Differential Equation Notes
Spring 2016
Professor Chris McCarthy

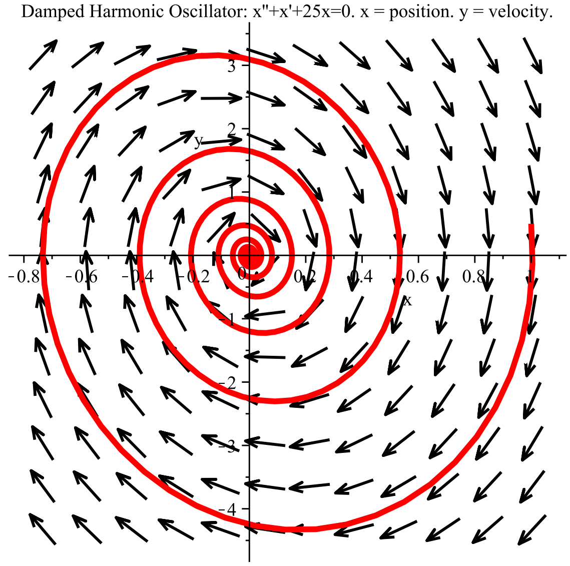

Figure 1: Phase portrait, with direction field, for the damped harmonic oscillator: $\ddot{x} + \dot{x} + 25x = 0$.

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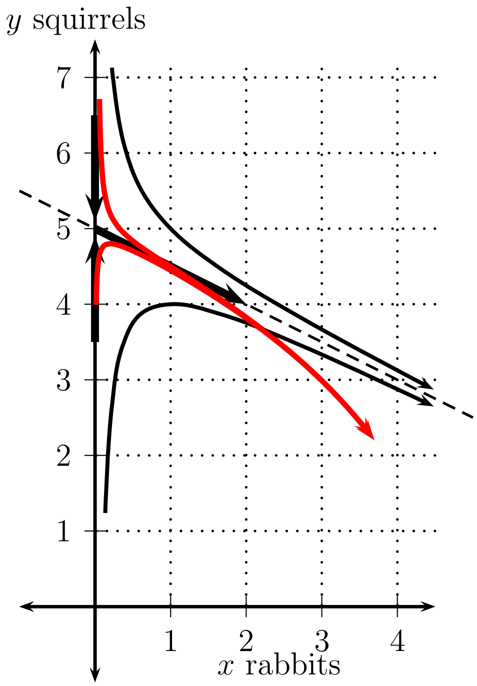


Differential equations and ecological models. The system of non linear differential equations:

$$\begin{aligned} \dot{x} &= 10\,x - xy - 2x^2 \qquad (x = \text{ rabbits}) \\ \dot{y} &= 5\,y - xy - y^2 \qquad (y = \text{ squirrels }) \end{aligned}$$

is an example of a model of a competitive ecological relationship, for example, between rabbits and squirrels, which eat some of the same foods. We can approximate the solution in the vicinity of the equilibrium point $(0, 5)$ by calculating the Jacobian matrix of the system at $(0, 5)$, solving the resulting linear system using the eigenvector method, and then translating the solution by $(0, 5)$. This calculation results in the approximate solution

$$\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 0 \\ 5 \end{pmatrix} + c_1 e^{-5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

valid in the vicinity of $(0, 5)$. The two black trajectories are representative of the approximate solutions generated by this linearizing technique, the two red trajectories are numerically generated from the original system of differential equations using an adaptive Runge-Kutta algorithm module built into Maple.

For more information on modeling ecological relationships using differential equation see out textbook.

Homework Fall 2016

Newest Homework appears at the bottom of the list.

Homework: Mixing Problem and Separable ODE's
To be handed in: the problem on the bottom of page 14 of these notes.

Homework: Mixing Problem and the Integrating Factor Method
To be handed in: the problem on the bottom of page 21 of these notes.

Homework: Differential Equations and Linear Differential Operators
Read Handout 5 (pages 22 to 31).
Do, on page 32, homework 10.
Note all the questions and solutions to homework 10 are on page 32 of these notes.

Homework: Euler Method (Numerical methods).
Read Handouts 11, 12, 13 (pages 51 to 67).
Hand in homework is page 67.

Homework: Laplace Transform Method I
Read Handouts 15, 16, 17 (pages 73 to 88).
Do, on page 74, homework 20A. Solutions to HW20A start on page 75.

Professor Chris McCarthy Mat 501 Notes Handout 1 (5^{th} Ed.)

Differential Equations is a huge subject. To help you focus on the most important topics (w.r.t. this course), I have written the following notes. When feasible, I cite relevant exercises and pages from our our text book, *Differential Equations and Boundary Value Problems: Computing and Modeling, 4^{th} Ed.*, by Edwards & Penney.

Chapter 1 - First Order ODE's

Section 1.1, page 6 of Edwards & Penney, 4^{th} Ed.

Notation: Suppose we have a function, y, of one independent variable, x. We will denote the first derivative of y w.r.t. x, $\frac{dy}{dx}$ as y'; the second derivative, $\frac{d^2y}{dx^2}$, as y'', and the higher order derivatives, $\frac{d^ny}{dx^n}$, as $y^{(n)}$. In this context, the parenthesis indicate differentiation, so $y^{(3)} = \frac{d^3y}{dx^3}$, but $y^3 = yyy$ (multiplication).

Order of the derivative. We say that the order of y' is one; the order of y'' is two; and the order of $y^{(n)}$ is n.

Order of the ODE: The order of a differential equation is the order of the highest derivative appearing in that differential equation. For example the differential equation $(y'')^5 = \sqrt{y^{(3)}} + 7.8xy - x^9$ has order 3 because the highest order derivative is $y^{(3)}$ which is third order. The two ODE's $y' = y$ and $(y')^5 - y = 0$ are both considered to be first order ODE's since the highest order derivative appearing is y', which is first order.

General Form of an n^{th} order ODE is: $F(x, y, y', y'', \ldots, y^{(n)}) = 0$. E.g., we can write the ODE $(y'')^5 = \sqrt{y^{(3)}} + 7.8xy - x^9$ in the form $F(x, y, y', y'', y^{(3)}) = 0$ by just by moving everything onto one side, e.g. $(y'')^5 - \sqrt{y^{(3)}} - 7.8xy + x^9 = 0$.

A Solution to an n^{th} order ODE. Given the n^{th} order ODE, $F(x, y, y', y'', \ldots, y^{(n)}) = 0$, a solution on the interval $I = (a, b) \subset \mathbb{R}$ is a function $u(x)$ which satisfies $F(x, u, u', u'', \ldots, u^{(n)}) = 0$ on the interval I. That said, usually when we solve an ODE, and we write it's solution, we will use the same letter as appears in the ODE itself, e.g. y, rather than u.

Section 1.2, page 10 of Edwards & Penney, 4^{th} Ed.

Methods for Solving ODE's Recall from Calculus that integration is more problematic than differentiation; and that your teacher taught you various methods (or tricks) to help you to find anti-derivatives, and perhaps some numerical techniques like Simpson's Rule, to allow efficient numerical approximations of definite integrals. Solving ODE's is very similar to integration, and a large part of this class will consist of learning these various methods.

The first method is for solving the simple first order ODE $y' - f(x) = 0$, or equivalently $y' = f(x)$. We will need:

Fundamental Theorem of Calculus. This theorem is the most important theorem in Calculus and by extension, it is central to the subject of ODE's. The Fundamental Theorem of Calculus (FTC) has two parts. I expect you to understand this Theorem and to be able to sketch its proof.

Fundamental Theorem of Calculus Part I.

$$\int_a^b \frac{d}{dx} f(x)\ dx = f(b) - f(a). \tag{1}$$

Proof (Sketch):

$$\int_a^b \frac{d}{dx} f(x) dx \approx \sum \frac{\Delta f_k}{\Delta x_k} \Delta x_k = \sum \Delta f_k = \Delta f = f(b) - f(a)$$

. Note that Δ means change or difference.

Fundamental Theorem of Calculus Part II.

$$\frac{d}{dx} \int_a^x f(w)\ dw = f(x). \tag{2}$$

Proof (Sketch):

$$\frac{d}{dx} \int_a^x f(w)\ dw = \lim_{h \to 0} \frac{\int_a^{x+h} f(w)\ dw\ -\ \int_a^x f(w)\ dw}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(w)\ dw}{h} \approx \lim_{h \to 0} \frac{f(x)h}{h} = f(x).$$

Using the FTC to solve the simple ODE $y' = f(x)$. By the FTC (see Formula (2)),

$$y(x) = \int_a^x f(w)\ dw$$

is a solution of $y' = f(x)$, Notice that, for each different value of a, the lower bound of the integral, we get a slightly function $y(x)$. However, no matter what $f(x)$ is

$$\int_a^a f(w)\ dw = 0.$$

So, if we have the ODE $y' = f(x)$ with initial value $y(a)$, meaning we know that value of $y(x)$ when x is equal to a, then we must have

$$y(x) = \int_a^x f(w)\ dw + y(a) \tag{3}$$

since $d/dx\ y(a) = 0$ and $\int_a^a f(w)\ dw = 0$.

However, in practice, everyone, including our text book, tends to say, to solve

$$y' = f(x)$$

we integrate both sides, i.e.

$$\int y'\ dx = \int f(x)\ dx$$

to get,

$$\underbrace{y(x) - y(a) = \int_a^x \frac{d}{dw}\ y(w)\ dw}_{\text{True by the FTC(see Formula (1))}} = \int_a^x f(w)\ dw,$$

Note that to apply the FTC we placed x and a in the limits of integration, a being an arbitrary initial condition; we also used a dummy variable, w in the integrand, since we shouldn't use x for both the limits of integration and in the integrand itself. So we get the solution

$$y(x) = \int_a^x f(w)\ dw + y(a)$$

just as before (compare with Formula (3)).

To summarize this method: To solve the ODE:

$$y' = f(x)$$

we integrate both sides and get,

$$y = \int f(x)\ dx + C$$

where C is a constant to be determined from the initial conditions. The solution $y = \int f(x)\ dx + C$ is called a **general solution**; and then when we find C, using initial conditions, we have what is called a **particular solution**.

Section 1.2, page 11 of Edwards & Penney, 4^{th} Ed.

Example 1, Page 11 Solve the initial value problem (IVP): $y' = 2x + 3$ with $y(1) = 2$.

Solution. Integrating both sides yields the general solution $y(x) = x^2 + 3x + C$. We find C by using the initial condition $y(1) = 2$, i.e. we plug $x = 1$ into $y(x) = x^2 + 3x + C$ and this must equal 2, since $y(1) = 2$. This allows us to solve for C. We get $y(1) = (1)^2 + 3(1) + C = 4 + C = 2$ implies $C = -2$, yielding our IVP's particular solution of $y(x) = x^2 + 3x - 2$.

Note that we could also have solved this by using the formula (3):

$$y(x) = \int_a^x f(w)\ dw + y(a)$$

with initial condition $y(a) = y(1) = 2$, so that the particular solution is

$$\begin{aligned} y(x) &= \int_a^x f(w)\ dw + y(a) \\ &= \int_1^x 2w + 3\ dw + y(1) \\ &= w^2 + 3w\Big|_1^x + 2 \\ &= \big([x^2 + 3x] - [(1)^2 + 3(1)]\big) + 2 \\ &= \big(x^2 + 3x - 4\big) + 2 \\ &= x^2 + 3x - 2. \end{aligned}$$

So we get the same particular solution as before. Of course integrating both sides and using C is faster and better, provided you have an easy to find anti-derivative. But if you can't find the anti-derivative, the more lengthy method (i.e. including the limits of integration) is better since it points us in the direction of a numerical solution. We will touch on numerical methods later in this course.

Note: An ODE may have many different solutions, or no solution at all. Note that the ODE $y' = 2x + 3$ had many different solutions, one for each value of C, i.e. one for each initial value. On the other hand, the ODE, $(y')^2 = -1$ has no solutions since $(y')^2$ can't be negative.

First Order Separable Differential Equations (Section 1.4 in Edwards & Penny) A first order separable [1] differential equation, is one which can be written in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

where y is of course a function of x, i.e. $y = y(x)$. The trick to solve separable DE's makes use of the Change of Variables Theorem (CVT, which is the basis for the u-substitution method for finding an anti-derivative:
Change of Variable Theorem (CVT) Let $y : [a, b] \to \mathbb{R}$ be differentiable, then

$$\int_{y(a)}^{y(b)} g(y)\ dy = \int_a^b g(y(x))\frac{dy}{dx}\ dx$$

Proof (First an important remark about the limits of integration): Generally speaking, we can interpret the limits of integration, a, b, in $\int_a^b f(x)\ dx$, as telling us how to orientate our integration: if $a < b$, then we start with $a = x_0$ and work our way up (in the positive direction) to reach b, so each $\Delta x_k = x_{k+1} - x_k$ will be positive. If $b < a$ then we will still start with $a = x_0$, but now, to get to b, we must work our way down (in the negative direction) so each $\Delta x_k = x_{k+1} - x_k$ will be negative. The result of this interpretation of the limits of integration a, b immediately gives us $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$.

Proof of CVT (Sketch): Let $a < b$ and let $a = x_0 < x_1 < \ldots < x_k \ldots < x_N = b$ be an "increasing" partition of the interval $[a, b]$. **Special Case of the CVT:** Suppose y is strictly increasing (or strictly decreasing) on $[a, b]$. Then $y(a) = y(x_0) = y_0, y(x_1), \ldots, y(x_k) = y_k \ldots, y_N = y(x_N) = y(b)$ will be an "ordered

[1] An equivalent formulation of being separable is to require that we can write $y' = f(x)g(y)$. The essential thing is being able to algebraically separate the x and the y terms.

partition" of the interval $[y(a), y(b)]$ (if y is increasing), or of the interval $[y(b), y(a)]$ (if y is decreasing). In both cases we have

$$\begin{aligned}
\int_{y(a)}^{y(b)} g(y)\ dy &\approx \sum_k g(y_k)\Delta y_k \\
&= \sum_k g(y_k)(y_{k+1} - y_k) = \sum_k g(y(x_k))(y(x_{k+1}) - y(x_k)) \\
&= \sum_k g(y(x_k)) \frac{y(x_{k+1}) - y(x_k)}{x_{k+1} - x_k}\ (x_{k+1} - x_k) \\
&\approx \sum_k g(y(x_k))\ \frac{dy}{dx}\bigg|_{x_k}\ \Delta x_k \approx \int_a^b g(y(x)) \frac{dy}{dx}\ dx.
\end{aligned}$$

To complete the proof, to prove the general case where y is not strictly increasing or decreasing, we note that we can partition the interval $[a, b]$ into subintervals such that on each subinterval y will be strictly increasing or strictly decreasing. So on each of these subintervals, the CVT is true (by the special case we proved above). To complete the proof we just note that we can glue these subintervals together by repeated usage of the following: if $\alpha < \beta < \gamma$ then $\int_\alpha^\gamma + \int_\gamma^\beta = \int_\alpha^\beta$. So the CVT is proven.

Using the CVT to solve separable first order ODE's. Suppose $g(y) \neq 0$ and we have:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \text{with IC} \quad y(a) = y_a. \tag{4}$$

Let's assume a solution $y(x)$ to the DE (4) exists on some interval $[a, b]$. Let's see if we can understand what that solution $y(x)$ must be like. A little algebra applied to (4) yields

$$g(y(x))\ \frac{dy}{dx} = f(x)$$

and so

$$\int_a^b g(y(x))\ \frac{dy}{dx}\ dx = \int_a^b f(x)\ dx. \tag{5}$$

Now apply the CVT to the left hand side (LHS) of (5) to get:

$$\int_{y(a)}^{y(b)} g(y)\ dy = \int_a^b f(x)\ dx \tag{6}$$

Note that in practice, we tend to write (6), as

$$\int g(y)\ dy = \int f(x)\ dx + C \tag{7}$$

and then solve for C. Note that if $\int g(y)\ dy$ has a complicated anti-derivative that this method will only define $y(x)$ implicitly. In the following example $\int g(y)\ dy$ is not too complicated, but even still, y is (at first) only defined implicitly, in terms of $|y|$, and we need to do some work to actually extract y by itself.

Example Solve the

$$\text{DE} \quad y' = yx \qquad \text{with Initial Condition (IC)} \quad y(0) = -1.$$

Solution This ODE is separable. We write

$$\frac{dy}{dx} = yx.$$

Then we separate the y terms and the x terms by moving the y and the dy onto one side of the equation, and the x and the dx onto the other side. We get

$$\frac{dy}{y} = x\ dx.$$

Integration yields

$$\int \frac{dy}{y} = \int x\ dx + C$$

and so

$$\ln|y| = \frac{x^2}{2} + C$$

Exponentiation of both sides yields

$$|y| = e^{\ln|y|} = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}}e^C = A_* e^{\frac{x^2}{2}}$$

where we are letting the constant $e^C = A_*$. We must have $A_* > 0$ since $e^C > 0$. From the IC we have $y(0) = -1$, which implies $|y(0)| = 1$; this and plugging $x = 0$ into

$$|y(x)| = A_* e^{\frac{x^2}{2}},$$

yields

$$1 = |y(0)| = A_* e^0 = A_*$$

so

$$|y(x)| = e^{\frac{x^2}{2}}, \qquad \text{which implies} \qquad y(x) = \pm e^{\frac{x^2}{2}}.$$

We can deal with $\pm$ as follows. We know that our solution $y(x)$ must be differentiable, since it is the solution of an ODE. Since differentiable implies continuous, $y(x)$ is continuous. Since $y(x)$ is continuous if $y(x)$ changes sign, going from negative to positive, or positive to negative, then $y(x)$ must pass through zero. However, $|y(x)| = e^{\frac{x^2}{2}}$ is never zero (it's always positive), so $|y(x)|$ is never zero, but that obviously implies that $y(x)$ is never zero as well. So $y(x)$ must always be positive, or always be negative. Since $y(0) = -1$ is negative, the particular solution $y(x)$ to this IVP must always be negative. So the particular solution must be:

$$y = -e^{\frac{x^2}{2}}.$$

Once you have seen this argument, you know what to expect, and similar problems can be solved rapidly.

Example. Solve $\dfrac{dy}{dx} = 2y$.

Solution. Separate the DE, get: $\dfrac{dy}{y} = 2dx$. Integrate both sides: $\displaystyle\int \frac{dy}{y} = \int 2\,dx$. Get $\ln|y| = 2x + c_1$. Eliminate the ln by apply e to both sides, get: $e^{\ln|y|} = e^{2x+c_1}$, which simplifies to: $|y| = e^{2x}\,e^{c_1}$. Since $y = y(x)$ is a differentiable function of x, y is continuous. So, y can only change sign if $y(x) = 0$ for some value of x. However, $|y(x)| = e^{2x}\,e^{c_1}$ is strictly positive, so $y(x)$ is never zero, so $y(x)$ cannot change sign. So $y(x)$ is either positive for all x or it is negative for all x. If $y(x)$ is positive for all x, then $y(x) = |y(x)|$, and so the solution must be $y(x) = e^{2x}\,e^{c_1}$. On the other hand, if $y(x)$ is negative for all x, then $y(x) = -|y(x)|$, and so the solution must be $y(x) = -e^{2x}\,e^{c_1}$. So we can write the solution compactly as $y(x) = Ce^{2x}$, where $C = e^{c_1}$ if $y(x)$ is positive for all x and where $C = -e^{c_1}$ if $y(x)$ is negative for all x. Here is the final answer:

$$y = Ce^{2x}.$$

Death, Immigration, Difference and Differential Equations. Mixing problems.

This worksheet is based upon "M&M - DEATH AND IMMIGRATION", which can be found online at `www.simiode.org`. Simiode.org is a great resource for free and innovative hands-on projects involving differential equations.

Physical scenario: We have some coins in a cup. We toss the coins onto our desk. The coins that land heads up, we count and put back into the cup. The coins that land tails up, we discard. We repeat this process, over and over.

Terminology: when we repeat a process, it is called iterating the process. Each repeat of the process is called an iterate.

For concreteness, let's suppose we start with 50 coins.

There are many different ways to mathematically model this process.

Mathematical model 1: deterministic difference equation: Let $n = 0, 1, 2, 3, \ldots$ refer to the number of times this processes has been iterated. Let $b(n) =$ how many coins we have in the cup at the start of the n^{th} iteration. So $b(0) = 50$. If we toss a coin once the probability of heads $= P(H) = 1/2$. So with each iteration we expect to keep 1/2 the coins. So we expect $b(n+1) = \frac{1}{2}b(n)$ or, equivalently $b(n+1) - b(n) = -\frac{1}{2}b(n)$. The $b(n+1) - b(n)$ term is a difference, so the equation $b(n+1) - b(n) = -\frac{1}{2}b(n)$ is called a difference equation.

Of course if you actually do this experiment, the observed $b(n)$ will typically not equal the $b(n)$ predicted by the model. After all, the model predicts: $b(0) = 50,\ b(1) = 25,\ b(2) = 12.5,\ b(3) = 6.25, \ldots,\ b(n) = \frac{b(0)}{2^n}$.

Mathematical model 2: deterministic differential equation: Let t refer to time, perhaps measured in minutes, and let b(t) be the number of coins surviving at time t.

We can imagine that instead of tossing all the coins at once, that in Δt time, with $0 < \Delta t < 1$ we apply the process to Δt of the coins. For example, if $\Delta t = \frac{1}{10}$, we'd take one tenth of the coins out of the cup, i.e., $\Delta t\ b(t)$, toss those coins on the table, remove the ones that came up tails (probably about 1/2 of them), and put the ones that came up heads back into the cup. The difference equation, becomes:

$$b(t + \Delta t) - b(t) = -\frac{1}{2}\ \Delta t\ b(t).$$

Why? Because in each Δt we flip $\Delta t\ b(t)$ coins; the probability of tails is 1/2 so we expect to remove 1/2 the coins we flip, hence we expect to remove $\frac{1}{2}\ \Delta t\ b(t)$ coins in Δt time. Since we are removing coins, the difference $b(n + \Delta t) - b(t)$ is negative. Hence the negative sign in the difference equation.

We can turn the difference equation into a differential equation by dividing both sides of the difference equation by Δt to get:

$$\frac{b(t + \Delta t) - b(t)}{\Delta t} = -\frac{1}{2}b(t),$$

and then letting $\Delta t \to 0$:

$$\lim_{\Delta t \to 0} \frac{b(t + \Delta t) - b(t)}{\Delta t} = \frac{db}{dt} = -\frac{1}{2}b(t).$$

Of course when Δt is very small, we'd be "tossing" less than one coin, which doesn't make sense. But don't worry about that for now.

The DE $\dfrac{db}{dt} = -\dfrac{1}{2}b(t)$ is separable. Its solution is

$$b(t) = b(0)e^{-\dfrac{t}{2}} = 50e^{-\dfrac{t}{2}}.$$

This model predicts: $b(0) = 50,\ b(1) = 30.3,\ b(2) = 18.4.$

Death and Immigration

Same process as above, except now let's suppose that at each step in the process we add 10 coins. We start with 50 coins. We can think of the 10 coins being added as immigrants.

Model 1: As difference equation: As above: let $n = 0, 1, 2, 3, \ldots$ refer to the number of times this processes has been iterated. Let $b(n) =$ how many coins we have in the cup at the start of the n^{th} iteration. So $b(0) = 50$. If we toss a coin once the probability of heads $= P(H) = 1/2$. So with each iteration we expect to keep 1/2 the coins and then add 10 coins to our total (i.e. to $b(n)$) So we expect

$$b(n+1) = \frac{1}{2}b(n) + 10$$

or equivalently $b(n+1) - b(n) = -\dfrac{1}{2}b(n) + 10$. It will take a little work to figure out a closed form expression of $b(n)$.

$$\begin{aligned}
&b(n+1) = .5b(n) + 10\\
&b(n+2) = .5b(n+1) + 10 = .5(.5b(n) + 10) + 10 = .5^2b(n) + .5\cdot 10 + 10\\
&b(n+3) = .5b(n+2) + 10 = .5(.5^2b(n) + .5\cdot 10 + 10) + 10 = .5^3b(n) + .5^2\cdot 10 + .5\cdot 10 + 10
\end{aligned}$$

So, since we are starting with $n = 0$, we plug $n = 0$ into the above and get for $b(3)$:

$$b(0+3) = .5^3b(0) + .5^2\cdot 10 + .5\cdot 10 + 10 = .5^3b(0) + 10(.5^2 + .5^1 + .5^0) = .5^3b(0) + 10\sum_{k=0}^{2}.5^k$$

So, in general,

$$b(n) = .5^nb(0) + 10(.5^{n-1} + .5^{n-2} + \cdots 5^1 + .5^0) = .5^nb(0) + 10\sum_{k=0}^{n-1}.5^k$$

We can write the geometric series $\sum_{k=0}^{n-1} a^k$ in closed form. Here's the trick.

$$\begin{aligned}
S_{n-1} &= 1 + a + a^2 + \cdots + a^{n-1}\\
-\underline{aS_{n-1}} &= \underline{\qquad a + a^2 + \cdots + a^{n-1} + a^n}\\
(1-a)S_{n-1} &= 1 - a^n
\end{aligned}$$

Dividing both sides by $1 - a$ yields: $S_{n-1} = \dfrac{1 - a^n}{1 - a}$, so letting $a = .5$ we get:

$$\sum_{k=0}^{n-1}.5^k = \frac{1 - .5^n}{1 - .5} = 2(1 - .5^n)$$

so that

$$b(n) = .5^n(50) + 10\cdot 2(1 - .5^n)$$

Since $.5^n \to 0$ as $n \to \infty$ we have

$$\lim_{n\to\infty} b(n) = \lim_{n\to\infty} .5^n(50) + 10 \cdot 2(1 - .5^n) = 0(50) + 10 \cdot 2(1 - 0) = 0 + 10(2) = 20.$$

So, if we iterate this process many times, our initial population size $b(0)$ doesn't matter, eventually we'll have (about) 20 coins. Of course if you actually do this experiment, the observed $b(n)$ will typically not equal the $b(n)$ predicted by the model. The model predicts: $b(0) = 50,\ b(1) = 35,\ b(2) = 27.5,\ b(3) = 23.75,\ b(4) = 21.875,\ b(5) = 20.9375,\ \ldots,\ b(n) = .5^n(50) + 10 \cdot 2(1 - .5^n)$

As a differential equation: Let t refer to time, perhaps measured in minutes, and let b(t) be the number of coins surviving at time t.

We can imagine that instead of tossing all the coins at once and then adding 10 coins, that in Δt time, with $0 < \Delta t < 1$ we apply the process to Δt of the coins. For example, if $\Delta t = \frac{1}{10}$, we'd take one tenth of the coins out of the cup, i.e., $\Delta t\ b(t)$, toss those coins on the table, remove the ones that came up tails (probably about 1/2 of them), and put the ones that came up heads back into the cup. Then we'd add one tenth of the ten immigrating coins to the total. The difference equation, becomes:

$$b(t + \Delta t) - b(t) = \left(-\frac{1}{2}\ b(t) + 10\right)\ \Delta t$$

We can turn this difference equation into a differential equation by dividing both sides by Δt to get:

$$\frac{b(t + \Delta t) - b(t)}{\Delta t} = -\frac{1}{2}b(t) + 10,$$

and then letting $\Delta t \to 0$:

$$\lim_{\Delta t\to 0} \frac{b(t + \Delta t) - b(t)}{\Delta t} = \frac{db}{dt} = -\frac{1}{2}b(t) + 10.$$

Of course when Δt is very small, we'd be "tossing" less than one coin and having less than one coin immigrate, which doesn't make sense. But don't worry about that for now.

The DE $\frac{db}{dt} = -\frac{1}{2}b(t) + 10$ is separable. We can solve it as follows.

Let $k = 1/2$ to make the algebra quicker and write $b(t)$ as just b: $\frac{db}{dt} = -kb + 10$. Separate the b and the t parts: $\frac{db}{-kb + 10} = dt$. Integrate both sides: $\int \frac{db}{-kb + 10} = \int dt = t + c_1$. To integrate the first integral, use u-substitution with $u = -kb + 10$ so that $\frac{du}{db} = -k$ and $\frac{du}{-k} = db$ so that $\int \frac{db}{-kb + 10} = -\frac{1}{k} \int \frac{du}{u} = -\frac{1}{k} \ln|u| = -\frac{1}{k} \ln|-kb+10|$. So $-\frac{1}{k} \ln|-kb+10| = t+c_1$. Move the $-k$ to the left side of the equation to get: $\ln|-kb+10| = -kt - kc_1$. Let $-kc_1 = c_2$ and apply e to the base of both sides of the equation (since e^x and $\ln x$ are inverses of each other) to get: $e^{\ln|-kb+10|} = e^{-kt+c_2} = e^{-kt}e^{c_2}$, which simplifies to: $|-kb+10| = e^{-kt}e^{c_2}$. By the continuity argument, we can remove the absolute value sign by replacing e^{c_2} with c_3, where c_3 is a constant which might be negative, depending on the initial conditions. We get: $-kb+10 = c_3e^{-kt}$. We isolate b by moving the 10 to the other side and then dividing by $-k$, to get: $b = \frac{c_3}{-k}e^{-kt} + \frac{10}{k}$. Let $c = \frac{c_3}{-k}$, and we get: $b = ce^{-kt} + \frac{10}{k}$. Finally, let $k = 1/2 = .5$ so that: $b = ce^{-kt} + 20$. We can use our initial condition $b(0) = 50$ to find c. $50 = b(0) = c + 20$ so $c = 30$. We get our solution:

$$b(t) = 30e^{-.5t} + 20$$

Notice that as $t \to \infty$ that $b(t) \to 20$. This model predicts: $b(0) = 50,\ b(1) = 38.2,\ b(2) = 31.04,\ b(3) = 26.7,\ b(4) = 24.06, \ldots, b(15) = 20.02, \ldots$

Homework for Handout 2. Questions 2 and 3 are mixing problems.

1. We start with 60 dice in a cup. We shake the cup and toss the dice onto our desk. All the dice that show a 6 we keep and put back into our cup along with an additional 3 dice (immigrants). The other dice we throw away.

A. Model this as a difference equation. How many die should we expect to have after 1 and 2 iterations? I.e, what is $b(1)$ and $b(2)$. What is happens to $b(n)$ when n is very large?

B. Model this as a differential equation. What happens when $t \to \infty$. Note: the difference equation is the better model for this scenario.

2. We start with 1000 liters of seawater in a tank. In seawater there are 35 g of salt per liter. Suppose we continually pour almost fresh water (2 g salt/liter) into the top of the tank at the rate of 4 liters/minute and at the bottom of the tank, we continually drain off 4 liters/minute. So the amount of water in the tank is always 1000 liters. Assume that the water in the tank is being stirred so that the saltwater and the freshwater mix immediately. How long until the concentration of salt in the tank is 4 g/liter?

Hints for 2. To start: let $b(t) = g$ of salt in the tank at time t, t measured in minutes. The concentration of the salt at time t is just $\dfrac{b(t)\ g}{1000\ L}$. So we want to solve for t: $\dfrac{b(t)\ g}{1000\ L} = \dfrac{4\ g}{1\ L}$ or simply $b(t) = 4000$. To find $b(t)$ we set up and then solve the DE:

The amount of salt flowing into the tank each minute is $\dfrac{2\ g}{L}\ \dfrac{4\ L}{\text{min}} = \dfrac{8\ g}{\text{min}}$; the salt flowing out of the tank is $\dfrac{b(t)\ g}{1000\ L}\ \dfrac{4\ L}{\text{min}} = \dfrac{b(t)\ g}{250\ \text{min}}$. So the inflow rate minus the outflow rate gives us the DE:

$$\frac{db}{dt} = \frac{8\ g}{\text{min}} - \frac{b(t)\ g}{250\ \text{min}}$$

There are $(1000\ L)\dfrac{35\ g}{1\ L} = 35,000g$ of salt in the tank to begin with. So we have the Initial Value Problem (IVP):

$$\frac{db}{dt} = 8 - kb(t), \quad b(0) = 3500, \quad k = 1/250.$$

We can solve the above DE by separation of variable and then by plugging in the initial value $b(0) = 35,000$ we can the IVP. Finally, we solve $b(t) = 4,000$ for t.

3. We start with 1000 liters of seawater in a tank. In seawater there are 35 g of salt per liter. Suppose we continually pour almost fresh water (1.5 g salt/liter) into the top of the tank at the rate of 5 liters/minute and at the bottom of the tank, we continually drain off 5 liters/minute. So the amount of water in the tank is always 1000 liters. Assume that the water in the tank is being stirred so that the saltwater and the freshwater mix immediately. How long until the concentration of salt in the tank is 3 g/liter?

4. Solve the following initial value problem: $\dfrac{dy}{dx} = 2 - 3y \;\; y(0) = 8.$

5. Solve the following initial value problem: $y^2\,\dfrac{dy}{dx} = x^3, \;\; y(0) = 8.$

6. Solve the following initial value problem: $\dfrac{dy}{dx} = 3y \;\; y(0) = 8.$

7. Solve the following initial value problem: $\dfrac{dy}{dx} = 3y + 2 \;\; y(0) = 8.$

Solutions to the HW from Handout 2. $\boxed{\textbf{Answers are boxed}}$.

1. We start with 60 dice in a cup. We shake the cup and toss the dice onto our desk. All the dice that show a 6 we keep and put back into our cup along with an additional 3 dice (immigrants). The other dice we throw away.

A. Model this as a difference equation. How many die should we expect to have after 1 and 2 iterations? I.e, what is $b(1)$ and $b(2)$. What is happens to $b(n)$ when n is very large?

B. Model this as a differential equation. What happens when $t \to \infty$. Note: the difference equation is the better model for this scenario.

Solution to 1A. Each die has six faces. Since each face is equally likely, the probability of any particular face showing (i.e. facing up) is 1/6. So the probability of any individual die surviving an iteration is 1/6. Hence:

$$b(n+1) = \frac{1}{6}\,b(n) + 3 \quad \text{with} \quad b(0) = 60. \tag{8}$$

By subtracting $b(n)$ from both sides of Equation (8) we get the difference equation:

$$\boxed{b(n+1) - b(n) = -\frac{5}{6}\,b(n) + 3 \quad \text{with} \quad b(0) = 60.} \tag{9}$$

Using Equation (8) we calculate: $\boxed{b(1) = 13}$ and $\boxed{b(2) = 5.17}$. If we calculate $b(3), b(4), \ldots$, it looks like $b(n) \to 3.6$ as $n \to \infty$.

We prove that $b(n) \to 3.6$ as $n \to \infty$ as follows.

Let $a = 1/6,\ c = 3$, and $b_0 = 60$. With these substitutions Equation (8) becomes

$$b(n+1) = a\,b(n) + c \quad \text{with} \quad b(0) = b_0.$$

Writing $b(n+1)$ in the above manner makes the algebra easier. Iterating the formula for $b(n+1)$ yields:

$$\begin{aligned}
b(1) &= a\,b_0 + c \\
b(2) &= a\,b(1) + c = a^2 b_0 + ac + c = a^2 b_0 + c(a+1) \\
b(3) &= a\,b(2) + c = a^3 b_0 + a^2 c + ac + c = a^3 b_0 + c(a^2 + a + 1)
\end{aligned}$$

So

$$b(n) = a^n b_0 + c(a^{n-1} + a^{n-2} + \cdots + a + 1) = a^n b_0 + c\,\frac{1-a^n}{1-a} \tag{10}$$

where the last equality follows from the formula for summing the geometric series:

$$1 + a + a^2 + \cdots + a^{n-1} = \frac{1-a^n}{1-a}.$$

Plugging $a = 1/6, b_0 = 60$, and $c = 3$ into our formula for $b(n)$, i.e., into Equation (10), we get:

$$b(n) \;=\; 60\left(\frac{1}{6}\right)^n + 3\,\frac{1-(\frac{1}{6})^n}{1-\frac{1}{6}} \;=\; 60\left(\frac{1}{6}\right)^n + 3\,\frac{1-(\frac{1}{6})^n}{\frac{5}{6}} \;=\; 60\left(\frac{1}{6}\right)^n + \frac{18}{5}\left(1-\left(\frac{1}{6}\right)^n\right) \tag{11}$$

Since $\lim_{n\to\infty}\left(\frac{1}{6}\right)^n = 0$ it follows from Equation (11) that

$$\boxed{\lim_{n\to\infty} b(n) \;=\; \lim_{n\to\infty}\ 60\left(\frac{1}{6}\right)^n + \frac{18}{5}\left(1-\left(\frac{1}{6}\right)^n\right) \;=\; (60)(0) + \frac{18}{5}\,(1-0) \;=\; \frac{18}{5} \;=\; 3.6}$$

Solution to 1B. In 1A, we found the difference equation: Equation (9):

$$b(n+1) - b(n) = -\frac{5}{6}\,b(n) + 3 \quad \text{with} \quad b(0) = 60. \tag{12}$$

If we model this discrete process as a continuous one, and we assume that in Δt time, with $0 < \Delta t < 1$, that Δt of the change $b(n+1) - b(n)$ occurs, we have, from Equation (12):

$$b(t + \Delta t) - b(t) = \left(-\frac{5}{6}\,b(t) + 3\right)\Delta t \quad \text{with} \quad b(0) = 60. \tag{13}$$

Dividing Equation (13) by Δt and taking the limit as $\Delta t \to 0$ gives us the initial value problem (IVP):

$$\lim_{\Delta t \to 0} \frac{b(t+\Delta t) - b(t)}{\Delta t} = \frac{db}{dt} = -\frac{5}{6}\,b(t) + 3 \quad \text{with IC} \;\; b(0) = 60. \tag{14}$$

Notation: IC = initial conditions. We solve the IVP

$$\boxed{\frac{db}{dt} = -\frac{5}{6}\,b + 3 \quad \text{with} \quad b(0) = 60}$$

by separation of variables.

Let $k = \dfrac{5}{6} = k$ and $q = 3$. The DE becomes:

$$\frac{db}{dt} = -k\,b + q. \tag{15}$$

Separation yields $\dfrac{db}{-k\,b + q} = dt$.

Integrate both sides: $\displaystyle\int \frac{db}{-k\,b+q} = \int dt$.

Using u-substitution we find $\displaystyle\int \frac{db}{-k\,b+q}$. Let $u = -k\,b + q$ so $\dfrac{du}{-k} = db$ and

$$\int \frac{db}{-k\,b+q} = -\frac{1}{k}\int \frac{du}{u} = -\frac{1}{k}\,\ln|u| = -\frac{1}{k}\,\ln|-k\,b+q|$$

So

$$\int \frac{db}{-k\,b+q} = \int dt$$

$$-\frac{1}{k}\,\ln|-k\,b+q| = t + c_1$$

Some algebra allows us to solve for b: Multiplying by $-k$ gives us: $\ln|-k\,b+q| = -kt - kc_1$. Applying e gives us: $|-k\,b+q| = e^{-kt}e^{-kc_1}$. We eliminate the absolute value by replacing the constant e^{-kc_1} by the constant c_2 which will be positive (or negative) depending on the the IC. We get: $-k\,b + q = c_2 e^{-kt}$. Subtracting q and then dividing by $-k$ yields: $b = \dfrac{c_2}{-k}\,e^{-kt} + \dfrac{q}{k}$. Letting $c = \dfrac{c_2}{-k}$ gives us the general solution:

$$b = c\,e^{-kt} + \frac{q}{k} \tag{16}$$

Plugging in our $k = 5/6$ and $q = 3$ gives us $b = c\,e^{-\frac{5}{6}t} + \dfrac{3}{\frac{5}{6}}$. So the general solution to the DE is:

$$b(t) = c\,e^{-\frac{5}{6}t} + 3.6$$

Since $b(0) = 60 = c + 3.6$ we know $c = 56.4$. So the particular solution to the IVP is:

$$\boxed{b(t) = 56.4\,e^{-\frac{5}{6}t} + 3.6}$$

Clearly $\boxed{\lim_{t\to\infty} b(t) = 3.6}$

2. We start with 1000 liters of seawater in a tank. In seawater there are 35 g of salt per liter. Suppose we continually pour almost fresh water (2 g salt/liter) into the top of the tank at the rate of 4 liters/minute and at the bottom of the tank, we continually drain off 4 liters/minute. So the amount of water in the tank is always 1000 liters. Assume that the water in the tank is being stirred so that the saltwater and the freshwater mix immediately. How long until the concentration of salt in the tank is 4 g/liter?

Solution to 2. Let $b(t)$ = g of salt in the tank at time t, t measured in minutes. The concentration of the salt at time t is $\dfrac{b(t)\text{ g}}{1000\text{ L}}$. So to figure out how long it will take for the concentration of salt in the tank to be 4 g salt/liter, we solve for t: $\dfrac{b(t)\text{ g}}{1000\text{ L}} = \dfrac{4\text{ g}}{1\text{ L}}$, which simplifies to solving $b(t) = 4000$, for t. So the first thing we have to do is find $b(t)$.

The rate at which salt flows into the tank is $\left(\dfrac{2\text{ g}}{\text{L}}\right)\left(\dfrac{4\text{ L}}{\text{min}}\right) = \dfrac{8\text{ g}}{\text{min}}$.

The rate at which the salt flows out of the tank is $\left(\dfrac{b(t)\text{ g}}{1000\text{ L}}\right)\left(\dfrac{4\text{ L}}{\text{min}}\right) = \dfrac{b(t)\text{ g}}{250\text{ min}}$.

$$\begin{aligned}\frac{db}{dt} &= \text{ the rate at which the amount of salt in the tank changes}\\ &= (\text{salt inflow rate}) - (\text{salt outflow rate})\\ \frac{db}{dt} &= \frac{8\text{ g}}{\text{min}} - \frac{b(t)\text{ g}}{250\text{ min}}\end{aligned}$$

There are $(1000\text{ L})\left(\dfrac{35\text{ g}}{\text{L}}\right) = 35,000$ g of salt in the tank to begin with. So $b(0) = 35,000$. If we let $q = 8$ (= salt inflow rate) and $k = \frac{1}{250}$ (= fraction of tank that empties each minute). Our DE becomes:

$$\frac{db}{dt} = -k\,b + q, \quad b(0) = 35,000$$

Which is identical in form to the differential equation (15). Using separation of variables we found the general solution to be Equation (16):

$$b = c\,e^{-kt} + \frac{q}{k} \quad \text{with } c \text{ a constant which we can determine from the initial conditions.} \tag{17}$$

Plugging $q = 8$ and $k = \frac{1}{250}$ into Equation (17) we get the general solution:

$$b(t) = c\,e^{-\frac{1}{250}t} + 2,000$$

We find c by using the initial condition. $b(0) = 35,000 = c + 2,000$ so $c = 33,000$ and the particular solution is:

$$b(t) = 33,000\,e^{-\frac{1}{250}t} + 2,000$$

Finally, we solve $b(t) = 4,000$, for t.

$$\begin{aligned}33,000\,e^{-\frac{1}{250}t} + 2,000 &= 4,000\\ e^{-\frac{1}{250}t} &= \frac{2,000}{33,000} = \frac{2}{33}\\ -\frac{1}{250}t &= \ln\left(\frac{2}{33}\right)\\ t &= -(250)\ln\left(\frac{2}{33}\right)\\ &= \boxed{700.84\text{ min}}\end{aligned}$$

3. We start with 1000 liters of seawater in a tank. In seawater there are 35 g of salt per liter. Suppose we continually pour almost fresh water (1.5 g salt/liter) into the top of the tank at the rate of 5 liters/minute and at the bottom of the tank, we continually drain off 5 liters/minute. So the amount of water in the tank is always 1000 liters. Assume that the water in the tank is being stirred so that the saltwater and the freshwater mix immediately. How long until the concentration of salt in the tank is 3 g/liter?

Solution to 3. Question 3. is just Question 2. with different numbers. Let $q = 7.5$ (1.5 g/L $\times$ 5 L/min = 7.5 g/min = salt inflow rate) and $k = \frac{1}{200}$ (1/1000 tank/L $\times$ 5 L/min = 1/200 tank/min = fraction of tank that empties each minute). We get the DE:

$$\frac{db}{dt} = -k\,b + q, \quad b(0) = 35,000$$

Which is identical in form to the differential equation (15). Using separation of variables we found the general solution to be Equation (16):

$$b = c\,e^{-kt} + \frac{q}{k} \quad \text{with } c \text{ a constant which we can determine from the initial conditions.} \tag{18}$$

Plugging $q = 7.5$ and $k = \frac{1}{200}$ into Equation (18) we get the general solution:

$$b(t) = c\,e^{-\frac{1}{200}t} + 1,500$$

We find c by using the initial condition. $b(0) = 35,000 = c + 1,500$ so $c = 33,500$ and the particular solution is:

$$b(t) = 33,500\,e^{-\frac{1}{200}t} + 1,500$$

Finally, we solve $b(t) = 3,000$ (3,000 g of salt = (3 g/L)(1,000 L)), for t.

$$\begin{aligned} 33,500\,e^{-\frac{1}{200}t} + 1,500 &= 3,000 \\ e^{-\frac{1}{200}t} &= \frac{1,500}{33,500} = \frac{15}{335} \\ -\frac{1}{200}\,t &= \ln\left(\frac{15}{335}\right) \\ t &= -(200)\ln\left(\frac{15}{335}\right) \\ &= \boxed{621.22 \text{ min}} \end{aligned}$$

4. Solve the following initial value problem: $\dfrac{dy}{dx} = 2 - 3y \;\; y(0) = 8.$

Solution to 4. Separation of variables. $\displaystyle\int \frac{dy}{2-3y} = \int dx$. Use u-substitution with $u = 2 - 3y$ and $\dfrac{du}{-3} = dy$. The integrals become $\dfrac{1}{-3}\displaystyle\int \frac{du}{u} = \int dx$. The integrals evaluate to: $-\dfrac{1}{3}\ln|2-3y| = x + c_1$. Apply some algebra to get: $\ln|2-3y| = -3x + c_2$. Apply e to get: $2 - 3y = c_3 e^{-3x}$. Some algebra yields the general solution: $y = c_4 e^{-3x} + \dfrac{2}{3}$. Use IC to find c_4: $8 = y(0) = c_4 + \dfrac{2}{3}$ implies $c_4 = \dfrac{22}{3}$. So the particular solution is: $\boxed{y = \dfrac{22}{3}\,e^{-3x} + \dfrac{2}{3}}$.

5. Solve the following initial value problem: $y^2 \dfrac{dy}{dx} = x^3, \quad y(0) = 8.$

Solution to 5. Separation of variables. $\int y^2\, dy = \int x^3\, dx$. The integrals evaluate to: $\dfrac{y^3}{3} = \dfrac{x^4}{4} + c_1$. Apply some algebra to get: $y^3 = \dfrac{3x^4}{4} + c_2$. Take cube roots of both sides to get general solution:

$$y = \left(\frac{3x^4}{4} + c_2\right)^{1/3}.$$

Use IC to find c_2: $8 = y(0) = (0 + c_2)^{1/3}$. Cubing both sides implies $c_2 = 512$. So the particular solution is: $\boxed{y = \left(\dfrac{3x^4}{4} + 512\right)^{1/3}}$.

6. Solve the following initial value problem: $\dfrac{dy}{dx} = 3y \quad y(0) = 8.$

Solution to 6. Separation of variables. $\int \dfrac{dy}{y} = \int 3\, dx$. The integrals evaluate to: $\ln|y| = 3x + c_1$. Apply e to get the general solution: $y = c_2 e^{3x}$. Use IC to find c_2: $8 = y(0) = c_2$. So the particular solution is: $\boxed{y = 8e^{3x}}$.

7. Solve the following initial value problem: $\dfrac{dy}{dx} = 3y + 2 \quad y(0) = 8.$

Solution to 7. Separation of variables. $\int \dfrac{dy}{3y+2} = \int dx$. Use u-substitution with $u = 3y + 2$ and $\dfrac{du}{3} = dy$. The integrals become $\dfrac{1}{3}\int \dfrac{du}{u} = \int dx$. The integrals evaluate to: $\dfrac{1}{3}\ln|3y + 2| = x + c_1$. Apply some algebra to get: $\ln|3y + 2| = 3x + c_2$. Apply e to get: $3y + 2 = c_3 e^{3x}$. Some algebra yields the general solution: $y = c_4 e^{-3x} - \dfrac{2}{3}$. Use IC to find c_4: $8 = y(0) = c_4 - \dfrac{2}{3}$ implies $c_4 = \dfrac{26}{3}$. So the particular solution is: $\boxed{y = \dfrac{26}{3} e^{3x} - \dfrac{2}{3}}$.

Mixing problem homework to be handed in. Just one question:

We start with 2000 liters of seawater in a tank. In seawater there are 35 g of salt per liter. Suppose we continually pour fresh water (0.5 g salt/liter) into the top of the tank at the rate of 10 liters/minute and at the bottom of the tank, we continually drain off 10 liters/minute. So the amount of water in the tank is always 2000 liters. Assume that the water in the tank is being stirred so that the saltwater and the freshwater mix immediately. How long until the concentration of salt in the tank is 6 g/liter?

Circle or box your final answer (of how many minutes until the salt concentration is 6 g/L).

Show all work. Don't skip steps. Show the integration.
Use a pencil (so you can erase your mistakes).
Be neat.

Note about freshwater. The American Meteorological Society defines freshwater to be water containing less that 1 g/L of dissolved salts. See `http://glossary.ametsoc.org/wiki/Freshwater`. Freshwater containing less than .5 g/L of dissolved salts is drinkable, however, a salinity content of less than .25 g salt/L is preferred.

Professor Chris McCarthy Mat 501 Notes Handout 3 (5^{th} Ed.)

Linear First Order Differential Equations (Section 1.5 in Edwards & Penney)

The concept of linearity is very important in mathematics. In different contexts it has different formulations. For example, the solution set of the equation $ax + by = c$ is a straight line, hence $ax + by = c$ is called a linear equation. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ which satisfies $f(ax + by) = af(x) + bf(y)$ is called linear – such functions (which can be represented by matrices) are the study of linear algebra. We say that the derivative and the integral are linear (operators) since $(af + bg)' = af' + bg'$ and $\int_\alpha^\beta (af + bg)\, dx = a\int_\alpha^\beta f\, dx + b\int_\alpha^\beta g\, dx$. The first order DE $A(x)y' + B(x)y = C(x)$ is considered to be a **first order linear differential equation**. This is because the $A(x), B(x)$ and $C(x)$ end up acting a little bit like constants, at least relative to the solution y. We'll talk more about this later, time permitting.

More importantly, to solve the **first order linear differential equation**

$$A(x)y' + B(x)y = C(x)$$

we first divide by $A(x)$ to get

$$y' + \frac{B(x)}{A(x)}y = \frac{C(x)}{A(x)}.$$

From here on, I will let $\frac{B(x)}{A(x)} = P(x)$ and $\frac{C(x)}{A(x)} = Q(x)$, so as to match the (standard) notation used in text books:

$$y' + P(x)y = Q(x). \tag{19}$$

Somebody figured out the following neat trick! If we multiply Equation (19) by $e^{\int P(x)\,dx}$, (or if we want to be overly careful, by $e^{\int_a^x P(w)\,dw}$), we get

$$y'e^{\int P(x)\,dx} + P(x)e^{\int P(x)\,dx}y = Q(x)e^{\int P(x)\,dx}. \tag{20}$$

But

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = y'e^{\int P(x)\,dx} + P(x)e^{\int P(x)\,dx}y = Q(x)e^{\int P(x)\,dx}. \tag{21}$$

And so, integrating both ends of (21) yields:

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C \tag{22}$$

and so

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C$$

and, furthermore,

$$y = e^{-\int P(x)\,dx}\left(\int Q(x)e^{\int P(x)\,dx}\,dx + C\right).$$

Which is difficult to remember! So, instead, just remember the trick, the process, multiplying by the the integrating factor $e^{\int P(x)\,dx}$, etc.

Example (Example 1, p49 of Edwards & Penney): Solve the IVP

$$y' - y = \frac{11}{8}e^{-x/3}, \text{ with } y(0) = -1.$$

Solution: $P(x) = -1$ and $Q(x) = \frac{11}{8}e^{-x/3}$. So the integrating factor is

$$e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x+C}.$$

We will always take $C = 0$ in the integrating factor to simplify matters, so the integrating factor is e^{-x}. So

$$\left(ye^{-x}\right)' = e^{-x}\frac{11}{8}e^{-x/3} = \frac{11}{8}e^{-4x/3}$$

integrating both sides of $(ye^{-x})' = \frac{11}{8}e^{-4x/3}$, see [2], yields:

$$ye^{-x} = \int \frac{11}{8}e^{-4x/3}\,dx + C = \left(\frac{11}{8}\right)\left(\frac{-3}{4}\right)e^{-4x/3} + C = \frac{-33}{32}e^{-4x/3} + C =$$

Moving the e^{-x} to the RHS, by multiplying by e^x, yields the general solution:

$$y = \frac{-33}{32}e^{-x/3} + Ce^x.$$

Plugging in the IC $y(0) = -1$ gives us

$$-1 = y(0) = \frac{-33}{32}e^{-(0)/3} + Ce^0 = \frac{-33}{32} + C,$$

so that with some arithmetic we get

$$-1 = \frac{-32}{32} = \frac{-33}{32} + C \quad \Rightarrow \quad C = \frac{1}{32}.$$

Hence the IVP particular solution is:

$$y(x) = \frac{-33}{32}e^{-x/3} + \frac{1}{32}e^x$$

which we can write nicely in the form:

$$y(x) = \frac{1}{32}\left(e^x - 33e^{-x/3}\right).$$

Separable Equation: A Numerical Example and dimensional analysis. (Based on Example 5 from Edwards & Penney 4 Ed., page 39). In 1999 the world population is 6 billion people and each day the population is growing by 212,000 people. Assume the population at time t, with t measured in years, satisfies the DE: $P' = kP$. Find k and the particular solution to this IVP. Let the units of P be billions (b).

Solution: Since $P' = kP$ we have, $dP/P = k\,dt$ which integrates to $P(t) = P(0)e^{kt}$ with $P(0) = 6$ billion. So the only difficulty is to find k. The units of k must be 1/year since $\frac{dP}{dt}$ has units of billions/year so to make the units in

$$\frac{dP}{dt}\frac{\text{billions}}{\text{year}} = kP(t)\text{ billions.}$$

match up, we must have the units of k be 1/year. Since $P' = kP$ we have $\frac{P'}{P} = k$ so $\frac{P'(0)}{P(0)} = k$. We approximate $P'(0)$: The units of $P'(0)$ are $\frac{\text{billions}}{\text{year}}$ so, using dimensional analysis we get

$$\left.\frac{dP}{dt}\right|_{t=0}\frac{\text{billion}}{\text{year}} = \frac{212,000\text{ people}}{\text{day}}\,\frac{365.25\text{ days}}{\text{year}}\,\frac{\text{billion}}{1,000,000,000} = 0.07743\frac{\text{billion}}{\text{year}}.$$

So

$$k = \frac{P'(0)}{P(0)} = \frac{P'(0)}{P(0)} = \frac{0.07743\text{ billion}}{\text{year}}\Big/ 6\text{ billion} = 0.0129/\text{year and so } P(t) = 6e^{0.0129t/\text{year}}\text{ billion.}$$

HW 3: page 56: #1,5. page 43: #19, 35, 36.

[2] To find the anti-derivative of $\int \frac{11}{8}e^{-4x/3}\,dx$ use u-substitution of $u = -4x/3$ and $du = -(3/4)\,dx$

Example. Mixing Problem. A tank is initially filled with 1000 L of sea water (35 g salt/L). At time $t = 0$ an inflow of 4 L water/min with a concentration 2 g salt/L is started. Simultaneously the tank's drain is opened and the water goes down the drain at a rate of 7 L water/min. Let $b(t)$ be the number of grams of salt in the tank as a function of t, t in minutes. Find $b(t)$. **Hint. Use the integrating factor method.**

Solution. See Figure 2.

The net rate at which the tank is emptying is

$$4\frac{\text{L}}{\text{min}} - 7\frac{\text{L}}{\text{min}} = -3\frac{\text{L}}{\text{min}}.$$

This implies that the amount of water in the tank, in liters, at time t is given by

$$1000 - 3t$$

where t is measured in minutes. The concentration of salt in the tank is given by

$$\frac{\text{g salt in tank}}{\text{L of water left in tank}} = \frac{b(t)\ g}{(1000-3t)\ L}.$$

We get the ODE

$$b'(t) = \left(\frac{4\ \text{L}}{\text{min}}\right)\left(\frac{2\ \text{g}}{\text{L}}\right) - \left(\frac{7\ \text{L}}{\text{min}}\right)\left(\frac{b(t)\ \text{g}}{(1000-3t)\ \text{L}}\right)$$

which we condense into:

$$b' = 8 - \frac{7}{1000-3t}b$$

and write as

$$b' + \frac{7}{1000-3t}b = 8$$

We use the integrating factor method. If $b' + P(t)b = Q(t)$ then $\left(b(t)e^{\int P}\right)' = Q(t)e^{\int P}$. For this problem, $P(t) = \dfrac{7}{1000-3t}$ and $Q(t) = 8$.

$$\int P = \int \frac{7\,dt}{(1000-3t)} = -\frac{7\ln|1000-3t|}{3}$$

Since $1000 - 3t \geq 0$ in our problem, since when $1000 - 3t = 0$ the tank is empty, we can ignore the absolute value in $\ln|1000 - 3t|$. The integrating factor is

$$e^{\int P} = e^{-\dfrac{7\ln(1000-3t)}{3}} = \left(e^{\ln(1000-3t)}\right)^{-\frac{7}{3}} = (1000-3t)^{-\frac{7}{3}}$$

So $\left(b(t)e^{\int P}\right)' = Q(t)e^{\int P}$ becomes

$$\left(b(t)\,(1000-3t)^{-\frac{7}{3}}\right)' = 8\,(1000-3t)^{-\frac{7}{3}}$$

Integrating the above equation with respect to dt we get

$$b(t)\,(1000-3t)^{-\frac{7}{3}} = \frac{8\,(1000-3t)^{-\frac{4}{3}}}{4} + C$$

A little algebra yields:

$$b(t) = \frac{8\,(1000-3t)^{-\frac{4}{3}}}{4}(1000-3t)^{\frac{7}{3}} + (1000-3t)^{\frac{7}{3}}\,C$$

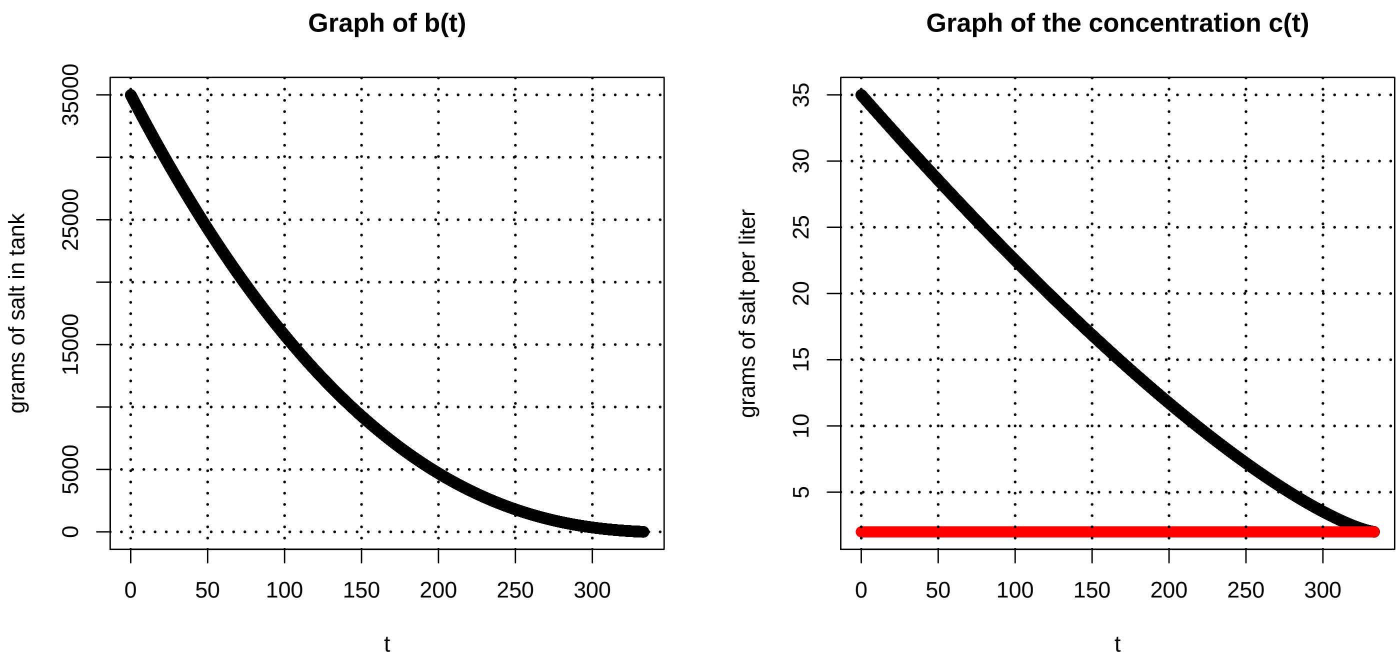


Figure 2: Mixing Problem. On the left is the graph of $b(t)$, the amount of salt in the tank at time t, $b(t) = 2\,(1000 - 3t) + .0033\,(1000 - 3t)^{\frac{7}{3}}$. On the right is the graph of $c(t)$, the concentration of salt in the tank at time t, $c(t) = \dfrac{b(t)}{1000 - 3t}$. The horizontal red line is the concentration of salt in the inflow, 2 g/L. Note that the tank will be empty when $t = 1000/3 \approx 333.33$ minutes.

and some more algebra give us:

$$b(t) = 2\,(1000 - 3t) + (1000 - 3t)^{\frac{7}{3}}\,C.$$

The initial condition is $b(0) = 35,000$ g salt. We use this to solve for C.

$$b(0) = 35,000 = 2000 + 1000^{\frac{7}{3}}C$$

which implies $33,000 = 1000^{\frac{7}{3}}C = 10^7C$. So $C = \dfrac{33 \times 10^3}{10^7} = 33 \times 10^{-4} = .0033$ and so the solution is

$$b(t) = 2\,(1000 - 3t) + .0033\,(1000 - 3t)^{\frac{7}{3}} \tag{23}$$

See Figure 2.

We can solve the ODE and the IVP using Maple:

```
de := diff(b(t), t)+7*b(t)/(1000-3*t) = 8;
dsolve(de, b(t));
ics := b(0) = 35000;
dsolve({de, ics});
```

Maple gives us the solution

$$b\,(t) = -\frac{33}{1000000}\,(-1000 + 3\,t)^{7/3}\,(-1000)^{2/3} + 2000 - 6\,t \tag{24}$$

The equivalence of our solution, Equation (23), and Maple's solution, Equation (24), follows from $100 = (-1000)^{2/3}$ and $(1000 - 3\,t)^{7/3} = -\,(-1000 + 3\,t)^{7/3}$.

HW 3 (with solutions): Edwards & Penney, 4^{th} Ed. Page 56: #1,5. Page 43: #19, 34, 35.

(Page 56: #1.) Solve the IVP:

$$y' + y = 2, \quad y(0) = 0.$$

Solution: We identify this ODE as being "Linear First Order" with $P(x) = 1$ and $Q(x) = 2$. It is also separable. First we'll solve this IVP using the separation of variables method:

$$\frac{dy}{dx} = 2 - y \;\Rightarrow\; \frac{dy}{2-y} = dx \;\Rightarrow\; \int \frac{dy}{2-y} = \int dx + C$$

Using u-substitution with $u = 2 - y$ so $-du = dy$ we get

$$\int \frac{dy}{2-y} = -\int \frac{du}{u} = -\ln|2-y|$$

so $-\ln|2-y| = x + C$ or $\ln|2-y| = -x + C$ so $|2 - y(x)| = Ae^{-x}$. Since $y(0) = 0$ we have $A = 2$ and so $2 - y(x) = \pm 2e^{-x}$ implying $y(x) = 2(1 \pm e^{-x})$ and finally, using $y(0) = 0$ again, that $y(x) = 2(1 - e^{-x})$ or equivalently

$$y(x) = 2 - 2e^{-x}.$$

It is easy to check that that this solves the IVP.

We solve (Page 56: #1) again, this time using the integrating factor method:

$$y' + y = 2, \quad y(0) = 0.$$

Solution: $P(x) = 1$ and $Q(x) = 2$ so the integrating factor is

$$e^{\int P(x)\,dx} = e^{\int 1\,dx} = e^x$$

and so

$$\left(ye^{\int P(x)\,dx}\right)' = Q(x)e^{\int P(x)\,dx} \text{ becomes } (ye^x)' = 2e^x$$

and thus

$$\int (ye^x)'\,dx = \int 2e^x\,dx + C$$

which implies $ye^x = 2e^x + C$ and so $y = 2 + Ce^{-x}$. Using the IC $y(0) = 0$ we see that $C = -2$, and so $y = 2 - 2e^{-x}$.

(Page 56: #5) Using the integrating factor method solve:

$$xy' + 2y = 3x, \quad y(1) = 5.$$

Solution: We divide by x to get

$$y' + (2/x)y = 3, \quad y(1) = 5$$

So $P(x) = 2/x$ and $Q(x) = 3$ so the integrating factor is

$$e^{\int P(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = x^2$$

and so

$$\left(ye^{\int P(x)\,dx}\right)' = Q(x)e^{\int P(x)\,dx} \text{ becomes } \left(yx^2\right)' = 3x^2$$

and thus

$$\int (yx^2)'\,dx = \int 3x^2\,dx + C$$

which implies $yx^2 = x^3 + C$ and so

$$y(x) = x + \frac{C}{x^2}.$$

Using the IC $y(1) = 5$ we see that

$$5 = y(1) = 1 + \frac{C}{1^2}$$

implying $C = 4$, and so

$$y(x) = x + \frac{4}{x^2}.$$

(Page 43: #19). Solve the following IVP by separation of variables:

$$\frac{dy}{dx} = ye^x, \qquad y(0) = 2e.$$

Solution.

$$\int \frac{dy}{y} = \int e^x\,dx + C \Rightarrow \ln|y| = e^x + C \Rightarrow |y| = Ae^{e^x}$$

From the IC $y(0) = 2e$:

$$|y(0)| = Ae^{e^0} \text{ becomes } |2e| = Ae \Rightarrow A = 2 \Rightarrow |y(t)| = 2e^{e^x} \Rightarrow y(t) = \pm 2e^{e^x}.$$

Since $y(0) = 2e$ we can resolve the $\pm$ to be $+$, so the solution is

$$y(t) = 2e^{e^x}.$$

(Page 43: #35). (Radiocarbon dating) Carbon in an ancient skull only contains only 1/6 the amount of ^{14}C as a present-day bone. How old is the skull?

Solution: We know, see page 38, that radioactive decay follows $y(t) = y(0)e^{-kt}$, where, for the decay of ^{14}C, $k = 0.0001216$, if time t is measured in years. The units of y, the amount of ^{14}C, can be any measure of mass. So we want to solve the following equation for t_{skull}:

$$\frac{1}{6}y(0) = y(0)e^{-kt_{\text{skull}}} \quad \text{where}$$

t_{skull} = age of skull in years;
$y(0)$ = original amount of ^{14}C in the skull t_{skull} years ago;
$y(t_{\text{skull}})$ = amount of ^{14}C remaining in the skull after t_{skull} years: We get

$$\frac{1}{6} = e^{-kt_{\text{skull}}} \quad \Rightarrow \quad \ln\left(\frac{1}{6}\right) = -kt_{\text{skull}} \quad \Rightarrow \quad \frac{\ln\left(\frac{1}{6}\right)}{-k} = t_{\text{skull}} \quad \Rightarrow \quad \frac{\ln(6)}{k} = t_{\text{skull}}$$

so

$$\frac{\ln(6)}{0.0001216} = 14734.86406 = t_{\text{skull}}$$

The skull is about 14,734.9 years old.

Page 43: #34. (Population Growth) Suppose bacteria increase by 6 fold in 10 hours. How long did it take for the population to double?

Solution: Assuming exponential growth, we have $y(t) = y(0)e^{kt}$. We can measure time in hours, but maybe with bacteria, it is better to use minutes. So 10 hours = 600 minutes. We have

$$6y(0) = y(0)e^{k\,600\text{ minutes}},$$

where the units of k are 1/minutes and the units of y is proportional to the number of bacteria, e.g. mass, colony area, bacteria count, etc. A little algebra lets us solve for k:

$$k = \frac{\ln(6)}{600\text{ min}}$$

and then solving

$$2y(0) = y(0)e^{kt_{\text{double}}}$$

for t_{double} give us

$$\frac{\ln(2)}{k} = t_{\text{double}}.$$

Substituting our formula for k yields

$$t_{\text{double}} = \frac{\ln(2)}{\frac{\ln(6)}{600\text{ min}}} \approx 232\text{ min}.$$

So it takes about 232 min (≈ 3.87 hr) for the population to double.

Homework. HAND IN THIS ONE PROBLEM. Mixing Problem. A tank with a 4000 L capacity is initially filled with 1000 L of sea water (35 g salt/L). At time $t = 0$ an inflow 8 L water/min with a concentration 2 g salt/L is started. Simultaneously the tank's drain is opened and the water goes down the drain at a rate of 3 L water/min. Let $b(t)$ be the number of grams of salt in the tank as a function of t, t in minutes.

(a) Find the differential equation for $b(t)$ and then solve that differential equation to find $b(t)$. Circle your answer. **Hint. Use the integrating factor method.**

(b) What will the concentration of salt be in the tank when the tank is filled? Circle answer.

(c) Sketch $b(t)$ with t between 0 and how long it takes to fill tank. (You can use Maple to do the graph if you want. Then attach print out, or sketch it by hand).

Vector Spaces.

The Vector Spaces of n-tuples.
When students first learn about vectors, vectors tend to be presented as being n-tuples of numbers:

$$v = (v_1, v_2, \ldots, v_n).$$

Example 1. The 2-tuple representation of $\mathbb{R}^2$
Here we represent the vector space $\mathbb{R}^2$ as the set of 2-tuples of real numbers:

$$\mathbb{R}^2 = \{(x, y) \,|\, x, y \in \mathbb{R}\}.$$

Definition of vector addition for $\mathbb{R}^2$: if

$$(x, y), (a, b) \in \mathbb{R}^2 \;\text{ then }\; (x, y) + (a, b) = (x + a, y + b).$$

Definition of scalar multiplication for $\mathbb{R}^2$: if

$$(x, y) \in \mathbb{R}^2,\; c \in \mathbb{R}, \;\text{ then }\; c(x, y) = (cx, cy).$$

Concrete numerical example: $(1, 2) + 10(5, 6) = (1, 2) + (50, 60) = (51, 62)$.

Function spaces: functions into $\mathbb{R}$ (or $\mathbb{C}$) as vectors.

A generalization of the vector spaces of n-tuples of real numbers $\mathbb{R}^n$ (or complex numbers $\mathbb{C}^n$), are the vector spaces whose vectors are functions that map into $\mathbb{R}$ (or $\mathbb{C}$) from some domain. Such mathematical objects are sometimes called function spaces. It is easy to see that the vector spaces of n-tuples, like $\mathbb{R}^n$ (or $\mathbb{C}^n$), are function spaces: explicitly, $\mathbb{R}^n = \{v \,|\, v : \{1, 2, \ldots, n\} \to \mathbb{R}\}$ and ($\mathbb{C}^n = \{v \,|\, v : \{1, 2, \ldots, n\} \to \mathbb{C}\}$), with addition and scalar multiplication defined point-wise. See example 2 below.

Example 2. $\mathbb{R}^2$ as a function space.

Here we represent the vector space $\mathbb{R}^2$ as a set of functions from a finite set with two elements, say $\{1, 2\}$ into the real numbers $\mathbb{R}$. In set theoretic notation:

$$\mathbb{R}^2 = \{v \,|\, v : \{1, 2\} \to \mathbb{R}\}.$$

Definition of vector addition for $\mathbb{R}^2$: if $v, w \in \mathbb{R}^2$ so that $v : \{1, 2\} \to \mathbb{R}$ and $w : \{1, 2\} \to \mathbb{R}$ then we define

$$\begin{aligned}&(v + w) : \{1, 2\} \to \mathbb{R} \text{ by}\\ &\qquad (v + w)(1) = v(1) + w(1)\\ &\qquad (v + w)(2) = v(2) + w(2).\end{aligned}$$

Definition of scalar multiplication for $\mathbb{R}^2$: if $c \in \mathbb{R}$ then we define

$$\begin{aligned}&(cv) : \{1, 2\} \to \mathbb{R} \text{ by}\\ &\qquad (cv)(1) = cv(1)\\ &\qquad (cv)(2) = cv(2).\end{aligned}$$

Concrete numerical example: So if $v, w \in \mathbb{R}^2$ are defined by

$$\begin{aligned}v(1) = 1,\; v(2) = 2\\ w(1) = 5,\; w(2) = 6\end{aligned}$$

then $(v+10w):\{1,2\}\to\mathbb{R}$ by

$$(v+10w)(1)=v(1)+10w(1)=1+10\cdot 5=51$$
$$(v+10w)(2)=v(2)+10w(2)=1+10\cdot 6=62.$$

The connection between Example 1 and 2:
If v is a vector in $\mathbb{R}^2$ then we have $v=(v_1,v_2)=(v(1),v(2))$.

Linear Differential Operators.

Linear Operators: We say that $\mathcal{L}$ is a linear operator on the vector space V if $\mathcal{L}:V\to V$ and if $c_1,c_2\in\mathbb{R}$ and $y_1,y_2\in V$ implies:

$$\mathcal{L}(c_1y_1+c_2y_2)=c_1\mathcal{L}(y_1)+c_2\mathcal{L}(y_2).$$

Important example: if $\mathcal{L}$ is a linear operator on $\mathbb{R}^n$, then $\mathcal{L}$ can be represented as an $n\times n$ matrix of real numbers. Proof. See footnote [3].

Linear Differential Operators The linear differential operators are linear operators which are built out of linear combinations of derivatives of various orders. The vector space V, on which the linear differential operators will operate, will consist of (sufficiently) differentiable functions [4] defined on some subset of $\mathbb{R}$, or possible all of $\mathbb{R}$ itself.

We will write D to denote $\frac{d}{dx}$ and D^k to denote $\frac{d^k}{dx^k}$. If $\mathcal{L}$ is a linear differential operator, then $\mathcal{L}$ can be expressed in the form

$$\mathcal{L}=A_n(x)D^n+A_{n-1}(x)D^{n-1}+\ldots+A_2(x)D^2+A_1(x)D+A_0(x)D^0,$$

where the $A_i(x),\ i=1,2,\ldots,n$, are sufficiently differentiable and where

$$D^0[y]=y,\qquad D[y]=y',\qquad D^{(k)}[y]=\overbrace{D\circ D\circ\cdots\circ D}^{k\text{ times}}[y].$$

[3] Let $\mathcal{B}=\{b_n\}_{i=1}^n$ be a basis for $\mathbb{R}^n$. Then $v\in\mathbb{R}^n$ is uniquely expressible as $v=\sum_{i=1}^n v_ib_i$, with each $v_i\in\mathbb{R}$. Using the linearity of $\mathcal{L}$ we have $\mathcal{L}(v)=\mathcal{L}\left(\sum_{k=1}^i v_ib_i\right)=\sum_{i=1}^n v_i\mathcal{L}(b_i)$. Since $\mathcal{L}(b_i)\in\mathbb{R}^n$, $\exists$ real numbers $L_{ij}, j=1,2,\ldots,n$, such that $\mathcal{L}(b_i)=\sum_{j=1}^n L_{ij}b_j$. So the $n\times n$ matrix L whose ij entry is L_{ij} contains all the information needed to describe the operator $\mathcal{L}$ (provided we have been told which basis we are using for $\mathbb{R}^n$). Explicitly, $\mathcal{L}(v)=\left(Lv^T\right)^T$, where: T denotes transpose; the vectors $v,\mathcal{L}(v)$, and $\left(Lv^T\right)^T$ are n-tuples whose entries are uniquely determined by $\mathcal{B}$; Lv^T is the $n\times 1$ column vector created by the matrix multiplication of the $n\times n$ matrix L with the $n\times 1$ column vector v^T.

[4] It is difficult (lengthy) to say exactly what sufficiently differentiable means! In the strictest sense the functions (i.e. the vectors) should be infinitely differentiable to ensure that $\mathcal{L}:V\to V$, i.e. we need to insure that if $f\in V$ then $\mathcal{L}\circ\mathcal{L}\circ\cdots\circ\mathcal{L}[y]$ is differentiable (and so in V). The problem is that the derivatives of differentiable functions are not always themselves differentiable. For example, let f be the everywhere differentiable function:

$$f(x)=\begin{cases}0, & x<0;\\ x^2, & x\ge 0\ .\end{cases}\qquad \text{It is easy to see that its derivative}\qquad f'(x)=\begin{cases}0, & x<0;\\ 2x, & x\ge 0\ .\end{cases}$$

is not differentiable at $x=0$. However, restricting ourselves to only the infinitely differentiable functions means we would not have access to many functions that might be useful in important applications. So we will be sloppy about this issue of infinite differentiability, and we will use terms like "sufficient" to avoid dealing with these issues. This is one example of where engineers and physicists have an advantage over mathematicians. Engineers and physicists tend to be more willing to bend mathematical rules, provided this bending results in something really useful, whereas mathematicians tend to be more "fastidious" and tend to want the mathematical machinery that they are working with to be perfectly well defined and logical.

See [5] for a note about the origin of the D notation.

Notation and remark regarding composition: The symbol $\circ$ means composition. The multiplication of two operators is not always defined because the multiplication of vectors is not always defined. You can always add two vectors, or multiply a vector by a scalar, however multiplying vectors with each other – that is a whole different story.

So, when working with operators one is typically composing them (rather than multiplying them) and so it is common to simply write $\mathcal{L}\mathcal{M}$, to indicate composition, rather than $\mathcal{L} \circ \mathcal{M}$. That said, I'll use $\circ$ occasionally to avoid ambiguity. Also some authors write $\mathcal{L}\mathcal{M}$ to mean both multiplication and composition (assuming that multiplication of vectors is defined within the vector space that the operators are operating on), and then you have to be "clever" enough to figure out which one the author means. (Hopefully you can figure out which is intended by the context.) As we will see below, the composition of linear operators algebraically resembles multiplication in many ways.

Notation: It is customary to use square brackets, like [ ] to indicated that an operator is operating on a function space. So a linear differential operator operating on the function y will look like:

$$\begin{aligned}\mathcal{L}[y] &= \big(A_n(x)D^n + A_{n-1}(x)D^{n-1} + \ldots + A_2(x)D^2 + A_1(x)D + A_0(x)\big)[y] \\ &= A_n(x)D^n[y] + A_{n-1}(x)D^{n-1}[y] + \ldots + A_2(x)D^2[y] + A_1(x)D[y] + A_0(x)[y] \\ &= A_n(x)y^{(n)} + A_{n-1}(x)y^{(n-1)} + \ldots + A_2(x)y'' + A_1(x)y' + A_0(x)y\end{aligned}$$

where $y^{(n)}$ means the n^{th} derivative of y w.r.t. x.

Example 1: Let $\mathcal{L}$ be the linear differential operator $(D^2 + x^2D^0)$, which would usually be written as $(D^2 + x^2)$, (since D^0 acts like multiplication by 1) and let y be a sufficiently differentiable function. Then applying $\mathcal{L}$ to y we get

$$\mathcal{L}[y] = (D^2 + x^2D^0)[y] = (D^2 + x^2)[y] = D^2[y] + x^2[y] = y'' + x^2y.$$

For example, if $y = x^5$ then

$$\mathcal{L}[x^5] = (x^5)'' + x^2(x^5) = 20x^3 + x^7.$$

Example 2: Let $\mathcal{L}$ be the linear differential operator $(D - 1)$. Applying $\mathcal{L}$ to y we get

$$\mathcal{L}[y] = y' - y.$$

Example 3: The ODE describing proportional growth (or decay) is $y' = ky$. You need to memorize that this DE has general solution $y = Ce^{kx}$ (and particular solution $y(x) = y(0)e^{kx}$). The DE $y' = ky$ is equivalent to $y' - ky = 0$, so if we let

$$\mathcal{L} = D - k$$

then the differential equation $y' = ky$ is equivalent to $(D - k)[y] = 0$, which is to say: the DE is equivalent to

$$\mathcal{L}[y] = 0.$$

Theorem: Linear combinations of linear operators are linear operators

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear operators operating on the same vector space V and $c_1, c_2 \in \mathbb{R}$ it is easy to see $c_1\mathcal{L}_1 + c_2\mathcal{L}_2$ will also be a linear operator on V.

[5] The D notation, as pertains to differential operators, seems to have been invented by Oliver Heaviside, who made many important contributions to electrical engineering; physics, and math. His life story is quite interesting.

Theorem: The composition of two or more linear operators will be a linear operator

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear operators operating on the same vector space V then it is easy to see [6], that $\mathcal{L}_2\mathcal{L}_1$ will also be a linear operator on V. Sometimes we will use the composition symbol $\circ$ to denote composition.

It is extremely important to realize that $\mathcal{L}_2\mathcal{L}_1$ denotes composition, not multiplication.
It is not always possible to multiply two vectors and so it is not always possible to multiply two linear operators. That said, the composition of linear operators acts like multiplication in some ways:

Suppose we have four linear operators $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ and four numbers $c_1, c_2, c_3, c_4 \in \mathbb{R}$. Then $(c_1\mathcal{L}_1+c_2\mathcal{L}_2)$ and $(c_3\mathcal{L}_3 + c_4\mathcal{L}_4)$ are also linear operators. We have

$$\begin{aligned}(c_1\mathcal{L}_1 + c_2\mathcal{L}_2) \circ (c_3\mathcal{L}_3 + c_4\mathcal{L}_4) =& c_1\mathcal{L}_1 \circ (c_3\mathcal{L}_3 + c_4\mathcal{L}_4) + c_2\mathcal{L}_2 \circ (c_3\mathcal{L}_3 + c_4\mathcal{L}_4)\\ =& c_1c_3(\mathcal{L}_1 \circ \mathcal{L}_3) + c_1c_4(\mathcal{L}_1 \circ \mathcal{L}_4) + c_2c_3(\mathcal{L}_2 \circ \mathcal{L}_3) + c_2c_4(\mathcal{L}_2 \circ \mathcal{L}_4),\end{aligned}$$

which symbol-wise, is exactly what you would get from multiplication. Most mathematicians and engineers would write the above as

$$\begin{aligned}(c_1\mathcal{L}_1 + c_2\mathcal{L}_2)(c_3\mathcal{L}_3 + c_4\mathcal{L}_4) =& c_1\mathcal{L}_1(c_3\mathcal{L}_3 + c_4\mathcal{L}_4) + c_2\mathcal{L}_2(c_3\mathcal{L}_3 + c_4\mathcal{L}_4)\\ =& c_1c_3\mathcal{L}_1\mathcal{L}_3 + c_1c_4\mathcal{L}_1\mathcal{L}_4 + c_2c_3\mathcal{L}_2\mathcal{L}_3 + c_2c_4\mathcal{L}_2\mathcal{L}_4.\end{aligned}$$

Which is much neater looking! Just remember, we are composing, not multiplying! It just looks like multiplication, because of the linearity.

Warning! The composition of linear operators is not commutative, i.e. sometimes: $\mathcal{L}_1\mathcal{L}_2 \neq \mathcal{L}_2\mathcal{L}_1$. A few lines down you will see an example of a linear differential operators not being commutative.

Note about the multiplication of matrices (especially for Linear Algebra students)
In linear algebra you multiply matrices (which are representations of linear operators on $\mathbb{R}^n$) by each other. As explained earlier (see page 23), once a basis $\mathcal{B}$ for $\mathbb{R}^n$ is chosen, we can represent the linear operators on $\mathbb{R}^n$ by $n \times n$ matrices. So, if $\mathcal{L}$ is any linear operator on $\mathbb{R}^n$ we will write $\mathcal{L}_\mathcal{B}$ to denote its matrix representation w.r.t. $\mathcal{B}$. Now, suppose that $\mathcal{L}_1, \mathcal{L}_2$ are two linear operators on $\mathbb{R}^n$. We have the matrix equality:

$$(\underbrace{\mathcal{L}_1\mathcal{L}_2}_{\text{composition}})_\mathcal{B} \quad = \quad \underbrace{(\mathcal{L}_1)_\mathcal{B}(\mathcal{L}_2)_\mathcal{B}}_{\text{matrix multiplication}} \quad .$$

Linear differential operators and composition.

Linear differential operators don't always commute with one another, as the following example shows.
Commute: A and B commute if $AB = BA$.

Example 1: Consider the linear differential operators $D = \frac{d}{dx}$ and xD. So $D[y] = y'$ and $xD[y] = xy'$.

$$(D \circ xD)[y] = D[xD[y]] = D[xy'] = xD[y'] + D[x]y' = xy'' + 1y' = xy'' + y' \tag{25}$$
$$(xD \circ D)[y] = xD[D[y]] = xD[y'] = xy''. \tag{26}$$

So we see that linear differential operators don't always commute.

Theorem: Two linear differential operator with constant coefficients will always commute.

[6] $\mathcal{L}_2\mathcal{L}_1[c_1y_1 + c_2y_2] = \mathcal{L}_2[\mathcal{L}_1[c_1y_1 + c_2y_2]] = \mathcal{L}_2[c_1\mathcal{L}_1[y_1] + c_2\mathcal{L}_1[y_2]] = c_1\mathcal{L}_2[\mathcal{L}_1[y_1]] + c_2\mathcal{L}_2[\mathcal{L}_1[y_2]] = c_1\mathcal{L}_2\mathcal{L}_1[y_1] + c_2\mathcal{L}_2\mathcal{L}_1[y_2]$.

Proof: Let $\mathcal{L} = \sum A_k D^k$ and $\mathcal{M} = \sum B_j D^j$ be two linear differential operators with constant coefficients $A_i, B_j \in \mathbb{R}$ (or $\mathbb{C}$). We need to show that $\mathcal{L}\mathcal{M} = \mathcal{M}\mathcal{L}$. We will show this in stages. Since $D = \frac{d}{dx}$ is linear we have

$$D\mathcal{L} = D\sum A_k D^k = \sum A_k DD^k = \sum A_k D^{k+1} = \sum A_k D^k D = (\sum A_k D^k)D = \mathcal{L}D.$$

An identical argument shows that D^j and $B_j D^j$ will commute with $\mathcal{L}$. Now, suppose that the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$ both commute with $\mathcal{L}$ then the following argument shows that the sum, $\mathcal{L}_1 + \mathcal{L}_2$, will commute with $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$(\mathcal{L}_1 + \mathcal{L}_2)\mathcal{L} = \mathcal{L}_1\mathcal{L} + \mathcal{L}_2\mathcal{L} = \mathcal{L}\mathcal{L}_1 + \mathcal{L}\mathcal{L}_2 = \mathcal{L}(\mathcal{L}_1 + \mathcal{L}_2).$$

Repeating the above argument shows that if $\mathcal{L}_j,\ j = 1, 2, \ldots, n$, commutes with $\mathcal{L}$ then $\sum_{j=1}^n \mathcal{L}_j$ commutes with $\mathcal{L}$. Letting $\mathcal{L}_j = B_j D^j$ completes the proof.

The Fundamental Theorem of Algebra
(Applied to linear differential operators with constant coefficients)

Suppose $\mathcal{L} = \sum_{k=0}^n A_k D^k$ is a linear differential operator with constant coefficients $A_k \in \mathbb{R}$. We can algebraically manipulate $\sum_{k=0}^n A_k D^k$ like it is an n^{th} degree polynomial and so, by the Fundamental Theorem of Algebra, uniquely factor it over the complex numbers $\mathbb{C}$:

$$\sum_{k=0}^{n} A_k D^k = A_n \overbrace{(D - r_1)^{n_1} \cdots (D - r_i)^{n_i} \cdots (D - r_m)^{n_m}}^{\text{remember this is composition!}}$$

where $n = n_1 + \cdots + n_i + \cdots n_m$. The power n_i is called the multiplicity of the root r_i. If r_i is complex and has multiplicity n_i then there will be a root, say r_j, with r_j being the complex conjugate [7] of r_i and having the same multiplicity r_i. The theorem on page 25 (that we just proved) assures us that the factors [8] $(D - r_i)$, commute with each w.r.t. composition, in the same way that the binomials $(x - r_i)$ commute with each other w.r.t. multiplication. There is a technical way to express this, but we don't need that level of abstraction.

Example 1. Suppose $\mathcal{L} = D^2 + 5D + 6$, so that

$$\mathcal{L}[y] = (D^2 + 5D + 6)[y] = D^2[y] + 5D[y] + 6[y] = y'' + 5y' + 6y.$$

Factoring $\mathcal{L}$ yields

$$\mathcal{L} = D^2 + 5D + 6 = \overbrace{(D+3)(D+2)}^{\text{composition}} = \overbrace{(D+2)(D+3)}^{\text{composition}}$$

So $\mathcal{L}$, as a **polynomial** [9], has distinct real roots -3 and -2. In other words, if D were a variable (and it isn't! $D = \frac{d}{dx}$) then $D = -3,\ D = -2$ would be the solutions of the quadratic equation $D^2 + 5D + 6 = 0$. We make use of this in the next example.

Example 2. Suppose $\mathcal{L} = D^2 + 5D + 6$, like in the previous example, (Example 1.). Then

$$\mathcal{L}[y] = 0$$

---

[7] **complex conjugate** Let $a, b \in \mathbb{R}$ and $i^2 = -1$ so that $z = a + bi \in \mathbb{C}$. The complex conjugate of z, denoted $\overline{z}$, is $a - bi$. I.e.

$$\overline{a + bi} = a - bi.$$

Note: $(a + bi)\overline{(a + bi)} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$
$(a + bi) + \overline{(a + bi)} = (a + bi) + (a - bi) = 2a$
$(a + bi) - \overline{(a + bi)} = (a + bi) - (a - bi) = 2bi$

[8] Each factor $(D - r_i)$ is, of course, a linear differential operators with constant coefficients, as is each term $(D - r_i)^{n_i}$.

[9] $\mathcal{L}$ is not a polynomial in the usual sense, it is a linear differential operator with constant coefficients. But, if we pretend that D is a variable (and not $\frac{d}{dx}$) and we pretend that composition is multiplication, then the resulting object is a polynomial in D. As a result, such operators are sometimes called polynomial differential operators.

is the DE

$$y'' + 5y' + 6y = 0.$$

We know that

$$\mathcal{L}[y] = (D+2) \circ (D+3)[y] = (D+3) \circ (D+2)[y]$$

We can use this to solve the DE as follows. Suppose

$$\begin{aligned} y_1 &\text{ satisfies } (D+2)[y_1] = 0 \\ y_2 &\text{ satisfies } (D+3)[y_2] = 0 \end{aligned}$$

then

$$\begin{aligned} \mathcal{L}[y_1] &= (D+3) \circ (D+2)[y_1] = (D+3)[0] = 0 \quad \text{because}(D+2)[y_1] = 0 \\ \mathcal{L}[y_2] &= (D+2) \circ (D+3)[y_2] = (D+2)[0] = 0 \quad \text{because}(D+3)[y_2] = 0 \end{aligned}$$

But, then by the linearity of the operator $\mathcal{L}$, any linear combination of y_1, y_2:

$$y = c_1 y_1 + c_2 y_2$$

will be a solution of the DE $\mathcal{L}[y] = 0$ because:

$$\mathcal{L}[c_1 y_1 + c_2 y_2] = c_1 \mathcal{L}[y_1] + c_2 \mathcal{L}[y_2] = 0 + 0 = 0.$$

So, which functions are y_1 and y_2? Well,

$$\begin{aligned} (D+2)[y_1] &= 0 \;\Rightarrow \\ D[y_1] + 2[y_1] &= 0 \\ y_1' + 2y_1 &= 0 \\ y_1' &= -2y_1 \\ \frac{dy_1}{dx} &= -2y_1 \\ \frac{dy_1}{y_1} &= -2\,dx \\ \int \frac{dy_1}{y_1} &= -2 \int dx + C_0 \\ \ln|y_1| &= -2x + C_a \\ y_1 &= C_1 e^{-2x}. \end{aligned}$$

Similarly,

$$y_2 = C_2 e^{-3x}.$$

So the general solution of $y'' + 5y' + 6y = 0$ is $y = c_1 y_1 + c_2 y_2$, which is:

$$y = c_1 C_1 e^{-2x} + c_2 C_2 e^{-3x}$$

of course, we should just let $c_1 C_1$ be called c_1, and let $c_2 C_2$ be called c_2. With this simplification the general solution becomes

$$y = c_1 e^{-2x} + c_2 e^{-3x}.$$

Homogenous linear differential equation: If a DE can be put in the form $\mathcal{L}[y] = 0$ where $\mathcal{L}$ is a linear operator, we call the DE a **homogenous linear differential equation**. The homogenous refers to it being equal to zero.

This usage of homogenous doesn't have anything to do with our earlier definition of homogenous, where we said that a DE's was homogenous if it could be put in the form $y' = F(y/x)$. Mathematicians like the word homogenous, so they use it in many different contexts.

Non-repeating real roots. If a DE can be put into the form $\mathcal{L}[y] = 0$ where $\mathcal{L}$ is a linear differential operator with constant coefficients and if we can "factor" $\mathcal{L}$ as

$$\mathcal{L} = (D - r_1)(D - r_2)\cdots(D - r_n)$$

with none of the roots $r_1, r_2, \ldots, r_n$, repeating, and all of the roots real (i.e. no complex roots) then the general solution will be

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \cdots + c_n e^{r_n x}$$

. Using this technique, the following DE's can be quickly solved.:

Example 1a. Find the general solution of $y'' + 8y' + 12y = 0$.
Answer: Write $y'' + 8y' + 12y = 0$ as $(D^2 + 8D + 12)[y] = 0$.
Then, since $D^2 + 8D + 12 = (D + 2)(D + 6)$, the general solution is $y = c_1 e^{-2x} + c_2 e^{-6x}$.
Example 1b. Find the general solution of $y'' - 10y' + 16y = 0$.
Answer: Write $y'' - 10y' + 16y = 0$ as $(D^2 - 10D + 16)[y] = 0$.
Then, since $D^2 - 10D + 16 = (D - 2)(D - 8)$, the general solution is $y = c_1 e^{2x} + c_2 e^{8x}$.
Example 1c. Find the general solution of $y'' - 10y' - 24y = 0$.
Answer: Write $y'' - 10y' - 24y = 0$ as $(D^2 - 10D - 24)[y] = 0$.
Then, since $D^2 - 10D - 24 = (D + 2)(D - 12)$, the general solution is $y = c_1 e^{-2x} + c_2 e^{12x}$.

Tricky Example 1 Find the general solution of $(y' - 1)(y' + 1) = 0$.
Correct Answer: Clearly $y' = \pm 1$. If $y' = 1$ then $y = x + C_1$. If $y' = -1$ then $y = -x + C$.
So even if we are given an Initial Condition $y(x_0) = y$ for this first degree ODE we won't have a unique particular solution (unless we are also told some information about y', i.e. basically whether y' is $+1$ or -1. This lack of uniqueness is not surprising because of the $(y')^2$ term, i.e. $(y' - 1)(y' + 1) = (y')^2 - 1 = 0$. This is similar to the equation from elementary algebra $x^2 = 25$ which doesn't have a unique solution as $x = \pm 5$.
Wrong Answer: Do not solve $(y' - 1)(y' + 1) = 0$ by writing:
$(D - 1)(D + 1) = 0$ and then claiming the solution is $y = c_1 e^x + c_2 e^{-x}$. This would be wrong! Why? First of all, $y' - 1 = 0$ does not correspond to $(D - 1)[y] = 0$, it corresponds to $D[y] - 1 = 0$. Moreover: $(D - 1) \circ (D + 1)[y] = D \circ (D + 1)[y] - 1 \circ (D + 1)[y] = (D^2 - 1)[y] = y'' - y$, which is different than the DE in the problem!

Tricky Example 2 Find the general solution of $(y' - y)(y' + y) = 0$.
Correct Answer: $y' - y = 0$ can be written as $(D - 1)[y] = 0$. As a polynomial the operator $(D - 1)$ has a single root, $r = +1$, so $(D - 1)[y] = 0$ has as its solution $y_1 = y_1(0)e^x$. Similarly, $y' + y = 0$ can be written as $(D + 1)[y] = 0$ and its solution is $y_2 = y_2(0)e^{-x}$. So the solution is y_1 or y_2 and we need some information about y' to determine whether y_1 or y_2 is the solution.
Wrong Answer: Do not solve $(y' - y)(y' + y) = 0$ by writing:
$(D - 1)(D + 1) = 0$ and then claiming that the solution is $y = c_1 e^x + c_2 e^{-x}$. This would be wrong! Why? Even though $y' - y = 0$ can be written as $(D - 1)[y]$ and $y' + y = 0$ can be written as $(D + 1)[y]$. It does not follow that $(y' - y)(y' + y) = 0$ can be written as $(D - 1)(D + 1)[y]$, why? Because $(D - 1)(D + 1)[y]$ is really $(D - 1) \circ (D + 1)[y]$. If we want to write $(y' - y)(y' + y) = 0$ using operator notation, it would be something like $(D - 1)[y] \cdot (D + 1)[y]$, with $\cdot$ meaning multiplication. Also, $(D - 1) \circ (D + 1)[y] = (D^2 - 1)[y] = y'' - y$, which is different than $(y' - y)(y' + y) = (y')^2 - y^2 = 0$.

Repeated roots. Consider the DE

$$(D - r)^n[y] = 0.$$

So the root r has multiplicity n. If $n > 1$, we say the root r is repeated. The trick to solve this kind of DE is to guess that the solution will be something like $y = x^k e^{rx}$ and then to plug y into the DE and see what happens. So we need to calculate $(D-r)^n[x^k e^{rx}]$. Let's start off with $n = 1$.

$$\begin{aligned}(D-r)^1[x^k e^{rx}] = (D-r)[x^k e^{rx}] &= D[x^k e^{rx}] - r[x^k e^{rx}] \\ &= D[x^k]e^{rx} + x^k D[e^{rx}] - rx^k e^{rx} \\ &= kx^{k-1}e^{rx} + \underbrace{x^k re^{rx} - rx^k e^{rx}}_{=0} \\ &= kx^{k-1}e^{rx}.\end{aligned}$$

So

$$(D-r)^2[x^k e^{rx}] = k(k-1)x^{k-2}e^{rx}$$

and finally:

$$(D-r)^n[x^k e^{rx}] = k(k-1)\cdots(k-n+1)x^{k-n}e^{rx}$$

The solution $y = x^k e^{rx}$ must satisfy $(D-r)^n[y] = 0$, and this will happen provided

$$k(k-1)\cdots(k-n+1) = 0$$

which happens if and only if one of the terms $k, (k-1), \ldots, (k-n+1)$ is 0. This in turn happens if and only if $k = 0, 1, 2, \ldots n-1$. So, the differential equation $(D-r)^n[y] = 0$ has the n (linearly independent [10]) solutions:

$$x^0 e^{rx},\ xe^{rx},\ x^2 e^{rx},\ \ldots, x^{n-1}e^{rx}.$$

So, by the linearity of the linear differential operator $(D-r)^n$ the differential equation

$$(D-r)^n[y] = 0$$

will have as its general solution:

$$y = c_0 e^{rx} + c_1 xe^{rx} + c_2 x^2 e^{rx} + c_3 x^3 e^{rx} + \cdots + c_{n-1}x^{n-1}e^{rx}.$$

Notice that if $n = 1$ this solution corresponds to $y = y(0)e^{rx}$ with $c_0 = y(0)$. Also, see [11] for what happens when r is a complex number.

Example 1. Find the general solution of $y'' + 6y' + 9y = 0$.

Answer. We can write this DE as $(D^2 + 6D + 9)[y] = 0$, which factors as $(D+3)^2[y] = 0$. So the general solution is

$$y = c_0 e^{-3x} + c_1 xe^{-3x}.$$

Example 2. Find the general solution of $y''' + 6y'' + 9y' = 0$.

Answer. We can write this DE as $(D^3 + 6D^2 + 9D)[y] = 0$, which factors as $D(D+3)^2[y] = 0$, which we can write as $(D-0)(D+3)^2 = 0$. The solution of $(D-0)[y] = 0$ is $y = ce^{0x} = c$, which makes sense

[10] Linear independent means that none of these n solutions is a linear combination of the other $n-1$ solutions.

[11] If a homogenous linear DE has real coefficients, and its associated "polynomial" linear differential operator has a complex root $r = a + bi$ of multiplicity n then its conjugate $\overline{r} = a - bi$ will also be a root of multiplicity n. For simplicity, let's assume that the only roots of the operator are r and its conjugate $\overline{r}$, and that they have multiplicity n. It turns out the general solution will be

$$e^{ax}((c_0 \cos bx + d_0 \sin bx) + x(c_1 \cos bx + d_1 \sin bx) + \cdots + x^{n-1}(c_{n-1}\cos bx + d_{n-1}\sin bx)),$$

where $c_0, c_1, \ldots, c_{n-1}$ and $d_0, d_1, \ldots, d_{n-1}$ are real constants to be determined from the initial conditions. We will address the case of complex roots in more detail below.

since $(D-0)[y]=0$ is just $y'=0$, so of course $y=$ constant. So the general solution consists of all linear combinations of c, e^{-3x}, xe^{-3x}. So the general solution is

$$y = c + c_0 e^{-3x} + c_1 x e^{-3x}.$$

Complex roots

Consider the DE $(D^2+1)[y]=0$. This factors as $(D-i)(D+i)[y]=0$ where $i=\sqrt{-1}$. So the solution of $(D^2+1)[y]=0$ should be

$$y = c_1 e^{ix} + c_2 e^{-ix}.$$

However, what does e^{ix} mean?

The all important Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The proof of this formula follows from looking at the Taylor series for e^x, $\cos x$, and $\sin x$ and plugging in $i\theta$ for x and using the fact that: $i^{4k}=+1; \quad i^{4k+1}=i; \quad i^{4k+2}=-1; \quad i^{4k+3}=-i$, with $k=0,1,\ldots$ Here are the Taylor series, and the calculations:

$$\begin{aligned}
e^x &= 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5! + \ldots \\
\cos x &= 1 - x^2/2! + x^4/4! - \cdots \\
\sin x &= x - x^3/3! + x^5/5! - \cdots \\
e^{i\theta} &= 1 + i\theta + (i\theta)^2/2! + (i\theta)^3/3! + (i\theta)^4/4! + (i\theta)^5/5! + \cdots \\
&= 1 + i\theta - \theta^2/2! - i\theta^3/3! + \theta^4/4! + i\theta^5/5! = \cdots \\
&= \underbrace{(1 - \theta^2/2! + \theta^4/4! - \cdots)}_{\cos\theta} + i\underbrace{(\theta - \theta^3/3! + \theta^5/5! - \cdots)}_{\sin\theta}
\end{aligned}$$

So, using Euler's formula, we see that

$$e^{ix} = \cos x + i\sin x.$$

So, since e^{ix} is a solution to the DE, we have, by linearity,

$$(D^2+1)[e^{ix}] = (D^2+1)[\cos x + i\sin x] = (D^2+1)[\cos x] + i(D^2+1)[\sin x] = 0.$$

Now, if any complex number $a+bi=0$ it must be the case that both $a=0$ and $b=0$. So, both $(D^2+1)[\cos x] = 0$ and $(D^2+1)[\sin x]=0$. So we have two linearly independent solutions; $\cos x$ and $\sin x$. So the general solution is $y = c_1\cos x + c_2\sin x$. If we would have used e^{-ix} we would have gotten the same result.

Complex roots. General case: If we have the DE:

$$\mathcal{L}[y] = (D-r)(D-\bar{r})[y] = 0$$

where $r=a+bi$ and $\bar{r}=a-bi$, see [12], this DE's general solution will be:

$$y = c_1 e^{rx} + c_3 e^{\bar{r}x} = c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x}.$$

[12] $\mathcal{L}[y] = (D^2 - 2aD + (a^2+b^2))[y] = 0$

This works out to be:

$$\begin{aligned}
y &= c_1 e^{(a+bi)x} + c_2 e^{(a-bi)x} \\
&= c_1 e^{ax} e^{ibx} + c_2 e^{ax} e^{-ibx} \\
&= c_1 e^{ax} \underbrace{(\cos(bx) + i\sin(bx))}_{e^{ibx}} + c_2 e^{ax} \underbrace{(\cos(-bx) + i\sin(-bx))}_{e^{-ibx}} \\
&= c_1 e^{ax}(\cos bx + i\sin bx) + c_2 e^{ax}(\cos bx - i\sin bx) \\
&= e^{ax}(c_1(\cos bx + i\sin bx) + c_2(\cos bx - i\sin bx)) \\
&= e^{ax}((c_1 + c_2)\cos bx + i(c_1 - c_2)\sin bx) \\
&= e^{ax}(A\cos bx + iB\sin bx), \quad \text{where } (c_1 + c_2) = A;\ (c_1 - c_2) = B
\end{aligned}$$

Since $y = e^{ax}(A\cos bx + iB\sin bx)$ is a solution to $\mathcal{L}[y] = 0$ we have, by the linearity of $\mathcal{L}$:

$$\begin{aligned}
\mathcal{L}[Ae^{ax}\cos bx + iBe^{ax}\sin bx] &= \mathcal{L}[Ae^{ax}\cos bx] + i\mathcal{L}[Be^{ax}\sin bx] = 0 \\
&= A\mathcal{L}[e^{ax}\cos bx] + iB\mathcal{L}[e^{ax}\sin bx] = 0.
\end{aligned}$$

Now, if any complex number $a+bi = 0$ it must be the case that both $a = 0$ and $b = 0$. So, both $\mathcal{L}[Ae^{ax}\cos bx] = 0$ and $\mathcal{L}[Be^{ax}\sin bx] = 0$, which means that $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$ are solutions of $\mathcal{L}[y] = 0$. By the linearity of $\mathcal{L}$ any linear combination of y_1, y_2 will be a solution of $\mathcal{L}[y] = 0$. Hence the general solution is:

$$y = e^{ax}(c_1\cos bx + c_2\sin bx)$$

where c_1 and c_2 are constants (perhaps different than the previous c_1, c_2) to be determined from the initial conditions.

Example 1. Solve $y'' + 4y = 0$.

Answer Write the DE as $(D^2 + 4)[y] = 0$. This factors as $(D - 2i)(D + 2i)[y] = 0$. So the roots are the complex conjugates $a + bi = 0 + 2i$ and $a - bi = 0 - 2i$. The general solution is $y = e^{ax}(c_1\cos bx + c_2\sin bx)$ which for this DE works out to be:

$$y = c_1\cos 2x + c_2\sin 2x,$$

since $e^{ax} = e^0 = 1$.

Example 2. Solve $y'' - 4y' + 13y = 0$.

Answer Write the DE as $(D^2 - 4D + 13)[y] = 0$. Using the quadratic formula (see below) we see that the roots are the complex conjugates $a + bi = 2 + 3i$ and $a - bi = 2 - 3i$. The general solution is $y = e^{ax}(c_1\cos bx + c_2\sin bx)$ which for this DE works out to be:

$$y = e^{2x}(c_1\cos 3x + c_2\sin 3x).$$

To find the roots of $D^2 - 4D + 13$ we apply the quadratic formula which says if $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So plugging in $a = 1,\ b = -4$, and $c = 13$ we get the roots of $D^2 - 4D + 13$:

$$\frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{16 - 52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

Homework 10: In Edwards and Penney 4^{th} Ed:
Read Section 3.3. Quickly skim through sections 3.1 and 3.2.
Do questions: 1, 2, 3, 5, 7, 8, 9, 10 on page 183 (at the end of Section 3.3).

Solutions to HW 10

Homework 10: Edwards and Penney 4^{th} Ed: Page 183 (End of §3.3) 1, 2, 3, 5, 7, 8, 9, 10

#1 p183. Find the general solution of the DE: $y'' - 4y = 0$.

Solution: Write $y'' - 4y = 0$ as $(D^2 - 4)[y] = 0$, which factors as $(D-2)(D+2)[y]$, and so has roots $r = 2, -2$. So the general solution is $y = c_1 e^{2x} + c_2 e^{-2x}$.

#2 p183. Find the general solution of the DE: $2y'' - 3y' = 0$.

Solution: Write $2y'' - 3y' = 0$ as $(2D^2 - 3D)[y] = 0$, which factors as $D(2D-3)[y]$, and so has roots $r = 0, \frac{3}{2}$. So the general solution is $y = c_1 e^{0x} + c_2 e^{\frac{3}{2}x} = c_1 + c_2 e^{\frac{3}{2}x}$.

#3 p183. Find the general solution of the DE: $y'' + 3y' - 10y = 0$.

Solution: Write $y'' + 3y' - 10y = 0$ as $(D^2 + 3D = 10)[y] = 0$, which factors as $(D+5)(D-2)[y]$, and so has roots $r = -5, 2$. So the general solution is $y = c_1 e^{-5x} + c_2 e^{2x}$.

#5 p183. Find the general solution of the DE: $y'' + 6y' + 9y = 0$.

Solution: Write $y'' + 6y' + 9y = 0$ as $(D^2 + 6D + 9)[y] = 0$, which factors as $(D+3)^2[y]$, and so has repeated root $r = -3$ with multiplicity $n = 2$. So the general solution is $y = c_0 e^{-3x} + c_1 x e^{-3x}$.

#7 p183. Find the general solution of the DE: $4y'' - 12y' + 9y = 0$.

Solution: Write $4y'' - 12y' + 9y = 0$ as $(4D^2 - 12D + 9)[y] = 0$. Applying the quadratic formula we get $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} = \frac{12 \pm \sqrt{144 - 144}}{8} = \frac{3}{2}$. So $r = \frac{3}{2}$ is a repeated root with multiplicity $n = 2$. So the general solution is $y = c_0 e^{\frac{3}{2}x} + c_1 x e^{\frac{3}{2}x}$.

#8 p183. Find the general solution of the DE: $y'' - 6y' + 13y = 0$.

Solution: Write $y'' - 6y' + 13y = 0$ as $(D^2 - 6D + 13)[y] = 0$. Applying the quadratic formula we get $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(13)}}{2(1)} = \frac{6 \pm \sqrt{36 - 52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$. So the roots $r = 3 \pm 2i = a \pm bi$ are complex conjugates. So the general solution is $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$.

#9 p183. Find the general solution of the DE: $y'' + 8y' + 25y = 0$.

Solution: Write $y'' + 8y' + 25y = 0$ as $(D^2 + 8D + 25)[y] = 0$. Applying the quadratic formula we get $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(8) \pm \sqrt{(8)^2 - 4(1)(25)}}{2(1)} = \frac{-8 \pm \sqrt{64 - 100}}{2} = \frac{-8 \pm \sqrt{-36}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$. So the roots $r = -4 \pm 3i = a \pm bi$ are complex conjugates. So the general solution is $y = e^{-4x}(c_1 \cos 3x + c_2 \sin 3x)$.

#10 p183. Find the general solution of the DE: $5y^{(4)} + 3y^{(3)} = 0$.

Solution: Write $5y^{(4)} + 3y^{(3)} = 0$ as $(5D^4 + 3D^2)[y] = 0$, which factors as $D^3(5D+3)[y]$, and so has roots $r = 0, -\frac{3}{5}$. The repeated root $r = 0$ has multiplicity $n = 3$. The root $r = -\frac{3}{5}$ is not repeated. So the general solution is $y = c_0 e^{0x} + c_1 x e^{0x} + c_2 x^2 e^{0x} + d e^{-\frac{3}{5}x} = c_0 + c_1 x + c_2 x^2 + d e^{-\frac{3}{5}x}$.

Linear Independent Solutions

An important theorem you need to know, and which is easy to remember is:

A homogenous linear differential equation of order n will have n linearly independent solutions.

*A set of vectors is **linearly independent** if none of the vectors in the set can be written as a linear combination of the $n-1$ other vectors in that set.*

Remember, that in differential equations, the vectors are (sufficiently) differentiable functions, some of which will be the solutions to the differential equations that we want to solve.

Example. It is easy to see that the functions (vectors) $y_1 = x$ and $y_2 = x^2$ are linearly independent:

Suppose they are not. Suppose that the set $\{x, x^2\}$ is not linearly independent. Then we can find a real number c such that $x^2 = cx$. But then $x^2 - cx = 0$. This implies $x(x-c) = 0$, i.e. $x = 0$ or $x = c$. However c is constant and x is a variable, which is a contradiction. So, we're done.

Text book note: In Sections 3.1 and 3.2 the text book discusses many important theorems relating to the linear independence of solutions.

The following 5^{th} order linear DE has, as predicted, 5 linearly independent solutions:

Example 1. Find the general solution of the following 5^{th} order, homogenous, linear DE with constant coefficients :

$$5y^{(5)} + 3y^{(3)} = 0$$

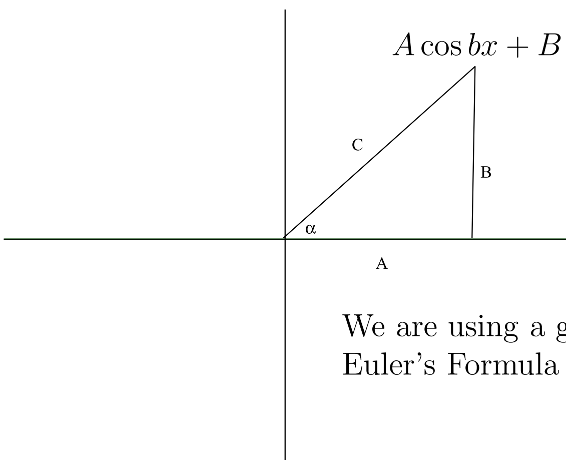
.

Solution: Write $5y^{(5)}+3y^{(3)} = 0$ as $(5D^5+3D^3)[y] = 0$, which factors as $D^3(5D^2+3)[y]$, and so has roots $r = 0,\ \pm i\sqrt{\frac{3}{5}}$. The repeated root $r = 0$ has multiplicity $n = 3$. The complex roots $r = 0 \pm i\sqrt{\frac{3}{5}} = a \pm bi$ are conjugates of each other and purely imaginary (purely imaginary means the real part $a = 0$). So the general solution is $y = c_0e^{0x}+c_1xe^{0x}+c_2x^2e^{0x}+e^{0x}(d_1\cos\sqrt{\frac{3}{5}}x+d_2\sin\sqrt{\frac{3}{5}}x) = c_0+c_1x+c_2x^2+d_1\cos\sqrt{\frac{3}{5}}x+d_2\sin\sqrt{\frac{3}{5}}x$.

So, the DE: $5y^{(5)} + 3y^{(3)} = 0$ has the 5 linearly independent solutions:

$$y_1 = e^{0x} = 1; \quad y_2 = x; \quad y_3 = x^2, \quad y_4 = \cos\sqrt{\frac{3}{5}}\,x; \quad y_5 = \sin\sqrt{\frac{3}{5}}\,x.$$

Alternative form of $A\cos bx + B\sin bx$.



$$\begin{aligned} A\cos bx + B\sin bx &= \frac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2}}(A\cos bx + B\sin bx) \\ &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos bx + \frac{B}{\sqrt{A^2+B^2}}\sin bx\right) \\ &= \sqrt{A^2+B^2}\,(\cos\alpha\cos bx + \sin\alpha\sin bx) \\ &= \sqrt{A^2+B^2}\cos(bx-\alpha) \end{aligned}$$

We are using a general Trig Identity, which we can derive easily from Euler's Formula and the identities $i^2 = -1;\ \ \cos(-\theta) = \cos\theta;\ \ \sin(-\theta) = -\sin\theta$:

$$\begin{aligned} e^{-i\alpha}e^{ibx} &= e^{i(bx-\alpha)} \\ (\cos\alpha - i\sin\alpha)(\cos bx + i\sin bx) &= \cos(bx-\alpha) + i\sin(bx-\alpha) \\ (\cos\alpha\cos bx + \sin\alpha\sin bx) + i(\cos\alpha\sin bx - \sin\alpha\cos bx) &= \cos(bx-\alpha) + i\sin(bx-\alpha) \end{aligned}$$

Based on Example 1. p 189 of Edwards and Penney 4^{th} Ed:

Solve the IVP:

$$y'' + 100y = 0 \qquad\qquad y(0) = 1\ \text{ m} \qquad y'(0) = -5\ \text{ m/s}\ .$$

Answer: We write the DE as $(D^2 + 100)[y] = 0$ which has roots $\pm 10i$. So the general solution is

$$y = A\cos 10\ \text{rad/sec}\ x + B\sin 10\ \text{rad/sec}\ x.$$

Note: m = meters; m/s = meters/second; x is time and measured in seconds. To find the constants A and B:

$y(0) = 1 = A\cos 0 + B\sin 0 = A + 0$ implies $A = 1$ m .

$y' = -10A\sin 10x + 10B\cos 10x$ so $y'(0) = -5 = 10B$, which implies $B = -1/2$ m .

To find the phase angle α, I think it is easiest to work in the first quadrant, by finding the first quadrant angle defined by $(|A|, |B|)$. and then to use that to find α:

$$\arctan\frac{|B|}{|A|} = \arctan\frac{|-1/2|}{|1|} = \arctan .5 = 0.4636476090 \text{ rad.}$$

But then $\alpha = 2\pi - 0.4636476090 = 6.283185308 - 0.4636476090 = 5.819537699$ radians.

The amplitude $\sqrt{A^2+B^2} = \sqrt{1^2+.5^2} = \sqrt{1.25} = 1.118033989$ m.

So the particular solution, $y = \sqrt{A^2+B^2}\,\cos(bx-\alpha)$ becomes:

$$y = 1.118033989\cos(10x \text{ rad/sec } - 5.819537699 \text{ rad }) \text{ m .}$$

Notation:

$\omega =$ the angular (or circular) frequency, measured in radians/time, (often rad/sec).

$\nu = \omega\frac{\text{cycles}}{2\pi\text{ rad}} =$ the frequency, measured in cycles/time (often measured in Hertz (Hz) where 1 Hz = 1 cycle/sec).

$T = \frac{1}{\nu} =$ the period, measured in units of time/cycle, (often sec/cycle).

Usually the letter A denotes amplitude, however in this example the amplitude is $\sqrt{A^2+B^2}$.

Often the solution to this sort of DE will be given in the form:

$$y = A\cos(2\pi ft + \phi) \quad \text{or} \quad y = A\cos(2\pi\nu t + \phi),$$

where $A =$ amplitude; $2\pi = 2\pi$ radians/cycle; $f = \nu =$ cycles/time; $t =$ time; and $\phi =$ a phase angle $(= -\alpha)$, measured in radians.

So, in this example:

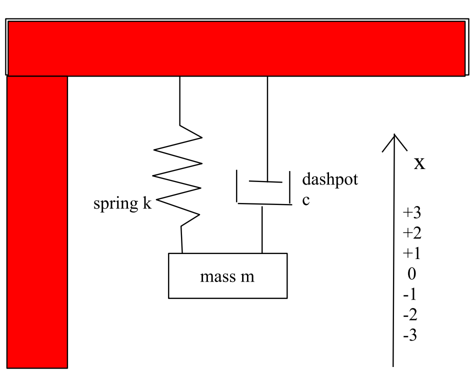
$\omega =$ angular frequency $= 10$ rad/sec.

$\nu = \omega\frac{\text{cycles}}{2\pi\text{ rad}} = 10\frac{\text{rad}}{\text{sec}}\frac{\text{cycles}}{2\pi\text{ rad}} = \frac{10}{2\pi}\frac{\text{cycles}}{\text{sec}} = 1.591549430$ Hz $=$ frequency.

$T = \frac{1}{\nu} = 0.6283185311$ sec = the period.

The DE $ay'' + by' + cy = 0$ is extraordinarily important in engineering and physics as it serves as a model for many different phenomena. See **Section 3.4** in Edwards & Penney 4^{th} Ed.

Perhaps, most importantly is the **harmonic oscillator**, which can be realized as a mass-spring-dashpot system. Imagine one end of a metal spring attached to the ceiling and the other to a weight.



The weight of the mass hanging on the spring forces the spring to extend until the internal forces in the spring balance the gravitational forces pulling the spring downward (towards the negative x direction). We'll locate our vertical number line so that $x = 0$ is at this equilibrium point where the force of gravity pulling downward is exactly counter balanced by the internal electrical forces within the spring.

Now, suppose the mass height is at equilibrium (i.e. at $x = 0$). If we push the mass up, or pull it down, the internal forces of the spring in combination with gravity will work in such a way that the mass will feel a restorative force K forcing the spring back to its equilibrium position (back towards $x = 0$). We can imagine this restorative force being a function of the position of the mass, so $K = K(x)$ since x denotes the position of the mass. Then we write $K(x)$ as a Taylor series: $K(x) = k_0 + k_1x + k_2x^2 + \ldots$ and we try to figure what the coefficients, $k_0, k_1, \ldots$ could be.

At equilibrium (at $x = 0$), the restorative force is 0, so we must have $k_0 = 0$ (because if we substitute $x = 0$ into $K(x) = k_0 + k_1 x + k_2 x^2 + \ldots$ we get $K(0) = k_0$ and since $K(0) = 0$, we must have $k_0 = 0$). When $x > 0$ the restorative force K points down (so $K($ positive $x) < 0$); when $x < 0$ the restorative force points up (so $K($ negative $x) > 0$), so $K(x)$ is oppositely signed to x, which means $k_1 < 0$. Why? Because when $|x|$ is close to zero, the $k_1 x$ term dominates the Taylor series since $k_0 = 0$. We don't bother to analyze the higher order coefficients, $k_2, k_3, \ldots$, for that same reason, i.e. when $|x|$ is small, the $k_1 x$ term dominates the Taylor series. So, for small displacements of the mass, we can approximate the restorative force by its first non-zero Taylor series term, i.e. $K(x) \approx -kx$ where we are letting $k = |k_1|$. This restorative force is traditional represented by the letter k, which in this example would be called the **spring constant**.

Now, as the spring and weight move through space they will encounter resistance to this motion due to internal and external frictional forces (represented by a dashpot). E.g. as the metal in the spring bends heat is given off; as the mass (or the system) moves through space it collides with atoms slowing it down. These frictional type forces act in the direction opposite to the direction the mass is traveling. If the mass is moving upwards, this frictional force will point downwards, and vice a versa. Moreover, the faster the system moves, the greater the frictional force [13]. We lump all the various frictional type forces into a single force called a damping force [14]. A similar argument, as given for the restorative force being $\approx -kx$, will indicate that the damping force $\approx -c\dot{x}$ with $c \geq 0$. In this example, c would be called the **damping constant**.

From Newton we know that $F = ma$ (Force = mass $\times$ acceleration). So $ma = F = F_r + F_d$ becomes $m\ddot{x} = -kx - c\dot{x}$, which we can write as $m\ddot{x} + c\dot{x} + kx = 0$ or as $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$. We solve this, as usual, by writing the DE in differential operator notation: as $(mD^2 + cD + k)[x] = 0$ or as $\left(D^2 + \frac{c}{m}D + \frac{k}{m}\right)[x] = 0$. Either way, after applying the quadratic formula we see that the roots r will be

$$r = \frac{-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}}}{2} = -\left(\frac{c}{2m}\right) \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)} = -p \pm \sqrt{p^2 - \omega_0^2},$$

where we are letting

$p = \frac{c}{2m}$ = the **damping factor**; and

$\omega_0 = \sqrt{\frac{k}{m}}$ = the system's (natural) **undamped angular frequency**. ω is pronounced omega. If we let

$\omega_1 = \sqrt{p^2 - \omega_0^2} = \omega_0 \sqrt{\left(\frac{p}{w_0}\right)^2 - 1} = \omega_0\sqrt{\zeta^2 - 1}$, where

$\zeta = \frac{p}{w_0}$ = the **damping ratio** (ζ is pronounced zeta), then we can write the roots of the DE, in very compact form as $r = -p \pm w_1$.

The solution to the harmonic oscillator problem will depend on whether ζ is $>$, $=$, or < 1.

Over damped case: $\zeta > 1$

$\zeta > 1$ implies that $\omega_1 = \omega_0\sqrt{\zeta^2 - 1}$ is a positive real number, so both roots will be distinct real numbers. $\zeta > 1$ also implies that both roots are negative. The root $r_- = -p - \omega_1$ is negative since both p and ω_1 are positive. The root $r_+ = -p + \omega_1$ is negative since we have $0 < \omega_1 = \sqrt{p^2 - \omega_0^2} < \sqrt{p^2} = p$. So in the over damped case the general solution is

$$x(t) = c_1 e^{(-p+\omega_1)t} + c_2 e^{(-p-\omega_1)t}$$

Critically damped case: $\zeta = 1$

$\zeta = 1$ implies that $\omega_1 = \omega_0\sqrt{\zeta^2 - 1} = 0$ so the root $r = -p \pm \omega_1$ is $-p$. This means the root is a repeated (negative) number. So in the critically damped case the general solution is

$$x(t) = (c_1 + c_2 t)e^{-pt}.$$

[13] We are not going to consider sticking type forces which exert a force only when the object wants to start moving. These sticking type forces are why when you start pushing something you often have to give it a little extra push at the start.

[14] One of the main triumphs of classical mechanics was the paradigm shift that came about from viewing constant velocity as the natural state of an object in motion; that objects tended to slow down due to damping forces, rather than because deceleration, and then stopping, was the natural state for objects put into motion.

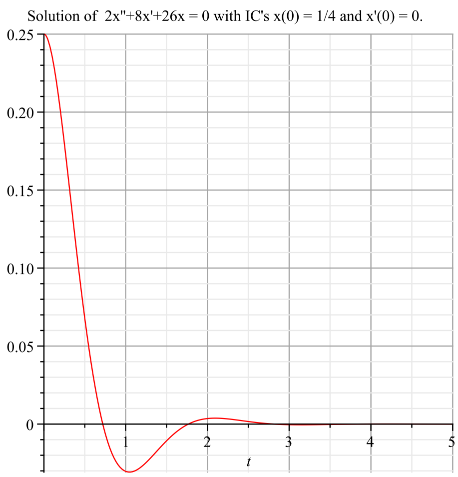
Under damped case: $\zeta < 1$

$\zeta < 1$ implies that $\omega_1 = \omega_0\sqrt{\zeta^2 - 1}$ will be imaginary. So the two roots are complex conjugates of each other. So in the under damped case the general solution is

$$x(t) = e^{-pt}(c_1 \cos\omega_1 t + c_2 \sin\omega_1 t).$$

Under damped example Suppose that a weight (mass) of 2 kg is attached to spring-dashpot system hanging from a ceiling. Suppose the spring constant $k = 26\ N/m$ and that dashpot provides $c = 8\ N/(m/s)$ of resistance (frictional forces). If the weight is pushed upwards 1/4 m above equilibrium and then released, find the equation of motion for the system. m = meters; s = seconds; N = Newton = unit of force = $kg\ m/s^2$.

Answer The DE is $(mD^2 + cD + k)[x] = 0$ which becomes



$$(2D^2 + 8D + 26)[x] = 0.$$

The roots are

$$\frac{-8 \pm \sqrt{64 - 208}}{4} = \frac{-8 \pm \sqrt{-144}}{4} = \frac{-8 \pm 12}{4} = -2 \pm 3i$$

So the general solution is

$$x(t) = (c_1 \cos 3t + c_2 \sin 3t)e^{-2t}.$$

This implies that

$$\dot{x} = (-3c_1 \sin 3t + 3c_2 \cos 3t)e^{-2t} - 2(c_1 \cos 3t + c_2 \sin 3t)e^{-2t}.$$

Using $x(0) = 1/4$ m and $x'(0) = 0$ m/s yields $1/4 = c_1$ (from $x(0) = 1/4$) and $0 = 3c_2 - 2c_1 = 3c_2 - 1/2$ (from $\dot{x}(0) = 0$), which implies $c_2 = 1/6$. Since $\sqrt{c_1^2 + c_2^2} = \sqrt{(1/4)^2 + (1/6)^2} = 0.3004626$ and since $(c_1, c_2) \in$ first quadrant and $\arctan(|c_2|/|c_1|) = \arctan((1/6)/(1/4)) = 0.5880026$ we get the particular solution:

$$x(t) = \left(\frac{1}{4}\cos 3t + \frac{1}{6}\sin 3t\right)\ e^{-2t} \text{ meters } = 0.3004626\ \cos(3t - 0.5880026)\, e^{-2t} \text{ meters.}$$

Non homogenous equations and the method of undetermined coefficients: section 3.5.

To solve the non-homogenous linear ODE with constant coefficients, $\mathcal{L}[y] = f(x)$, e.g. $y' - y = x^2$, what we do is notice that, by the linearity of $\mathcal{L}$, the solution will consist of two parts, the homogenous part, y_h, which solves $\mathcal{L}[y] = 0$ and the a particular part, y_p that solves the non-homogenous $\mathcal{L}[y] = f(x)$. The entire solution will be $y = y_h + y_p$. The trick is to solve the homogenous part as usual, and then to guess at a solution for the non-homogenous part. Basically if the non homogenous part is a polynomial, we'll guess y_p = a polynomial of the same degree. If the non-homogenous part is $\cos ax$ or $\sin bx$ or $c_1 \cos ax + c_2 \sin bx$ then we'll guess that y_p is $A\cos ax + B \sin bx$.

Example: The solution of $y' - y = x^2$ has homogenous part $y_h = e^x$. We'll guess that $y_p = Ax^2 + Bx + C$. Then $y_p' - y_p = x^2$ becomes $(2Ax + B) - (Ax^2 + Bx + C) = x^2$ and hence: $(-A)x^2 + (2A - B)x + (B - C) = x^2$, so that $A = -1$; $(2A - B)x = (2(-1) - B)x = 0$, so $B = -2$; $(B - C) = 0$ so $C = B = -2$. We get

$$y = c_1e^x - x^2 - 2x - 2.$$

We check our solution: $y' - y = x^2$ becomes $(c_1e^x - 2x - 2) - (c_1e^x - x^2 - 2x - 2) = x^2$. It checks!

Homework #11 Do problems # 15 – 20 on page 197 at the end of § 3.4 in Edwards & Penney 4^{th} Ed. Do problems # 1, 2, 6 on page 210 at the end of § 3.5 in Edwards & Penney 4^{th} Ed.

Selected Solutions to HW 11.

Notes for the undamped spring problem: $m,\ c = 0,\ k,\ x_0,\ v_0$.
$m\ddot{x} + k = 0$ Write this in linear operator notation: $(mD^2 + k)[x] = 0$. The roots are $\pm i\sqrt{\frac{k}{m}} = \pm\omega_0$. The general solution is $x(t) = A\cos\omega_0 t + B\sin\omega_0 t$, or even better:

$$\sqrt{A^2 + B^2}\ \cos(\omega_0 t - \alpha).$$

Finding the Particular solution: $\dot{x} = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t$. $x_0 = x(0) = A$ and $v_0 = \dot{x}(0) = B\omega_0 \Rightarrow \frac{v_0}{\omega_0} = B$. So the Amplitude $= \sqrt{A^2 + B^2} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2}$. Phase angle α: $(A, B) = (x_0, v_0/\omega_0)$ and $\arctan\frac{|B|}{|A|} = \arctan\frac{|B|}{|A|}$, then we find $\alpha \in [0, 2\pi)$ depending on the quadrant that (A, B) is in.

So the important calculations are:

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad A = x_0 \qquad B = \frac{v_0}{\omega_0} \qquad \text{Amplitude } = \sqrt{A^2 + B^2} \qquad \arctan\frac{|B|}{|A|}.$$

Some Solutions to HW #11:
Which was: problems # 15 – 20 on page 197 at the end of § 3.4 in Edwards & Penney 4^{th} Ed. and problems # 1, 2, 6 on page 210 at the end of § 3.5 in Edwards & Penney 4^{th} Ed.

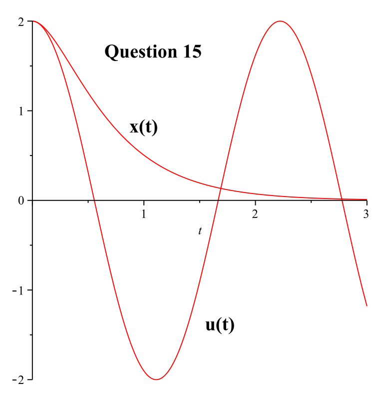
(Page 197 #15) $m = 1/2,\ c = 3,\ k = 4,\ x_0 = 2,\ v_0 = 0$.
$m\ddot{x} + c\dot{x} + k = 0$ becomes $(1/2)\ddot{x} + 3\dot{x} + 4 = 0$. Write this in linear operator notation: $(.5D^2 + 3D + 4)[x] = 0$. Multiply by 2 get: $(D^2 + 6D + 8)[x] = 0$, This factors as $(D + 2)(D + 4)[x] = 0$. The roots are $r = -2, -4$. The general solution is

$$x(t) = c_1 e^{-2t} + c_2 e^{-4t},$$

which is over damped.

Finding the particular solution. First: $\dot{x} = -2c_1 e^{-2t} - 4c_2 e^{-4t}$. Then we use the initial conditions: $x_0 = x(0) = 2 = c_1 + c_2$ and $v_0 = \dot{x}(0) = 0 = -2c_1 - 4c_2$. This implies: $c_1 = -2c_2$ and $2 = (-2c_2) + c_2 = -c_2$. So: $c_2 = -2$ and $c_1 = 4$. Hence the particular solution is

$$x(t) = 4e^{-2t} - 2e^{-4t}.$$



Undamped version of the same problem:
$m = 1/2,\ c = 0,\ k = 4,\ x_0 = 2,\ v_0 = 0$.
$\omega_0 = \sqrt{\frac{4}{1/2}} = \sqrt{8}$, $A = x_0 = 2$, $B = \frac{v_0}{\omega_0} = \frac{0}{\sqrt{8}} = 0$, Amplitude $= \sqrt{A^2 + B^2} = \sqrt{2^2 + 0^2} = \sqrt{4} = 2$, $\arctan\frac{|B|}{|A|} = \arctan\frac{|0|}{|2|} = \arctan\frac{0}{2} = 0$. Since $(A, B) = (2, 0)$, $\alpha = 0$ rad. So the particular solution is $u(t) = 2\cos\sqrt{8}\ t$.

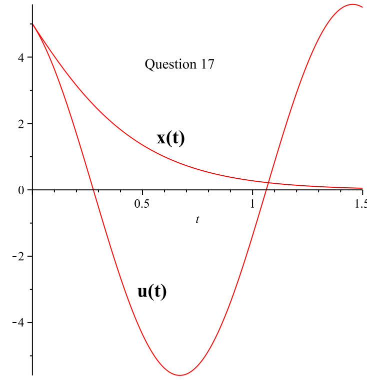
(Page 197 #17) $m = 1,\ c = 8,\ k = 16,\ x_0 = 5,\ v_0 = -10$.
$m\ddot{x} + c\dot{x} + k = 0$ becomes $\ddot{x} + 8\dot{x} + 16 = 0$. Write this in linear operator notation: $(D^2 + 8D + 16)[x] = 0$. This factors as $(D + 4)(D + 4)[x] = 0$. The roots are $r = -4$ with multiplicity 2. The general solution is

$$x(t) = c_0 e^{-4t} + c_1 t e^{-4t},$$

which is critically damped.

Finding the particular solution. First: $\dot{x} = -4c_0e^{-4t}+c_1e^{-4t}(1-4t)$. Then we use the initial conditions: $x_0 = x(0) = 5 = c_0$ and $v_0 = \dot{x}(0) = -10 = -4c_0 + c_1$. This implies: $c_0 = 5$ and $-10 = -4(5) + c_1 = -20 + c_1$. So: $c_0 = 5$ and $c_1 = 10$. Hence the particular solution is

$$x(t) = 5e^{-4t} + 10te^{-4t}.$$



Undamped version of the same problem:

$m = 1,\ c = 0,\ k = 16,\ x_0 = 5,\ v_0 = -10.$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{1}} = 4$, $A = x_0 = 5$, $B = \frac{v_0}{\omega_0} = \frac{-10}{4} = -2.5$, Amplitude $= \sqrt{A^2 + B^2} = \sqrt{5^2 + (-2.5)^2} = \sqrt{31.25} = 5.590169944$, $\arctan\frac{|B|}{|A|} = \arctan\frac{|-2.5|}{|5|} = \arctan\frac{1}{2} = 0.4636476090$ Since $(A, B) = (5, -2.5)$, $\alpha = 2\pi - 0.4636476090 = 5.819537699$ rad. So the particular solution is $u(t) = 5.590169944\cos(4t - 5.819537699)$.

(Page 197 #19) $m = 4,\ c = 20,\ k = 169,\ x_0 = 4,\ v_0 = 16.$

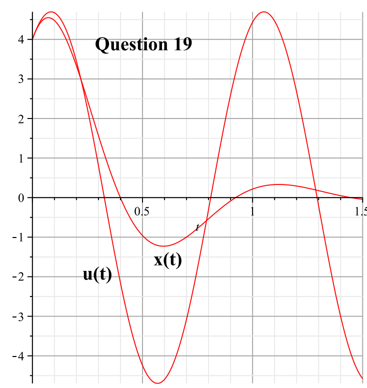
$m\ddot{x}+c\dot{x}+k = 0$ becomes $4\ddot{x}+20\dot{x}+169 = 0$. Write this in linear operator notation: $(4D^2+20D+169)[x] = 0$. Using the quadratic formula $r = \frac{-20\pm\sqrt{400-2704}}{8} = \frac{-20\pm\sqrt{-2304}}{8} = \frac{-20\pm 48i}{8} = -\frac{5}{2} \pm 6i$, which are complex conjugates. The general solution is

$$x(t) = (A\cos 6t + B\sin 6t)e^{-\frac{5}{2}t}.$$

which is under damped.

Finding the particular solution. First:

$$\dot{x} = x(t) = \left((-6A\sin 6t + 6B\cos 6t) - \frac{5}{2}(A\cos 6t + B\sin 6t)\right)e^{-\frac{5}{2}t}.$$



Then we use the initial conditions: $x_0 = x(0) = 4 = A$ and $v_0 = \dot{x}(0) = 16 = 6B - \frac{5}{2}A$. This implies: $16 = 6B - \frac{5}{2}4 = 6B - 10$. So: $A = 4$ and $B = \frac{26}{6} = \frac{13}{3}$. Hence the particular solution is

$$x(t) = (4\cos 6t + \frac{13}{3}\sin 6t)e^{-\frac{5}{2}t}.$$

The amplitude $= \sqrt{A^2 + B^2} = \sqrt{4^2 + \left(\frac{13}{3}\right)^2} = \frac{\sqrt{313}}{3} = 5.897268669$. $\arctan\frac{|13/3|}{|4|} = 0.8253768504$ rad. Since (A,B) = (4, 13/3) is in the first quadrant $\alpha = 0.8253768504$ rad. So we get

$$x(t) = \frac{\sqrt{313}}{3}e^{-\frac{5}{2}t}\cos(6t - 0.8253768504)$$

.

Undamped version of the same problem:

$m = 4,\ c = 0,\ k = 169,\ x_0 = 4,\ v_0 = 16.$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{169}{4}} = \frac{13}{2} = 6.5$, $A = x_0 = 4$, $B = \frac{v_0}{\omega_0} = \frac{16}{\frac{13}{2}} = \frac{32}{13} = 2.461538462$, Amplitude $= \sqrt{A^2 + B^2} = \sqrt{4^2 + (32/13)^2} = (4/13)\sqrt{233} = 4.696719237$, $\arctan\frac{|B|}{|A|} = \arctan\frac{2.461538462}{4} = 0.5516549826$ Since $(A, B) = (4, 2.461538462)$, $\alpha = 0.5516549826$ rad. So the particular solution is

$$u(t) = \frac{4}{13}\sqrt{233}\cos\left(\frac{13}{2}t - 0.5516549826\right).$$

Non homogenous equations and the method of undetermined coefficients: section 3.5.

Reprinted from earlier handout To solve the non-homogenous linear ODE with constant coefficients, $\mathcal{L}[y] = f(x)$, e.g. $y' - y = x^2$, what we do is notice that, by the linearity of $\mathcal{L}$, the solution will consist of two parts, the homogenous part, y_h, which solves $\mathcal{L}[y] = 0$ and the a particular part, y_p that solves the non-homogenous $\mathcal{L}[y] = f(x)$. The entire solution will be $y = y_h + y_p$. The trick is to solve the homogenous part as usual, and then to guess at a solution for the non-homogenous part. Basically if the non homogenous part is a polynomial, we'll guess $y_p =$ a polynomial of the same degree. If the non-homogenous part is $\cos ax$ or $\sin bx$ or $c_1 \cos ax + c_2 \sin bx$ then we'll guess that y_p is $A \cos ax + B \sin bx$.

Reprinted from earlier handout: **Example:** The solution of $y' - y = x^2$ has homogenous part $y_h = e^x$. We'll guess that $y_p = Ax^2 + Bx + C$. Then $y_p' - y_p = x^2$ becomes $(2Ax + B) - (Ax^2 + Bx + C) = x^2$ and hence: $(-A)x^2 + (2A - B)x + (B - C) = x^2$, so that $A = -1$; $(2A - B)x = (2(-1) - B)x = 0$, so $B = -2$; $(B - C) = 0$ so $C = B = -2$. We get

$$y = c_1 e^x - x^2 - 2x - 2.$$

We check our solution: $y' - y = x^2$ becomes $(c_1 e^x - 2x - 2) - (c_1 e^x - x^2 - 2x - 2) = x^2$. It checks!

Note: when used to in the method of undetermined coefficients the homogenous solution y_h of $\mathcal{L}[y] = 0$ is sometimes called the complementary solution, y_c.

The following table gives forcing terms together with guesses for the particular solution.

Forcing Term	Guess for particular solution y_p
polynomial of degree n	$A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$
e^{kt}	Ae^{kt}
$\sin kt$ or $\cos kt$	$A \cos kt + B \sin kt$

(#1 p210) from HW 11. Find the solution to $y'' + 16y = e^{3x}$.

Answer $(D^2 + 16)[y] = e^{3x}$. First the homogenous: $(D^2 + 16)[y] = 0$. The roots are $\pm 4i$ so $y_h = A \cos 4x + B \sin 4x$. The particular: we guess at particular solution $y_p = ce^{3x}$ (the derivatives of y_p are multiples of y_p so we don't need to include them in our guess for y_p. Substitution of y_p of this into the linear operator yields $(D^2 + 16)[ce^{3x}] = 9ce^{3x} + 16ce^{3x} = 25ce^{3x} = e^{3x}$. So $c = \frac{1}{25}$. So the general solution is

$$y = y_h + y_p = A \cos 4x + B \sin 4x + \frac{1}{25} e^{3x}.$$

(#6 p210) from HW 11. Find the solution to $2y'' + 4y' + 7y = x^2$.

Answer $(2D^2 + 4D + 7)[y] = x^2$. First the homogenous: $(2D^2 + 4D + 7)[y] = 0$. Using the quadratic formula we find that the roots are $\frac{-4 \pm \sqrt{16-56}}{4} = -1 \pm \frac{\sqrt{-40}}{4} = -1 \pm \frac{\sqrt{10}}{2} i$ so $y_h = (A \cos \frac{\sqrt{10}}{2} x + B \sin \frac{\sqrt{10}}{2} x)e^{-x}$. The particular: we guess at particular solution $y_p = c_0 + c_1 x + c_2 x^2$. Substitution of y_p of this into the linear operator yields

$$\begin{aligned}(2D^2 + 4D + 7)[c_0 + c_1 x + c_2 x^2] &= 2(2c_2) + 4(c_1 + 2c_2 x) + 7(c_0 + c_1 x + c_2 x^2) \\ &= (7c_0 + 4c_1 + 4c_2) + (8c_2 + 7c_1)x + (7c_2)x^2 \\ &= x^2.\end{aligned}$$

So $7c_2 = 1$ and $(8c_2 + 7c_1) = 0$ and $(7c_0 + 4c_1 + 4c_2) = 0$. This implies $c_2 = \frac{1}{7}$. $(8c_2 + 7c_1) = 0$ implies $-\frac{8}{7} c_2 = c_1$ implies $\frac{-8}{49} = c_1$ and after some algebra, $c_0 = 4/343$. Hence $y_p = \frac{4}{343} - \frac{8}{49} x + \frac{1}{7} x^2$.

So the general solution is

$$y = y_h + y_p = (A\cos\frac{\sqrt{10}}{2}x + B\sin\frac{\sqrt{10}}{2}x)e^{-x} + \frac{4}{343} - \frac{8}{49}x + \frac{1}{7}x^2.$$

Note: $\frac{-8}{49} = \frac{-56}{343}$ and $\frac{1}{7} = \frac{49}{343}$.

Method of Undetermined Coefficients to solve the non-homogenous linear DE - interference. (See page 205 in Edwards & Penney, 4th Ed.)

Consider $\mathcal{L}[y] = f(x)$. Suppose y_h is a solution of $\mathcal{L}[y_h] = 0$. Suppose that f and y are linearly dependent. So the expected trial solution y_t will not be independent of y_h. So we fix the expected trial solution y_E to make it independent of y_h. What we do is multiply y_E by x^s with s big enough to ensure linear independence. I.e. $y_{trial} = x^s y_E$. This is confusing to read and somewhat vague, but the example below, and in the book, will clear things up, hopefully.

Note: If the derivatives of $f(x)$, i.e. if $D^n[f]$ are linear independent for infinitely many n the method of undetermined coefficient will not work, and then you would use something called variation of parameters (which we may learn later this semester if we have time).

Example for undetermined coefficients with interference, page 210 #12:

$y^{(3)} + y' = 2 - \sin x$. So $f(x) = 2 - \sin x$. As a linear operator DE this is just $(D^3 + D)[y] = 2 - \sin x$. Factoring to solve the homogenous equation we get: $D(D^2+1)[y] = 0$. So the roots are $0, \pm i$. The homogenous (also called complementary) solution is $c_0 e^{0x} + c_1 \cos x + c_2 \sin x$ which is more nicely written as:

$$y_h = y_c = c_0 + c_1 \cos x + c_2 \sin x.$$

We would expect the trial solution to be of the form:

$$A + B \sin x + C \cos x$$

based upon the form of $f(x) = 2 - \sin x$.

However, notice that the A term in the expected trial solution is is not linearly independent of the constant term, c_0, in the homogenous solution (i.e. in the complementary solution); also notice that the $\sin x, \cos x$ in the expected trial solution is not linearly independent of those terms in y_h either. So we need to fix up the expected trial solution. Instead of: $A + B \sin x + C \cos x$ we have

$$y_p = Ax + Bx \sin x + Cx \cos x.$$

So in both parts, we multiplied by $x^s = x^1$. This is just a coincidence. Usually you will multiply the different parts by different powers of s.

Homework #12 Do questions 3, 5, 7, 9, 12, 15, 16 on p210 in Edwards & Penney 4^{th} Ed.
Hints:
(#5, p 210) Use trig identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$.
(#7, p 210) $\sinh x = \frac{e^x - e^{-x}}{2}$.

Selected Solutions to HW 12.

Homework #12 was questions 3, 5, 7, 9, 12, 15, 16 on p210 in Edwards & Penney 4^{th} Ed.

(# 3 p210) $y'' - y' - 6y = 2\sin 3x$

Answer $(D^2 - D - 6)[y]$ factors as $(D-3)(D+2)[y]$, hence the roots are $r = 3, -2$ and $y_h = y_c = c_1e^{3x} + c_2e^{-2x}$. The trial solution is $y_t = A\cos 3x + B\sin 3x$. so

$$\begin{aligned} D[y_t] &= -3A\sin 3x + 3B\cos 3x \\ D^2[y_t] &= -9A\cos 3x - 9B\sin 3x \\ &= -9y_t \text{ but then:} \\ (D^2 - D - 6)[y_t] &= -9y_t - (-3A\sin 3x + 3B\cos 3x) - 6y_t \\ &= -15y_t + 3A\sin 3x - 3B\cos 3x \\ &= -15(A\cos 3x + B\sin x) + 3A\sin 3x - 3B\cos 3x \\ &= \underbrace{(-15A - 3B)}_{=\,0}\cos 3x + \underbrace{(3A - 15B)}_{=\,2}\sin 3x = 2\sin 3x \text{ (we equate coefficients).} \end{aligned}$$

We get a system of two linear equations in two unknowns:

$$\begin{aligned} -15A - 3B &= 0 \quad \text{(eq 1)} \\ 3A - 15B &= 2 \quad \text{(eq 2)} \end{aligned}$$

(eq 1) + 5(eq 2) $\Rightarrow -78B = 10$ so $B = -\frac{5}{39}$. (eq 1) $\Rightarrow A = -\frac{1}{5}B = -\frac{1}{5}\left(-\frac{5}{39}\right)$ so $A = \frac{1}{39}$. So

$$y_p = \frac{1}{39}\cos 3x - \frac{5}{39}\sin 3x.$$

The general solution, $y = y_h + y_p$, is

$$y = c_1e^{3x} + c_2e^{-2x} + \frac{1}{39}\cos 3x - \frac{5}{39}\sin 3x.$$

(# 5 p210) $y'' + y' + y = \sin^2 x$

Answer To avoid dealing directly with $\sin^2 x$, we can use the trig identity [15]:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

So the DE: $y'' + y' + y = \sin^2 x$ can be written as:

$$y'' + y' + y = \frac{1}{2} - \frac{1}{2}\cos 2x,$$

[15] Proof that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ using Euler's Formula: To save space, let $cx = \cos x, c2x = \cos 2x, sx = \sin x, s2x = \sin 2x$. Then $\underbrace{(c2x)}_{\text{Re}} + i\underbrace{(s2x)}_{\text{Im}} = e^{i2x} = e^{ix}e^{ix} = (cx + isx)^2 = \underbrace{(cx)^2 - (sx)^2}_{\text{Real part}} + i\underbrace{2(cx)(sx)}_{\text{Imaginary part}}$. Equating the real parts on either end of the above equality we get $c2x = (cx)^2 - (sx)^2$. But $(cx)^2 + (sx)^2 = 1$ so $(cx)^2 = 1 - (sx)^2$. But then then $c2x = (cx)^2 - (sx)^2$ can be written as $c2x = 1 - (sx)^2 - (sx)^2 = 1 - 2(sx)^2$. This implies $\frac{1}{2}(1 - c2x) = (sx)^2$.

or using operator notation, as: $(D^2+D+1)[y] = \frac{1}{2}(1-\cos 2x)$. First we solve the associated homogenous DE: $(D^2+D+1)[y]=0$. Using the quadratic formula we find that the roots $r = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are complex conjugates. So the homogenous (complementary) solution is

$$y_h = y_c = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}\,x + c_2 \sin\frac{\sqrt{3}}{2}\,x\right).$$

The trial solution is

$$y_t = A\cos 2x + B\sin 2x + C$$

based upon the form of $\frac{1}{2} - \frac{1}{2}\cos 2x$. So

$$\begin{aligned}
D[y_t] &= -2A\sin 2x + 2B\cos 2x \\
D^2[y_t] &= -4A\cos 2x - 4B\sin 2x, \qquad \text{but then:} \\
(D^2+D+1)[y_t] &= (-4A\cos 2x - 4B\sin 2x) + (-2A\sin 2x + 2B\cos 2x) + (A\cos 2x + B\sin 2x + C) \\
&= \underbrace{(-4A+2B+A)}_{=-\frac{1}{2}}\cos 2x + \underbrace{(-4B-2A+B)}_{=0}\sin 2x + \underbrace{C}_{=\frac{1}{2}} = \frac{1}{2} - \frac{1}{2}\cos 2x
\end{aligned}$$

By equating coefficients we get a system of three linear equations in three unknowns:

$$\begin{aligned}
-4A+2B+A &= -\frac{1}{2} \quad (\text{eq } 1) \\
-4B-2A+B &= 0 \quad (\text{eq } 2) \\
C &= \frac{1}{2} \quad (\text{eq } 3)
\end{aligned}$$

Since $C = 1/2$ we have two equations with two unknowns:

$$\begin{aligned}
-3A+2B &= -\frac{1}{2} \quad (\text{eq } 1) \\
-2A-3B &= 0 \quad (\text{eq } 2)
\end{aligned}$$

3(eq 1) + 2(eq 2) $\Rightarrow$ $-13A = -\frac{3}{2}$ so $A = \frac{3}{26}$. (eq 2) $\Rightarrow$ $B = -\frac{2}{3}A = -\frac{2}{3}\left(\frac{3}{26}\right)$ so $B = -\frac{2}{26}$. So $y_p = A\cos 2x + B\sin 2x + C$ becomes:

$$y_p = \frac{1}{26}\left(3\cos 2x - 2\sin 2x\right) + \frac{1}{2}.$$

So the general solution $y = y_h + y_p$ is

$$y = e^{-\frac{1}{2}x}\left(c_1\cos\frac{\sqrt{3}}{2}\,x + c_2\sin\frac{\sqrt{3}}{2}\,x\right) + \frac{1}{26}\left(3\cos 2x - 2\sin 2x + 13\right),$$

since $\frac{13}{26} = \frac{1}{2} = C$.

(# 7 p210) $y'' - 4y = \sinh x$

Note: $\sinh x = \frac{e^x - e^{-x}}{2}$. So we want to solve: $(D^2-4)[y] = \frac{e^x-e^{-x}}{2}$. The roots are ± 2 so $y_h = y_c = c_1e^{2x} + c_2e^{-2x}$. The trial solution is $y_t = Ae^x + Be^{-x}$. We have $D[y_t] = Ae^x - Be^{-x}$ and $D^2[y_t] = Ae^x + Be^{-x} = y_t$. So

$$\begin{aligned}
(D^2-4)[y_t] &= y_t - 4y_t = -3y_t \\
&= \underbrace{-3A}_{=\frac{1}{2}}e^x - \underbrace{3B}_{=\frac{1}{2}}e^{-x} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}.
\end{aligned}$$

So $-3A = \frac{1}{2}$ implying $A = -\frac{1}{6}$; and $3B = \frac{1}{2}$ implying $B = \frac{1}{6}$. Thus

$$y_t = -\frac{1}{6}e^x + \frac{1}{6}e^{-x}$$

and so $y = y_h + y_p$ is:

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{6}e^x + \frac{1}{6}e^{-x}.$$

(# 9 p210) $y'' + 2y' - 3y = 1 + xe^x$

Answer Hints. $(D^2 + 2D - 3)[y] = 1 + xe^x$. Has roots $r = 1, -3$. So $y_h = y_c = c_1 e^x + c_2 e^{-3x}$. The trial solution would be of the form $A + Be^x + Cxe^x$ (because $D[xe^x] = e^x + xe^x$), however e^x is duplicated within y_h, and so we multiply $Be^x + Cxe^x$ by x to preserve the linear independence of the trial solution w.r.t. the homogenous (or complementary) solution. So we will use the trial solution: $y_t = A + x(Be^x + Cxe^x)$. Then calculate $D[y_t], D^2[y_t]$ and substitute these into $(D^2 + 2D - 3)[y] = 1 + xe^x$ and solve for A, B, C.

(# 12 p210) $y^{(3)} + y' = 2 - \sin x$

Answer Hints. $(D^{(3)} + D)[y] = 2 - \sin x$. Factors: $D(D^2 + 1)[y] = 2 - \sin x$. Has roots $r = \pm i, 0$. So $y_h = y_c = c_0 + c_1 \cos x + c_2 \sin x$. The trial solution would be of the form $c_0 + c_1 \cos x + c_2 \sin x$ but this would duplicate y_h. So we multiply by x to get $y_t = Ax + Bx \cos x + Cx \sin x$. Then plug y_t into the differential equation $(D^{(3)} + D)[y] = 2 - \sin x$ and solve for A, B, C.

Worked examples of interference (Good practice for Quiz 2)

1a. Find the general solution $y = y_h + y_p$ of the 4^{th} order linear differential equation:

$$y^{(4)} + 6y^{(3)} + 10y'' = -39 \cos x + 76 + 60x.$$

1b. Check that the particular part, y_p, of the general solution $y = y_h + y_p$ is correct.

Solution to 1a: Write the DE as: $(D^4 + 6D^3 + 10D^2)[y] = -39 \cos x + 76 + 60x$. First we factor, to get $D^2(D^2 + 6D + 10)$ then we apply the quadratic formula to find the roots:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i.$$

So the roots $r = -3 \pm i,\ 0$ (with 0 having multiplicity 2). So

$$y_h = e^{-3x}(c_1 \cos x + c_2 \sin x) + d_0 + d_1 x.$$

Examining $-39 \cos x + 76 + 60x$ our initial guess at a trial solution is $y_i = A \cos x + B \sin x + C + Ex$, However to avoid duplication (of terms in y_h) our trial solution will be $y_t = A \cos x + B \sin x + x^2(C + Ex)$, which I think is easier to differentiate written as: $y_t = A \cos x + B \sin x + Cx^2 + Ex^3$. Then

$$\begin{aligned}
D[y_t] &= -A \sin x + B \cos x + 2Cx + 3Ex^2 \\
D^2[y_t] &= -A \cos x - B \sin x + 2C + 6Ex \\
D^3[y_t] &= A \sin x - B \cos x + 0 + 6E \\
D^4[y_t] &= A \cos x + B \sin x + 0 + 0 \Rightarrow \\
(D^4 + 6D^3 + 10D^2)[y_t] &= (A - 6B - 10A) \cos x + (B + 6A - 10B) \sin x + (60E)x + (36E + 20C) \\
&= \underbrace{(-9A - 6B)}_{=\ -39} \cos x + \underbrace{(6A - 9B)}_{=\ 0} \sin x + \underbrace{(60E)}_{=\ 60} x + \underbrace{(36E + 20C)}_{=\ 76} \\
&= -39 \cos x + 76 + 60x
\end{aligned}$$

We solve for A, B, C, E. We have 4 equations in 4 unknowns:

$$\begin{aligned} -9A - 6B &= -39 \qquad (Eq\ 1) \\ 6A - 9B &= 0 \qquad (Eq\ 2) \\ 60E &= 60 \qquad (Eq\ 3) \\ 36E + 20C &= 76 \qquad (Eq\ 4) \end{aligned}$$

$(Eq\ 3) \Rightarrow E = 1$. Then $E = 1$ combined with $(Eq\ 4) \Rightarrow C = 2$. $(Eq\ 2) \Rightarrow \frac{2}{3}A = B$ which we substitute into $(Eq\ 1)$ to get $-13A = -39 \Rightarrow A = 3$. Then $A = 3$ combined with $\frac{2}{3}A = B \Rightarrow B = 2$. We get $y_p = 3\cos x + 2\sin x + 2x^2 + x^3$. So the general solution $y = y_h + y_p$ is

$$\underbrace{y = e^{-3x}(c_1 \cos x + c_2 \sin x) + d_0 + d_1 x + 3\cos x + 2\sin x + 2x^2 + x^3}_{\text{Answer to 1a.}}.$$

1b. Check that the particular part, y_p, of the general solution $y = y_h + y_p$ is correct.

$$\begin{aligned} y_p &= 3\cos x + 2\sin x + 2x^2 + x^3 \\ D[y_p] &= -3\sin x + 2\cos x + 4x + 3x^2 \\ D^2[y_p] &= -3\cos x - 2\sin x + 4 + 6x \\ D^3[y_p] &= 3\sin x - 2\cos x + 0 + 6 \\ D^4[y_p] &= 3\cos x + 2\sin x + 0 + 0 \Rightarrow \\ (D^4 + 6D^3 + 10D^2)[y_t] &= (3 - 12 - 30)\cos x + (2 + 18 - 20B)\sin x + (60)x + (36 + 40) \\ &= -39\cos x + 0\sin x + 60x + 76 \end{aligned}$$

So y_p checks! You could also check y_h, but that would take many steps so there is a better chance that you'd make an error checking y_h, than you made finding it in the first place!

Q4A Find the general solution $y = y_h + y_p$ of the 2^{nd} order linear differential equation:

$$y'' + 4y' = 16x - 40\cos 2x$$

Write Answer on Line.

General solution $y = y_h + y_p$ is: $y(x) =$ ____________________________

Q4A Find the solution of the IVP:

$$y'' + 4y' = 16x - 40\cos 2x, \qquad y(0) = 3, \quad y'(0) = 7$$

Write Answer on Line.

Solution to the IVP is: $y(x) =$ ____________________________

Show work below.

Part A.

$(D^2 + 4D)[y] = 16x - 40\cos 2x$ factors as $D(D+4)[y] = 16x - 40\cos 2x$ and has roots $0, -4$. So $y_h = c_0 + c_1 e^{-4x}$. The forcing term $16x - 40\cos 2x$ suggests $y_i = A\cos 2x + B\sin 2x + C + Ex$, however to avoid duplication, we choose $y_t = A\cos 2x + B\sin 2x + Cx + Ex^2$. But then

$$\begin{aligned} y_t &= A\cos 2x + B\sin 2x + Cx + Ex^2 \\ D[y_t] &= -2A\sin 2x + 2B\cos 2x + C + 2Ex \\ D^2[y_t] &= -4A\cos 2x - 4B\sin 2x + 0 + 2E \end{aligned}$$

which implies

$$(D^2+4D)[y_t] = (-4A\cos 2x - 4B\sin 2x + 0 + 2E) + 4(-2A\sin 2x + 2B\cos 2x + C + 2Ex) = 16x - 40\cos 2x.$$

Collecting like terms we get:

$$(-4A+8B)\cos 2x + (-8A-4B)\sin 2x + (2E+4C) + 8Ex = (0) + (16)x + (-40)\cos 2x + (0)\sin 2x.$$

So $8Ex = 16x \Rightarrow E = 2$ and this combined with $2E + 4C = 0$ implies $C = -1$. In the forcing term, the coefficient of $\sin 2x$ is 0 and so $(-8A - 4B) = 0$ this implies $-2A = B$. Substituting this into the coefficient of $\cos 2x$ in the trial solution, which is $(-4A+8B)$, implies $-4A+8(-2A) = -20A$. The coefficient of $\cos 2x$ in the forcing term is -40 and so $A = 2$, and so $B = -2A = -2(2) = -4$. So we get, plugging the values for A, B, C, and E, into y_t:

$$y_p = 2\cos 2x - 4\sin 2x - x + 2x^2.$$

The general solution $y = y_h + y_p$ is thus

$$y = c_0 + c_1e^{-4x} + 2\cos 2x - 4\sin 2x - x + 2x^2.$$

Part B.

$$\begin{aligned} y &= c_0 + c_1e^{-4x} + 2\cos 2x - 4\sin 2x - x + 2x^2 \quad \text{and so} \\ y' &= \quad -4c_1e^{-4x} - 4\sin 2x - 8\cos 2x - 1 + 4x \end{aligned}$$

Using the IC we get:

$$\begin{aligned} y(0) = 3 &= c_0 + c_1 + 2 \quad \text{and} \\ y'(0) = 7 &= -4c_1 - 8 - 1 \end{aligned}$$

which becomes

$$\begin{aligned} 1 &= c_0 + c_1 \\ 16 &= -4c_1 \end{aligned}$$

So $c_1 = -4$ and $c_0 = 5$. So the solution to the IVP is:

$$y = 5 - 4e^{-4x} + 2\cos 2x - 4\sin 2x - x + 2x^2$$

HW # 13: Edwards & Penney 4th Ed. Page 210. Questions: 19 – 23.

Systems of Equations – Chapter 4

Example 1. Solve the initial value problem:

$$\begin{array}{lll} (Eq1) & x' = x + y, & x(0) = 5 \\ (Eq2) & y' = 2x - y, & y(0) = 4 \end{array}$$

Answer to Example 1. (Eq 1) implies:

$$\begin{aligned} x' - x &= y, \\ x'' - x' &= y' \end{aligned}$$

and combining this with (Eq 2) we get

$$\begin{aligned} x'' - x' &= 2x - (x' - x) \\ x'' - 3x &= 0 \\ (D^2 - 3)[x] &= 0 \end{aligned}$$

So the roots $r = \pm\sqrt{3}$, which implies

$$\begin{aligned} x(t) &= c_1 e^{-\sqrt{3}\ t} + c_2 e^{\sqrt{3}\ t} \\ x'(t) &= -c_1\sqrt{3}\ e^{-\sqrt{3}\ t} + c_2\sqrt{3}\ e^{\sqrt{3}\ t} \end{aligned}$$

But $y = x' - x$ and so

$$\begin{aligned} y &= (-c_1\sqrt{3}\ e^{-\sqrt{3}\ t} + c_2\sqrt{3}\ e^{\sqrt{3}\ t}) - (c_1 e^{-\sqrt{3}\ t} + c_2 e^{\sqrt{3}\ t}) \\ &= (-c_1\sqrt{3} - c_1)\ e^{-\sqrt{3}\ t} + (c_2\sqrt{3} - c_2) e^{\sqrt{3}\ t} \end{aligned}$$

So we get the general solution:

$$\begin{aligned} x(t) &= c_1 e^{-\sqrt{3}\ t} + c_2 e^{\sqrt{3}\ t} \\ y(t) &= (-\sqrt{3} - 1)c_1\ e^{-\sqrt{3}\ t} + (\sqrt{3} - 1)c_2 e^{\sqrt{3}\ t} \end{aligned}$$

Using the IC we can find the particular solution:

$$\begin{aligned} x(0) &= c_1 + c_2 = 5 \\ y(0) &= (-\sqrt{3} - 1)c_1 + (\sqrt{3} - 1)c_2 = 4 \end{aligned}$$

Multiply $x(0)$ by $(\sqrt{3} + 1)$ to get

$$\begin{aligned} (\sqrt{3} + 1)c_1 + (\sqrt{3} + 1)c_2 &= 5(\sqrt{3} + 1) \\ (-\sqrt{3} - 1)c_1 + (\sqrt{3} - 1)c_2 &= 4 \end{aligned}$$

then add the above two equations to eliminate c_1, getting: $(\sqrt{3} + 1)c_2 + (\sqrt{3} - 1)c_2 = 5(\sqrt{3} + 1) + 4$ which simplifies to $2\sqrt{3}c_2 = 5\sqrt{3} + 9$ and so $c_2 = \frac{5\sqrt{3}+9}{2\sqrt{3}}$.

From $x(0)$ we have $c_1 = 5 - c_2 = 5 - \frac{5\sqrt{3}+9}{2\sqrt{3}} = \frac{10\sqrt{3}}{2\sqrt{3}} - \frac{5\sqrt{3}+9}{2\sqrt{3}} = \frac{5\sqrt{3}-9}{2\sqrt{3}}$. So, we get the particular solution:

$$\begin{aligned} x(t) &= \frac{5\sqrt{3} - 9}{2\sqrt{3}} e^{-\sqrt{3}\ t} + \frac{5\sqrt{3} + 9}{2\sqrt{3}} e^{\sqrt{3}\ t} \\ y(t) &= (-\sqrt{3} - 1)\frac{5\sqrt{3} - 9}{2\sqrt{3}}\ e^{-\sqrt{3}\ t} + (\sqrt{3} - 1)\frac{5\sqrt{3} + 9}{2\sqrt{3}} e^{\sqrt{3}\ t} \end{aligned}$$

p266 #7 (From HW 14 B (see HW 14 below)) Solve: $x' = 4x + y + 2t$ and $y' = -2x + y$

Solution: From the x' equation we get $y = x' - 4x - 2t$ and so $y' = x'' - 4x' - 2$. Combining this with the y' equation yields $x'' - 4x' - 2 = -2x + (x' - 4x - 2t)$ which is equivalent to $x'' - 5x' + 6x = 2 - 2t$. We can write this as $(D^2 - 5D + 6)[x] = 2 - 2t$ which factors as $(D - 2)(D - 3)[x] = 2 - 2t$ so the roots are $r = 2, 3$ and so $x_h = c_1e^{2t} + c_2e^{3t}$. Based upon the forcing term of $2 - 2t$ the trial solution should be: $x_t = A + Bt$ and so $D[x_t] = B$ and $D^2[x_t] = 0$. So

$$(D^2 - 5D + 6)[x_t] = 0 - 5B + 6(A + Bt) = (6A - 5B) + (6B)t = 2 + (-2)t.$$

So $6B = -2$ which implies $B = -\frac{1}{3}$. Substituting $B = -\frac{1}{3}$ into $(6A - 5B) = 2$ gives us $6A - 5(-1/3) = 6A + \frac{5}{3} = 2$ which implies $6A = 2 - \frac{5}{3} = \frac{6}{3} - \frac{5}{3} = \frac{1}{3}$ so $A = \frac{1}{18}$. This implies that $x_p = \frac{1}{18} - \frac{1}{3}t$ and so $x = x_h + x_p$ must be:

$$x(t) = c_1e^{2t} + c_2e^{3t} + \frac{1}{18} - \frac{1}{3}t$$

which implies

$$x'(t) = 2c_1e^{2t} + 3c_2e^{3x} - \frac{1}{3}.$$

Then,

$$\begin{aligned} y &= x' - 4x - 2t \\ &= \left(2c_1e^{2t} + 3c_2e^{3t} - \frac{1}{3}\right) - 4\left(c_1e^{2t} + c_2e^{3t} + \frac{1}{18} - \frac{1}{3}t\right) - 2t \\ &= (2c_1 - 4c_1)\, e^{2t} + (3c_2 - 4c_2)\, e^{3t} + \left(-\frac{1}{3} - \frac{4}{18}\right) + \left(\frac{4}{3} - 2\right) t \\ &= -2c_1e^{2t} - c_2e^{3t} - \frac{5}{9} - \frac{2}{3}t. \end{aligned}$$

So we get the particular solution:

$$\begin{aligned} x(t) &= c_1e^{2t} + c_2e^{3t} + \frac{1}{18} - \frac{1}{3}t \\ y(t) &= -2c_1e^{2t} - c_2e^{3t} - \frac{5}{9} - \frac{2}{3}t. \end{aligned}$$

HW 14

From Edwards and Penney 4th Ed:

HW 14 A: p 210 #25-30; p266 #1-6. (From: Method of Undetermined Coefficients Section 3.5)

HW 14 B: p 266 #7-11 See solution to #7 p266 above. (From: Linear Systems 4.2)

HW for Quiz 1 - Hand this in.
Print on Printer. Put answer in space provided. Put work on back or attach with staple.

1. Solve the initial value problem (IVP).

$$y'' + 2y' + 5y = 0, \quad y(0) = 2, \; y'(0) = 4.$$

Answer:

2. Solve the IVP $y' = -7y, \; y(0) = -4$.

Answer (no work needed): ______________________.

3. Solve the IVP:

$$\begin{aligned} x' &= -x + y & x(0) &= 3 \\ y' &= -24x + 9y & y(0) &= 14 \end{aligned}$$

Be sure to find both $x(t)$ and $y(t)$.

Answer:

4. Find the solution to the 3^{rd} order IVP:

$$y''' + 4y'' = 96x \qquad \text{IC} \quad y(0) = 7, \quad y'(0) = -15, \quad y''(0) = 58$$

Answer:

5. Solve the IVP $y' = xy, \quad y\left(\sqrt{2}\right) = 5\text{e}$.

Answer:

Maple Solutions to HW for Quiz 1. Check your answers using Maple.

1. $y'' + 2y' + 5y = 0,\ \ y(0) = 2,\ y'(0) = 4$. Solution using Maple.

```
de := diff(y(x), x$2) + 2*(diff(y(x), x)) + 5*y(x) = 0; # defining the diff eq.
dsolve(de, y(x));                                       # General Solution
ics := y(0) = 2, (D(y))(0) = 4;                         # defining Initial Conditions
dsolve({de, ics});                                      # Particular solution to the IVP
```

---

2. $y' = -7y,\ y(0) = -4$. Please memorize that $y' = ky$ has solution $y = y(0)e^{kt}$. Solution using Maple.

```
de := diff(y(x), x) = -7*y(x);  # defining the diff eq.
ics := y(0) = -4;               # defining the Initial Conditions
dsolve({de, ics});              # Particular solution to the IVP
```

3.

$$
\begin{aligned}
x' &= -x + y & x(0) &= 3 \\
y' &= -24x + 9y & y(0) &= 14
\end{aligned}
$$

Solution using Maple.

```
de := diff(x(t), t) = -x(t)+y(t), diff(y(t), t) = -24*x(t)+9*y(t);  # Define ODE's
dsolve([de]);                                                        # General solution
ics := x(0) = 3, y(0) = 14;                                          # Define IC's
dsolve([de, ics])                                                    # Solution to IVP
```

---

4. $y''' + 4y'' = 96x$ IC $y(0) = 7,\ \ y'(0) = -15,\ \ y''(0) = 58$. Solution using Maple.

```
de := diff(y(x), x$3)+4*(diff(y(x), x$2)) = 96*x;                       # Define ODE
ics := ((D@@2)(y))(0) = 58, ((D@@1)(y))(0) = -15, ((D@@0)(y))(0) = 7; # Define IC's
dsolve({de, ics})                                                       # Solution to IVP
```

---

5. $y' = xy,\ \ y\left(\sqrt{2}\right) = 5\mathrm{e}$. Solution using Maple

```
de := diff(y(x), x) = x*y(x);   # Define ODE
ics := y(sqrt(2)) = 5*exp(1);   # Define IC's
dsolve({de, ics})               # Solution to IVP
```

Numerical Methods - Section 2.4, p. 112, Edwards & Penney 4^{th} Ed.

Often differential equations can not be exactly solved and instead we need to approximate their solutions using numerical methods.

Introduction to the Euler Method

Let $y = y(x)$. Consider the first order differential equation

$$\frac{d}{dx}\, y = f(x, y).$$

Example 1: if $y' = x + y$ then $f(x, y) = x + y$.
Example 2: if $y' = \sin\left(xy + \sqrt{y}\right)$ then $f(x, y) = \sin\left(xy + \sqrt{y}\right)$.

The definition of $\dfrac{dy}{dx}$ is

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

So if h is small enough, then (approximately)

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h}.$$

and the differential equation $\dfrac{d}{dx}\, y = f(x, y)$ becomes (approximately)

$$\frac{y(x+h) - y(x)}{h} \approx f(x, y).$$

If we multiply both sides by h we get:

$$y(x+h) - y(x) \approx f(x, y)\ h,$$

which allows us to approximate $y(x+h)$ as:

$$y(x+h) \approx y(x) + f(x, y)\ h. \tag{27}$$

The Euler Method approximates the solution y by iterating (27).

The Euler Method

Consider the differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$, which we'll write as $IC = (x_0, y_0)$. We call h the step size and we'll hope that the step size h is small enough so that $y(x+h) \approx y(x) + f(x, y)h$.

The initial (or 0^{th}) Euler point is the IC:

$$(x_0, y_0)$$

Then, assuming we have defined the 0^{th} through n^{th} Euler points:

$$(x_0, y_0), \quad (x_1, y_1), \quad (x_2, y_2), \quad (x_3, y_3), \quad \ldots, \quad (x_n, y_n)$$

the $n+1$ Euler point is defined the $n+1$ Euler step, which is:

$$\begin{aligned} y_{n+1} &= y_n + f(x_n, y_n)h \\ x_{n+1} &= x_n + h \end{aligned}$$

The point (x_{n+1}, y_{n+1}) is called the $n+1$ Euler point. The Euler points approximate the solution, i.e:

$$\text{exact value} \quad \rightarrow \quad (x_n, y(x_n)) \approx (x_n, y_n) \quad \leftarrow \quad \text{Euler approximation}$$

Example 1. For the IVP $y' = y,\ \ y(0) = 1$ calculate the sequence of (x_n, y_n), $n = 0, 1, 2$ if the Euler step size $h = 0.5$.

Solution: Note that $f(x, y) = y$ for this problem.

$$\begin{aligned}
(x_0, y_0) &= (0, 1) \qquad \text{(Initial Conditions)}\\
x_1 &= x_0 + h\\
&= 0 + 0.5 = 0.5\\
y_1 &= y_0 + f(x_0, y_0)h\\
&= 1 + f(0, 1)0.5 = 1 + 1(0.5) = 1.5\\
(x_1, y_1) &= (0.5, 1.5) \qquad \text{(Euler approximation of } y(0.5) = 1.5)\\
x_2 &= x_1 + h\\
&= 0.5 + 0.5 = 1.0\\
y_2 &= y_1 + f(x_1, y_1)h\\
&= 1.5 + f(.5, 1.5)0.5 = 1.5 + 1.5(0.5) = 2.25\\
(x_2, y_2) &= (1.0, 2.25) \qquad \text{(Euler approximation of } y(1.0) = 2.25)
\end{aligned}$$

Error analysis for Example 1. The actual (or exact) solution of the IVP $y' = y$ with IC $y(0) = 1$ is $y(x) = e^x$, since the DE $y' = ay$ has solution $y = y(0)e^{ax}$.

The errors are the differences between the Euler approximation and the actual true value of y:

IVP: $y' = y,\ y(0) = 1.$ Step size $= 0.5.$ Euler approximates of y for $x = 0.5$ and $x = 1.0.$

step	x_n	Euler approximation y_n	actual solution $y = e^x$	error $= y(x_n) - y_n$
0	0.0	1.0	$e^{0.0} = 1.0$	$1.0 - 1.0 = 0.0$
1	0.5	1.5	$e^{0.5} = 1.648721271$	$1.648721271 - 1.5 = 0.148721271$
2	1.0	2.25	$e^{1.0} = 2.718281828$	$2.718281828 - 2.25 = 0.468281828$

Notice that as we get further from the initial conditions, as we go from x_0, to x_1, to x_2, the Euler approximation got worse. I.e. the errors grew larger, going from 0.0, to 0.149, to 0.468. This is fairly typical, as errors often compound (build upon each other), rather than cancel.

If we choose a smaller step size, say $h = 0.01 = \frac{1}{100}$ we would get more accurate approximations. See Figure 3 (page 53). However, a smaller step size means it takes more steps (and more computing time) to get anywhere. For example, to go from the initial $x_0 = 0.0$ to $x = 1.0$, using a step size of $\frac{1}{100}$ requires 100 steps. In other words, there is no free lunch! More accuracy requires more computational time.

IVP: $y' = y,\ y(0) = 1.$ Step size $= 0.01.$ Euler approximates of y for $x = 0.5$ and $x = 1.0.$

step	x_n	Euler approximation y_n	actual solution $y = e^x$	error $= y(x_n) - y_n$
0	0.0	1.0	$e^{0.0} = 1.0$	$1.0 - 1.0 = 0.0$
50	0.50	1.644632	$e^{0.5} = 1.648721271$	$1.648721271 - 1.644632 = 0.004089$
100	1.0	2.704814	$e^{1.0} = 2.718281828$	$2.718281828 - 2.704814 = 0.013468$

It turns out that the Euler Method is a **first order method**. A numerical method being first order for an IVP on an interval $[a, b]$ means that there exists a number $C > 0$, such that on $[a, b]$, the maximal error (meaning the maximum difference between the actual solution and the approximate solution) will be less than

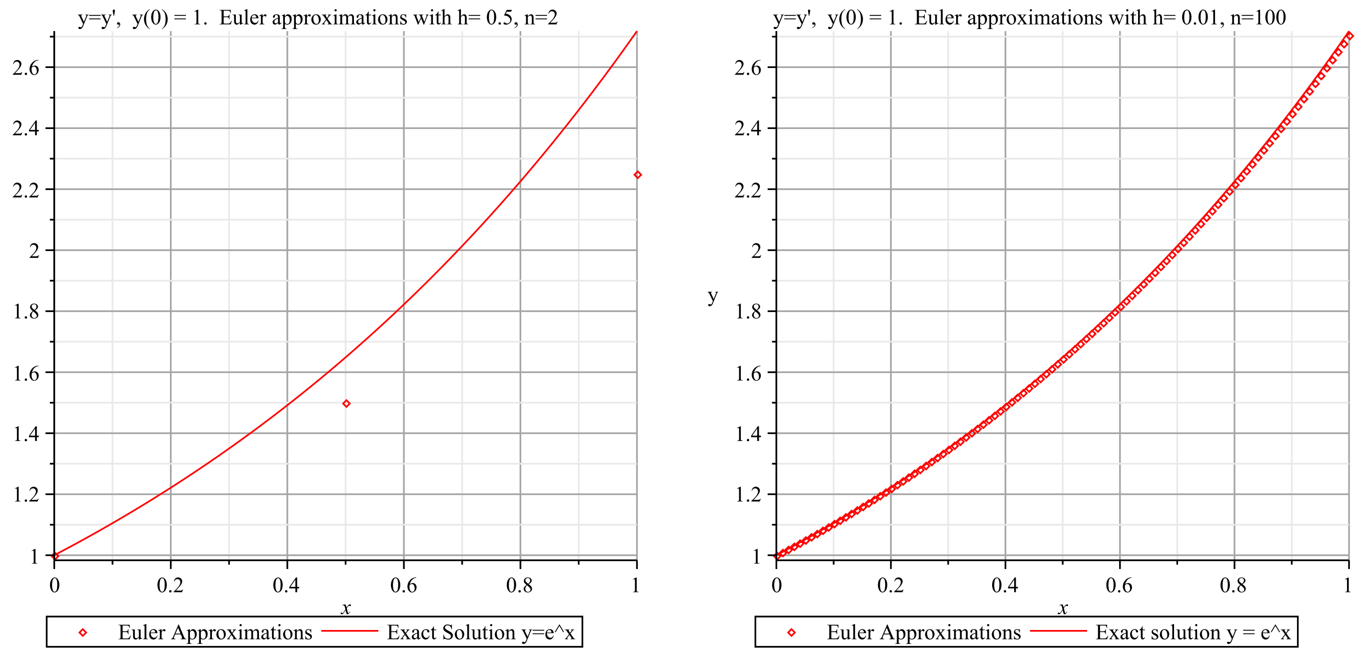


Figure 3: The solution to the IVP $y' = y,\ y(0) = 1$ is $y = e^x$. We graph the exact and approximate solutions on the interval $x \in [0, 1]$ in the above two figures. In the left figure, the Euler approximations are calculated with $h = 0.5$, so $n = 2$. As you can see, in the left figure, the approximations (the points) hint at the the actual solution (the solid curve), but not very closely. In the right figure, $h = 0.01$, so n $= 100$, and the Euler approximations are, at this resolution, indistinguishable from the exact solution.

hC on that interval. The key thing is that the C is fixed (w.r.t. the interval and the IVP), but we can let h get as small as we like. So the approximation can be made as accurate as we desire (in theory).

In practice, the problem is that C can be huge, and even worse, it is often impossible to figure out a numerical upper bound for C. As a result it is somewhat impossible to know how accurate your numerical solution to the IVP is. What tends to be done, whether you are using the Euler or some other numerical method, is the following. Two solutions to the IVP are numerically approximated using two step sizes, h_1 and h_2. If both approximate solutions are close enough to each other then usually (but not always!) the actual solution will be close to the approximate solutions you found, and you're done. But, if the two approximate solutions aren't close enough to each other, then what you have to do is to pick two even smaller values for h, and repeat the process.

In practice, one also has to worry about round off errors when working with floating point numbers (e.g. if we represent the fraction $1/3$ by 0.33 we have introduced a round off error of $1/3 - 0.33 = 0.00\overline{33}$). These round off errors will, with luck, cancel each other out, but it is quite possible for them to accumulate.

Another problem that can occur, and can be caused by either the round off error, or the approximation method's inherent error, is that your solution can switch to a divergent trajectory. We'll learn more about this later. For now, you can think of it in terms of stability. If a pencil is balancing on its point, if you push it just the tiniest bit, say to the left, it will fall in that direction. In the same way, if an IVP is very sensitive to small perturbations, then a small error can lead to a drastically different (and incorrect) approximate solution.

These and other considerations make numerical analysis a very interesting, deep, and important subject – or a head-ache, depending on your perspective! It is very important to be aware of these issues.

Maple. The following Maple code allows a computer to implement the Euler Method calculations quickly. As written below, the code is set for $y' = y,\ y(0) = 1$ on the interval $[0, 1]$ with 2 steps. You should test this code as is (make sure it gives you correct results, i.e. (0.5, 1.5); (1.0, 2.25)); then use it to solve some IVP's.

```
##########  user input #####################################
f := (x, y) -> y; #Define the slope function f(x,y) as in y' = f(x,y)
x0 := 0; y0 := 1; #Initial Conditions
xf := 1;          # Final x (end point of interval [x0, xf])
n := 2;           # number of steps to reach end point xf
#########  end user input ##################################
x := x0: y := y0:       # initialize variables x and y
h := evalf((xf-x0)/n):  # calculate step size
for i to n do: # start of for loop
k := f(x, y):  # left hand slope
y := y+h*k:    # Euler step updating y
x := x+h:      # Euler step updating x
print(x, y):   # print result to screen
od:            # end of for loop
```

Important note: other numerical methods are much more efficient than the Euler method, see sections 2.5 and 2.6, Edwards & Penney 4^{th} Ed. and also [16]. However, that said, none of the other methods are as simple and as beautiful as the Euler method; and none are as fundamental. Moreover, these other methods are built upon the Euler method. The Euler method is also important in theoretical settings, especially in the analysis of iterative systems.

In this class you need to know the Euler Method. See homework.

Classical example showing the Euler approximations, in the limit, yields an exact solution. Consider the Euler approximations for the solution of the IVP $y' = y,\ y(0) = 1$. Note: to reach the Euler approximation of $y(x)$ requires n steps if we make the step size $h = x/n$. Also note: in this IVP, the Euler step $y_n = y_{n-1} + f(x,y)h$ becomes $y_n = y_{n-1} + y_{n-1}h$ since for this IVP $f(x,y) = y$. So, with a little algebra we get $y_n = y_{n-1}(1 + h)$ and so $y_n = y_0(1 + h)^n = (1 + h)^n$. So the Euler approximation of $y(x)$, y_n, is $\left(1 + \frac{x}{n}\right)^n$. We can evaluate $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ by applying ln to both sides and using L'hopital's rule. Let $L = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ then $\ln L = \lim_{n\to\infty} n \ln\left(1 + \frac{x}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{x}{n}\right)}{n^{-1}}$. Differentiation of numerator and denominator w.r.t. n yields $\ln L = \lim_{n\to\infty} \frac{\left(1+\frac{x}{n}\right)^{-1}(-xn^{-2})}{-n^{-2}} = x$. So $L = e^x$, the solution of the IVP.

HW #15: Euler Method.

End of Section 2.4, page 121 (Edwards & Penney 4^{th} Ed.), do #1, 3, 5. Do these problems by hand and using Maple.

---

[16] For example, in the 4^{th} order Runge-Kutta method (Section 2.6, Edwards & Penney 4^{th} Ed.) the maximal error will be less than h^4C. Since the step size h can be chosen to be between 0 and 1, we have $h^4 << h$. So in that sense fourth order is much better than first order. In practice, professional numerical ODE solvers, such as built into Maple and Matlab, do not use the Euler method, rather they tend to use modern adaptive-step size versions of Runge-Kutta approaches, especially the Runge-Kutta-Fehlberg Method (RKF45) which try (and often succeed) at quickly approximating a solution with a high degree of accuracy. That said, deep down, RKF45 is just the Euler method, tweaked a little.

The Euler method for higher order differential equations.

To use the Euler method the differential equation needs to be first order. In this section you'll learn how to convert a higher order differential equation to first order.

Converting a 2^{nd} order ODE to a 2 dimensional system of 1^{st} order ODE's.

Consider the simple mass-spring-dashpot second order ODE:

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{28}$$

We can convert (28) to a first order system by letting $v = \dot{x}$. I choose the letter v since $\dot{x}$ is the velocity, see [17]. With this substitution the second order DE (28) becomes the first order system

$$\begin{aligned} \dot{x} &= v \\ m\dot{v} + cv + kx &= 0. \end{aligned} \tag{29}$$

Notice that in each of the two equations of (29) only one of the variables has a derivative. In other words, even though $m\dot{v} + c\dot{x} + kx = 0$ would also be first order, we use cv instead of $c\dot{x}$, and write $m\dot{v} + cv + kx = 0$. We do this avoid having both a $\dot{v}$ and an $\dot{x}$ in the same equation. If we do this, it will make it easier for us to numerically solve the system.

We put the derivatives on one side and rewrite (29) as:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x - \frac{c}{m}v. \end{aligned} \tag{30}$$

and then, using column vector notation, we write (30) as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\frac{k}{m}x - \frac{c}{m}v \end{pmatrix}.$$

Now, we can easily apply Euler's method. Recall, that in the 1-dimensional case we numerically approximated the solution to the DE $y' = f(x, y)$ by iterating the recursion formula

$$\begin{aligned} y_{n+1} &= y_n + f(x, y)h \\ x_{n+1} &= x_n + h, \end{aligned} \tag{31}$$

to get a sequence of points (x_n, y_n) (called Euler points).

When dealing with higher dimensional differential equations it is common to let t be the independent variable. So, if we replace x by t then the the 1-dimensional Euler method formula (31) becomes

$$\begin{aligned} y_{n+1} &= y_n + f(t, y)h \\ t_{n+1} &= t_n + h. \end{aligned} \tag{32}$$

Note: h is called the "step size" or the "time step" and sometimes Δt is used rather than h.

We can easily extend (32) to higher dimensions: So, in the mass-spring-dashpot example, the Euler method iteration scheme would be:

$$\begin{aligned} \begin{pmatrix} x_{n+1} \\ v_{n+1} \end{pmatrix} &= \begin{pmatrix} x_n \\ v_n \end{pmatrix} + \begin{pmatrix} v_n \\ -\frac{k}{m}x_n - \frac{c}{m}v_n \end{pmatrix} h \\ t_{n+1} &= t_n + h \end{aligned} \tag{33}$$

[17] When the order of the DE is low, I think that using letters, like x, v is easier than using subscripts like x_0, x_1.

We can write (33) entirely in column vector form:

$$\begin{pmatrix} x_{n+1} \\ v_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ v_n \\ t_n \end{pmatrix} + \begin{pmatrix} v_n \\ -\frac{k}{m}x_n - \frac{c}{m}v_n \\ 1 \end{pmatrix} h \tag{34}$$

which makes sense since $\dfrac{d}{dt}\, t = 1$. Furthermore, since $\dfrac{d}{dt}\, v(t) = a(t)$, where $a(t) =$ acceleration at time t, we can write (34) as:

$$\begin{pmatrix} x_{n+1} \\ v_{n+1} \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ v_n \\ t_n \end{pmatrix} + \begin{pmatrix} v_n \\ a_n \\ 1 \end{pmatrix} \Delta t \tag{35}$$

where we've let $a_n = -\frac{k}{m}x_n - \frac{c}{m}v_n$ be the Euler approximation of the acceleration at the n^{th} step. Actually (35) is quite general and if you think about it for a moment, makes perfect sense.

HW 16: p. 279 Problems # 1, 2, 12, 13. Only use the Euler Method on these problems. For #12, only approximate up to $x(.3)$. For #13, set up the problem as a first order system. Then, if you can, using Maple, with $h \le .1$, numerically solve the system and then look for when the velocity changes from positive to negative. It takes between 7 and 8 seconds to reach maximum height (at least using Runge-Kutta).

From HW 16: p. 279 #13 Edwards & Penney 4th Ed.

A crossbow bolt (arrow) is shot upwards with an initial velocity of 288 ft/sec. Its deceleration from air resistance is $0.04v$. Then its height x satisfies the IVP:

$$x'' = -32 - (0.04)x'; \qquad x(0) = 0, \quad x'(0) = 288.$$

Find the maximum height of the bolt and the time it takes to reach that height. Solve this numerically using Euler's method.

Exact Answer: We can find the exact answer: $x'' + .04x' = -32$ as an operator DE is $(D^2 + .04D)[x] = -32$. This easily factors into $D(D + .04)[x] = -32$ which has roots $r = 0, -.04$. So $x_h = c_0 + c_1e^{-.04t}$. The forcing term -32 suggests $x_i = A$, however to avoid duplication $x_t = At$. $D[x_t] = A$ and $D^2[x_t] = 0$ so $(D^2 + .04D)[x_t] = 0 + .04A = -32$ so $A = -800$ yielding $x_p = -800t$. We get $x(t) = c_0 + c_1e^{-.04t} - 800t$. The initial conditions allow us to find c_0 and c_1: $x(0) = c_0 + c_1 = 0$ and $x' = -.04c_1e^{-.04t} - 800$ so $x'(0) = -.04c_1 - 800 = 288$ imply that $c_1 = \frac{800+288}{-.04} = -27,200$ and $c_0 = 27,200$. So $x(t) = 27,200 - 27,200e^{-.04t} - 800t$. To maximize x we find the t where $x' = 0$ and plug that value of t into x. We do that calculation: $x' = (.04)27,200e^{-.04t} - 800 = 1088e^{-.04t} - 800 = 0$ if $t = \frac{1}{-.04}\ln\frac{800}{1088} = 7.687117495$. Then we plug $t = 7.687117495$ into $x(t) = 27,200 - 27,200e^{-.04t} - 800t$ and we get the maximum height: $x(7.687117495) = 1050.306004$.

Answer by converting to first order ODE and using the Euler method.

Let $v = x'$ then the DE becomes

$$\begin{aligned} x' &= v \\ v' &= -.04v - 32 \end{aligned}$$

So

$$\begin{aligned} x_{n+1} &= x_n + v_n h \\ v_{n+1} &= v_n + (-.04v_n - 32)h \\ t_{n+1} &= t_n + h. \end{aligned}$$

We can iterate and solve for the maximum height using the following Maple code:

```
# Euler Method solution of Problem #13 on Page 279 of Edwards and Penney 4th Ed.

x0 :=    0:  #initial position
v0 :=  288:  #initial velocity
t0 :=    0:  #initial time
tf :=   10:  #final time
h  := 0.01:  #time step
#-----------------------End of user input ----------
NumOfSteps := (tf-t0)/h:  # number of steps to reach tf
xn := x0:  # initialize xn, vn, tn
vn := v0:
tn := t0:
tmax:=0:    #initialize tmax, the time when x is maximized
xmax:=0:    #initialize xmax, the maximum (height) value of x
#--------------------- Euler Iteration Loop ----------------------
for k from 1 by 1 to NumOfSteps do
xnplus1:= xn + vn*h:
vnplus1:= vn + (-32 - .04*vn)*h:
tnplus1 := tn+ h:
if (vn > 0) and (vnplus1 < 0) then tmax:= tnplus1; xmax:=xnplus1; end if;
# the above line stores max x and t when this occurs when v changes sign
xn := xnplus1: # re-initialize xn,vn,tn
vn := vnplus1:
tn := tnplus1:
od:            # end of Euler iteration
#------------- Output Result ----------------------
 printf( "Euler Method with time step h = %a over the interval t = %a to %a seconds.", h,t0,tf);
 printf( "Maximum approximate height is %a feet which occurs at %a seconds.\n", xmax, tmax );
```

In less than 2 seconds (on my 2010 Toshiba NB205 netbook), the above code outputs:

```
Euler Method with time step h = .1e-1 over the interval t = 0 to 10 seconds.
Maximum approximate height is 1051.536419 feet which occurs at 7.69 seconds.
```

This closely agrees with the exact answer.

From HW 16: p. 279 #12 Edwards & Penney 4th Ed.
Using Euler's method numerically find x(.3) using a step size of 0.1 and 0.05.

$$x'' + x = \sin t; \qquad x(0) = 0.$$

In my version of Edwards and Penney 4^{th} Ed. the second initial condition $x'(0)$ is missing. However, the exact solution is given: $x(t) = \frac{1}{2}(\sin t - t\cos t)$, so $x'(t) = \frac{1}{2}(\cos t - \cos t + t\sin t) = \frac{1}{2}t\sin t$ and so $x'(0) = 0$.

Let $v = x'$ then $x'' + x = \sin t$ becomes $v' = -x + \sin t$ and so we can represent $x'' + x = \sin t$ as the following system of first order DE's:

$$\begin{aligned} x' &= v \\ v' &= -x + \sin t. \end{aligned}$$

The Euler method equations are:

$$\begin{aligned} x_{n+1} &= x_n + v_n h \\ v_{n+1} &= v_n + (-x_n + \sin t_n)h \\ t_{n+1} &= t_n + h. \end{aligned}$$

Initialization:

$$x_0 = 0$$
$$v_0 = 0$$
$$t_0 = 0.$$

If we let $h = .1$ it will take 3 iterations to reach $t = .3$:

Iteration 1:

$$\begin{aligned}
x_1 &= x_0 + v_0 h = 0 + 0(.1) = 0 \\
v_1 &= v_0 + (-x_0 + \sin t_0)h = 0 + (-0 + \sin 0)(.1) = 0 \\
t_1 &= t_0 + h = 0 + .1 = .1
\end{aligned}$$

Iteration 2:

$$\begin{aligned}
x_2 &= x_1 + v_1 h = 0 + 0(.1) = 0 \\
v_2 &= v_1 + (-x_1 + \sin t_1)h = 0 + (-0 + \sin .1)(.1) = .009983341665 \\
t_2 &= t_1 + h = .1 + .1 = .2
\end{aligned}$$

Iteration 3:

$$\begin{aligned}
x_3 &= x_2 + v_2 h = 0 + 0(.1) = 0 + .009983341665(.1) = .0009983341665 \\
v_3 &= v_2 + (-x_2 + \sin t_2)h = .009983341665 + (-0 + \sin .2)(.1) = .02985027474 \\
t_3 &= t_2 + h = .2 + .1 = .3
\end{aligned}$$

So with a step size of $h = .1$ we get the Euler approximations of

$$x(.3) \approx 0009983341665 \text{ and } v(.3) \approx .02985027474.$$

If we let $h = .05$ it will take 6 iterations to reach $t = .3$. I've tabulated the results of the Euler approximations of $x(.3)$ for $h = .1, .05, .001$, and $.0001$ below (see next page for tables).

Note that the approximations when $h = .001$ and $h = .0001$ closely agree with each other.

The approximation of $x(.3)$ using $h = .0001$ took roughly 30 seconds in Maple on my 2010 Toshiba NB205 Netbook (processor = Intel 1.66-GHz Atom N280) including outputting over 3000 lines of code to produce a Latex table with 3000 rows.

See next page for tables.

Euler points for $x'' + x = \sin t$ over $t = [0, .3]$ if the time step $h = .1$.

n	x_n	v_n	t_n
0	0	0	0
1	0.	0.	.1
2	0.	.9983341665e-2	.2
3	.9983341665e-3	.2985027474e-1	.3

Euler points for $x'' + x = \sin t$ over $t = [0, .3]$ if the time step $h = .05$.

n	x_n	v_n	t_n
0	0	0	0
1	0.	0.	.05
2	0.	.2498958464e-2	.10
3	.1249479232e-3	.7490629296e-2	.15
4	.4994793880e-3	.1495628853e-1	.20
5	.1247293814e-2	.2486478110e-1	.25
6	.2490532869e-2	.3717261438e-1	.30

Euler points for $x'' + x = \sin t$ over $t = [0, .3]$ if the time step $h = .001$.

n	x_n	v_n	t_n
0	0	0	0
1	0.	0.	.001
2	0.	.9999998333e-6	.002
⋮	⋮	⋮	⋮
298	.4328222689e-2	.4360742299e-1	.298
299	.4371830112e-2	.4389670371e-1	.299
300	.4415726816e-2	.4418689660e-1	.300

Euler points for $x'' + x = \sin t$ over $t = [0, .3]$ if the time step $h = .0001$.

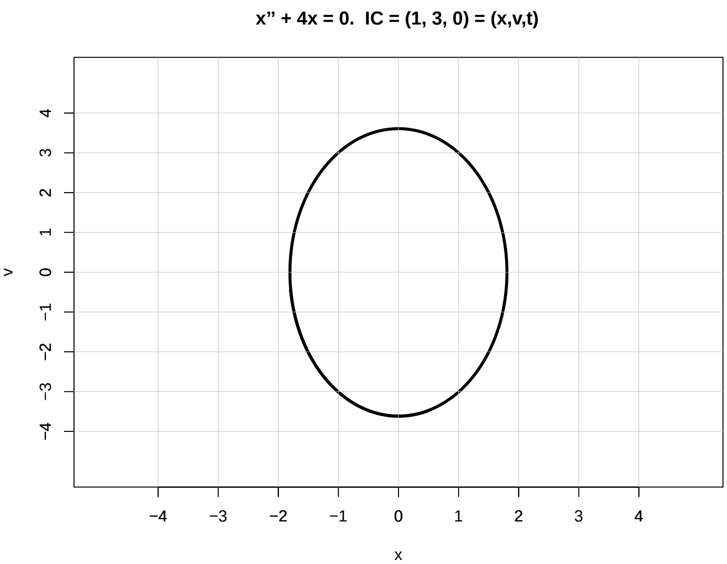
n	x_n	v_n	t_n
0	0	0	0
1	0.	0.	.0001
2	0.	.9999999983e-8	.0002
3	.9999999983e-12	.2999999985e-7	.0003
⋮	⋮	⋮	⋮
2998	.4446377457e-2	.4425573592e-1	.2998
2999	.4450803031e-2	.4428482420e-1	.2999
3000	.4455231513e-2	.4431392159e-1	.3000

Professor Chris McCarthy Mat 501 Notes Handout 13 (3^{nd} Ed.)

Euler Tangent Method

Example 1. Consider the IVP $\ddot{x} + 4x = 0$ with IC: $(x_0, v_0, t_0) = (1, 3, 0)$. As usual we let $\dot{x} = v$. Approximate $(x(1.0), v(1.0))$ using the Euler tangent method with a step size of $\Delta t = 0.5$. Solve the IVP exactly and then find $(x(1.0), v(1.0))$. **Note** that the figure below shows the trajectory of the IVP in (x, v) phase space. Notice that it passes through the IC $(x_0, v_0) = (1, 3)$. Recall that trajectories in (x, v) phase space are the graphs of $(x(t), v(t))$ where x is the solution to IVP's.

Instructions: Put the Euler points (x_n, v_n, t_n) $n = 0,\ 1,\ 2$ on the lines below. Put the tangent vector $T(x_0, v_0, t_0)$ on the indicated line. Put the Euler approximation and the actual values of x(1.0) and $v(1.0)$ on the indicated lines. Put the trajectory of the IVP, i.e, $x(t)$ and $v(t)$ on the indicated lines. Plot/draw the Euler points (x_n, v_n) $n = 0,\ 1,\ 2$; the tangent vector $T(x_0, v_0)$; and the exact values of $(x(1), v(1))$ in the appropriate locations in the (x, v) phase space diagram (below).



$(x_0, v_0, t_0) =$ ______________________

$(x_1, v_1, t_1) =$ ______________________

$(x_2, v_2, t_2) =$ ______________________

$\underbrace{T(x_0, v_0, t_0) = \text{______________________}}_{T \text{ is for tangent vector}}$

$x(1) \approx$ __________ $v(1) \approx$ __________

$x(1) =$ __________ $v(1) =$ __________

$x(t) =$ ______________________________ $v(t) =$ ______________________________

Solution. (a) The tangent vector to (x, v, t) is just its time derivative $(\dot{x}, \dot{v}, \dot{t})$. Since $\dot{x} = v$ and $\dot{v} = -4x$ and $\dot{t} = 1$ the tangent vector is given by $T(x, v, t) = (\dot{x}, \dot{v}, \dot{t}) = (v, -4x, 1)$. The T is for tangent. $T(x, v, t)$ gives the tangent to the trajectory that passes through the point (x, v, t).

Our IC is $(x_0, v_0, t_0) = (1,\ 3,\ 0)$. Then we apply the Euler process:

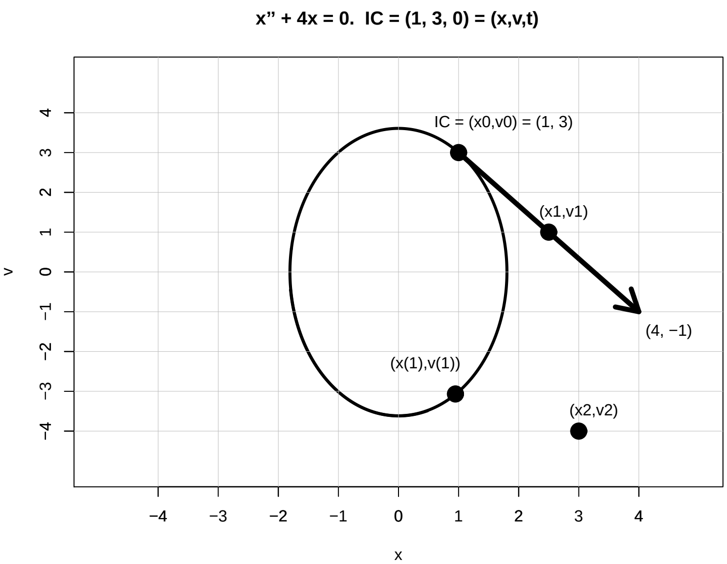
$$\begin{aligned}
(x_1, v_1, t_1) &= (x_0, v_0, t_0) + T(x_0, v_0, t_0) \cdot \Delta t \\
&= (x_0, v_0, t_0) + (v_0, -4x_0, 1) \cdot \Delta t \\
&= (1, 3, 0) + (3, -4, 1) \cdot (0.5) \quad \text{Note: } (3, -4) = \text{the tangent vector at } (x_0, v_0), \text{ it} \\
&= (2.5,\ 1,\ 0.5) \qquad\qquad\qquad \text{goes from } (1, 3) \text{ to } (4, -1) = (1, 3) + (3, -4). \\
(x_2, v_2, t_2) &= (x_1, v_1, t_1) + T(x_1, v_1, t_1) \cdot \Delta t \\
&= (x_1, v_1, t_1) + (v_1, -4x_1, 1) \cdot \Delta t \\
&= (2.5, 1, 0.5) + (1, -10, 1) \cdot (0.5) \\
&= (3,\ -4,\ 1.0)
\end{aligned}$$

(c) We can write the DE as $(D^2 + 4)[x] = 0$ which has imaginary roots $\pm 2i$ so the general solution is $x(t) = c_1 \cos 2t + c_2 \sin 2t$ and $v(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$. Solving for the constants we plug in the IC. We get $x(0) = 1 = c_1$ and $v(0) = 3 = 2c_2$ so $c_2 = 1.5$. We get the exact solution and trajectory,

$$x(t) = \cos 2t + 1.5 \sin 2t$$
$$v(t) = 3\cos 2t - 2\sin 2t$$

which we will evaluate at $t = 1.0$. Plugging in $t = 1.0$ we get: $(x(1), v(1)) = (0.9477993, -3.067035)$.

So we have:



$(x_0, v_0, t_0) = \underline{\quad (1, 3, 0) \quad}$

$(x_1, v_1, t_1) = \underline{\quad (2.5,\ 1,\ 0.5) \quad}$

$(x_2, v_2, t_2) = \underline{\quad (3,\ -4,\ 1.0) \quad}$

$\underbrace{T(x_0, v_0, t_0) = \underline{\quad (3,\ -4,\ 1) \quad}}_{T \text{ is for tangent vector}}$

$x(1.0) \approx \underline{\quad 3 \quad}$ $v(1.0) \approx \underline{\quad -4 \quad}$ from (x_2, v_2, t_2)

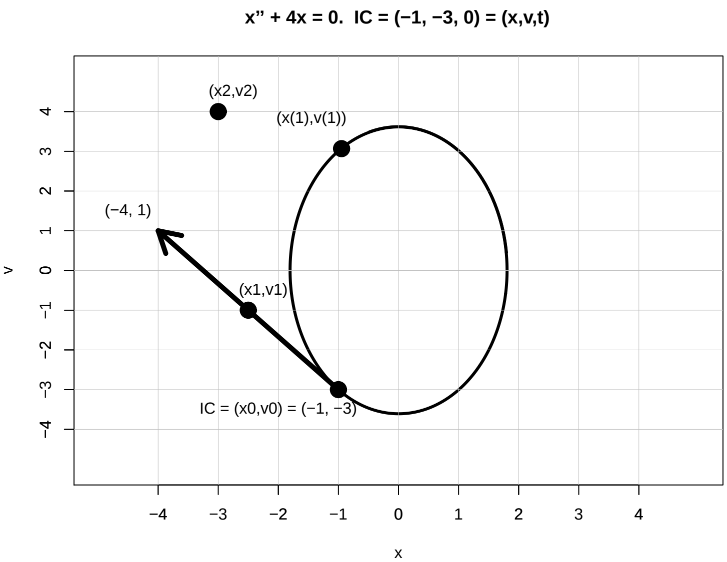
$x(1.0) = \underline{\quad 0.9477993 \quad}$ $v(1.0) = \underline{\quad -3.067035 \quad}$

$x(t) = \underline{\quad \cos 2t + 1.5 \sin 2t \quad}$ $v(t) = \underline{\quad 3\cos 2t - 2\sin 2t \quad}$

See next page →

Example 2. Consider the IVP $\ddot{x} + 4x = 0$ with IC: $(x_0, v_0, t_0) = (-1, -3, 0)$. As usual we let $\dot{x} = v$. Approximate $(x(1.0), v(1.0))$ using the Euler tangent method with a step size of $\Delta t = 0.5$.

Plot and put on the indicated lines: the Euler points for $n = 0, 1, 2$, $T(x_0, v_0, t_0)$, the exact and approximate values for $x(1.0)$ and $v(1.0)$. Put the exact solution to the IVP on the indicated lines. Note: the trajectory through the IC: $(-1, -3)$ is drawn below.



$(x_0, v_0, t_0) =$ $\underline{\quad(-1, -3, 0)\quad}$

$(x_1, v_1, t_1) =$ $\underline{\quad(-2.5,\ -1,\ 0.5)\quad}$

$(x_2, v_2, t_2) =$ $\underline{\quad(-3,\ 4,\ 1.0)\quad}$

$$\underbrace{T(x_0, v_0, t_0) = \underline{\quad(-3,\ 4,\ 1)\quad}}_{T \text{ is for tangent vector}}$$

$x(1.0) \approx$ $\underline{\quad -3 \quad}$ $v(1.0) \approx$ $\underline{\quad 4 \quad}$ from (x_2, v_2, t_2)

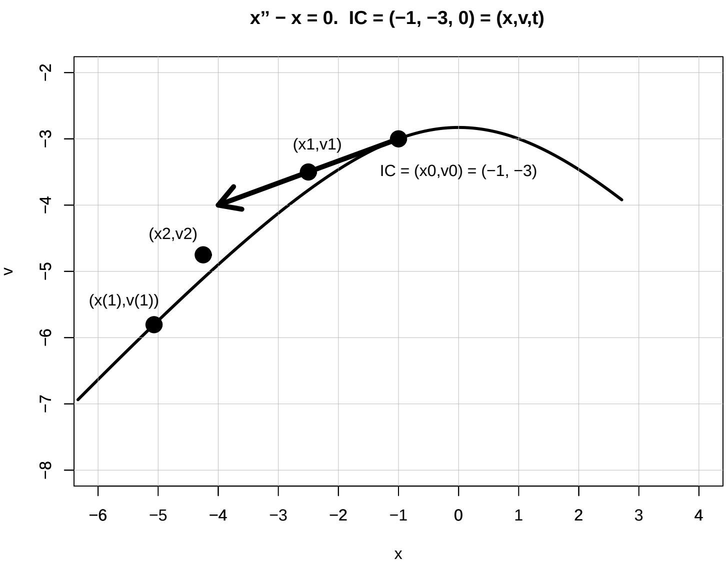
$x(1.0) =$ $\underline{\quad -0.9477993 \quad}$ $v(1.0) =$ $\underline{\quad 3.067035 \quad}$

$x(t) =$ $\underline{\quad -\cos 2t - 1.5 \sin 2t \quad}$ $v(t) =$ $\underline{\quad -3\cos 2t + 2\sin 2t \quad}$

See next page →

Example 3. Consider the IVP $\ddot{x} - x = 0$ with IC: $(x_0, v_0, t_0) = (-1, -3, 0)$. As usual we let $\dot{x} = v$. Approximate $(x(1.0), v(1.0))$ using the Euler tangent method with a step size of $\Delta t = 0.5$.

Plot and put on the indicated lines: the Euler points for $n = 0, 1, 2$, $T(x_0, v_0, t_0)$, the exact and approximate values for $x(1.0)$ and $v(1.0)$. Put the exact solution to the IVP on the indicated lines. Note: a section of the trajectory through the IC: $(-1, -3)$ is drawn below.



$(x_0, v_0, t_0) = \underline{\quad (-1, -3, 0) \quad}$

$(x_1, v_1, t_1) = \underline{\quad (-2.5,\ -3.5,\ 0.5) \quad}$

$(x_2, v_2, t_2) = \underline{\quad (-4.25,\ -4.75,\ 1.0) \quad}$

$\underbrace{T(x_0, v_0, t_0) = \underline{\quad (-3,\ -1,\ 1) \quad}}_{T \text{ is for tangent vector}}$

$x(1.0) \approx \underline{\ -4.25\ }$ $v(1.0) \approx \underline{\ -4.75\ }$ from (x_2, v_2, t_2)

$x(1.0) = \underline{\quad -5.068684 \quad}$ $v(1.0) = \underline{\quad -5.804443 \quad}$

$x(t) = \underline{\quad -2e^t + e^{-t} \quad}$ $v(t) = \underline{\quad -2e^t - e^{-t} \quad}$

Be careful with numerical methods! In Figure 4, we see the results of applying the Euler method to find an approximate solution to $x'' + x = 0$ with IC: $(x, v) = (3, 4)$ for $t \in [0, 35]$ On the left, the time step is 0.1 on the right, the time step is 0.01, but everything else is the same. Recall that the differential equation $x'' + x = 0$ models an undamped harmonic oscillator and that its solution is $x = A\cos(t) + B\sin(t)$ which is periodic. So the phase diagram on the left can't be correct. The problem with the diagram on the left, is that the time step used, $h = \Delta t = 0.1$ is not small enough. The errors in the approximations (i.e, in the Euler points), though small, are large enough that they have the effect of shifting the Euler points (x_n, v_n) onto trajectories each a little further away from the origin than they should be. The result is the misleading trajectory which spirals outwards. On the other hand, in the phase diagram on the right, the trajectory is much more accurate. It shows an ellipse as it should.

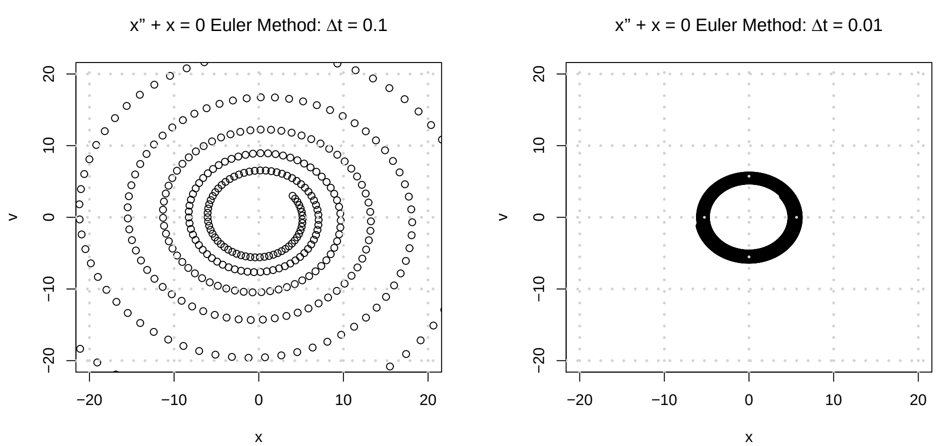


Figure 4: The Euler method applied to the undamped harmonic oscillator $x'' + x = 0$ with IC: $(x, v) = (3, 4)$.

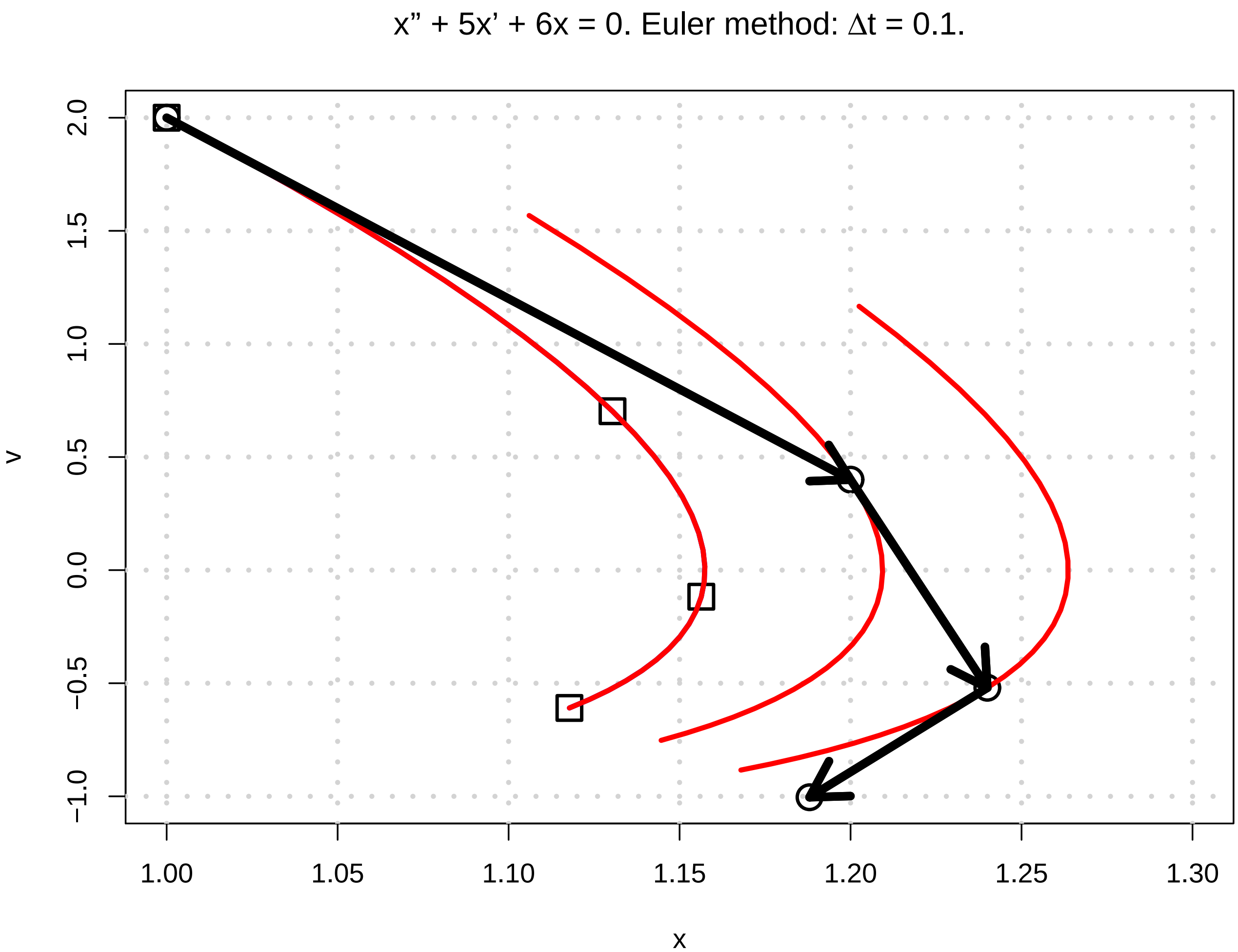
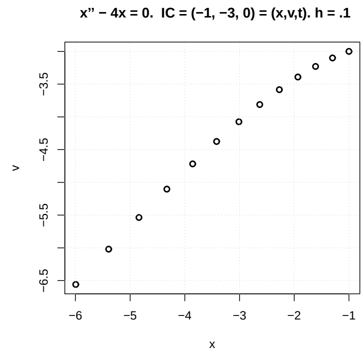


Figure 5: The above Figure illustrates using the Euler method to approximate $(x(.3), \dot{x}(.3))$ for the IVP: $\ddot{x} + 5\dot{x} + 6x = 0$ with IC: $(x_0, v_0, t_0) = (1, 2, 0)$ using a time step of $\Delta t = 0.1$. The red curves are trajectories $(x(t), v(t))$ for $\ddot{x} + 5\dot{x} + 6x = 0$ in (x, v) phase space. The left most trajectory corresponds to the IC from the IVP that we are applying Euler to: $(1, 2, 0)$. It is easy enough to find that trajectory directly: $x(t) = 5e^{-2t} - 4e^{-3t}$ and $v(t) = \dot{x}(t) = -10e^{-2t} + 12e^{-3t}$ and then to plot $(x(t), v(t))$, which we do for $t \in [0, 0.3]$, to produce the left most trajectory. The little squares $\square$ are the exact values of $(x(t), v(t))$ for $t = 0,\ .1,\ .2\ .3$. Notice that trajectory starts at $(1, 2)$, which is the IC. The little circles $\circ$ are the Euler points and the arrows are the tangent vectors scaled by Δt. $T(x, v, t) = (\dot{x}, \dot{v}, \dot{t}) = (v, -6x-5v, 1)$ is the Euler tangent vector, but in (x, v) phase space we only draw the $(\dot{x}, \dot{v})$ part (which is technically called the projection). The Euler point for $n = 1$ is $(x_1, v_1, t_1) = (x_0, v_0, t_0) + T(x_0, v_0, t_0)\Delta t = (1,\ 2,\ 0) + (2,\ -16,\ 1)(0.1) = (1.2,\ 0.4,\ .1)$, which when projected into (x, v) phase space, is $(x_1, v_1) = (1.2,\ 0.4)$, which is on the middle trajectory. The equations for the middle trajectory were found by solving: $\ddot{x} + 5\dot{x} + 6x = 0$ with IC: $(x_1, v_1, t_1) = (1.2,\ 0.4,\ .1)$ to get: $x(t) = 4e^{1/5}e^{-2t} - \frac{14}{5}e^{3/10}e^{-3t}$ and $v(t) = -8e^{1/5}e^{-2t} + \frac{42}{5}e^{3/10}e^{-3t}$ These were plotted for $t \in [0, 0.3]$ to get the middle trajectory. This process was repeated to get the second and third Euler points; the other tangents; and trajectory.

R Script for Euler Tangent Method.

Output: list of Euler Points, Euler Tangents, and plots Euler points. See small figure:




```
titleString = "  x'' - 4x = 0.  IC = (-1, -3, 0) = (x,v,t). h = .1";
x0 = -1; v0 =-3; t0 = 0;  # IC
tf = 1.2;                 # tf = end time = t_final
h = .1;                   # Delta t = time step
#---------------------------------------------------------
T = function(IC){x = IC[1];
                 v = IC[2];
                 t = IC[3];
                 c(v, 4*x , 1);  # put tangent function here!
                 };              # outputs tangent
#--------------------------------------------------------
# Don't change anything below here unless you know what
# you are doing!   :-)
#--------------------------------------------------------
EulerTangentMethod = function(IC,tf,h,T){  # Euler Method Function Start
  En = IC;    # Initialize En (Euler Pts) to IC
  Tn = T(En); # Initialize Tn (Euler Tangents)
  n = 0;      # Initialize n  (Counting Euler pts)
  xn = En[1];  vn = En[2];  tn = En[3];   # More Initialization
  Txn = Tn[1]; Tvn = Tn[2]; Ttn = Tn[3];  # More Initialization
  NumOfSteps = abs(tf-tn)/h; NumOfSteps;  # Number of steps to tf
  h = h*(tf - tn)/abs(tf-tn);             # to allow tf < t0
  for (k in 1:NumOfSteps){   # for loop start
   EnPlus1 = En + h*Tn;        # Euler Method
   En = EnPlus1;               # Iterate
   Tn = T(En);                 # Store results
   n[k+1] = k;
   xn[k+1] =  En[1]; vn[k+1] = En[2]; tn[k+1] = En[3];
   Txn[k+1] = Tn[1]; Tvn[k+1] = Tn[2]; Ttn[k+1] = Tn[3];
   };                       # for loop end
out =   cbind(n,xn,vn,tn,Txn,Tvn);  # output
 }                                  # end of function
#----------------------------------------------
IC = c(x0,v0,t0);
a = EulerTangentMethod(IC,tf,h,T);    # Store output in a
plot(a[,2],a[,3],  lwd = 2,           # plot Euler points
     xlab = "x", ylab = "v",          # a[,2] = x and a[,3] = v
     main = titleString);
grid();                               # add grid to plot
a;                                    # put "a" on screen.
```

You can download R for free from `www.r-project.org`. R is open source, professional quality statistics software. In R, open a new script (menu: File, New script). Copy and paste the above into script. Run the script (menu: Edit, Run all).

Equilibrium solutions.

An equilibrium solution (or point) is an "initial condition" which when plugged into a first order differential equation makes the derivative (or tangent vector) equal to zero. If we are thinking in terms of a physical situation modeled by an ODE, an equilibrium solution is a state, which if the system is in, the system will stay in. For example if we have a harmonic oscillator $x'' + kx = 0$ the equilibrium solution is $(0, 0) = (x, v)$, i.e., the state in which the system has no potential energy ($x = 0$) and no kinetic energy ($v = 0$).

Equilibrium solutions are extremely important since (1) it is important to know which states are fixed; (2) often the system will tend to an equilibrium solution as $t \to \infty$; (3) usually it is easy to find the equilibrium solution; and (4) to understand the geometry of phase diagrams (i.e., pictures of phase space with a lot of trajectories and arrows drawn) you need to know where the equilibrium solutions are.

Equilibrium solutions can be classified as (1) stable: meaning if you perturb the system a little bit away from the equilibrium solution, the system will return to equilibrium. E.g. a damped harmonic oscillator); (2) unstable: meaning if you slightly perturbation of the system away from the equilibrium solution the state of the system will move away from the an equilibrium solution. E.g. if $x(t)$ counts how many bacteria are present on an isolated food source, then $x = 0$ is is the equilibrium solution. However, if a single bacteria is introduced to the food source x will increase at an exponential rate; (3) other classifications which we won't discuss.

A wonderful resource for understanding phase diagrams and equilibrium solutions and the relationship of linear algebra to differential equations is: *Differential Equations, Dynamical Systems, and an Introduction to Chaos* by Morris W. Hirsch, Stephen Smale, and Robert L. Devaney. Smale is one of the giants of modern differential equations and geometry. Robert Devaney did quite a bit of work with fractal geometry and dynamical systems.

To find the equilibrium solution just set the first derivative equal to zero (or the zero vector) and then solve for the variables. Here are some examples.

Example 1: $y' = (y - 2)(y + 3)$ has equilibrium solutions $y = 2$ and $y = -3$ because $0 = (y - 2)(y + 3)$ has solutions $y = 2$ and $y = -3$.

Example 2: The system

$$\begin{aligned} \dot{x} &= v - 5 \\ \dot{v} &= x + v + 1 \end{aligned}$$

has equilibrium solution $(x, v) = (-6, 5)$ because

$$\begin{aligned} 0 &= v - 5 \\ 0 &= x + v + 1 \end{aligned}$$

implies $v = 5$ which, when plugged into the second equation, indicates $x = -6$.

Example 3. The harmonic oscillator $x'' + x = 0$ has equilibrium solution $x, x' = (0, 0)$ because if we write it as a first order ODE:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x \end{aligned}$$

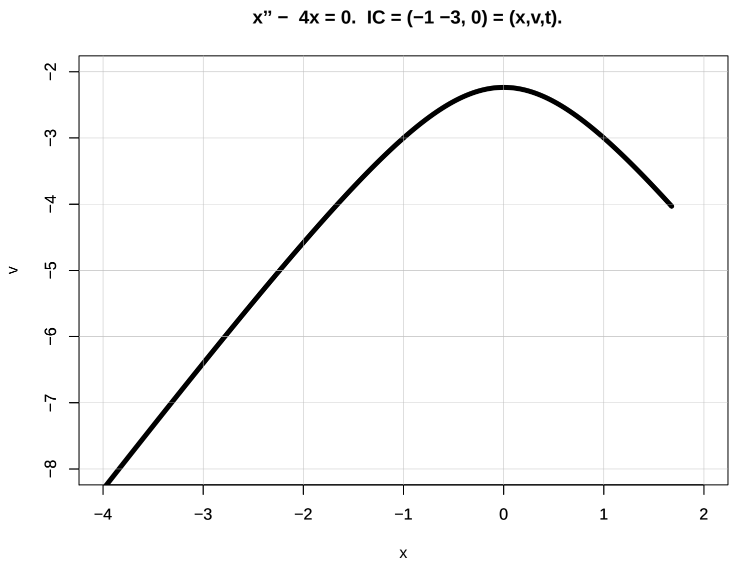
and then set $(\dot{x}, \dot{v}) = (0, 0)$ we get:

$$\begin{aligned} 0 &= v \\ 0 &= -x \end{aligned}$$

Part I of Homework for Quiz II. Hand this in! **Name:**

Q1. Consider the IVP $\ddot{x} - 4x = 0$ with IC: $(x_0, v_0, t_0) = (-1, -3, 0)$. As usual we let $\dot{x} = v$. Approximate $(x(1.0), v(1.0))$ using the Euler tangent method with a step size of $\Delta t = 0.5$.

Plot and put on the indicated lines: the Euler points for $n = 0, 1, 2$, $T(x_0, v_0, t_0)$, the exact and approximate values for $x(1.0)$ and $v(1.0)$. Put the exact solution to the IVP on the indicated lines. Note: a section of the trajectory through the IC: $(-1, -3)$ is drawn below.



$(x_0, v_0, t_0) =$ ________________

$(x_1, v_1, t_1) =$ ________________

$(x_2, v_2, t_2) =$ ________________

$\underbrace{T(x_0, v_0, t_0) = \text{________________}}_{T \text{ is for tangent vector}}$

$x(1) \approx$ ________ $v(1) \approx$ ________

$x(1) =$ ________ $v(1) =$ ________

$x(t) =$ ________________________ $v(t) =$ ________________________

Show work. If extra page is used please STAPLE it. Thanks!

Q2. Find the equilibrium solutions of $y' = (1 - y)y$ Answer ________________________

The logistic DE. Slope fields.

For ODE's, an equilibrium solution is a solution which is a constant, i.e. $y(t) = c$. We can find the equilibrium solution(s) by setting $y' = 0$ in the differential equation and solving for y.

A good way to understand about equilibrium solutions is to consider the following examples from mathematical biology.

Exponential Growth Model. Let $y(t) =$ the number of bacteria in a liter of sea water. Let $t =$ time in hours. Suppose it takes 1 hour for each of those bacteria to reproduce by binary fission[18]. If you focus on 1 of these bacteria, then, after 1 hour you will see it undergo fission and turn into 2 bacteria; after 2 hours that original bacterium will have turned into 4 bacteria, after 3 hours, into $8 = 2^3$ descendants, and so on. So if you start with $y(0)$ bacteria, then after t hours each of those original bacteria will have turned into 2^t descendants. So

$$y(t) = y(0)2^t \tag{36}$$

We can write y in terms of a differential equation: plug $2 = e^{\ln 2}$ into (36) to get

$$y(t) = y(0)2^t = y(0)\left(e^{\ln 2}\right)^t = y(0)\, e^{(\ln 2)t} = y(0)e^{r_0 t} \quad \text{if we let } r_0 = \ln 2;$$

then recall that the differential equation $y' = r_0 y$ has solution $y(t) = y(0)e^{r_0 t}$. So we can model the exponential growth of the bacteria as the ODE

$$y' = r_0 y \tag{37}$$

where, if the bacteria undergo binary fission once every hour, $r_0 = \ln 2$. See [19].

We'll call r_0 the growth constant. Some people call r_0 the growth rate, even though technically it isn't. The growth rate technically is y'. In any case, a bigger r_0 means the population will grow faster. If r_0 is negative, it means the population is decreasing. If r_0 is zero, it means the population size is constant.

Logistic Growth Model. The exponential growth model is fairly accurate when there is only a small population of bacteria, but as the population of bacteria grows, they will start to exhaust the resources of their environment. It is impossible to have exponential growth for any significant length of time because very quickly $e^{r_0 t}$ becomes huge, even if $r_0 > 0$ is quite small. Perhaps the simplest way to improve the exponential growth model is to introduce a constant K called the carrying capacity. For a liter of sea water K might be on the order of 1 billion, meaning that a liter of sea water can support a population of about 1 billion bacteria. If there are less than K bacteria in that liter, the bacteria population will increase, but if there are more than K bacteria, the population will decrease, as there isn't enough food[20]. So, the growth constant r_0 should depend on y. In particular, as y increases r_0 should decrease due to there being less food/resourses for each bacterium. To turn the exponential model (37) into the logistic model (39) we replace r_0 in (37) by the linear function

$$r(y) = r_0 - \frac{r_0}{K}\, y = r_0\left(1 - \frac{y}{K}\right), \tag{38}$$

where r_0 is the growth constant (from the exponential model) under optimal conditions, see [21]. We get the logistic differential equation:

$$y' = r_0\left(1 - \frac{y}{K}\right)\, y \tag{39}$$

See [22]. If we set $y' = 0$ in (39) we get $0 = r_0\left(1 - \dfrac{y}{K}\right)\, y$. Solving for y we get the logistic differential equation's two equilibrium solutions: $y = 0$ (extinction) and $y = K$ (carrying capacity). Note: $y(t) = 0$ and $y(t) = K$

[18]Not all bacteria reproduce by binary fission, however it is the most common method.

[19]If the bacteria undergo binary fission every τ hours, then $y(t) = 2^{t/\tau}$ and so $r_0 = \dfrac{\ln 2}{\tau}$.

[20]The introduction of K is of course an over simplification. There are a huge number of factors which govern the population dynamics of bacteria in sea water.

[21]Note that (a) $r(0) = r_0$, (b) if $y \in (0, K)$, then $r_0 > r(y) > 0$, (c) $r(K) = 0$, (d) if $y > K$, then $r(y) < 0$.

[22]The logistic differential equation $y' = r_0\left(1 - \dfrac{y}{K}\right)\, y = r_0 y - \dfrac{r_0}{K}\, y^2$ is non-linear due to the y^2 term.

are solutions to (39). This is easy to verify by direct substitution: e.g. $0 = \frac{d}{dt}K = r_0\left(1 - \frac{K}{K}\right)K = 0$.

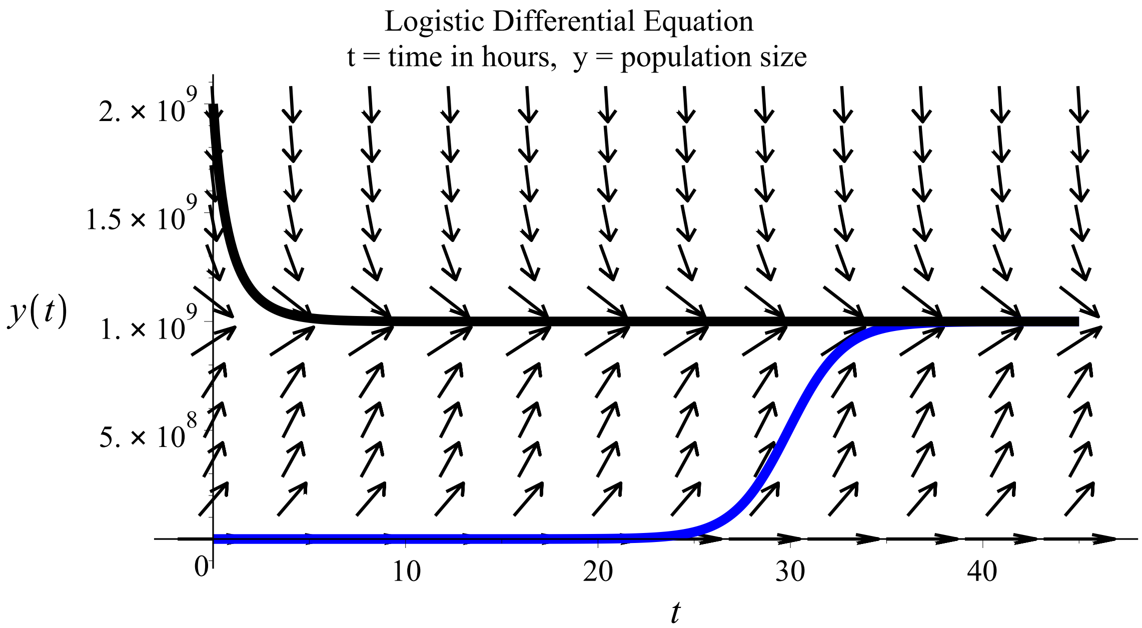


Figure 6: Slope (Direction) field plot of the logistic differential equation $y' = r_0\left(1 - \frac{y}{K}\right)y$. The arrow at (t, y) is tangent to the solution curve passing through the point (t, y). Parameter values: the carrying capacity $K = 1$ billion $= 1 \times 10^9$; $r_0 = \ln 2$ (the population doubles each hour under optimal conditions); $t = 0$ to 45 hours. The initial conditions are $y(0) = 1$ for the lower solution curve, and $y(0) = 2$ billion for the upper solution curve. Notice that the solution curves are asymptotic to the equilibrium solution $y(t) = K$.

Solving the logistic differential equation: Let $\frac{r_0}{K} = \kappa$. Then the logistic differential equation (39) becomes $\frac{dy}{dt} = \kappa(K - y)y$ which separates into $\frac{dy}{y(K - y)} = \kappa\, dt$. Using partial fraction expansion we get $\frac{dy}{y(K-y)} = \frac{1}{K}\left(\frac{1}{y} + \frac{1}{K-y}\right)dy = \kappa\, dt$. Integrating, we get $\frac{1}{K}(\ln|y| - \ln|K - y|) = \kappa t + c_1$, a little algebra and we get

$$\ln\left|\frac{y}{K-y}\right| = K\kappa t + Kc_1. \tag{40}$$

Since $K\kappa = r_0$, if we let $Kc_1 = c_2$ we can rewrite (40) as

$$\ln\left|\frac{y}{K-y}\right| = r_0 t + c_2. \tag{41}$$

We apply e to both sides of (41) and we get (after a little algebra):

$$\left|\frac{y}{K-y}\right| = e^{c_2}\, e^{r_0 t}. \tag{42}$$

It can be shown that if for some t_* that $y(t_*) = 0$ (resp. K), that $y(t)$ is the equilibrium solution $y(t) = 0$ (resp. $y(t) = K$). If y(t) is not one of the equilibrium solutions, then it can be shown that (42) implies

$$\frac{y}{K-y} = c\, e^{r_0 t}. \tag{43}$$

where c is a constant equal to one of e^{c_2} or $-e^{c_2}$. We can now solve for y.

$$
\begin{aligned}
\frac{y}{K-y} &= c\,e^{r_0 t} \\
y &= c(K-y)e^{r_0 t} \\
y &= cKe^{r_0 t} - cye^{r_0 t} \\
y + cye^{r_0 t} &= cKe^{r_0 t} \\
y + cye^{r_0 t} &= cKe^{r_0 t} \\
y\left(1 + ce^{r_0 t}\right) &= cKe^{r_0 t} \\
y &= \frac{cKe^{r_0 t}}{1 + ce^{r_0 t}} \quad \text{then multiply by } \frac{e^{-r_0 t}}{e^{-r_0 t}} \text{ to get:} \\
y &= \frac{cK}{e^{-r_0 t} + c} \quad \text{notice that } \lim_{t\to\infty} \frac{cK}{e^{-r_0 t} + c} = K \qquad (44)
\end{aligned}
$$

If we plug $t = 0$ into (43) and let $y_0 = y(0)$ we get $c = \dfrac{y_0}{K - y_0}$, which if we plug into (44), yields the non-equilibrium solution of the logistic differential equation:

$$
y(t) \;=\; \frac{\dfrac{y_0}{K-y_0}K}{e^{-r_0 t} + \dfrac{y_0}{K-y_0}} \;=\; \frac{\dfrac{y_0}{K-y_0}K}{e^{-r_0 t} + \dfrac{y_0}{K-y_0}}\,\frac{K-y_0}{K-y_0} \;=\; \frac{y_0 K}{(K-y_0)\;e^{-r_0 t} + y_0}. \qquad (45)
$$

If $y(0) = y_0 = 0$ or K, then (45) also gives us the equilibrium solutions. Hence, the solution to the logistic equation for any IC $y(0) = y_0$ is:

$$
y(t) \;=\; \frac{y_0 K}{(K-y_0)\;e^{-r_0 t} + y_0}. \qquad (46)
$$

The Maple code use to produce Figure 6 is:

```
with(plots); with(DEtools);
r0 := log(2);
K := 1000000000;
LogisticDE := [diff(y(t), t) = r0*(1-y(t)/K)*y(t)];
ic := [y(0) = 1, y(0) = (8*(1/4))*K];
t0 := 0;
tf := 45;
DEplot(LogisticDE, y(t), t = t0 .. tf, ic,
  thickness = 6, numpoints = 400, arrows = smalltwo,
  dirfield = [12, 12], color = black, linecolor = [blue, black],
  title = "Logistic Differential Equation \n t = time in hours,  y = population size",
  titlefont = ["ROMAN", "NORMAL", 15],
  axesfont = ["ROMAN", "NORMAL", 15],
  labelfont = ["ROMAN", "NORMAL", 18] )
```

(Qualitative Theory) – Nonlinear systems and phenomena – Chapter 6

The qualitative theory of differential equations is very beautiful and very useful to mathematicians, scientists, and engineers. Qualitative theory focuses on understanding the general behavior of solutions rather on finding a particular solution.

For example, consider the undamped harmonic oscillator $m\ddot{x} + kx = 0$.

If we let $\dot{x} = v$ then $\ddot{x} = \dot{v}$ and we can write the harmonic oscillator DE as $m\dot{v} + kx = 0$, which yields the system

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x \end{aligned} \tag{47}$$

We know that the general solution of $m\ddot{x} + kx = 0$ is $x(t) = c_1 \cos\sqrt{\frac{k}{m}}\ t + c_2 \sin\sqrt{\frac{k}{m}}\ t$, or, using the relationships discussed on page 33, we can write the general solution in the form, $x(t) = A\cos(\omega t + \phi)$ where A is an amplitude; ϕ is a phase angle; and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency. So, since $v = \dot{x}$, the above system (47), has general solution

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi) \\ v(t) &= -A\omega\sin(\omega t + \phi). \end{aligned}$$

So $x(t)$ and $v(t)$ are cyclic with period $\frac{2\pi}{\omega}$. We can calculate what the solution $(x(t), v(t))$ of the system (47) will look like if graphed:

$$\left(\frac{x(t)}{A}\right)^2 + \left(\frac{v(t)}{A\omega}\right)^2 = \cos^2(\omega t + \phi) + \sin^2(\omega t + \phi) = 1.$$

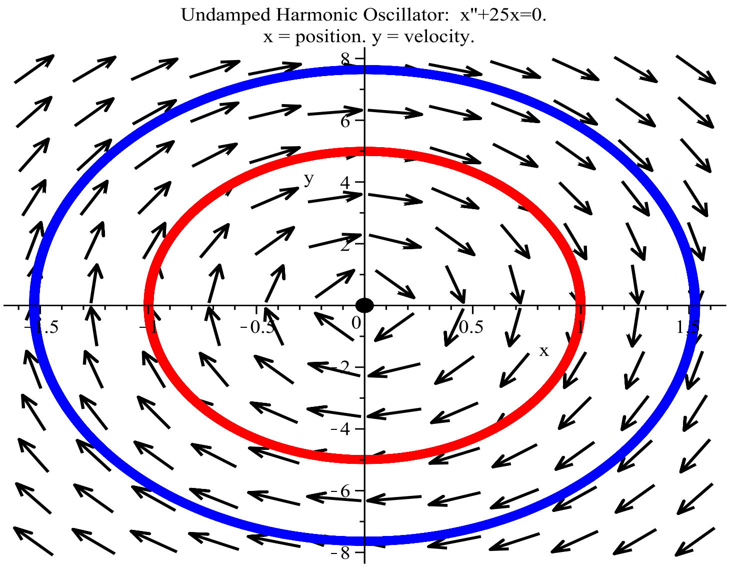
So if we plot a solution $(x(t), v(t))$ we will get an ellipse, see Figure 7a on page 72, provided $A \neq 0$. If $A = 0$ then we get the constant solution $(x(t), v(t)) = (0, 0)$ for all t.

Figure 7a on page 72, was produced using the following Maple code[23]:

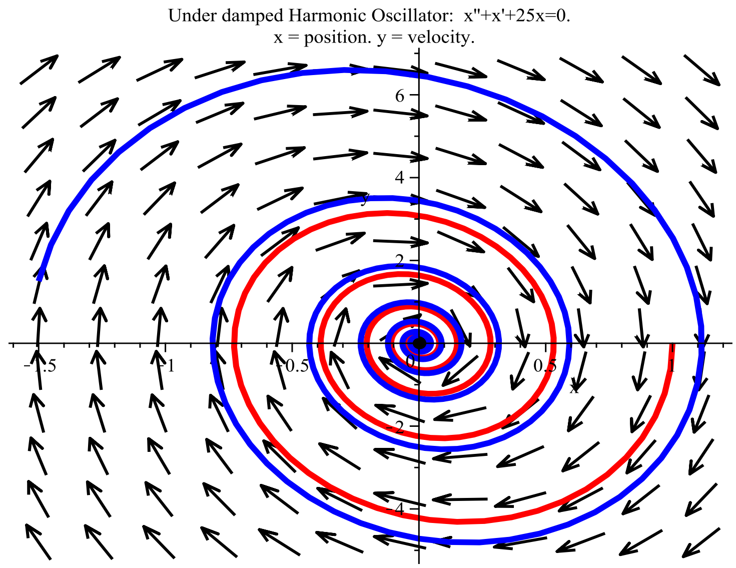
```
with(plots): with(DEtools):
c:=0; # Remember to change title if you change c!
DampedOscDE := [diff(x(t), t) = y(t), diff(y(t), t) = -25*x(t)-c*y(t)];
ic := [[x(0) = 1, y(0) = 0], [x(0) = -1.5, y(0) = 1.5], [x(0) = 0, y(0) = .1]]; #initial conditions
t0 := 0; tf := 10;
DEplot(DampedOscDE, [x(t), y(t)], t = t0 .. tf, ic, thickness = 6, numpoints = 400,
    arrows = smalltwo, dirfield = [12, 12], color = black, linecolor = [red, blue, black],
    title = "Undamped Harmonic Oscillator:  x''+25x=0. \n x = position. y = velocity.")
```

---

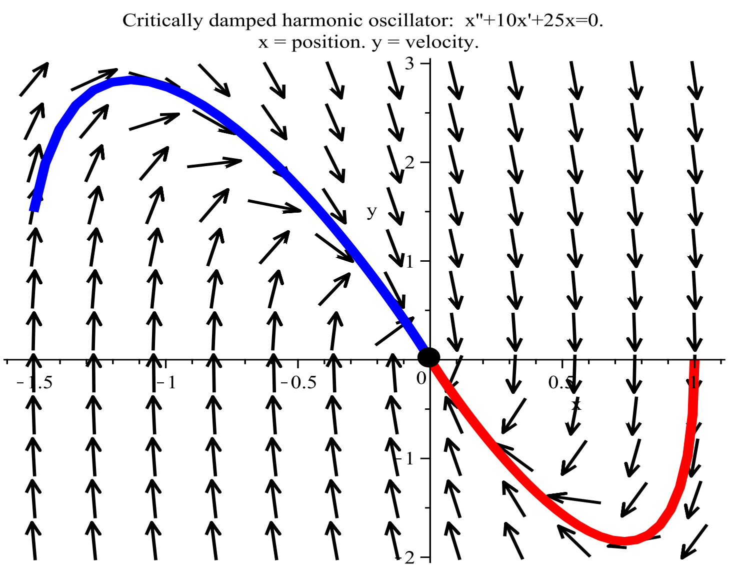
[23]Maple can produce phase portraits with direction fields (the arrows), like in Figure 7a if the DE is a first order autonomous system in x and y, meaning that it can be put into the form $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. Note: autonomous means that DE itself is independent of time. So to get Maple to produce a phase portrait for $x'' + 25x = 0$ we had to convert the DE into the first order system: $x' = y$ and $y' = -25x$. Note: this is exactly the same conversion as we would have done had we wanted to apply Euler's Method, the only difference being that for Maple, we let $x' = y$ rather than letting $x' = v$ – for this kind of plotting, Maple expects us to use the variables x and y.



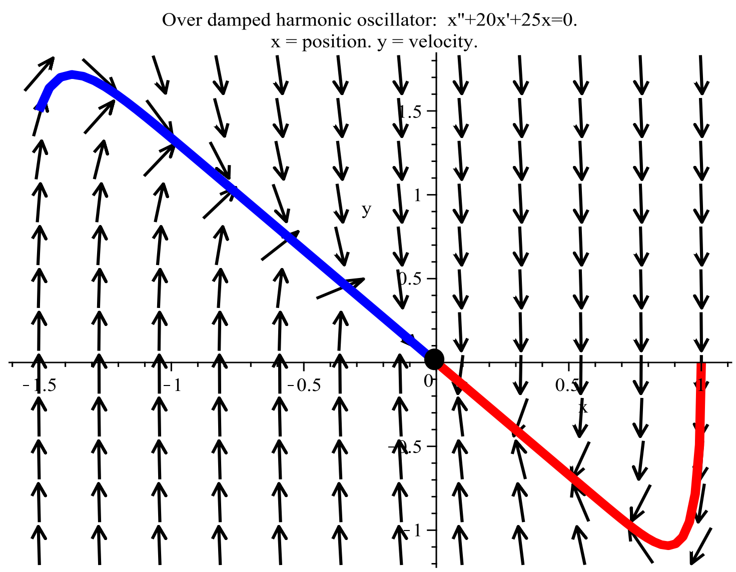
(a) Undamped Oscillator.



(b) Under-damped Oscillator.



(c) Critically-damped Oscillator.



(d) Over-damped Oscillator.

Figure 7: Phase portraits for the harmonic oscillator $m\ddot{x} + c\dot{x} + kx = 0$ with $m = 1, k = 25$ and c varying. The trajectories (solutions) shown are for the initial conditions $(x(0), y(0)) = (1, 0)$, $(-1.5, 1.5)$, and $(0, 0)$;. In (a) the damping constant $c = 0$ so the system is undamped; In (b) the damping constant $c = 1$ and so $c^2 - 4mk < 0$ and the system is under damped, and so the system will oscillate but with smaller and smaller amplitude (as energy is lost due to damping). In (c) the damping constant $c = 10$ and so $c^2 - 4mk = 0$ and so the system is critically damped. In (d) the damping constant $c = 20$ and so $c^2 - 4mk > 0$ and so the system is over damped. Both critically and over damped systems have no over shooting of $x = 0$ and no oscillations. If we have mass m and spring k and we choose c so that the system is critically damped ($c = \sqrt{4mk}$), that choice of c will yield a system in which the mass approaches $x = 0$ fastest without overshoot and without any oscillation.

Short list of important Laplace transforms.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^n (integer $n \geq 0$)	$\frac{n!}{s^{n+1}}$ $(s > 0)$	e^{at}	$\frac{1}{s-a}$ $(s > a)$
$\cos kt$	$\frac{s}{s^2+k^2}$ $(s > 0)$	$\sin kt$	$\frac{k}{s^2+k^2}$ $(s > 0)$
$u(t-a)$	$\frac{e^{-as}}{s}$ $(s > 0)$	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
$\delta_a(t)$ $(a \geq 0)$	e^{-as}	$e^{at} f(t)$	$F(s-a)$
$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2+k^2}$ $(s > a)$	$e^{at} \sin kt$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$
$e^{at} t^n$ (integer $n \geq 0$)	$\frac{n!}{(s-a)^{n+1}}$ $(s > a)$	$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$
$tf(t)$	$-\frac{d}{ds} F(s)$	$f(at)$ $(a > 0)$	$\frac{1}{a} F(s/a)$
$u(t-a)f(t-a)$	$e^{-as} F(s)$		

Laplace Transform Methods – Chapter 8

The Laplace transform is used extensively in engineering. If you intend on being an engineer, you need to know this material. The definition of the Laplace transform $\mathscr{L}$ is as follows:

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

It is traditional to use capital letters to denote the Laplace transforms of functions, e.g. $\mathscr{L}\{x(t)\} = X(s)$ and $\mathscr{L}\{y(t)\} = Y(s)$, and so on. So $\mathscr{L} : f(t) \mapsto F(s)$.

Engineers say the Laplace transform converts a function (or a problem) in the time domain t, to the frequency domain s. This is because s must have units of $\frac{1}{\text{time}}$ because we need for the the $-st$ in the e^{-st} of the definition of the Laplace transform to be dimensionless (to have no units). Since the units of frequency are basically $\frac{1}{\text{units of time}}$, engineers say that the Laplace transform converts a function in the time domain t into a function in the frequency domain s.

Calculating the $\mathscr{L}\{f(t)\}$ can be difficult. Often calculating the Laplace transform involves integration by parts and special techniques. To avoid wasting time calculating these complicated integrals, there are tables of Laplace transforms. However for some simple functions, it is easy to calculate the Laplace transform. For example:

$$\begin{aligned}\mathscr{L}\left\{e^{at}\right\} &= \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{(-s+a)t}\, dt = \lim_{x\to\infty} \int_0^x e^{(-s+a)t}\, dt \\ &= \lim_{x\to\infty} \frac{1}{-s+a} e^{(-s+a)x} \bigg|_0^x = \lim_{x\to\infty} \left(\frac{1}{-s+a} e^{(-s+a)x} - \frac{1}{-s+a} \right) = \frac{1}{s-a}.\end{aligned}$$

provided $s > a$ since $e^{-\infty} = 0$. Note that the above formula implies, if we let $a = 0$ that

$$\mathscr{L}\left\{e^0\right\} = \mathscr{L}\{1\} = \frac{1}{s}$$

provided $s > 0$. Also, note that it is obvious that $\mathscr{L}\{0\} = 0$.

Important facts about the Laplace transform: (1) $\mathscr{L}$ is a linear operator so

$$\mathscr{L}\{af(t) + bg(t)\} = a\mathscr{L}\{f(t)\} + b\mathscr{L}\{g(t)\}$$

(2) $\mathscr{L}$ is invertible in the following sense: given the Laplace transform $F(s)$ we can figure out (at least in theory) its inverse $\mathscr{L}^{-1}\{F(s)\} = f(t)$ where $f(t)$ is uniquely defined on $[0, \infty]$ and $\mathscr{L}\{f(t)\} = F(s)$.

$\mathscr{L}^{-1}$ is linear so that

$$\mathscr{L}^{-1}\{aF(s) + bG(s)\} = a\mathscr{L}^{-1}\{F(s)\} + b\mathscr{L}^{-1}\{G(s)\}.$$

We will mostly figure out transforms, and especially their inverses, by simply looking at a transform table. You can find such tables in your text book, or online. Directly calculating an inverse of a transform (using an integral formula such as Mellin's Formula is not something we will focus on.

Example Find $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}$. **Answer**: since $\mathscr{L}\left\{e^{at}\right\} = \frac{1}{s-a}$ it follows that

$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Laplace transforms of derivatives: The most important formula for Laplace transforms w.r.t. solving IVP differential equations is:

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots - s^{n-k}f^{(k-1)}(0) - \cdots f^{(n-1)}(0).$$

By combining the linearity of the Laplace transform together with the above formula we can solve linear IVP.

Example Find the particular solution of $y'' - 4y = 0$ with IC $y(0) = 3,\ y'(0) = 5$.

Answer We apply $\mathscr{L}$ to the DE and use its linear property to write:

$$\begin{aligned}
\mathscr{L}\left\{y'' - 4y\right\} &= \mathscr{L}\left\{0\right\} \\
\mathscr{L}\left\{y''\right\} - 4\mathscr{L}\left\{y\right\} &= \mathscr{L}\left\{0\right\} \\
\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathscr{L}\{y''\}} - 4Y(s) &= 0 \\
\underbrace{s^2Y(s) - 3s - 5}_{\text{using IC}} - 4Y(s) &= 0 \\
s^2Y(s) - 4Y(s) &= 3s + 5 \\
Y(s)(s^2 - 4) &= 3s + 5 \\
Y(s) &= \frac{3s+5}{s^2-4} = \frac{3s+5}{(s-2)(s+2)}
\end{aligned}$$

Then using partial fraction expansion we have

$$\frac{3s+5}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}$$

Multiply both sides by $(s-2)(s+2)$ to get:

$$3s + 5 = A(s+2) + B(s-2) = \underbrace{(A+B)}_{3}s + \underbrace{(2A-2B)}_{5}$$

There are many different ways to find A and B. One ways is to equation coefficients: $3 = A + B$ and $2A - 2B = 5$. Another way, would be to plug in $s = 0$, to get $5 = 2A - 2B$, and then to differentiate both sides, to get $3 = A + B$. And yet a third way would be to plug in $s = -2$ and $s = 2$: if we plug in $s = -2$ we get $-1 = -4B$ which implies $B = \frac{1}{4}$. If we plug in $s = 2$ we get $11 = 4A$ which implies $A = \frac{11}{4}$. So we have

$$Y(s) = \frac{\frac{11}{4}}{s-2} + \frac{\frac{1}{4}}{s+2}$$

. Next we apply $\mathscr{L}^{-1}$ and use its linearity to get:

$$\begin{aligned}
\mathscr{L}^{-1}\{Y(s)\} &= \mathscr{L}^{-1}\left\{\frac{\frac{11}{4}}{s-2}\right\} + \mathscr{L}^{-1}\left\{\frac{\frac{1}{4}}{s+2}\right\} \\
&= \frac{11}{4}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} \\
&= \frac{11}{4}e^{2t} + \frac{1}{4}e^{-2t}
\end{aligned}$$

because

$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}.$$

Solving this same problem as a linear operator DE: $y'' - 4y = 0$ with IC $y(0) = 3,\ y'(0) = 5$ becomes $(D^2 - 4)[y] = 0$, which has roots $r = \pm 2$, so the general solution is $y = c_1 e^{2t} + c_2 e^{-2t}$ and so $y' = 2c_1 e^{2t} - 2c_2 e^{-2t}$. Using the IC we get $y(0) = c_1 + c_2 = 3$ and $y'(0) = 2c_1 - 2c_2 = 5$. So $y(0)$ implies $c_1 = 3 - c_2$; we plug this into $y'(0)$ and get $2(3 - c_2) - 2c_2 = 6 - 4c_2 = 5$ which implies $1 = 4c_2$ so $c_2 = \frac{1}{4}$ and $c_1 = 3 - \frac{1}{4} = \frac{11}{4}$. So we get $y(t) = \frac{11}{4}e^{2t} + \frac{1}{4}e^{-2t}$ which agrees with our answer using the Laplace transform.

Homework 20A. Page 462 Do problems 1,2, 5, 10, 17, 19

Selected Solutions to HW 20A: p 462 problems 1,2, 5, 10, 17, 19 (Edwards & Penny 4th Ed)

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p462 #1. $x'' + 4x = 0; x(0) = 5, x'(0) = 0.$

Answer using Laplace transform method. Apply $\mathscr{L}$ to the DE. Get $s^2X(s) - sx(0) - x'(0) + 4X(s) = 0$ then some algebra and the IC's yield $(s^2 + 4)X(s) = 5s$ which gives us $X(s) = \frac{5s}{s^2+4} = 5\frac{s}{s^2+2^2}$. This is directly invertible, see table of Laplace transforms, and we get $x(t) = 5\cos 2t$.

Answer using the linear operator D: $x'' + 4x = 0$ is $(D^2 + 4)[x] = 0$ which has roots $r = \pm 2i$ so the general solution is $x(t) = c_1 \cos 2t + c_2 \sin 2t$ and $x'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t$. Using the IC we get $x'(0) = 0 = 2c_2$ so $c_2 = 0$; and $x(0) = 5 = c_1$. So the particular solution is $x(t) = 5\cos 2t$, which agrees with the solution we got using the Laplace transform method.

---

p462 #2. $x'' + 9x = 0; x(0) = 3, x'(0) = 4.$

Answer using Laplace transform method. Apply $\mathscr{L}$ to the DE. Get $s^2X(s) - sx(0) - x'(0) + 9X(s) = 0$ then some algebra and the IC's yield $(s^2 + 9)X(s) = 3s + 4$ which gives us $X(s) = \frac{3s+4}{s^2+9} = 3\frac{s}{s^2+3^2} + 4\frac{1}{s^2+3^2} = 3\frac{s}{s^2+3^2} + \frac{4}{3}\frac{3}{s^2+3^2}$. This is directly invertible, see table of Laplace transforms, and we get $x(t) = 3\cos 3t + \frac{4}{3}\sin 3t$.

---

p462 #5. $x'' + x = \sin 2t; x(0) = 0, x'(0) = 0.$

Answer using Laplace transform method. Apply $\mathscr{L}$ to the DE. Get

$$s^2X(s) - sx(0) - x'(0) + X(s) = \frac{2}{s^2 + 2^2}$$

then some algebra and the IC's yield

$$(s^2 + 1)X(s) = \frac{2}{s^2 + 4}$$

which, after some algebra becomes and partial fraction expansion becomes:

$$X(s) = \frac{2}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 4} + \frac{Ds + E}{s^2 + 1}. \tag{48}$$

Multiplying by $(s^2 + 4)(s^2 + 1)$ yields

$$\begin{aligned} 2 &= (As + B)(s^2 + 1) + (Ds + E)(s^2 + 4) \\ &= As^3 + Bs^2 + As + B + Ds^3 + Es^2 + 4Ds + 4E \\ &= \underbrace{(A + D)}_{=\,0} s^3 + \underbrace{(B + E)}_{=\,0} s^2 + \underbrace{(A + 4D)}_{=\,0} s + \underbrace{(B + 4E)}_{=\,2} \end{aligned}$$

So we have 4 equations in the four unknowns A, B, C, D. $A + 4D = 0$ combined with $A + D = 0$ imply that $A = D = 0$. $B + E = 0$ implies $B = -E$ and this plugged into $B + 4E = 2$ gives us $-E + 4E = 2$, so $3E = 2$, hence $E = \frac{2}{3}$ and $B = -\frac{2}{3}$. So Equation (48) becomes

$$\begin{aligned} X(s) = \frac{-\frac{2}{3}}{s^2+4} + \frac{\frac{2}{3}}{s^2+1} &= \left(-\frac{2}{3}\right)\left(\frac{1}{2}\,\frac{2}{s^2+2^2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{s^2+1^2}\right) \\ &= \left(-\frac{1}{3}\right)\left(\frac{2}{s^2+2^2}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{s^2+1^2}\right) \end{aligned} \tag{49}$$

I wrote

$$\frac{-\frac{2}{3}}{s^2+4} \text{ as } \left(-\frac{1}{3}\right)\left(\frac{2}{s^2+2^2}\right)$$

because, from the Laplace transform table I know that: $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$ so

$$\mathscr{L}^{-1}\left\{\left(-\frac{1}{3}\right)\left(\frac{2}{s^2+2^2}\right)\right\} = \left(-\frac{1}{3}\right)\mathscr{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \left(-\frac{1}{3}\right)\sin 2t.$$

So, in this manner we apply $\mathscr{L}^{-1}$ to Equation (49) and get the IVP's solution:

$$x(t) = -\frac{1}{3}\sin 2t + \frac{2}{3}\sin t. \tag{50}$$

We could have also solved this same IVP, $x'' + x = \sin 2t;\ x(0) = 0, x'(0) = 0$, by entering the following commands into Maple:

```
de:= diff(x(t),t,t) + x(t) = sin(2*t);
dsolve({de, x(0) = 0, D(x)(0)=0} ,x(t));
```

Then, in a few seconds, Maple outputs the solution:

```
             2            1
    x(t) =  - sin(t) - - sin(2 t)
             3            3
```

which matches the solution we obtained, Equation (50), using the Laplace transform.

---

p462 #10. $x'' + 3x' + 2x = t; x(0) = 0, x'(0) = 2$.
Answer using Laplace transform method. Apply $\mathscr{L}$ to the DE. Get

$$s^2X(s) - sx(0) - x'(0) + 3(sX(s) - x(0)) + 2X(s) = \frac{1}{s^2}$$

which after plugging in the IC becomes

$$\begin{aligned} s^2X(s) - 2 + 3sX(s) + 2X(s) &= \frac{1}{s^2} \\ (s^2 + 3 + 2)X(s) &= 2 + \frac{1}{s^2} \end{aligned}$$

Then some algebra yields:

$$\begin{aligned} X(s) &= \frac{2}{s^2+3s+2} + \frac{1}{s^2(s^2+3s+2)} = \frac{2s^2+1}{s^2(s^2+3s+2)} \\ &= \frac{2s^2+1}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \end{aligned} \tag{51}$$

using partial fraction expansion. We solve for A, B, C, D by multiplying both sides by $s^2(s+1)(s+2)$:

$$2s^2+1 = As(s+1)(s+2)+B(s+1)(s+2)+Cs^2(s+2)+Ds^2(s+1) \tag{52}$$

If we let $s=0$, then Equation (52) implies that $1=2B$, so $B=\frac{1}{2}$. If we let $s=-2$ then Equation (52) implies $9=-4D$ so $D=-\frac{9}{4}$. If we let $s=-1$ then Equation (52) implies $3=C$. So Equation (52) becomes:

$$2s^2+1 = As(s+1)(s+2)+\frac{1}{2}(s+1)(s+2)+3s^2(s+2)-\frac{9}{4}s^2(s+1) \tag{53}$$

If we let $s=1$ then Equation (53) becomes $3=6A+3+9-\frac{18}{4}$ which implies $6A=\frac{9}{2}-9$. Clearing the denominator we get: $12A=9-18=-9$, so $A=-\frac{9}{12}=-\frac{3}{4}$. So, with these values for A,B,C,D, Equation (51) becomes:

$$X(s)=\frac{-\frac{3}{4}}{s}+\frac{\frac{1}{2}}{s^2}+\frac{3}{s+1}+\frac{-\frac{9}{4}}{s+2}$$

Using the Laplace transform table we see: $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\}=\frac{1}{t^n}$ and $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$. So we write $X(s)$ as

$$X(s)=\left(-\frac{3}{4}\right)\left(\frac{1}{s}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{s^2}\right)+3\left(\frac{1}{s+1}\right)-\left(\frac{9}{4}\right)\left(\frac{1}{s+2}\right)$$

to make it easy to invert $X(s)$. We get, using the linearity $\mathscr{L}^{-1}$:

$$\begin{aligned}
\mathscr{L}^{-1}\{X(s)\} &= \mathscr{L}^{-1}\left\{\left(-\frac{3}{4}\right)\left(\frac{1}{s}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{s^2}\right)+3\left(\frac{1}{s+1}\right)-\left(\frac{9}{4}\right)\left(\frac{1}{s+2}\right)\right\}\\
&= \left(-\frac{3}{4}\right)\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+\left(\frac{1}{2}\right)\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}+3\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}-\left(\frac{9}{4}\right)\mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\}\\
&= \left(-\frac{3}{4}\right)1+\left(\frac{1}{2}\right)t+3e^{-t}-\left(\frac{9}{4}\right)e^{-2t}\\
&= -\frac{3}{4}+\frac{t}{2}+3e^{-t}-\frac{9}{4}e^{-2t}
\end{aligned}$$

We can also solve this same IVP, $x''+3x'+2x=t;\ x(0)=0, x'(0)=2$ in Maple using the following commands:

```
de:= diff(x(t),t,t) + 3*diff(x(t),t)+2*x(t) = t;
dsolve({de, x(0) = 0, D(x)(0)=2} ,x(t));
```

which yields the solution:

```
                  3   1     9
        x(t) = - - + - t - - exp(-2 t) + 3 exp(-t)
                  4   2     4
```

which matches the solution we obtained using Laplace transform methods.

---

p463 #17. Find

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-3)}\right\}$$

Answer: Theorem 2 (on page 460 of Edwards and Penny 4th Ed) tells us that

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\}=\frac{F(s)}{s} \text{ and } \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}=\int_0^t f(\tau)\,d\tau.$$

So

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} d\tau = \int_0^t e^{3\tau}\, d\tau = \frac{1}{3}e^{3\tau}\bigg|_0^t = \frac{1}{3}e^{3t} - \frac{1}{3}.$$

We could have gotten this answer via partial fraction expansion:

$$\begin{aligned}\frac{1}{s(s-3)} &= \frac{A}{s} + \frac{B}{s-3} \Rightarrow \\ 1 &= A(s-3) + Bs\end{aligned}$$

Letting $s = 3$ implies $B = \frac{1}{3}$ and letting $s = 0$ implies $A = -\frac{1}{3}$, so

$$\frac{1}{s(s-3)} = -\frac{1}{3}\frac{1}{s} + -\frac{1}{3}\frac{1}{s-3}$$

and then using the Laplace transform table we see that

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-3)}\right\} = \mathscr{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s} + -\frac{1}{3}\frac{1}{s-3}\right\} = -\frac{1}{3} + \frac{1}{3}e^{3t}.$$

We could have also found $\mathscr{L}^{-1}\left\{\frac{1}{s(s-3)}\right\}$ using Maple:

```
with(inttrans): #loads integral transform module
F := 1/(s*(s-3));
f := invlaplace(F, s, t)
```

which outputs

```
f := (1/3)*exp(3*t)-1/3
```

Maple can also find the Laplace transform of $f(t)$. For example, Maple can find

$$\mathscr{L}\left\{-\frac{1}{3} + \frac{1}{3}e^{3t}\right\}$$

as follows.

```
> with(inttrans):
f := (1/3)*exp(3*t)-1/3;
F := laplace(f, t, s);
```

Maple outputs:

```
        1
    ---------
    s (s - 3)
```

Homework 20B. Page 463. Do problems 10-12, 21 (E&P 4th Ed.)

Short list of important Laplace transforms.

$f(t)$	$F(s)$	$f(t)$	$F(s)$
t^n (integer $n \geq 0$)	$\frac{n!}{s^{n+1}}$ $(s > 0)$	e^{at}	$\frac{1}{s-a}$ $(s > a)$
$\cos kt$	$\frac{s}{s^2+k^2}$ $(s > 0)$	$\sin kt$	$\frac{k}{s^2+k^2}$ $(s > 0)$
$u(t-a)$	$\frac{e^{-as}}{s}$ $(s > 0)$	$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$
$\delta_a(t)$ $(a \geq 0)$	e^{-as}	$e^{at} f(t)$	$F(s-a)$
$e^{at} \cos kt$	$\frac{s-a}{(s-a)^2+k^2}$ $(s > a)$	$e^{at} \sin kt$	$\frac{k}{(s-a)^2+k^2}$ $(s > a)$
$e^{at} t^n$ (integer $n \geq 0$)	$\frac{n!}{(s-a)^{n+1}}$ $(s > a)$	$\int_0^t f(\tau)\, d\tau$	$\frac{F(s)}{s}$
$t f(t)$	$-\frac{d}{ds} F(s)$	$f(at)$ $(a > 0)$	$\frac{1}{a} F(s/a)$
$u(t-a) f(t-a)$	$e^{-as} F(s)$		

Selected solutions to HW 20B, which was: p. 463 10-12, 21 E&P 4th Ed.
p463 #12

$$x' = x + 2y; \qquad y' = x + e^{-t}; \qquad IC: x(0) = y(0) = 0.$$

Answer: Apply $\mathscr{L}$ to both equations, get:

$$\begin{aligned} sX(s) - x(0_-) &= X(s) + 2Y(s) \\ sY(s) - y(0_-) &= X(s) + \frac{1}{s+1} \end{aligned}$$

using the IC, get

$$\begin{aligned} sX(s) &= X(s) + 2Y(s) \\ sY(s) &= X(s) + \frac{1}{s+1} \end{aligned}$$

Starting with the first equation, we do some algebra to solve for $X(s)$ and $Y(s)$:

$$\begin{aligned} sX(s) &= X(s) + 2Y(s) \Rightarrow \\ (s-1)X(s) &= 2Y(s) \Rightarrow \\ X(s) &= \frac{2}{(s-1)} Y(s). \text{ So the second equation:} \\ sY(s) &= X(s) + \frac{1}{s+1}, \text{ becomes} \\ sY(s) &= \frac{2}{(s-1)} Y(s) + \frac{1}{s+1} \Rightarrow \\ sY(s) - \frac{2}{(s-1)} Y(s) &= \frac{1}{s+1} \Rightarrow \\ \left(s - \frac{2}{(s-1)}\right) Y(s) &= \frac{1}{s+1} \Rightarrow \\ \left(s\frac{(s-1)}{(s-1)} - \frac{2}{(s-1)}\right) Y(s) &= \frac{1}{s+1} \Rightarrow \\ \left(\frac{s^2 - s - 2}{(s-1)}\right) Y(s) &= \frac{1}{s+1}. \text{ But } s^2 - s - 2 = (s-2)(s+1) \Rightarrow \\ Y(s) &= \frac{1}{s+1} \frac{(s-1)}{(s-2)(s+1)} = \frac{s-1}{(s-2)(s+1)^2} \\ X(s) &= \frac{2}{(s-1)} Y(s) = \frac{2}{(s-2)(s+1)^2} \end{aligned}$$

Partial fraction expansion:

$$X(s) = \frac{2}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiply by $(s-2)(s+1)^2$, get:

$$2 = A(s+1)^2 + B(s+1)(s-2) + C(s-2)$$

Letting $s=-1$ implies $2=-3C$, so $C=-\frac{2}{3}$.
Letting $s=2$ implies $2=9A$, so $A=\frac{2}{9}$.
We can figure out B by differentiation:

$$\begin{aligned} 2 &= A(s+1)^2 + B(s+1)(s-2) + C(s-2) \\ &= A(s+1)^2 + B(s^2-s-2) + C(s-2) \end{aligned}$$

Differentiating $2 = A(s+1)^2 + B(s^2-s-2) + C(s-2)$ twice give us $0 = 2A+2B$, so $B=-A=-\frac{2}{9}$. So

$$X(s) = \frac{\frac{2}{9}}{s-2} + \frac{-\frac{2}{9}}{s+1} + \frac{-\frac{2}{3}}{(s+1)^2}$$

so

$$x(t) = \frac{2}{9}e^{2t} - \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t}$$

Note that

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a),$$

(translation in the s domain) and that, if $n \geq 0$ we have

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}.$$

Combining these two, we get:

$$\mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}},$$

which also appears in the Laplace transform table, and so

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}.$$

Partial fraction expansion of $Y(s)$:

$$Y(s) = \frac{s-1}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiply by $(s-2)(s+1)^2$, get:

$$s-1 = A(s+1)^2 + B(s+1)(s-2) + C(s-2)$$

Letting $s=-1$ implies $-2=-3C$, so $C=\frac{2}{3}$.
Letting $s=2$ implies $1=9A$, so $A=\frac{1}{9}$.
We can figure out B by differentiation. Differentiating $s-1 = A(s+1)^2 + B(s^2-s-2) + C(s-2)$ twice give us $0=2A+2B$, so $B=-A=-\frac{1}{9}$. So

$$Y(s) = \frac{\frac{1}{9}}{s-2} + \frac{-\frac{1}{9}}{s+1} + \frac{\frac{2}{3}}{(s+1)^2}$$

so

$$y(t) = \frac{1}{9}e^{2t} - \frac{1}{9}e^{-t} + \frac{2}{3}te^{-t}$$

We get the final answer:

$$x(t) = -\frac{2}{9}e^{2t} + \frac{2}{9}e^{-t} - \frac{2}{3}te^{-t}$$
$$y(t) = \frac{1}{9}e^{2t} - \frac{1}{9}e^{-t} + \frac{2}{3}te^{-t}$$

Based on example 2, page 466 E&P 4th Ed. Find $x(t)$ if

$$X(s) = \frac{3s+19}{s^2+6s+34}.$$

Answer: $s^2+6s+34$ is not factorable over the reals since its determinant is $b^2-4a = 6^2-4(1)(34) < 0$. So we write $s^2+6s+34$ in the form $(s-a)^2+k^2$, because $\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+k^2}\right\} = e^{at}\cos kt$ and $\mathscr{L}^{-1}\left\{\frac{k}{(s-a)^2+k^2}\right\} = e^{at}\sin kt$.

We divide the 6 in the $6s$ by 2 to get 3, so $(s+3)^2 = s^2+6s+9$ hence:

$$s^2+6s+34 = (s+3)^2 + (-9+34) = (s+3)^2+25 = (s+3)^2+5^2$$

and so

$$\begin{aligned}
\frac{3s+19}{s^2+6s+34} &= \frac{3s+19}{(s+3)^2+5^2} \\
&= \frac{3s}{(s+3)^2+5^2} + \frac{19}{(s+3)^2+5^2} \\
&= 3\frac{s+3-3}{(s+3)^2+5^2} + \frac{19}{(s+3)^2+5^2} \\
&= 3\frac{s+3}{(s+3)^2+5^2} - \frac{9}{(s+3)^2+5^2} + \frac{19}{(s+3)^2+5^2} \\
&= 3\frac{s+3}{(s+3)^2+5^2} + \frac{10}{(s+3)^2+5^2} \\
&= 3\frac{s+3}{(s+3)^2+5^2} + 2\frac{5}{(s+3)^2+5^2}
\end{aligned}$$

So

$$x(t) = 3e^{-3t}\cos 5t + 2e^{-3t}\sin 5t.$$

HW 21
p. 472 #1,3,5,9,15,16, 27, 28,31 and
p. 491 #1,3,11, and
p. 502 #1,3

Impulse and the Dirac Delta function $\delta_a(t)$

In physics and engineering an impulse is a force or action applied over a relatively short time interval. Two typical examples: (1) a collision, (2) a voltage spike in an an electric circuit.

Let's analyze a collision, e.g. kicking a ball. Suppose that we kick a ball of mass of m_b at time $t = 0$ seconds. Let's neglect gravity, bouncing, friction, air resistance, the spinning of the ball, the noise the kick makes, and so on, and imagine that this ball will simply travel in a straight line and that its position at time t is given by $x(t)$. When the ball is kicked, the foot collides with the ball, and then, a moment later, the ball is moving with momentum $m_b v_b$ where v_b is the velocity of the ball after it is kicked. In terms of Newton's equation of motion, $F = ma$ we have $f(t) = m_b\ddot{x}$ where $f(t)$ is the force acting on the ball at time t. So $f(t) = 0$, except for that brief moment (at $t = 0$) when the foot is in contact with the ball. If we substitute $v(t)$ for $\dot{x}(t)$, the equation of motion becomes: $f(t) = m_b\dot{v}$.

If we integrate both sides of $f(t) = m_b\dot{v}$ over the interval $t \in [a, b]$, and if we assume $\dot{v}(t)$ exists at every point in $[a, b]$, then the 2nd part of the fundamental theorem of calculus (FTC) tells us that

$$\begin{aligned} \int_a^b f(t)\,dt &= \overbrace{\int_a^b m_b\dot{v}\,dt = m_b v(b) - m_b v(a)}^{\text{if fundamental theorem of calculus applies}} \\ &= \begin{cases} 0, & \text{if } a < b < 0 \text{ or } 0 < a < b\ ; \\ m_b v_b, & \text{if } a < 0 < b. \end{cases}, \end{aligned} \tag{54}$$

since before being kicked the ball's velocity is 0 and after it is kicked (at $t = 0$) the ball's velocity is v_b. However, if the velocity $v(t)$ jumps from from 0 to v_b at $t = 0$, then v is discontinuous at $t = 0$. But then $\dot{v}(0)$ does not exist. This implies that we can not expect the FTC to apply to $\dot{v}$ w.r.t. intervals that contain $t = 0$. Moreover, no ordinary function $f(t)$ can make (54) true.

However, if we really want the FTC to apply, even when it shouldn't; and if we really want an $f(t)$ that satisfies (54), even though no functions do – we are in luck, since there is a work around.

We can use the Dirac delta function $\delta_a(t)$ (which isn't exactly a function) as it has the exact property that we need. In particular: if $g(t)$ is any function, then

$$\int_\alpha^\beta \delta_a(t)g(t)\,dt = \begin{cases} 0, & \text{if } a \notin [\alpha, \beta]; \\ g(a), & \text{if } a \in [\alpha, \beta]. \end{cases}$$

Three notes:

(1) $\delta_0(t)$ is sometimes simply denoted $\delta(t)$ so that $\delta_a(t) = \delta_0(t - a) = \delta(t - a)$.

(2) We have, if $a \geq 0$,

$$\mathscr{L}\{\delta_a(t)\} = \int_0^\infty e^{-st}\delta_a(t)\,dt = e^{-as} \qquad \text{and} \qquad \mathscr{L}\{\delta_0(t)\} = 1.$$

(3) $\delta_a(t)$ has units of $\frac{1}{\text{time}}$.

Returning to the kicked ball problem:

Using the Dirac delta function we can define

$$f(t) = m_b v_b \delta_0(t).$$

Then Newton's equation of motion becomes

$$m_b v_b \delta_0(t) = m_b\ddot{x}.$$

Dividing both sides by m_b and taking the Laplace transform we get:

$$\begin{aligned}
\mathscr{L}\{v_b\delta_0(t)\} &= \mathscr{L}\{\ddot{x}\} \\
v_b \cdot 1 &= s^2X(s) - sx(0) - \dot{x}(0) \\
v_b &= s^2X(s) \\
v_b\frac{1}{s^2} &= X(s) \\
\mathscr{L}^{-1}\left\{v_b\frac{1}{s^2}\right\} &= \mathscr{L}^{-1}\{X(s)\} \\
v_b t &= x(t)
\end{aligned}$$

which of course is the correct solution. Notice that IC are $x(0) = \dot{x}(0) = 0$, so, even though the impact occurs at $t = 0$, we think of the initial conditions as happening first. Sometimes engineers denote this by writing

$$\mathscr{L}\{f''\} = s^2F(s) - sf(0_-) - f'(0_-).$$

A very useful formula is the Heaviside function or step function $u(t-a) = \begin{cases} 0, & \text{if } t < a; \\ 1, & \text{if } t \geq a. \end{cases}$ We have

$$\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s).$$

So to find $\mathscr{L}^{-1}\{e^{-as}F(s)\}$ what you do is first find $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and then you "shift" by a and get $u(t-a)f(t-a)$.

Example. Find

$$\mathscr{L}^{-1}\left\{e^{-3s}\frac{2!}{s^3}\right\}$$

Answer. $\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} = t^2$ so $\mathscr{L}^{-1}\left\{e^{-3s}\frac{2!}{s^3}\right\} = u(t-3)(t-3)^2$

For a second example see the solution of #3 p502 (E&P 4th Ed.) below.

Selected solutions to HW # 21, which was:

p. 472 #1,3,5,9,15,16, 27, 28,31 and p. 491 #1,3,11, and p. 502 #1,3 all in E&P 4th Ed.

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p. 472 #1 Combining $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$, and $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for integers $n \geq 0$, we get: if $f(t) = e^{\pi t}t^4$ then $F(s) = \frac{4!}{(s-\pi)^5} = 24/(s-\pi)^5$.

p. 472 #3 Combining $\mathscr{L}\{e^{at}f(t)\} = F(s-a)$, and $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$, we get: if $f(t) = e^{-2t}\sin 3\pi t$ then $F(s) = \frac{3\pi}{(s+2)^2+(3\pi)^2}$

p. 472 #5 $F(s) = \frac{3}{2s-4} = \frac{3}{2}\frac{1}{s-2}$ so $f(t) = \frac{3}{2}e^{2t}$ because $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$.

p. 472 #9 $F(s) = \frac{3s+5}{s^2-6s+25}$ The determinant of $s^2 - 6s + 25$ is $(6)^2 - 4(1)(25) < 0$ so $s^2 - 6s + 25$ is not factorable over the reals, i.e. its roots are complex. So we write $s^2 - 6s + 25 = (s-a)^2 + k^2$. Since $-6/2 = -3$ we have $a = 3$ and so $s^2 - 6s + 25 = (s-3)^2 + k^2 = (s^2 - 6s + 9) - 9 + 25 = (s-3)^2 + 16$. So $F(s) = \frac{3s+5}{s^2-6s+25} = \frac{3s+5}{(s-3)^2+4^2} = \frac{3s}{(s-3)^2+4^2} + \frac{5}{(s-3)^2+4^2}$. Firstly: $\frac{3s}{(s-3)^2+4^2} = 3\frac{(s-3)+3}{(s-3)^2+4^2} = 3\frac{(s-3)}{(s-3)^2+4^2} + \frac{9}{(s-3)^2+4^2}$. Secondly, $\frac{9}{(s-3)^2+4^2} + \frac{5}{(s-3)^2+4^2} = \frac{14}{(s-3)^2+4^2} = \frac{14}{4}\frac{4}{(s-3)^2+4^2}$. So $F(s) = 3\frac{(s-3)}{(s-3)^2+4^2} + \frac{14}{4}\frac{4}{(s-3)^2+4^2}$ But then $f(t) = 3e^{3t}\cos 4t + \frac{14}{4}e^{3t}\sin 4t$.

p. 472 #15 $F(s) = \frac{1}{s^3-5s^2} = \frac{1}{s^2(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-5}$. Multiplying by $s^2(s-5)$ yields $1 = As(s-5) + B(s-5) + Cs^2$. Letting $s = 0$ implies $1 = -5B$, so $B = -1/5$. Letting $s = 5$ implies $1 = 25C$, so $C = 1/25$. Differentiating once we get $0 = A(2s-5) + B + 2Cs$. Differentiating again yields $0 = 2A + 2C$, so $A = -C = -1/25$. So $F(s) = \frac{-1/25}{s} + \frac{-1/5}{s^2} + \frac{1/25}{s-5}$. So $f(t) = (-1/25)\cdot 1 + (-1/5)t + (1/25)e^{5t} = \frac{1}{25}(-1-5t+e^{5t})$.

p. 472 #27 $x'' + 6x' + 25x = 0$; $x(0) = 2$, $x'(0) = 3$. Applying the Laplace transform we get: $s^2X(s) - sx(0) - x'(0) + 6(sX(s) - x(0)) + 25X(s) = 0$ substituting in the IC and a little algebra yields $(s^2+6s+25)X(s) - (2)s - (3) - 6(2) = 0$. Some algebra: $X(s) = \frac{2s+15}{s^2+6s+25} = \frac{2s+15}{(s+3)^2+4^2}$, we've written the denominator as $(s+3)^2+4^2$ since $s^2+6s+25$ has complex roots and does not factor over the reals. $\frac{2s+15}{(s+3)^2+4^2} = 2\frac{s}{(s+3)^2+4^2} + \frac{15}{(s+3)^2+4^2} = 2\frac{(s+3)-3}{(s+3)^2+4^2} + \frac{15}{(s+3)^2+4^2} = 2\frac{(s+3)}{(s+3)^2+4^2} - \frac{6}{(s+3)^2+4^2} + \frac{15}{(s+3)^2+4^2} = 2\frac{(s+3)}{(s+3)^2+4^2} + \frac{9}{4}\frac{4}{(s+3)^2+4^2}$. Taking the inverse Laplace transform we get: $x(t) = 2e^{-3t}\cos 4t + \frac{9}{4}e^{-3t}\sin 4t$

p. 491 #1 $F(s) = \frac{e^{-3s}}{s^2}$. Use $e^{-as}F(s) = \mathscr{L}\{u(t-a)f(t-a)\}$ and $\mathscr{L}\{t^m\} = \frac{n!}{s^{n+1}}$. Get $f(t) = u(t-3)(t-3)$, so $f(t) = \begin{cases} 0, & t < 3; \\ t-3, & 3 \le t. \end{cases}$

p. 491 #3 $F(s) = \frac{e^{-s}}{s+2}$. We have $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$ so $\mathscr{L}\{e^{-2t}\} = \frac{1}{s+2}$ so $f(t) = u(t-1)e^{-2(t-1)}$.

p. 491 #11 $f(t) = 2$ if $0 \le t < 3$; $f(t) = 0$ everywhere else. Write $f(t) = 2u(t) - 2u(t-3)$. Then $\mathscr{L}\{f(t)\} = 2\frac{1}{s} - 2\frac{e^{-3s}}{s}$.

p. 502 #1 $x'' + 4x = \delta(t); x(0) = x'(0) = 0$. Apply the Laplace transform. Get:

$$s^2X(s) - sx(0_-) - x'(0_-) + 4X(s) = 1$$

which, after substitution of the IC, becomes $s^2X(s) + 4X(s) = 1$, then some algebra: $X(s) = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+2^2}$. Applying the inverse Laplace transform yields $x(t) = \frac{1}{2}\sin 2t$.

p. 502 #3 $x'' + 4x' + 4x = 1 + \delta(t-2); x(0) = x'(0) = 0$. Apply the Laplace transform. Get:

$$s^2X(s) - sx(0_-) - x'(0_-) + 4(sX(s) - x(0_-)) + 4X(s) = \frac{1}{s} + e^{-2s}$$

which, after substitution of the IC, becomes $s^2X(s) + 4sX(s) + 4X(s) = \frac{1}{s} + e^{-2s}$, then some algebra: $(s^2+4s+4)X(s) = \frac{1}{s} + e^{-2s}$, but $s^2+4s+4 = (s+2)^2$, so we have:

$$X(s) = \frac{1}{s(s+2)^2} + \frac{e^{-2s}}{(s+2)^2}.$$

We'll apply the inverse Laplace transform, first to: $F_1(s) = \frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$ Clear denominators, get $1 = A(s+2)^2 + Bs(s+2) + Cs$. Let $s = -2$,get $C = -1/2$. Let $s = 0$,get $A = 1/4$. We'll find B by differentiating twice. First some algebra: $1 = A(s+2)^2 + B(s^2+2s) + Cs$. Differentiating once yields: $0 = A2(s+2) + B(2s+2) + C$. Differentiating again yields: $0 = A2 + B2$, which implies $B = -A = -1/4$. So

$$F_1(s) = \frac{1}{s(s+2)^2} = \frac{1/4}{s} + \frac{-1/4}{s+2} + \frac{-1/2}{(s+2)^2},$$

so $f_1(t) = (1/4) + (-1/4)e^{-2t} + (-1/2)te^{-2t}$. Now we'll apply the inverse Laplace transform to $F_2(s) = \frac{e^{-2s}}{(s+2)^2}$. Since $\mathscr{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$ and since $\mathscr{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = e^{at}t^n$ we have

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{(s+2)^2}\right\} = u(t-2)(t-2)e^{-2(t-2)} = f_2(t).$$

So

$$\begin{aligned} x(t) &= f_1(t) + f_2(t) = (1/4) + (-1/4)e^{-2t} + (-1/2)te^{-2t} + u(t-2)(t-2)e^{-2(t-2)} \\ &= \frac{1}{4}\left(1 - e^{-2t} - 2te^{-2t}\right) + u(t-2)(t-2)e^{-2(t-2)} \end{aligned}$$

Question 1. Using the Laplace transform solve the IVP:

$$x'' + 6x' + 10x = \delta(t-1) + u(t-2) - u(t-3); \quad x(0_-) = 1,\ x'(0_-) = 2.$$

Answer to Question 1. Applying the Laplace transform to the DE yields:

$$\begin{aligned}\mathscr{L}\{x'' + 6x' + 10x\} &= \mathscr{L}\{\delta(t-1) + u(t-2) - u(t-3)\}\\ \mathscr{L}\{x''\} + 6\mathscr{L}\{x\}' + 10\mathscr{L}\{x\} &= \mathscr{L}\{\delta(t-1)\} + \mathscr{L}\{u(t-2)\} - \mathscr{L}\{u(t-3)\}\end{aligned}$$

We let $\mathscr{L}\{x(t)\} = X(s)$ and make use of the following formulas:

$$\begin{aligned}\mathscr{L}\{x''\} &= s^2X(s) - sx(0_-) - x'(0_-)\\ &= s^2X(s) - s - 2 \qquad (\text{Because}: x(0_-) = 1,\ x'(0_-) = 2.)\\ \mathscr{L}\{x'\} &= sX(s) - x(0_-)\\ &= sX(s) - 1\end{aligned}$$

$$\begin{aligned}\mathscr{L}\{\delta(t-a)\} = e^{-as} \quad &\Rightarrow \quad \mathscr{L}\{\delta(t-1)\} = e^{-s}\\ \mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s} \quad &\Rightarrow \quad \mathscr{L}\{u(t-2)\} = \frac{e^{-2s}}{s}\\ &\qquad\ \ \mathscr{L}\{u(t-3)\} = \frac{e^{-3s}}{s}\end{aligned}$$

Hence,

$$\begin{aligned}\underbrace{\left(s^2X(s) - s - 2\right)}_{\mathscr{L}\{x''\}} + 6\underbrace{\left(sX(s) - 1\right)}_{\mathscr{L}\{x'\}} + 10\underbrace{X(s)}_{\mathscr{L}\{x\}} &= \underbrace{e^{-s}}_{\mathscr{L}\{\delta(t-1)\}} + \underbrace{\frac{e^{-2s}}{s}}_{\mathscr{L}\{u(t-2)\}} - \underbrace{\frac{e^{-3s}}{s}}_{\mathscr{L}\{u(t-3)\}}\\ s^2X(s) + 6sX(s) + 10X(s) - s - 2 - 6 &= e^{-s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\\ \left(s^2 + 6s + 10\right)X(s) - (s+8) &= e^{-s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\\ \left(s^2 + 6s + 10\right)X(s) &= s + 8 + e^{-s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}\end{aligned}$$

Then dividing by $s^2 + 6s + 10$ we get:

$$X(s) = \frac{s+8}{s^2+6s+10} + \frac{e^{-s}}{s^2+6s+10} + \frac{e^{-2s}}{s(s^2+6s+10)} - \frac{e^{-3s}}{s(s^2+6s+10)}$$

So,

$$\begin{aligned}x(t) &= \mathscr{L}^{-1}\{X(s)\}\\ &= \mathscr{L}^{-1}\left\{\frac{s+8}{s^2+6s+10} + \frac{e^{-s}}{s^2+6s+10} + \frac{e^{-2s}}{s(s^2+6s+10)} - \frac{e^{-3s}}{s(s^2+6s+10)}\right\}\\ &= \mathscr{L}^{-1}\left\{\frac{s+8}{s^2+6s+10}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-s}}{s^2+6s+10}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+6s+10)}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+6s+10)}\right\}\end{aligned}$$

Finding:

$$\mathscr{L}^{-1}\left\{\frac{s+8}{s^2+6s+10}\right\}.$$

First: we try to factor $s^2+6s+10$, for the partial fraction expansion, but we quickly realize that $s^2+6s+10$ doesn't factor over the reals, since its determinant, $b^2-4ac=(6)^2-4(1)(10)<0$. So we know to write $s^2+6s+10$ in the form $(s-a)^2+k^2$ because $(s-a)^2+k^2$ shows up in the denominator of both

$$\mathscr{L}\left\{e^{at}\cos kt\right\}=\frac{s-a}{(s-a)^2+k^2}\quad\text{and}\quad\mathscr{L}\left\{e^{at}\sin kt\right\}=\frac{k}{(s-a)^2+k^2}.$$

So $s^2+6s+10=(s+3)^2+1$, i.e. $a=-3$ and $k=1$. Then

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+8}{s^2+6s+10}\right\}&=\mathscr{L}^{-1}\left\{\frac{s+8}{(s+3)^2+1}\right\}\\&=\mathscr{L}^{-1}\left\{\frac{(s+3)+5}{(s+3)^2+1}\right\}\\&=\mathscr{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1^2}+5\frac{1}{(s+3)^2+1^2}\right\}\\&=e^{-3t}\left(\cos t+5\sin t\right)\end{aligned}\tag{55}$$

Note: $e^{-3t}\left(\cos t+5\sin t\right)$ is the solution to the homogenous IVP $x''+6x'+10=0;\ x(0_-)=1,\ x'(0_-)=2$.
Finding

$$\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s^2+6s+10}\right\}.$$

The trick to finding this one, is to combine the Laplace transform rule

$$\mathscr{L}^{-1}\left\{e^{-as}F(s)\right\}=\mathscr{L}^{-1}\left\{e^{-as}\mathscr{L}\left\{f(t)\right\}\right\}=u(t-a)f(t-a)\quad\text{with}$$
$$\mathscr{L}\left\{e^{at}\sin kt\right\}=\frac{k}{(s-a)^2+k^2}.$$

So

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-s}}{s^2+6s+10}\right\}&=\mathscr{L}^{-1}\left\{e^{-s}\frac{1}{(s+3)^2+1^2}\right\}\\&=\mathscr{L}^{-1}\left\{e^{-s}\mathscr{L}\left\{e^{-3t}\sin t\right\}\right\}=u(t-1)e^{-3(t-1)}\sin(t-1).\end{aligned}$$

Finding

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+6s+10)}\right\}\quad\text{and}\quad\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s(s^2+6s+10)}\right\}.$$

The trick to finding these is to use partial fraction expansion, together with the methods used already in this same problem. We start by noting that:

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+6s+10)}\right\}=\mathscr{L}^{-1}\left\{e^{-2s}\frac{1}{s\left((s+3)^2+1^2\right)}\right\}$$

So we need to find

$$\frac{1}{s\left((s+3)^2+1^2\right)}$$

which we do using partial fraction expansion:

$$\begin{aligned}\frac{1}{s\left((s+3)^2+1^2\right)}&=\frac{A}{s}+\frac{Bs+C}{(s+3)^2+1^2}\\1&=A\left((s+3)^2+1^2\right)+\underbrace{(Bs+C)s}_{Bs^2+Cs}\end{aligned}$$

Letting $s=0$ implies $1=10A$ so $A=1/10$. Differentiating once yields: $0=2(s+3)A+2Bs+C$. Letting $s=-3$ implies $0=-6B+C$ and so $C=6B$. Differentiating again yields: $0=2A+2B$ which implies $B=-A=-1/10$; and so $C=6B=6(-1/10)=-6/10$. Hence:

$$\begin{aligned}\frac{1}{s\left((s+3)^2+1^2\right)} &= \frac{1/10}{s}+\frac{(-1/10)s-(6/10)}{(s+3)^2+1^2}\\ &= \frac{1}{10}\left(\frac{1}{s}+\frac{-s-6}{(s+3)^2+1^2}\right)\\ &= \frac{1}{10}\left(\frac{1}{s}-\frac{s+6}{(s+3)^2+1^2}\right)\\ &= \frac{1}{10}\left(\frac{1}{s}-\frac{(s+3)+3}{(s+3)^2+1^2}\right)\\ &= \frac{1}{10}\left(\frac{1}{s}-\frac{s+3}{(s+3)^2+1^2}-3\frac{1}{(s+3)^2+1^2}\right)\end{aligned}$$

So

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s\left((s+3)^2+1^2\right)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{10}\left(\frac{1}{s}-\frac{s+3}{(s+3)^2+1^2}-3\frac{1}{(s+3)^2+1^2}\right)\right\}\\ &= \frac{1}{10}\mathscr{L}^{-1}\left\{\frac{1}{s}-\frac{s+3}{(s+3)^2+1^2}-3\frac{1}{(s+3)^2+1^2}\right\}\\ &= \frac{1}{10}\left(1-e^{-3t}\cos t-3e^{-3t}\sin t\right)\end{aligned}$$

So

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s\left(s^2+6s+10\right)}\right\} &= \frac{u(t-2)}{10}\left(1-e^{-3(t-2)}\cos(t-2)-3e^{-3(t-2)}\sin(t-2)\right)\\ \mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s\left(s^2+6s+10\right)}\right\} &= \frac{u(t-3)}{10}\left(1-e^{-3(t-3)}\cos(t-3)-3e^{-3(t-3)}\sin(t-3)\right)\end{aligned}$$

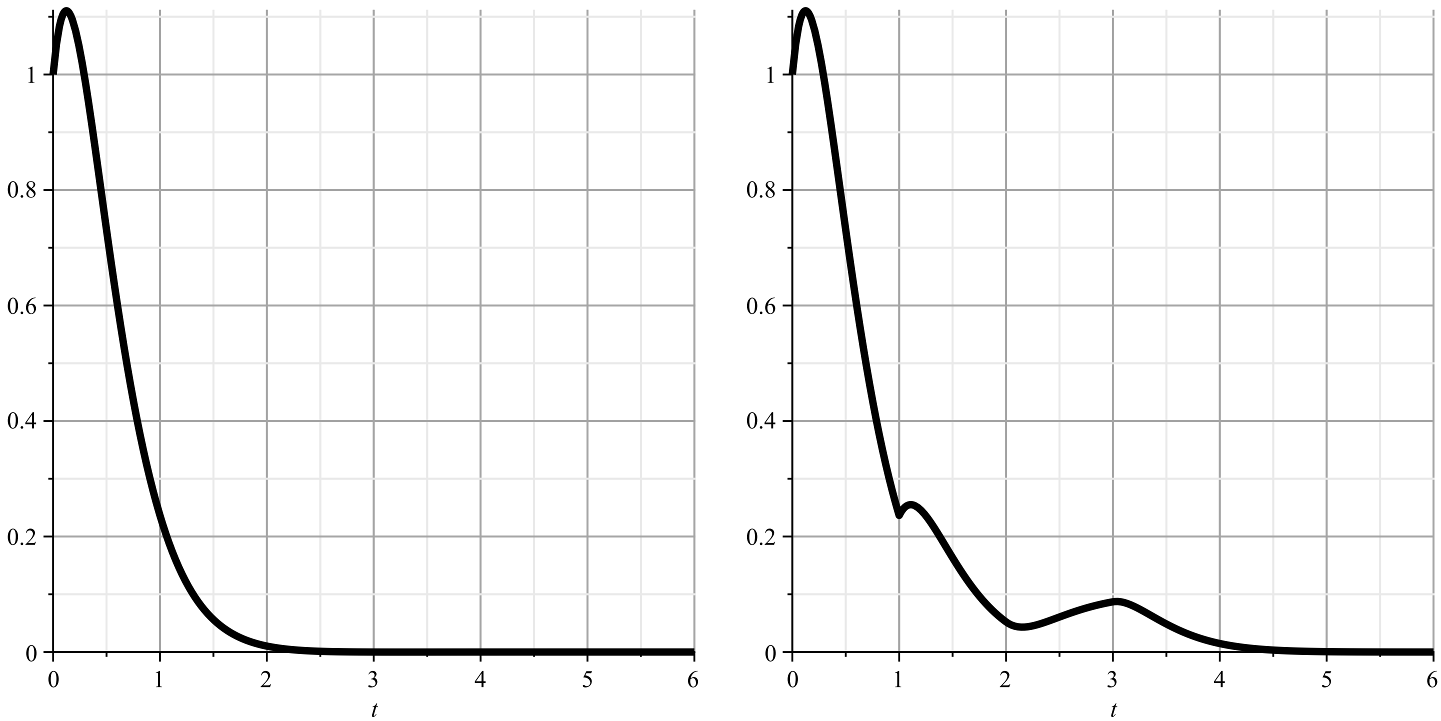
So

$$\begin{aligned}x(t) = {}& e^{-3t}\left(\cos t+5\sin t\right)\\ &+u(t-1)e^{-3(t-1)}\sin(t-1)\\ &+\frac{u(t-2)}{10}\left(1-e^{-3(t-2)}\cos(t-2)-3e^{-3(t-2)}\sin(t-2)\right)\\ &-\frac{u(t-3)}{10}\left(1-e^{-3(t-3)}\cos(t-3)-3e^{-3(t-3)}\sin(t-3)\right)\end{aligned}\tag{56}$$

See Figure 8, on page 88 of these notes, for a graph of $x(t)$, and for an a discussion regarding the effects that the initial conditions and the forcing terms have on $x(t)$.

Also, you can and should check the solution obtained above (i.e. $x(t)$), by solving the DE in Maple with the following commands:

```
ode1 := diff(x(t), t, t)+6*(diff(x(t), t))+10*x(t) =
                Dirac(t-1)+Heaviside(t-2)-Heaviside(t-3);
dsolve({ode1, x(0) = 1, (D(x))(0) = 2}, x(t));
plot(rhs(%), t = 0 .. 6, color = "black", thickness = 4, gridlines);
```



(a) $x'' + 6x' + 10x = 0.$

(b) $x'' + 6x' + 10x = \delta(t-1) + u(t-2) - u(t-3)$.

Figure 8: Comparison of the DE $x''+6x'+10x = f(t)$ with two different forcing terms (or "inputs"), $f(t)$. In (8a) the forcing term is $f(t) = 0$. In (8b) the forcing term is $f(t) = \delta(t-1)+u(t-2)-u(t-3)$. In both (8a) and (8b) the initial conditions are identical: $x(0) = 1,\ x'(0) = 2$. We can think of the DE: $x'' + 6x' + 10x = f(t)$, as modeling the behavior of the mass-dashpot-spring mechanical system: $m\ddot{x} + c\dot{x} + kx = f(t)$ having parameters $m = 1, c = 6$, and $k = 10$. **Regarding (8a):** the values $m = 1,\ c = 6$, and $k = 10$ result in an under-damped system (the definition of "under damped" is on page 36) because the homogenous IVP $(D^2+6D+10)[x] = 0;\ x(0_-) = 1,\ x'(0_-) = 2$ has complex roots $r = -3\pm i$ and solution (see (55) on page 86): $x(t) = e^{-3t}(c_1 \cos t + c_2 \sin t) = e^{-3t}(\cos t + 5 \sin t) = e^{-3t}\sqrt{26}\cos(t - 1.373400767)$, since $1^2 + 5^2 = 26$ and $\arctan(5/1) = 1.373400767$ radians. Here we're using $\sqrt{A^2 + B^2}\cos(bx - \alpha) = A\cos bx + B\sin bx$ which is derived on page 33. The first positive value for which $x(t) = 0$ is when $t - 1.373400767 = \pi/2$, i.e. when $t = 2.944197094$. By that time the damping term e^{-3t} dominates the $\sqrt{26}$ "amplitude" term as $\sqrt{26}e^{-3t} = 0.0007439453256$ at $t = 2.944197094$. As a result, even though $x(t)$ oscillates, the oscillations are so small that the oscillatory motion of $x(t)$ is not visible in (8a). **Regarding (8b):** notice how the initial conditions and the forcing terms in (8b) effect the graph of the solution (which represents the position, $x(t)$, of the mass). At $t = 0$ the solution starts at $x = 1$ with an initial velocity of 2 in the positive direction. This accounts for the solution initially going upwards. However the "spring" pulls the solution (the position of the mass) towards zero. At $t = 1$ the Dirac delta function, which acts as an impulse in the positive x direction, causes the slope (i.e. the velocity, x') to be discontinuous at precisely $t = 1$, where the slope switches instantaneously from being negative to positive. But then, as at all t, the the restorative force of the spring pulls the solution towards 0. At $t = 2$ the Heaviside function $u(t-2)$ starts exerting a unit force in the positive x direction (pushing the solution upwards), but then at $t = 3$ the forcing term $-u(t-3)$ exactly cancels the $u(t-2)$ force (so for $t \geq 3,\ f(t) = 0$) and the spring pulls the solution towards zero without the interference of an opposing force. In (8b), the solution $x(t)$, see (56) on page 87, has the damping terms $e^{-3t}, e^{-3(t-1)}, e^{-3(t-2)}$, and $e^{-3(t-3)}$. These damping terms dominate (8b), just as e^{-3t} dominated in (8a), insuring that $x(t)$'s oscillations will have such small amplitudes that when we plot $x(t)$ it will seem like $x(t) = 0$ once $t > 4.5$, or so.

Chapter 9. Fourier Series and Differential Equations

Notation: in this section vectors in $\mathbf{R^n}$ are denoted by a **boldface** font.
If $\mathbf{v} = (v_1,\ v_2,\ \ldots,\ v_n)$ and $\mathbf{w} = (w_1,\ w_2,\ \ldots,\ w_n)$ are vectors in $\mathbf{R^n}$ the dot product of $\mathbf{v}$ and $\mathbf{w}$ is:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n.$$

Properties of the dot product. If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{R^n}$ and a, b are scalars in $\mathbf{R}$ then:

$$\begin{aligned}
\text{commutative: } & \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \\
\text{bi-linear: } & \mathbf{u} \cdot (a\mathbf{v} + b\mathbf{w}) = a\ (\mathbf{u} \cdot \mathbf{v})\ +\ b\ (\mathbf{u} \cdot \mathbf{w}) \\
& (a\mathbf{v} + b\mathbf{w}) \cdot \mathbf{u} = a\ (\mathbf{v} \cdot \mathbf{u})\ +\ b\ (\mathbf{w} \cdot \mathbf{u}) \\
\text{positive definite: } & \mathbf{v} \cdot \mathbf{v} = 0 \text{ if and only if } \mathbf{v} = \mathbf{0} = (0, 0, \ldots, 0)
\end{aligned}$$

The norm of a vector $\mathbf{v}$ is denoted $||\mathbf{v}||$. The norm of a vector is its length. There are many different ways to measure length (at least in abstract mathematics). In $\mathbf{R^n}$ the most common and important way to measure length is using the Euclidean distance. The Euclidean distance is just the one we use everyday when we measure lengths using a straight ruler. By combining the Pythagorean Theorem (which gives us a formula for Euclidean length) with the notation of the dot product, we get the important formula:

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = (\mathbf{v} \cdot \mathbf{v})^{1/2}$$

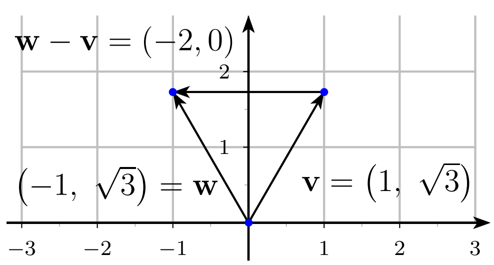
Let $\angle\,\mathbf{vw}$ be the angle formed by the vectors $\mathbf{v}$ and $\mathbf{w}$. A consequence of the the law of cosines from elementary trigonometry[24] is the very important dot product cosine formula:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}||\ \cos(\angle\,\mathbf{vw})$$

For example if $\mathbf{v} = \left(1, \sqrt{3}\right)$ and $\mathbf{w} = \left(-1, \sqrt{3}\right)$ then

$$\begin{aligned}
\mathbf{v} \cdot \mathbf{w} &= \left(1, \sqrt{3}\right) \cdot \left(-1, \sqrt{3}\right) = (1)(-1) + \left(\sqrt{3}\right)\left(\sqrt{3}\right) = -1 + 3 = 2 \\
||\mathbf{v}|| &= (\mathbf{v} \cdot \mathbf{v})^{1/2} = \left(\left(1, \sqrt{3}\right) \cdot \left(1, \sqrt{3}\right)\right)^{1/2} = \left((1)^2 + \left(\sqrt{3}\right)^2\right)^{1/2} = (1 + 3)^{1/2} = \sqrt{4} = 2 \\
||\mathbf{w}|| &= (\mathbf{w} \cdot \mathbf{w})^{1/2} = \left(\left(-1, \sqrt{3}\right) \cdot \left(-1, \sqrt{3}\right)\right)^{1/2} = \left((-1)^2 + \left(\sqrt{3}\right)^2\right)^{1/2} = (1 + 3)^{1/2} = \sqrt{4} = 2 \\
||\mathbf{w} - \mathbf{v}|| &= \left|\left|\left(-1, \sqrt{3}\right) - \left(1, \sqrt{3}\right)\right|\right| = ||(-2, 0)|| = ((-2, 0) \cdot (-2, 0))^{1/2} = (4 + 0)^{1/2} = 2
\end{aligned}$$

We typically draw vectors as arrows. The vector $\mathbf{v}$ is drawn as the arrow from the origin $(0, 0)$ to the point $\left(1, \sqrt{3}\right)$. The vector $\mathbf{w} - \mathbf{v} = (-2, 0)$ is said to be the vector going from $\mathbf{v}$ to $\mathbf{w}$ because $\mathbf{v} + \mathbf{w} - \mathbf{v} = \mathbf{w}$. In terms of arrows, we represent adding $\mathbf{w} - \mathbf{v}$ to $\mathbf{v}$ by translating the arrow representing $\mathbf{w} - \mathbf{v}$ so that the arrow for $\mathbf{w} - \mathbf{v}$ starts at the end of the arrow representing $\mathbf{v}$. See picture:



[24] The law of cosines formula is: $c^2 = a^2 + b^2 - 2ab\cos(\angle\,ab)$

The distance from $\mathbf{v}$ to $\mathbf{w}$ is $||\mathbf{w}-\mathbf{v}||=2$. So the triangle formed by the arrows $\mathbf{v},\mathbf{w}$ and the translation of $\mathbf{w}-\mathbf{v}$ is an equilateral triangle, with each side having length $=2$. All the angles of an equilateral triangle are equal $60°$, so $\angle\mathbf{vw}=60°$, but then $\cos(\angle\,\mathbf{vw})=\cos 60°=1/2$. The values we calculated for $\mathbf{v}\cdot\mathbf{w},\mathbf{v},\mathbf{w},\mathbf{w}-\mathbf{v}$ and $\cos(\angle\,\mathbf{vw})$ are in agreement with the dot product cosine formula:

$$\underbrace{\mathbf{v}\cdot\mathbf{w}}_{=\,2}=\underbrace{||\mathbf{v}||}_{=\,2}\,\underbrace{||\mathbf{w}||}_{=\,2}\,\underbrace{\cos(\angle\,\mathbf{vw})}_{=1/2}.$$

If $\mathbf{v},\ \mathbf{w}$ are perpendicular then $\cos(\angle\mathbf{vw})=0$ and so $\mathbf{v}\cdot\mathbf{w}=||\mathbf{v}||\ ||\mathbf{w}||\ \cos(\angle\mathbf{vw})=0$.

The vectors $\mathbf{v},\ \mathbf{w}$ are called orthogonal if $\mathbf{v}\cdot\mathbf{w}=0$. So two vectors are orthogonal if they are perpendicular to each other, or if one or both of them are the zero vector. For example, the standard basis for $\mathbf{R^2}$, $\mathbf{e_1},\ \mathbf{e_2}$, is orthogonal since:

$$\mathbf{e_1}\cdot\mathbf{e_2}=(1,0)\cdot(0,1)=(1)(0)+(0)(1)=0.$$

If $\mathcal{O}$ is a set of non-zero orthogonal vectors and $\mathbf{v},\mathbf{w}\in\mathcal{O}$ then $\mathbf{v}\cdot\mathbf{w}\neq 0$ if and only if $\mathbf{v}=\mathbf{w}$.

Theorem: If $\mathcal{O}$ is a set of non-zero orthogonal vectors, then the vectors in $\mathcal{O}$ are linearly independent.

Proof: Suppose not. Suppose that $\mathcal{O}$ is linearly dependent. $\mathcal{O}$ being linearly dependent means that at least one vector $\mathbf{v}\in\mathcal{O}$ can be written as a finite linear combination of other vectors $\mathbf{v_i}\in\mathcal{O}, i=1,2,\ldots,n$, with each of the $\mathbf{v_i}\neq\mathbf{v}$. I.e. we can write:

$$\mathbf{v}=a_1\mathbf{v_1}\ +\ a_2\mathbf{v_2}\ +\ \cdots\ +\ a_n\mathbf{v_n} \tag{57}$$

with $a_i\in\mathbf{R}$. To see that Equation (57) is impossible, we take the dot product of $\mathbf{v}$ and Equation (57) and get:

$$\underbrace{\mathbf{v}\cdot\mathbf{v}}_{\neq\,0}\ =\ (a_1\mathbf{v_1}\ +\ a_2\mathbf{v_2}+\cdots\ +\ a_n\mathbf{v_n})\cdot\mathbf{v}\ =\ a_1\underbrace{\mathbf{v_1}\cdot\mathbf{v}}_{=\,0}+a_2\underbrace{\mathbf{v_2}\cdot\mathbf{v}}_{=\,0}+\ \cdots\ +\ a_n\underbrace{\mathbf{v_n}\cdot\mathbf{v}}_{=\,0}=0, \tag{58}$$

which is a contradiction as the left hand side of Equation (58) $=\mathbf{v}\cdot\mathbf{v}\neq 0$ but the right hand side of Equation (58) $=0$. So $\mathcal{O}$ must have been linearly independent.

Orthogonal sets of non-zero vectors are extremely important for the following reason.

Suppose that $\mathcal{O}$ is an orthogonal set of non-zero vectors and that for $i=1,2,\ldots$ we have $\mathbf{v_i}\in\mathcal{O}$. Further suppose that $\mathbf{w}=\sum a_i\mathbf{v_i}$. Then we can calculate the a_i by using the dot product:

$$\mathbf{w}\cdot\mathbf{v_i}=\left(\sum a_i\mathbf{v_i}\right)\cdot\mathbf{v_i}=\sum a_i\left(\mathbf{v_i}\cdot\mathbf{v_i}\right)=a_i\left(\mathbf{v_i}\cdot\mathbf{v_i}\right)$$

But then

$$a_i=\mathbf{w}\cdot\frac{1}{(\mathbf{v_i}\cdot\mathbf{v_i})}\,\mathbf{v_i}$$

We can think of vectors in $\mathbf{R^n}$ as being functions from the set $\{1,2,3,\ldots,n\}$ to $\mathbf{R}$. For example, the vector $\mathbf{v}=\left(1,\sqrt{3}\right)\in\mathbf{R^2}$ is the function

$$\begin{aligned} v:\ \{1,2\}\to\ \mathbf{R}\quad &\text{defined by:}\\ v(1)=v_1&=1\\ v(2)=v_2&=\sqrt{3}\end{aligned}$$

If we think of the vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R^n}$ as functions v and w from $\{1, 2, 3, \ldots, n\}$ to $\mathbf{R}$, then the dot product of $\mathbf{v}$ and $\mathbf{w}$ is the sum

$$v \cdot w = \sum_{k=1}^{n} v(k) w(k) = \int_{\{1,2,3,\ldots,n\}} v(k)\, w(k)\, dk$$

where the integral appearing in the above equality could be justified by noting that the integrals are sums and by letting dk be the the counting measure on $\{1, 2, 3, \ldots, n\}$. The important idea is this, we can, think of the dot product as integrating the product of two functions.

Function spaces.

The set of functions that map some fixed set $\mathcal{S}$ into the real numbers $\mathbf{R}$ form a vector space over $\mathbf{R}$. Meaning that the vectors are the functions and the scalars are the real numbers. If $\mathcal{S} = \{1, 2, 3, \ldots, n\}$ the resulting vector space has n dimensions and is called $\mathbf{R^n}$.

If f and g are two functions, defined on the interval $[a, b]$, we define their dot product and norm to to be

$$f \cdot g = \int_a^b f(t)\, g(t)\, dt \quad \text{and} \quad ||f|| = (f \cdot f)^{1/2}$$

We say that two functions f and g are orthogonal if $f \cdot g = 0$.

If f and g are members of a collection of functions that are periodic with period P we typically define their dot product and norm thusly:

$$f \cdot g = \int_a^{a+P} f(t) g(t)\ dt \quad \text{and} \quad ||f|| = (f \cdot f)^{1/2}$$

By periodicity, the value we choose for the lower limit of integration, a, is unimportant, as long as the upper limit is $a+P$. We'll call the interval $[a, a+P]$ the fundamental interval for that collection. The most important set of periodic orthogonal functions are the sines and cosines. It is traditional to take the fundamental interval to be $[-L, L]$, implying a period of $2L$.

$$\begin{aligned}
\text{fundamental interval} &= [-L, L] \\
\text{Fourier cosines} &= \left\{ \cos\frac{n\pi\, t}{L} \;\middle|\; n = 0, 1, 2, \ldots \right\} \\
\text{Fourier sines} &= \left\{ \sin\frac{n\pi\, t}{L} \;\middle|\; n = 1, 2, \ldots \right\} \\
\text{Fourier sines and cosines} &= \left\{ \cos\frac{n\pi\, t}{L} \;\middle|\; n = 0, 1, 2, \ldots \right\} \bigcup \left\{ \sin\frac{n\pi\, t}{L} \;\middle|\; n = 1, 2, \ldots \right\} \\
&= \text{Fourier sines} \quad \bigcup \quad \text{Fourier cosines}
\end{aligned}$$

We can remember the above definitions by noting that as t goes from $-L$ to L, if $n = 1$, the angle $\dfrac{n\pi\, t}{L}$ goes from $-\pi$ to π, which is one of the standard intervals used in describing the sine and cosine functions. Also, note, if $n = 0$, then $\cos\dfrac{n\pi\, t}{L} = \cos\dfrac{0\pi\, t}{L} = 1$, the constant function.

We say that the function f is even if $f(-x) = f(x)$. Geometrically this means that the graph of f is symmetric about the y-axis.

Examples of even functions: the constant functions, the function $f(x) = 0$, polynomials with only even powers, $\cos cx$ where c is any constant. See Figure 9 on page 100.

We say that the function f is odd if $f(-x) = -f(x)$. Geometrically this means that, to see what the graph of f looks like for negative x, we reflect the positive x part of the graph of f about the y-axis (so that that f looks even) and then we reflect the negative x part about the x-axis (which is the same as multiplying the negative x part by -1).

Examples of odd functions: the function $f(x) = 0$, polynomials with only odd powers, $\sin cx$ where c is any constant. See Figure 9 on page 100.

The following facts about odd and even functions are obvious, especially if you think about them geometrically.

1. An odd function times an odd function is even. Example: $x^3 \times x^5 = x^8$.
2. An odd function times an even function is odd. Example: $x^3 \times x^2 = x^5$.
3. An even function times an even function is even. Example: $x^4 \times x^2 = x^6$.
4. If f is even then
$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$
5. If f is odd then
$$\int_{-L}^{L} f(x)\,dx = 0.$$

Unless otherwise noted, from now on, we will assume that the dot product will be of the form:

$$f \cdot g = \int_{-L}^{L} f(x)g(x)\,dx$$

where $L > 0$.

It is obvious that the even and odd functions are orthogonal to each other with respect to the dot product $f \cdot g = \int_{-L}^{L} f(x)g(x)\,dx$. In particular: if f is even and g is odd (or vice er versa) so that $f(x)g(x)$ is an odd function, then $f \cdot g = 0$, which is the definition of being orthogonal. Example: x^4 and $\sin 4x$ are orthogonal to each other since $x^4 \cdot \sin 4x = \displaystyle\int_{-L}^{L} x^4 \sin 4x\,dx = 0$.

The first main theorem of Fourier analysis is that the Fourier sines and cosines form an orthogonal set of vectors:

Theorem:
The Fourier sines and cosines form an orthogonal set of vectors, with respect to the dot product

$$f \cdot g = \int_{-L}^{L} fg\,dt.$$

The dot products of the Fourier sines and cosines are as follows.

Let m, n be non negative integers, i.e. $m, n \in\ 0, 1, 2, 3, \ldots$ then:

$$\cos\frac{n\pi\, t}{L}\ \cdot\ \cos\frac{m\pi\, t}{L} = \begin{cases} 0, & m \neq n; \\ 2L, & m = n = 0. \\ L, & m, n > 0 \text{ and } m = n; \end{cases} \tag{59}$$

Let m, n be positive negative integers, i.e. $m, n \in\ 1, 2, 3, \ldots$ then:

$$\sin\frac{n\pi\, t}{L}\ \cdot\ \sin\frac{m\pi\, t}{L} = \begin{cases} 0, & m \neq n; \\ L, & m = n. \end{cases} \tag{60}$$

Let $n \in\ 0, 1, 2, 3, \ldots$ and $m \in\ 1, 2, 3, \ldots$ then:

$$\cos\frac{n\pi\, t}{L}\ \cdot\ \sin\frac{m\pi\, t}{L}\ =\ 0 \tag{61}$$

Note: Although this theorem might seem difficult to remember, all it says are two or three things: (1) that the Fourier sines and cosines are orthogonal; (2) that the dot product of a Fourier sine or cosine with itself is L, except for the constant function 1 (which is a Fourier cosine since if we let $n = 0$ in $\cos\dfrac{n\pi\, t}{L}$ we get the constant function 1) which has the property that $1 \cdot 1 = 2L$.

Proof: Since the Fourier cosines are even and the Fourier sines are odd, and since even $\times$ odd = odd, Equation (61) is true. Since $\displaystyle\int_{-L}^{L} (1)(1)\, dt = 2L$ the $n = m = 0$ part of Equation (59) is true.

To prove the rest of theorem requires more work.

From Euler formula:

$$e^{\theta i} = \cos\theta + i\sin\theta$$

it follows that:

$$e^{-\theta i}\ =\ \cos(-\theta) + i\sin(-\theta)\ =\ \cos\theta - i\sin\theta$$

which implies

$$\cos\theta\ =\ \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \qquad \text{and} \qquad \cos\frac{n\pi\, t}{L} = \frac{1}{2}\left(e^{i\dfrac{n\pi\, t}{L}} + e^{-i\dfrac{n\pi\, t}{L}}\right) \tag{62}$$

$$\sin\theta\ =\ \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \qquad \text{and} \qquad \sin\frac{n\pi\, t}{L} = \frac{1}{2i}\left(e^{i\dfrac{n\pi\, t}{L}} - e^{-i\dfrac{n\pi\, t}{L}}\right). \tag{63}$$

We will prove Equation (60). Equation (59) is proved similarly.

$$\begin{aligned}
\sin\frac{n\pi\, t}{L}\ \cdot\ \sin\frac{m\pi\, t}{L} &= \int_{-L}^{L} \sin\frac{n\pi\, t}{L}\ \sin\frac{m\pi\, t}{L}\, dt \\
&= \int_{-L}^{L} \frac{1}{2i}\left(e^{i\dfrac{n\pi\, t}{L}} - e^{-i\dfrac{n\pi\, t}{L}}\right)\ \frac{1}{2i}\left(e^{i\dfrac{m\pi\, t}{L}} - e^{-i\dfrac{m\pi\, t}{L}}\right)\, dt \\
&= \frac{1}{-4}\int_{-L}^{L}\left(e^{i\dfrac{(n+m)\pi\, t}{L}} - e^{i\dfrac{(n-m)\pi\, t}{L}} - e^{i\dfrac{(m-n)\pi\, t}{L}} + e^{i\dfrac{(n+m)\pi\, t}{L}}\right)\, dt
\end{aligned} \tag{64}$$

The integral on line (64) is actually easy to evaluate because for any integer k, if $k \neq 0$ then

$$\int_{-L}^{L} e^{i\frac{k\pi\, t}{L}}\, dt = \frac{L}{i\pi k}\, e^{i\frac{k\pi\, t}{L}}\Bigg|_{-L}^{L} = \frac{L}{i\pi k}\left(e^{i\frac{\pi k\, L}{L}} - e^{i\frac{-\pi k L}{L}}\right) = \frac{L}{i\pi k}\left(e^{i\pi k} - e^{-i\pi k}\right) = \frac{L}{i\pi k}\left(\left(e^{i\pi}\right)^k - \left(e^{-i\pi}\right)^k\right) = 0 \tag{65}$$

because $e^{i\pi} = e^{-i\pi} = -1$. On the other hand, if $k = 0$ then

$$\int_{-L}^{L} e^{i\frac{k\pi\, t}{L}}\, dt = \int_{-L}^{L} e^{0}\, dt = \int_{-L}^{L} dt = 2L \tag{66}$$

Now we can easily evaluate the integral on line (64). We are assuming that m and n are positive integers.

If $n \neq m$, then $(n-m), (m-n)$ and $(n+m)$ are all non-zero integers, and so by (65) the integral on line (64) must be zero.

If $n = m$, then $(n-m) = (m-n) = 0$ and $(n+m) \neq 0$, so by (65) and (66)

$$\text{the integral on line (64)} = \frac{1}{-4}\left(0 - 2L - 2L + 0\right) = \frac{-4L}{-4} = L$$

Hence

$$\sin\frac{n\pi\, t}{L} \cdot \sin\frac{m\pi\, t}{L} = \begin{cases} 0, & m \neq n; \\ L, & m = n. \end{cases}$$

Suppose that $f(t)$ is a function on $\mathbf{R}$ and that

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi\, t}{L} + b_n \sin\frac{n\pi\, t}{L}\right) \qquad \leftarrow \text{Fourier series of } f \tag{67}$$

then, since the Fourier sines and cosines are orthogonal, by Equations (59), (60), and (61), we have:

$$f \cdot 1 = \int_{-L}^{L} f(t)1\, dt = \int_{-L}^{L} \frac{a_0}{2}\, dt = La_0$$

$$\text{for } n = 1, 2, 3, \ldots \qquad f \cdot \cos\frac{n\pi\, t}{L} = \int_{-L}^{L} f(t)\cos\frac{n\pi\, t}{L}\, dt = \int_{-L}^{L} a_n \cos\frac{n\pi\, t}{L}\cos\frac{n\pi\, t}{L}\, dt = La_n$$

$$\text{for } n = 1, 2, 3, \ldots \qquad f \cdot \sin\frac{n\pi\, t}{L} = \int_{-L}^{L} f(t)\sin\frac{n\pi\, t}{L}\, dt = \int_{-L}^{L} b_n \sin\frac{n\pi\, t}{L}\sin\frac{n\pi\, t}{L}\, dt = Lb_n$$

which allows us to calculate the Fourier series coefficients a_n and b_n:

$$\text{for } n = 0, 1, 2, 3, \ldots \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi\, t}{L}\, dt$$

$$\text{for } n = 1, 2, 3, \ldots \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi\, t}{L}\, dt.$$

Theorem. Suppose that $f(t)$ is has period $2L$ and is piecewise continuous on $[-L, L]$ then the Fourier series for f converges to $f(t)$ except at those t where $f(t)$ is not continuous. If f is discontinuous at t then, at t, the Fourier series for f will converge to $\frac{1}{2}\left(f(t_-) + f(t_+)\right)$ where $f(t_-)$ and $f(t_+)$ are the left and right handed limits of f at t.

Note: Piecewise continuous means that on the interval $[-L, L]$ there are at most finitely many discontinuities and that every discontinuity is a jump discontinuity. In particular $f(t)$ is bounded on $[-L, L]$.

Proof. See advanced texts on Fourier Analysis.

Example 1:
See Figure 10 on page 101. Find the Fourier series of the square wave function of period 2:

$$f(t) = \begin{cases} -1, & \text{if } t \in (-1,0); \\ 0, & \text{if } t = 0; \\ +1, & \text{if } t \in (0,1). \end{cases}$$
$$f(t+2) = f(t)$$

Solution: The Fourier series of $f(t)$ is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L} \right) \tag{68}$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\frac{n\pi t}{L}\, dt \qquad n = 0, 1, 2, 3, \ldots \tag{69}$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\frac{n\pi t}{L}\, dt \qquad n = 1, 2, 3, \ldots \tag{70}$$

In the formulas the period is assumed to be $2L$. The square wave $f(t)$ has period 2. So, for $f(t)$, $L = 1$ and so $\frac{1}{L} = 1$ hence:

$$\begin{aligned}
a_0 &= \frac{1}{L} \int_{-L}^{L} f(t)\, dt = \int_{-1}^{0} (-1)\, dt + \int_{0}^{1} (+1)\, dt = -1 + 1 = 0 \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(t) \cos\frac{n\pi t}{L}\, dt \\
&= \int_{-1}^{0} (-1) \cos n\pi t\, dt + \int_{0}^{1} (+1) \cos n\pi t\, dt \\
&= \left[-\frac{1}{n\pi} \sin n\pi t \right]_{-1}^{0} + \left[\frac{1}{n\pi} \sin n\pi t \right]_{0}^{1} \\
&= [\,-0 - -0\,] \qquad + \qquad [\,0 - 0\,] \qquad = \qquad 0; \qquad n = 1, 2, 3, \ldots
\end{aligned}$$

Of course, since the square wave is an odd function, all the a_n must be zero. Why? Because odd functions are orthogonal to the even functions and the cosines are even. So the above calculation showing $a_n = 0$ was unnecessary.

$$
\begin{aligned}
b_n &= \frac{1}{L}\ \int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt \\
&= \int_{-1}^{0}(-1)\sin n\pi t\,dt + \int_{0}^{1}(+1)\sin n\pi t\,dt \\
&= \left[--\frac{1}{n\pi}\ \cos n\pi t\right]_{-1}^{0} + \left[-\frac{1}{n\pi}\ \cos n\pi t\right]_{0}^{1} \\
&= \left[\frac{1}{n\pi}\ \cos n\pi t\right]_{-1}^{0} - \left[\frac{1}{n\pi}\ \cos n\pi t\right]_{0}^{1} \\
&= \left[\frac{1}{n\pi}\ \cos n\pi t\right]_{-1}^{0} - \left[-\frac{1}{n\pi}\ \cos n\pi t\right]_{1}^{0} \qquad \text{because } \Big|_{0}^{1} = -\Big|_{1}^{0} \\
&= \frac{1}{n\pi}\ \left(\cos n\pi t\Big|_{-1}^{0} +\ \cos n\pi t\Big|_{1}^{0}\right) \\
&= \frac{1}{n\pi}\ \left(\cos n\pi t\Big|_{1}^{0} +\ \cos n\pi t\Big|_{1}^{0}\right) \qquad \text{note: } \cos n\pi t\Big|_{-1}^{0} = \cos n\pi t\Big|_{1}^{0} \text{ because } \cos(x) = \cos(-x) \\
&= \frac{2}{n\pi}\ \cos n\pi t\Big|_{1}^{0} \\
&= \frac{2}{n\pi}\ (\cos 0 - \cos n\pi) \\
&= \begin{cases} \dfrac{2}{n\pi}\ (1 - -1) = \dfrac{4}{n\pi}, & \text{if } n = 1,\ 3,\ 5,\ \text{ odd, } \ldots \\[2ex] \dfrac{2}{n\pi}\ (1-1) \quad = \quad 0, & \text{if } n = 2,\ 4,\ 6,\ \text{ even, } \ldots \end{cases}
\end{aligned}
$$

So, keeping in mind that for the square wave $f(t)$ of period 2: $L = 1$, $a_n = 0$, and b_n is as described immediately above, the Fourier series for $f(t)$ is

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right) \\
f(t) &= \frac{0}{2} + \sum_{n=1}^{\infty}(0\cos n\pi t + b_n\sin n\pi t) \\
&= \sum_{n=1}^{\infty} b_n\sin n\pi t \\
&= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{4}{n\pi}\ \sin n\pi t \qquad \leftarrow \quad \textit{answer} \\
&= \frac{4}{\pi}\ \sin\pi t + \frac{4}{3\pi}\ \sin 3\pi t + \frac{4}{5\pi}\ \sin 5\pi t + \cdots \qquad \leftarrow \quad \textit{answer}
\end{aligned}
$$

See Figure 10 on page 101.

Some useful integration formulas, which can be derived using integration by parts, for finding the Fourier series of polynomial $f(t)$ are:

$$\int u^n \cos u\, du = u^n \sin u - n \int u^{n-1} \sin u\, du \tag{71}$$

$$\int u^n \sin u\, du = -u^n \cos u + n \int u^{n-1} \cos u\, du \tag{72}$$

Homework 24a. (E & P 4th Ed.) Read Example 2 on Page 587. Do p. 588, # 1, 15, 21.

Example. Find the Fourier series of $f(t) = t,\ t \in (-L, L)$
Solution. $f(t)$ is odd, so $a_n = 0$ for all n. We want to find

$$b_m = \int_{-L}^{L} t \sin \frac{n\pi\, t}{L}\, dt$$

Using odd $\times$ odd is even, and

$$\int_{-L}^{L} \text{even}\, dt = 2 \int_0^L \text{even}\, dt$$

$$b_m = \frac{2}{L} \int_0^L t \sin \frac{n\pi\, t}{L}\, dt$$

Using Equation (72):

$$\int u^n \sin u\, du = -u^n \cos u + n \int u^{n-1} \cos u\, du$$

with

$$u = \frac{n\pi t}{L} \text{ so that } t = \frac{L}{n\pi} u \text{ and } dt = \frac{L}{n\pi} du$$

we have

$$\begin{aligned}
b_m &= \frac{2}{L} \int_0^L t \sin \frac{n\pi\, t}{L}\, dt \\
&= \frac{2}{L} \int_0^{n\pi} \left(\frac{L}{n\pi} u\right) (\sin u) \left(\frac{L}{n\pi} du\right) \\
&= \frac{2}{L} \left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi} u \sin u\, du \\
&= \frac{2}{L} \left(\frac{L}{n\pi}\right)^2 [-u \cos u + \sin u]_0^{n\pi} \\
&= \frac{2}{L} \left(\frac{L}{n\pi}\right)^2 [-(n\pi) \cos(n\pi)] \\
&= \frac{2}{L} \left(\frac{L}{n\pi}\right)^2 (-1)(n\pi)(-1)^n \\
&= (-1)^{n+1} \frac{2L}{n\pi}
\end{aligned}$$

So the Fourier series for $f(t)$ is:

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi\, t}{L} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi}\ \sin \frac{n\pi\, t}{L} \tag{73}$$

which, when expanded looks like:

$$\frac{2L}{\pi}\left(\frac{1}{1}\sin\frac{1\pi\, t}{L} - \frac{1}{2}\sin\frac{2\pi\, t}{L} + \frac{1}{3}\sin\frac{3\pi\, t}{L} - \cdots\right)$$

Example: Solve the end point value ODE

$$x'' + 4x = 4t \qquad x(0) = x(1) = 0$$

on the interval $[0, 1]$.

Note. The end point values are also called boundary values.

Solution: Since the boundary conditions are $x(0) = x(1) = 0$ we guess at a solution of the form

$$x(t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi\, t}{L} \qquad \text{with } L = 1$$

since $\sin n\pi t = 0$ if $t = 0$ or 1. Plugging this sine series representation of x into $x''+4x = 4t$ and differentiating term by term yields

$$\begin{aligned}
x'' + 4x &= 4t \\
\left(\sum_{n=1}^{\infty} c_n \sin n\pi t\right)'' + 4\left(\sum_{n=1}^{\infty} c_n \sin n\pi t\right) &= 4t \\
\left(-\sum_{n=1}^{\infty} c_n n^2\pi^2 \sin n\pi t\right) + 4\left(\sum_{n=1}^{\infty} c_n \sin n\pi t\right) &= 4t \\
\sum_{n=1}^{\infty} c_n(4 - n^2\pi^2)\sin n\pi t &= 4t \qquad (74)
\end{aligned}$$

On the other hand, from Equation (73) we know that the Fourier series for $f(t) = t$ on the interval $[-L, L]$ is

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2L}{n\pi}\sin\frac{n\pi\, t}{L} \qquad (75)$$

and so by letting $L = 1$ and multiplying the above series by 4, we can represent $f(t) = 4t$, as the Fourier sine series,

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{8}{n\pi}\sin n\pi t$$

on $[-L, L] = [-1, 1]$. We actually only need to represent $4t$ on the interval $[0, 1]$ but it is OK if we represent $4t$ on the longer interval $[-1, 1]$. Technically this is called an odd extension of $4t$ because we are using sine functions, which are odd. If we were using even functions, the cosines, it would be called an even extension.

Substituting the Fourier sine series for $4t$ into Equation (74) we get

$$\sum_{n=1}^{\infty} c_n(4 - n^2\pi^2)\sin n\pi t = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{8}{n\pi}\sin n\pi t$$

Equating like terms we get

$$c_n(4 - n^2\pi^2) = (-1)^{n+1}\frac{8}{n\pi} \qquad \Rightarrow \qquad c_n = (-1)^{n+1}\frac{8}{n\pi(4 - n^2\pi^2)}$$

which implies

$$x(t) = \sum_{n=1}^{\infty} c_n \sin n\pi t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{8}{n\pi(4 - n^2\pi^2)} \ \sin n\pi t$$

Note: If we solve the homogenous equation we get $(D^2+4)[x_h] = 0$ which has roots $r = \pm 2i$ which implies $x_h = c_1 \cos 2t + c_2 \sin 2t$. So

$$x(t) = c_1 \cos 2t + c_2 \sin 2t \ + \ \sum_{n=1}^{\infty} c_n \sin n\pi t$$

Since $x(0) = 0$, we have $c_1 = 0$. Since $x(1) = 0$ and $\sin n\pi = 0$ we have $0 = x(1) = c_2 \sin(2) + 0$ which implies $c_2 = 0$. So $x_h(t) = 0$ and

$$x(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{8}{n\pi(4 - n^2\pi^2)} \ \sin n\pi t$$

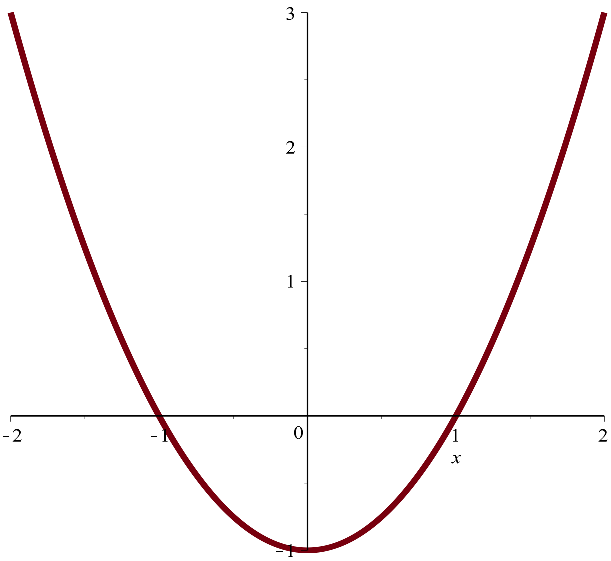
Note on integration by parts.

Integration by parts formula: $\int u\, dv = uv - \int v\, du$.
Example: Find $\int t \sin \omega t\, dt$.
Let $u = t$ so $du = dt$.
Let $dv = \sin \omega t\, dt$ so $v = \int dv = \int \sin \omega t\, dt = -\dfrac{\cos \omega t}{\omega}$.
Hence:

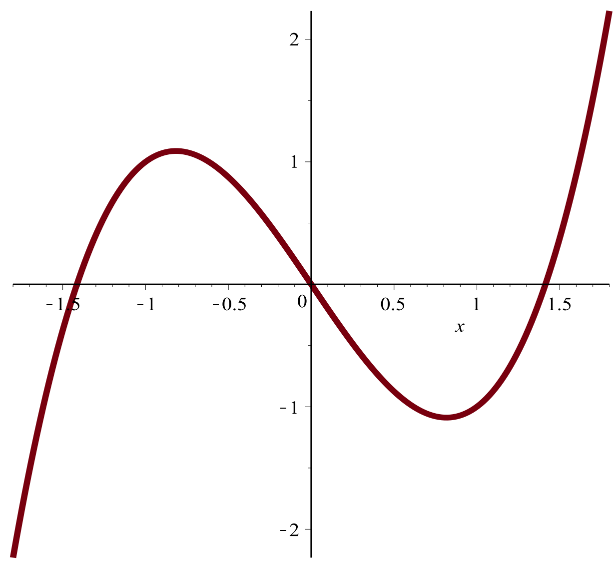
$$\int t \sin \omega t\, dt = (t)\left(-\frac{\cos \omega t}{\omega}\right) - \int \left(-\frac{\cos \omega t}{\omega}\right) dt = -\frac{t \cos \omega t}{\omega} + \frac{\sin \omega t}{\omega^2} + C$$

.

Homework 24b. (Edwards and Penny 4th Ed) Page 606 # 11 - 15.

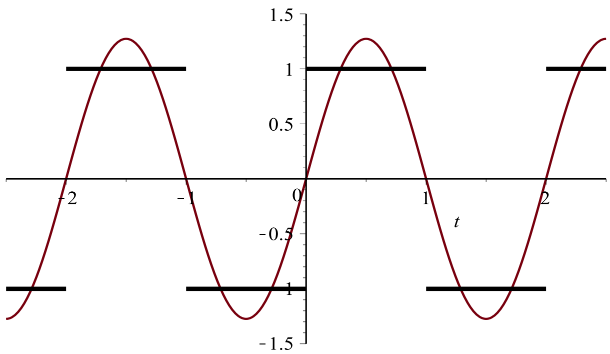


(a) $f(x) = x^2 - 1$ is an even function. It is symmetric about the y-axis.

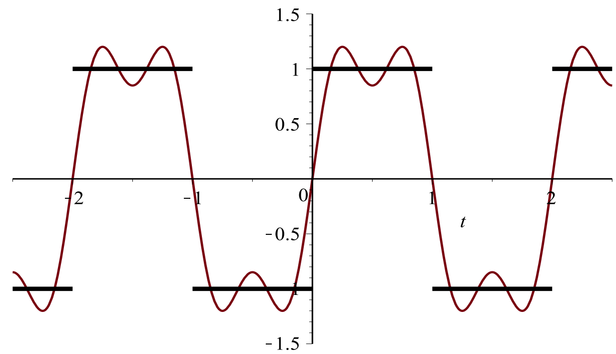


(b) $f(x) = x^3 - 2x$ is an odd function. It is symmetric about the origin.

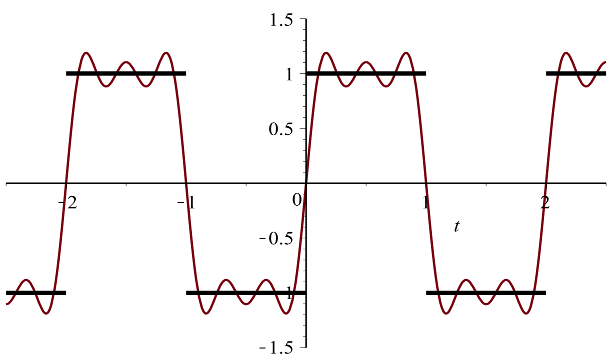
Figure 9: Examples of even and odd functions.



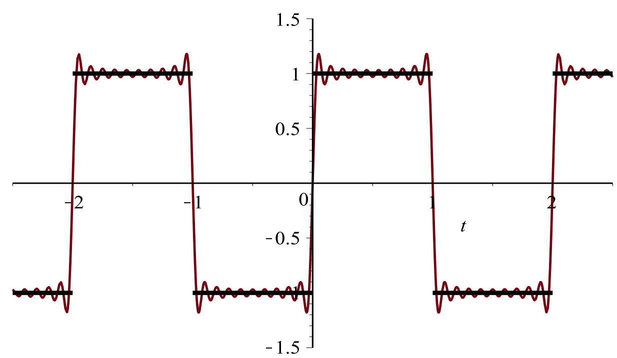
(a) Period 2 square wave $f(t)$, together with the first term of its Fourier series: $\frac{4}{\pi}\,\sin \pi t$



(b) Period 2 square wave $f(t)$, together with the sum of the first two non-zero terms in its Fourier series: $\frac{4}{\pi}\,\sin \pi t + \frac{4}{3\pi}\,\sin 3\pi t$



(c) Period 2 square wave $f(t)$, together with the sum of the first three non-zero terms in its Fourier series: $\frac{4}{\pi}\,\sin \pi t + \frac{4}{3\pi}\,\sin 3\pi t + \frac{4}{5\pi}\,\sin 5\pi t$



(d) Period 2 square wave $f(t)$, together with the sum of the first ten non-zero terms in its Fourier series: $\frac{4}{\pi}\,\sin \pi t + \frac{4}{3\pi}\,\sin 3\pi t + \cdots + \frac{4}{19\pi}\,\sin 19\pi t$

Figure 10: As we sum more of the terms in the Fourier series of $f(t)$, the square wave of period 2, we see that the approximation of $f(t)$ becomes more and more accurate. The over-shooting and under-shooting at the jump discontinuities is called the Gibb's phenomenon. This phenomena is most apparent in sub-figure (d), where we plot the sum the first 10 non-zero terms in the Fourier series. Note that the under and over-shooting at the jump discontinuities does not disappear or die out as we increase the number of terms being summed (sometimes called the frequencies or the harmonics). Instead, it can be shown, that the amount of under and over-shoot actually tends to some finite amount. This does not contradict the fact that the partial sums of the Fourier series converge to $f(t)$, as the convergence to $f(t)$ is point-wise (rather than uniform). What happens is this: as we sum more and more terms in the Fourier series, the Gibb's phenomenon moves closer and closer to the jump discontinuity. The Gibb's phenomenon is responsible for rippling effects sometimes seen in image compression.

Selected Solutions to Homework from Handout 24

Homework 24b was: Page 606 # 11 - 15 (Edwards and Penny 4th Ed).

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p 606 # 11 (E & P 4th Ed):

$$x'' + 2x = 1 \qquad x(0) = x(\pi) = 0$$

Solution. We assume that the solution will take the form of a sine series $\displaystyle\sum_{n=1}^{\infty} c_n \sin \frac{n\pi\, t}{L}$ with $L = \pi$ since the boundary conditions are $x(0) = x(\pi) = 0$. Plugging $x(t) = \displaystyle\sum_{n=1}^{\infty} c_n \sin nt$, into the DE yields:

$$\begin{aligned} x'' + 2x &= 1 \\ \left(\sum_{n=1}^{\infty} c_n \sin nt\right)'' + 2\left(\sum_{n=1}^{\infty} c_n \sin nt\right) &= 1 \\ \sum_{n=1}^{\infty} c_n n^2(-1)\sin nt + 2\sum_{n=1}^{\infty} c_n \sin nt &= 1 \\ \sum_{n=1}^{\infty} (2-n^2)c_n \sin nt &= 1 \end{aligned} \tag{76}$$

This suggests that we represent the $f(t) = 1$ on the right hand side of Equation (76) as a Fourier sine series: $\sum_{n=1}^{\infty} b_n \sin nt$. Of course such a representation, if it converges, will converge to an odd function of period 2π, since the Fourier sines are all odd, and since $\sin nt$ has period $\dfrac{2\pi}{n}$. This means that if we are to represent $f(t) = 1$ as a Fourier sine series which converges to $f(t) = 1$ on the interval $(0, \pi)$, the function $f(t)$ must be extended as an odd function of period 2π. I.e, as

$$f(t) = \begin{cases} -1, & t \in (-\pi, 0); \\ +1, & t \in (0, \pi). \end{cases}$$

Note: if we wanted to represent $f(t)$ as a Fourier cosine series, the extension would be called even, since the cosines are even. In that case, we'd define $f(t) = 1$ on all of $(-\pi, \pi)$.

The full Fourier series for the odd extension of $f(t) = 1$, consisting, in theory of both cosines and sines, will actually be a Fourier sine series, since all the cosine coefficients

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi\, t}{L}\, dt = 0.$$

This is because $f(t)$ is odd and $\cos\dfrac{n\pi\, t}{L}$ is even, and odd $\times$ even is odd, and integrating an odd function on the symmetric about 0 interval $[-L, L]$ results in 0. Moreover, since $f(t)$ is odd and $\sin\dfrac{n\pi\, t}{L}$ is odd, and odd $\times$ odd is even, we have:

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi\, t}{L}\, dt = \frac{2}{L}\int_{0}^{L} f(t)\sin\frac{n\pi\, t}{L}\, dt$$

So, since $L = \pi$ and since $f(t) = 1$ on $(0, \pi)$ we have:

$$b_n \;=\; \frac{2}{L}\int_0^{\pi} \sin nt\, dt \;=\; -\frac{2}{n\,\pi}\cos nt\Big|_0^{\pi} \;=\; -\frac{2}{n\,\pi}[\cos n\pi - \cos 0] = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \dfrac{4}{n\,\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

Hence, the Fourier sine series for $f(t) = 1$, oddly extended, is:

$$\sum_{n \text{ odd}}^{\infty} \frac{4}{n\,\pi} \sin nt$$

If we substitute the Fourier sine series for $f(t) = 1$ into Equation (76) we get:

$$\sum_{n=1}^{\infty}(2-n^2)c_n \sin nt = \sum_{n \text{ odd}}^{\infty} \frac{4}{n\,\pi} \sin nt, \qquad \text{which implies:}$$

$$(2-n^2)c_n = \begin{cases} 0, & \text{if } n \text{ is even;} \\ \dfrac{4}{n\,\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

$(2-n^2)c_n = 0$ for n even, implying $c_n = 0$ for n even.
$(2-n^2)c_n = \dfrac{4}{n\,\pi}$ for n odd, implying $c_n = \dfrac{4}{(2-n^2)n\,\pi}$ for n odd. So,

$$x(t) = \sum_{n \text{ odd}}^{\infty} \frac{4}{(2-n^2)n\,\pi} \sin nt \quad \leftarrow \quad \textit{answer}$$

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p 606 # 15 (E & P 4th Ed):

$$x'' + 2x = t \qquad x'(0) = x'(\pi) = 0$$

Solution. We assume that the solution will take the form of a cosine series $\displaystyle\sum_{n=0}^{\infty} c_n \cos\frac{n\pi\,t}{L}$ with $L = \pi$ since the boundary conditions are $x'(0) = x'(\pi) = 0$, since the derivative of cosine is sine.

Plugging $x(t) = \displaystyle\sum_{n=0}^{\infty} c_n \cos nt$, into the DE yields:

$$x'' + 2x = t$$

$$\left(\sum_{n=0}^{\infty} c_n \cos nt\right)'' + 2\left(\sum_{n=0}^{\infty} c_n \cos nt\right) = t$$

$$\sum_{n=0}^{\infty} c_n n^2(-1)\cos nt + 2\sum_{n=0}^{\infty} c_n \cos nt = t$$

$$\sum_{n=0}^{\infty}(2-n^2)c_n \cos nt = t \tag{77}$$

This suggests that we represent the $f(t) = t$ on the right hand side of Equation (77) as a Fourier cosine series: $\sum_{n=0}^{\infty} a_n \cos nt$. Of course such a representation, if it converges, will converge to an even function of period 2π, since the Fourier cosines are all odd, and since $\cos nt$ has period $\dfrac{2\pi}{n}$. This means that if we are to represent $f(t) = t$ as a Fourier cosine series which converges to $f(t) = 1$ on the interval $(0, \pi)$, the function $f(t)$ must be extended as an even function of period 2π. I.e, as

$$f(t) = \begin{cases} -t, & t \in (-\pi, 0); \\ +t, & t \in (0, \pi). \end{cases}$$

Note: if we wanted to represent $f(t)$ as a Fourier sine series, the extension would be called odd, since the sines are odd. In that case, we'd define $f(t) = t$ on all of $(-\pi, \pi)$.

The full Fourier series for the even extension of $f(t) = t$, consisting, in theory of both cosines and sines, will actually be a Fourier cosine series, since all the sine coefficients

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi\, t}{L}\, dt = 0.$$

This is because $f(t)$ is even and $\sin \dfrac{n\pi\, t}{L}$ is odd, and even $\times$ odd is odd, and integrating an odd function on the symmetric about 0 interval $[-L, L]$ results in 0. Moreover, since $f(t)$ is even and $\cos \dfrac{n\pi\, t}{L}$ is even, and even $\times$ even is even, we have:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi\, t}{L}\, dt = \frac{2}{L} \int_{0}^{L} f(t) \cos \frac{n\pi\, t}{L}\, dt$$

So, since $L = \pi$, and since $f(t) = t$ on $(0, \pi)$, and since

$$\int u^n \cos u\, du = u^n \sin u - n \int u^{n-1} \sin u\, du$$
$$\int u \cos u\, du = u \sin u - \int \sin u\, du = u \sin u + \cos u,$$

if we let $u = nt$ so that $t = \dfrac{u}{n}$ and $dt = \dfrac{du}{n}$ we have, for $n = 1, 2, 3, \ldots$

$$\begin{aligned}
a_n &= \frac{2}{L} \int_0^{\pi} t \cos nt\, dt \\
&= \frac{2}{\pi} \int_0^{n\pi} \frac{u}{n} \cos u \, \frac{du}{n} \\
&= \frac{2}{n^2\pi} \int_0^{n\pi} u \cos u\, du \\
&= \frac{2}{n^2\pi} (u \sin u + \cos u)\Big|_0^{n\pi} \\
&= \frac{2}{n^2\pi} [(n\pi \sin n\pi + \cos n\pi) - (0 \sin 0 + \cos 0)] \\
&= \frac{2}{n^2\pi} [(0 + (-1)^n) - (0 + 1)] \\
&= \frac{2}{n^2\pi} [(-1)^n - 1] \\
&= \begin{cases} 0, & \text{if } n \text{ is even;} \\ \\ -\dfrac{4}{n^2\pi}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

When $n = 0$,

$$a_0 \;=\; \frac{2}{L}\int_0^\pi t\,dt \;=\; \frac{2}{\pi}\,\frac{t^2}{2}\bigg|_0^\pi \;=\; \frac{t^2}{\pi}\bigg|_0^\pi = \pi$$

Hence, the Fourier cosine series for $f(t) = t$, evenly extended, is:

$$\frac{\pi}{2} + \sum_{n \text{ odd}}^{\infty} -\frac{4}{n^2\pi}\,\cos nt$$

So, if we substitute the Fourier cosine series for $f(t) = t$, into Equation (77), we get:

$$\sum_{n=0}^{\infty}(2-n^2)c_n\cos nt = \frac{\pi}{2} - \sum_{n \text{ odd}}^{\infty}\frac{4}{n^2\pi}\,\cos nt, \qquad \text{which implies:}$$

$$(2-n^2)c_n = \begin{cases} \dfrac{\pi}{2}, & \text{if } n = 0; \\[2ex] 0, & \text{if } n > 0 \text{ is even;} \\[2ex] -\dfrac{4}{n^2\pi}, & \text{if } n \text{ is odd.} \end{cases}$$

This implies

$$c_n = \begin{cases} \dfrac{\pi}{4}, & \text{if } n = 0; \\[2ex] 0, & \text{if } n > 0 \text{ is even;} \\[2ex] -\dfrac{4}{(2-n^2)n^2\pi} & \text{if } n \text{ is odd.} \end{cases}$$

So,

$$x(t) = \frac{\pi}{4} - \sum_{n \text{ odd}}^{\infty}\frac{4}{(2-n^2)n^2\pi}\,\cos nt \quad \leftarrow \quad \textit{answer}$$

or prettied up a little:

$$x(t) = \frac{\pi}{4} + \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty}\frac{\cos nt}{n^2(n^2-2)} \quad \leftarrow \quad \textit{answer}$$

Partial Differential Equations (PDE's)

A PDE is like an ODE, except instead of regular derivatives, there are partial derivatives. PDE's are extremely important, but usually, extremely difficult to solve (as they involve more than one independent variable).

Let's suppose $u(x,t)$ is a function of x and t. You can think of x as being the position along the x-axis and t as being the time. Notation: We write u_t to represent $\frac{\partial u}{\partial t}$ and u_{xx} to represent $\frac{\partial}{\partial x}\frac{\partial}{\partial x}u = \frac{\partial^2 u}{\partial x^2}$.

The PDE that we will study first is $u_t = ku_{xx}$ where k is some positive constant. This PDE is called the heat equation or the diffusion equation.

The heated rod problem. See Edwards and Penney 4th Ed, Section 9.5, page 616. We have a thin cylindrical rod of length L situated along the x-axis. We will orientate the rod along the line segment $[0, L]$. Let $u(x,t)$ be the temperature of the rod at x at time t. It can be shown that the temperature function $u(x,t)$ will satisfy the heat equation:

$$u_t = ku_{xx} \qquad \text{where } k > 0 \text{ is some positive constant.}$$

The diffusion in a tube problem. We have a thin uniform cylindrical tube of length L situated along the x-axis. The tube is filled with a porus substance or liquid through which some other substance, like a dye or a sugar, is diffusing. We will orientate the tube along the line segment $[0, L]$. Let $u(x,t)$ be the concentration of the diffusing substance (e.g. the dye or salt) at x at time t. It can be shown that the concentration function $u(x,t)$ will satisfy the diffusion equation:

$$u_t = ku_{xx} \qquad \text{where } k > 0 \text{ is some positive constant.}$$

Notice that the heat equation and the diffusion equation are the same PDE.

Derivation of the diffusion equation. Diffusion is the net movement of a substance from a region of higher concentration to a region of lower concentration. Diffusion is driven by the random motion of the molecules that are diffusing. The diffusing molecules move in all directions. For concreteness, let's suppose that the substance diffusing is sugar in water. Suppose that one region of water has a higher concentration of sugar and an adjacent region has a lower concentration of sugar. Random motion will result in sugar molecules moving in both directions across the interface of the two regions. However, as there are more sugar molecules in the region with a higher concentration, there will be more sugar molecules moving across the interface in the direction of the lower concentration. This will result in a net flow of sugar molecules into the region of lower concentration. Moreover, the larger the difference in concentrations, the larger will be the net amount of sugar moving across the interface per unit time (i.e. the net flow rate). On the other hand, if the concentration of sugar is the same on both sides of the interface, then there will be no net flow of sugar across the interface. Individual sugar molecules will still be moving across the interface, but the same number of sugar molecules will be going in both directions. Hence the net flow rate of sugar will be zero.

We are modeling the diffusion that takes place in a narrow uniform cylinder of cross sectional area A. This is sometimes called 1 dimensional diffusion since we are assuming that the concentration varies only with x (and t) but not in the y or z directions. We let $u(x,t)$ equal the concentration (of sugar) at x at the time t. $u(x,t)$ has units of mass/volume. In the MKS measurement system[25] $u(x,t)$ has units of $\frac{kg}{m^3}$ and A has units of m^2. $\frac{\partial}{\partial x}u(x,t) = u_x$ is the (1 dimensional) gradient of the concentration. If $u_x > 0$ it means the concentration is increasing as x increases. This means that the net flow rate (of the sugar) will be in the negative x direction. Moreover, as magnitude of the gradient u_x increases the magnitude of the flow rate should also increase. These two observations suggest what is called Fick's Law for 1 dimensional diffusion.

[25] Meter (m). Kilogram (kg). Second (s).

Fick's Law: the net rate at which a diffusing substance flows in the positive x direction past a point x is $-kA\ u_x(x,t)$, where k is a positive constant having units of $\frac{\text{area}}{\text{time}}$ and A is the cross sectional area. In the MKS system k has units of $\frac{m^2}{s}$ and so $-kA\ u_x(x,t)$ has units of $\frac{m^2}{s}\, m^2\, \frac{1}{m}\, \frac{kg}{m^3} = \frac{kg}{s}$, which makes sense since $-kA\ u_x(x,t)$ is the net flow rate.

Now, consider a section of the tube that goes from x_0 to x_1. The mass of sugar in this section of tube is $\int_{x_0}^{x_1} A\ u(x,t)\,dx$ where A is the cross sectional area of the tube. Notice that in the MKS system that A would have units of m^2 and dx units of m, and so $\int_{x_0}^{x_1} A\ u(x,t)\,dx$ would have units of mass (kg), as it should. The rate at which the amount of sugar in that section of tube changes can be found two different ways. The first way is by differentiation with respect to t: $\frac{\partial}{\partial t}\ \int_{x_0}^{x_1} A\ u(x,t)\,dx = \int_{x_0}^{x_1} A\ u_t(x,t)\,dx$. The second way uses Fick's Law applied to the ends of that section of tube, i.e. to x_0 and x_1. Fick's Law gives the net flow relative to the positive x direction, so a positive net flow means the net flow is moving to the right. So at x_0 the net flow into the section of tube between x_0 and x_1 is given directly by Fick's Law: $-kA\ u_x(x_0,t)$. At x_1 the net flow into the section of tube between x_0 and x_1 is given by the negative of Fick's Law: $kA\ u_x(x_1,t)$ because to go into that section of tube at the x_1 end, means the flow has to be moving in the negative x direction (to the left). Hence, the total net flow of sugar into the section of tube between x_0 and x_1 is given by $kA\,u_x(x_1,t) - kA\,u_x(x_0,t)$. Since the total net flow of sugar into (or out of) a section of tube equals the rate at which the amount of sugar in that section of tube changes we have:

$$\int_{x_0}^{x_1} A\ u_t(x,t)\,dx = kA\ u_x(x_1,t) - kA\ u_x(x_0,t). \tag{78}$$

If we let $x_1 \to x_0$, in Equation (78) above, then $dx \approx x_1 - x_0$ and $u(x,t) \approx u(x_0,t)$ on the interval $[x_0, x_1]$. Moreover, as x_1 gets closer to x_0 the approximations become closer and closer, and in the limit, we have equality. Hence:

$$\begin{aligned}
A\,u_t(x_0,t)\,(x_1 - x_0) &\approx \int_{x_0}^{x_1} A\,u_t(x,t)\,dx = kA\,u_x(x_1,t) - kA\,u_x(x_0,t)\\
A\,u_t(x_0,t)\,(x_1 - x_0) &\approx kA\,u_x(x_1,t) - kA\,u_x(x_0,t)\\
A\,u_t(x_0,t) &\approx \frac{kA\,u_x(x_1,t) - kA\,u_x(x_0,t)}{x_1 - x_0} \quad \text{divide by } A \text{ and take the limit as } x_1 \to x_0\\
u_t(x_0,t) &= k\ \lim_{x_1 \to x_0} \frac{u_x(x_1,t) - u_x(x_0,t)}{x_1 - x_0}\\
&= k\ \frac{\partial}{\partial x} u_x(x,t)\bigg|_{x=x_0}\\
&= k u_{xx}(x_0,t). \quad \text{Now let } x = x_0 \text{ to get:}\\
u_t(x,t) &= k u_{xx}(x,t) \quad \leftarrow \quad \text{the diffusion equation.}
\end{aligned}$$

Alternatively, we can finish the derivation of the diffusion equation by differentiating equation Equation (78) with respect to x_1; invoke the Fundamental Theorem of Calculus: $\frac{d}{dx}\int_a^x f(w)\,dw = f(x)$; note that the derivative of a constant is zero; cancel out the A's; and then let $x_1 = x$:

$$\begin{aligned}
\frac{\partial}{\partial x_1}\int_{x_0}^{x_1} A\,u_t(x,t)\,dx &= \frac{\partial}{\partial x_1}\ (kA\,u_x(x_1,t) - kA\,u_x(x_0,t))\\
A\,u_t(x_1,t) &= kA\,u_{xx}(x_1,t) + 0\\
u_t(x,t) &= k\,u_{xx}(x,t) \quad \leftarrow \quad \text{the diffusion equation.}
\end{aligned}$$

Solving PDE's using separation of variables.

The most important technique in solving PDE's is "separation of variables". This technique works well on some important PDE's, like the heat equation, the diffusion equation, and the wave equation, which is $u_{tt} = ku_{xx}$, which we may study later. To apply the method of separation of variables we assume that the solution $u(x,t)$ of the PDE can be written in separated form:

$$u(x,t) = X(x)T(t)$$

where $X(x)$ is a function of x alone and $T(t)$ is a function of t alone. We then plug $X(x)T(t)$ into the PDE and try to figure out $X(x)$ and $T(t)$ by combining mathematical reasoning with our understanding of what the PDE is modeling.

We will solve the heat equation for two different sets of boundary conditions: (1) assuming that the ends of the rod is kept at a temperature of 0 degrees (zero endpoint temperature); and (2) assuming that the ends of the rods are insulated.

(1) Solving the heat equation with zero endpoint temperature via separation of variables.

Suppose the ends of the rod are inserted into an icy bath that keeps the temperature at the ends of the rod at 0 degrees for all t. This assumption gives the (endpoint) boundary conditions: $u(0,t) = u(L,t) = 0$. The initial condition is just the initial temperature in the rod: $u(x,0) = f(x)$. It is traditional to write the PDE, together with its boundary and initial conditions in the following manner:

$$\begin{aligned} u_t = ku_{xx} &\quad 0 < x < L, \quad t > 0 \\ \text{(Boundary Conditions)} &\quad u(0,t) = u(L,t) = 0 \\ \text{(Initial Conditions)} &\quad u(x,0) = f(x) \end{aligned}$$

Separation of variables: if we let $u(x,t) = X(x)T(t)$, then

$$u_t = \frac{\partial}{\partial t} X(x)T(t) = X(x)T'(t) = XT' \tag{79}$$

$$u_{xx} = \frac{\partial^2}{\partial x^2} X(x)T(t) = X''(x)T(t) = X''T. \tag{80}$$

Plugging Equations (79) and (80) into the heat equation, $u_t = ku_{xx}$ yields:

$$XT' = kX''T \quad \text{which implies} \quad \frac{X''}{X} = \frac{T'}{kT} = -\lambda \tag{81}$$

where $\lambda > 0$ is a constant. Why are the ratios in (81) equal to a constant? Because: $\dfrac{X''(x)}{X(x)}$ is a function of x alone and $\dfrac{T'(t)}{kT(t)}$ is a function of t alone, so if these two ratios are equal for all possible values of x and t, the ratios must be a constant. To see this, just fix t_0, then $\dfrac{X''(x)}{X(x)} = \dfrac{T'(t_0)}{kT(t_0)}$ for all x, i.e. $\dfrac{X''(x)}{X(x)}$ is a constant. Below we will see why it must be the case that $\lambda > 0$.

The relationship $\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda$ gives us two ODE's: $X'' + \lambda X = 0$ and $T' + \lambda kT = 0$.

We'll first concentrate on: $X'' + \lambda X = 0$.

Consider the boundary condition $u(0,t) = 0$. Since $u(x,t) = X(x)T(t)$ it follows that $u(0,t) = X(0)T(t) = 0$. If $X(0) \neq 0$ we can divide $u(0,t) = X(0)T(t) = 0$ by $X(0)$ to get $T(t) = 0$ for all t. However, if $T(t) = 0$ then $u(x,t) = X(x)T(t) = 0$ for all t, which doesn't make sense since that means the temperature of the rod will always be 0. So it must be the case that $X(0) = 0$. An identical argument applied to the boundary condition $u(L,t) = 0$ tells us that $X(L) = 0$.

Showing that $\lambda > 0$. Suppose not. Suppose $\lambda < 0$, then $-\lambda > 0$ and $\sqrt{-\lambda}$ is a real positive number. Let $a = \sqrt{-\lambda}$ so that $a^2 = -\lambda$. Using a and the differential operator D we can re-write the differential equation $X'' + \lambda X = 0$ as $X'' - a^2 X = 0$ and as $(D^2 - a^2)[X] = 0$. The roots of $D^2 - a^2$ are just $\pm a$ and so the general solution of $X'' + \lambda X = 0$ is $X(x) = c_1 e^{ax} + c_2 e^{-ax}$.

Plugging the boundary condition $X(0) = 0$ into the general solution $X(x) = c_1 e^{ax} + c_2 e^{-ax}$ gives us: $X(0) = c_1 + c_2 = 0$, which implies $c_2 = -c_1$. So

$$X(x) = c_1 \left(e^{ax} - e^{-ax}\right). \tag{82}$$

Plugging the second boundary condition, $X(L) = 0$ into (82) gives us $X(L) = c_1 \left(e^{aL} - e^{-aL}\right) = 0$ which implies[26] c_1, a, or L equal zero. Physically, $L = 0$ makes no sense, since then the rod would have no length. If $c_1 = 0$ then the solution is $X(x) = 0$, which also makes no sense, since that would mean the temperature of the rod $u(x,t) = X(x)T(t)$ is identically zero along its entire length for all t. If $a = 0$ then $\lambda = 0$ (since $a^2 = -\lambda$) and the ODE $X'' + \lambda X = 0$ becomes $X'' = 0$. We solve $X'' = 0$ by integrating twice to get $X(x) = Ax + B$ with A and B being constants. Plugging the boundary condition $X(0) = 0$ into $X(x) = Ax + B$ implies $B = 0$ so that $X(x) = Ax$. Plugging the boundary condition $X(L) = 0$ into $X(x) = Ax$ implies $X(L) = 0 = AL$ which implies $A = 0$ (since $L \neq 0$), so $X(x) = 0$, which doesn't make sense since then the temperature function of the rod, $u(x,t) = X(x)T(t)$ is zero everywhere always. So we've shown that $\lambda \leq 0$ leads to rod's temperature function $u(x,t)$ being identically 0, which doesn't make sense. So it must be the case that $\lambda > 0$.

Since $\lambda > 0$ the differential equation $X'' + \lambda X = 0$ is that of the harmonic oscillator (which we have studied extensively); its roots are $\pm i\sqrt{\lambda}$; and its general solution is: $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$. Plugging the boundary condition $X(0) = 0$ into $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$ implies that $A = 0$ so that $X(x) = B\sin\sqrt{\lambda}\,x$. Plugging the boundary condition $X(L) = 0$ into $X(x) = B\sin\sqrt{\lambda}\,x$ give us: $B\sin\sqrt{\lambda}\,L = 0$.

Now, $B = 0$ gives us the trivial solution $X(x) = 0$, which physically doesn't make sense (as discussed above), so it must be the case $B \neq 0$. But then it must be the case that $\sin\sqrt{\lambda}\,L = 0$, which implies $\sqrt{\lambda}\,L = n\pi$ with n being an integer, so $\sqrt{\lambda} = \dfrac{n\pi}{L}$, which is actually a function of n. So instead of just writing λ, we should now write λ_n, with $\lambda_n = \dfrac{n^2\pi^2}{L^2}$. Note, since $\dfrac{n\pi}{L} = \sqrt{\lambda_n} > 0$ if follows that $n > 0$, so $n = 1, 2, 3, \ldots$, see [27].

So, for each n, $n = 1, 2, 3, \ldots$, we get a solution $X_n(x) = B_n \sin\dfrac{n\pi x}{L}$ of the boundary value problem $X_n'' + \lambda_n X_n = 0$; $X_n(0) = X_n(L) = 0$. The λ_n are called eigenvalues and the solutions $X_n(x) = B_n \sin\dfrac{n\pi x}{L}$ are called eigenfunctions[28].

The second ODE implied by (81) was $T' + \lambda k T = 0$, which we now realize is actually a family of ODE's $T_n' + \lambda_n k T_n = 0$ indexed by $n = 1, 2, 3, \ldots$. These ODE's are easy to solve since they're equivalent to $T_n' = -\lambda_n k T_n$ which has solution $T_n(t) = c_n e^{-\lambda_n k t}$, with c_n being a constant (typically we would solve for c_n by using initial conditions). However, we won't explicitly find c_n since we'll be using the initial conditions (the initial temperature in the rod) to solve for the B_n, and we'll just let B_n absorb the constant c_n. So summing up our work so far: our guess at a solution of the form $u(x,t) = X(x)T(t)$ led us to a family of solutions:

$$X_n(x)T_n(t) = B_n e^{-\lambda_n k t} \sin\frac{n\pi x}{L} \quad \text{where } \lambda_n = \frac{n^2\pi^2}{L^2} \text{ and } n = 1, 2, 3, \ldots$$

[26] The exponential function e^x is one to one so if $e^{aL} = e^{-aL}$ then $aL = -aL$, which implies $2aL = 0$, which implies a or L equals 0.

[27] An additional argument against allowing $n < 0$, is that $\sin\dfrac{n\pi x}{L}$ and $\sin\dfrac{-n\pi x}{L} = -\sin\dfrac{n\pi x}{L}$ are linearly dependent and do not truly give us additional solutions.

[28] $X_n'' + \lambda_n X_n = 0$ is the same as $X_n'' = -\lambda_n X_n$ which is the same as $D^2[X_n] = -\lambda_n X_n$ and so $-\lambda_n$ plays the same role as the eigenvalues that you may have learned about in linear algebra. Moreover the solution $X_n(x) = B_n \sin\dfrac{n\pi x}{L}$ plays the same role as the eigenvector. So, with this analogy in mind we like to call λ_n an eigenvalue and the solution $X_n(x) = B_n \sin\dfrac{n\pi x}{L}$ the corresponding eigenfunction.

It is easy to check that each product $X_n(x)T_n(x)$ satisfies the heat equation $u_t = ku_{xx}$ and the boundary conditions $u(0,t) = u(L,t) = 0$ by direct calculation and substitution.

Let $D_t = \dfrac{\partial}{\partial t}$ and $D_x = \dfrac{\partial}{\partial x}$. We can write the heat equation in the form $u_t - ku_{xx} = 0$ and then use the linear operators D_t and D_x to express the heat equation as the linear operator equation $(D_t - kD_x^2)[u] = 0$. So, just as with homogenous linear ODE's, linear combinations (sums) of solutions will also be solutions. Moreover since the boundary conditions are zero, linear combinations of solutions will still satisfy the boundary conditions. So the sum

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n kt} \sin\frac{n\pi x}{L} \tag{83}$$

will satisfy the heat equation and its boundary conditions $u(0,t) = u(L,t) = 0$. Finally we use the initial condition $u(x,0) = f(x)$ (which is the initial temperature in the rod) to solve for the coefficients B_n. Letting $t = 0$ in (83) we get:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \tag{84}$$

Equation (84) is just the Fourier sine series for $f(x)$ oddly extended to $[-L, L]$. So

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Hence we arrive at the solution to the heat equation $u_t = ku_{xx}$ with boundary conditions $u(0,t) = u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L} \tag{85}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

You should memorize how to derive the above solution. On your exam I will expect you to use separation of variables to derive the above equation. You do not need to show the discarded cases of $\lambda \le 0$. But you will have to show the case of $\lambda > 0$. Do not just memorize the solution!!

Some values of k. Silver (k = 1.7 cm^2/sec). Iron (k = .15 cm^2/sec). Concrete (k = .005 cm^2/sec). See page 622 in E & P 4th Ed.

HW: page 627 #9 E& P 4th Ed.

$$\begin{aligned} 10u_t &= u_{xx} && 0 < x < 5,\ t > 0 \\ \text{(Boundary Conditions)} & && u(0,t) = u(5,t) = 0 \\ \text{(Initial Conditions)} & && u(x,0) = 25 \end{aligned}$$

Solution: plugging $u = X(x)T(t)$ into $10\,u_t = u_{xx}$ yields $10XT' = X''T$ which implies $\dfrac{10T'}{T} = \dfrac{X''}{X} = -\lambda$. Since the boundary conditions are $u(0,t) = u(5,t) = 0$ (endpoint zero boundary conditions) we know that $\lambda > 0$. A little algebra gives us $X'' + \lambda X = 0$ and $T' + \dfrac{\lambda}{10}T = 0$.

First we address the DE $X'' + \lambda X = 0$ with BC $X(0) = X(5) = 0$

Since $\lambda > 0$ we have $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. Using $X(0) = 0$ we get $A = 0$, and so $X(x) = B\sin\sqrt{\lambda}x$. Using $X(5) = 0$ we get $X(5) = B\sin\sqrt{\lambda}\ 5 = 0$. Since $B = 0$ results in $X(x) = 0$, the trivial solution, we must have $\sin\sqrt{\lambda}\ 5 = 0$, which is true if $\sqrt{\lambda}\ 5 = n\pi$, which implies $\lambda_n = \dfrac{n^2\pi^2}{25}$. Since $\sqrt{\lambda} > 0$, it follows that $n = 1, 2, 3, \ldots$ So $X_n(x) = B_n \sin\dfrac{n\pi\, x}{5}$ for $n = 1, 2, 3, \ldots$

Now we focus on the differential equation $T_n' + \dfrac{\lambda_n}{10}T_n = 0$ which has solution $T_n(t) = c_n e^{-\dfrac{\lambda_n t}{10}}$. We suppress the c_n as that will be absorbed into the B_n. Summing the products $X_n(x)T_n(t)$ we get the general solution

$$u(x,t) = \sum_{n=1}^{\infty} B_n\, e^{-\dfrac{n^2\pi^2\ t}{250}} \sin\frac{n\pi\, x}{5}$$

Letting $t = 0$ the IC implies:

$$u(x,0) = 25 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi\, x}{5}$$

which is just the Fourier Sine series for $f(x) = 25$ oddly extended. For an oddly extended function we have

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\, dx$$

Hence

$$B_n = \frac{2}{5}\int_0^5 25\sin\frac{n\pi x}{5}\, dx = -\ \frac{2}{5}\, 25\, \frac{5}{n\pi}\cos\frac{n\pi x}{5}\bigg|_0^5 = -\frac{50}{n\pi}((-1)^n - 1)$$

So $B_n = -\ -\dfrac{100}{n\pi} = \dfrac{100}{n\pi}$ if n is odd, zero otherwise. Hence, the we come to the particular solution:

$$u(x,t) = \sum_{n=\text{odd}}^{\infty} \frac{100}{n\pi}\, e^{-\dfrac{n^2\pi^2\ t}{250}} \sin\frac{n\pi\, x}{5}$$

Homework (Heat Equation PDE)

Read Section 9.5, page 623 - 625 E & P 4th Ed on heated rod with insulated ends.

Do pages 626 - 627 # 1,2, 9 (see solution above), 10, 11

Heated rod with insulated end points.

See E & P 4th Ed. page 623. Same heated rod as before, except this time the ends of the rod are insulated, mathematically this translates into the boundary conditions being $u_x(0,t) = u_x(L,t) = 0$. The PDE, together with its BC and IC is thus:

$$\begin{aligned} u_t = ku_{xx} &\quad 0 < x < L,\ t > 0 \\ \text{(Boundary Conditions)} &\quad u_x(0,t) = u_x(L,t) = 0 \\ \text{(Initial Conditions)} &\quad u(x,0) = f(x) \end{aligned}$$

Solution.

Separation of variables: as before, plugging $u(x,t) = X(x)T(t)$ into the heat equation $u_t = ku_{xx}$ gives us:

$$XT' = kX''T \quad \text{which implies} \quad \frac{X''}{X} = \frac{T'}{kT} = -\lambda \tag{86}$$

where $\lambda \geq 0$ is a constant. As before, the relationship $\dfrac{X''}{X} = \dfrac{T'}{kT} = -\lambda$ gives us two ODE's: $X'' + \lambda X = 0$ and $T' + \lambda kT = 0$.

We'll first concentrate on: $X'' + \lambda X = 0$.

Since $u_x(x,t) = X'(x)T(t)$ the boundary condition $u_x(0,t) = 0$ implies $u_x(0,t) = X'(0)T(t) = 0$. If $X'(0) \neq 0$ we can divide $u_x(0,t) = X'(0)T(t) = 0$ by $X'(0)$ to get $T(t) = 0$ for all t, which would give us the trivial solution, hence we must have $X'(0) = 0$. An identical argument applied to the boundary condition $u_x(L,t) = 0$ tells us that $X'(L) = 0$.

Showing that $\lambda \geq 0$.

Suppose not. Suppose $\lambda < 0$, then $-\lambda > 0$ and $\sqrt{-\lambda}$ is a real positive number. Let $a = \sqrt{-\lambda}$ so that $a^2 = -\lambda$. As before, if $\lambda < 0$ the general solution of $X'' + \lambda X = 0$ is $X(x) = c_1 e^{ax} + c_2 e^{-ax}$.

Plugging the boundary condition $X'(0) = 0$ into the derivative of the general solution:

$$X'(x) = c_1 a e^{ax} - c_2 a e^{-ax}$$

gives us: $X'(0) = a(c_1 - c_2) = 0$, which implies $a = 0$ or $c_1 - c_2 = 0$. We are assuming $\lambda < 0$, so $a = \sqrt{-\lambda} \neq 0$. So $c_1 - c_2 = 0$ which implies that $c_1 = c_2$ and $X'(x) = c_1 a\left(e^{ax} - e^{-ax}\right)$. Plugging the boundary condition $X'(L) = 0$ into $X'(x) = c_1 a\left(e^{ax} - e^{-ax}\right)$ gives us $X'(L) = c_1 a\left(e^{aL} - e^{-aL}\right) = 0$. Since $a \neq 0$ and since $c_1 = 0$ gives us the trivial solution, we must have $e^{aL} - e^{-aL} = 0$, which implies $aL = -aL$ which is implies $L = 0$ (which doesn't make sense) or $a = 0$ (which would imply $\lambda = 0$, which we assuming is not the case). So it must be the case that $\lambda \geq 0$.

Consider the case of $\lambda = 0$. If $\lambda = 0$ it is convenient to let $\lambda = \lambda_0 = 0$ and to write the differential equation $X'' + \lambda X = 0$ as

$$X_0'' + \lambda_0 X_0 = 0 \tag{87}$$

Since $\lambda_0 = 0$ the differential equation (87) becomes $X_0'' = 0$, which we easily solve by integrating twice. We get $X_0(x) = A + Bx$, which when differentiated becomes: $X_0'(x) = B$. Plugging the boundary condition $X'(0) = 0$ into $X_0'(x) = B$ gives us $X_0'(0) = B = 0$. Plugging the boundary condition $X'(L) = 0$ into $X_0'(x) = B$ also gives us $X_0'(L) = B = 0$. So, both boundary conditions tell us that $B = 0$. So, $\lambda_0 = 0$ is an eigenvalue with non-trivial solution (eigenfunction) $X_0(x) = A$. It will be convenient to define $A_0 = 2A$ so that $X_0(x) = \dfrac{A_0}{2}$.

If $\lambda > 0$ the differential equation $X'' + \lambda X = 0$ is that of the harmonic oscillator; its roots are $\pm i\sqrt{\lambda}$; and its general solution is: $X(x) = A\cos\sqrt{\lambda}\ x + B\sin\sqrt{\lambda}\ x$. Plugging the boundary condition $X'(0) = 0$ into $X'(x) = -A\sqrt{\lambda}\ \sin\sqrt{\lambda}\ x + B\sqrt{\lambda}\ \cos\sqrt{\lambda}\ x$ implies that $B = 0$ so that

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\, x \tag{88}$$

Plugging the boundary condition $X'(L) = 0$ into (88) give us: $-A\sqrt{\lambda}\ \sin\sqrt{\lambda}\, L = 0$. If $A = 0$ we get the trivial solution (that the temperature of the rod is uniformly and always 0). So $A \neq 0$ which implies $\sin\sqrt{\lambda}\ L = 0$ which implies $\sqrt{\lambda}\ L = n\pi$, with n being an integer, so $\sqrt{\lambda} = \dfrac{n\pi}{L}$, which is actually a function of n. So instead of just writing λ, we write λ_n, with $\lambda_n = \dfrac{n^2\pi^2}{L^2}$. Note, since $\dfrac{n\pi}{L} = \sqrt{\lambda_n} > 0$ we have $n > 0$, i.e. $n = 1, 2, 3, \ldots$.

So, for each n, $n = 0, 1, 2, 3, \ldots$, we get an eigenvalue $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ with corresponding solution (and eigenfunction)

$$X_n(x) = \begin{cases} \dfrac{A_0}{2}, & \text{if } n = 0; \\ A_n \cos\dfrac{n\pi x}{L}, & \text{if } n = 1, 2, 3, \ldots. \end{cases}$$

of the boundary value problem $X_n'' + \lambda_n X_n = 0$; $X'(0) = X'(L) = 0$.

The second ODE implied by (86) was $T' + \lambda kT = 0$, which we now realize is actually a family of ODE's $T_n' + \lambda_n kT_n = 0$ indexed by $n = 0, 1, 2, 3, \ldots$. These ODE's are easy to solve since they're equivalent to $T_n' = -\lambda_n kT_n$ which has solution $T_n(t) = c_n e^{-\lambda_n kt}$, with c_n being a constant. We won't explicitly find c_n since we'll be using the initial conditions (the initial temperature in the rod) to solve for the A_n, and we'll just let A_n absorb the constant c_n. Note that $T_0(t) = c_0 \times 1$ and we'll let c_0 be absorbed into A_0 and let $T_0(t) = 1$, the constant function.

As before, linear combinations of solutions will again be a solution. Moreover since the boundary conditions are zero, linear combinations of solutions will still satisfy the boundary conditions. So the sum

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \cos\frac{n\pi x}{L} \tag{89}$$

will satisfy the heat equation and its boundary conditions $u_x(0,t) = u_x(L,t) = 0$. Finally we use the initial condition $u(x,0) = f(x)$ (which is the initial temperature in the rod) to solve for the coefficients A_n. Letting $t = 0$ in (89) we get:

$$u(x,0) = f(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \tag{90}$$

Equation (90) is just the Fourier cosine series for $f(x)$ evenly extended to $[-L, L]$. So

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \ldots$$

Hence we arrive at the solution to the heat equation $u_t = ku_{xx}$ with boundary conditions $u(0,t) = u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 kt/L^2} \cos\frac{n\pi x}{L}$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n = 0, 1, 2, 3, \ldots$$

You should memorize how to derive the above solution. On your exam I will expect you to use separation of variables to derive the above equation. You do not need to show the discarded cases of $\lambda < 0$. But you will have to show the cases of $\lambda \geq 0$. Do not just memorize the solution!!

HW: page 627 #11 E& P 4th Ed.

$$\begin{aligned}
5u_t &= u_{xx} & &0 < x < 10,\ t > 0 \\
\text{(Boundary Conditions)} & & &u_x(0,t) = u_x(10,t) = 0 \\
\text{(Initial Conditions)} & & &u(x,0) = 4x
\end{aligned}$$

Solution: plugging $u = X(x)T(t)$ into $5\,u_t = u_{xx}$ yields $5XT' = X''T$ which implies $\dfrac{5T'}{T} = \dfrac{X''}{X} = -\lambda$. Since the boundary conditions are $u_x(0,t) = u_x(10,t) = 0$ (insulated endpoints) we know that $\lambda \geq 0$. A little algebra gives us $X'' + \lambda X = 0$ and $T' + \dfrac{\lambda}{5}T = 0$.

First we address the DE $X'' + \lambda X = 0$ with BC $X'(0) = X'(10) = 0$

$\lambda \geq 0$, when $\lambda > 0$, the general solution is $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ and $X'(x) = -A\sqrt{\lambda}\ \sin\sqrt{\lambda}x + B\sqrt{\lambda}\ \cos\sqrt{\lambda}x$. Using $X'(0) = 0$ we get $B = 0$, and so $X'(x) = A\sqrt{\lambda}\ \sin\sqrt{\lambda}x$. Using $X'(10) = 0$ we get $X'(10) = A\sqrt{\lambda}\ \sin\sqrt{\lambda}\ 10 = 0$. Since $A = 0$ results in $X(x) = 0$, the trivial solution, we must have $\sin\sqrt{\lambda}\ 10 = 0$, which is true if $\sqrt{\lambda}\ 10 = n\pi$, which implies $\dfrac{n\pi}{10} = \sqrt{\lambda} \geq 0$ which implies that $n = 0, 1, 2, \ldots$, and so we will write λ_n, instead of just λ; so $\lambda_n = \dfrac{n^2\pi^2}{100}$. When $\lambda = \lambda_0 = 0$ the general solution is $X(x) = A + Bx$, so $X'(x) = B$. The boundary conditions $X'(0) = X'(10) = 0$ imply $B = 0$ and $X(x) = A = \dfrac{A_0}{2}$. So

$$X_n(x) = \begin{cases} \dfrac{A_0}{2}, & \text{if } n = 0; \\ A_n \cos\dfrac{n\pi\, x}{10}, & \text{if } n = 1, 2, 3, \ldots. \end{cases}$$

Now we focus on the differential equation $T_n' + \dfrac{\lambda_n}{5}T_n = 0$ which has solution $T_n(t) = c_n e^{-\frac{\lambda_n t}{5}}$. We suppress the c_n, or let $c_n = 1$ as they will be absorbed into the A_n. Summing the products $X_n(x)T_n(t)$ we get the general solution

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n\, e^{-\frac{n^2\pi^2\ t}{500}} \cos\frac{n\pi\, x}{10}$$

Letting $t = 0$ the IC implies:

$$u(x,0) = 4x = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi\, x}{10}$$

which is just the Fourier Cosine series for $f(x) = 4x$ evenly extended. For an evenly extended function we have

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3, \ldots$$

Hence, if n = 1,2,3,...

$$\begin{aligned}
A_n &= \frac{2}{10}\int_0^{10} 4x\cos\frac{n\pi x}{10}\,dx = \frac{8}{10}\int_0^{10} x\cos\frac{n\pi x}{10}\,dx \\
\frac{10}{8}A_n &= \int_0^{10} x\cos\frac{n\pi x}{10}\,dx \\
&= \frac{x\ 10}{n\pi}\sin\frac{n\pi\ x}{10}\Big|_0^{10} - \int \frac{10}{n\pi}\sin\frac{n\pi\ x}{10}\,dx \\
&= \left(\frac{x\ 10}{n\pi}\sin\frac{n\pi\ x}{10} + \frac{100}{n^2\pi^2}\cos\frac{n\pi\ x}{10}\right)\Big|_0^{10} \\
&= \left(0 + \frac{100}{n^2\pi^2}(-1)^n\right) - \left(0 + \frac{100}{n^2\pi^2}\right) \\
&= \frac{100}{n^2\pi^2}((-1)^n - 1) \\
&= -\frac{200}{n^2\pi^2} \text{ if } n \text{ is odd, and } 0 \text{ if } n \text{ is even. Multiply by } \frac{8}{10} \text{ to get} \\
A_n &= -\frac{160}{n^2\pi^2} \text{ if } n \text{ is odd, and } 0 \text{ if } n \text{ is even.}
\end{aligned}$$

In the above we used integration by parts. Let $u = x$ so $du = dx$. Let $dv = \cos\dfrac{n\pi x}{10}$ so $v = \dfrac{10}{n\pi}\sin\dfrac{n\pi\ x}{10}$.

For $n = 0$, for an evenly extended function $f(x)$ we have:

$$A_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{10}\int_0^{10} 4x\,dx = \left.\frac{4x^2}{10}\right|_0^{10} = 40. \text{ So } \frac{A_0}{2} = 20.$$

Putting this all together, we have the particular solution:

$$u(x,t) = 20 - \sum_{n=\text{odd}}^{\infty} \frac{160}{n^2\pi^2}\, e^{-\dfrac{n^2\pi^2\,t}{500}} \cos\frac{n\pi\,x}{10}$$

Part II: The Heat Equation and Equilibrium (Steady State) Solutions

A system is in equilibrium if it doesn't change with time. The equilibrium (steady state) solution of a PDE is found by setting the time derivative of the solution equal to zero, i.e. $u_t = 0$. If $u_t = 0$ then the heat equation becomes $0 = ku_{xx}$, which has solution $u(x,t) = A + Bx$ (which makes sense, as we are assuming that the system is not changing with time). For the heat equation, $u(x,t) = A + Bx$ is the steady state solution.

Equilibrium for zero endpoint temperature. Plugging the boundary conditions $u(0,t) = u(L,t) = 0$ into the steady state solution $u(x,t) = A + Bx$ implies both A and B are zero. This means the steady state solution is $u(x,t) = 0$. This makes sense since the ends of the rod are kept at 0 degrees, and since no heat or energy is being put into the system, eventually all the heat in the rod will diffuse out of the ends of the rod and the temperature of the rod will become zero for the entire rod. We can get this same result, that at equilibrium the temperature of the rod will be uniformly zero, by plugging $t = \infty$ into the general solution for the heated rod with zero endpoint temperature (91) below, and noting that $e^{-\infty} = 0$:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L} \tag{91}$$

Equilibrium for the rod with insulated ends. As shown above, the steady state solution is $u(x,t) = A + Bx$, so $u_x(x,t) = \frac{\partial}{\partial x}\, u(x,t) = Bx$. Plugging the insulated rod's boundary conditions, $u_x(0,t) = u_x(L,t) = 0$ into the gradient of the steady state solution, i.e into $u_x(x,t) = Bx$ implies B is zero. This means the steady state solution is $u(x,t) = A$. So at equilibrium the heat energy will be uniformly distributed in the rod and all parts of the rod will have the same equilibrium temperature A. What value should A have?

The ends of the rod are insulated so no heat energy can diffuse out of the rod through its ends. So the total amount of heat energy in the rod will be constant. So, the total amount of heat energy in the rod at any time t will be the same as the total amount of heat energy in the rod at $t = 0$. Temperature is a type of concentration: temperature $= \dfrac{\text{heat energy}}{\text{volume}}$. So the total amount of heat energy in the rod at time $t = 0$ (and for all $t > 0$), is

$$E_{\text{total}} = \int_0^L \mathcal{A}\, f(x)\,dx \qquad \text{since } d(\text{volume}) = \mathcal{A}\,dx$$

where $f(x) = u(x,0) =$ the initial temperature in the rod and $\mathcal{A} =$ the cross sectional area of the rod.

Since the total volume of the rod is $\mathcal{A}\,L$, the average temp. (concentration of heat energy) in the rod is:

$$\frac{E_{\text{total}}}{\text{vol}} = \frac{1}{\text{vol}}\int_0^L \mathcal{A}\, f(x)\,dx = \frac{1}{\mathcal{A}\,L}\int_0^L \mathcal{A}\, f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{2}\,\frac{2}{L}\int_0^L f(x)\,dx = \frac{A_0}{2}$$

When the temperature of the rod is at equilibrium all parts of the rod will have the same temperature, which must be the average temperature: $\dfrac{E_{\text{total}}}{\text{vol}}$. So the equilibrium (steady state) temperature $A = \dfrac{A_0}{2}$.

We can get the same result by plugging $t = \infty$ into the general solution for the heated rod with insulated ends (92) below, and noting that $e^{-\infty} = 0$:

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 kt/L^2} \cos\frac{n\pi x}{L} \tag{92}$$

Solutions to the Heat Equations PDE HW questions:
page 626 #'s 1, 2, 9 (solved above), 10, 11 (solved above) (E&P 4th ED)

Page 626 # 1. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0, \quad$ BC $u(0,t) = u(\pi,t) = 0, \quad$ IC $u(x,0) = 4\sin 2x$

Solution. Let $u = X(x)T(t)$. Then $XT' = 3X''T$ and so $\dfrac{T'}{3T} = \dfrac{X''}{X} = -\lambda$, with $\lambda > 0$ due to the BC. This in turn implies two DE's: the harmonic oscillator DE $X'' + \lambda X = 0$ with BC $X(0) = X(\pi) = 0$ and the exponential decay DE $T' = -3\lambda T$. The solution to $X'' + \lambda X = 0$ is $X = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$, plugging in the BC $X(0) = 0$ implies $X = B\sin\sqrt{\lambda}\,x$. Plugging in the BC $X(\pi) = 0$ implies that $\sin\sqrt{\lambda}\,\pi = 0$ which implies $\sqrt{\lambda}\,\pi = n\pi$. So $\sqrt{\lambda}$ is actually a sequence of constants: $\sqrt{\lambda_n} = n$ with $n = 1, 2, 3, \ldots$, also $\lambda_n = n^2$ and $X_n(x) = B_n \sin nx$, both with $n = 1, 2, 3, \ldots$. Pugging λ_n into $T' = -3\lambda T$ gives us the sequence of DE's: $T_n' = -3\lambda_n T_n$ which has solutions $T_n(t) = c_n e^{-3\lambda_n t}$. Letting B_n absorb the constant c_n and applying the linearity of the heat equation to the solutions $X_n(x)T_n(x)$ we get the general solution: $u(x,t) = \displaystyle\sum_{n=1}^{\infty} B_n e^{-3\lambda_n t} \sin nx$ where $\lambda_n = n^2$. Plugging the IC's $u(x,0) = 4\sin 2x$ into the general solution gives us: $4\sin 2x = \displaystyle\sum_{n=1}^{\infty} B_n \sin nx$. We can easily find the B_n if we remember that the Fourier sines are orthogonal (which implies that they are linearly independent). We can simply write:

$$(0\,\sin x) + (4\sin 2x) + (0\,\sin 3x) + \cdots = 4\sin 2x = \sum_{n=1}^{\infty} B_n \sin nx = B_1 \sin 1x + B_2 \sin 2x + B_3 \sin 3x + \cdots$$

and equate coefficients. We get $B_2 = 4$ and $B_n = 0$ if $n \neq 2$. Note. $B_2 = 4$ because the coefficient of $4\sin 2x$ is 4. So we get the particular solution (and answer to this question): $u(x,t) = 4e^{-3\lambda_2 t}\sin 2x$ with $\lambda_2 = 2^2 = 4$.

$$u(x,t) = 4e^{-12t}\sin 2x \quad \leftarrow \quad \textit{answer}$$

In the previous problem we were able to avoid using the formulas from the Fourier series to calculate the B_n. Suppose we (for some reason) wanted to calculate the B_n in $4\sin 2x = \displaystyle\sum_{n=1}^{\infty} B_n \sin nx$ using the Fourier formulas. This is what we'd do:

Recall that the Fourier (sine) series for $f(x)$ oddly extended to the interval $[-L, L]$ is $f(x) = \displaystyle\sum_{n=1}^{\infty} B_n \sin\frac{n\pi\, x}{L}x$ with $B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi\, x}{L}\,dx$. So $f(x) = 4\sin 2x = \displaystyle\sum_{n=1}^{\infty} B_n \sin nx$ is just the Fourier sine series for $f(x) = 4\sin 2x$, oddly extended, with $L = \pi$. So $B_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi (4\sin 2x)(\sin nx)\,dx = \dfrac{8}{\pi}\displaystyle\int_0^\pi (\sin 2x)(\sin nx)\,dx$. Using the product to sum trig identity: $(\sin a)(\sin b) = \dfrac{\cos(a-b) - \cos(a+b)}{2}$ we get $B_n =$ $\dfrac{8}{\pi}\displaystyle\int_0^\pi (\sin 2x)(\sin nx)\,dx = \dfrac{8}{\pi}\displaystyle\int_0^\pi \frac{\cos(2x - nx) - \cos(2x + nx)}{2}\,dx = \dfrac{4}{\pi}\displaystyle\int_0^\pi \cos((2-n)x) - \cos((2+n)x)\,dx$. If $n \neq 2$ then $(2-n) \neq 0$ and the integral gives us $B_n = \dfrac{4}{\pi}\left[\dfrac{\sin((2-n)x)}{2-n} - \dfrac{\sin((2+n)x)}{2+n}\right]_0^\pi = 0$ because

$\sin m\pi = 0$ for all integers m. If $n = 2$ then $(2-n) = 0$ and $\cos((2-n)x) = \cos(0) = 1$ and so the integral gives us $B_2 = \dfrac{4}{\pi}\displaystyle\int_0^{\pi} 1 - \cos(4x)\,dx$ which evaluates to $B_2 = \dfrac{4}{\pi}\left[x - \dfrac{\sin(4x)}{4}\right]_0^{\pi} = \dfrac{4}{\pi}[\pi] = 4$.

So $B_n = 0$ unless $n = 2$, in which case, $B_2 = 4$, as determined previously.

Page 626 # 2. $u_t = 10u_{xx}$, $0 < x < 5$, $t > 0$, BC $u_x(0,t) = u_x(5,t) = 0$, IC $u(x,0) = 7$

Solution. Let $u = X(x)T(t)$. Then $XT' = 10X''T$ and so $\dfrac{T'}{10T} = \dfrac{X''}{X} = -\lambda$, with $\lambda \geq 0$ due to the BC. This in turn implies two DE's: the harmonic oscillator DE $X'' + \lambda X = 0$ with BC $X'(0) = X'(\pi) = 0$ and the exponential decay DE $T' = -10\lambda T$. The solution to $X'' + \lambda X = 0$ is $X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$, and so $X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}\,x + B\sqrt{\lambda}\cos\sqrt{\lambda}\,x$. Plugging in the BC $X'(0) = 0$ into $X'(x)$ implies $B = 0$ and $X = A\cos\sqrt{\lambda}\,x$. Plugging in the BC $X'(5) = 0$ into X' implies that $-A\sqrt{\lambda}\sin 5\sqrt{\lambda} = 0$ which implies $5\sqrt{\lambda} = n\pi$. So $\sqrt{\lambda}$ is actually a sequence of constants: $\sqrt{\lambda_n} = \dfrac{n\pi}{5}$ with $n = 0, 1, 2, 3, \ldots$, so $\lambda_n = \dfrac{n^2\pi^2}{25}$ and $X_n(x) = A_n \cos\dfrac{n\pi\,x}{5}$, both with $n = 1, 2, 3, \ldots$. When $n = 0$ we have $\lambda_n = 0$ and the DE for X_0 becomes $X_0'' = 0$. Integrating twice we get $X_0(x) = A + Bx$. Plugging in the IC $X'(0) = 0$ and $X'(5) = 0$ into $X_0'(x) = B$ tell us that $B = 0$. So $X_0(x) = A = \dfrac{A_0}{2}$, we write X_0 in this way to match the formula used for the Fourier series. Pugging λ_n into $T' = -10\lambda T$ gives us the sequence of DE's: $T_n' = -10\lambda_n T_n$ which has solutions $T_n(t) = c_n e^{-10\lambda_n t}$, $n = 0, 1, 2, \ldots$. Note: $T_0 = e^0 = 1$. Letting A_n absorb the constant c_n and applying the linearity of the heat equation to the solutions $X_n(x)T_n(x)$ we get the general solution: $u(x,t) = \dfrac{A_0}{2} + \displaystyle\sum_{n=1}^{\infty} A_n e^{-10\lambda_n t}\cos\dfrac{n\pi x}{5}$ where $\lambda_n = \dfrac{n^2\pi^2}{25}$. Plugging the IC's $u(x,0) = 7$ into the general solution gives us: $7 = \dfrac{A_0}{2} + \displaystyle\sum_{n=1}^{\infty} A_n\cos\dfrac{n\pi\,x}{5}$. We can easily find the A_n if we remember that the Fourier cosines are orthogonal (which implies that they are linearly independent). We must have $\dfrac{A_0}{2} = 7$ and $A_n = 0$ if $n = 1, 2, 3, \ldots$ So we get the particular solution (and answer to this question): $u(x,t) = 7$

Note: In the above problem the solution was $u(x,t) = 7$. So the temperature of the rod $u(x,t)$ (or its concentration) is uniform and constant. This makes sense because the BC were of the type $u_x(0,t) = u_x(L,t) = 0$, which means that the ends of the rod were insulated so no heat (or concentrate) could leave or enter the rod. Diffusion requires non-uniformity, however the initial state of the rod was uniform (so no net diffusion could happen there) and the ends are insulated (sealed) so no net diffusion there either. Hence the tube will remain in its uniform initial state.

Note: we could also do the problem the long way calculating the A_n using the Fourier series formula for an evenly extended function:

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

If $n = 1, 2, 3, \ldots$

$$A_n = \frac{2}{5}\int_0^5 7\cos\frac{n\pi x}{L}\,dx = \frac{2}{5}\left[7\frac{5}{n\pi x}\sin\frac{n\pi x}{5}\right]_0^5 = 0$$

If $n = 0$ we get

$$A_0 = \frac{2}{5}\int_0^5 7\,dx = \frac{2}{5}[7x]_0^5 = \frac{2}{5}[7(5) - 0] = 14$$

Plugging these values for A_n into the general solution: $u(x,t) = \dfrac{A_0}{2} + \displaystyle\sum_{n=1}^{\infty} A_n e^{-10\lambda_n t} \cos \dfrac{n\pi x}{5}$ with $\lambda_n = \dfrac{n^2\pi^2}{25}$, gives us, as before, the same answer: $u(x,t) = \dfrac{A_0}{2} = \dfrac{14}{2} = 7$

One trick to integrate $\int (\sin ax)(\sin bx)\, dx$ and $\int (\cos ax)(\cos bx)\, dx$ and $\int (\cos ax)(\sin bx)\, dx$ is to rewrite the integrands using the following product to sum trig identities:

$$\sin a \sin b = \frac{1}{2}\left(\cos(a-b) - \cos(a+b)\right) \tag{93}$$
$$\cos a \cos b = \frac{1}{2}\left(\cos(a-b) + \cos(a+b)\right) \tag{94}$$
$$\sin a \cos b = \frac{1}{2}\left(\sin(a-b) + \sin(a+b)\right) \tag{95}$$

which are easily derived if we start with Euler's formula: $e^{ix} = \cos x + i \sin x$. We use Euler's formula to derive the angle sum trig identities:

$$\begin{aligned} \cos(a+b) + i\sin(a+b) = e^{i(a+b)} &= e^{ia}e^{ia} = (\cos a + i \sin a)(\cos b + i \sin b) \\ &= \cos a \cos b - \sin a \sin b + i(\cos a \sin b + \cos b \sin a) \end{aligned}$$

Equating the real and the imaginary parts gives us the angle sum trig identities:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{96}$$
$$\sin(a+b) = \cos a \sin b + \cos b \sin a \tag{97}$$

Since $\cos(-b) = \cos b$ and $\sin(-b) = -\sin(b)$, if in (96) and (97) we replace b with $-b$, we get the angle difference formulas:

$$\cos(a-b) = \cos a \cos b + \sin a \sin b \tag{98}$$
$$\sin(a-b) = -\cos a \sin b + \cos b \sin a \tag{99}$$

Then

$$\text{(98)} - \text{(96) gives us:} \qquad \cos(a-b) - \cos(a+b) = 2\sin a \sin b \tag{100}$$
$$\text{(98)} + \text{(96) gives us:} \qquad \cos(a-b) + \cos(a+b) = 2\cos a \cos b \tag{101}$$
$$\text{(97)} + \text{(99) gives us:} \qquad \sin(a+b) + \sin(a-b) = 2\cos b \sin a \tag{102}$$

Dividing (100), (101), and (102) by 2 yields the product to sum trig identities (93), (94), and (95).
