QM HW11

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Problem 1
We need to solve the equation:

$$\begin{cases} i\frac{\partial}{\partial t}G(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,t) \\ G(x,0) = -i\delta(x). \end{cases} \tag{1.1}$$

Let the Laplace transform of $G(x,t)$ with respect to t be

$$\widetilde{G}(x,s) = \mathcal{L}\{G(x,t)\} = \int_0^\infty e^{-st}\,G(x,t)\,\mathrm{d}t\,, \tag{1.2}$$

where s is a complex parameter with $\mathrm{Re}(s) > 0$.

$$\mathcal{L}\left\{i\frac{\partial G}{\partial t}\right\} = i\big[s\widetilde{G}(x,s) - G(x,0)\big] = is\widetilde{G}(x,s) - \delta(x). \tag{1.3}$$

The right-hand side transforms as:

$$\mathcal{L}\left\{-\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2}\right\} = -\frac{\hbar^2}{2m}\frac{\partial^2\widetilde{G}(x,s)}{\partial x^2}. \tag{1.4}$$

Thus the transformed equation is:

$$\frac{\partial^2\widetilde{G}}{\partial x^2} - k^2\widetilde{G} = \frac{2m}{\hbar^2}\,\delta(x). \tag{1.5}$$

where $k = \sqrt{\frac{2mis}{\hbar^2}}$. We choose the branch such that $\mathrm{Re}(k) > 0$.

The solution decaying as $|x| \to \infty$ is:

$$\widetilde{G}(x,s) = -\frac{m}{\hbar^2 k}e^{-k|x|} = -\sqrt{\frac{m}{2i\hbar^2 s}}\,e^{-\sqrt{\frac{2mis}{\hbar^2}}\,|x|}. \tag{1.6}$$

We recognize the Laplace transform pair (from tables):

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}\,e^{-a^2/(4t)}, \quad a > 0. \tag{1.7}$$

So,

$$\boxed{G(x,t) = -e^{-i\pi/4}\sqrt{\frac{m}{2\pi\hbar^2 t}}\ \exp\left(\frac{imx^2}{2\hbar^2 t}\right)}. \tag{1.8}$$

Problem 2

We start from the definition of the propagator in quantum mechanics:

$$iG(x_b, t_b; x_a, t_a) = \langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle, \tag{2.1}$$

with $H = \frac{p^2}{2m}$ for a free particle.

Divide the time interval into N equal segments:

$$\Delta t = \frac{t_b - t_a}{N}, \quad t_j = t_a + j\Delta t, \quad j = 0, 1, \ldots, N, \tag{2.2}$$

where $t_0 = t_a,\ t_N = t_b$, and correspondingly $x_0 = x_a,\ x_N = x_b$.

Insert $N-1$ completeness relations $\int dx_j |x_j\rangle\langle x_j| = 1$:

$$iG(x_b, t_b; x_a, t_a) = \int dx_1 \ldots dx_{N-1} \prod_{j=1}^{N} \langle x_j | e^{-iH\Delta t} | x_{j-1} \rangle. \tag{2.3}$$

For small Δt, we use the approximation $e^{-iH\Delta t} \approx e^{-i\frac{p^2}{2m}\Delta t}$ (since $V = 0$):

$$\langle x_j | e^{-i\frac{p^2}{2m}\Delta t} | x_{j-1} \rangle = \int \frac{dp}{2\pi}\, e^{ip(x_j - x_{j-1})} e^{-i\frac{p^2}{2m}\Delta t}. \tag{2.4}$$

This is a Gaussian integral. Using the formula

$$\int_{-\infty}^{\infty} dp\, e^{-ap^2 + bp} = \sqrt{\frac{\pi}{a}}\, e^{b^2/(4a)}, \quad \mathrm{Re}(a) > 0,$$

with $a = \frac{i\Delta t}{2m},\ b = i(x_j - x_{j-1})$, we get:

$$\langle x_j | e^{-i\frac{p^2}{2m}\Delta t} | x_{j-1} \rangle = \sqrt{\frac{m}{2\pi i\Delta t}}\ \exp\left[i\frac{m}{2}\frac{(x_j - x_{j-1})^2}{\Delta t}\right]. \tag{2.5}$$

Thus the propagator becomes:

$$iG(x_b, t_b; x_a, t_a) = \left(\frac{m}{2\pi i\Delta t}\right)^{N/2} \int \prod_{j=1}^{N-1} dx_j\ \exp\left[i\sum_{j=1}^{N} \frac{m}{2}\frac{(x_j - x_{j-1})^2}{\Delta t}\right]. \tag{2.6}$$

We now separate the classical path from fluctuations. Let

$$x_j = x_{\mathrm{cl}}(t_j) + \delta x_j, \tag{2.7}$$

where the classical path for a free particle is linear:

$$x_{\mathrm{cl}}(t) = x_a + \frac{x_b - x_a}{t_b - t_a}(t - t_a). \tag{2.8}$$

The boundary conditions are $\delta x_0 = \delta x_N = 0$. The discretized action is:

$$S_N = \sum_{j=1}^{N} \frac{m}{2} \frac{(x_j - x_{j-1})^2}{\Delta t} = S_{\text{cl}} + S_{\text{fl}}, \tag{2.9}$$

where

$$S_{\text{cl}} = \frac{m}{2} \frac{(x_b - x_a)^2}{t_b - t_a}, \tag{2.10}$$

and the fluctuation part is

$$S_{\text{fl}} = \sum_{j=1}^{N} \frac{m}{2} \frac{(\delta x_j - \delta x_{j-1})^2}{\Delta t}. \tag{2.11}$$

The linear cross term vanishes because the classical path satisfies the equation of motion.

The propagator now factorizes:

$$iG(x_b, t_b; x_a, t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} e^{iS_{\text{cl}}} \int \prod_{j=1}^{N-1} d(\delta x_j)\, e^{iS_{\text{fl}}}. \tag{2.12}$$

The fluctuation integral is Gaussian:

$$\int \prod_{j=1}^{N-1} d(\delta x_j)\, \exp\left[i \frac{m}{2\Delta t} \sum_{j=1}^{N} (\delta x_j - \delta x_{j-1})^2\right] = \left(\frac{2\pi i \Delta t}{m}\right)^{(N-1)/2} \frac{1}{\sqrt{N}}. \tag{2.13}$$

This result follows from evaluating the determinant of the tridiagonal matrix

$$M_{jk} = \frac{m}{i\Delta t} \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}_{(N-1)\times(N-1)},$$

with $\det M = N \left(\frac{m}{i\Delta t}\right)^{N-1}$.

Putting everything together:

$$iG(x_b, t_b; x_a, t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \left(\frac{2\pi i \Delta t}{m}\right)^{(N-1)/2} \frac{1}{\sqrt{N}} e^{iS_{\text{cl}}}. \tag{2.14}$$

Using $N\Delta t = t_b - t_a$, we simplify:

$$iG(x_b, t_b; x_a, t_a) = \sqrt{\frac{m}{2\pi i (t_b - t_a)}}\, \exp\left[i \frac{m}{2} \frac{(x_b - x_a)^2}{t_b - t_a}\right]. \tag{2.15}$$

Thus the free propagator obtained via the path integral method is

$$\boxed{G(x_b, t_b; x_a, t_a) = -i\sqrt{\frac{m}{2\pi i(t_b - t_a)}}\ \exp\left[i\frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\right]}. \tag{2.16}$$

This matches the result from Problem 1 (solving the Schrödinger equation directly).

Problem 3

For infinitesimal ε, we have

$$e^{-\varepsilon H} = e^{-\frac{\varepsilon V(x)}{2}} e^{-\frac{\varepsilon p^2}{2m}} e^{-\frac{\varepsilon V(x)}{2}} + \mathcal{O}(\varepsilon^3). \tag{3.1}$$

$$\langle x_{k+1}|e^{-\varepsilon H}|x_k\rangle \approx \sqrt{\frac{m}{2\pi\varepsilon}} \exp\left[-\frac{m}{2\varepsilon}(x_{k+1} - x_k)^2 - \varepsilon V(x_k)\right]. \tag{3.2}$$

So,

$$G(x_N, x_0; \beta) = \left(\frac{m}{2\pi\varepsilon}\right)^{\frac{N}{2}} \int \prod_{k=1}^{N-1} dx_k\ \exp\left[-\sum_{k=0}^{N-1}\left(\frac{m}{2\varepsilon}(x_{k+1} - x_k)^2 + \varepsilon V(x_k)\right)\right]. \tag{3.3}$$

Taking the limit when $\varepsilon \to 0$, $N \to +\infty$, and let x_N to be same as x_0, we get

$$G(x_0, x_0; \beta) = \int_{x(0)=x_0}^{x(\beta)=x_0} \mathcal{D}x(\tau)\ e^{-S[x]}. \tag{3.4}$$

Where,

$$S[x] = \int_0^\beta \mathrm{d}t\ \left[\frac{m}{2}\dot{x}^2 + V(x)\right]. \tag{3.5}$$

Then, by definition,

$$Z(\beta) = \int \mathrm{d}x\ G(x, x; \beta) = \int \mathrm{d}x \int_{x(0)=x}^{x(\beta)=x} \mathcal{D}x(\tau)\ e^{-S[x]}. \tag{3.6}$$

Problem 4

The matrices of spin-1 operators are:

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad S_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}, \quad S_z = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix} \tag{4.1}$$

Let

$$|\hat{n}, m\rangle = e^{-i\phi S_z} e^{-i\theta S_y}\,|\hat{z}, m\rangle. \tag{4.2}$$

Then,

$$\begin{aligned}\nabla_{\hat{n}}\,|\hat{n}, m\rangle &= \left(\frac{\partial}{\partial\theta}\,|\hat{n}, m\rangle, \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\,|\hat{n}, m\rangle\right)\\ &= \left(-iS_y\,|\hat{n}, m\rangle, -\frac{i}{\sin\theta}S_z\,|\hat{n}, m\rangle\right).\end{aligned} \tag{4.3}$$

Since

$$[S_z, S_y] = -iS_x, [S_x, S_y] = iS_z. \tag{4.4}$$

By Baker-Hausdorff lemma,

$$e^{i\theta S_y} S_z e^{-i\theta S_y} = \cos\theta S_z - i\sin\theta S_x. \tag{4.5}$$

So,

$$A_\theta = -i\,\langle \hat{n}, m| -iS_y |\hat{n}, m\rangle = -i\,\langle \hat{z}, m| e^{i\phi S_z}(-iS_y)e^{-i\phi S_z} |\hat{z}, m\rangle = 0. \tag{4.6}$$

$$A_\phi = -i\,\langle \hat{n}, m| -\frac{i}{\sin\theta} S_z |\hat{n}, m\rangle = -\frac{1}{\sin\theta}\,\langle \hat{n}, m| S_z |\hat{n}, m\rangle = -m\cot\theta. \tag{4.7}$$

Thus,

$$\hat{\mathbf{r}} \cdot (\nabla \times \mathbf{A}) = -m. \tag{4.8}$$

Therefore,

$$q = -4\pi m. \tag{4.9}$$