

QM HW6

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Problem 1 (Current, gauge transformation)

(1) Use the substitution:

$$-i\hbar\nabla \longrightarrow -i\hbar\nabla - \frac{q\vec{A}}{c}, \quad (1.1)$$

the probability current will be:

$$\vec{j} = \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi) \longrightarrow \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi) - \frac{q}{mc} \vec{A} |\psi|^2. \quad (1.2)$$

Hence,

$$\boxed{\vec{j} = \frac{\hbar}{m} \text{Im}(\psi^* \nabla \psi) - \frac{q}{mc} \vec{A} |\psi|^2.} \quad (1.3)$$

(2)

$$A'_\mu = A_\mu + \partial_\mu f. \quad (1.4)$$

Since the commutator is antisymmetric and the derivative is commutative,

$$\partial_{[\mu} \partial_{\nu]} f = 0. \quad (1.5)$$

Thus,

$$F'_{\mu\nu} = \partial_{[\mu} A'_{\nu]} = \partial_{[\mu} A_{\nu]} + \partial_{[\mu} \partial_{\nu]} f = F_{\mu\nu}. \quad (1.6)$$

So,

$$\mathbf{E}' = \mathbf{E}, \quad \mathbf{B}' = \mathbf{B}. \quad (1.7)$$

(3)

$$\frac{\partial \psi'}{\partial t} = i \frac{\partial \varphi}{\partial t} \psi' + e^{i\varphi} \frac{\partial \psi}{\partial t} \quad (1.8)$$

Cancelate $e^{i\varphi}$, and plug in Schrödinger equation, we should take

$$\boxed{\varphi = \frac{q}{\hbar c} f.} \quad (1.9)$$

(4)

$$\rho' = e^{i\varphi} \psi e^{-i\varphi} \psi^* = \psi \psi^* = \rho. \quad (1.10)$$

If we let

$$\psi = \sqrt{\rho} e^{\frac{iS}{\hbar}}, \quad (1.11)$$

then,

$$\mathbf{j} = \frac{\rho}{m} \left(\nabla S - \frac{q\mathbf{A}}{c} \right). \quad (1.12)$$

$$S' = S + \hbar\varphi, \quad \nabla S' - \frac{q\mathbf{A}'}{c} = \nabla S - \frac{q\mathbf{A}}{c}. \quad (1.13)$$

So the probability current is invariant under the gauge transformation.

Problem 2 (Landau gauge)

(1)

$$\left(p_x - \frac{qBy}{c} \right)^2 e^{ik_x x} = \left(\hbar k_x - \frac{qBy}{c} \right)^2 e^{ik_x x}. \quad (2.1)$$

$$\left[\frac{\left(\hbar k_x - \frac{qBy}{c} \right)^2}{2m} + (V(y) - E) \right] \phi_{k_x}(y) = \frac{\hbar^2}{2m} \phi_{k_x}''(y). \quad (2.2)$$

We can define $H_y(k_x)$ as

$$H_y(k_x) = \frac{\left(\hbar k_x - \frac{qBy}{c} \right)^2}{2m} + \frac{p_y^2}{2m} + V(y). \quad (2.3)$$

(2) Note that

$$\frac{\partial H_y(k_x)}{\partial k_x} = \frac{\hbar k_x - qBy/c}{m}. \quad (2.4)$$

$$I_x(x, k_x) = qJ = \frac{q\rho}{m} \left(\hbar k_x - \frac{qBy}{c} \right) = \frac{q}{L_x} \frac{\partial E_n}{\hbar \partial k_x}. \quad (2.5)$$

(3)

$$I_{x,n} = \frac{1}{2\pi} \int dk_x I_x(n, k_x) = \frac{q^2}{h} \Delta V / L_x. \quad (2.6)$$

$$\sigma_{xy} = \frac{I_x}{E_y} = \frac{q^2}{h} \nu. \quad (2.7)$$

This result is not related to V_{imp} .

Problem 3 (Spherical coordinates)

I will use Einstein summation convention if there's neither special announcement nor summation symbol.

In spherical coordinates, Lamé coefficients are

$$A_r = 1, \quad A_\theta = r, \quad A_\phi = r \sin \theta. \quad (3.1)$$

(1)

$$\nabla f = \mathbf{g}^i \partial_i f = \sum_i \frac{1}{A_i} \mathbf{e}^i \partial_i f. \quad (3.2)$$

So,

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi. \quad (3.3)$$

(2) Note that

$$\frac{\partial \sqrt{g}}{\partial x^i} = \Gamma_{ji}^j \sqrt{g}. \quad (3.4)$$

We have,

$$\nabla \cdot \mathbf{V} = \partial_i V^i + V^m \Gamma_{im}^i = \partial_i V^i + V^m \frac{1}{\sqrt{g}} \partial_m \sqrt{g} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} V^i). \quad (3.5)$$

That is

$$\nabla \cdot \mathbf{V} = \sum_i \frac{1}{A_r A_\theta A_\phi} \partial_i \left(\frac{A_r A_\theta A_\phi}{A_i} V^{\langle i \rangle} \right). \quad (3.6)$$

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V^{\langle r \rangle}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta V^{\langle \theta \rangle}) + \frac{1}{r \sin \theta} \frac{\partial V^{\langle \phi \rangle}}{\partial \phi}. \quad (3.7)$$

(3)

$$\nabla \times \mathbf{V} = \epsilon^{ijk} \nabla_i V_j \mathbf{g}_k = \epsilon^{ijk} (\partial_i V_j - V_m \Gamma_{ij}^m) \mathbf{g}^k = \epsilon^{ijk} \partial_i V_j \mathbf{g}^k. \quad (3.8)$$

$$\nabla \times \mathbf{V} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \mathbf{e}_r & r \mathbf{e}_\theta & r \sin \theta \mathbf{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ V^{\langle r \rangle} & r V^{\langle \theta \rangle} & r \sin \theta V^{\langle \phi \rangle} \end{vmatrix}. \quad (3.9)$$

(4) By (1) and (2),

$$\nabla^2 f = \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \quad (3.10)$$

Problem 4 (Angular momentum operators)

(1)

$$\mathbf{l} = \mathbf{r} \times \mathbf{p} := -i\hbar \mathbf{r} \times \nabla = \frac{\partial}{\partial \theta} \hat{\phi} - \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \hat{\theta}. \quad (4.1)$$

Then, project on $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, we obtain

$$l_z = -i\hbar \frac{\partial}{\partial \phi}, \quad (4.2)$$

$$l_x = -i\hbar \left(-\sin \phi \frac{\partial}{\partial \theta} - \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right), \quad (4.3)$$

$$l_y = -i\hbar \left(\cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right). \quad (4.4)$$

(2)

$$l_x = (l_+ + l_-) / 2, \quad l_y = (l_+ - l_-) / 2i, \quad (4.5)$$

$$l^2 = l_x^2 + l_y^2 + l_z^2 = l_z^2 + l_+ l_- + l_- l_+. \quad (4.6)$$

By (4.3) and (4.4),

$$l^2 = -\hbar^2 \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]. \quad (4.7)$$

(3) To avoid confusion, we use \mathbf{L} and \mathbf{x} to represent \mathbf{l} and \mathbf{r} respectively.

$$\begin{aligned} \mathbf{L}^2 &= \sum_{ijklmk} \varepsilon_{ijk} x_i p_j \varepsilon_{lmk} x_l p_m \\ &= \sum_{ijlm} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) x_i p_j x_l p_m \\ &= \sum_{ijlm} [\delta_{il} \delta_{jm} x_i (x_l p_j - i\hbar \delta_{jl}) p_m - \delta_{im} \delta_{jl} x_i p_j (p_m x_l + i\hbar \delta_{lm})] \\ &= \mathbf{x}^2 \mathbf{p}^2 - i\hbar \mathbf{x} \cdot \mathbf{p} - \sum_{ijlm} \delta_{im} \delta_{jl} [x_i p_m (x_l p_j - i\hbar \delta_{jl}) + i\hbar \delta_{lm} x_i p_j] \\ &= \mathbf{x}^2 \mathbf{p}^2 - (\mathbf{x} \cdot \mathbf{p})^2 + i\hbar \mathbf{x} \cdot \mathbf{p}. \end{aligned} \quad (4.8)$$

That is

$$-\frac{\hbar^2}{2m} \nabla^2 = -\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{\mathbf{L}^2}{2mr^2}. \quad (4.9)$$

Problem 5 (Associated Legendre Polynomials)

(1)

$$\frac{d}{dz} P^{|m|}(z) = (1 - z^2)^{\frac{|m|}{2}-1} [-z|m|G(z) + (1 - z^2)G'(z)]. \quad (5.1)$$

$$\begin{aligned} \frac{d}{dz} \left[(1 - z^2) \frac{d}{dz} P^{|m|} \right] &= (1 - z^2)^{\frac{|m|}{2}-1} ((z^2 - 1)|m|(2zG'(z) + G(z)) \\ &\quad + z^2|m|^2G(z) + (z^2 - 1)((z^2 - 1)G''(z) + 2zG'(z))). \end{aligned} \quad (5.2)$$

Hence,

$$(1 - z^2)G'' - 2(|m| + 1)zG' + [\beta - |m|(|m| + 1)]G = 0. \quad (5.3)$$

(2)

$$(1 - z^2)G'' = \sum_{n=2}^{+\infty} n(n-1)a_n (z^{n-2} - z^n), \quad (5.4)$$

$$zG' = \sum_{n=1}^{+\infty} n a_n z^n. \quad (5.5)$$

Thus,

$$\begin{aligned} \sum_{n=0}^{+\infty} \{ (n+2)(n+1)a_{n+2} - n(n+1)a_n - 2(|m|+1)na_n \\ + [\beta - |m|(|m|+1)]a_n \} z^n = 0. \end{aligned} \quad (5.6)$$

Therefore,

$$a_{n+2} = \frac{(n+|m|)(n+|m|+1) - \beta}{(n+1)(n+2)} a_n. \quad (5.7)$$

(3) If $\beta = l(l+1)$, then, when $n+|m| \geq l$, $a_{n+2} = 0$. So G becomes a polynomial.

Problem 6 (Generation function of Legendre Polynomials)

(1) One has

$$\frac{\partial T}{\partial t} = \sum_{n=0}^{+\infty} l P_l(z) t^{l-1}. \quad (6.1)$$

$$\frac{\partial}{\partial t} \left(\frac{1}{\sqrt{1-2tz+t^2}} \right) = (z-t)(1-2z+t^2)^{-\frac{3}{2}} = \frac{z-t}{1-2zt+t^2} T. \quad (6.2)$$

Compare the coefficients of t^l , we obtain,

$$(l+1)P_{l+1}(z) - (2l+1)zP_l(z) + lP_{l-1}(z) = 0. \quad (6.3)$$

(2)

$$\frac{\partial T}{\partial z} = \sum_{n=0}^{+\infty} P'_l(z) t^l. \quad (6.4)$$

$$\frac{\partial}{\partial z} \left(\frac{1}{\sqrt{1-2tz+t^2}} \right) = \frac{t}{1-2zt+t^2} T. \quad (6.5)$$

Hence,

$$P'_{l+1} - 2zP'_l + P'_{l-1} = P_l. \quad (6.6)$$

(3) $l \times (6.6) - \frac{d}{dz} (6.3)$:

$$zP'_l - P'_{l-1} = lP_l. \quad (6.7)$$

(6.6)+(6.7):

$$P'_{l+1} - zP'_l = (l+1)P_l. \quad (6.8)$$

(4) (6.7): $l \rightarrow l+1$, (6.8) $\times z$:

$$(z^2 - 1)P'_l = (l+1)(P_{l+1} - zP_l). \quad (6.9)$$

So,

$$\frac{d}{dz} \left[(1-z^2) \frac{d}{dz} P_l \right] = (l+1) \frac{d}{dz} (zP_l - P_{l+1}) = (l+1)(P_l + zP'_l - P'_{l+1}). \quad (6.10)$$

Plug in (6.8), we obtain

$$\frac{d}{dz} \left[(1-z^2) \frac{dP_l(z)}{dz} \right] + l(l+1)P_l(z) = 0 \quad (6.11)$$

By (6.11)

$$\frac{d}{dz} \left[(1-z^2) \left(P_n \frac{dP_m}{dz} - P_m \frac{dP_n}{dz} \right) \right] + [m(m+1) - n(n+1)] P_m P_n = 0 \quad (6.12)$$

Since $[1-z^2]_{z=\pm 1} = 0$, the first term in (6.12) will be zero after integrating. So the second term must be zero after integrating if $n \neq m$, exact,

$$\int_{-1}^1 P_n(z) P_m(z) dz = 0, \quad n \neq m. \quad (6.13)$$

(5) By (6.3),

$$lP_l - (2l-1)zP_{l-1} + (l-1)P_{l-2} = 0, \quad zP_l = \frac{(l+1)P_l + lP_{l-1}}{2l+1}. \quad (6.14)$$

Multiplying both sides by P_l , and integrate, we obtain,

$$l \int_{-1}^1 P_l^2 dz = (2l-1) \int_{-1}^1 zP_{l-1}P_l dz = \frac{l(2l-1)}{2l+1} \int_{-1}^1 P_{l-1}^2 dz. \quad (6.15)$$

Thus,

$$(2l+1) \int_{-1}^1 P_l^2(z) dz = (2(l-1)+1) \int_{-1}^1 P_{l-1}^2(z) dz = \int_{-1}^1 P_0^2(z) dz = 2. \quad (6.16)$$

Therefore,

$$\int_{-1}^1 P_l^2(z) dz = \frac{2}{2l+1}. \quad (6.17)$$

Problem 7 (Associated Legendre Polynomials)

(1)

$$\frac{d}{dz} P_l^{|m|} = (1-z^2)^{\frac{|m|-1}{2}} \left[(1-z^2) \frac{d^{|m|+1}}{dz^{|m|+1}} P_l - z|m| \frac{d^{|m|}}{dz^{|m|}} P_l \right]. \quad (7.1)$$

$$\frac{d^{|m|}}{dz^{|m|}} \left[(1-z^2) \frac{dP_l}{dz} \right] = (1-z^2) \frac{d^{|m|}}{dz^{|m|}} P_l - 2|m|z \frac{d^{|m|-1}}{dz^{|m|-1}} \frac{dP_l}{dz} - |m|(|m|-1) \frac{d^{|m|-2}}{dz^{|m|-2}} \frac{dP_l}{dz}. \quad (7.2)$$

By (6.11),

$$-l(l+1) \frac{d^{|m|}}{dz^{|m|}} P_l = \frac{d}{dz} \frac{d^{|m|}}{dz^{|m|}} \left[(1-z^2) \frac{dP_l}{dz} \right]. \quad (7.3)$$

So,

$$\frac{d}{dz} \left[(1-z^2) \frac{d}{dz} P_l^{|m|}(z) \right] + \left[l(l+1) - \frac{m^2}{1-z^2} \right] P_l^{|m|}(z) = 0. \quad (7.4)$$

(2)

$$\frac{d}{dz} \left[(1-z^2) \left(\frac{d}{dz} P_l^{|m|} P_{l'}^{|m|} - P_l^{|m|} \frac{d}{dz} P_{l'}^{|m|} \right) \right] + [l(l+1) - l'(l'+1)] P_l^{|m|} P_{l'}^{|m|} = 0. \quad (7.5)$$

So,

$$\int_{-1}^1 P_l^{|m|} P_{l'}^{|m|} dz = 0. \quad (7.6)$$

Problem 8 (Laguerre polynomials)

(3) One has

$$\frac{\partial U}{\partial u} = \sum_{m=0}^{+\infty} \frac{L_{m+1}(\xi)}{m!} u^m. \quad (8.1)$$

$$\frac{\partial}{\partial u} \left[\frac{1}{1-u} e^{-\frac{\xi u}{1-u}} \right] = \frac{1}{1-u} e^{-\frac{\xi u}{1-u}} \frac{\partial}{\partial u} \ln \left[\frac{1}{1-u} e^{-\frac{\xi u}{1-u}} \right] = U \frac{1-u-\xi}{(1-u)^2}. \quad (8.2)$$

Hence,

$$L_{m+1}(\xi) + (\xi - 2m - 1) L_m(\xi) + m^2 L_{m-1}(\xi) = 0. \quad (8.3)$$

$$\frac{\partial U}{\partial \xi} = U \frac{\partial}{\partial \xi} \ln U = U \frac{-u}{1-u}, \quad (8.4)$$

so,

$$L'_m(\xi) - m L'_{m-1}(\xi) + m L_{m-1}(\xi) = 0. \quad (8.5)$$

(4) Derivative of (8.3) shows that

$$L'_{m+1} + L_m + (\xi - 1 - 2m) L'_m + m^2 L'_{m-1} = 0. \quad (8.6)$$

By (8.5) and (8.3)

$$L'_{m+1} - L_{m+1} + (2m + 2 - \xi) L_m + (\xi - 1 - m) L'_m = 0, \quad (8.7)$$

$$L''_{m+1} - L'_{m+1} + L'_m(2m + 2 - \xi) + (\xi - 1 - m) L''_m = 0. \quad (8.8)$$

Plug in (8.5) after $m \rightarrow m + 1$:

$$\xi L''_m + (1 - \xi) L'_m + (m + 1) L_m = 0. \quad (8.9)$$

(5)

$$\frac{d^s}{d\xi^s} [\xi L''_m(\xi)] = s \frac{d^{s-1}}{d\xi^{s-1}} [L''_m(\xi)] + \xi \frac{d^s}{d\xi^s} [\xi L''_m(\xi)]. \quad (8.10)$$

$$\frac{d^s}{d\xi^s} [(1 - \xi) L'_m(\xi)] = -s \frac{d^{s-1}}{d\xi^{s-1}} [L'_m(\xi)] + (1 - \xi) \frac{d^s}{d\xi^s} [\xi L'_m(\xi)]. \quad (8.11)$$

Therefore,

$$\xi L_m^{s''}(\xi) + (s + 1 - \xi) L_m^{s'}(\xi) + (m - s) L_m^s(\xi) = 0. \quad (8.12)$$