QM HW1

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Problem 1 (Canonical Transformation)

(1)

$$d\Phi(q, P, t) = pdq + QdP + (H' - H)dt$$
(1.1)

$$\frac{\partial \Phi}{\partial a} = p, \frac{\partial \Phi}{\partial P} = Q, \frac{\partial \Phi}{\partial t} + H = H'. \tag{1.2}$$

(2) We have proved:

$$\frac{\partial S}{\partial q_t} = -p_t, \frac{\partial S}{\partial q_{t+\tau}} = p_{t+\tau}, \tag{1.3}$$

First, we can check:

$$\frac{\partial \Psi}{\partial q_t} = -\frac{\partial S}{\partial q_t} = p_t \tag{1.4}$$

Second ,by chain rule, we obtain

$$\frac{\partial \Psi}{\partial p_{t+\tau}} = q_{t+\tau} + \left(p_{t+\tau} - \frac{\partial S}{\partial p_{t+\tau}} \right) \frac{\partial q_{t+\tau}}{\partial p_{t+\tau}} = q_{t+\tau} \tag{1.5}$$

(3)

$$dp_t \wedge dq_t = dp_{t+\tau} \wedge dq_{t+\tau}, \frac{\partial (q_{t+\tau}, p_{t+\tau})}{\partial (q_t, p_t)} = 1$$
(1.6)

Problem 2 (Hamilton-Jacobi equation)

(1)

$$\frac{\partial S}{\partial t} = \frac{\mathrm{d}S}{\mathrm{d}t} - \frac{\partial S}{\partial q}\dot{q} = L - p\dot{q} = -H \tag{2.1}$$

(2)

$$\beta = \frac{\partial S}{\partial \alpha} \tag{2.2}$$

$$H' = H + \frac{\partial S}{\partial t} = 0 \tag{2.3}$$

Remark. Another way to calculate $\frac{\partial S}{\partial q_f}$ and $\frac{\partial S}{\partial t_f}$.

$$q = q(q_f, t_f, t), dq = \frac{\partial q}{\partial q_f} dq_f + \frac{\partial q}{\partial t_f} dt_f + \frac{\partial q}{\partial t} dt$$
 (2.4)

We have $q|_{t=t_0} = q_0, q|_{t=t_f}$, hence,

$$\left(\frac{\partial q}{\partial q_f}\right)_{t_f, t}\Big|_{t=t_0} = 0, \left(\frac{\partial q}{\partial q_f}\right)_{t_f, t}\Big|_{t=t_f} = 1$$
(2.5)

Now,let $t \equiv t_0, dq = dq_0 = 0$. Thus,

$$\left. \frac{\partial q}{\partial t_f} \right|_{t=t_0} = 0 \tag{2.6}$$

Let t be a function of t_f and take the form $t = t_f.dq = dq_f$,so

$$\left(\frac{\partial q}{\partial t_f} + \frac{\partial q}{\partial t}\right) dt_f = 0 \tag{2.7}$$

$$\left. \frac{\partial q}{\partial t_f} \right|_{t=t_f} = -\left. \frac{\partial q}{\partial t} \right|_{t=t_f} = -\dot{q}|_{t=t_f} \tag{2.8}$$

$$\frac{\partial S}{\partial q_f} = \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \dot{q}} \frac{\partial}{\partial q_f} \left(\frac{\partial q}{\partial t} \right) + \frac{\partial L}{\partial q} \frac{\partial q}{\partial q_f} \right] dt$$

$$= \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \dot{q}} \frac{\partial}{\partial t} \left(\frac{\partial q}{\partial q_f} \right) + \frac{\partial L}{\partial q} \frac{\partial q}{\partial q_f} \right] dt$$

$$= \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial q_f} \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial q} - \frac{\partial}{\partial t} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \frac{\partial q}{\partial q_f} dt$$

$$= p(t_f)$$
(2.9)

$$\frac{\partial S}{\partial t_f} = L(t_f) + \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial \dot{q}} \frac{\partial}{\partial t_f} \left(\frac{\partial q}{\partial t} \right) + \frac{\partial L}{\partial q} \frac{\partial q}{\partial t_f} \right] dt$$

$$= L(t_f) + p \frac{\partial q}{\partial t_f} \Big|_{t_0}^{t_f}$$

$$= L - p\dot{q} = -H$$
(2.10)

Problem 3 (Harmonic Oscillator)

(1) easy to check:

$$\frac{\partial S}{\partial x} = p, \frac{\partial S}{\partial t} = -E \tag{3.1}$$

 $\frac{\partial S}{\partial t_f} = -E \tag{3.2}$

$$\frac{\partial S}{\partial x_f} = p = \pm m\omega \sqrt{A^2 - x^2} \tag{3.3}$$

(3)
$$S = \pm m\omega \int \sqrt{\frac{2E}{m\omega^2} - x^2} \, dx - Et + \text{const.}$$
 (3.4)

$$\frac{\partial S}{\partial E} = \pm \frac{\arcsin(\sqrt{\frac{m\omega^2}{2E}}x)}{\omega} - t \tag{3.5}$$

New Hamitonian:

$$H' = H + \frac{\partial S}{\partial t} = 0 \tag{3.6}$$

By Hamilton equation

$$\dot{\beta} = \frac{\partial H'}{\partial E} = 0 \tag{3.7}$$

Therefore β is a constant. It means the initial phase of oscillator.

Problem 4 (Planck's derivation of black body radiation)

(1) $\omega \to +\infty$:

$$\ln u = -\frac{\hbar\omega}{k_B T} + \text{const.} \tag{4.1}$$

$$\frac{\partial(\ln u)}{\partial\left(\frac{1}{T}\right)} = -\frac{\hbar\omega}{k_B} \tag{4.2}$$

 $\omega \to 0$:

$$\frac{1}{u}\frac{\partial u}{\partial\left(\frac{1}{T}\right)} = -T = -\frac{u}{k_B} \tag{4.3}$$

We assume that

$$\frac{\partial (\ln u)}{\partial \left(\frac{1}{T}\right)} = -\frac{\hbar \omega + u}{k_B} \tag{4.4}$$

That leads to

$$\frac{u}{\hbar\omega + u} = Ce^{-\frac{\hbar\omega}{k_BT}} \tag{4.5}$$

By the expression at low frequency, we can deduce that ${\cal C}=1.$

$$u = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_BT}} - 1} \tag{4.6}$$

Problem 5 (Heisenberg's magic)

Quantization condition:

$$\oint p \mathrm{d}q = \int_0^T m \dot{x}^2 \mathrm{d}t = nh$$
(5.1)

$$x_{n \leftarrow m}(t) = x_{n \leftarrow m} e^{i\omega_{n \leftarrow m}t} \tag{5.2}$$

$$\dot{x}_{n \leftarrow m}^{2}(t) = \sum_{k = -\infty}^{+\infty} x_{n \leftarrow k} x_{k \leftarrow m} \omega_{n \leftarrow k} \omega_{k \leftarrow m} e^{i\omega_{n \leftarrow m}t}$$
 (5.3)

Let m = n + l, we obtain f-sum rule

$$\sum_{l=0}^{+\infty} \left(\omega_{n \to n+l} |x_{n+l,n}|^2 - \omega_{n-l \to n} |x_{n,n-l}|^2 \right) = \frac{\hbar}{2m}$$
 (5.4)

In classic mechanics ,we have

$$\ddot{x} + \omega_0^2 x = 0 \tag{5.5}$$

Plug in $x_{n\pm l\leftarrow n}(t) = x_{n\pm l\leftarrow n}e^{i\omega_{n\pm l\leftarrow n}t}$

$$(\omega_0^2 - \omega_{n \pm l \leftarrow n}^2) x_{n \pm l \leftarrow n} = 0 \tag{5.6}$$

Only when $l = \pm 1, \omega_0^2 - \omega_{n \pm l \leftarrow n}^2 = 0, x_{n \pm l \leftarrow n} \neq 0$

$$\frac{\hbar}{2m} = \omega_0 \left(|x_{n+1,n}|^2 - |x_{n,n-1}|^2 \right)$$
 (5.7)

There must exists a lower bound , or $\left|x_{n,n-1}\right|^2 \geq 0$ can not hold. Thus,

$$x_{n+1,n} = \sqrt{\frac{\hbar}{2m\omega_0}(n+1)} \tag{5.8}$$

Therefore,

$$(\dot{x})_{nn}^2 = (n+1/2)\frac{\hbar\omega_0}{m} \tag{5.9}$$