QM HW1

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Problem 1 (Canonical Transformation)
(1)

$$\mathrm{d}\Phi(q,P,t) = p\mathrm{d}q + Q\mathrm{d}P + (H' - H)\mathrm{d}t \tag{1.1}$$

$$\frac{\partial \Phi}{\partial q} = p, \frac{\partial \Phi}{\partial P} = Q, \frac{\partial \Phi}{\partial t} + H = H'. \tag{1.2}$$

(2) We have proved:

$$\frac{\partial S}{\partial q_t} = -p_t, \frac{\partial S}{\partial q_{t+\tau}} = p_{t+\tau}, \tag{1.3}$$

First, we can check:

$$\frac{\partial \Psi}{\partial q_t} = -\frac{\partial S}{\partial q_t} = p_t \tag{1.4}$$

Second ,by chain rule,we obtain

$$\frac{\partial \Psi}{\partial p_{t+\tau}} = q_{t+\tau} + \left(p_{t+\tau} - \frac{\partial S}{\partial p_{t+\tau}}\right)\frac{\partial q_{t+\tau}}{\partial p_{t+\tau}} = q_{t+\tau} \tag{1.5}$$

(3)

$$\mathrm{d}p_t \wedge \mathrm{d}q_t = \mathrm{d}p_{t+\tau} \wedge \mathrm{d}q_{t+\tau}, \frac{\partial(q_{t+\tau}, p_{t+\tau})}{\partial(q_t, p_t)} = 1 \tag{1.6}$$

Problem 2 (Hamilton-Jacobi equation)
(1)

$$\frac{\partial S}{\partial t} = \frac{\mathrm{d}S}{\mathrm{d}t} - \frac{\partial S}{\partial q}\dot{q} = L - p\dot{q} = -H \tag{2.1}$$

(2)

$$\beta = \frac{\partial S}{\partial \alpha} \tag{2.2}$$

$$H' = H + \frac{\partial S}{\partial t} = 0 \tag{2.3}$$

Remark. Another way to calculate $\frac{\partial S}{\partial q_f}$ and $\frac{\partial S}{\partial t_f}$.

$$q = q(q_f, t_f, t), \mathrm{d}q = \frac{\partial q}{\partial q_f}\mathrm{d}q_f + \frac{\partial q}{\partial t_f}\mathrm{d}t_f + \frac{\partial q}{\partial t}\mathrm{d}t \tag{2.4}$$

We have $q|_{t=t_0} = q_0, q|_{t=t_f}$, hence,

$$\left(\frac{\partial q}{\partial q_f}\right)_{t_f,t}\bigg|_{t=t_0} = 0, \left(\frac{\partial q}{\partial q_f}\right)_{t_f,t}\bigg|_{t=t_f} = 1 \tag{2.5}$$

Now,let $t \equiv t_0, \mathrm{d}q = \mathrm{d}q_0 = 0$.Thus,

$$\frac{\partial q}{\partial t_f}\bigg|_{t=t_0} = 0 \tag{2.6}$$

Let t be a function of t_f and take the form $t = t_f.\mathrm{d}q = \mathrm{d}q_f$,so

$$\left(\frac{\partial q}{\partial t_f} + \frac{\partial q}{\partial t}\right)\mathrm{d}t_f = 0 \tag{2.7}$$

$$\frac{\partial q}{\partial t_f}\bigg|_{t=t_f} = -\left.\frac{\partial q}{\partial t}\right|_{t=t_f} = -\dot{q}|_{t=t_f} \tag{2.8}$$

$$\begin{aligned}\frac{\partial S}{\partial q_f} &= \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial q_f}\left(\frac{\partial q}{\partial t}\right) + \frac{\partial L}{\partial q}\frac{\partial q}{\partial q_f}\right]\mathrm{d}t \\ &= \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial t}\left(\frac{\partial q}{\partial q_f}\right) + \frac{\partial L}{\partial q}\frac{\partial q}{\partial q_f}\right]\mathrm{d}t \\ &= \left.\frac{\partial L}{\partial \dot{q}}\frac{\partial q}{\partial q_f}\right|_{t_0}^{t_f} + \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial q} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\frac{\partial q}{\partial q_f}\,\mathrm{d}t \\ &= p(t_f)\end{aligned} \tag{2.9}$$

$$\begin{aligned}\frac{\partial S}{\partial t_f} &= L(t_f) + \int_{t_0}^{t_f}\left[\frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial t_f}\left(\frac{\partial q}{\partial t}\right) + \frac{\partial L}{\partial q}\frac{\partial q}{\partial t_f}\right]\mathrm{d}t \\ &= L(t_f) + p\left.\frac{\partial q}{\partial t_f}\right|_{t_0}^{t_f} \\ &= L - p\dot{q} = -H\end{aligned} \tag{2.10}$$

Problem 3 (Harmonic Oscillator)
(1) easy to check:

$$\frac{\partial S}{\partial x} = p, \frac{\partial S}{\partial t} = -E \tag{3.1}$$

(2)

$$\frac{\partial S}{\partial t_f} = -E \tag{3.2}$$

$$\frac{\partial S}{\partial x_f} = p = \pm m\omega\sqrt{A^2 - x^2} \tag{3.3}$$

(3)

$$S = \pm m\omega \int \sqrt{\frac{2E}{m\omega^2} - x^2}\,\mathrm{d}x - Et + \text{const.} \tag{3.4}$$

$$\frac{\partial S}{\partial E}=\pm\frac{\arcsin(\sqrt{\frac{m\omega^2}{2E}}x)}{\omega}-t \tag{3.5}$$

New Hamitonian:

$$H'=H+\frac{\partial S}{\partial t}=0 \tag{3.6}$$

By Hamilton equation

$$\dot{\beta}=\frac{\partial H'}{\partial E}=0 \tag{3.7}$$

Therefore β is a constant.It means the initial phase of oscillator.

Problem 4 (Planck's derivation of black body radiation)
(1) $\omega\rightarrow+\infty$:

$$\ln u=-\frac{\hbar\omega}{k_BT}+\text{const.} \tag{4.1}$$

$$\frac{\partial(\ln u)}{\partial\left(\frac{1}{T}\right)}=-\frac{\hbar\omega}{k_B} \tag{4.2}$$

$\omega\rightarrow 0$:

$$\frac{1}{u}\frac{\partial u}{\partial\left(\frac{1}{T}\right)}=-T=-\frac{u}{k_B} \tag{4.3}$$

We assume that

$$\frac{\partial(\ln u)}{\partial\left(\frac{1}{T}\right)}=-\frac{\hbar\omega+u}{k_B} \tag{4.4}$$

That leads to

$$\frac{u}{\hbar\omega+u}=Ce^{-\frac{\hbar\omega}{k_BT}} \tag{4.5}$$

By the expression at low frequency,we can deduce that $C=1$.

$$u=\frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_BT}}-1} \tag{4.6}$$

Problem 5 (Heisenberg's magic)
Quantization condition:

$$\oint p\mathrm{d}q=\int_0^T m\dot{x}^2\mathrm{d}t=nh \tag{5.1}$$

$$x_{n\leftarrow m}(t)=x_{n\leftarrow m}e^{i\omega_{n\leftarrow m}t} \tag{5.2}$$

$$\dot{x}^2_{n\leftarrow m}(t)=\sum_{k=-\infty}^{+\infty}x_{n\leftarrow k}x_{k\leftarrow m}\omega_{n\leftarrow k}\omega_{k\leftarrow m}e^{i\omega_{n\leftarrow m}t} \tag{5.3}$$

Let $m=n+l$, we obtain f-sum rule

$$\sum_{l=0}^{+\infty}\left(\omega_{n\rightarrow n+l}\left|x_{n+l,n}\right|^2-\omega_{n-l\rightarrow n}\left|x_{n,n-l}\right|^2\right)=\frac{\hbar}{2m} \tag{5.4}$$

In classic mechanics ,we have

$$\ddot{x} + \omega_0^2 x = 0 \tag{5.5}$$

Plug in $x_{n\pm l\leftarrow n}(t) = x_{n\pm l\leftarrow n} e^{i\omega_{n\pm l\leftarrow n} t}$

$$(\omega_0^2 - \omega_{n\pm l\leftarrow n}^2) x_{n\pm l\leftarrow n} = 0 \tag{5.6}$$

Only when $l = \pm 1, \omega_0^2 - \omega_{n\pm l\leftarrow n}^2 = 0, x_{n\pm l\leftarrow n} \neq 0$

$$\frac{\hbar}{2m} = \omega_0 \left(|x_{n+1,n}|^2 - |x_{n,n-1}|^2 \right) \tag{5.7}$$

There must exists a lower bound ,or $|x_{n,n-1}|^2 \geq 0$ can not hold.Thus,

$$x_{n+1,n} = \sqrt{\frac{\hbar}{2m\omega_0}(n+1)} \tag{5.8}$$

Therefore,

$$(\dot{x})^2_{nn} = (n + 1/2)\frac{\hbar\omega_0}{m} \tag{5.9}$$