# Comparison of adaptive and model-free methods for dynamic measurement

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***Abstract*—Dynamic measurement aims to improve the speed and accuracy characteristics of measurement devices by signal processing. State-of-the-art dynamic measurement methods are model-based adaptive methods, *i.e.*, 1) they estimate model parameters in real-time and 2) based on the identified model perform model-based signal processing. The proposed model-free method belongs to the class of the subspace identification methods. It computes directly the quantity of interest without an explicit parameter estimation. This allows efficient computation as well as applicability to general high order multivariable processes.**

***Index Terms*—subspace methods, total least squares, adaptive filtering, model-free signal processing.**

## I. Introduction

Dynamic measurement aims to improve the speed and accuracy characteristics of measurement devices by real-time signal processing. A well known special case of dynamic measurement, used in this paper as an illustrative example, is the dynamic weighing problem [2]. We formalize dynamic measurement as an input estimation problem for a dynamical system with step input and use ideas from subspace identification in order to solve it. In our problem formulation, the step level is the unknown (to-be-measured) quantity, the output is the known (measured) quantity, and the input-output relation represents the dynamics of the measurement process.

Two situations are distinguished: known dynamics and unknown dynamics of the measurement process. The case of known dynamics is often unrealistic. For example, in the dynamic weighing problem, the process dynamics depends on the to-be-measured variable (see Section V-B), so that it is unknown a priori. In this paper, we consider the case of unknown measurement process dynamics.

Approaches based on adaptive filtering, system identification, neural networks, *etc*. are proposed in the literature for solving the dynamic weighing problem. A good introduction to the subject and overview of approaches is given in [7]. The approach based on adaptive filtering offers state-of-the-art methods for dynamic weighing, however, these methods tend to be computationally more expensive than simpler methods based on linear time-invariant compensation. In this paper, we compare the model-free method, presented in Section IV with the adaptive filtering algorithm of [9].

Adaptive filters estimate the measurement process dynamics as well as the quantity of interest. Note that although the measurement process dynamics is not needed, it is estimated by the adaptive filter as a by product. An attractive feature of the model-free approach is that parameter estimation of the measurement process dynamics is not required. Such methods are known as model-free or data-driven [6], [5], [4]. Apart from the above mentioned conceptual difference, an advantage of the model-free algorithm for dynamic measurement over the existing ones is that it is applicable to general high-order multivariable measurement processes.

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## II. Problem formulation

The general problem considered is defined as follows.

**Problem 1.** Given p-dimensional vector output observations

$$y = \big(y(1), \ldots, y(T)\big), \qquad y(t) \in \mathbb{R}^{\mathtt{p}}$$

of a stable linear time-invariant system with known dc-gain $G \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, generated by step input $u = \bar{u}s$, where $s$ is the unit step function $s(t) = 1$ if $t \geq 0$ and $s(t) = 0$ otherwise, find the input step value $\bar{u} \in \mathbb{R}^{\mathtt{m}}$.

In steady-state, the output of the device is equal to

$$y(\infty) = \bar{y} := G\bar{u}.$$

In order to be able to determine uniquely the value of interest $\bar{u}$ from $\bar{y}$, the dc-gain matrix $G$ must be known and must have a left inverse. A necessary and sufficient condition for existence of a left inverse of $G$ is that $G$ has full column rank, *i.e.*, the following condition must be satisfied

$$\operatorname{rank}(G) = \mathtt{m}. \tag{A}$$

Assuming that (A) is satisfied, and choosing the least-squares left inverse $G^{+}_{\mathrm{ls}} = (G^{\top}G)^{-1}G^{\top}$, of $G$, we obtain an unbiased estimator $\widehat{u} := G^{+}_{\mathrm{ls}}y$, which we call the "naive estimator". The naive estimator is a static correction, which uses only knowledge of the dc-gain.

If properly designed, dynamic compensators can significantly outperform the naive estimator. Due to the unknown process dynamics, however, the dynamic compensators are adaptive systems: they perform on-line process identification and model based design. The model-free approach bypasses the parameter identification and compensator design steps and derives directly the quantity of interest—the estimate of the measured variable—by processing $y$.

## III. Notation and preliminaries

The model-free method for dynamic measurement is based on properties of the class of dynamical systems considered: responses of linear time-invariant systems to step inputs. First, we introduce the notation used in the paper and then state the properties used in the derivation of the algorithm.

### A. Notation

A dynamical system is defined by the set $\mathscr{B}$ of its trajectories. The statement "$w$ is a trajectory of the system $\mathscr{B}$" is concisely written as "$w \in \mathscr{B}$". The system under consideration has an input-output partitioning $w = (u, y)$, *i.e.*, the first components of the trajectory are inputs and the remaining ones are outputs. The set of functions from $\mathbb{R}$ to $\mathbb{R}^q$ is denoted by $(\mathbb{R}^q)^{\mathbb{R}}$. The restriction of $w \in (\mathbb{R}^q)^{\mathbb{R}}$ on the set $\mathscr{T} \subset \mathbb{R}$ is denoted by $w|_{\mathscr{T}}$ and, similarly, the restriction of $\mathscr{B} \subset (\mathbb{R}^q)^{\mathbb{R}}$ on $\mathscr{T}$ is denoted by $\mathscr{B}|_{\mathscr{T}}$.

Linear time-invariant systems $\mathscr{B}$ admit an input-state-output representation

$$\begin{aligned}\mathscr{B} = \mathscr{B}_{\mathrm{ss}}(A,B,C,D) := \{ w = (u,y) \mid \text{there is } x, \text{ such that}\\ \frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu,\ y = Cx + Du \quad \text{in continuous-time or}\\ \sigma x = Ax + Bu,\ y = Cx + Du \quad \text{in discrete-time}\},\end{aligned}$$

where

$$(\sigma^\tau y)(t) := y(t+\tau), \qquad \text{for all } t.$$

A state space representation of an autonomous linear time-invariant system $\mathscr{B}$ is denoted by $\mathscr{B}_{\mathrm{ss}}(A,C)$.

### B. Behavior of a system with step input

Let the unknown measurement process dynamics $\mathscr{B}$ be a linear time-invariant system of order $\mathtt{n}$ with $\mathtt{m}$ inputs and $\mathtt{p}$ outputs. We define the behavior $\mathscr{B}_{\mathrm{s}}$ of the outputs of $\mathscr{B}$, generated by step inputs, *i.e.*,

$$\mathscr{B}_{\mathrm{s}} := \{ y \mid (\bar{u}s, y) \in \mathscr{B}, \text{ for some } \bar{u} \in \mathbb{R}^{\mathtt{m}} \}.$$

It can be shown that $\mathscr{B}_{\mathrm{s}}$ is a linear time-invariant autonomous system of order $\mathtt{n}+\mathtt{m}$. Moreover, $\mathtt{m}$ poles of $\mathscr{B}_{\mathrm{s}}$ are at zero, in the continuous-time case, or at one, in the discrete-time case, *i.e.*,

$$\mathscr{B}_{\mathrm{s}} = \mathscr{B}'_{\mathrm{s}} + \bar{y}s,$$

for a vector $\bar{y} \in \mathbb{R}^{\mathtt{p}}$ and a system $\mathscr{B}'_{\mathrm{s}}$, which is autonomous linear time-invariant of order $\mathtt{n}$.

Define the unit difference operator

$$\Delta := (1 - \sigma^{-1}).$$

The behavior $\Delta\mathscr{B}_{\mathrm{s}}$ is linear time-invariant autonomous of order $\mathtt{n}$ and

$$\mathscr{B}_{\mathrm{s}} = \Delta\mathscr{B}_{\mathrm{s}} + \bar{y}s, \qquad \text{for some } \bar{y} \in \mathbb{R}^{\mathtt{p}}.$$

Therefore, for any $y \in \mathscr{B}_{\mathrm{s}}$ that corresponds to a step input $\bar{u}s$, we have

$$y = \bar{y}s + \Delta y, \quad \text{where } \bar{y} = G\bar{u} \text{ and } \Delta y \in \Delta\mathscr{B}_{\mathrm{s}}. \tag{1}$$

## IV. Model-free algorithm

Assume that an upper bound $\mathtt{n}_{\max}$ for the order $\mathtt{n}$ of $\Delta\mathscr{B}_{\mathrm{s}}$ is known and consider the block-Hankel matrix

$$\mathscr{H}(\Delta y) := \begin{bmatrix} \Delta y(1) & \Delta y(2) & \cdots & \Delta y(\mathtt{n}_{\max}) \\ \Delta y(2) & \Delta y(3) & \cdots & \Delta y(\mathtt{n}_{\max}+1) \\ \Delta y(3) & \iddots & & \vdots \\ \vdots & & & \vdots \\ \Delta y(T-\mathtt{n}_{\max}) & \cdots & \cdots & \Delta y(T) \end{bmatrix},$$

with $\mathtt{n}_{\max}$ columns, constructed from $\Delta y$. Since the observed output $y$ is exact and the order of the system $\Delta\mathscr{B}$ is $\mathtt{n}$, we have that image$\big(\mathscr{H}(\Delta y)\big) \subseteq \Delta\mathscr{B}|_{[1,\ldots,T-\mathtt{n}_{\max}]}$ and

$$\operatorname{rank}\big(\mathscr{H}(\Delta y)\big) \leq \mathtt{n}. \tag{2}$$

Assuming that equality holds in (2) (persistency of excitation of $\Delta y$, see [10]), we have that

$$\operatorname{image}\big(\mathscr{H}(\Delta y)\big) = \Delta\mathscr{B}|_{[1,\ldots,T-\mathtt{n}_{\max}]}.$$

Then, using (1), we obtain the system of linear equations

$$\begin{bmatrix} \begin{bmatrix} G \\ \vdots \\ G \end{bmatrix} & \mathscr{H}(\Delta y) \end{bmatrix} \begin{bmatrix} \bar{u} \\ \ell \end{bmatrix} = \begin{bmatrix} y(\mathtt{n}_{\max}+1) \\ \vdots \\ y(T) \end{bmatrix}. \tag{3}$$

$\ell := \lceil \mathtt{n}/\mathtt{p} \rceil$, where $\lceil \cdot \rceil$ denotes rounding to the nearest bigger integer. The matrix in the left-hand-side and the right-hand-side of (3) depend on the given output data $y$ and the dc-gain $G$ only. Therefore, the quantity of interest $\bar{u}$ can be computed directly from the data by solving (3). The resulting model-free method is summarized in Algorithm 1.

**Algorithm 1** Model-free method for dynamic measurement.

**Input:** $y \in (\mathbb{R}^{\mathtt{p}})^T$, $G \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, and natural number $\mathtt{n}_{\max}$.

1: Compute the difference signal $\Delta y$.
2: Solve the system of linear equations (3).

**Output:** $\bar{u}$.

Under the generic assumption (A), given

$$T \geq \mathtt{n}_{\max} + \mathtt{m} \tag{4}$$

exact output samples, Algorithm 1 computes the exact input step value $\bar{u}$. (Condition (4) ensures that system (3) has at least as many unknowns as equations.)

In the case of noisy data, the system (3) generically has no solution. A heuristic modification of Algorithm 1 for the noisy case is to replace the exact solution on step 2 with the least squares approximation. The resulting estimation method is unbiased, however, it is not statistically efficient. An efficient maximum likelihood method for known process dynamics is the Kalman filter. To the best of our knowledge, there are no maximum likelihood methods for dynamic measurement proposed in the literature for the case of unknown process dynamics.

A recursive version of Algorithm 1 is obtained by using a recursive method for solution of the system of linear equation (3). Details for an on-line implementation, using the recursive least squares algorithm, are given in the appendix. The computational cost for updating the estimate with a new measurement is $O\big((\mathtt{m}+n)^2\mathtt{p}\big)$, which is similar to the one of dynamic measurement with a fixed compensator of order $\mathtt{n}$.

In the next section, the on-line model-free estimation method is tested empirically. Matlab software reproducing the reported results is available from

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http://homepages.vub.ac.be/~imarkovs/
publications/sensor-ieee-code.tar
```

The results show that the model-free algorithm is effective in solving dynamic weighing problems.

## V. Examples

### A. Simulation setup

In the simulations, we use the output error model

$$y = y_0 + \widetilde{y}, \tag{5}$$

where the exact output $y_0 = \big(y_0(1), \ldots, y_0(T)\big)$ is a uniformly sampled output trajectory of a continuous-time system $\mathscr{B} = \mathscr{B}_{\mathrm{ss}}(A,B,C,D)$, obtained with step input $\bar{u}_0 s$ and initial condition $x_{\mathrm{ini}}$. The measurement noise $\widetilde{y}$ is white Gaussian process and the sampling rate is chosen, so that sufficiently many output samples are collected during the observed transient process.

The adaptive and model-free methods are applied on the data $y$ and the corresponding estimates are plotted. In addition, we show the result of the naive estimator $\widehat{u} := G_{\mathrm{ls}}^{+} y$. The convention used in the figures to denote the different estimates is as follows:

- naive estimator — dotted line,
- recursive model-free algorithm — dashed line,
- adaptive filter of [9] — dashed-dotted line.

The true parameter value $\bar{u}$ is denoted by dashed line and the true output trajectory $y_0$ by a solid line.

In all experiments, the results for a specific noise realization are presented along with the average estimation errors

$$e = \frac{1}{N} \sum_{i=1}^{N} \| \bar{u} - \widehat{u}^{(i)} \|_1,$$

where $\| \cdot \|_1$ is the 1-norm and $\widehat{u}^{(i)}(t)$ is the estimate of $\bar{u}$, using the data up to time $t$ in an $i$th Monte Carlo repetition.

### B. Dynamic weighing

We use the dynamic weighing problem for the empirical comparison of the methods, because it is an important and well studied special case of the dynamic measurement problem. In addition, the adaptive method of [9] is designed and optimized for this problem. Application of adaptive filtering methods to other dynamic measurement problems would require a nontrivial modification of the methods and is outside the scope of this paper.

The setup of the problem is as follows: an object with mass $M$ is placed on a platform with mass $m$. The platform is modeled as a mass, spring, damper system and its vertical deviation from the equilibrium point is the measured signal $y$. The dynamical response of the weighing system starts from a certain (in general nonzero) initial condition. Note that by placing the object on the platform results in a step change of the total mass parameter, as well as a step input to the system.

The measurement process dynamics is governed by the differential equation

$$(M+m)\frac{\mathrm{d}^2}{\mathrm{d}t^2} y = -ky - d\frac{\mathrm{d}}{\mathrm{d}t} y - Mg,$$

where $g$ is the gravitational constant, $k$ is the elasticity, and $d$ is the damping. Choosing the state vector as position and velocity and taking as an input $u_0 = Ms$, we obtain a state space model $\mathscr{B}(A,b,c,0)$ of the weighting process with the following parameters

$$A = \begin{bmatrix} 0 & 1 \\ \frac{k}{M+m} & \frac{d}{M+m} \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ -\frac{g}{M+m} \end{bmatrix}, \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The performance of the recursive model-free algorithm is compared on test examples with the performance of the adaptive filtering method of [9]. Following the empirical observations in Section III of [9] about the algorithm's speed of convergence, we use the "simple data vector (11–2)" and pre-filter the data by a low-pass filter. Apart from the low-pass filter the adaptive algorithm requires specification of the initial conditions—the past values of $y$, the initial parameter vector $\theta(0)$, and the initial covariance matrix $P(0)$. These parameters are not specified in [9] and their choice affects the performance of the method. In the simulation examples, reported in this paper, we set $y(-1) = y(0) = 0$ and assign $\theta(0)$ and $P(0)$ to the batch solution of the estimation problem for the first three data points $y(1), y(2), y(3)$. The same initialization is used for the recursive model-free algorithm, see (6).

Numerical results with object's masses 1, 10, and 100 are shown in Figures 1–3, respectively. They indicate that, on the average, the model-free method achieves better performance. Note, however, that neither the model-free nor the adaptive method are statistically optimal. Therefore, the question of computation of the limit of performance as well as design of improved computational methods remains open. For on-line implementation the computational cost (efficiency) is an important limiting factor that should be taken into account. In this respect, the proposed model-free method achieves good statistical accuracy vs computational efficiently trade-off.

## VI. Conclusions and future works

A new method for speeding up measurement processes with slow dynamics was developed. Knowledge of the process dynamics is not required and the process dynamics is not explicitly identified. The method is applicable to general high-order multivariable linear time-invariant processes. Recursive implementation of the method, based on ordinary least squares approximation, has computational complexity per time step, comparable with the one of linear time-invariant compensation with precomputed filter gain and performance that is comparable to the one of the adaptive filtering methods. The recursive least squares method, however, is suboptimal in the output error setup. Future work aims at modification of the method to make it optimal (maximum-likelihood) in the output error case and generalization of the algorithm to estimation under different noise assumptions.

## VII. Acknowledgments

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013) / ERC Grant agreement number 258581 "Structured low-rank approximation: Theory, algorithms, and applications".

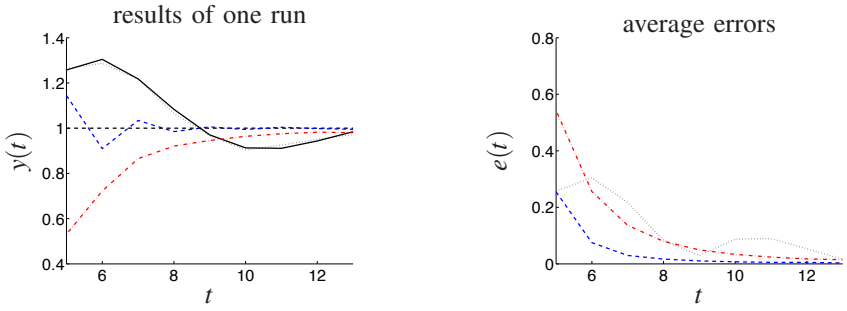


Fig. 1. Simulation 1: $M = 1$

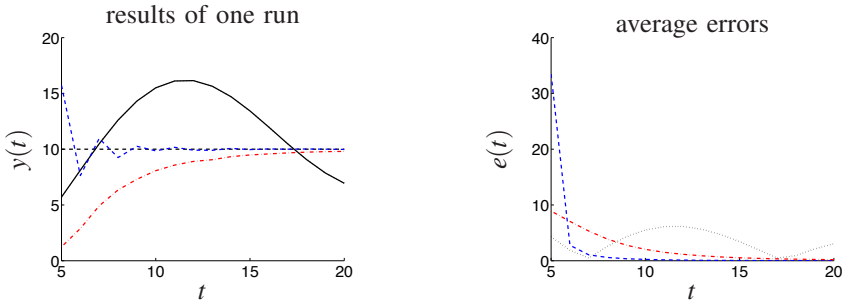


Fig. 2. Simulation 2: $M = 10$

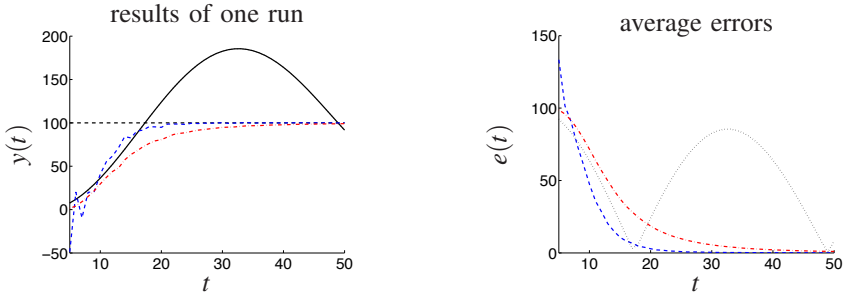


Fig. 3. Simulation 3: $M = 100$

## APPENDIX

The implementation of Algorithm 1 requires only solution of the system of linear equations (3). In the case of noisy data, the ordinary least squares approximation is used instead. Each data point $y(t)$ corresponds to p equations. The recursive least squares algorithm, updates the solution available on the current iteration by a new data point solution. In order to simplify the notation, let $Ax \approx b$, where $A \in \mathbb{R}^{m\times n}$, be the system (3). Let $a(i) = a_{i,:}$ be the $i$th row of $A$, $A_{1:i,:}$ the submatrix of $A$ formed by the first $i$ rows, $b(i) := b_i$, and $b_{1:i}$ the subvector of $b$ formed by the first $i$ elements. An algorithm (see, [3, Lemma 2.6.1 and Problem 2.6]) for computing the (exponentially weighted) least squares approximate solutions $x(n),\ldots,x(m)$ of the sequence of problems $A_{1:i,:}x(i) \approx b_{1:i}$, for $i = n,\ldots,m$ is

$$
\begin{aligned}
K &:= \frac{1}{\lambda}Pa^\top\left(1+\frac{1}{\lambda}aPa^\top\right)^{-1}\\
\sigma x &= x + K(b-ax), \qquad x(0) = x_{\text{ini}}\\
\sigma P &= \frac{1}{\lambda}\big(P - KaP\big), \qquad P(0) = P_{\text{ini}}.
\end{aligned}
$$

The forgetting factor $\lambda \in (0,1]$ specifies an exponential weighting $\lambda^i r_i$ of the residual $r := Ax - b$ in the least squares approximation criterion. The algorithm is initialized with the solution of the system formed by the first $n$ equations

$$
x_{\text{ini}} := A^{-1}_{1:n,:}b_{1:n}, \qquad P_{\text{ini}} := (A^\top_{1:n,:}A_{1:n,:})^{-1}. \tag{6}
$$

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