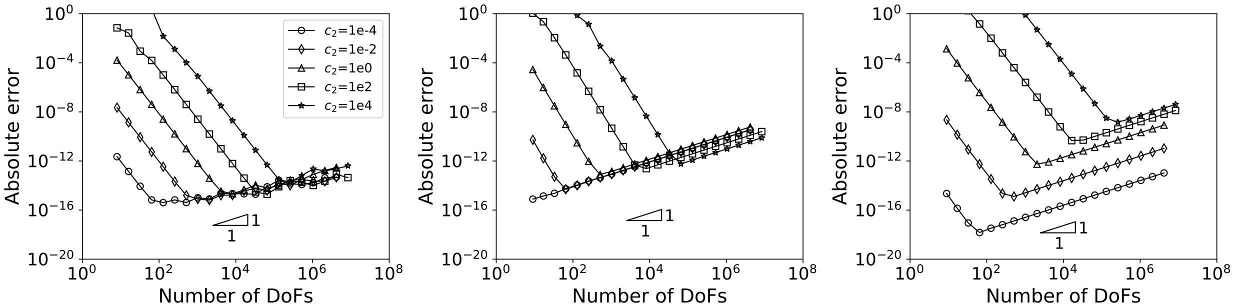
Fig. 32. Absolute errors for different c_1 using the mixed FEM with scheme M_1.



(a) Solution **(b)** First derivative **(c)** Second derivative

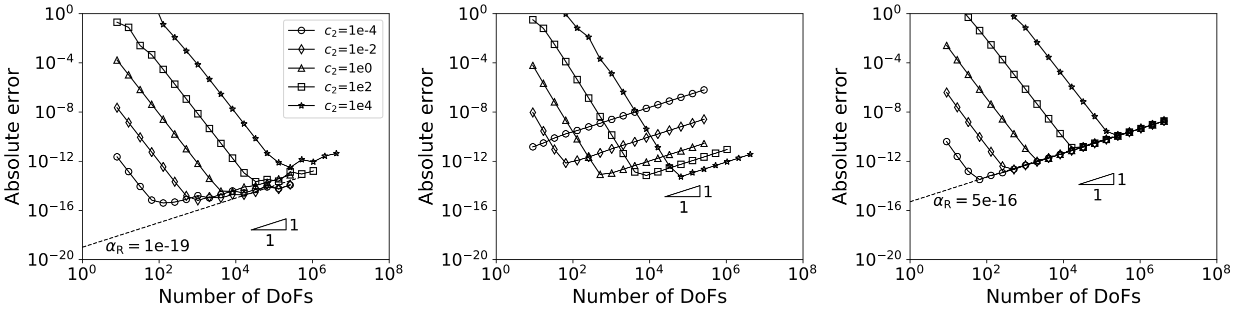
Fig. 33. Absolute errors for different c_1 using the mixed FEM with scheme M_2.

Case 2. For Case 2, using the mixed FEM without scaling the right-hand side and schemes M_1 and M_2, the absolute errors are shown in Figs. 34–36.



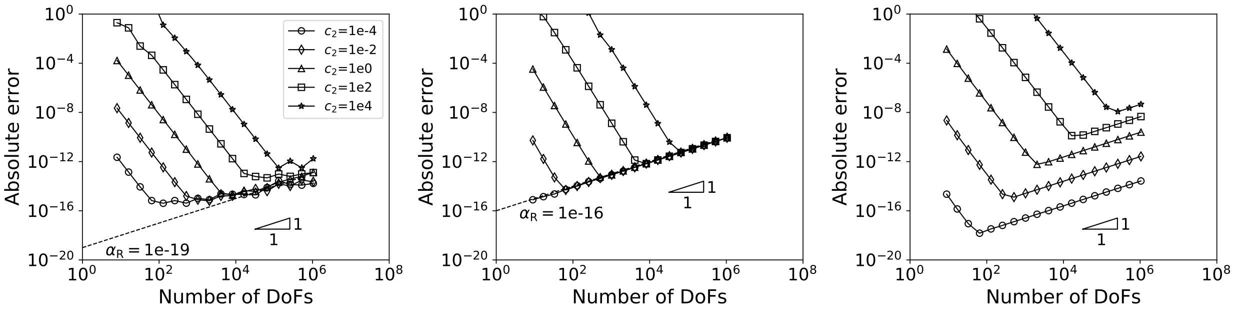
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 34. Absolute errors for Case 2 using the mixed FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

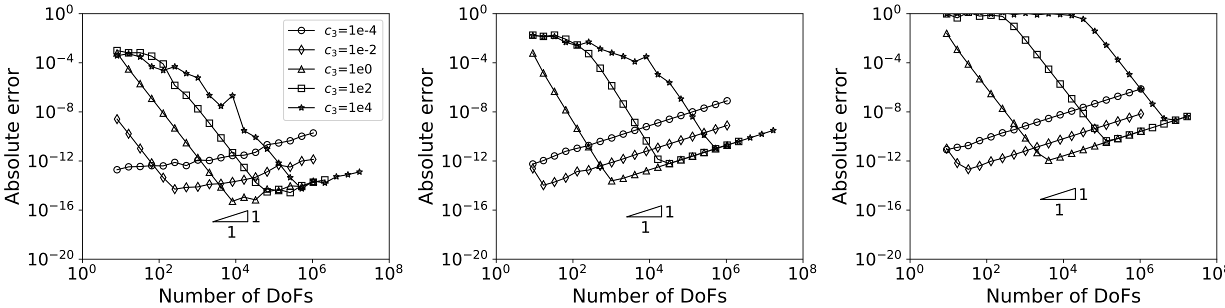
Fig. 35. Absolute errors for Case 2 using scheme M_1.



(a) Solution **(b)** First derivative **(c)** Second derivative

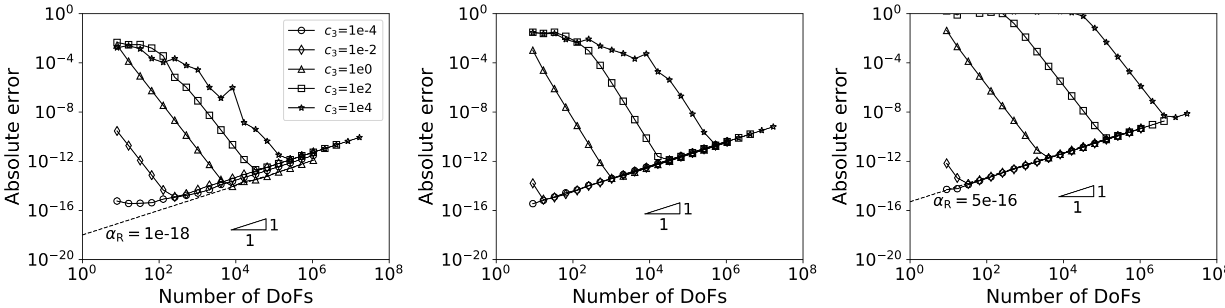
Fig. 36. Absolute errors for Case 2 using scheme M_2.

Case 3. For Case 3, using the mixed FEM without scaling the right-hand side and schemes M_1 and M_2, the absolute errors are shown in Figs. 37–39.



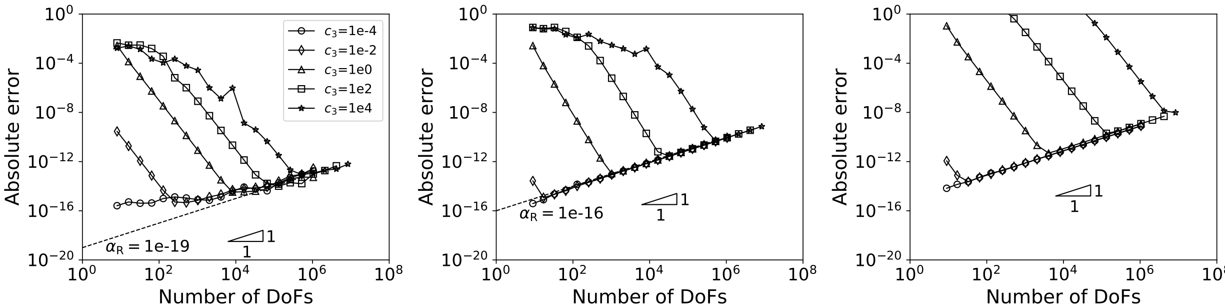
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 37. Absolute errors for Case 3 using the mixed FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

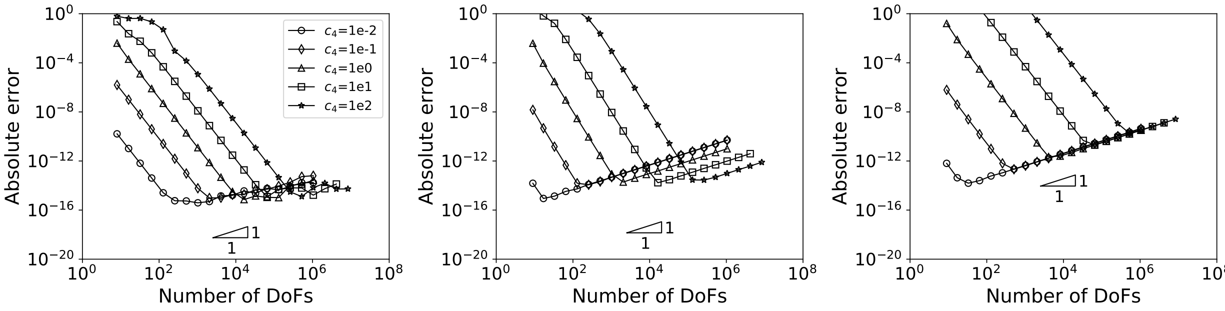
Fig. 38. Absolute errors for Case 3 using scheme M_1.



(a) Solution **(b)** First derivative **(c)** Second derivative

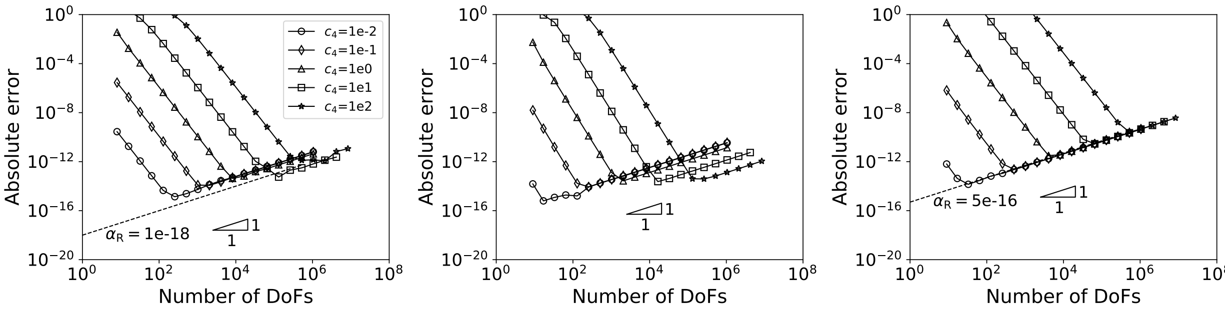
Fig. 39. Absolute errors for Case 3 using scheme M_2.

Case 4. For Case 4, using the mixed FEM without scaling the right-hand side and schemes M_1 and M_2, the absolute errors are shown in Figs. 40–42.



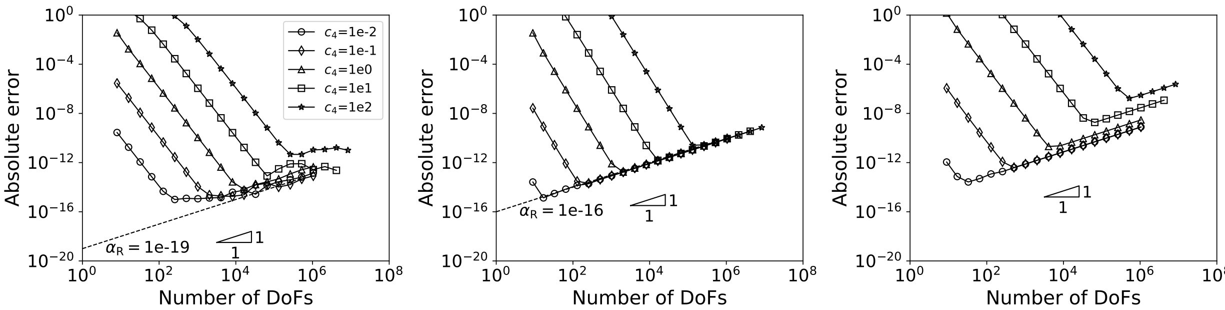
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 40. Absolute errors for Case 4 using the mixed FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

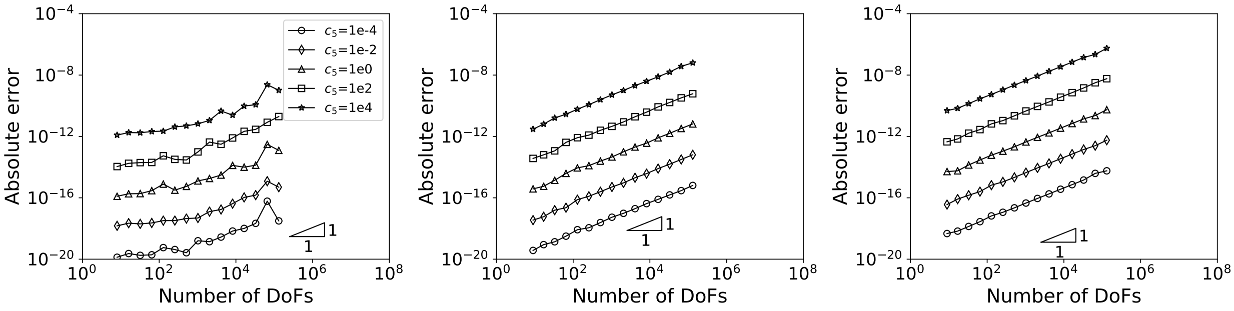
Fig. 41. Absolute errors for Case 4 using scheme M_1.



(a) Solution **(b)** First derivative **(c)** Second derivative

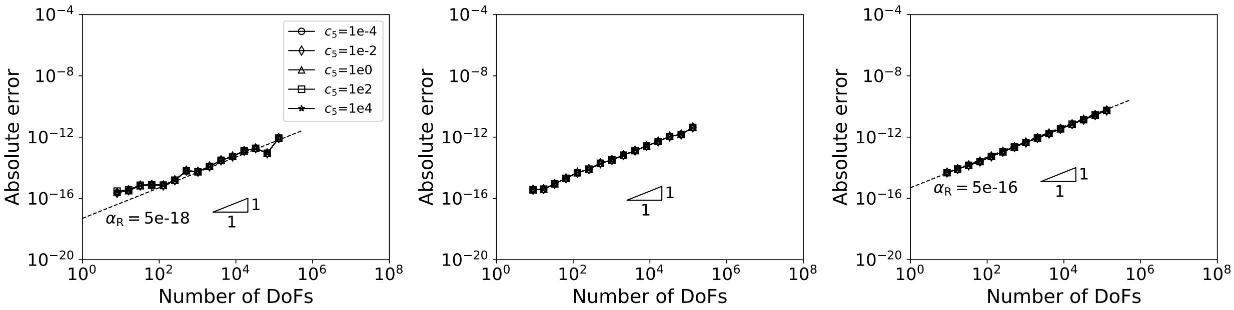
Fig. 42. Absolute errors for Case 4 using scheme M_2.

Case 5. For Case 5, using the mixed FEM without scaling the right-hand side and schemes M_1, the absolute errors are shown in Figs. 43–44.



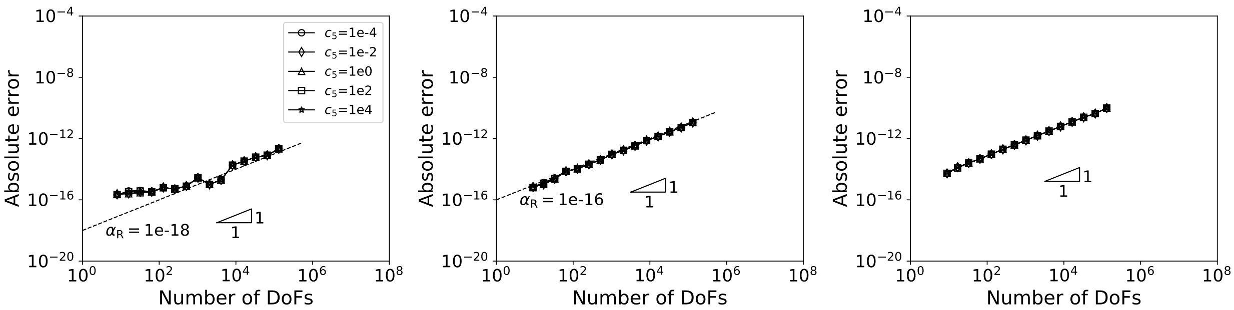
(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 43. Absolute errors for Case 5 using the mixed FEM without scaling the right-hand side.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 44. Absolute errors for Case 5 using scheme M_1.



(a) Solution **(b)** First derivative **(c)** Second derivative

Fig. 45. Absolute errors for Case 5 using scheme M_2.

References

[1] Jie Liu, Matthias Möller, and Henk M Schuttelaars. Balancing truncation and round-off errors in practical fem: one-dimensional analysis. *arXiv preprint arXiv:1912.08004*, 2019.