QM HW6

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Problem 1 (Current, gauge transformation)
(1) Use the substitution:

$$-i\hbar\nabla \longrightarrow -i\hbar\nabla - \frac{q\vec{A}}{c}, \tag{1.1}$$

the probability current will be:

$$\vec{j} = \frac{\hbar}{m}\text{Im}\left(\psi^*\nabla\psi\right) \longrightarrow \frac{\hbar}{m}\text{Im}\left(\psi^*\nabla\psi\right) - \frac{q}{mc}\vec{A}\left|\psi\right|^2. \tag{1.2}$$

Hence,

$$\boxed{\vec{j} = \frac{\hbar}{m}\text{Im}\left(\psi^*\nabla\psi\right) - \frac{q}{mc}\vec{A}\left|\psi\right|^2.} \tag{1.3}$$

(2)

$$A'_\mu = A_\mu + \partial_\mu f. \tag{1.4}$$

Since the commutator is antisymmetric and the derivative is commutative,

$$\partial_{[\mu}\partial_{\nu]}f = 0. \tag{1.5}$$

Thus,

$$F'_{\mu\nu} = \partial_{[\mu}A'_{\nu]} = \partial_{[\mu}A_{\nu]} + \partial_{[\mu}\partial_{\nu]}f = F_{\mu\nu}. \tag{1.6}$$

So,

$$\mathbf{E}' = \mathbf{E},\ \mathbf{B}' = \mathbf{B}. \tag{1.7}$$

(3)

$$\frac{\partial\psi'}{\partial t} = i\frac{\partial\varphi}{\partial t}\psi' + e^{i\varphi}\frac{\partial\psi}{\partial t} \tag{1.8}$$

Cancelate $e^{i\varphi}$, and plug in Schrödinger equation, we should take

$$\boxed{\varphi = \frac{q}{\hbar c}f.} \tag{1.9}$$

(4)

$$\rho' = e^{i\varphi}\psi e^{-i\varphi}\psi^* = \psi\psi^* = \rho. \tag{1.10}$$

If we let

$$\psi = \sqrt{\rho}e^{\frac{iS}{\hbar}}, \tag{1.11}$$

then,

$$\mathbf{j} = \frac{\rho}{m}\left(\nabla S - \frac{q\mathbf{A}}{c}\right). \tag{1.12}$$

$$S' = S + \hbar\varphi, \quad \nabla S' - \frac{q\mathbf{A}'}{c} = \nabla S - \frac{q\mathbf{A}}{c}. \tag{1.13}$$

So the probability current is invariant under the gauge transformation.

Problem 2 (Landau gauge)
(1)

$$\left(p_x - \frac{qB}{c}y\right)^2 e^{ik_x x} = \left(\hbar k_x - \frac{qB}{c}y\right)^2 e^{ik_x x}. \tag{2.1}$$

$$\left[\frac{\left(\hbar k_x - \frac{qBy}{c}\right)^2}{2m} + (V(y) - E)\right]\phi_{k_x}(y) = \frac{\hbar^2}{2m}\phi''_{k_x}(y). \tag{2.2}$$

We can define $H_y(k_x)$ as

$$H_y(k_x) = \frac{\left(\hbar k_x - \frac{qBy}{c}\right)^2}{2m} + \frac{p_y^2}{2m} + V(y). \tag{2.3}$$

(2) Note that

$$\frac{\partial H_y(k_x)}{\partial k_x} = \frac{\hbar k_x - qBy/c}{m}. \tag{2.4}$$

$$I_x(x, k_x) = qJ = \frac{q\rho}{m}\left(\hbar k_x - \frac{qBy}{c}\right) = \frac{q}{L_x}\frac{\partial E_n}{\hbar\partial k_x}. \tag{2.5}$$

(3)

$$I_{x,n} = \frac{1}{2\pi}\int \mathrm{d}k_x \, I_x(n, k_x) = \frac{q^2}{h}\Delta V/L_x. \tag{2.6}$$

$$\sigma_{xy} = \frac{I_x}{E_y} = \frac{q^2}{h}\nu. \tag{2.7}$$

This result is not related to V_{imp}.

Problem 3 (Spherical coordinates)
I will use Einstein summation convention if there's neither special announcement nor summation symbol.

In spherical coordinates, Lamé coefficients are

$$A_r = 1, \quad A_\theta = r, \quad A_\phi = r\sin\theta. \tag{3.1}$$

(1)

$$\nabla f = \mathbf{g}^i\partial_i f = \sum_i \frac{1}{A_i}\mathbf{e}^i\partial_i f. \tag{3.2}$$

So,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{3.3}$$

(2) Note that

$$\frac{\partial \sqrt{g}}{\partial x^i} = \Gamma^j_{ji}\sqrt{g}. \tag{3.4}$$

We have,

$$\nabla \cdot \mathbf{V} = \partial_i V^i + V^m \Gamma^i_{im} = \partial_i V^i + V^m \frac{1}{\sqrt{g}}\partial_m \sqrt{g} = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}V^i\right). \tag{3.5}$$

That is

$$\nabla \cdot \mathbf{V} = \sum_i \frac{1}{A_r A_\theta A_\phi}\partial_i\left(\frac{A_r A_\theta A_\phi}{A_i}V\langle i\rangle\right). \tag{3.6}$$

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V\langle r\rangle\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta V\langle\theta\rangle\right) + \frac{1}{r\sin\theta}\frac{\partial V\langle\phi\rangle}{\partial \phi}. \tag{3.7}$$

(3)

$$\nabla \times \mathbf{V} = \epsilon^{ijk}\nabla_i V_j \mathbf{g}_k = \epsilon^{ijk}\left(\partial_i V_j - V_m\Gamma^m_{ij}\right)\mathbf{g}^k = \epsilon^{ijk}\partial_i V_j \mathbf{g}^k. \tag{3.8}$$

$$\nabla \times \mathbf{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ V\langle r\rangle & rV\langle\theta\rangle & r\sin\theta V\langle\phi\rangle \end{vmatrix}. \tag{3.9}$$

(4) By (1) and (2),

$$\nabla^2 f = \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}. \tag{3.10}$$

Problem 4 (Angular momentum operators)
(1)

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} := -i\hbar\mathbf{r}\times\nabla = \frac{\partial}{\partial\theta}\hat{\phi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\hat{\theta}. \tag{4.1}$$

Then, project on $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, we obtain

$$l_z = -i\hbar\frac{\partial}{\partial\phi}, \tag{4.2}$$

$$l_x = -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \tag{4.3}$$

$$l_y = -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right). \tag{4.4}$$

(2)

$$l_x = \left(l_+ + l_-\right)/2, \quad l_y = \left(l_+ - l_-\right)/2i, \tag{4.5}$$

$$l^2 = l_x^2 + l_y^2 + l_z^2 = l_z^2 + l_+l_- + l_-l_+. \tag{4.6}$$

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By (4.3) and (4.4),

$$l^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]. \tag{4.7}$$

(3) To avoid confusion, we use $\mathbf{L}$ and $\mathbf{x}$ to represent $\mathbf{l}$ and $\mathbf{r}$ respectively.

$$
\begin{aligned}
\mathbf{L}^2 &= \sum_{ijlmk} \varepsilon_{ijk} x_i p_j \varepsilon_{lmk} x_l p_m \\
&= \sum_{ijlm} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x_i p_j x_l p_m \\
&= \sum_{ijlm} [\delta_{il}\delta_{jm} x_i (x_l p_j - i\hbar\delta_{jl}) p_m - \delta_{im}\delta_{jl} x_i p_j (p_m x_l + i\hbar\delta_{lm})] \\
&= \mathbf{x}^2\mathbf{p}^2 - i\hbar\mathbf{x}\cdot\mathbf{p} - \sum_{ijlm} \delta_{im}\delta_{jl}[x_i p_m (x_l p_j - i\hbar\delta_{jl}) + i\hbar\delta_{lm} x_i p_j] \\
&= \mathbf{x}^2\mathbf{p}^2 - (\mathbf{x}\cdot\mathbf{p})^2 + i\hbar\mathbf{x}\cdot\mathbf{p}.
\end{aligned}
\tag{4.8}
$$

That is

$$-\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} \right) + \frac{\mathbf{L}^2}{2mr^2}. \tag{4.9}$$

Problem 5 (Associated Legendre Polynomials)

(1)

$$\frac{\mathrm{d}}{\mathrm{d}z} P^{|m|}(z) = \left(1 - z^2\right)^{\frac{|m|}{2} - 1} \left[-z|m|G(z) + \left(1 - z^2\right) G'(z) \right]. \tag{5.1}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left[(1 - z^2)\frac{\mathrm{d}}{\mathrm{d}z} P^{|m|} \right] &= \left(1 - z^2\right)^{\frac{|m|}{2} - 1} \left(\left(z^2 - 1\right) |m| \left(2zG'(z) + G(z)\right) \right. \\
&\quad \left. + z^2|m|^2 G(z) + \left(z^2 - 1\right) \left(\left(z^2 - 1\right) G''(z) + 2zG'(z) \right) \right).
\end{aligned}
\tag{5.2}
$$

Hence,

$$(1 - z^2)G'' - 2(|m| + 1)zG' + [\beta - |m|(|m| + 1)] G = 0. \tag{5.3}$$

(2)

$$(1 - z^2)G'' = \sum_{z=2}^{+\infty} n(n-1)a_n \left(z^{n-2} - z^n \right), \tag{5.4}$$

$$zG' = \sum_{n=1}^{+\infty} n a_n z^n. \tag{5.5}$$

Thus,

$$
\begin{aligned}
\sum_{n=0}^{+\infty} \{ (n+2)(n+1)a_{n+2} - n(n+1)a_n - 2(|m|+1)na_n \\
+ [\beta - |m|(|m|+1)] a_n \} z^n = 0.
\end{aligned}
\tag{5.6}
$$

Therefore,

$$a_{n+2} = \frac{(n + |m|)(n + |m| + 1) - \beta}{(n+1)(n+2)} a_n. \tag{5.7}$$

(3) If $\beta = l(l+1)$, then, when $n + |m| \geq l$, $a_{n+2} = 0$. So G becomes a polynomial.

Problem 6 (Generation function of Legendre Polynomials)

(1) One has

$$\frac{\partial T}{\partial t} = \sum_{n=0}^{+\infty} l P_l(z) t^{l-1}. \tag{6.1}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right) = (z-t)\left(1-2z+t^2\right)^{\frac{3}{2}} = \frac{z-t}{1-2zt+t^2}T. \tag{6.2}$$

Compare the coefficients of t^l, we obtain,

$$(l+1)P_{l+1}(z) - (2l+1)zP_l(z) + lP_{l-1}(z) = 0. \tag{6.3}$$

(2)

$$\frac{\partial T}{\partial z} = \sum_{n=0}^{+\infty} P_l'(z)t^l. \tag{6.4}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right) = \frac{t}{1-2zt+t^2}T. \tag{6.5}$$

Hence,

$$P_{l+1}' - 2zP_l' + P_{l-1}' = P_l. \tag{6.6}$$

(3) $l\times$(6.6)$-\frac{\mathrm{d}}{\mathrm{d}z}$ (6.3):

$$zP_l' - P_{l-1}' = lP_l. \tag{6.7}$$

(6.6)+(6.7):

$$P_{l+1}' - zP_l' = (l+1)P_l. \tag{6.8}$$

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