

# QM HW8

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**Problem 1** (Derivative  $x^H, p^H$  in harmonic oscillator case)

We have

$$[H, x] = -i\hbar \frac{p}{m}, \quad [H, p] = i\hbar m\omega^2 x. \quad (1.1)$$

Thus,

$$\underbrace{[H, \dots [H, x]]}_{n \text{ copies}} = \begin{cases} -i\hbar^{2k-1} \omega^{2k-2} \frac{p}{m} & , n = 2k - 1 \\ \hbar^{2k} \omega^{2k} x & , n = 2k \end{cases} \quad (1.2)$$

$$\underbrace{[H, \dots [H, p]]}_{n \text{ copies}} = \begin{cases} i\hbar^{2k-1} m\omega^{2k} \frac{p}{m} & , n = 2k - 1 \\ \hbar^{2k} \omega^{2k} x & , n = 2k \end{cases} \quad (1.3)$$

By Baker-Hausdorff lemma, we have

$$\begin{aligned} x(t) &= e^{\frac{iHt}{\hbar}} x e^{-\frac{iHt}{\hbar}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n \underbrace{[H, \dots [H, x]]}_{n \text{ copies}} \\ &= \frac{p}{m\omega} \sum_{k=0}^{+\infty} \frac{1}{(2k-1)!} (-1)^{k-1} \omega^{2k-1} + \sum_{k=0}^{+\infty} \frac{1}{(2k)!} \omega^{2k} x \\ &= x \cos(\omega t) + \frac{p}{m\omega} \sin(\omega t). \end{aligned} \quad (1.4)$$

Similarly,

$$p(t) = e^{\frac{iHt}{\hbar}} p e^{-\frac{iHt}{\hbar}} = -m\omega x \sin(\omega t) + p \cos(\omega t). \quad (1.5)$$

**Problem 2**

We decompose the Hamiltonian  $H^S$  of the Schrödinger picture into the free part  $H_0$  and the perturbative part  $V$  as

$$H^S = H_0 + V,$$

where  $H_0$  is independent of time;  $V$  may depend on time. We define the state vector evolution with time as

$$|\Psi^I(t)\rangle = e^{iH_0 t/\hbar} |\Psi^S(t)\rangle = e^{iH_0 t/\hbar} T(t, 0) |\Psi^S(0)\rangle,$$

and correspondingly the operator

$$F^I(t) = e^{iH_0t/\hbar} F^S e^{-iH_0t/\hbar}.$$

In such a convention, we keep the inner product invariant:

$$\langle \Psi_A^I(t) | F^I(t) | \Psi_B^I(t) \rangle = \langle \Psi_A^S(t) | F^S(t) | \Psi_B^S(t) \rangle.$$

Now let us derive the equation of motion. We have

$$\frac{d}{dt} F^I(t) = \frac{1}{i\hbar} [F^I(t), H_0] + e^{iH_0t/\hbar} \frac{\partial F^S(t)}{\partial t} e^{-iH_0t/\hbar}.$$

For the state vector, we have

$$\begin{aligned} \frac{\partial}{\partial t} |\Psi^I(t)\rangle &= \frac{i}{\hbar} H_0 e^{iH_0t/\hbar} |\Psi^S(t)\rangle + e^{iH_0t/\hbar} \frac{1}{i\hbar} H^S |\Psi^S(t)\rangle \\ &= e^{iH_0t/\hbar} \frac{i}{\hbar} (H_0 - H^S) e^{-iH_0t/\hbar} |\Psi^I(t)\rangle \\ &= \frac{1}{i\hbar} V^I(t) |\Psi^I(t)\rangle. \end{aligned}$$

From the above equation, we can derive the time-evolution operator  $U(t, t_0)$  in the interaction picture as

$$\begin{aligned} |\Psi^I(t)\rangle &= U(t, t_0) |\Psi^I(t_0)\rangle, \\ U(t, t_0) &= T \exp \left\{ -\frac{i}{\hbar} \int_{t_0}^t dt' V^I(t') \right\}. \end{aligned}$$