Homework 4: Partial Differential Equation

A. The Advection Equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

$$0 \leq x \leq \infty, 0 \leq t$$

$$u(0,t) = 0$$

$$u(x,0) = \begin{cases} sin8\pi x & (0 \leq x \leq 0.5) \\ 0 & (0.5 < x) \end{cases}$$

As the requirement, the advection equation can be discretized as followed, where the time is approximated as forward differentials and spatial is approximated as center difference.

$$\frac{\partial u}{\partial t} = \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

$$\frac{\partial u}{\partial x} = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad => \quad \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left(u_{j+1}^n - u_{j-1}^n\right) \qquad (\lambda = \frac{\Delta t}{\Delta x})$$

As for the stability and its relationship with λ, we can know by decomposing it in accord with Fourier transformation.

$$u_j^n = g^n e^{ij\xi\Delta x}$$

$$g^{n+1}e^{ij\xi\Delta x} = g^n e^{ij\xi\Delta x} - \frac{\lambda}{2}(g^n e^{i(j+1)\xi\Delta x} - g^n e^{i(j-1)\xi\Delta x})$$

$$g = 1 - \frac{\lambda}{2}(e^{i\xi\Delta x} - e^{-i\xi\Delta x})$$

$$g = 1 - \lambda\cos(\xi\Delta x)$$

Because of $|g| \leq 1 + \mathrm{k}\Delta\mathrm{x}$,

$$|g| = \sqrt{1 + \lambda^2 \sin^2(\xi\Delta x)}$$

$$\max|g| = \sqrt{1 + \lambda^2}$$

As the calculated above, there exists the possibility of instability, because the value of $|g|$ cannot grantee the stability condition. The bigger the value of λ is, the more possible to reach instability, such as 0.2.

B. The Poisson Equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x^2} = 6x - 3y & (0 < x < 1, 0 < y < 1) \\ u(x, 0) = 0 & (0 < x < 1) \\ u(x, 1) = 3x - \dfrac{3}{2}x^2 & (0 < x < 1) \\ u(0, y) = 0 & (0 < y < 1) \\ u(1, y) = 3y^2 - \dfrac{3}{2}y & (0 < x < 1) \end{cases}$$

With the discretization, we can transform it into followed form.

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(\Delta y)^2} = 6 \times i\Delta x - 3 \times j\Delta y$$

As $h = \Delta x = \Delta y = \frac{1}{16}$, we can further simplify the equation above.

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = (6i - 3j) \times h^3$$

Then we can write down the simultaneous equations and it can be solved by using the iteration methods, such as Jacobi method.

By the way, I have answered the class questionnaire.