Dirac Delta Function

Definition 1.
$\delta(x)$ is a generalized function satisfies:

$$\delta(x) := \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}, \tag{1}$$

$$\int_{-\epsilon}^{+\epsilon} \delta(x)\,\mathrm{d}x = 1, \quad \epsilon > 0. \tag{2}$$

All the equalities should be understand under integration.

Property 1.

$$\delta(-x) = \delta(x). \tag{3}$$

Proof.

$$\delta(-x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}, \tag{4}$$

$$\int_{-\epsilon}^{+\epsilon} \delta(-x)\,\mathrm{d}x = -\int_{+\epsilon}^{-\epsilon} \delta(-x)\ \mathrm{d}(-x) = \int_{-\epsilon}^{+\epsilon} \delta(x)\,\mathrm{d}x, \quad \epsilon > 0. \tag{5}$$

Property 2.

$$\delta(ax) = \frac{1}{|a|}\delta(x) \quad (a \neq 0). \tag{6}$$

Proof.

$$\delta(ax) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases} = \frac{\delta(x)}{|a|}, \tag{7}$$

$$\int_{-\epsilon}^{+\epsilon} \delta(ax)\,\mathrm{d}x = \begin{cases} \frac{1}{a}\int_{-\epsilon}^{+\epsilon} \delta(ax)\ \mathrm{d}(ax)\,,\ a > 0 \\ \frac{1}{a}\int_{+\epsilon}^{-\epsilon} \delta(ax)\ \mathrm{d}(ax)\,,\ a < 0 \end{cases} = \frac{1}{|a|}\int_{-\epsilon}^{+\epsilon} \delta(x)\,\mathrm{d}x. \tag{8}$$

Property 3.

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{9}$$

Property 4.

$$\int \delta(x-y)\delta(y-a)\,\mathrm{d}y = \delta(x-a). \tag{10}$$

Property 5.

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}\,\mathrm{d}k. \tag{11}$$

Proof.
Add a convergence factor to soft cutoff the divergent integration.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}e^{-\epsilon k^2}\,\mathrm{d}k = \frac{e^{-\frac{x^2}{4\epsilon}}}{2\sqrt{\pi\epsilon}}. \tag{12}$$

Then check it like above proof.

Property 6.

$$\delta[g(x)] = \sum_n \frac{\delta(x-x_n)}{|g'(x_n)|},\ g(x_n)=0, g'(x_n)\neq 0. \tag{13}$$

Proof.
Around $x = x_n$,

$$g(x) = g(x_n) + g'(x_n)(x-x_n) = g'(x_n)(x-x_n) \tag{14}$$

Thus,

$$\delta[g(x)] = \delta(\sum_n g'(x_n)(x-x_n)) = \sum_n \delta[g'(x_n)(x-x_n)]. \tag{15}$$

By Property 2,

$$\delta[g(x)] = \sum_n \frac{1}{|g'(x_n)|}\delta(x-x_n),\ g(x_n)=0, g'(x_n)\neq 0. \tag{16}$$

For those $g(x_n)=0, g'(x_n)=0$, take $2 \leq m = \min(\{m\in\mathbb{N} \mid g^m(x_n)\neq 0\})$. Then we induct on m that

$$\delta[g^m(x_n)(x-x_n)^m] = 0. \tag{17}$$

For $m=2$,

$$\delta[g''(x_n)(x-x_n)^2] = \tag{18}$$