Quantum Mechanics Technology Processing and Conclusions

Jiete XUE

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Contents

1 Quantization

$$\oint p \, \mathrm{d}q = nh + C, \; n = 1, 2, 3, \ldots \tag{1.1}$$

This is not always right, but it's a good approximation.

2 Heisenberg's Breakthrough

Heisenberg use matrices to represent quantum states.[1]

$$x_{n,m}(t) = x_{n,m} e^{i\omega_{n,m} t}. \tag{2.1}$$

With the **Ritz combination law**[2]

$$\omega_{n,m} = \omega_{n,k} + \omega_{k,m}, \tag{2.2}$$

we can calculate x^2 like

$$\left(x^2\right)_{n,m} e^{i\omega_{n,m} t} = \sum_k x_{n,k}(t) x_{k,m}(t) = \sum_k x_{n,k} x_{k,m} e^{i\omega_{n,m} t}. \tag{2.3}$$

Problem 2.1 (Harmonic Oscillator)

By solving this example, we can get how to deal with the problems in quantum situations. The equation of the harmonic oscillator in the classic mechanics is:

$$\ddot{x} + \omega_0^2 x = 0. \tag{2.4}$$

We can deduce that

$$\left(\omega_0^2 - \omega_{n,m}^2\right) x_{n,m}(t) = 0. \tag{2.5}$$

Their's a strange assumption, we may treat[3] $\omega_{n,n+1}$ as ω_0, then only for those $x_{n,n\pm 1}$ can not be zero. Then we can use (1.1) to obtain other information. (Use t as parameter to do the integration.)

Only when $n = m$, the integration of

$$-m\omega_{n,k}\omega_{k,m} x_{n,k} x_{k,m} e^{i\omega_{n,m} t} \tag{2.6}$$

in a period can not be zero.[4] Then we obtain [5]

$$\frac{\hbar}{2m} = \omega_0 \left[\left|x_{n+1,n}\right|^2 - \left|x_{n,n-1}\right|^2 \right]. \tag{2.7}$$

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[1] Here use the similar notation $x(t)$ and x, but they represent different things. In some cases, we will ignore (t) but still represent $x(t)$ for convenience.

[2] You can imagine a electron moving from state m to state n passing state k.

[3] Their must exists a $\omega_{n,m} = \omega_0$, or $x_{n,m} = 0$, or the physics phenomenon can not be detected.

[4] $\omega_{n,n} = 0$.

[5] Here I used f-sum rule exactly. But I think there's some details need to check, about where the $\frac{1}{2}$ comes from. It appear as C in quantization condition.

3 De Broglie's & Schrödinger's Equation

De Broglie use

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} = v_g = \frac{pc^2}{E} \tag{3.1}$$

and

$$E^2 = p^2c^2 + m_0^2c^4 \tag{3.2}$$

with

$$E = \hbar\omega \tag{3.3}$$

to derive that

$$p = \hbar k. \tag{3.4}$$

It is wave-like. The Debye told to Schrödinger that: If it is a wave, then you can try to get a wave equation to describe it.
If we consider a plane wave, the it take the form of

$$\psi \sim e^{\frac{i}{\hbar}(px-Et)}. \tag{3.5}$$

The exponent is similar to S- the action. So we can let $\psi = e^{iS/\hbar}$, and try to explore what we can get after plugging it into the Hamilton equation.

$$\frac{\hbar^2}{2m}(\nabla\psi)^2 + (E - V(r))\,\psi^2 = 0. \tag{3.6}$$

We want a linear equation. Then we let ∇ act on it, and suppose $V(r)$ is smooth at the $\hbar$ scale. Then we get

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right)\psi = E\psi. \tag{3.7}$$

Born gave ψ a physics meaning: The probability density for the particle that appear at position x.

Problem 3.1 (The relation between image of ψ and E)
We know that, the second order derivative can represent the curvature. So with the energy growing, the image of the wave function will twist and cross the average line more and more.

4 Dirac's Notation

4.1 Bra & Ket

P. A. M Dirac divided the word "bracket" into two parts, one is the bra, and the other is the ket. We call left vector as bra, and right vector as ket. We can denote[1] the state ψ as $|\psi\rangle$, with its dual form $\langle\psi|$. Each of these vectors contains

[1] Clear and simple, its my favorite.

a complete information about the state. We have to clarify the computation law about the bras and kets.

$$a\left|\alpha\right\rangle + b\left|\beta\right\rangle \xleftrightarrow{DC.} a^*\left\langle\alpha\right| + b^*\left\langle\beta\right|,\ \forall a,b\in\mathbb{C}. \tag{4.1}$$

Inner product:

$$\langle\alpha|\beta\rangle := \left\langle\alpha\right|\ \left|\beta\right\rangle \tag{4.2}$$

is a complex number. Here we postulate two fundamental properties of inner products:

$$\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \tag{4.3}$$

and

$$\langle\alpha|\alpha\rangle \geq 0. \tag{4.4}$$

The property of the second one is because we can take $\left|\beta\right\rangle = \left|\alpha\right\rangle$ in (4.3), then we can obtain $\langle\alpha|\alpha\rangle$ is a real number.

In fact, all the states forms a vector space V over $\mathbb{C}$.

4.2 Operator

All the operator[1] forms a unitary ring K, it can be a K-module structure acting on V. If X is an operator, then we define an operator (**Hermitian adjoint**) $X^\dagger$ as

$$X\left|\psi\right\rangle \xleftrightarrow{DC.} \left\langle\psi\right| X^\dagger. \tag{4.5}$$

We say X is **Hermitian** iff.

$$X^\dagger = X. \tag{4.6}$$

Easy to prove:

$$\begin{aligned}(XY)^\dagger &= Y^\dagger X^\dagger, && (4.7)\\ (\left|\alpha\right\rangle\left\langle\beta\right|)^\dagger &= \left|\beta\right\rangle\left\langle\alpha\right|, && (4.8)\\ \left\langle\beta\right| X\left|\alpha\right\rangle &= \left(\left\langle\alpha\right| X^\dagger \left|\beta\right\rangle\right)^*. && (4.9)\end{aligned}$$

Theorem 4.1

The eigenvalues of a Hermitian operator A are real; the eigenkets of A corresponding to different eigenvalues are orthogonal.

Proof. We have

$$A\left|a_1\right\rangle = a_1\left|a_1\right\rangle,\ \left\langle a_2\right| A = a_2^*\left\langle a_2\right|. \tag{4.10}$$

Hence,

$$(a_1 - a_2^*)\langle a_2|a_1\rangle = 0. \tag{4.11}$$

Let $a_2 = a_1$, then we obtain

$$a_1 = a_2^*. \tag{4.12}$$

[1]Some where will use hat to distinguish the operator, but not always for convenience.

So a_1 is real. If $a_1 \neq a_2$, then

$$\langle a_2 | a_1 \rangle = 0. \tag{4.13}$$

i.e. the eigenkets are orthogonal. □

All the observables must be real, so its operator must be Hermitian.[1]
If $\{|a\rangle\}$ is complete, then

$$\sum_a |a\rangle \langle a| = 1. \tag{4.14}$$

Let it act on a state, then we can get the state represent as the bases of $\{|a\rangle\}$. And for a particular a, we can define the **projection operator** as

$$\Lambda_a = |a\rangle \langle a| . \tag{4.15}$$

If we use (4.14) to act on both left and right of an operator X, then we can make it represent in the matrix form.

We write an observable X as

$$X = \sum_{a,b} |a\rangle \langle a| X |b\rangle \langle b| . \tag{4.16}$$

Since $\langle a| X |b\rangle$ is real,

$$X^\dagger = \sum_{a,b} |b\rangle \langle b| X |a\rangle \langle a| = X. \tag{4.17}$$

So X is Hermitian.

5 Commutator and Anticommutator

For any two operators A and B, we define their **commutator** as

$$[A, B] = AB - BA. \tag{5.1}$$

And the **anticommutator** is

$$\{A, B\} = AB + BA. \tag{5.2}$$

Its easy to check

$$[AB, C] = A[B, C] + [A, C]B, \tag{5.3}$$

$$[A, BC] = [A, B]C + B[A, C], \tag{5.4}$$

$$[A, [B, C]] + [B, [A, C]] + [C, [A, B]] = 0, \tag{5.5}$$

$$[A, B]^\dagger = [B^\dagger, A^\dagger]. \tag{5.6}$$

[1] We will explain it soon.

$$\{AB, C\} = A\{B, C\} - [A, C]B, \tag{5.7}$$

$$\{A, BC\} = \{A, B\}C - B[A, C]. \tag{5.8}$$

Suppose $f(A), g(B)$ are differentiable functions of A and B, let

$$f(A) = \sum_{n=0}^{\infty} a_n A^n, \tag{5.9}$$

we have

$$[A^n, B] = \sum_{k=0}^{n-1} A^{n-k-1}[A, B]A^k. \tag{5.10}$$

If $[A, B]$ is commutable with A, then,

$$[f(A), B] = f'(A)[A, B]. \tag{5.11}$$

Similarly, if $[A, B]$ is commutable with B, then,

$$[A, g(B)] = g'(B)[A, B]. \tag{5.12}$$

6 Compatible Observables

Observables A and B are defined to be **compatible** when the corresponding operators commute,

$$[A, B] = 0. \tag{6.1}$$

Theorem 6.1
Suppose that A and B are compatible observables, and the eigen-values of A are nondegenerate. Then the matrix elements $\langle a''| B |a'\rangle$ are all diagonal.

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7 Uncertainty Relation

Define ΔA as

$$\Delta A := A - \langle A \rangle , \tag{7.1}$$

where $\langle A \rangle$ is expectation value of A. And the **dispersion** of A is

$$\left\langle (\Delta A)^2 \right\rangle = \left\langle \left(A^2 - 2A \langle A \rangle + \langle A \rangle^2 \right) \right\rangle = \left\langle A^2 \right\rangle - \langle A \rangle^2 . \tag{7.2}$$

Theorem 7.1
For any state, we must have the following inequality:

$$\left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq \frac{1}{4} \left| \langle [A, B] \rangle \right|^2 . \tag{7.3}$$

Proof. The Schwarz inequality[1]:

$$\langle\alpha|\alpha\rangle \langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 . \tag{7.4}$$

Take[2]

$$|\alpha\rangle = \Delta A \,|\rangle \,, \; |\beta\rangle = \Delta B \,|\rangle \,,$$

then,

$$\left\langle (\Delta A)^2 \right\rangle \left\langle (\Delta B)^2 \right\rangle \geq |\langle \Delta A \Delta B\rangle|^2 . \tag{7.5}$$

Here we use a trick: Any operator can be divided into Hermitian part and anti-Hermitian part,

$$\Delta A \Delta B = \frac{1}{2}[\Delta A, \Delta B] + \frac{1}{2}\{\Delta A, \Delta B\}. \tag{7.6}$$

Easy to prove that the expect value of Hermitian operator is real, and the anti-Hermitian's is purely imaginary. Thus,

$$|\langle \Delta A \Delta B\rangle|^2 = \frac{1}{4} |\langle [\Delta A, \Delta B]\rangle|^2 + \frac{1}{4} |\langle \{\Delta A, \Delta B\}\rangle|^2 . \tag{7.7}$$

One have $[\Delta A, \Delta B] = [A, B]$ and $|\dots|^2 \geq 0$, then we get the result. □

8 Position and Momentum

I will follow J. J. Sakurai's steps to derive some famous results. As continuous spectra, they have following changes [3]:

$$\langle a'|a''\rangle = \delta_{a',a''} \longrightarrow \langle \xi'|\xi''\rangle = \delta\left(\xi' - \xi''\right). \tag{8.1}$$

Where $\delta\left(\xi' - \xi''\right)$ is the Dirac delta function.

$$\sum_a |a\rangle \langle a| = 1 \longrightarrow \int \mathrm{d}\xi \,|\xi\rangle \langle\xi| = 1. \tag{8.2}$$

8.1 Position & Wave Function

For position, we can measure its position, which means we can get its components at the same time, they are compatible. So

$$[\hat{x}_i, \hat{x}_j] = 0. \tag{8.3}$$

$$|\mathbf{x}\rangle = |x, y, z\rangle . \tag{8.4}$$

$$\hat{x}\,|\mathbf{x}\rangle = x\,|\mathbf{x}\rangle \,, \; \hat{y}\,|\mathbf{x}\rangle = y\,|\mathbf{x}\rangle \,, \; \hat{z}\,|\mathbf{x}\rangle = z\,|\mathbf{x}\rangle . \tag{8.5}$$

We call

$$\psi_\alpha(x) = \langle x|\alpha\rangle \tag{8.6}$$

the wave function.

[1] Consider $|\alpha\rangle + \lambda\,|\beta\rangle$ and take $\lambda = -\langle\beta|\alpha\rangle \langle\beta|\beta\rangle$.

[2] Blank ket $|\rangle$ emphasizes the fact that our consideration may be applied to any ket.

[3] We use ξ to represent a continuous variable.

8.2 Translation

Consider an operator $\mathscr{J}(\mathrm{d}\mathbf{x})$ satisfying

$$\mathscr{J}(\mathrm{d}\mathbf{x})\,|\mathbf{x}\rangle = |\mathbf{x}\rangle\,. \tag{8.7}$$

Then

$$\mathscr{J}^{\dagger}(\mathrm{d}\mathbf{x})\mathscr{J}(\mathrm{d}\mathbf{x}) = 1 \tag{8.8}$$

needs to be guaranteed. We expect the translation does not depend on the trace, which means

$$\mathscr{J}(\mathrm{d}\mathbf{x}_1 + \mathrm{d}\mathbf{x}_2) = \mathscr{J}(\mathrm{d}\mathbf{x}_1) + \mathscr{J}(\mathrm{d}\mathbf{x}_2). \tag{8.9}$$

For the third property,suppose we consider a translation in the opposite direction; we expect the opposite-direction translation to be the same as the inverse of the original translation:

$$\mathscr{J}(-\,\mathrm{d}\mathbf{x}) = \mathscr{J}^{-1}(\mathrm{d}\mathbf{x}). \tag{8.10}$$

For the fourth property,

$$\lim_{\mathrm{d}\mathbf{x}\to 0}\mathscr{J}(\mathrm{d}\mathbf{x}) = 1. \tag{8.11}$$

Consider

$$\mathscr{J}(\mathrm{d}\mathbf{x}) = 1 - i\hat{\mathbf{K}}\cdot\mathrm{d}\mathbf{x}\,, \tag{8.12}$$

it let all the properties we want hold, if $\hat{\mathbf{K}}$ is Hermitian. We can check that

$$[\hat{\mathbf{x}}, \mathscr{J}(\mathrm{d}\mathbf{x})] = \mathrm{d}\mathbf{x}\,. \tag{8.13}$$

Plug in (8.12), we obtain

$$[\hat{x}_i, \hat{K}_j] = i\delta_{ij}\,. \tag{8.14}$$

8.3 Momentum

An infinitesimal translation in classical mechanics can be regarded as a canonical transformation,

$$\mathbf{x}_{\mathrm{new}} = \mathbf{x} + \mathrm{d}\mathbf{x}\,,\ \mathbf{p}_{\mathrm{new}} = \mathbf{p}. \tag{8.15}$$

obtainable from the generating function

$$F(\mathbf{x},\mathbf{p}) = \mathbf{x}\cdot\mathbf{p}_{\mathrm{new}} + \mathbf{p}\cdot\mathrm{d}\mathbf{x}\,. \tag{8.16}$$

This equation has a striking similarity to the infinitesimal translation operator. Let

$$\mathscr{J}(\mathrm{d}\mathbf{x}) = 1 - i\hat{\mathbf{p}}\cdot\mathrm{d}\mathbf{x}\,/\hbar. \tag{8.17}$$

The commutation relation (8.14) now becomes

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{i,j}\,. \tag{8.18}$$

By (7.3), we obtain the **position-momentum uncertainty relation**:

$$\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p)^2 \right\rangle \geq \frac{\hbar^2}{4} \tag{8.19}$$

As for finite position,

$$\mathscr{J}(\Delta \mathbf{x}) = \lim_{N\to\infty}\left(1 - \frac{i\hat{\mathbf{p}}\cdot\Delta\mathbf{x}}{N\hbar}\right)^N = \exp\left(-\frac{i\hat{\mathbf{p}}\cdot\Delta\mathbf{x}}{\hbar}\right). \tag{8.20}$$

By computation,

$$[\hat{p}_i, \hat{p}_j] = 0. \tag{8.21}$$

8.4 Momentum Operator in the Position Basis

By computing $\mathscr{J}(\mathrm{d}\mathbf{x})\,|\alpha\rangle$ in the representation where the position eigenkets are used as base kets, and plug in (8.17), we obtain

$$\hat{\mathbf{p}}\,|\alpha\rangle = \int \mathrm{d}\mathbf{x}\,|\mathbf{x}\rangle\left(-i\hbar\nabla\langle\mathbf{x}|\alpha\rangle\right). \tag{8.22}$$

That is also

$$\langle\mathbf{x}|\,\hat{\mathbf{p}}\,|\alpha\rangle = -i\hbar\nabla\langle\mathbf{x}|\alpha\rangle. \tag{8.23}$$

8.5 Momentum-Space Wave Function

The notation $\phi_\alpha(p)$ is often used

$$\langle p|\alpha\rangle = \phi_\alpha(p). \tag{8.24}$$

To transform the bases from position to momentum, we need to calculate $\langle x|p\rangle$. One have

$$\langle x|\,\hat{p}\,|p\rangle = p\,\langle x|p\rangle. \tag{8.25}$$

By (8.23),

$$\langle x|\,\hat{p}\,|p\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle. \tag{8.26}$$

Thus,

$$\langle x|p\rangle = A\exp\left(\frac{ipx}{\hbar}\right). \tag{8.27}$$

Do the derivation will lose a constant, but integration won't.[1]

$$\delta(x-x') = \int \mathrm{d}p\,\langle x|p\rangle\langle p|x'\rangle = 2\pi\hbar AA^*\delta(x-x'). \tag{8.28}$$

For convenience, we take A as a real number, then

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipx}{\hbar}\right). \tag{8.29}$$

[1]Is it an easy integration?

For three dimension case,

$$\langle \mathbf{x} | \mathbf{p} \rangle = \left[\frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \right] \exp\left(\frac{i\mathbf{p}\cdot\mathbf{x}}{\hbar} \right). \tag{8.30}$$

8.6 Dirac's Corresponding Rule

Replace classical Poission bracket by commutator as follows[1]

$$[\quad,\quad]_{\text{classical}} \longrightarrow \frac{[\quad,\quad]}{i\hbar}. \tag{8.31}$$

9 Exponential Operators and Baker-Hausdorff Lemma

If it exists, then we denote[2]

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{9.1}$$

Easy to check that if $[A, B] = 0$, then

$$\exp(A)\exp(B) = \exp(A + B). \tag{9.2}$$

Define[3]

$$O(\lambda) = e^{i\lambda G} A e^{-i\lambda G}, \tag{9.3}$$

where A and G are operators, and λ is a complex parameter. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} O(\lambda) = i e^{i\lambda G}[G, A]e^{-i\lambda G} = i[G, O(\lambda)]. \tag{9.4}$$

Hence,

$$O(\lambda) = O(0) + i \int_0^\lambda \mathrm{d}\lambda_1\, [G, O(\lambda_1)]. \tag{9.5}$$

We have $O(0) = A$. And we replace $O(\lambda_1)$ similarly, also for $\lambda_2, \lambda_3, \ldots$, then we obtain,

$$O(\lambda) = A + i \int_0^\lambda \mathrm{d}\lambda_1\, [G, A] + i^2 \int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2\, [G, [G, A]] + \ldots \tag{9.6}$$

Easy to check

$$\int_0^\lambda \mathrm{d}\lambda_1 \int_0^{\lambda_1} \mathrm{d}\lambda_2 \cdots \int_0^{\lambda_{n-1}} \mathrm{d}\lambda_n = \frac{\lambda^n}{n!}. \tag{9.7}$$

[1] I think the $i\hbar$ comes from the translation where setting a constant as $i\hbar$.
[2] Sometimes we denote as $\exp(A)$.
[3] We write extra i because we often use the Hermitian conjugate, and i will give a negative sign to fit the form if G is Hermitian.

Therefore[1],

$$\boxed{e^{i\lambda G}Ae^{-i\lambda G} = \sum_{n=0}^{\infty}\frac{(i\lambda)^n}{n!}C_G^{(n)}(A)}, \tag{9.8}$$

where[2],

$$C_G(A) = [G, A],\ C_G^{(n)}(A) = [G, C_G^{(n-1)}(A)]. \tag{9.9}$$

Or we can write in the form

$$e^{\Omega}Ae^{-\Omega} = e^{\mathrm{Ad}_\Omega}(A). \tag{9.10}$$

Let us consider a special case: $[A, B]$ is commutable with A and B. By Baker-Hausdorff lemma, we have

$$e^A e^B e^{-A} = e^B + [A, e^B] + \dots. \tag{9.11}$$

By (5.4), we can induct on n that

$$[A, [A, B]^{n-1}e^B] = [A, B]^n e^B. \tag{9.12}$$

So,

$$e^A e^B = e^B e^A e^{[A,B]}. \tag{9.13}$$

We also have another more symmetrical form[3]:

Theorem 9.1

$$[A, [A, B]] = [B, [A, B]] = 0 \Rightarrow e^A e^B = e^{A+B}e^{\frac{1}{2}[A,B]}. \tag{9.14}$$

Proof. Let

$$O(\lambda) = e^{\lambda A}e^{\lambda B}, \tag{9.15}$$

then,

$$\frac{\mathrm{d}O}{\mathrm{d}\lambda} = e^{\lambda A}\left(A + B\right)e^{\lambda B}. \tag{9.16}$$

We want to get a differential equation of $o(\lambda)$, so we have to take $(A + B)$ to the left of $\exp(\lambda A)$[4].

$$e^{\lambda A}Be^{\lambda B} = e^{\lambda A}Be^{-\lambda A}e^{\lambda A}e^{\lambda B}. \tag{9.17}$$

By Baker-Hausdorff lemma, we obtain

$$\frac{\mathrm{d}O}{\mathrm{d}\lambda} = \left[(A + B) + \lambda[A, B]\right]O. \tag{9.18}$$

[1] It is easy to take the wrong signature, remember it is something like derivative and note the first term, like $[G, A]$, then it won't be wrong.

[2] This notation is not formal, but the following notation "Ad" (get from Deepseek) may be better to use in articles.

[3] In fact we do not need any condition, then we can derive a general form, but that is a bit complex and unnecessary.

[4] Trick: If there is not we want, then create one, like $+1\ -1$, here we times 1.

The solution is

$$O = \exp\left[\lambda(A+B) + \frac{1}{2}\lambda^2[A,B]\right]. \tag{9.19}$$

□

10 Harmonic Oscillator

We discuss the 1D case first. We have the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{10.1}$$

10.1 Method of Analysis (Hermite polynomials)

We solve the wave function in position representation, then we obtain many information. Let $z = \frac{x}{\sqrt{\frac{\hbar}{m\omega}}}$, then

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + \frac{1}{2}z^2\right]\psi(z) = \frac{E}{\hbar\omega}\psi(z). \tag{10.2}$$

A must to satisfy is

$$\lim_{z\to\infty}\psi(z) = 0. \tag{10.3}$$

So we first analyze the behavior of $\psi(z)$ at $z \to \pm\infty$. For $\psi'' = z^2\psi$, we try the solution[1] $\psi(z) = e^S(z)$

$$\psi' = S'\psi,\ \psi'' = \left(S'' + S'^2\right)\psi. \tag{10.4}$$

Comparison of two expression yields

$$S'' + S'^2 = z^2. \tag{10.5}$$

The leading order leads to $s' = \pm z$, $s = \pm\frac{1}{2}z^2$. So we can try the solution

$$\psi(z) = e^{-\frac{z^2}{2}}u(z). \tag{10.6}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) - 2z\frac{\mathrm{d}}{\mathrm{d}z}u(z) - (\lambda - 1)\,u(z) = 0. \tag{10.7}$$

We consider the series solution, let

$$u(z) = \sum_{k=0}^{+\infty} a_k z^k. \tag{10.8}$$

The we can deduce that

$$\frac{a_{k+2}}{a_k} = \frac{2k+1-\lambda}{(k+1)(k+2)}. \tag{10.9}$$

[1] This may keep the form.

We have $u(\pm\infty) = 1$, so there must exists an $N \in \mathbb{N}$, for any $k > N$,

$$a_k = 0. \tag{10.10}$$

Therefore,

$$\lambda_n = 2n + 1,\ \ n = 0, 1, 2, \dots \tag{10.11}$$

which means

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{10.12}$$

The solution of (10.7) is Hermite polynomials[1] $H_n(z)$.

$$\psi(x) = e^{-\frac{1}{2}\frac{m\omega x^2}{\hbar}} \sum_{n=0}^{+\infty} \frac{c_n}{n!} H_n\left(x/\sqrt{\frac{\hbar}{m\omega}}\right). \tag{10.13}$$

And for a fixed n, the normalized wavefunction is

$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{x^2}{2} H_n(x)}. \tag{10.14}$$

10.2 Method of Algebra

In usual computing law,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2\left(x - \frac{ip}{m\omega}\right)\left(x + \frac{ip}{m\omega}\right). \tag{10.15}$$

Let [2]

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right),\ \ \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right). \tag{10.16}$$

Then,

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{10.17}$$

Let

$$\hat{N} = \hat{a}^\dagger \hat{a}, \tag{10.18}$$

then,

$$\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right). \tag{10.19}$$

So $\hat{N}$ is compatible with $\hat{H}$. We denote an energy eigenket of $\hat{N}$ by its eigenvalue[3] n, so

$$\hat{N}\left|n\right\rangle = n\left|n\right\rangle, \tag{10.20}$$

$$\hat{H}\left|n\right\rangle = \left(n + \frac{1}{2}\right)\hbar\omega\left|n\right\rangle. \tag{10.21}$$

[1] There's some properties about Hermite polynomial.

[2] If we denote $l = \frac{\hbar}{m\omega}$, then it will be $a = \frac{1}{\sqrt{2}}\left(\frac{x}{l} + i\frac{pl}{\hbar}\right)$. $\frac{1}{\sqrt{2}}$ seems like to normalize.

[3] We will later show that n must be a nonnegative integer.

To appreciate the physical significance of $\hat{a}, \hat{a}^\dagger$, and $\hat{N}$, let us first note that

$$[\hat{N}, \hat{a}] = -\hat{a}, \; [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{10.22}$$

As a result, we have

$$\hat{N}\hat{a}^\dagger |n\rangle = [\hat{N}, \hat{a}^\dagger] |n\rangle + \hat{a}^\dagger \hat{N} |n\rangle = (n+1)\, \hat{a}^\dagger |n\rangle , \tag{10.23}$$

$$\hat{N}\hat{a} |n\rangle = [\hat{N}, \hat{a}] |n\rangle + \hat{a}\hat{N} |n\rangle = (n-1)\, \hat{a} |n\rangle . \tag{10.24}$$

We write

$$\hat{a} |n\rangle = c |n-1\rangle , \tag{10.25}$$

then,

$$\langle n| \hat{N} |n\rangle = \langle n| \hat{a}^\dagger \hat{a} |n\rangle = |c|^2 . \tag{10.26}$$

Thus,

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle . \tag{10.27}$$

Similarly, it is easy to show that[1]

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle . \tag{10.28}$$

From (10.26) we know that $n \geq 0$. If we act $\hat{a}$ on a state for many times, we can conclude that the sequence must terminate with $n = 0$, or it won't stop and n will become not real. Use

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{a} + \hat{a}^\dagger\right) , \; \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}} \left(-\hat{a} + \hat{a}^\dagger\right) , \tag{10.29}$$

we can get more information. To solve the wave function, we can deduce $\langle x| \hat{a} |0\rangle = 0$, which leads to a differential equation for the wave function.

10.3 Further Discussion of 1D Case: Coherent State

> We have seen that an energy eigenstate does not behave like the classical oscillator—in the sense of oscillating expectation values for x and p—no matter how large n may be. We may logically ask, How can we construct a superposition of energy eigenstates that most closely imitates the classical oscillator? In wave-function language, we want a wave packet that bounces back and forth without spreading in shape. It turns out that a coherent state defined by the eigenvalue equation for the non-Hermitian annihilation operator a,
>
> $$a |\lambda\rangle = \lambda |\lambda\rangle ,$$
>
> with, in general, a complex eigenvalue λ does the desired job.
>
> —*Modern Quantum Mechanics*

[1] Memory trick: the square root of a lager one.

We want it to be related to the states we have known. And the simplist one is $|0\rangle$. Let $|\lambda\rangle = A|0\rangle$. Then what we expect is $aA|0\rangle = \lambda A|0\rangle$. Or in another form:

$$(A^{-1}aA - \lambda)|0\rangle = f(a)|0\rangle ,$$

where f is a arbitrary function. We want to make it easy, let us set $f(a) = a$. Then by the Baker-Hausdorff lemma, the following let what we want holds

$$A = e^{\lambda a^\dagger}. \tag{10.30}$$

use (9.13) to do the normalization[1],

$$\langle 0|e^{\lambda^* a}e^{\lambda a^\dagger}|0\rangle = e^{|\lambda|^2}. \tag{10.31}$$

Therefore[2],

$$|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle = e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}|n\rangle . \tag{10.32}$$

We have discussed in 8.3 that the operator $e^{i\alpha p}$ is a dispalcement operator.

[1] Expand the series we get $\exp(\lambda a)|0\rangle = |0\rangle$.

[2] A Poission distribution.