Problem 4. Angular momentum operators

1) Prove that in the spherical coordinates,

$$\begin{aligned} l_x &= -i\hbar(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}) \\ l_y &= -i\hbar(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}) \\ l_z &= -i\hbar\frac{\partial}{\partial\phi}. \end{aligned} \tag{9}$$

2) Define $l_+ = l_x + il_y$, and $l_- = l_x - il_y$. Prove that $l^2 = l_z^2 + \frac{1}{2}(l_+l_- + l_-l_+)$, and

$$l^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \tag{10}$$

3) Prove that $l^2 = r^2p^2 - (\mathbf{r}\cdot\mathbf{p})^2 + i\hbar(\mathbf{r}\cdot\mathbf{p})$. Based on this relation prove that

$$-\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\mathbf{l}^2}{2mr^2}. \tag{11}$$

Problem 5. Associated Legendre Polynomials $P_l^{|m|}$

Set $\cos\theta = z$, consider the following equation

$$\frac{d}{dz}\left\{(1-z^2)\frac{d}{dz}P^{|m|}(z)\right\} + \left\{\beta - \frac{m^2}{1-z^2}\right\}P^{|m|}(z) = 0 \tag{12}$$

To remove the singular point at $z = \pm 1$, define $P(z) = (1-z^2)^{\frac{|m|}{2}}G(z)$.

1) Prove that the differential equation changes to

$$(1-z^2)G'' - 2(|m|+1)zG' + \{\beta - |m|(|m|+1)\}G = 0. \tag{13}$$

2) Plug in $G = \sum_{n=0}^{\infty} a_n z^n$ in the above equation. Derive the recursion formula

$$a_{\nu+2} = \frac{(\nu+|m|)(\nu+|m|+1)-\beta}{(\nu+1)(\nu+2)}a_\nu. \tag{14}$$

3) Show that when $\beta = l(l+1)$, we arrive at polynomial solutions.

Problem 6. Generation function of Legendre Polynomials

Define the generation function of Legendre polynomials $T(t,z) = \sum_{l=0}^{\infty} P_l(z)t^l = \frac{1}{\sqrt{1-2tz+t^2}}$.

1) Calculate $\frac{\partial T}{\partial t}$, and then prove that $(1-2zt+t^2)\sum_l lP_l t^{l-1} = (z-t)\sum_l P_l t^l$. Prove that

$$(l+1)P_{l+1}(z) - (2l+1)zP_l(z) + lP_{l-1}(z) = 0. \tag{15}$$

2) Calculate $\frac{\partial T}{\partial z}$. Prove that

$$\tfrac{d}{dz}P_{l+1}(z) - 2z\tfrac{d}{dz}P_l(z) + \tfrac{d}{dz}P_{l-1}(z) = P_l(z). \tag{16}$$

3) Prove that

$$\begin{aligned} z\tfrac{d}{dz}P_l(z) - \tfrac{d}{dz}P_{l-1}(z) &= lP_l(z). \\ \tfrac{d}{dz}P_{l+1}(z) - z\tfrac{d}{dz}P_l(z) &= (l+1)P_l(z) \end{aligned} \tag{17}$$

4) Prove that

$$\tfrac{d}{dz}\left\{(1-z^2)\tfrac{d}{dz}P_l(z)\right\} + l(l+1)P_l(z) = 0. \tag{18}$$

5) Prove that if $l \neq l'$,

$$\int_{-1}^{+1} P_{l'}(z)P_l(z) = 0. \tag{19}$$

(Hint: Multiply $P_{l'}(z)$ to the equation in 1).)

6) Based on the results in 1), prove that

$$\int_{-1}^{+1} dz(P_l(z))^2 = \tfrac{2l-1}{2l+1}\int_{-1}^{+1} dz(P_{l-1}(z))^2 \tag{20}$$

And finally

$$\int_{-1}^{+1} dz(P_l(z))^2 = \tfrac{2}{2l+1}. \tag{21}$$

Problem 7. Associated Legendre Polynomials

Define the Associated Legendre polynomial

$$P_l^{|m|}(z) = (1-z^2)^{|m|/2}\tfrac{d^{|m|}}{dz^{|m|}}P_l(z). \tag{22}$$

1) Prove that $P_l^{|m|}(z)$ satisfies

$$\tfrac{d}{dz}\left\{(1-z^2)\tfrac{d}{dz}P_l^{|m|}(z)\right\} + \left\{l(l+1) - \tfrac{m^2}{1-z^2}\right\}P_l^{|m|}(z) = 0. \tag{23}$$

2) Prove that if $l \neq l'$,

$$\int_{-1}^{+1} P_{l'}^{|m|}(z)P_l^{|m|}(z) = 0. \tag{24}$$

(Hint: Multiply $P_{l'}^{|m|}(z)$ to the equation in 1).)

3) Prove that

$$\int_{-1}^{+1} dz(P_l^{|m|+1}(z))^2 = (l-|m|)(l+|m|+1)\int_{-1}^{+1} dz(P_{l-1}^{|m|}(z))^2, \tag{25}$$

such that

$$\int_{-1}^{+1} dz(P_l^{|m|}(z))^2 = \tfrac{2}{2l+1}\tfrac{(l+|m|)!}{(l-|m|)!}. \tag{26}$$

(Hint: You can use the definition of $P_l^{|m|}$ and also the equation in 1)

4) Prove that

$$zP_l^{|m|}(z) = \tfrac{l+|m|}{2l+1}P_{l-1}^{|m|}(z) + \tfrac{l-|m|+1}{2l+1}P_{l-1}^{|m|}(z) \tag{27}$$

Problem 8. Laguerre polynomials

1) Consider the differential equation

$$\xi u'' + (2(l+1) - \xi) - ru'' + (\lambda - l - 1)u = 0 \tag{28}$$

Expand the expression of u as

$$u(\xi) = \sum_{\nu=0}^{+\infty} a_\nu \xi^\nu. \tag{29}$$

Plug it into the above equation and find the recursion relation between $a_{\nu+1}$ and a_ν. Set $a_0 = 1$, please find the expression of $u(\xi)$.

2) Please show that in the general case $u(\xi) \sim e^\xi$ as $\xi \to +\infty$. Please find that under what condition u can be truncated as a polynomial.

3) Define the generation function

$$U(\xi, u) = \sum_{m=0}^{+\infty} \tfrac{L_m(\xi)}{m!} u^m = \tfrac{1}{1-u} e^{-\frac{\xi u}{1-u}}. \tag{30}$$

Calculate $\frac{\partial U}{\partial u}$ based on the above equation, and prove that

$$L_{m+1}(\xi) + (\xi - 1 - 2m)L_m(\xi) + \xi^2 L_{m-1}(\xi) = 0. \tag{31}$$

Calculate $\frac{\partial U}{\partial \rho}$ based on the above equation, and prove that

$$L'_m(\xi) - mL'_{m-1}(\xi) + mL_{m-1}(\xi) = 0. \tag{32}$$

4) Prove that

$$\xi L''_m(\xi) + (1 - \xi)L'_m(\xi) + mL_m(\xi) = 0. \tag{33}$$

5) Define the associated Laguerre polynomials as $L^s_m(\xi) = \frac{d^s}{d\xi^s} L_m(\xi)$. Prove that

$$\xi L^{s,\prime\prime}_m(\xi) + (s + 1 - \xi)L^{s,\prime}_m(\xi) + (m - s)L^s_m(\xi) = 0. \tag{34}$$