応用数値計算法

期末試験問題

2014年7月28日実施　担当　土屋

<u>注意</u>：　問題は3問ある．日本語と英語で問題が記載してあるがどちらかを読んで回答すること．両者は同じにしてあるが，異なる場合は選択した言語の問題文に従うこと．解答は日本語でも英語でも構わない．

<u>Note</u>:　There are three questions. Each is written both in Japanese and English. Select one of them to answer. They should be same. Follow the questions in the language that you have selected, if they seem different. You may answer either in Japanese or English.

１．(LU 分解)

LU 分解を用いて次の連立１次方程式を解け．(ピボット選択は不要)

$$\begin{cases} 2a + b - 2c + d + 2e = -38 \\ -4a + 4b + 7c - d - 3e = 55 \\ 6a - 9b - 19c + 4d + e = -91 \\ -8a + 2b + 18c - 8d - 2e = 130 \\ 4a - 10b + 11c - 13d + 23e = -81 \end{cases}$$

１．(LU factorization)

Solve the following system of linear equations using LU factorization (without pivoting)

$$\begin{cases} 2a + b - 2c + d + 2e = -38 \\ -4a + 4b + 7c - d - 3e = 55 \\ 6a - 9b - 19c + 4d + e = -91 \\ -8a + 2b + 18c - 8d - 2e = 130 \\ 4a - 10b + 11c - 13d + 23e = -81 \end{cases}$$

2．(べき乗法を用いた固有値計算)

べき乗法を用いた行列$\boldsymbol{A}$の固有値計算では適当な初期ベクトル$\boldsymbol{x}^{(0)}$を与え，反復

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{A}\boldsymbol{x}^{(k)}$$

を繰り返すことで，固有値λ_1，固有ベクトル$\boldsymbol{v}_1$を得る．べき乗法に関する以下の問いに答えよ．なお，$\boldsymbol{A} \in \mathbb{R}^{n\times n}$の固有値，固有ベクトルを$\lambda_j$，$\boldsymbol{v}_j$ $(j = 1,2 \dots n)$，$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$とする．

a) $k \to \infty$において，$\boldsymbol{x}^{(k)}$が固有ベクトルに収束することを示せ．

b) べき乗法の基本アルゴリズム（最大値ノルムによる規格化）を簡潔に説明せよ．

c) 対称行列$\boldsymbol{A}$についてレーリー商$R(\boldsymbol{x})$を用いたべき乗法を考える．

$$R(\boldsymbol{x}) = \frac{(\boldsymbol{x},\boldsymbol{A}\boldsymbol{x})}{(\boldsymbol{x},\boldsymbol{x})} = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}}$$

ここで，$(\boldsymbol{a},\boldsymbol{b})$はベクトル$\boldsymbol{a},\boldsymbol{b}$の内積である．b)の基本アルゴリズムに対してレーリー商を用いたべき乗法のアルゴリズムの違いを説明し，収束の速さの違いについて議論せよ．

2．(Power iteration for Eigenvalues calculation)

In the power iteration method of matrix $\boldsymbol{A}$, a calculation starts with a vector $\boldsymbol{x}^{(0)}$ and repeats an iteration;

$$\boldsymbol{x}^{(k+1)} = \boldsymbol{A}\boldsymbol{x}^{(k)},$$

then obtains the eigenvalue λ_1, and the corresponding eigenvector $\boldsymbol{v}_1$. Let the eigenvalues and eigenvectors of matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ as λ_j and $\boldsymbol{v}_j$ $(j = 1,2 \dots n)$, $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$, respectively, and answer the following questions.

a) Show the vector $\boldsymbol{x}^{(k)}$ converges to the eigenvalue at $k \to \infty$.

b) Explain the basic algorism of the power iteration method (normalized with the maximum norm) briefly.

c) In case that the matrix $\boldsymbol{A}$ is symmetric, the power iteration method using Rayleigh's quotient $R(\boldsymbol{x})$:

$$R(\boldsymbol{x}) = \frac{(\boldsymbol{x},\boldsymbol{A}\boldsymbol{x})}{(\boldsymbol{x},\boldsymbol{x})} = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}},$$

can be used, where $(\boldsymbol{a},\boldsymbol{b})$ expresses the dot product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Explain the difference in the algorisms of the method using $R(\boldsymbol{x})$ by comparing with the basic method, and discuss the convergence rate of these two methods.

$$\frac{u_j^{(n+1)} - 2u_j^{(n)} + u_j^{(n-1)}}{\Delta x^2} =$$

3．(偏微分方程式，解の安定性)

a) 空間x，時間tの関数$u(x,t)$に対する波動方程式は，cを定数として

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

で示される．これを近似するもっとも一般的な差分方程式を示せ．なお，格子点(x_j, t_n)は

$$\begin{cases} x_j = x_0 + j\Delta x \\ t_n = t_0 + n\Delta t \end{cases}$$

で定義され，この格子点での関数値をu_j^nとする．

$$\frac{\partial u}{\partial x} \quad \frac{u_{j+1}^{(n+1)} - u_j^{(n)}}{\Delta x}$$

b) 上記で求めた差分方程式は波数ξの特解

$$u_j^n = g^n e^{-i\xi j \Delta x}$$

~~u_{j+1}~~

を持つ．$i = \sqrt{-1}$．このとき，増幅率gが満たす関係式を導出せよ．

$$\frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta x} =$$

c) 上記の関係式をgについて解き，差分方程式の解の安定性を議論せよ．

3．(partial differential equations, stability of numerical solution)

a) The wave equation in one space dimension for a scalar function $u(x,t)$, where t is time, x is a special variable, is;

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

c is a constant. Describe a typical differential equation that approximates the wave equation. Note that the grid point (x_j, t_n) is defined as;

$$\begin{cases} x_j = x_0 + j\Delta x \\ t_n = t_0 + n\Delta t \end{cases}$$

The value of the functions at the point is defined as u_j^n.

b) The differential equation has a special solution with a wave number ξ;

$$u_j^n = g^n e^{-i\xi j \Delta x}$$

where $i = \sqrt{-1}$. Derive the equation of the gain g by substituting it to the differential equation.

c) By using the solution of the equation derived in b), discuss the stability of numerical solution of the differential equation.