Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 7 - 2 : Limits – series (II)

1. The aim of this exercise is to study the so-called **Stolz–Cesàro theorem**. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two series, which satisfy $\lim\limits_{n\to\infty} a_n = +\infty$ or 0 and $\lim\limits_{n\to\infty} b_n = +\infty$ or 0 at the same time. In this exercise, we will propose a method to compute $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ for some particular cases.

(1) Suppose $(b_n)_{n\in\mathbb{N}}$ is strictly increasing, $\lim\limits_{n\to\infty} b_n = +\infty$ and $-\infty \leqslant l < +\infty$. If

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\},$$

prove that there exists a $p \in]l, +\infty[$ and an $n_0 \in \mathbb{N}$, such that for all $n > n_0$, we have

$$\frac{a_{n+1}}{b_{n+1}} < p\left(1 - \frac{b_{n_0}}{b_{n+1}}\right) + \frac{a_{n_0}}{b_{n+1}}.$$

Hint : We can prove $a_{n+1} = (a_{n+1}-a_n)+(a_n-a_{n-1})+\cdots+(a_{n_0+1}-a_{n_0}) + a_{n_0} < p(b_{n+1}-b_{n_0}) + a_{n_0}$ for some particular $p, n_0 \in \mathbb{R}$.

(2) We keep all the notations and conditions in (1). Let

$$c_n = p\left(1 - \frac{b_{n_0}}{b_n}\right) + \frac{a_{n_0}}{b_n}.$$

Prove that for all $q > p$, there exist $n_1 \in \mathbb{N}$, such that for all $n > n_1$, we have $c_n < q$. Then deduce that there exists an $n_2 \in \mathbb{N}$, such that for all $n > n_2$, we have $\frac{a_{n+1}}{b_{n+1}} < c_{n+1} < q$.

(3) We keep all the notations and conditions in (1). For all real number $q' < l$, there exists an $n_3 \in \mathbb{N}$, such that for all $n > n_3$, we have $q' < \frac{a_{n+1}}{b_{n+1}}$.

Hint : By the same strategy as that in (1) and (2).

(4) Prove that $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$ under the assumption in (1).

(5) Suppose $(b_n)_{n\in\mathbb{N}}$ is strictly decreasing, $\lim\limits_{n\to\infty} a_n = 0$, $\lim\limits_{n\to\infty} b_n = 0$, and $-\infty \leqslant l < +\infty$. Prove $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$ by a similar strategy as above.

(6) Use Stolz–Casàro theorem to study the following limits.

i. Let $(a_n)_{n\in\mathbb{N}}$ be a series satisfying $\lim\limits_{n\to\infty} a_n = b \in \mathbb{R}$. Prove

$$\lim_{n\to\infty} \frac{a_1 + \cdots + a_n}{n} = b.$$

ii. Find $\lim\limits_{n\to\infty} \frac{1!+2!+\cdots+n!}{n!}$.

iii. $\lim\limits_{n\to+\infty} \dfrac{n^2}{a^{2n}} = 0 \ (|a| > 1)\,;$

iv. $\lim\limits_{n\to+\infty} \dfrac{1^k + 2^k + \cdots + n^k}{n^{k+1}} = \dfrac{1}{k+1} \ (\forall k \in \mathbb{N})\,;$

v. $\lim\limits_{n\to+\infty} \dfrac{1^k + 2^k + \cdots + n^k}{n^k} - \dfrac{n}{k+1} = \dfrac{1}{2} \ (\forall k \in \mathbb{N});$

vi. $\lim\limits_{n\to+\infty} \dfrac{a_1 + 2a_2 \cdots + na_n}{n(n+1)} = \pm\infty$ if $\lim\limits_{n\to+\infty} a_n = \pm\infty$.

2. In this exercise, we will study the limit $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

(1) Prove that the series $\left(\left(1 + \frac{1}{n}\right)^n\right)_{n\in\mathbb{N}}$ is increasing.

(2) Prove that the series $\left(\left(1 + \frac{1}{n}\right)^{n+1}\right)_{n\in\mathbb{N}}$ is decreasing.

(3) Deduce that the existences of $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ and $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1}$, and prove

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n+1}.$$

Note : We denote $e = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, which is an irrational number. We call e the base (or bottom number) of the natural logarithm, and usually we denote $\ln(\cdot) = \log_e(\cdot)$. In fact, we have $e = 2.718281828459\cdots$.

3. For any $a \in \mathbb{R}$, we denote by $|a|$ the element $\max\{a, -a\}$ in $\mathbb{R}$.

(1) Show that, for any $(a, b) \in \mathbb{R} \times \mathbb{R}$, one has

$$|a + b| \leqslant |a| + |b|, \quad |a - b| \leqslant |a| + |b|.$$

(2) Show that, for any $(a, b) \in \mathbb{R} \times \mathbb{R}$,

$$\big||a| - |b|\big| \leq |a - b|$$

(3) Let I be an infinite subset of $\mathbb{N}$ and $(x_n)_{n\in I}$ be a sequence of real numbers and $\ell \in \mathbb{R}$.

(a) Show that $(x_n)_{n\in I}$ converges to ℓ if and only if

$$\limsup_{n\in I,\, n\to+\infty} |x_n - \ell| = 0.$$

(b) Show that $(x_n)_{n\in I}$ has a subsequence that converges to ℓ if and only if

$$\liminf_{n\in I,\, n\to+\infty} |x_n - \ell| = 0.$$

(4) Let a and b be two real numbers. Show that

$$\max\{|a|,|b|\} \leqslant \max\{|a+b|,|a-b|\} \leqslant 2\max\{|a|,|b|\}.$$

(5) Let a and b be real numbers. Show that

$$\frac{1}{2} \leqslant \max\{|a+b|,|a-b|,|1-a|\}.$$

(6) Let a and b be real numbers. Show that

$$\max\{a,b\} = \frac{a+b+|a-b|}{2}, \quad \min\{a,b\} = \frac{a+b-|a-b|}{2}$$

4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{\mathbb{R}}\setminus\{0\}$. Assume that $(|x_n|)_{n\in\mathbb{N}}$ tends to $+\infty$.

(1) Let ε be a positive real number. Show that there exists $N\in\mathbb{N}$ such that

$$\forall\, n\in\mathbb{N},\ n\geqslant N,\ \text{ one has } |x_n| > \varepsilon^{-1}.$$

(2) Show that the sequence $(|x_n^{-1}|)_{n\in\mathbb{N}}$ tends to 0.

5. Let ε be a positive real number.

(1) Show that, for any $n\in\mathbb{N}$, $(1+\varepsilon)^n \geqslant 1+n\varepsilon$.
(2) Prove that the sequence $\big((1+\varepsilon)^n\big)_{n\in\mathbb{N}}$ tends to $+\infty$.
(3) Let a be a non-zero real number such that $|a|<1$. Show that

$$\lim_{n\to+\infty} a^n = 0.$$

(4) Let a be a non-zero real number such that $|a|<1$. Determine the limit of the series

$$\sum_{n\in\mathbb{N}} a^n.$$

6. In this exercice, we consider the following two sequences $\alpha = (a_n)_{n\in\mathbb{N},\, n\geqslant 1}$ and $\beta = (b_n)_{n\in\mathbb{N},\, n\geqslant 1}$ defined as

$$a_n = \Big(1+\frac{1}{n}\Big)^n, \quad b_n = \Big(1+\frac{1}{n}\Big)^{n+1}.$$

(1) Prove the inequality $(1+t)^n \geqslant 1+nt$ for $t \geqslant -1$ and $n \in \mathbb{N}$. Show that the sequence α is increasing and the sequence β is decreasing.

(2) Show that the series

$$\sum_{n\in\mathbb{N}} \frac{1}{(n+1)(n+2)}$$

is convergente and its limit is 1.

(3) Show that

$$a_n \leqslant \sum_{k=0}^{n} \frac{1}{k!} \leqslant 3.$$

(4) Prove that the sequences α and β converge to the same limit, which belongs to $[2,3]$. We denote by e this limit.

(5) Show that, for any $(n,m) \in \mathbb{N}\times\mathbb{N}$ such that $\min\{n,m\} \geqslant 1$, one has

$$a_n \leqslant b_m.$$

7. Consider the sequence $(a_n)_{n\in\mathbb{N},\, n\geqslant 1}$ defined as

$$a_n = \frac{\sin(n)}{n}.$$

(1) Show that

$$|a_n| \leqslant \frac{1}{n}.$$

(2) Show that

$$\limsup_{n\to+\infty} |a_n| \leq 0.$$

(3) Does the sequence $(a_n)_{n\in\mathbb{N},\, n\geqslant 1}$ converge ?

8. In this exercice, we study the convergence of the series

$$H(\alpha) = \sum_{n\in\mathbb{N},\, n\geqslant 1} \frac{1}{n^\alpha},$$

where α is a real number.

(1) Show that, if $\alpha \leqslant 0$, then the series $H(\alpha)$ diverges.

(2) Let N be a positive integer, show that

$$\sum_{k=N+1}^{2N} \frac{1}{k} \geqslant \frac{1}{2}.$$

(3) Show that the series $H(1)$ diverges.

(4) Let n and ℓ be positive integers. Show that

$$\frac{1}{(n+1)^{1+\frac{1}{\ell}}} \leqslant \ell\left(\frac{1}{n^{\frac{1}{\ell}}} - \frac{1}{(n+1)^{\frac{1}{\ell}}}\right).$$

One can use a result of Exercice 4.

(5) Show that, for any $\alpha > 1$, the series $H(\alpha)$ converges.

(6) Show that, for any positive integer n, one has

$$\frac{1}{n+1} \leqslant \ln(n+1) - \ln(n) \leqslant \frac{1}{n},$$

where for any $t > 0$, $\ln(t)$ is the unique real number such that

$$e^{\ln(t)} = t.$$

(7) Show that, for any positive integer n, one has

$$\ln(n+1) \leqslant \sum_{k=1}^{n} \frac{1}{k} \leqslant \ln(n) + 1.$$

(8) Show that the sequence

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n), \quad n \in \mathbb{N},\, n \geqslant 1$$

is decreasing and is bounded from below by 0.

(9) Show that the sequence in the previous question converges in $\mathbb{R}$.
Note : Usually we denote

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n)\right),$$

which is an irrational number. We call it Euler constant. In fact, we have $\gamma = 0.577216\cdots$.

9. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers defined in a recursive way as

$$u_0 = 1, \quad u_{n+1} = \frac{u_n}{\sqrt{u_n^2+1}}.$$

(1) Show that the mapping $(n \in \mathbb{N}) \mapsto u_n$ is decreasing.
(2) Show that $0 \leqslant u_n \leqslant 1$ for any $n \in \mathbb{N}$.
(3) Compute u_1 and u_2.
(4) Conjecture an expression of u_n. Prove your conjecture by induction on n.
(5) Show that the sequence $(u_n)_{n\in\mathbb{N}}$ has a limite and determine its limit.

10. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers parametrized by $\mathbb{N}$ defined in a recursive way as

$$u_0 = 3, \quad u_{n+1} = \frac{u_n - 2}{2u_n + 5}.$$

Show that

$$\forall\, n \in \mathbb{N} \quad u_n = \frac{9-8n}{3+8n}.$$

Show that the sequence $(u_n)_{n\in\mathbb{N}}$ converges and determine its limit.

11. Consider Fibonacci's sequence $(F_n)_{n\in\mathbb{N}}$ defined as

$$F_0 = 1, \quad F_1 = 1, \quad F_{n+2} = F_{n-1} + F_n.$$

(1) Show that, for any $n \in \mathbb{N}$,

$$F_nF_{n+2} - F_{n+1}^2 = (-1)^n.$$

(2) Show that the function

$$(n \in \mathbb{N}) \longmapsto \frac{F_{2n+1}}{F_{2n}}$$

is increasing.

(3) Show that the function

$$(n \in \mathbb{N}) \longmapsto \frac{F_{2n+2}}{F_{2n+1}}$$

is decreasing.

(4) Let

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

Show that, for any $n \in \mathbb{N}$,

$$\alpha^{n+2} = \alpha^{n+1} + \alpha^n, \quad \beta^{n+2} = \beta^{n+1} + \beta^n.$$

(5) Find real numbers λ and μ such that

$$\lambda + \mu = 1, \quad \lambda\alpha + \mu\beta = 1.$$

(6) Prove that

$$\forall\, n \in \mathbb{N}, \quad F_n = \lambda\alpha^n + \mu\beta^n.$$

(7) Show that the sequence $(F_n/\alpha^n)_{n\in\mathbb{N}}$ converges. Determine its limit.

12. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of real numbers defined as

$$u_n = \sqrt{n+1} - \sqrt{n}.$$

(1) Show that, for any $n \in \mathbb{N}$, one has

$$u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

(2) Prove that the sequence $(u_n)_{n\in\mathbb{N}}$ is convergente. Determine its limit.

13. Let I be an infinte subset of $\mathbb{N}$, $(a_n)_{n\in I}$ and $(b_n)_{n\in I}$ be bounded sequences in $\mathbb{R}$.

(1) Show that

$$\limsup_{n\in I,\, n\to+\infty} (a_n + b_n) \leqslant \Big(\limsup_{n\in I,\, n\to+\infty} a_n\Big) + \Big(\limsup_{n\in I,\, n\to+\infty} b_n\Big).$$

(2) Show that

$$\liminf_{n\in I,\, n\to+\infty} (a_n + b_n) \geqslant \Big(\liminf_{n\in I,\, n\to+\infty} a_n\Big) + \Big(\liminf_{n\in I,\, n\to+\infty} b_n\Big).$$

(3) Assume that $(b_n)_{n\in I}$ converges to a real number ℓ. Show that

$$\limsup_{n\in I,\, n\to+\infty} (a_n + b_n) = \ell + \limsup_{n\in I,\, n\to+\infty} a_n,$$
$$\liminf_{n\in I,\, n\to+\infty} (a_n + b_n) = \ell + \liminf_{n\in I,\, n\to+\infty} a_n.$$

(4) Determine the limit superior and the limit inferior of the sequence

$$u_n = (-1)^n + \frac{1}{n}, \quad n \in \mathbb{N},\ n \geqslant 1.$$

Does this sequence converge?

14. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. We assume that

$$\lim_{n\to+\infty} |u_{n+1} - u_n| = 0.$$

Let λ be an element of $\mathbb{R}$ such that

$$\liminf_{n\to+\infty} u_n < \lambda < \limsup_{n\to+\infty} u_n.$$

(1) Let n be a natural number. Show that there exist $p \in \mathbb{N}_{\geqslant n}$ such that $u_p > \lambda$ and $q \in \mathbb{N}_{\geqslant n}$ such that $u_q < \lambda$.

(2) For any n, let $\psi(n)$ be the smallest natural number in $\mathbb{N}_{\geqslant n}$ such that $u_{\psi(n)} > \lambda$ and $\varphi(n)$ be the smallest natural number in $\mathbb{N}_{\geqslant \psi(n)}$ such that $u_{\varphi(n)} < \lambda$. Show that the sequences $(\varphi(n))_{n\in\mathbb{N}}$ and $(\psi(n))_{n\in\mathbb{N}}$ are increasing and tend to $+\infty$.

(3) Show that

$$\forall\, n \in \mathbb{N}, \quad u_{\varphi(n)} < \lambda \leqslant u_{\varphi(n)-1}.$$

(4) Show that the sequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ converges to λ.

(5) Show that there exists a subsequence of $(u_n)_{n\in\mathbb{N}}$ that converges to λ.

(6) Is the conclusion of the previous question necessarily true without the condition

$$\lim_{n\to+\infty} |u_{n+1} - u_n| = 0?$$

Justify your answer.

15. Let a be a positive real number and let N be an integer such that $N \geqslant 2a$.

(1) Show that, for $n \geqslant N$, one has

$$\frac{a^n}{n!} \leqslant \frac{a^N}{N!}\frac{1}{2^{n-N}}.$$

(2) Deduce that $a^n = o(n!)$, $n \to +\infty$.

(3) Show that the series

$$\sum_{n\in\mathbb{N}} \frac{a^n}{n!}$$

is convergent.

16. The purpose of this exercise is to compare the sequences $(n!)_{n\in\mathbb{N}_{\geqslant 1}}$ and $(n^n)_{n\in\mathbb{N}_{\geqslant 1}}$.

(1) Let a be a real number such that $0 < a < 1$. Show that, for any positive integer n, one has

$$\frac{n!}{n^n} \leqslant \frac{a^{an}}{a}.$$

(2) Show that $n! = o(n^n)$, $n \to +\infty$.

(3) Show that the series

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} \frac{n!}{n^n}$$

is convergent.

17. In this exercise, we study the convergence of the sequence

$$u_n = \left(\frac{1}{n!}\right)^{\frac{1}{n}}, \quad n \in \mathbb{N}_{\geqslant 1}.$$

(1) Let a be a real number such $0 < a < 1$. Show that, for any $n \in \mathbb{N}_{\geqslant a^{-1}}$, one has

$$n! \geqslant (an)^{n-an}.$$

(2) Deduce that

$$\lim_{n\to+\infty} \frac{1}{(n!)^{1/n}} = 0.$$

(3) Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is decreasing. Deduce that the series

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} (-1)^n u_n$$

converges.

(4) Does the series

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} u_n$$

converge?

18. Study the convergence and the absolute convergence of the following series.

$$\sum_{n\in\mathbb{N}} \frac{\ln(n)}{2^n}, \quad \sum_{n\in\mathbb{N}_{\geqslant 2}} \frac{(-1)^n}{\ln(n)}, \quad \sum_{n\in\mathbb{N}_{\geqslant 1}} \frac{\sin(n)}{n^2}, \quad \sum_{n\in\mathbb{N}} \frac{n^2+1}{3n^4+2}, \quad \sum_{n\in\mathbb{N}} \binom{2n}{n}^{-1}.$$

19. (1) Let a and b be positive real numbers. Show that there exists a positive integer N such that

$$\forall\, n \in \mathbb{N}_{\geqslant N}, \quad \ln(n) \leqslant an^b.$$

(2) Show that, for any $\alpha \in \mathbb{R}_{>1}$ and $b \in \mathbb{R}_{>0}$, there exists a positive integer N' such that

$$\forall\, n \in \mathbb{N}_{\geqslant N'}, \quad n^2 \leqslant \alpha^{n^b}$$

(3) Let $\alpha \in \mathbb{R}_{>1}$ and $b \in \mathbb{R}_{>0}$. Show that, the series

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} \frac{1}{\alpha^{n^b}}$$

converges.

20. For $n \in \mathbb{N}_{\geqslant 2}$, let

$$u_n = \frac{(-1)^n}{n+(-1)^n}.$$

(1) For any $n \in \mathbb{N}_{\geqslant 1}$, let $v_n = u_{2n} + u_{2n+1}$. Show that $v_n = O(n^{-2})$.

(2) Prove that the series

$$\sum_{n\in\mathbb{N}_{\geqslant 2}} u_n$$

converges and determine its limit. We can use the fact that the sequence

$$\sum_{k=1}^{n} \frac{1}{k} - \ln(n), \quad n \in \mathbb{N}_{\geqslant 1}$$

converges in $\mathbb{R}$.

(3) Does the series

$$\sum_{n\in\mathbb{N}_{\geqslant 2}} u_n$$

converge absolutely ?

21. Let $(u_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathbb{R}_{>0}$ which converges to 0. For any $n \in \mathbb{N}$, let

$$S_n = \sum_{k=0}^{n} (-1)^k u_k.$$

Recall that we have proved in the course that the sequence $(S_n)_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ to a limit, which we denote as ℓ. For any $n \in \mathbb{N}$, let $R_n = \ell - S_n$. We assume in addition that the following conditions are satisfied :

$$\forall\, n \in \mathbb{N}, \quad u_{n+2} + u_n \geqslant 2u_{n+1}, \quad \lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = 1.$$

(1) Show that, for any $n \in \mathbb{N}$, $(-1)^{n+1} R_n > 0$.
(2) Show that, for any $n \in \mathbb{N}$, $|R_n| + |R_{n+1}| = u_{n+1}$.
(3) For any $n \in \mathbb{N}$, let $v_n = u_n - u_{n+1}$. Show that the sequence $(v_n)_{n\in\mathbb{N}}$ is decreasing and converges to 0.
(4) Show that

$$\forall\, n \in \mathbb{N}, \quad |R_n| - |R_{n+1}| = (-1)^{n+1} \sum_{k\in\mathbb{N}_{>n}} (-1)^k v_k.$$

Deduce that $|R_n| - |R_{n+1}| \geqslant 0$ for any $n \in \mathbb{N}$.
(5) Show that, for any $n \in \mathbb{N}$,

$$\frac{u_{n+1}}{2} \leqslant |R_n| \leqslant \frac{u_n}{2}.$$

(6) Show that

$$\lim_{n\to+\infty} (-1)^{n+1} \frac{2R_n}{u_n} = 1.$$

22. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ which converges to some $x \in \mathbb{R}$. Suppose that, for any $n \in \mathbb{N}$, $u_n \neq x$, and that the sequence

$$\frac{u_{n+1} - x}{u_n - x}, \quad n \in \mathbb{N}$$

converges in $\mathbb{R}$ to some $\lambda \in [-1, 1[$.

(1) For any $n \in \mathbb{N}$, let

$$v_n = \frac{u_{n+1} - \lambda u_n}{1 - \lambda}.$$

Show that $v_n - x = o(u_n - x)$, $n \to +\infty$.

(2) Suppose that $u_n \neq u_{n+1}$ for any $n \in \mathbb{N}$. Show that

$$\lim_{n \to +\infty} \frac{u_{n+2} - u_{n+1}}{u_{n+1} - u_n} = \lambda.$$

(3) Suppose that $u_n \neq u_{n+1}$ and $u_n + u_{n+2} \neq 2u_{n+1}$ for any $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let

$$w_n = \frac{u_n u_{n+2} - u_{n+1}^2}{u_n + u_{n+2} - 2u_{n+1}}.$$

Show that $w_n - x = o(u_n - x)$, $n \to +\infty$.

23. For any real number x, we denote by $\lfloor x \rfloor$ the largest integer that is bounded from above by x. Note that

$$\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1.$$

In this exercice, we fix an integer d which is $\geqslant 2$.

(1) Let n be an integer. Show that

$$0 \leqslant n - d \left\lfloor \frac{n}{d} \right\rfloor \leqslant d - 1.$$

(2) For any $n \in \mathbb{N}_{\geqslant 1}$, let

$$u_n = \frac{n - d\lfloor \frac{n}{d} \rfloor}{n(n+1)}.$$

Show that the series

$$\sum_{n \in \mathbb{N}_{\geqslant 1}} u_n$$

converges.

(3) For any $n \in \mathbb{N} \geqslant 1$, let

$$S_n = \sum_{k=1}^{n} u_k, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Show that

$$S_{nd} = H_{nd} - H_n.$$

(4) Show that

$$\lim_{n \to +\infty} S_{nd} = \ln(d).$$

(5) Prove that

$$\lim_{n\to+\infty} S_n = \ln(d).$$

(6) Deduce that $\ln(d) \leqslant d-1$.

24. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_{>0}$. We assume that the series

$$\sum_{n\in\mathbb{N}} u_n$$

converges and we denote by S its sum.

(1) Show that the series

$$\sum_{n\in\mathbb{N}} u_n^2$$

converges to a limit T.

(2) Show that T belongs to $]0, S^2[$.

(3) Suppose that the series $(u_n)_{n\in\mathbb{N}}$ is of the form $u_n = q^n$, where $0 < q < 1$. Express S and T in terms of q. Show that, for any $\delta \in\,]0,1[$ there exists $q \in\,]0,1[$ such that $T/S^2 = \delta$.

25. Let $(u_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence of real numbers. We suppose that $(u_n)_{n\in\mathbb{N}_{\geqslant 1}}$ converges to a limit $\ell \in \mathbb{R}$. For any $n \in \mathbb{N}_{\geqslant 1}$, let $S_n = u_1 + \cdots + u_n$.

(1) For any $N \in \mathbb{N}_{\geqslant 1}$, let

$$\varepsilon_N = \sup_{n\in\mathbb{N}_{\geqslant N}} |u_n - \ell|.$$

Show that, for any natural number n such that $n \geqslant N$, one has

$$|S_n - n\ell| \leqslant \sum_{k=1}^{N} |u_k - \ell| + (n-N)\varepsilon_N.$$

(2) Show that, for any $N \in \mathbb{N}_{\geqslant 1}$, one has

$$\limsup_{n\to+\infty} \left|\frac{S_n}{n} - \ell\right| \leqslant \varepsilon_N.$$

(3) Show that the sequence $(S_n/n)_{n\in\mathbb{N}_{\geqslant 1}}$ converges to ℓ.

26. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_{>0}$. For any $n \in \mathbb{N}$, let

$$S_n = \sum_{k=0}^{n} u_k.$$

(1) Show that the series

$$\sum_{n\in\mathbb{N}} \frac{u_n}{S_n^2}$$

converges. We can use the inequality $S_n \geqslant S_{n-1}$.

(2) Suppose that the sequence $(S_n)_{n\in\mathbb{N}}$ converges to a limit S. Show that the series

$$\sum_{n\in\mathbb{N}} \frac{u_n}{S_n}$$

converges.

27. Let $(u_n)_{n\in\mathbb{N}_{\geqslant 1}}$ be a sequence in $\mathbb{R}_{>0}$. For any $n \in \mathbb{N}_{\geqslant 1}$, let

$$P_n = \prod_{k=1}^{n}(1+u_k), \quad v_n = \frac{u_n}{P_n}.$$

(1) Show that the series

$$\sum_{n\geqslant 1} v_n$$

converges.

(2) Suppose that the series

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} u_n$$

diverges.

(a) Show that $(P_n)_{n\in\mathbb{N}_{\geqslant 1}}$ tends to $+\infty$.

(b) Show that

$$\sum_{n=1}^{N} v_n = 1 - \frac{1}{P_N}.$$

(c) Deduce that

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} v_n = 1.$$