QM HW3

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Problem 1 (A quantum particle in an infinitely deep potential well)
(1) When $|x| > \frac{L}{2}$, $\psi(x) = 0$. Now devoted exclusively to the case where $|x| < \frac{L}{2}$.By Schrödinger equation,

$$E\psi + \frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = 0. \tag{1.1}$$

Let $k = \sqrt{2mE/\hbar^2}$,

$$\psi(x) = A\sin(kx) + B\cos(kx). \tag{1.2}$$

For odd parity, $B = 0$. For even parity, $A = 0$. Take the boundary condition

$$\psi(\pm\frac{L}{2}) = 0. \tag{1.3}$$

$$\cos\left(\frac{k^+L}{2}\right) = 0, \sin\left(\frac{k^-L}{2}\right) = 0. \tag{1.4}$$

So, we have

$$k_n^+ = \frac{(2n-1)\,\pi}{L},\ k_n^- = \frac{2n\pi}{L}. \tag{1.5}$$

That is

$$\boxed{E_n^+ = \frac{\hbar^2\pi^2(2n-1)^2}{2mL^2},\ E_n^- = \frac{\hbar^2\pi^2(2n)^2}{2mL^2}}. \tag{1.6}$$

$$\psi_n^+ = A_n^+\cos\left(\frac{(2n-1)\,\pi x}{L}\right),\ \psi_n^- = A_n^-\sin\left(\frac{2n\pi x}{L}\right). \tag{1.7}$$

Normalize the wave functions,

$$\boxed{\psi_n^+ = \sqrt{\frac{2}{L}}\cos\left(\frac{(2n-1)\,\pi x}{L}\right),\ \psi_n^- = \sqrt{\frac{2}{L}}\sin\left(\frac{2n\pi x}{L}\right)}. \tag{1.8}$$

(2)

$$\Psi(x,t) = \sqrt{\frac{1}{L}}\left[e^{-iE_1^+t}\cos\left(\frac{\pi x}{L}\right) + e^{-iE_1^-t}\sin\left(\frac{2\pi x}{L}\right)\right] \tag{1.9}$$

$$= \boxed{\sqrt{\frac{1}{L}}\left[e^{-i\frac{\hbar^2\pi^2}{2mL^2}t}\cos\left(\frac{\pi x}{L}\right) + e^{-i\frac{2\hbar^2\pi^2}{mL^2}t}\sin\left(\frac{2\pi x}{L}\right)\right]}. \tag{1.10}$$

(3) By symmetry,

$$\langle x \rangle = 0,\ \langle p \rangle = 0. \tag{1.11}$$

$$\left\langle x^2 \right\rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi^* x^2 \psi \ \mathrm{d}x = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}. \tag{1.12}$$

$$\left\langle p^2 \right\rangle = {}^{1}2mE = \frac{n^2\pi^2\hbar^2}{L^2}. \tag{1.13}$$

So,

$$\boxed{\sqrt{\Delta x^2} = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}}, \tag{1.14}$$

$$\boxed{\sqrt{\Delta p^2} = \frac{n\pi\hbar}{L}}, \tag{1.15}$$

$$\boxed{\sqrt{\Delta x^2}\sqrt{\Delta p^2} = \hbar\sqrt{\frac{n^2\pi^2}{12} - \frac{1}{2}} \geq \hbar\sqrt{\frac{\pi^2}{12} - \frac{1}{2}} > \frac{\hbar}{2}}. \tag{1.16}$$

[1] By Schrödinger equation: $\frac{p^2}{2m}\left|\psi\right\rangle = E\left|\psi\right\rangle$, then $\left\langle p^2 \right\rangle = \left\langle \psi \middle| p^2 \middle| \psi \right\rangle = 2mE$.