

# Hermite Ploynomial

## 1 Explicit Expression

$$H_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!} (2z)^{n-2k}. \quad (1)$$

## 2 Generating Function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \quad (2)$$

## 3 Hermite Equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \quad (3)$$

This is a Sturm-Liouville type equation with weight function  $w(x) = e^{-x^2}$ .

**Proof.**

$$\frac{\partial G}{\partial s} = \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+1}(z) s^n. \quad (4)$$

$$2sG = \sum_{n=1}^{\infty} 2n \frac{1}{n!} H_{n-1} s^n. \quad (5)$$

Compare the coefficients of  $s^n$ ,

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0. \quad (6)$$

## 4 Recurrence Relations

$$\frac{\partial G}{\partial z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d}{dz} H_n(z) s^n. \quad (7)$$

Hence,

$$\frac{d}{dz} H_n = 2nH_{n-1}. \quad (8)$$

## 5 Rodrigues Formula

$$e^{-(s-z)^2} = \sum_{n=0}^{\infty} \frac{H_n(z)e^{-z^2}}{n!} s^n. \quad (9)$$

$$H_n(z)e^{-z^2} = \left. \frac{d^n}{ds^n} e^{-(s-z)^2} \right|_{s=0}. \quad (10)$$

$ds = -d(z-s)$ , hence

$$\boxed{H_n(z) = (-)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}.} \quad (11)$$

## 6 Parity Property

$$H_n(-x) = (-)^n H_n(x). \quad (12)$$

## 7 Special Values

$$H_{2m}(0) = (-)^m \frac{(2m)!}{m!}. \quad (13)$$

$$H_{2m+1}(0) = 0. \quad (14)$$

## 8 Orthogonality Relation

$$\int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi} 2^n n! \delta_{mn} \quad (15)$$

**Proof.**

$$\int_{-\infty}^{+\infty} G_1(s, z) G_2(t, z) dz = e^{-(z-(s+t))^2} e^{2st} = \Gamma\left(\frac{1}{2}\right) e^{2st}. \quad (16)$$

Hence,

$$\int_{-\infty}^{+\infty} G_1(s, z) G_2(t, z) dz = \sqrt{\pi} e^{2st}. \quad (17)$$

$$G_1(s, z) G_2(t, z) = \sum_{(n,m) \in \mathbb{N}^2} \frac{1}{n! m!} H_n H_m s^n t^m. \quad (18)$$

$$e^{2st} = \sum_{n=0}^{+\infty} \frac{(2st)^n}{n!}. \quad (19)$$

Therefore,

$$\boxed{\int_{-\infty}^{+\infty} H_n(z) H_m(z) e^{-z^2} dz = \delta_{nm} 2^n n! \sqrt{\pi}.} \quad (20)$$

## 9 The First Several Hermite Polynomials

$$\begin{aligned}H_0(x) &= 1 \\H_1(x) &= 2x \\H_2(x) &= 4x^2 - 2 \\H_3(x) &= 8x^3 - 12x \\H_4(x) &= 16x^4 - 48x^2 + 12\end{aligned}$$