

Nabla Operator

This article aim to give a proof¹ of the expression² of nabla operator in different coordinate system and explain its “meaning”. And this will be a cheat sheet.

1 The Nabla Operator ∇

Nabla operator has three action modules: “association”, “inner product”, “cross product”³. Or we can say we denote three different actions as a same notation . They have different meanings, and so have the different expressions.

2 Physical Component and Orthogonal Basis

$\mathbf{g}^i, \mathbf{g}_i$ as the natural basis have different units. That is inconvenient for physics calculation. So we construct another covariant basis:

$$\mathbf{g}_{(i)} = \frac{\mathbf{g}_i}{\sqrt{g_{ii}}} = \beta_{(i)}^j \mathbf{g}_j, \quad (1)$$

where, underline means do not take summation, and

$$\beta_{(i)}^j = \frac{\delta_i^j}{\sqrt{\mathbf{g}_i \cdot \mathbf{g}_i}}. \quad (2)$$

Then the physical component can be written as

$$v^{(i)} = \sqrt{g_{ii}} v^i, \quad v_{(i)} = \frac{1}{\sqrt{g_{ii}}} v_i. \quad (3)$$

If \mathbf{g}_i is orthogonal, let

$$|\mathbf{g}_i| = A_i, \quad (4)$$

which are called the **Lamé coefficient**, then

$$g_{ij} = \begin{cases} 0, & i \neq j \\ A_i^2, & i = j. \end{cases} \quad (5)$$

Let

$$\mathbf{e}_i = \frac{\mathbf{g}_i}{A_i}, \quad \mathbf{e}^i = A_i \mathbf{g}^i, \quad (6)$$

then,

$$\mathbf{e}_i = \mathbf{e}^i = \mathbf{e}(i). \quad (7)$$

$$\Gamma_{ij}^k = 0, (i \neq j \neq k), \quad \Gamma_{ij}^i = \frac{1}{A_i} \frac{\partial A_i}{\partial x^j}, \quad \Gamma_{ii}^j = -\frac{A_i}{A_j^2} \frac{\partial A_i}{\partial x^j}, (i \neq j). \quad (8)$$

¹I will use tensor notation.

²Mainly for 3D vectors.

³In my words.

3 Gradient

Let \mathbf{T} be a tensor, then we define the gradient of the tensor as

$$\nabla \mathbf{T} = \mathbf{g}^i \frac{\partial \mathbf{T}}{\partial x^i} = \frac{\mathbf{e}^i}{A_i} \frac{\partial \mathbf{T}}{\partial x^i}. \quad (9)$$

4 Divergence

Note that

$$\frac{\partial \sqrt{g}}{\partial x^i} = \Gamma_{ji}^j \sqrt{g}, \quad (10)$$

we have

$$\nabla \cdot \mathbf{F} = \partial_i F^i + F^m \Gamma_{im}^i = \partial_i F^i + F^m \frac{1}{\sqrt{g}} \partial_m \sqrt{g} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} F^i). \quad (11)$$

In orthogonal coordinate system, we have

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^3 \frac{1}{A_1 A_2 A_3} \partial_i \left(\frac{A_1 A_2 A_3}{A_i} F(i) \right). \quad (12)$$

5 Curl

$$\nabla \times \mathbf{F} = \epsilon^{ijk} \nabla_i F_j \mathbf{g}_k = \epsilon^{ijk} (\partial_i F_j - F_m \Gamma_{ij}^m) \mathbf{g}^k = \epsilon^{ijk} \partial_i F_j \mathbf{g}^k \quad (13)$$

$$= \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{g}_1 & \mathbf{g}_1 & \mathbf{g}_1 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}. \quad (14)$$

In orthogonal coordinate system, we have

$$\nabla \times \mathbf{F} = \frac{1}{A_1 A_2 A_3} \begin{vmatrix} A_1 \mathbf{e}_1 & A_2 \mathbf{e}_2 & A_3 \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 F(1) & A_2 F(2) & A_3 F(3) \end{vmatrix}. \quad (15)$$

6 Laplacian

For a scalar function f , we have

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{A_1 A_2 A_3} \sum_{i=1}^3 \partial_i \left(\frac{A_1 A_2 A_3}{A_i^2} \partial_i f \right). \quad (16)$$

7 Special 3D Coordinates

7.1 Rectangular coordinates

$$A_1 = A_2 = A_3 = 1. \quad (17)$$

7.2 Cylindrical Coordinates

$$A_1 = A_3 = 1, \quad A_2 = \rho. \quad (18)$$

7.3 Spherical Coordinates

$$A_1 = 1, \quad A_2 = r, \quad A_3 = r \sin \theta. \quad (19)$$