QM HW6

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Problem 1 (Current, gauge transformation)
(1) Use the substitution:

$$-i\hbar\nabla \longrightarrow -i\hbar\nabla - \frac{q\vec{A}}{c}, \tag{1.1}$$

the probability current will be:

$$\vec{j} = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right) \longrightarrow \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right) - \frac{q}{mc}\vec{A}\left|\psi\right|^2. \tag{1.2}$$

Hence,

$$\boxed{\vec{j} = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right) - \frac{q}{mc}\vec{A}\left|\psi\right|^2.} \tag{1.3}$$

(2)

$$A'_\mu = A_\mu + \partial_\mu f. \tag{1.4}$$

Since the commutator is antisymmetric and the derivative is commutative,

$$\partial_{[\mu}\partial_{\nu]}f = 0. \tag{1.5}$$

Thus,

$$F'_{\mu\nu} = \partial_{[\mu}A'_{\nu]} = \partial_{[\mu}A_{\nu]} + \partial_{[\mu}\partial_{\nu]}f = F_{\mu\nu}. \tag{1.6}$$

So,

$$\mathbf{E}' = \mathbf{E},\ \mathbf{B}' = \mathbf{B}. \tag{1.7}$$

(3)

$$\frac{\partial\psi'}{\partial t} = i\frac{\partial\varphi}{\partial t}\psi' + e^{i\varphi}\frac{\partial\psi}{\partial t} \tag{1.8}$$

Cancelate $e^{i\varphi}$, and plug in Schrödinger equation, we should take

$$\boxed{\varphi = \frac{q}{\hbar c}f.} \tag{1.9}$$

(4)

$$\rho' = e^{i\varphi}\psi e^{-i\varphi}\psi^* = \psi\psi^* = \rho. \tag{1.10}$$

If we let
$$\psi = \sqrt{\rho} e^{\frac{iS}{\hbar}}, \tag{1.11}$$
then,
$$\mathbf{j} = \frac{\rho}{m}\left(\nabla S - \frac{q\mathbf{A}}{c}\right). \tag{1.12}$$
$$S' = S + \hbar\varphi, \quad \nabla S' - \frac{q\mathbf{A}'}{c} = \nabla S - \frac{q\mathbf{A}}{c}. \tag{1.13}$$
So the probability current is invariant under the gauge transformation.

Problem 2 (Landau gauge)
(1)
$$\left(p_x - \frac{qB}{c}y\right)^2 e^{ik_x x} = \left(\hbar k_x - \frac{qB}{c}y\right)^2 e^{ik_x x}. \tag{2.1}$$
$$\left[\frac{\left(\hbar k_x - \frac{qBy}{c}\right)^2}{2m} + (V(y) - E)\right]\phi_{k_x}(y) = \frac{\hbar^2}{2m}\phi''_{k_x}(y). \tag{2.2}$$
We can define $H_y(k_x)$ as
$$H_y(k_x) = \frac{\left(\hbar k_x - \frac{qBy}{c}\right)^2}{2m} + \frac{p_y^2}{2m} + V(y). \tag{2.3}$$
(2) Note that
$$\frac{\partial H_y(k_x)}{\partial k_x} = \frac{\hbar k_x - qBy/c}{m}. \tag{2.4}$$
$$I_x(x, k_x) = qJ = \frac{q\rho}{m}\left(\hbar k_x - \frac{qBy}{c}\right) = \frac{q}{L_x}\frac{\partial E_n}{\hbar \partial k_x}. \tag{2.5}$$
(3)
$$I_{x,n} = \frac{1}{2\pi}\int \mathrm{d}k_x\, I_x(n, k_x) = \frac{q^2}{h}\Delta V / L_x. \tag{2.6}$$
$$\sigma_{xy} = \frac{I_x}{E_y} = \frac{q^2}{h}\nu. \tag{2.7}$$
This result is not related to V_{imp}.

Problem 3 (Spherical coordinates)
I will use Einstein summation convention if there's neither special announcement nor summation symbol.

In spherical coordinates, Lamé coefficients are
$$A_r = 1, \quad A_\theta = r, \quad A_\phi = r\sin\theta. \tag{3.1}$$
(1)
$$\nabla f = \mathbf{g}^i \partial_i f = \sum_i \frac{1}{A_i}\mathbf{e}^i \partial_i f. \tag{3.2}$$

So,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi. \tag{3.3}$$

(2) Note that

$$\frac{\partial \sqrt{g}}{\partial x^i} = \Gamma^j_{ji}\sqrt{g}. \tag{3.4}$$

We have,

$$\nabla \cdot \mathbf{V} = \partial_i V^i + V^m \Gamma^i_{im} = \partial_i V^i + V^m \frac{1}{\sqrt{g}}\partial_m \sqrt{g} = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}V^i\right). \tag{3.5}$$

That is

$$\nabla \cdot \mathbf{V} = \sum_i \frac{1}{A_r A_\theta A_\phi}\partial_i\left(\frac{A_r A_\theta A_\phi}{A_i} V\langle i\rangle\right). \tag{3.6}$$

$$\nabla \cdot \mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 V\langle r\rangle\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta V\langle \theta\rangle\right) + \frac{1}{r\sin\theta}\frac{\partial V\langle\phi\rangle}{\partial\phi}. \tag{3.7}$$

(3)

$$\nabla \times \mathbf{V} = \epsilon^{ijk}\nabla_i V_j \mathbf{g}_k = \epsilon^{ijk}\left(\partial_i V_j - V_m \Gamma^m_{ij}\right)\mathbf{g}^k = \epsilon^{ijk}\partial_i V_j \mathbf{g}^k. \tag{3.8}$$

$$\nabla \times \mathbf{V} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\mathbf{e}_\phi \\ \partial_r & \partial_\theta & \partial_\phi \\ V\langle r\rangle & rV\langle\theta\rangle & r\sin\theta V\langle\phi\rangle \end{vmatrix}. \tag{3.9}$$

(4) By (1) and (2),

$$\nabla^2 f = \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}. \tag{3.10}$$

Problem 4 (Angular momentum operators)

(1)

$$\mathbf{l} = \mathbf{r}\times\mathbf{p} := -\mathrm{i}\hbar\mathbf{r}\times\nabla = \frac{\partial}{\partial\theta}\hat{\phi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\hat{\theta}. \tag{4.1}$$

Then, project on $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$, we obtain

$$l_z = -\mathrm{i}\hbar\frac{\partial}{\partial\phi}, \tag{4.2}$$

$$l_x = -\mathrm{i}\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \tag{4.3}$$

$$l_y = -\mathrm{i}\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right). \tag{4.4}$$

(2)

$$l_x = \left(l_+ + l_-\right)/2, \quad l_y = \left(l_+ - l_-\right)/2\mathrm{i}, \tag{4.5}$$

$$l^2 = l_x^2 + l_y^2 + l_z^2 = l_z^2 + l_+ l_- + l_- l_+. \tag{4.6}$$

By (4.3) and (4.4),

$$l^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{4.7}$$

(3) To avoid confusion, we use $\mathbf{L}$ and $\mathbf{x}$ to represent $\mathbf{l}$ and $\mathbf{r}$ respectively.

$$\begin{aligned}
\mathbf{L}^2 &= \sum_{ijlmk} \varepsilon_{ijk} x_i p_j \varepsilon_{lmk} x_l p_m \\
&= \sum_{ijlm} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) x_i p_j x_l p_m \\
&= \sum_{ijlm} [\delta_{il}\delta_{jm} x_i (x_l p_j - \mathrm{i}\hbar\delta_{jl}) p_m - \delta_{im}\delta_{jl} x_i p_j (p_m x_l + \mathrm{i}\hbar\delta_{lm})] \\
&= \mathbf{x}^2\mathbf{p}^2 - \mathrm{i}\hbar\mathbf{x}\cdot\mathbf{p} - \sum_{ijlm}\delta_{im}\delta_{jl}[x_i p_m (x_l p_j - \mathrm{i}\hbar\delta_{jl}) + \mathrm{i}\hbar\delta_{lm} x_i p_j] \\
&= \mathbf{x}^2\mathbf{p}^2 - (\mathbf{x}\cdot\mathbf{p})^2 + \mathrm{i}\hbar\mathbf{x}\cdot\mathbf{p}.
\end{aligned} \tag{4.8}$$

That is

$$-\frac{\hbar^2}{2m}\nabla^2 = -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\mathbf{L}^2}{2mr^2}. \tag{4.9}$$

Problem 5 (Associated Legendre Polynomials)

(1)

$$\frac{\mathrm{d}}{\mathrm{d}z}P^{|m|}(z) = \left(1-z^2\right)^{\frac{|m|}{2}-1}\left[-z|m|G(z) + \left(1-z^2\right)G'(z)\right]. \tag{5.1}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}P^{|m|}\right] &= \left(1-z^2\right)^{\frac{|m|}{2}-1}\Big(\left(z^2-1\right)|m|\left(2zG'(z)+G(z)\right) \\
&\quad + z^2|m|^2G(z) + \left(z^2-1\right)\left(\left(z^2-1\right)G''(z) + 2zG'(z)\right)\Big).
\end{aligned} \tag{5.2}$$

Hence,

$$(1-z^2)G'' - 2(|m|+1)zG' + [\beta - |m|(|m|+1)]\,G = 0. \tag{5.3}$$

(2)

$$(1-z^2)G'' = \sum_{z=2}^{+\infty} n(n-1)a_n\left(z^{n-2} - z^n\right), \tag{5.4}$$

$$zG' = \sum_{n=1}^{+\infty} na_n z^n. \tag{5.5}$$

Thus,

$$\begin{aligned}
\sum_{n=0}^{+\infty}\{(n+2)(n+1)a_{n+2} - n(n+1)a_n - 2(|m|+1)na_n \\
+ [\beta - |m|(|m|+1)]\,a_n\}\,z^n = 0.
\end{aligned} \tag{5.6}$$

Therefore,

$$a_{n+2} = \frac{(n+|m|)\,(n+|m|+1) - \beta}{(n+1)(n+2)}a_n. \tag{5.7}$$

(3) If $\beta = l(l+1)$, then, when $n+|m| \geq l$, $a_{n+2} = 0$. So G becomes a polynomial.

Problem 6 (Generation function of Legendre Polynomials)
(1) One has

$$\frac{\partial T}{\partial t} = \sum_{n=0}^{+\infty} l P_l\left(z\right) t^{l-1}. \tag{6.1}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right) = (z-t)\left(1-2z+t^2\right)^{\frac{3}{2}} = \frac{z-t}{1-2zt+t^2}T. \tag{6.2}$$

Compare the coefficients of t^l, we obtain,

$$(l+1)P_{l+1}\left(z\right) - (2l+1)zP_l(z) + lP_{l-1}(z) = 0. \tag{6.3}$$

(2)

$$\frac{\partial T}{\partial z} = \sum_{n=0}^{+\infty} P_l'(z)t^l. \tag{6.4}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{1-2tz+t^2}}\right) = \frac{t}{1-2zt+t^2}T. \tag{6.5}$$

Hence,

$$P_{l+1}' - 2zP_l' + P_{l-1}' = P_l. \tag{6.6}$$

(3) $l\times$(6.6)$-\frac{\mathrm{d}}{\mathrm{d}z}$ (6.3):

$$zP_l' - P_{l-1}' = lP_l. \tag{6.7}$$

(6.6)+(6.7):

$$P_{l+1}' - zP_l' = (l+1)P_l. \tag{6.8}$$

(4) (6.7):$l \to l+1$, (6.8)$\times z$:

$$(z^2-1)P_l' = (l+1)(P_{l+1} - zP_l). \tag{6.9}$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}P_l\right] = (l+1)\frac{\mathrm{d}}{\mathrm{d}z}\left(zP_l - P_{l+1}\right) = (l+1)(P_l + zP_l' - P_{l+1}'). \tag{6.10}$$

Plug in (6.8), we obtain

$$\frac{d}{dz}\left[(1-z^2)\frac{\mathrm{d}P_l(z)}{\mathrm{d}z}\right] + l(l+1)P_l(z) = 0 \tag{6.11}$$

By (6.11)

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\left(P_n\frac{\mathrm{d}P_m}{\mathrm{d}z} - P_m\frac{\mathrm{d}P_n}{\mathrm{d}z}\right)\right] + \left[m(m+1) - n(n+1)\right]P_mP_n = 0 \tag{6.12}$$

Since $\left[1-z^2\right]_{z=\pm 1} = 0$, the first term in (6.12) will be zero after integrating. So the second term must be zero after integrating if $n \neq m$, exact,

$$\int_{-1}^{1} P_n(z)P_m(z) = 0, \quad n \neq m. \tag{6.13}$$

(5) By (6.3),

$$lP_l - (2l-1)zP_{l-1} + (l-1)P_{l-2} = 0, \quad zP_l = \frac{(l+1)P_l + lP_{l-1}}{2l+1}. \tag{6.14}$$

Multiplying both sides by P_l, and integrate, we obtain,

$$l\int_{-1}^{1} P_l^2 \,\mathrm{d}z = (2l-1)\int_{-1}^{1} zP_[l-1]P_l \,\mathrm{d}z = \frac{l(2l-1)}{2l+1}\int_{-1}^{1} P_{l-1}^2 \,\mathrm{d}z\,. \tag{6.15}$$

Thus,

$$(2l+1)\int_{-1}^{1} P_l^2(z)\,\mathrm{d}z = (2(l-1)+1)\int_{-1}^{1} P_{l-1}^2(z)\,\mathrm{d}z = \int_{-1}^{1} P_0^2(z)\,\mathrm{d}z = 2. \tag{6.16}$$

Therefore,

$$\int_{-1}^{1} P_l^2(z)\,\mathrm{d}z = \frac{2}{2l+1}. \tag{6.17}$$

Problem 7 (Associated Legendre Polynomials)
(1)

$$\frac{\mathrm{d}}{\mathrm{d}z}P_l^{|m|} = \left(1-z^2\right)^{\frac{|m|}{2}-1}\left[(1-z^2)\frac{\mathrm{d}^{|m|+1}}{\mathrm{d}z^{|m|+1}}P_l - z|m|\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}P_l\right]. \tag{7.1}$$

$$\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}\left[(1-z^2)\frac{\mathrm{d}P_l}{\mathrm{d}z}\right] = (1-z^2)\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}P_l - 2|m|z\frac{\mathrm{d}^{|m|-1}}{\mathrm{d}z^{|m|-1}}\frac{\mathrm{d}P_l}{\mathrm{d}z} - |m|(|m|-1)\frac{\mathrm{d}^{|m|-2}}{\mathrm{d}z^{|m|-2}}\frac{\mathrm{d}P_l}{\mathrm{d}z}. \tag{7.2}$$

By (6.11),

$$-l(l+1)\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}P_l = \frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}\left[(1-z^2)\frac{\mathrm{d}P_l}{\mathrm{d}z}\right]. \tag{7.3}$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}P_l^{|m|}(z)\right] + \left[l(l+1) - \frac{m^2}{1-z^2}\right]P_l^{|m|}(z) = 0. \tag{7.4}$$

(2)

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[(1-z^2)\left(\frac{\mathrm{d}}{\mathrm{d}z}P_l^{|m|}P_{l'}^{|m|} - P_l^{|m|}\frac{\mathrm{d}}{\mathrm{d}z}P_{l'}^{|m|}\right)\right] + \left[l(l+1) - l'(l'+1)\right]P_l^{|m|}P_{l'}^{|m|} = 0. \tag{7.5}$$

So,

$$\int_{-1}^{1} P_l^{|m|}P_{l'}^{|m|}\,\mathrm{d}z = 0. \tag{7.6}$$

(4) Let $\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}$ map at (6.3), we obtain,

$$z\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}P_l + |m|\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}P_l = \frac{l+1}{2l+1}\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}P_{l+1} + \frac{1}{2l+1}\frac{\mathrm{d}^{|m|}}{\mathrm{d}z^{|m|}}P_{l-1}. \tag{7.7}$$

Map $\frac{\mathrm{d}^{|m|-1}}{\mathrm{d}z^{|m|-1}}$ at $P'_{l+1} - P_{l-1} = (2l+1)P_l$, and deduce the term with $\frac{\mathrm{d}^{|m|-1}}{\mathrm{d}z^{|m|-1}}$, we obtain,

$$zP_l^{|m|} = \frac{l+|m|}{2l+1}P_{l-1}^{|m|} + \frac{l-|m|+1}{2l+1}P_{l+1}^{|m|}. \tag{7.8}$$

Problem 8 (Laguerre polynomials)
(1)

$$\sum_{\nu=0}^{+\infty}\left[a_{\nu+1}\nu(\nu+1)+2(l+1)a_{\nu+1}(\nu+1)-a_\nu\nu+(\lambda-l-1)\,a_\nu\right]\xi^\nu=0. \tag{8.1}$$

$$a_{\nu+1}=\frac{\nu+l+1-\lambda}{(\nu+1)(2l+2+\nu)}a_\nu. \tag{8.2}$$

$$u(\xi)=\sum_{\nu=0}^{+\infty}\prod_{m=0}^{\nu-1}\frac{m+l+1-\lambda}{(m+1)(2l+2+m)}\xi^\nu. \tag{8.3}$$

(2) When ν gets large, we have

$$\frac{\nu+l+1-\lambda}{(\nu+1)(2l+2+\nu)}\sim O(\frac{1}{\nu}). \tag{8.4}$$

Its action is like $\frac{1/(n+1)!}{1/n!}$, so in the general case,

$$u(\xi)\sim \mathrm{e}^\xi. \tag{8.5}$$

When there exists ν, such that $\nu+l+1-\lambda=0$, then the following term will be zero, and u becomes a polynomial. So λ should be a integer lager than $l+1$.
(3) One has

$$\frac{\partial U}{\partial u}=\sum_{m=0}^{+\infty}\frac{L_{m+1}(\xi)}{m!}u^m. \tag{8.6}$$

$$\frac{\partial}{\partial u}\left[\frac{1}{1-u}e^{-\frac{\xi u}{1-u}}\right]=\frac{1}{1-u}e^{-\frac{\xi u}{1-u}}\frac{\partial}{\partial u}\ln\left[\frac{1}{1-u}e^{-\frac{\xi u}{1-u}}\right]=U\frac{1-u-\xi}{(1-u)^2}. \tag{8.7}$$

Hence,

$$L_{m+1}(\xi)+(\xi-2m-1)\,L_m(\xi)+m^2L_{m-1}(\xi)=0. \tag{8.8}$$

$$\frac{\partial U}{\partial \xi}=U\frac{\partial}{\partial \xi}\ln U=U\frac{-u}{1-u}, \tag{8.9}$$

so,

$$L_m'(\xi)-mL_{m-1}'(\xi)+mL_{m-1}(\xi)=0. \tag{8.10}$$

(4) Derivative of (8.8) shows that

$$L_{m+1}'+L_m+(\xi-1-2m)L_m'+m^2L_{m-1}'=0. \tag{8.11}$$

By (8.10) and (8.8)

$$L_{m+1}'-L_{m+1}+(2m+2-\xi)L_m+(\xi-1-m)L_m'=0, \tag{8.12}$$

$$L_{m+1}''-L_{m+1}'+L_m'(2m+2-\xi)+(\xi-1-m)L_m''=0. \tag{8.13}$$

Plug in (8.10) after $m \to m+1$:

$$\xi L_m'' + (1-\xi)L_m' + (m+1)L_m = 0. \tag{8.14}$$

(5)

$$\frac{\mathrm{d}^s}{\mathrm{d}\xi^s}\left[\xi L_m''(\xi)\right] = s\frac{\mathrm{d}^{s-1}}{\mathrm{d}\xi^{s-1}}\left[L_m''(\xi)\right] + \xi\frac{\mathrm{d}^s}{\mathrm{d}\xi^s}\left[\xi L_m''(\xi)\right]. \tag{8.15}$$

$$\frac{\mathrm{d}^s}{\mathrm{d}\xi^s}\left[(1-\xi) L_m'(\xi)\right] = -s\frac{\mathrm{d}^{s-1}}{\mathrm{d}\xi^{s-1}}\left[L_m'(\xi)\right] + (1-\xi)\frac{\mathrm{d}^s}{\mathrm{d}\xi^s}\left[\xi L_m'(\xi)\right]. \tag{8.16}$$

Therefore,

$$\xi {L_m^s}''(\xi) + (s+1-\xi)\, {L_m^s}'(\xi) + (m-s)\, L_m^s(\xi) = 0. \tag{8.17}$$