QM HW10

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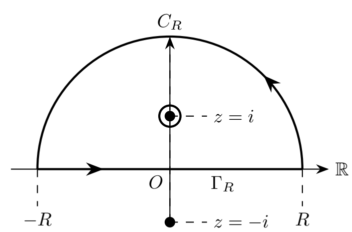
Problem 1

$$H'(t) = -\frac{F_0\tau/\omega}{\tau^2+t^2}x = -\frac{F_0\tau/\omega}{\tau^2+t^2}\frac{a+a^\dagger}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}. \tag{1.1}$$

$$c^{(1)} = \frac{-\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty}\mathrm{d}t\exp(\mathrm{i}\omega t)\langle 1|H'|0\rangle = \frac{\mathrm{i}F_0}{\sqrt{2\hbar m\omega^3}}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega\tau z}\frac{\mathrm{d}z}{1+z^2}. \tag{1.2}$$



By residue theorem,

$$\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega\tau z}\frac{\mathrm{d}z}{1+z^2} = \pi\mathrm{e}^{-\omega\tau}. \tag{1.3}$$

So,

$$P = |c^{(1)}|^2 = \frac{\pi^2F_0^2\mathrm{e}^{-2\omega\tau}}{2\hbar m\omega^3}. \tag{1.4}$$

$\tau \gg 1/\omega$ means low frequency of transition, it is proper.

Problem 2

$$H'(t) = eE_0 z\exp(-t/\tau),\ t>0. \tag{2.1}$$

$$c_f^{(1)}(+\infty) = \frac{-\mathrm{i}}{\hbar}\int_0^{+\infty}\mathrm{d}t'\langle f|H'(t)|i\rangle\mathrm{e}^{\mathrm{i}\omega_{fi}t'}. \tag{2.2}$$

$$\langle f|z|i\rangle = \sqrt{\frac{4\pi}{3}}\langle R_{nl}|r|R_{10}\rangle\cdot\langle Y_l^m|Y_1^0|Y_0^0\rangle. \tag{2.3}$$

Since Y_1^0 is odd and by Wigner-Eckart theorem, only $l = 1$ and $m = 0$, the element can not be zero.

$$\langle Y_l^m | \, Y_1^0 \, | Y_0^0 \rangle = \frac{1}{\sqrt{4\pi}} \int |Y_1^0|^2 \, \mathrm{d}\Omega = \frac{1}{\sqrt{4\pi}}. \tag{2.4}$$

$$\int_0^{+\infty} \mathrm{e}^{-t/\tau + \mathrm{i}\omega_{fi} t} \mathrm{d}t = \frac{1}{1/\tau - \mathrm{i}\omega_{fi}}. \tag{2.5}$$

So,

$$P_{210} = \frac{e^2 E_0^2 |\, \langle R_{nl} | \, r \, | R_{10} \rangle \, |^2 \tau^2}{2\hbar^2 \left(1 + \omega^2 \tau^2\right)}. \tag{2.6}$$

where E_0 is the energy of ground state, and $\omega = \frac{3}{4} \frac{E_0}{\hbar}$.

Problem 3

(a) Let

$$\Delta\omega = \omega - \omega_{21}. \tag{3.1}$$

Define new variables:

$$\tilde{c}_1(t) = c_1(t) e^{-i\Delta\omega \, t/2}, \qquad \tilde{c}_2(t) = c_2(t) e^{i\Delta\omega \, t/2}. \tag{3.2}$$

Then,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} \Delta\omega & 2\gamma/\hbar \\ 2\gamma/\hbar & -\Delta\omega \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}. \tag{3.3}$$

Thus the effective Hamiltonian in the rotated frame is

$$H_{\text{eff}} = \frac{\hbar}{2} \begin{pmatrix} \Delta\omega & 2\gamma/\hbar \\ 2\gamma/\hbar & -\Delta\omega \end{pmatrix} = \frac{\hbar}{2} \Big(\Delta\omega \, \sigma_z + \frac{2\gamma}{\hbar} \, \sigma_x \Big). \tag{3.4}$$

$$U(t) = \exp\Big(-\frac{i}{\hbar} H_{\text{eff}} \, t \Big) = \exp\Big[-it \begin{pmatrix} \Delta\omega/2 & \gamma/\hbar \\ \gamma/\hbar & -\Delta\omega/2 \end{pmatrix} \Big]. \tag{3.5}$$

The eigenvalues of the matrix are $\pm\Omega_R$ with

$$\Omega_R = \sqrt{\Big(\frac{\gamma}{\hbar}\Big)^2 + \Big(\frac{\Delta\omega}{2}\Big)^2}. \tag{3.6}$$

Using the identity $\exp(-i\boldsymbol{n} \cdot \boldsymbol{\sigma} \, \theta) = \cos\theta \, I - i \sin\theta \, (\boldsymbol{n} \cdot \boldsymbol{\sigma})$, we obtain

$$U(t) = \cos(\Omega_R t) \, I - i \sin(\Omega_R t) \, \frac{1}{\Omega_R} \begin{pmatrix} \Delta\omega/2 & \gamma/\hbar \\ \gamma/\hbar & -\Delta\omega/2 \end{pmatrix}. \tag{3.7}$$

With $\tilde{c}_1(0) = 1, \ \tilde{c}_2(0) = 0$, equation (3.7) gives

$$\tilde{c}_2(t) = -i \, \frac{\gamma}{\hbar\Omega_R} \sin(\Omega_R t). \tag{3.8}$$

Returning to the original variables,

$$c_2(t) = \tilde{c}_2(t)e^{-i\Delta\omega\, t/2} = -i\,\frac{\gamma}{\hbar\Omega_R}e^{-i\Delta\omega\, t/2}\sin(\Omega_R t). \tag{3.9}$$

Hence the occupation probability of the excited state is

$$\boxed{|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega-\omega_{21})^2/4}\sin^2\Big[\sqrt{\frac{\gamma^2}{\hbar^2}+\frac{(\omega-\omega_{21})^2}{4}}\;t\Big]} \tag{3.10}$$

and, by normalization,

$$|c_1(t)|^2 = 1 - |c_2(t)|^2. \tag{3.11}$$

(b)

$$c_2^{(1)}(t) = -\frac{\mathrm{i}}{\hbar}\gamma\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega t'}\,\mathrm{d}t' = \frac{\gamma}{\hbar\Delta\omega}(1-\mathrm{e}^{-\mathrm{i}\Delta t}). \tag{3.12}$$

So,

$$P^{(1)}_{1\to 2}(t) = \frac{4\gamma^2}{\hbar^2\Delta\omega^2}\sin^2\left(\frac{\Delta\omega t}{2}\right). \tag{3.13}$$

For the cases where $\Delta\omega \gg \frac{\gamma}{\hbar}$, the approximation is valid. But for the cases where $\Delta\omega \ll \frac{\gamma}{\hbar}$, the approximation is not valid. This is because this resonance case.

Problem 4

$$H'(t) = eE_0(x\cos(\omega t) - y\sin(\omega t)),\ t>0. \tag{4.1}$$

$$x\cos(\omega t) - y\sin(\omega t) \sim (Y_1^{+1}+Y_1^{-1})\cos(\omega t) + \mathrm{i}(Y_1^{+1}-Y_1^{-1})\sin(\omega t). \tag{4.2}$$

That is

$$H'(t) \sim Y_1^{+1}\mathrm{e}^{\mathrm{i}\omega t} + Y_1^{-1}\mathrm{e}^{-\mathrm{i}\omega t}. \tag{4.3}$$

Since x, y has odd parity and by Wigner-Eckart theorem, only when $\Delta m = \pm 1$, $\Delta l = \pm 1$, the element can not be zero.

Problem 5

(a) We have shown that the possible final states should be $l' = 1$ and $m' = 0/\pm 1$. By symmetry, three banch have the same ratio.

(b) $l' = l \pm 1$, $m' = m$ or $m \pm 1$. By Wigner-Eckart theorem,

$$P(m') = |\,\langle l, m; 1, (m'-m)|l', m'\rangle\,|^2 \tag{5.1}$$

(1) For a fixed l', the ratio is

$$|\,\langle l,m;1,-1|l',m-1\rangle\,|^2 : |\,\langle l,m;1,0|l',m\rangle\,|^2 : |\,\langle l,m;1,1|l',m+1\rangle\,|^2. \tag{5.2}$$

(2)