Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 8–3 : Limit of functions, continuity

1. Prove the following statements.

(1) $\lim_{x\to 0} x \sin \frac{1}{x} = 0$;

(2) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ $(a > 0,\ n \in \mathbb{N})$;

(3) $\lim_{x\to 1}(3x^3 - 2) = 1$;

(4) $\lim_{x\to 3}(2x^2 - x) = 15$;

(5) $\lim_{x\to\pm\infty} \frac{[x]}{x} = 1$, where $[x]$ is the largest integer smaller than $x \in \mathbb{R}$;

(6) $\lim_{x\to\pm\infty} \frac{2x^2 - 1}{3x^2 - x} = \frac{2}{3}$;

(7) $\lim_{x\to\pm\infty} \frac{3x^2 - 1}{x^2 + x + 1} = 3$;

(8) $\lim_{x\to\pm\infty} \left(\sqrt{x^2 + x} - x\right) = \frac{1}{2}$.

2. Prove that $\lim_{x\to 0} \frac{1}{x}$ does not exist.

3. Prove that if $\lim_{x\to a} f(x) = A$, then $\lim_{x\to a} |f(x)| = |A|$. Here we assume that $a \in \mathbb{R}$ or $a = \pm\infty$.

4. Let the function f be defined as

$$f : x \mapsto x - [x], \quad x \in \mathbb{R}.$$

Prove that

(1) for any $n \in \mathbb{Z}$, $\lim_{x\to n} f(x)$ does not exist;

(2) for any $a \notin \mathbb{Z}$, $\lim_{x\to a} f(x)$ exists.

5. Calculate the following limits.

(1) $\lim_{x\to a} \frac{\sin x - \sin a}{x - a}$;

(2) $\lim_{x\to 0} \frac{\sin nx}{\sin mx}$ $(n, m \in \mathbb{N})$;

(3) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$;

(4) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{x}$;

(5) $\lim\limits_{x\to 0} \frac{(1+mx)^n-(1+nx)^m}{x^2}$ $(m, n \in \mathbb{N})$;

(6) $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{n}}-1}{x}$ $(n \in \mathbb{N})$;

(7) $\lim\limits_{x\to a} \frac{x^{\frac{1}{m}}-a^{\frac{1}{m}}}{x^{\frac{1}{n}}-a^{\frac{1}{n}}}$ $(a > 0, m, n \in \mathbb{N})$.

6. Determine the parameters a, b and c such that the following limit

$$\lim_{x\to+\infty} \Big(\frac{x^3+2}{x+1} - ax^2 - bx - c\Big)$$

equals to 0 or 1.

7. Suppose that $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are two functions such that $f(X) \subseteq Y$, $a \in \mathbb{R}$ is a limit point of X and $\lim\limits_{x\to a} f(x) = A$. Moreover, $A \in \mathbb{R}$ is a limit point of Y and $\lim\limits_{y\to A} g(y) = B$.

(1) Is it always true that $\lim\limits_{x\to a} g \circ f(x) = B$?

Hint : Consider the example where $f, g : \mathbb{R} \to \mathbb{R}$ are two functions defined by

$$f(x) = \begin{cases} 0 & x = 0, \\ x\sin(\frac{1}{x}) & x \neq 0, \end{cases} \quad \text{and} \quad g(y) = \begin{cases} 1 & y = 0, \\ 0 & y \neq 0. \end{cases}$$

(2) Prove that if there exists $\delta > 0$ such that $A \notin X \cap \big(]a-\delta, a+\delta[\setminus \{a\}\big)$, then $\lim\limits_{x\to a} g \circ f(x) = B$.

8. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ is an increasing function. Prove that the following three statements are all equivalent :

(1) $\lim\limits_{x\to+\infty} f(x)$ exists;

(2) the sequence $(f(n))$ converges;

(3) f is bounded above on $\mathbb{R}^+$.

9. Let $f : [0,1] \to [0,1]$ be an arbitrary increasing function.

(1) Prove that there exists $x \in [0,1]$ such that $f(x) = x$.

(2) Suppose $f(0) = 0$ and for any $x \in]0,1]$, $f(x) < x$. Define the sequence (x_n) by $x_{n+1} = f(x_n)$ for any $n \in \mathbb{N}$ and $x_0 \in]0,1]$. Does the sequence (x_n) converge?

10. Find the sets of the points that the following functions are continuous at. For the points of discontinuity, specify their kind of discontinuity.

(1) $f : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = \sqrt{x}\,$;

(2) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}; \end{cases}$$

(3) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin x}{|x|} & x \neq 0 \\ 1 & x = 0; \end{cases}$$

(4) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{x}{1-|x|} & |x| \neq 1 \\ 1 & |x| = 1; \end{cases}$$

(5) $f : [0, a^2] \to \mathbb{R}$ defined by $f(x) = \sin(\cos x + \sqrt{1 + a^2})$.

11. Prove the following properties about continuous functions.

(1) Prove that if f is continuous at a, then so is $|f|$. Is the converse true?

(2) Prove that every continuous function f can be written as $f = E + O$, where E is even and continuous, and O is odd and continuous.

(3) Prove that if f and g are continuous, then so are $\max(f, g)$ and $\min(f, g)$.

(4) Prove that every continuous function f can be written as $f = g - h$, where g and h and nonnegative and continuous.

12. Let $I \subseteq \mathbb{R}$ denote an interval, $\mathbb{Q}^c$ be the complement of $\mathbb{Q}$ in $\mathbb{R}$, and $f : I \to \mathbb{R}$ a function.

(1) Prove that if f is continuous on I, and $f(I) \subseteq \mathbb{Q}$, then f is a constant function.

(2) Prove that if f is continuous on I, and $f(I) \subseteq \mathbb{Q}^c$, then f is a constant function.

(3) Prove that if I is not a point set and $f(I \cap \mathbb{Q}) \subseteq \mathbb{Q}^c$, $f(I \cap \mathbb{Q}^c) \subseteq \mathbb{Q}$, then f is not continuous on I.

13. Suppose that $f, g : [a, b] \to \mathbb{R}$ are two continuous functions satisfying

$$f(x) \neq 0, \quad f^2(x) = g^2(x), \quad \forall x \in [a, b].$$

Prove that only one of the following cases hold :
1)$f(x) = g(x)$, $\forall x \in [a, b]$;
2)$f(x) = -g(x)$, $\forall x \in [a, b]$.

14. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Prove that

(1) if $a_0 < 0$, then $P_n(x) = 0$ has at least one positive root ;

(2) if $a_0 < 0$ and n is even, then $P_n(x) = 0$ has at least one negative root ;

(3) if n is even, then there exists $\tilde{x} \in \mathbb{R}$ such that $P_n(x) \geq P_n(\tilde{x})$ for any $x \in \mathbb{R}$;

(4) conclude from (c) that there exists $\beta \in \mathbb{R}$ such that for any $b \geq \beta$, $P_n(x) = b$ has a real root, and for any $b < \beta$, the equation $P_n(x) = b$ has no real roots.

15. Let $f : [a, b] \to \mathbb{R}$ be a function such that all points of discontinuity are removable discontinuities.

(1) Prove that for any $\epsilon > 0$, the set

$$E_\epsilon := \{x \in [a, b] : |\lim_{y \to x} f(y) - f(x)| > \epsilon\}$$

is a finite set.

(2) Prove that the set of discontinuities of f is countable. Conclude from this that there does not exist a function $f : [a, b] \to \mathbb{R}$ such that every point in $[a, b]$ is a removable discontinuity.

16. State whether the following functions are uniformly continuous, and provide a proof of your statement.

(1) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$;

(2) $f : [0, a] \to \mathbb{R}$ defined by $f(x) = x^2$;

(3) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$;

(4) $f :]a, 1[\to \mathbb{R}$ defined by $f(x) = \frac{1}{x^6}$, where $0 \leqslant a < 1$;

(5) $f :]0, +\infty] \to \mathbb{R}$ defined by $f(x) = \sin(\frac{1}{x})$.

17. Suppose that $a \in \mathbb{R}$ and $f : [a, +\infty[\to \mathbb{R}$ is a continuous function such that $\lim_{x \to \infty} f(x) = A$, where A is a finite real number or $+\infty$.

(1) Prove that f takes all values between $f(a)$ and A.

(2) Prove that if $A = +\infty$, then f is bounded from below and f attains its greatest lower bound.

(3) Prove that if A is a finite real number, then f is uniformly continuous on $[a, \infty)$.

18. Let X be a set and $f : X \to \mathbb{R}$ be a function. Determine whether the following statements are true or false. If true, please provide a proof; if false, give a counterexample.

(1) If X is a bounded set and f is continuous on X, then f is bounded on X.

(2) If X is a bounded set and f is uniformly continuous on X, then f is bounded on X.

(3) If X is closed, and f is bounded and continuous on X, then f is uniformly continuous.

(4) If X is bounded, and f is bounded and continuous on X, then f is uniformly continuous.

(5) If X is closed, and f is monotonic, bounded and continuous on X, then f is uniformly continuous.

(6) If X is bounded, and f is uniformly continuous on X, then f is bounded and continuous on X.

19. Let $E \subseteq \mathbb{R}$. The *modulus of continuity* of a function $f : E \to \mathbb{R}$ is the function $\omega(\delta)$ defined for $\delta > 0$ as follows :

$$\omega(\delta) = \sup_{\substack{x_1, x_2 \in E \\ |x_1 - x_2| < \delta}} |f(x_1) - f(x_2)|.$$

Show that

(1) the modulus of continuity is a nondecreasing nonnegative function having the limit
$$\omega(0^+) := \lim_{\delta \to +0} \omega(\delta);$$

(2) for every $\epsilon > 0$, there exists $\delta > 0$ such that for any points $x_1, x_2 \in E$, the relation $|x_1 - x_2| < \delta$ implies $|f(x_1) - f(x_2)| < \omega(0^+) + \epsilon$;

(3) if E is a closed interval, an open interval, or a half-open interval, the relation
$$\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$$
holds for the modulus of continuity of a function $f : E \to \mathbb{R}$;

(4) the moduli of continuity of the functions x and $\sin(x^2)$ on the whole real axis are respectively $\omega(\delta) = \delta$ and the constant $\omega(\delta) = 2$ in the domain $\delta > 0$;

(5) a function f is uniformly continuous on E if and only if $\omega(0^+) = 0$.

20. Let I denote a non-empty subset of $\mathbb{R}$, and f a function from I into I. A sequence (x_n) is defined by choosing an element x_0 of I and setting $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$. Such a sequence is called a **recursive sequence**.

(1) Prove that if the sequence (x_n) converges to an element l of I and if f is continuous at the point l, then $f(l) = l$.

(2) Assume that f is increasing on I.

  i. Show that the sequence (x_n) is monotonic.

  ii. Let l be an element of I such that $f(l) = l$. Prove that if $x_0 \leq l$ (respectively $x_0 \geq l$), then $(\forall n \in \mathbb{N})(x_n \leq l)$ (respectively $(\forall n \in \mathbb{N})(x_n \geq l)$).

  iii. Additionally, suppose that I is a closed and bounded interval of $\mathbb{R}$, and f is continuous on I. Prove that the sequence (x_n) converges to an element l of I such that $f(l) = l$.

(3) Assume that f is decreasing on I.

  i. Prove that the sequences (x_{2n}) and (x_{2n+1}) are monotonic and have opposite directions of variation.

  ii. Let I_0 be the closed interval with endpoints x_0 and x_1, and let l be an element of I such that $f(l) = l$. Prove that l belongs to I_0.

    A. Suppose that x_2 belongs to I_0. Prove that the sequences (x_{2n}) and (x_{2n+1}) are convergent. Let $l = \lim_{n\to+\infty} x_{2n}$ and $l' = \lim_{n\to+\infty} x_{2n+1}$. Prove that l and l' are elements of I, and if f is continuous at the points l and l', then $l = f(l')$ and $l' = f(l)$.

    B. Prove that if $x_2 \notin I_0$, the sequence (x_n) is divergent.

(4) i. Prove that the sequence (x_n), defined by $x_0 > -1$ and $x_{n+1} = \frac{x_n^2}{x_n+1}$ for $n \in \mathbb{N}$, has the limit 0.

  ii. Prove that the sequence (x_n), defined by $x_0 \in \mathbb{R}_+$ and $x_{n+1} = \frac{2x_n^2+3x_n}{x_n+1}$ for $n \in \mathbb{N}$, tends to 0 if $x_0 = 0$ and to $+\infty$ if $x_0 > 0$.

  iii. Prove that the sequence (x_n), defined by $x_0 \in [0, 2]$ and $x_{n+1} = -\frac{x_n^2}{2} + 2$ for $n \in \mathbb{N}$, diverges if $x_0 \neq -1 + \sqrt{5}$ and converges if $x_0 = -1 + \sqrt{5}$.

  iv. Prove that the sequence (x_n), defined by $x_0 \in [0, 2]$ and $x_{n+1} = 1 - \frac{x_n^2}{4}$ for $n \in \mathbb{N}$, converges to $2(\sqrt{2} - 1)$.

  v. Let $t \in [0, 1]$. Prove that the sequence (x_n), defined by $x_0 \in [0, 1]$ and $x_{n+1} = x_n + \frac{1}{2}(t - x_n^2)$ for $n \in \mathbb{N}$, converges to $\sqrt{t}$.

21. The goal of this problem is to determine the maximal ideals of the ring $A = C(I, \mathbb{R})$ of real-valued functions that are defined and continuous on a closed bounded interval I of $\mathbb{R}$. The finite covering theorem will be used.

For a subset $\mathcal{M} \subset A$, if it satisfies the following three conditions :

1) $\mathcal{M} \neq \emptyset$ and $\mathcal{M} \neq A$;

2) for any $p, q \in \mathcal{M}$, we have $p + q \in \mathcal{M}$;

3) for any $p \in \mathcal{M}$ and $f \in A$, we have $p \cdot f \in \mathcal{M}$;

then we call I an ideal of A. Suppose m is an ideal of A and there does not exist another ideal I such that $m \subsetneq I \subsetneq A$ (i.e., if I is an ideal and $m \subset I$, then $I = m$). Then we call m a maximal ideal of A.

(1) i. Let $x_0 \in I$. Prove that the set $\mathcal{M}_{x_0}$ of functions in A that vanish at x_0 is an ideal of A.

ii. Let J be an ideal of A such that $\mathcal{M}_{x_0} \subset J$ and $\mathcal{M}_{x_0} \neq J$. Let $f \in J \setminus \mathcal{M}_{x_0}$. Prove that there exists an element g of $\mathcal{M}_{x_0}$ such that $f^2 + g$ does not vanish on I. Deduce that $\mathcal{M}_{x_0}$ is a maximal ideal of A.

(2) Let $\mathcal{M}$ be a maximal ideal of A.

i. Let $f \in \mathcal{M}$. Prove that the equation $f(x) = 0$ has a solution in I.

ii. For each element $f \in A$, define

$$Z(f) = \{x \in I \mid f(x) = 0\}.$$

Suppose that $\bigcap_{f \in \mathcal{M}} Z(f) = \emptyset$.

Prove that for every element x of I, there exists an element $f_x \in \mathcal{M}$ such that $x \notin Z(f_x)$. Deduce that there exists a finite number $f_1, \ldots, f_n$ of functions in $\mathcal{M}$ such that

$$Z(f_1) \cap \cdots \cap Z(f_n) = \emptyset.$$

iii. Using the finite covering theorem, prove that there exists a finite number of functions $f_1, \ldots, f_n$ in A such that :

$$Z(f_1) \cap \cdots \cap Z(f_n) = \emptyset.$$

iv. Prove that $f_1^2 + \cdots + f_n^2 \notin \mathcal{M}$. Deduce that $Z\left(\bigcup_{f \in \mathcal{M}} Z(f)\right) \neq \emptyset$.

v. Prove that $\bigcap_{f \in \mathcal{M}} Z(f)$ consists of a single point x_0 in I, and that $\mathcal{M} = \mathcal{M}_{x_0}$.

A side remark : In this exercise the set I can be notably replaced by any metric space X that we introduced in the end of the chapter. We can imagine the maximal ideals in the ring $C(X)$ as points in the space, thereby studying the geometry on X through the algebraic objects in $C(X)$. This has served as the initial observation in the field of algebraic geometry, as well as the mathematical foundation theory of quantum mechanics.

22. Let $f : [0, +\infty[\to \mathbb{R}$ be a function.

(1) Suppose that f is bounded on any bounded and closed interval $[a, b]$ and
$$\lim_{x\to+\infty} f(x+1) - f(x) = l,$$
where l is a finite real number. Prove that $\lim_{x\to+\infty} \dfrac{f(x)}{x} = l$.

(2) Suppose that f is bounded from below on any bounded and closed interval $[a, b]$ and
$$\lim_{x\to+\infty} f(x+1) - f(x) = +\infty.$$
Prove that $\lim_{x\to+\infty} \dfrac{f(x)}{x} = +\infty$.

23. Prove the following identities about inverse functions of trigonometric functions.

(1) $\arccos(-x) = \arccos x$;
(2) $\arctan(-x) = -\arctan x$;
(3) $\arcsin x + \arccos x = \frac{\pi}{2}$;
(4) $\arctan x + \mathrm{arccot} x = \frac{\pi}{2}$.

24. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(y) = y - \lambda \sin y$, where $0 \le \lambda < 1$.

(1) Prove that f is a continuous and strictly increasing function ;
(2) Prove that there exists a unique continuous inverse function $g : \mathbb{R} \to \mathbb{R}$ such that
$$g(x) - \lambda \sin(g(x)) = x.$$

25. Suppose that $I \to \mathbb{R}$ is an nonempty interval, and $f : I \to \mathbb{R}$ is an increasing function. Prove that the following statements are equivalent :

(1) f is continuous on I ;
(2) $f(I)$ is an interval.

26. We consider the mapping $f : [0, +\infty[\to [0, +\infty[$ sending $x \in [0, +\infty[$ to x^2.

(1) Show that f is locally Lipschitzian but not Lipschitzian.
(2) Show that f is continuous but not uniformly continuous.
(3) Show that f is a bijection and its inverse mapping is continuous.
(4) Show that f^{-1} is not locally Lipschitzian at 0.
(5) Show that the restriction of f^{-1} to $[1, +\infty[$ is Lipschitzian.
(6) Show that the restriction of f^{-1} to $[0, 2]$ is uniformly continuous.
(7) Deduce that f^{-1} is uniformly continuous.

27. Let $f : \mathbb{R} \to \mathbb{R}$ be the mapping defined as

$$f(x) = \frac{1}{1 + |x|}.$$

Show that the mapping f is 1-Lipschitzian.

28. Let n be a positive integer, $(X_1, \mathscr{T}_1), \ldots, (X_n, \mathscr{T}_n)$ be topological spaces, and X be the product space $X_1 \times \cdots \times X_n$. We equip X with the product topology. For any $i \in \{1, \ldots, n\}$, let $p_i : X \to X_i$ be the mapping of projection to the i-th coordinate that sends $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$ to x_i.

(1) For any $i \in \{1, \ldots, n\}$, let U_i be an open subset of X_i. Show that $U_1 \times \cdots \times U_n$ is an open subset of X. One could write it as the intersection of $p_i^{-1}(U_i)$.
(2) Let

$$\mathcal{B} := \big\{ U_1 \times \cdots \times U_n \,|\, \forall\, i \in \{1, \ldots, n\},\ U_i \in \mathscr{T}_i \big\}.$$

Show that the intersection of two sets in $\mathcal{B}$ still belongs to $\mathcal{B}$.
(3) Let $\mathscr{T}$ be the set of elements $U \in \mathscr{P}(X)$ that can be written as a union of some sets in $\mathcal{B}$. Show that $\mathscr{T}$ is a topology and deduce that it identifies with the product topology of X.
(4) Let Y be a topological space and $p \in Y$. For any $i \in \{1, \ldots, n\}$, let $f_i : Y \to X_i$ be a mapping which is continuous at p. Let $f : Y \to X$ be the mapping sending $y \in Y$ to $(f_1(y), \ldots, f_n(y))$.
  (a) For any $i \in \{1, \ldots, n\}$, let V_i be a neighbourhood of $f_i(p)$. Show that $f^{-1}(V_1 \times \cdots \times V_n)$ is a neighbourhood of p.
  (b) Deduce that f is continuous at p.

29. Let $f : \mathbb{R} \to \mathbb{R}$ be the mapping which is defined as

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(1) Show that $\sqrt{2}$ is an irrational number.

(2) Deduce that, for any $n \in \mathbb{N}_{\geqslant 1}$, $\frac{\sqrt{2}}{n}$ is an irrational number.

(3) Let $p \in \mathbb{R}$.

(a) Show that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \setminus \mathbb{Q}$ that converges to p.

(b) Suppose that $p \in \mathbb{Q}$. Show that there exists a sequence in $\mathbb{Q}$ that converges to p.

(c) Suppose that $p \in \mathbb{R} \setminus \mathbb{Q}$. Show that the sequence of rational numbers

$$\frac{\lfloor np \rfloor}{n}, \quad n \in \mathbb{N}_{\geqslant 1}$$

converges to p.

(d) Deduce that f is not continuous at p.

(4) Let g_0 and g_1 be continuous mappings from $\mathbb{R}$ to $\mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be the mapping defined as

$$g(x) = \begin{cases} g_0(x), & \text{if } x \in \mathbb{Q}, \\ g_1(x), & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that g is continuous at some $p \in \mathbb{R}$ if and only if $g_0(x) = g_1(x)$.

30. (1) Show that the mapping from $\mathbb{R}$ to $\mathbb{R}$ sending $x \in \mathbb{R}$ to $|x|$ is 1-Lipschitzian.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping and $p \in \mathbb{R}$. We assume that f is continuous at p. Show that the function $|f|$ sending $x \in \mathbb{R}$ to $|f(x)|$ is continuous at p.

(3) Let $g : \mathbb{R} \to \mathbb{R}$ be a mapping such that $|g|$ is continuous at some $p \in \mathbb{R}$. Is the mapping g necessarily continuous at p?

(4) Let f and g be continuous mappings from $\mathbb{R}$ to $\mathbb{R}$. Show that the mappings

$$(x \in \mathbb{R}) \longmapsto \max\{f(x), g(x)\}$$

and

$$(x \in \mathbb{R}) \longmapsto \min\{f(x), g(x)\}$$

are both continuous.

31. We consider the mapping $g : \mathbb{R} \to \mathbb{R}$ defined as

$$g(x) = \begin{cases} \frac{1}{\ln|x|}, & \text{if } x \notin \{-1, -0, 1\}, \\ 0, & \text{if } x \in \{-1, -0, 1\}. \end{cases}$$

(1) Show that the mapping $x \mapsto x^{-1}$ from $\mathbb{R}\setminus\{0\}$ to $\mathbb{R}\setminus\{0\}$ is locally Lipschitzian. Deduce that it is a continuous mapping.
(2) Show that the mapping g is continuous at $\mathbb{R}\setminus\{-1,0,1\}$.
(3) Show that g is continuous at 0.
(4) Determine the left limit and the right limit of g at 1. Is the mapping g continuous at 1 ?
(5) Determine the left limit and the right limit of g at -1. Is the mapping g continuous at -1 ?

32. Let I be a non-empty open interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be an increasing mapping.

(1) Let $p \in I$, show that the mapping f has a left limit $f(p-)$ and a right limit $f(p+)$ at p.
(2) Show that, for any $p \in I$, $f(p-) \leqslant f(p+)$.
(3) Show that f is continuous at $p \in I$ if and only if $f(p-) = f(p+)$.
(4) Let n be a positive integer and

$$I_n = \{p \in I \mid f(p+) - f(p-) \geqslant 1/n\}.$$

Show that I_n is countable.
(5) Show that the set of $p \in I$ at which f is discontinuous is countable.

33. Let $\mathcal{F}$ be the Fréchet filter on $\mathbb{N}$ and $(X, \mathscr{T})$ be a topological space. Let $u = (u_n)_{n\in\mathbb{N}}$ be a sequence in X and x be an element of X. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to x if and only if the filter $u_*(\mathcal{F})$ contains $\mathcal{V}_x$.

34. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous mapping such that

$$\forall\,(x,y) \in \mathbb{R}^2, \quad f(x+y) = f(x) + f(y).$$

(1) Show that $f(0) = 0$.
(2) Show that, for any $x \in \mathbb{R}$, $f(-x) = -f(x)$.
(3) Show that, for any $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, one has $f(nx) = nf(x)$.
(4) Show that, for any $r \in \mathbb{Q}$ and $x \in \mathbb{R}$, one has

$$f(rx) = rf(x).$$

(5) Show that, for any $x \in \mathbb{R}$ one has

$$f(x) = f(1)x.$$

35. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous mapping.

(1) Show that the mapping $F : \mathbb{R}^2 \to \mathbb{R}$ defined as $F(x, y) = y - f(x)$ is continuous.

(2) Show that the graph of f, defined as

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = f(x)\},$$

is a closed subset of $\mathbb{R}^2$.

36. Let (X, d) be a metric space, $p \in X$ and $r > 0$. Let K be a compact subset of X such that $K \subseteq B(p, r)$. Show that there exists $r' \in \,]0, r[$ such that

$$\forall\, x \in K, \quad d(p, x) < r'.$$

37. Let (X, d) be a metric space and K_1, K_2 be two compact subsets of X that are disjoint. Show that

$$d(K_1, K_2) := \inf_{(x,y) \in K_1 \times K_2} d(x, y) > 0.$$

We could use the compactness of the product $K_1 \times K_2$.

38. Let V and W be vector spaces over $\mathbb{R}$, and $f : V \to W$ be an $\mathbb{R}$-linear mapping. We equip V and W with norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively. Recall that these norms define metrics on V and W, which also determine topologies on these vector spaces.

(1) Assume that f is continuous at $\mathbf{0}_V$. Show that it is continuous.

(2) We denote by $\|f\|$ the element

$$\sup_{x \in V \setminus \{\mathbf{0}_V\}} \frac{\|f(x)\|_W}{\|x\|_V}$$

in $[0, +\infty]$. Show that

$$\|f\| = \sup_{x \in V,\, \|x\|_V \leqslant 1} \|f(x)\|_W = \sup_{x \in V,\, \|x\|_V = 1} \|f(x)\|_W.$$

(3) Show that f is continuous if and only if $\|f\| < +\infty$.

(4) Let n be a positive integer. Show that the product topology on $\mathbb{R}^n$ identifies with the topology induced by the norm $\|\cdot\|_{\ell^\infty}$ defined as

$$\|(a_1, \ldots, a_n)\|_{\ell^\infty} := \max\{|a_1|, \ldots, |a_n|\}.$$

(5) Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$. Show that $\|\cdot\|$ is continuous, where we consider the product topology on $\mathbb{R}^n$. We could prove that there exists a constant $C_1 > 0$ such that $\|x\| \leqslant C_1 \|x\|_{\ell^\infty}$ for any $x \in \mathbb{R}^n$.

(6) Show that $\{x \in \mathbb{R}^n \mid \|x\|_{\ell^\infty} \leqslant 1\}$ is compact.
(7) Prove that there exists $C_2 > 0$ such that $\|x\| \geqslant C_2 \|x\|_{\ell^\infty}$.
(8) Show that the topology induced by $\|\cdot\|$ identifies with the product topology.
(9) Show that $\mathbb{R}^n$ is complete.

39. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a continuous mapping. We assume that f is injective.

(1) Let a, b and c be elements of I such that $a < b < c$. Show that

$$(f(b) - f(a))(f(c) - f(b)) > 0.$$

(2) Show that the mapping f is strictly monotone.

40. Show that the mapping

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad f(x) = \lfloor x \rfloor + \sqrt{x - \lfloor x \rfloor}$$

is well defined and is continuous.

41. We consider the mapping $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ defined as

$$\forall\, (x, y) \in \mathbb{R}^2, \quad |x + iy| = \sqrt{x^2 + y^2}.$$

For any $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$, we denote by $\overline{z}$ the complex number $x - iy$.

(1) We consider $\mathbb{C}$ is a vector space over $\mathbb{R}$. Show that $|\cdot| : \mathbb{C} \to \mathbb{R}$ is a norm and $|z|^2 = z\overline{z}$ for any $z \in \mathbb{C}$.
(2) For any $z \in \mathbb{C}$ and $r > 0$, let

$$D(z, r) := \{w \in \mathbb{C} \mid |w - z| \leqslant r\}.$$

Show that $D(z, r)$ is compact.
(3) Let $z \in \mathbb{C}$. Show that the sequence

$$\sum_{n \in \mathbb{N}} \frac{z^n}{n!}$$

converges in $\mathbb{C}$ to a limit which we denoted as $\exp(z)$.
(4) Show that $\exp(\cdot)$ is a morphism of groups from the additive group $(\mathbb{C}, +)$ to the multipicative group $(\mathbb{C}^\times, \times)$, where $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.
(5) Show that $\exp(\overline{z}) = \overline{\exp(z)}$.
(6) Show that, for any $\theta \in \mathbb{R}$ one has $|\exp(i\theta)| = 1$.

(7) Let G be a subgroup of $(\mathbb{R}, +)$. Show that, either $\overline{G} = \mathbb{R}$, or G is discrete, namely for any $x \in G$, x has a neighbourhood V such that $V \cap G = \{x\}$.

(8) Show that the kernel K of the group homomorphism

$$(\mathbb{R}, +) \longrightarrow (\mathbb{C}^{\times}, \times), \quad (\theta \in \mathbb{R}) \longmapsto \exp(i\theta)$$

is discrete.

(9) For any $\theta \in \mathbb{R}$, let $\cos(\theta)$ and $\sin(\theta)$ be real numbers such that

$$\exp(i\theta) = \cos(\theta) + i \sin(\theta).$$

Show that $\cos(2) < -\frac{1}{3}$.

(10) Deduce that $K \cap \mathbb{R}_{>0}$ is non-empty and there exists a unique real number $\pi > 0$ such that $K = 2\pi\mathbb{Z}$.

(11) Show that $\exp(i\pi) = -1$.

(12) Show that the image of $\cos(\cdot)$ is $[-1, 1]$.

(13) Show that $\cos(\cdot)$ is strictly decreasing on $[0, \pi/2]$ and $\sin(\cdot)$ is strictly increasing on $[0, \pi/2]$. Deduce that $\sin(\pi/2) = 1$.

(14) Show that the image of the mapping

$$\exp(i\cdot) : \mathbb{R} \longrightarrow \mathbb{C}, \quad (\theta \in \mathbb{R}) \longmapsto \exp(i\theta)$$

is equal to $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$. Deduce that S^1 is a subgroup of $\mathbb{C}^{\times}$.

(15) Show that the morphism of groups $\exp(\cdot) : \mathbb{C} \to \mathbb{C}^{\times}$ is surjective. What is its kernel ?