QM HW8

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Problem 1 (Derivative x^H, p^H in harmonic oscillator case)
We have

$$[H, x] = -\mathrm{i}\hbar \frac{p}{m}, \quad [H, p] = \mathrm{i}\hbar m\omega^2 x. \tag{1.1}$$

Thus,

$$\underbrace{[H, \dots [H, x]]}_{n \text{ copies}} = \begin{cases} -\mathrm{i}\hbar^{2k-1}\omega^{2k-2}\frac{p}{m} & , n = 2k-1 \\ \hbar^{2k}\omega^{2k}x & , n = 2k \end{cases} \tag{1.2}$$

$$\underbrace{[H, \dots [H, p]]}_{n \text{ copies}} = \begin{cases} \mathrm{i}\hbar^{2k-1}m\omega^{2k}\frac{p}{m} & , n = 2k-1 \\ \hbar^{2k}\omega^{2k}x & , n = 2k \end{cases} \tag{1.3}$$

By Baker-Hausdorff lemma, we have

$$\begin{aligned} x(t) = \mathrm{e}^{\frac{\mathrm{i}Ht}{\hbar}} x \mathrm{e}^{-\frac{\mathrm{i}Ht}{\hbar}} &= \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\frac{\mathrm{i}}{\hbar}\right)^n \underbrace{[H, \dots [H, x]]}_{n \text{ copies}} \\ &= \frac{p}{m\omega} \sum_{k=0}^{+\infty} \frac{1}{(2k-1)!}(-1)^{k-1}\omega^{2k-1} + \sum_{k=0}^{+\infty} \frac{1}{(2k)!}\omega^{2k}x \\ &= x\cos(\omega t) + \frac{p}{m\omega}\sin(\omega t). \end{aligned} \tag{1.4}$$

Similarly,

$$p(t) = \mathrm{e}^{\frac{\mathrm{i}Ht}{\hbar}} p \mathrm{e}^{-\frac{\mathrm{i}Ht}{\hbar}} = -m\omega x \sin(\omega t) + p\cos(\omega t). \tag{1.5}$$

Problem 2 (Interaction picture)
We decompose the Hamiltonian H^S of the Schrödinger picture into the free part H_0 and the perturbative part V as

$$H^S = H_0 + V,$$

where H_0 is independent of time; V may depend on time. We define the state vector evolution with time as

$$\left|\Psi^I(t)\right\rangle = e^{iH_0 t/\hbar} \left|\Psi^S(t)\right\rangle = e^{iH_0 t/\hbar} T(t, 0) \left|\Psi^S(0)\right\rangle,$$

and correspondingly the operator

$$F^I(t) = e^{iH_0t/\hbar} F^S e^{-iH_0t/\hbar}.$$

In such a convention, we keep the inner product invariant:

$$\left\langle \Psi_A^I(t) \right| F^I(t) \left| \Psi_B^I(t) \right\rangle = \left\langle \Psi_A^S(t) \right| F^S(t) \left| \Psi_B^S(t) \right\rangle.$$

Now let us derive the equation of motion. We have

$$\frac{d}{dt} F^I(t) = \frac{1}{i\hbar}[F^I(t), H_0] + e^{iH_0t/\hbar} \frac{\partial F^S(t)}{\partial t} e^{-iH_0t/\hbar}.$$

For the state vector, we have

$$\begin{aligned}
\frac{\partial}{\partial t} \left| \Psi^I(t) \right\rangle &= \frac{i}{\hbar} H_0 e^{iH_0t/\hbar} \left| \Psi^S(t) \right\rangle + e^{iH_0t/\hbar} \frac{1}{i\hbar} H^S \left| \Psi^S(t) \right\rangle \\
&= e^{iH_0t/\hbar} \frac{i}{\hbar} (H_0 - H^S) e^{-iH_0t/\hbar} \left| \Psi^I(t) \right\rangle \\
&= \frac{1}{i\hbar} V^I(t) \left| \Psi^I(t) \right\rangle.
\end{aligned}$$

From the above equation, we can derive the time-evolution operator $U(t, t_0)$ in the interaction picture as

$$\left| \Psi^I(t) \right\rangle = U(t, t_0) \left| \Psi^I(t_0) \right\rangle,$$

$$U(t, t_0) = T \exp\left\{ -\frac{i}{\hbar} \int_{t_0}^{t} dt' V^I(t') \right\}.$$

Problem 3 (Rotation Operator)
(1) Consider the commutator

$$[\hat{n} \cdot \vec{J}, r_i] = -\frac{\mathrm{i}}{\alpha} \frac{\partial g}{\partial \theta}\bigg|_{\theta=0;i,j} r_j. \tag{3.1}$$

Take $\hat{n}$ to be z-axis, it should be $-\frac{\mathrm{i}}{\hbar} \frac{\partial g}{\partial \theta}_{\theta=0;,i,j} r_j$. So, $\alpha = \hbar$.
(2) For a infinitesimal rotation

$$D(g(\hat{n}, \theta)) = 1 - \mathrm{i} \frac{\varepsilon}{\hbar} \hat{n} \cdot \vec{J},\ g_{ij} = \delta_{ij} - \varepsilon \epsilon_{ijk} \hat{n}_k. \tag{3.2}$$

So,

$$D^\dagger S_i D = S_i + \mathrm{i} \frac{\varepsilon}{\hbar} [\hat{n} \cdot \vec{J}, S_i],\ g_{ij} S_j = S_i - \varepsilon \epsilon_{ijk} \hat{n}_k S_j. \tag{3.3}$$

Take angular momentum to be $\vec{S}$, then

$$[S_i, S_j] = \mathrm{i}\hbar \epsilon_{ijk} S_k. \tag{3.4}$$

(3)

$$\begin{aligned}[L_i, L_j] =&[\epsilon_{imn}x_m p_n, \epsilon_{jkl}x_k p_l]\\ =&\epsilon_{imn}\epsilon_{jkl}\left(x_m x_k[p_n, p_l] + x_m[p_n, x_k]p_l + x_k[x_m, p_l]p_n + [x_m, x_k]p_l p_n\right)\\ =&i\hbar\epsilon_{imn}\epsilon_{jln}(x_m p_l - x_l p_m)\\ =&i\hbar(x_i p_j - x_j p_i)\\ =&i\hbar\epsilon_{ijk}\epsilon_{kmn}x_m p_n\\ =&i\hbar\epsilon_{ijk}L_k.\end{aligned} \tag{3.5}$$

(4) Take $\vec{J}$ as $\vec{L}$, Similar to (2).

Problem 4 (Pauli Matrices)
(1)

$$\sigma_i^2 = I,\ \sigma_i\sigma_j = -\sigma_j\sigma_i\ (i \neq j). \tag{4.1}$$

So,

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}. \tag{4.2}$$

(2)

$$\sigma_i\sigma_j = \delta_{ij}I + \mathrm{i}\epsilon_{ijk}\sigma_k. \tag{4.3}$$

Thus,

$$(\vec{\sigma}\cdot\vec{n})^2 = I. \tag{4.4}$$

Hence,

$$\exp\left[-\frac{\mathrm{i}}{2}\theta\vec{\sigma}\cdot\vec{n}\right] = \sum_{k=0}^{\infty}\frac{1}{k!}(-\frac{\mathrm{i}}{2}\theta\vec{\sigma}\cdot\vec{n})^k = \cos\left(\frac{\theta}{2}\right) - \mathrm{i}\frac{\vec{\sigma}\cdot\vec{n}}{2}\sin\left(\frac{\theta}{2}\right). \tag{4.5}$$

Problem 5 (Anti-unitary transformation)
(1)

$$R^{-1} = (UK)^{-1} = K^{-1}U{-1} = KU^{-1} = KU^{\dagger}. \tag{5.1}$$

So,

$$\langle R\psi|R\phi\rangle = \langle\psi^*|\,U^{\dagger}U\,|\phi^*\rangle = \langle\phi|\psi\rangle\,. \tag{5.2}$$

(2)

$$\left\langle R^{-1}\psi\middle|R^{-1}\phi\right\rangle = \left\langle U^{\dagger}\psi\middle|U^{\dagger}\phi\right\rangle^* = \langle\phi|\psi\rangle \tag{5.3}$$

Problem 6 (Time-reversal transformation)
(1)

$$\mathrm{i}\epsilon_{ijk}L_k = [L_i, L_j] = T[L_i, L_j]T^{-1} = T\mathrm{i}\epsilon_{ijk}L_kT^{-1} \tag{6.1}$$

Since L_k is arbitrary,

$$T\mathrm{i}T^{-1} = -\mathrm{i}. \tag{6.2}$$

(2) Let Π be the mechanical momentum, we expect $T\Pi T^{-1} = -\Pi$. So $H = \frac{\Pi^2}{2m}$ should not change under time-reversal transformation.

$$H^T = H. \tag{6.3}$$

(3) No, consider integer spin system, it is not always an energy level degeneracy.

Problem 7 (Parity Transformation)
(1) We expect any $|\psi\rangle$ satisfies:

$$\langle P\psi|P\psi\rangle = \langle\psi|\psi\rangle \,. \tag{7.1}$$

Let $|\psi\rangle = |\alpha\rangle + |\beta\rangle$, then

$$\mathrm{Re}\left(\langle P\alpha|P\beta\rangle\right) = \mathrm{Re}\left(\langle\alpha|\beta\rangle\right). \tag{7.2}$$

Let $|\psi\rangle = |\alpha\rangle + \mathrm{i}\,|\beta\rangle$, then

$$\mathrm{Im}\left(\langle P\alpha|P\beta\rangle\right) = \mathrm{Im}\left(\langle\alpha|\beta\rangle\right). \tag{7.3}$$

So,

$$\langle P\alpha|P\beta\rangle = \langle\alpha|\beta\rangle \,. \tag{7.4}$$

By Wigner theorem, P is unitary, we choose $P^2 = 1$. But $T^2 = \pm 1$ is depend on the spin of system.
(2) (a) Momentum eigenstate

Let the momentum eigenstate be:

$$\psi_p(x,t) = e^{i(px-\omega t)}$$

Time reversal transformation T:

$$\psi_p^T(x,t) = T\psi_p(x,t) = \psi_p^*(x,-t) = e^{-i(px+\omega t)} = e^{i[(-p)x-\omega t]} = \psi_{-p}(x,t)$$

Parity transformation P:

$$\psi_p^P(x,t) = P\psi_p(x,t) = \psi_p(-x,t) = e^{i[p(-x)-\omega t]} = e^{i[(-p)x-\omega t]} = \psi_{-p}(x,t)$$

Both T and P reverse the momentum: $p \to -p$.
(b) Angular momentum eigenstate
Let the angular momentum eigenstate be:

$$\psi_m(\varphi,t) = e^{im\varphi - i\omega t}$$

Considering only the spatial part at $t = 0$:

$$\psi_m(\varphi) = e^{im\varphi}$$

Time reversal transformation T:

$$\psi_m^T(\varphi) = T\psi_m(\varphi) = \psi_m^*(\varphi) = \psi_{-m}(\varphi)$$

Time reversal changes $m \to -m$.

Parity transformation P: In spherical coordinates, parity acts as $\varphi \to \varphi + \pi$:

$$\psi_m^P(\varphi) = P\psi_m(\varphi) = \psi_m(\varphi+\pi) = (-1)^m\psi_m(\varphi)$$

Parity gives a phase factor $(-1)^m$ but does not change m.