1．(LU 分解)

LU 分解を用いて次の連立 1 次方程式を解け．(ピボット選択は不要)

$$\begin{cases} 0.5a - b + 2c + d = 3 \\ a + 2b + 2c + d = 7 \\ 2a - 2b + 5c - 0.5d = 6.5 \\ 2a + 4b + 6c + d = 15 \end{cases}$$

1．(LU factorization)

Solve the following system of linear equations using LU factorization (without pivoting).

$$\begin{cases} 0.5a - b + 2c + d = 3 \\ a + 2b + 2c + d = 7 \\ 2a - 2b + 5c - 0.5d = 6.5 \\ 2a + 4b + 6c + d = 15 \end{cases}$$

2．(べき乗法を用いた固有値計算)

べき乗法を用いた行列$\boldsymbol{A}$の固有値計算では適当な初期ベクトル$\boldsymbol{x}^{(0)}$を与え，反復

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{A}\boldsymbol{x}^{(k)}$$

を繰り返すことで，固有値λ_1，固有ベクトル$\boldsymbol{v}_1$を得る．べき乗法に関する以下の問いに答えよ．なお，$\boldsymbol{A} \in \mathbb{R}^{n\times n}$の固有値，固有ベクトルは $\lambda_j, \boldsymbol{v}_j$ $(j = 1,2 \dots n)$，$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|$であると仮定する．

a) レーリー商$R(\boldsymbol{x})$は次の式で示される．なお，$(\boldsymbol{a},\boldsymbol{b})$はベクトル$\boldsymbol{a},\boldsymbol{b}$の内積である．

$$R(\boldsymbol{x}) = \frac{(\boldsymbol{x},\boldsymbol{A}\boldsymbol{x})}{(\boldsymbol{x},\boldsymbol{x})} = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}$$

対称行列$\boldsymbol{A}$の場合，固有値λ_1を計算する方法は2つある．ひとつは$R(\boldsymbol{x}^{(k)})$を用いる方法，もうひとつは各要素の比${x_i}^{(k+1)}/{x_i}^{(k)}$ $(i = 1,2 \dots n)$を用いる方法である．これらの値はいずれもλ_1に収束する．この2つの方法の収束の速さの違いについて議論せよ．

b) 絶対値最小の固有値λ_nを求める方法を示せ．

2．(Power iteration for Eigenvalues calculation)

In the power iteration of matrix $\boldsymbol{A}$, a calculation starts with an initial vector $\boldsymbol{x}^{(0)}$ and repeats an iteration;

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{A}\boldsymbol{x}^{(k)},$$

then obtains the eigenvalue λ_1, and the corresponding eigenvector $\boldsymbol{v}_1$. Let the eigenvalues and eigenvectors of matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ as λ_j and $\boldsymbol{v}_j$ $(j = 1,2 \dots n)$, respectively, and assume $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_{n-1}| > |\lambda_n|$. Answer the following questions related to the power iteration.

a) Rayleigh's quotient $R(\boldsymbol{x})$ is given as the following equation. Note that $(\boldsymbol{a},\boldsymbol{b})$ expresses the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$R(\boldsymbol{x}) = \frac{(\boldsymbol{x},\boldsymbol{A}\boldsymbol{x})}{(\boldsymbol{x},\boldsymbol{x})} = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}},$$

There are two methods to calculate λ_1 for a symmetric matrix $\boldsymbol{A}$. One is to use $R(\boldsymbol{x}^{(k)})$ and the other is to use the ratio of each component ${x_i}^{(k+1)}/{x_i}^{(k)}$, $(i = 1,2 \dots n)$. Both values converge to λ_1 after the iteration. Explain the difference in the convergence rate of these two methods.

b) Explain a method to calculate the eigenvalue λ_n of the least absolute value, based on the power iteration method.

3．(偏微分方程式，解の安定性)

空間x，時間tの関数$u(x,t)$に対する拡散方程式を差分法で解く．

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

格子点(x_j,t_n)は次式で定義し，この格子点での関数値をu_j^nとする．

$$x_j=x_0+j\Delta x,\quad t_n=t_0+n\Delta t$$

この偏微分方程式を近似する差分方程式の一つに Crank-Nicolson の差分方程式がある．

$$u_j^{n+1}=u_j^n+\frac{1}{2}\rho\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)+\frac{1}{2}\rho\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right)$$

なお，$\rho=\Delta t/(\Delta x)^2$である．以下の問いに答えよ．

a) 上記の差分方程式は波数ξの特解

$$u_j^n=g^n e^{-i\xi j\Delta x}$$

を持つ．$i=\sqrt{-1}$. このとき，増幅率gを求めよ．

b) 増幅率gを用いてこの差分方程式の解の安定性を議論せよ．

3．(partial differential equations, stability of numerical solution)

a) The diffusion equation in one space dimension for a scalar function $u(x,t)$, where t is time, x is a spatial variable, is;

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$$

The grid point (x_j,t_n) is defined as following equations and the value of the functions at the point is defined as u_j^n

$$x_j=x_0+j\Delta x,\quad t_n=t_0+n\Delta t$$

One of the difference equations, which approximate the diffusion equation, is the Crank-Nicolson difference equation as follows,

$$u_j^{n+1}=u_j^n+\frac{1}{2}\rho\left(u_{j+1}^n-2u_j^n+u_{j-1}^n\right)+\frac{1}{2}\rho\left(u_{j+1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}\right),$$

with $\rho=\Delta t/(\Delta x)^2$. Answer the following questions..

a) The difference equation has a particular solution with a wave number ξ;

$$u_j^n=g^n e^{-i\xi j\Delta x}$$

where $i=\sqrt{-1}$. Derive the equation of the gain g by substituting it to the difference equation.

b) Using the derived gain g, discuss the stability of numerical solution of the difference equation.