Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 9–2 : Differentiability : functions of one variable in $\mathbb{R}$

In this exercise sheet, we focus on functions defined on subsets of $\mathbb{R}$ and valued in $\mathbb{R}$ equipped with the subspace topology of $\mathbb{R}$.

1. Consider the Dirichlet function $f : \mathbb{R} \to \{0, 1\}$ defined by

$$D(x) = \begin{cases} 1 & \text{if } \; x \in \mathbb{Q} \\ 0 & \text{if } \; x \in \mathbb{Q}^c. \end{cases}$$

(a) Prove that D is not differentiable at any $x \in \mathbb{R}$.
(b) Let $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 D(x)$. Prove that g is differentiable at 0 and $g'(0) = 0$.

2. Let $a \in \mathbb{R}$ with $a > 0$. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|^a$ is differentiable at 0 if and only if $a > 1$.

3. Suppose that $f, g, h : X \to \mathbb{R}$ are functions satisfying the following conditions :

(a) $f(a) = g(a) = h(a)$, $f'(a) = g'(a)$;
(b) $f(x) \leq h(x) \leq g(x)$, for any $x \in X$.

Prove that h is differentiable at a and $h'(a) = f'(a) = g'(a)$.

4. Suppose that $I \subseteq \mathbb{R}$ is an open interval and $0 \in I$. Let $f : I \to \mathbb{R}$ be a function which is continuous at 0 and $f(0) = 0$. Prove that if

$$\lim_{x \to 0} \frac{f(2x) - f(x)}{x} = A \in \mathbb{R},$$

then f is differentiable at 0 and $f'(0) = A$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^{2n} \sin \frac{1}{x} & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0. \end{cases}$$

Prove that at $x = 0$, $f'(0), f''(0), \ldots, f^{(n)}(0)$ exist, but $f^{(n+1)}(0)$ does not exist.

6. Let $f : X \to \mathbb{R}$ be a function. Determine whether the following statements are true or false. If true, please provide a proof ; if false, give a counterexample.

(a) If f is differentiable at $x_0 \in X$, then f is continuous at x_0.

(b) If f is differentiable at x_0, then $|f|$ is also differentiable at x_0.

(c) If f is differentiable at x_0, then f is differentiable on a neighborhood of x_0.

(d) If f is differentiable on an interval $I \subseteq X$, then f' is also continuous on I.

7. Let f be a real-valued function defined on a neighborhood of $x_0 \in \mathbb{R}$, and $\alpha \in \mathbb{R}$. We say that f satisfies the *Lipschitz condition* of order α at x_0 if there exist $M, \delta > 0$ such that

$$|f(x) - f(x_0)| < M|x - x_0|^\alpha, \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$

(a) Prove that if f satisfies the Lipschitz condition of order α at x_0 for $\alpha > 0$, then f is continuous at x_0.

(b) Prove that if f satisfies the Lipschitz condition of order α at x_0 for $\alpha > 1$, then f is differentiable at x_0 and $f'(x_0) = 0$.

(c) Find a function f which satisfies the Lipschitz condition of order α, and is continuous at x_0 but not differentiable at x_0.

8. Let f be a function defined on a neighborhood of $x_0 \in \mathbb{R}$. We define the *symmetric derivative* of f at x_0 by

$$f'_s(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h}$$

provided the limit exists.

(a) Prove that if f has a right-hand derivative and a left-hand derivative at x_0, then it has a symmetric derivative at x_0. Calculate $f'_s(x_0)$ in terms of left-hand and right-hand derivatives at x_0.

(b) Prove that the function f defined by

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has a symmetric derivative at 0, but does not have left-hand or right-hand derivatives at 0.

(c) If $f : [a, b] \to \mathbb{R}$ is an increasing function and has a symmetric derivative at every point of this interval, then the symmetric derivative is positive.

(d) If a function has a local extreme at a point x_0, can one conclude that $f'_s(x_0) = 0$? If yes, please prove your statement ; if not, provide a counterexample.

9. (a) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a function which satisfies $f(0)=0$ and $f'(0)$ exists. Let (x_n) be a sequence defined by

$$x_n = f\left(\frac{1}{n^2}\right) + f\left(\frac{2}{n^2}\right) + \cdots + f\left(\frac{n}{n^2}\right).$$

Calculate $\lim\limits_{n\to+\infty} x_n$.

(b) Find the limits of the following sequences.

i. $\lim\limits_{n\to+\infty} \sin\dfrac{1}{n^2} + \sin\dfrac{2}{n^2} + \cdots + \sin\dfrac{n}{n^2}$;

ii. $\lim\limits_{n\to+\infty} \left(1+\dfrac{1}{n^2}\right)\left(1+\dfrac{2}{n^2}\right)\cdots\left(1+\dfrac{n}{n^2}\right)$.

10. (a) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Suppose that $x_0, x_1 \in [a,b]$ are two points where local maximums are attained and $x_0 < x_1$. Prove that there exists $\tilde{x}\in(x_0,x_1)$ such that $f(\tilde{x})$ is a local minimum.

(b) Let $f:[a,b]\to\mathbb{R}$ be a function that is continuously differentiable and only has one local extreme. Prove that if the local extreme is a local maximum, then it is the global maximum ; if the local extreme is a local minimum, then it is the global minimum.

11. Let $f: X\to\mathbb{R}$ be a differentiable function. Prove that

(a) if f is an even function, then f' is an odd function ;

(b) if f is an odd function, then f' is an even function ;

(c) if f is a periodic function with period T, so is f'.

12. Calculate the derivatives of the following functions.

(a) $f(x) = \sin^2((x+\sin x)^6)$;

(b) $f(x) = \log\frac{\sqrt{1+x^3}-1}{\sqrt{1+x^3}+1}$

(c) $f(x) = \arcsin(\cos x^2) + \arccos(\sin x^2)$;

(d) $f(x) = x^{\frac{1}{x}}$;

(e) $f(x) = (\log x)^x + x^{\alpha \log x}$;

(f) $f(x) = \sin\left(\frac{x^3}{\sin(\frac{x^3}{\sin x})}\right)$.

13. Find f' in terms of g' if

(a) $f(x) = g(x+g(a))$;

(b) $f(x) = g(x\cdot g(a))$;

(c) $f(x) = g(x+g(x))$;

(d) $f(x) = g(x)(x-a)$;

(e) $f(x) = g(a)(x-a)$;

(f) $f(x+3) = g(x^2)$.

14. Let $X \subseteq \mathbb{R}$. Suppose that a function $f : X \to \mathbb{R}$ is differentiable and satisfy the equation

$$f(x)^x = x^{f(x)},$$

for any $x \in X$. Calculate $f'(x)$.

15. (a) Prove that the function $f : [-1,1] \to \mathbb{R}$ defined by $f(x) = \arcsin x$ satisfies that

$$\arcsin^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k \\ (2k-1)(2k-3)\cdots 3\cdot 1 & \text{if } n = 2k+1. \end{cases}$$

(b) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

is infinitely differentiable and

$$f^{(n)}(0) = 0, \quad \forall n \in \mathbb{N}.$$

16. Use Leibniz rule for $(x-a)^n(x-b)^n$ to prove that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \frac{2n(2n-1)\cdots(n+1)}{n!}.$$

17. The following problem concerns properties about hyperbolic and inverse hyperbolic functions.

(a) The inverse function of $\sinh x$ is given by $\mathrm{arsinh} y = \ln(y + \sqrt{1+y^2}), \ \forall y \in \mathbb{R}$;

(b) The functions inverse to $\cosh x$ on $\mathbb{R}_+$ and $\mathbb{R}_-$ are respectively given by $\mathrm{arcosh}_+ y = \ln(y - \sqrt{y^2-1})$ and $\mathrm{arcosh}_- y = \ln(y + \sqrt{y^2-1}), \ \forall y \geq 1$;

(c) $\mathrm{arsinh}' y = \frac{1}{\sqrt{1+y^2}} \quad \forall y \in \mathbb{R}$;

(d) $\mathrm{arcosh}' y = -\frac{1}{\sqrt{y^2-1}} \quad \forall y \geq 1$;

(e) $\mathrm{artanh} y = \frac{1}{2}\ln\frac{1+y}{1-y}, \quad \forall |y| < 1$;

(f) $\mathrm{arcoth} y = \frac{1}{2}\ln\frac{y+1}{y-1}, \quad \forall |y| > 1$;

(g) $\mathrm{artanh}' y = \frac{1}{1-y^2}, \quad \forall |y| < 1$;

(h) $\operatorname{arcoth}' y = -\frac{1}{y^2-1}, \quad \forall |y| > 1\,;$

(i) $\sinh(x+y) = \sinh x \cdot \cosh y + \cosh x \cdot \sinh y\,;$

(j) $\cosh(x+y) = \cosh x \cdot \cosh y + \sinh x \cdot \sinh y.$

18. (Lorentz transform) Fix $c, v \geq 0$. Let $\alpha, \gamma, \delta \in \mathbb{R}$ and $\beta = \alpha v$. Suppose that the equations

$$\begin{pmatrix} \tilde{x} \\ \tilde{t} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

hold for all nonzero functions $x, \tilde{x}, \tilde{t} : [0, \infty) \to \mathbb{R}$ satisfying

$$x(t)^2 - c^2 t^2 = \tilde{x}(t)^2 - c^2 \tilde{t}(t)^2.$$

Find the the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in terms of c and v.

19. Let $y(t) = e^t$ and $x(t) = \ln t$. Prove that

$$\frac{dy}{dx} = te^t \quad \text{and} \quad \frac{d^2 y}{dx^2} = t(t+1)e^t.$$

20. Assume that f is C^∞ on an open interval $I \subset \mathbb{R}$ and $f'(x_0) \neq 0$ for a point $x_0 \in I$. Then there exists a neighborhood U of x_0 and a neighborhood V of $f(x_0)$ such that f has an inverse f^{-1} on U and $f^{-1} : V \to U$ is also C^∞.

21. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function, and $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to -\infty} f(x)$ (equal to either a finite number or $\pm\infty$). Prove that there exists $\xi \in \mathbb{R}$ such that $f'(\xi) = 0$.

(b) Let $f : [a, b] \to \mathbb{R}$ be a function satisfying

i. f is continuous on $[a, b]\,;$

ii. f is differentiable on $(a, b)\,;$

iii. $\lim_{x \to a^+} f(x) = \lim_{x \to b^-} f(x) = +\infty$.

Prove that for any $A \in \mathbb{R}$, there exists $\xi \in (a, b)$ such that $f'(\xi) = A$.

22. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function satisfying $f'(a) = f'(b)$. Prove that there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}.$$

23. Let $f : [a, b] \to \mathbb{R}$ be a function and $\alpha > 0$.

(a) If f is continuously differentiable on $[a, b]$, then f is Lipschitz of order 1.

(b) If f is differentiable on $[a, b]$ and is Lipschitz of order 1, then f' is bounded on $[a, b]$.

(c) If f is Lipschitz of order α for $\alpha > 1$, show that f is constant on $[a, b]$.

24. (a) Let f be a real-valued function, defined and continuous on the closed interval $[a, b]$, differentiable at every point of the open interval (a, b), except possibly at a point $x_0 \in (a, b)$. Prove that if the derivative f' has a limit at the point x_0, then the function f is differentiable at x_0 and

$$f'(x_0) = \lim_{x \to x_0} f'(x).$$

(b) Let $n \in \mathbb{N}$, and let $f : I \setminus \{a\} \to \mathbb{R}$ be a function of class C^n. If, for all $k \in [0, n]$, the derivative $f^{(k)}$ has a finite limit α_k at a, then f can be uniquely extended to $\tilde{f} : I \to \mathbb{R}$ such that $\tilde{f}$ is of class C^n.

25. Let $f : [a, \infty) \to \mathbb{R}$ be a differentiable function. If $\lim_{x \to +\infty} f(x)+f'(x) = l$. Prove that $\lim_{x \to +\infty} f(x) = l$. (Hint : assume without loss of generality that $l = 0$, and consider $g(x) := e^x f(x)$ for which one invokes Cauchy mean-value theorem.)

26. Find all the functions that satisfy the following conditions :

(a) $f(1) = 1, f'(0) > 0$;

(b) $\forall t \in \mathbb{R}, f'(t) \cdot f'(f(t)) = 1$.

(Hint : use Darboux theorem.)

27. (a) Let I be a closed interval, $K \in [0, 1)$, and $f : I \to I$ be a K-Lipschitz function (i.e. $|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in I$). Prove that any sequence $(u_n)_{n \in \mathbb{N}}$ defined by

$$u_0 \in I, \quad u_{n+1} = f(u_n) \quad \forall n \in \mathbb{N},$$

converges to a fixed point of f. (Hint : consider the cases where $I = [a, b]$, $I = [a, +\infty)$, $I = (-\infty, a]$, and $I = \mathbb{R}$.)

(b) Let $(u_n)_{n \in \mathbb{N}}$ be the sequence defined by

$$u_0 \in [0, 1], \quad u_{n+1} = 1 - \frac{u_n^2}{4}.$$

Does the sequence converge? If yes, please provide a proof and find the limit; if not, explain why.

28. Let f be a function of class C^n defined on I, which vanishes at $x_1 < \cdots < x_n$, with $n > 1$. Prove that for all $x \in I$, there exists $\xi \in I$ such that

$$f(x) = \frac{(x-x_1)\cdots(x-x_n)}{n!} f^{(n)}(\xi).$$

(Hint : for $x \in I \backslash \{x_1, \ldots, x_n\}$, consider $\varphi : t \mapsto f(t) - A(t-x_1)\cdots(t-x_n)$, where $A \in \mathbb{R}$ is chosen so that $\varphi(x) = 0$.)

29. Find the local extrema and global extrema of the following functions :

(a) $f(x) = x^3 + 2x^2 + x + 1, \quad x \in [-2, 2]$;

(b) $f(x) = 3x^4 - 8x^3 + 6x^2, \quad x \in [-1, 1]$;

(c) $f(x) = \frac{1}{x^5+x+1}, \quad x \in [-\frac{1}{2}, 1]$;

(d) $f(x) = \frac{x}{x^2-1}, \quad x \in [0, 5]$;

(e) $f(x) = \begin{cases} 0 & x \notin \mathbb{Q}, \\ x & x \in \mathbb{Q}; \end{cases}$

(f) $f(x) = \begin{cases} 0 & x \notin \mathbb{Q}, \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$

30. Prove the following inequalities :

(a) $\frac{2}{\pi}x < \sin x < x$ for any $x \in (0, \frac{\pi}{2})$;

(b) $x < \arcsin x < \frac{x}{\sqrt{1-x^2}} \quad (0 < x < 1)$;

(c) $e^x < \frac{1}{1-x}(x < 1)$;

(d) $(x^\alpha + y^\alpha)^{\frac{1}{\alpha}} < (x^\beta + y^\beta)^{\frac{1}{\beta}} \quad (x > 0, y > 0, \beta > \alpha > 0)$;

(e) $|1+x|^p \geq 1 + px + c_p\varphi_p(x)$, where c_p is a constant depending only on p. If $1 < p \leq 2$,

$$\varphi_p(x) = \begin{cases} |x|^2 & \text{for } |x| \leq 1, \\ |x|^p & \text{for } |x| > 1; \end{cases}$$

and if $p > 2$, $\varphi_p(x) = |x|^p$ on $\mathbb{R}$.

31. (Sufficient conditions for an extremum in terms of higher-order derivatives) Suppose a real-valued function f defined on a neighborhood of $x_0 \in \mathbb{R}$ has derivatives of order up to n inclusive at x_0 ($n \geq 1$). Suppose that $f'(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$.

(a) If n is odd, prove that there is no extremum at x_0 .

(b) If n is even, prove that x_0 is a local extremum. In particular, if $f^{(n)}(x_0) < 0$, x_0 is a local maximum and if $f^{(n)}(x_0) > 0$, x_0 is a local minimum.

32. (a) Let $f : [a, b] \to \mathbb{R}$ be a function that is differentiable on a neighborhood of a point $x_0 \in (a, b)$. Suppose that f' is continuous at x_0 and $f'(x_0) \neq 0$. Prove that there exists a neighborhood of x_0 such that f is monotonic on this neighborhood.

(b) If f' is not continuous at x_0, does the above conclusion still hold ?

(Hint : consider the function $f(x) = \begin{cases} \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$)

33. State whether the following functions are convex, concave or neither. If yes, please provide a proof ; if not, please explain why.

(a) $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^\alpha \quad (\alpha > 0)$;

(b) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x \sin x^2$;

(c) $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(x) = \log\left(\frac{x}{\sin x}\right)$;

(d) $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} (x-2)^2 & x \in [0, \infty) \\ (x+2)^2 & x \in (-\infty, 0]. \end{cases}$$

34. Let f and g be two convex, positive, and monotonically increasing functions on I.

(a) Assume f and g are twice differentiable. Show that $f \cdot g$ is convex.

(b) Show that the result holds in the general case.

35. Let I be an interval and $f : I \to \mathbb{R}$ a function satisfying

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(f(x) + f(y)), \quad \forall x, y \in I.$$

(a) Prove that $\forall k = \frac{m}{2^n} \in [0, 1]$, one has

$$f\left(kx + (1-k)y\right) \leq kf(x) + (1-k)f(y), \quad \forall x, y \in I.$$

(b) If f is continuous on I, then f is convex.

36. Let I be an open interval and $f : I \to \mathbb{R}$ be a convex function.

(a) Prove that for any $x \in I$, $\lim\limits_{y \to x^+} \dfrac{f(y) - f(x)}{y - x}$ and $\lim\limits_{y \to x^-} \dfrac{f(y) - f(x)}{y - x}$ exist, and

$$\lim_{y \to x^-} \frac{f(y) - f(x)}{y - x} \leq \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x}.$$

(b) Let $g, h : I \to \mathbb{R}$ be functions defined by $g(x) = \lim\limits_{y \to x^-} \dfrac{f(y) - f(x)}{y - x}$ and $h(x) = \lim\limits_{y \to x^+} \dfrac{f(y) - f(x)}{y - x}$. Prove that g and h are monotonically increasing, and

$$\forall a, b \in I, \quad h(a) \leq \frac{f(b) - f(a)}{b - a} \leq g(b).$$

37. Let I be an open interval and $f : I \to \mathbb{R}$ a function. Prove that f is convex if and only if f is continuous on I and the left-hand and right-hand derivatives exist and are monotonically increasing on I.

38. (a) Show that a function f is both convex and concave if and only if f is affine.

(b) Let f be a convex function that is bounded above on $\mathbb{R}$. Show that f is constant. Does the result hold if f is convex and bounded above on $[A, +\infty[$?

(c) Deduce that if f is a twice-differentiable function on $\mathbb{R}$, bounded and non-constant, then there exist t_1 and t_2 such that $f''(t_1)f''(t_2) < 0$.

39. Find the following limits of functions :

(a) $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3}$;

(b) $\lim\limits_{x \to 0} \dfrac{xe^x - \log(1 + x)}{x^2}$;

(c) $\lim\limits_{x \to 0} \sin x \log x$;

(d) $\lim\limits_{x \to 1} \dfrac{1}{\log x} - \dfrac{1}{x - 1}$;

(e) $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$;

(f) $\lim\limits_{x \to 0} \left(\dfrac{\cos x}{\cosh x}\right)^{\frac{1}{x^2}}$;

(g) $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$;

40. (a) Find the shortest distance from the point $(0, h)$ $(h > 0)$ to the parabola $y = x^2$.

(b) For all the isosceles triangles with a fixed perimeter, what is the ratio of the side to the base such that the volume of the solid generated by rotating the triangle about its base is maximized?

(c) Two rivers intersect perpendicularly, with widths a and b respectively. What is the maximum length the boat can have to transition from one river to the other ?

41. (a) Two bodes with masses m_1 and m_2 are moving in space under the action of their mutual gravitation alone. Using Newton's laws (i.e. the general law of motion and the law of universal gravitation), verify that the quantity

$$E = \left(\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2\right) + \left(-G\frac{m_1m_2}{r}\right) =: K + U,$$

where v_1 and v_2 are velocities of the bodies and r the distance between them, does not vary during this motion.

(b) Extend this result to the case of the motion of n bodies.

42. (Newton's method) Let α, β be two real numbers such that $\alpha < \beta$, and let $f : [\alpha, \beta] \to \mathbb{R}$ be a function satisfying the following conditions :

1) f is twice continuously differentiable on $[\alpha, \beta]$;
2) f' does not vanish on $[\alpha, \beta]$;
3) $f(\alpha)f(\beta) < 0$.

(a) Prove that the equation $f(x) = 0$ has a unique root l in $[\alpha, \beta]$.

(b) Let a be an element of the interval (α, β). Let $\widetilde{Q}_1$ be the linear function whose graph is the tangent to the graph of f at a. Show that the root x_1 of the equation $\widetilde{Q}_1(x) = 0$ is

$$x_1 = a - \frac{f(a)}{f'(a)}.$$

(c) Let $g : [\alpha, \beta] \to \mathbb{R}$ be the function defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

i. Show that g is differentiable on (α, β) and that

$$\forall x \in (\alpha, \beta),\ g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}.$$

ii. Show that there exist two points $a, b \in (\alpha, \beta)$ such that $a < l < b$, $l - a = b - l$, and

$$\forall x \in [a, b], \quad |g'(x)| < 1.$$

Deduce that $g([a, b]) \subseteq [a, b]$.

iii. We construct a sequence defined recursively by

$$x_0 = a, \quad x_{n+1} = g(x_n), \forall \ \ n \in \mathbb{N}.$$

Prove that the sequence converges to l.

(d) Use Newton's method to find he decimal approximation of $\sqrt{2}$, accurate to the third decimal place.

43. (Geometry of triangular functions) Let $S, C : \mathbb{R} \to \mathbb{R}$ be differential functions satisfying

$$\begin{cases} S' = C, & S(0) = 0 \\ C' = -S, & C(0) = 1 \end{cases}$$

Prove the following assertions :

(a) $S^2 + C^2 = 1$.

(b) There exists $A > 0$ such that

i. $S(0) = 0$, $S(A) = 1$ and S is strictly increasing on $[0, A]$;

ii. $C(0) = 1$, $C(A) = 0$ and c is strictly decreasing on $[0, A]$.

Moreover there exists $B > 0$ such that

i. $S(A) = 1$, $S(A + B) = 0$ and S is strictly decreasing on $[A, A + B]$;

ii. $C(0) = 0$, $C(A + B) = -1$ and S is strictly increasing on $[A, A + B]$.

(c) $\pi := A + B$ is the first zero of S on $\mathbb{R}_{>0}$.

(d) $S(x) = -S(x + \pi)$, $C(x) = -C(x + \pi)$ and 2π is the smallest period of S and C. Moreover, $A = B = \pi/2$.

(e) $\sin\theta < \theta < \tan\theta$ for $0 < \theta < \pi/2$.

(We will see in the next chapter that π is the length of the semicircle and θ is the length of the curve from $(0, 1)$ to $(\cos\theta, \sin\theta)$ on the semicircle for $0 \le \theta \le \pi$.)

44. Let X be a subset of $\mathbb{R}$ equipped with the subspace topology, and f, g be two bounded functions defined over X and valued in $\mathbb{R}$. For an $x \in X$, define $d(f(x), g(x)) = |f(x) - g(x)|$.

(a) Show that

$$\sup_{x \in X} d(f(x), g(x))$$

is a metric on the space of bounded functions defined over X.

(b) Prove that the uniform convergence of a sequence of bounded functions $(f_n)_{n\in\mathbb{N}}$ is equivalent to the convergence of $(f_n)_{n\in\mathbb{N}}$ (as a standard sequence) with respect to the above metric.