

Pauli Matrix

We define

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

We define $\sigma^0 = I_{2 \times 2}(\mathbb{C})$ just for convenience, it is used for balancing the index, you should not replace I any where. In the following part, we will use i, j, k to represent 1, 2, 3 and other Latin letters to represent 0, 1, 2, 3. Also, we will use Greek letters to represent 1, 2, which are the components of matrices. So there are something crazy:

$$\delta^\mu_\mu = 2, \delta^i_i = 3, \delta^a_a = 4. \quad (2)$$

Property 1.

$$\sigma^i \sigma^j = \delta^{ij} I + i \varepsilon^{ijk} \sigma_k. \quad (3)$$

If we define a general¹ ϵ^{abc} :

$$\epsilon_{abc} = \begin{cases} \varepsilon^{abc} & , 0 \notin \{a, b, c\} \\ 0 & , 0 \in \{a, b, c\} \end{cases} \quad (4)$$

Replace ε^{ijk} with ϵ^{abc} , the equality is still valid:

$$\boxed{\sigma^a \sigma^b = \delta^{ab} \sigma^0 + i \epsilon^{abc} \sigma_c.} \quad (5)$$

Property 2. For any $i \in \{1, 2, 3\}$,

$$\boxed{\det(\sigma^i) = -1, \text{tr}(\sigma^i) = 0, (\sigma^i)^\dagger = \sigma^i.} \quad (6)$$

Property 3 (Completeness).

Any $M \in M_{2 \times 2}(\mathbb{C})$ can be written as a linear combination of $\sigma^0, \sigma^1, \sigma^2, \sigma^3$. Since,

$$\sigma^0 + \sigma^3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \sigma^0 - \sigma^3 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \sigma^1 + i\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \sigma^1 - i\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \quad (7)$$

Let $M = \beta_a \sigma^a$, then

$$M \sigma^b = \beta_a \sigma^a \sigma^b = \beta_a (\delta^{ab} I + i \epsilon^{abc} \sigma_c). \quad (8)$$

We take its trace, the second term at right hand side is zero, so

$$\boxed{\beta^b = \frac{1}{2} \text{tr}(M \sigma^b).} \quad (9)$$

¹It is also anti-symmetric: $\forall a, b, c \in \{0, 1, 2, 3\}, (a-b)(b-c)(c-a) = 0 \Rightarrow \epsilon^{abc} = 0$.

That means

$$M_{\rho}^{\lambda} \delta_{\lambda}^{\mu} \delta_{\nu}^{\rho} = M_{\nu}^{\mu} = \frac{1}{2} M_{\rho}^{\lambda} (\sigma_a)^{\rho}_{\lambda} (\sigma^a)^{\mu}_{\nu}. \quad (10)$$

Since M_{ρ}^{λ} is arbitrary, we can deduce that

$$\boxed{(\sigma_a)^{\rho}_{\lambda} (\sigma^a)^{\mu}_{\nu} = 2 \delta_{\lambda}^{\mu} \delta_{\nu}^{\rho}.} \quad (11)$$

In particular,

$$(\sigma_i)^{\rho}_{\lambda} (\sigma^i)^{\mu}_{\nu} = 2 \delta_{\lambda}^{\mu} \delta_{\nu}^{\rho} - \delta_{\lambda}^{\rho} \delta_{\nu}^{\mu}. \quad (12)$$

Property 4 (Relation with cross product of vectors).

Let u^i, v^i be two vectors, then

$$\boxed{(u_i \sigma^i) (v_j \sigma^j) = i (u^i v^j \varepsilon_{ijk}) \sigma^k + u_k v^k I.} \quad (13)$$

In particular,

$$(u_i \sigma^i)^2 = (u^i u_i)^2 I. \quad (14)$$

Property 5 (Exponential form).

Let \mathbf{n} be a unitary vector, then we can deduce that

$$\boxed{\exp \left[i (n_i \sigma^i) \frac{\theta}{2} \right] = I \cos \frac{\theta}{2} + i (n_i \sigma^i) \sin \frac{\theta}{2}.} \quad (15)$$

Property 6 (Projection Operators).

Consider

$$P_{\pm} = \frac{1}{2} (1 + \sigma^i n_i), \quad (16)$$

we have

$$\boxed{P_{\pm}^2 = P_{\pm}}, \quad \boxed{(\sigma^i n_i) P_{\pm} = \pm P_{\pm}}, \quad \boxed{P_+ P_- = P_- P_+ = 0}. \quad (17)$$

It can be used to construct the eigenstates of the Hamiltonian of the form $(\sigma^i n_i)$.

Property 7 (Pauli Matrices in different coordinates).

Choose spherical coordinates as an example. We have

$$n_i \sigma^i = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}. \quad (18)$$

Then we choose \mathbf{n} as $\hat{r}, \hat{\theta}, \hat{\phi}$. Then we obtain

$$\sigma_r = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & -\cos \theta \end{pmatrix}, \quad (19)$$

$$\sigma_{\theta} = \begin{pmatrix} -\sin \theta & \cos \theta e^{-i\phi} \\ \cos \theta e^{i\phi} & \sin \theta \end{pmatrix}, \quad (20)$$

$$\sigma_{\phi} = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}. \quad (21)$$