Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 9–4 : Differentiability and Mean Value Theorems

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous mapping, such that $f(x) = 0$ if an only if $x = a$ or $x = b$. Show that $f(x) \geq 0$ for all $x \in [a, b]$ or $f(x) \leq 0$ for all $x \in [a, b]$.

2. Give an example of a continuous mapping $f :] - 1, 1[\to \mathbb{R}$ which attains a global maximum at 0, but is not differentiable at 0.

3. Give an example of a differentiable mapping $f :] - 1, 1[\to \mathbb{R}$ whose derivative at 0 equals 0, but such that 0 is neither a local minimum, nor a local maximum. .

4. Let $f :]a, b[\to \mathbb{R}$ be a differentiable mapping. Show that if f is increasing (decreasing), then $f'(x) \geq 0$ $(f'(x) \leq 0)$ for all $x \in]a, b[$.

5. Let $f :]a, b[\to \mathbb{R}$ be a differentiable mapping such that $f'(x) > 0$ $(f'(x) < 0)$ for all $x \in]a, b[$. Show that f is strictly increasing (decreasing).

6. Give an example of a function $f :] - 1, 1[\to \mathbb{R}$ which is continuous and strictly increasing, but not differentiable at 0.

7. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is strictly increasing and differentiable, but whose derivative at 0 is 0.

8. Give an example of a subset $X \subseteq \mathbb{R}$ and a mapping $f : X \to \mathbb{R}$, such that $f'(x) > 0$ for all $x \in X$, but f is not increasing.

9. Let $f :]a, b[\to \mathbb{R}$ be a differentiable mapping such that $f'(x) = 0$ for all $x \in]a, b[$. Show that f is constant on $]a, b[$.

10. Let $f, g :]a, b[\to \mathbb{R}$ be differentiable mappings such that $f'(x) = g'(x)$ for all $x \in]a, b[$. Show that there exists a constant $c \in \mathbb{R}$ such that $f = g$ on $]a, b[$.

11. Prove that the equation $x^3 + x - 1$ has exactly one solution.

12. Find an equation of a line tangent to the graph of the function $f(x) = x^3 - x$ parallel to the line passing through the points $f(0)$ and $f(2)$.

13. Let $M > 0$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function differentiable on $]a, b[$ such that $|f'(x)| \leq M$ for all $x \in]a, b[$. Show that f is Lipschitzian.

14. Let $f : \to$ be a differentiable mapping, such that f' is bounded. Show, that f is uniformly continuous.

15. We say that a function $f :]a,b[\to \mathbb{R}$ is concave upward on $]a,b[\subset \mathbb{R}$ if
$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

We say that a function $f :]a,b[\to \mathbb{R}$ is concave downward on $]a,b[\subset \mathbb{R}$ if
$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Assume that $f, Df :]a,b[\to \mathbb{R}$ are differentiable. Show that f is concave up, if and only if $f''(x) > 0$ for all $x \in]a,b[$ and that f is concave down if and only if $f''(x) < 0$ for all $x \in]a,b[$.

16. Formulate the definition of concavity of mappings in the language of convexity of sets.

17. (The First Derivative Test) Let $f :]a,b[\to \mathbb{R}$ be a differentiable mapping with a continuous derivative such that $f'(c) = 0$ for some $c \in]a,b[$. Prove the following statements :

    (a) if the sign of f' changes from positive to negative at c, then f has a local maximum at c

    (b) if the sign of f' changes from negative to positive at c, then f has a local minimum at c

    (c) if the sign of f' does not change at c, then f does not have a local minimum nor a local maximum at c. In this case we sax, that f has an inflection point at c.

18. (The Second Derivative Test) Let $f :]a,b[\to \mathbb{R}$ be a twice differentiable mapping with a continuous second derivative. Prove the following statements :

    (a) if $f'(c) = 0$ and $f''(c) > 0$ for $c \in]a,b[$, then f has a local minimum at c

    (b) if $f'(c) = 0$ and $f''(c) < 0$ for $c \in]a,b[$, then f has a local maximum at c

19. (Cauchy's Mean Value Theorem) Let $f, g :]a,b[\to \mathbb{R}$ be continuous mappings differentiable on $]a,b[$. then there exists $c \in]a,b[$ such that
$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

    (Hint : Consider the function $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x)$).

20. Let $(E, \|\cdot\|)$ and $(E, \|\cdot\|)$ be normed vector spaces over $\mathbb{R}$, let $U \subseteq E$ be an open, convex and connected set, and let $f : U \to F$ be a differentiable mapping such that
$$\forall x \in U \qquad \|Df(x)\| \leq K$$

Let $a, b \in U$. Show that

$$\|f(b) - f(a)\|_F \leq K\|b - a\|_E$$

21. We say, that a topological space X is connected if $X = A \cup B$ for $A, B \subseteq X$ open $A \cap B = \emptyset$ implies that $A = \emptyset$ and $B = X$ or $B = \emptyset$ and $A = X$. Show that the following conditions are equivalent :

   (a) X is connected

   (b) X cannot be presented as a disjoint sum of two non-empty open sets

   (c) X cannot be presented as a disjoint sum of two non-empty closed sets

   (d) The only sets that are simultaneously open and closed are X and $\emptyset$

   (e) All continuous functions $f : X \to \{0, 1\}$ are constant, where $\{0, 1\}$ is endowed with discrete topology.

   (f) All discrete-valued continuous maps $f : X \to \{1, \ldots, n\}$, for any natural n are constant

22. Show that any path connected topological space is connected.

23. Let $(E, \|\cdot\|)$ and $(E, \|\cdot\|)$ be normed vector spaces over $\mathbb{R}$, let $U \subseteq E$ be an open and connected set, and let $f : U \to F$ be a differentiable mapping such that

$$\forall x \in U \qquad Df(x) = 0.$$

Show that f is constant.

24. Let $(E, \|\cdot\|)$ and $(E, \|\cdot\|)$ be normed vector spaces over $\mathbb{R}$, let $U \subseteq E$ be an open and connected set, and let $f, g : U \to F$ be differentiable mappings such that

$$\forall x \in U \qquad Df(x) = Dg(x).$$

Show that $f = g + C$ on U for some constant C.

25. Show that the connectivity assumption in the exercises 23,25 and the convexity assumption in the exercise 20 is necessary.

26. Let $(E, \|\cdot\|)$ and $(E, \|\cdot\|)$ be normed vector spaces over $\mathbb{R}$, let $U \subseteq E$ be an open set, $f : U \to F$ be a mapping and $p \in U$. Show that f is differentiable at p if and only if, there exists a linear mapping $Df(p) \in \mathscr{L}(E, F)$ such that

$$\forall \epsilon > 0 \, \exists r > 0 (\|h\|_E < r, p + h \in U) \Rightarrow (\|f(p+h) - f(p) - Df(p)(h)\|_F \leq \epsilon \|h\|$$

27. Let U be an open subset of a valued field K, $(F, \|\cdot\|)$ be a normed vector space over K, and $f : U \to F$ be a mapping. If f is differentiable at $p \in U$, we denote by $f'(p)$ the element

$$Df(p)(1) \in F.$$

Show that :

$$f'(p) = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}.$$