

The Electromagnetic Radiation

1 Preamble and Notation

We start from D'Alembert equation considering the flat metric $\text{diag}(-1, 1, 1, 1)$:

$$\partial^\mu \partial_\mu A^\alpha = -\mu_0 J^\alpha, \quad (1)$$

where

$$x^\alpha = (ct, x^1, x^2, x^3), \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad (2)$$

$$A^\alpha = \left(\frac{\phi}{c}, \mathbf{A} \right), \quad J^\alpha = (\rho c, \mathbf{J}). \quad (3)$$

The solution of the above equation is (suppose a Green function G_R).

$$A^\mu = \mu_0 \int d^4 \mathbf{x} G_R(\mathbf{x}, \mathbf{x}') J^\mu(\mathbf{x}'). \quad (4)$$

G_R is the solution of

$$\partial^\mu \partial_\mu G_R(\mathbf{x}, \mathbf{x}') = \delta^4(\mathbf{x} - \mathbf{x}'). \quad (5)$$

Since it should satisfies the law of causality, we pick¹

$$G_R(\mathbf{x}, \mathbf{x}') = \frac{\delta(t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c})}{4\pi |\mathbf{r} - \mathbf{r}'|}. \quad (6)$$

Let

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \quad \tilde{t} = t - \frac{R}{c}. \quad (7)$$

We agree that any time dependent variable $f(t)$ with² \sim means replace t as \tilde{t} . Which means $\tilde{f}(t) = f(\tilde{t})$. Or³ we will write it into the form of $\mathcal{R}[f(t)](\mathbf{r}_1, \mathbf{r}_2) = f(\tilde{t})$ to remind us that it is not a simple function. For convenience, if there won't be any confusion, we neglect \mathbf{r}' that have appeared in the inner function $f(\mathbf{r}', t)$,

¹Another one is of the form: $t + r/c$, space and time is some sense dissymmetry since they take different sign, so when using Fourier transform to solve it, $1/(k^2 - \frac{\omega^2}{c^2})$ contribute a pair of pole.

²Some books or articles use $[\cdot]$ to denote the retarded term.

³Seldom do that, since this operator is not a linear operator in the whole space. But it will help us clarify the relation of variables when doing complex work.

and write the retarded term as $\mathcal{R}[f(\mathbf{r}', t)](\mathbf{r})$. We even neglect \mathbf{r} , which point out the position where we evaluate the retarded term. Also, sometimes we will use lower index “ret” to emphasize the expressions.

Deduce the term of time in (4), we obtain,

$$A^\mu(\mathbf{x}) = \mu_0 \int \frac{\tilde{J}^\mu(\mathbf{x}')}{4\pi R} d\mathbf{r}'. \quad (8)$$

2 Properties of Retardation Notation

Suppose $J = J(\mathbf{x})$, then

$$\frac{\partial \mathcal{R}[J(\mathbf{r}, t)]}{\partial x^\mu} = \frac{\partial \mathbf{r}}{\partial x^\mu} \cdot \nabla J(\mathbf{r}, \tilde{t}) + \frac{\partial \tilde{t}}{\partial x^\mu} \cdot \frac{\partial J(\mathbf{r}, \tilde{t})}{\partial \tilde{t}}. \quad (9)$$

In particular,

$$\frac{\partial}{\partial t} \mathcal{R}[J(\mathbf{r}, t)] = \mathcal{R} \left[\frac{\partial}{\partial t} J(\mathbf{r}, t) \right]. \quad (10)$$

$$\nabla \mathcal{R}[J(\mathbf{r}, t)] = \mathcal{R}[\nabla J(\mathbf{r}, t)] - \mathcal{R} \left[\frac{1}{c} \frac{\partial J(\mathbf{r}, t)}{\partial t} \right] \nabla R. \quad (11)$$

Note that \mathcal{R} is equipped with a position $\mathbf{r}_{\mathcal{R}}$, so we can define another derivative, which act on the position $\mathbf{r}_{\mathcal{R}}$.

$$\nabla_{r_{\mathcal{R}}} \mathcal{R}[J(\mathbf{r}, t)] = -\frac{1}{c} \frac{\partial J(\mathbf{r}, \tilde{t})}{\partial \tilde{t}} \cdot \nabla_{r_{\mathcal{R}}} R = -\mathcal{R} \left[\frac{1}{c} \frac{\partial J(\mathbf{r}, t)}{\partial t} \right] \nabla_{r_{\mathcal{R}}} R. \quad (12)$$

Since $\nabla R = -\nabla_{r_{\mathcal{R}}} R$, we have

$$\nabla \mathcal{R}[J(\mathbf{r}, t)] + \nabla_{r_{\mathcal{R}}} \mathcal{R}[J(\mathbf{r}, t)] = \mathcal{R}[\nabla J(\mathbf{r}, t)] \quad (13)$$

By symmetry of definition,

$$\nabla_1 \mathcal{R}[f](\mathbf{r}_1, \mathbf{r}_2) = -\nabla_2 \mathcal{R}[f](\mathbf{r}_1, \mathbf{r}_2). \quad (14)$$

Example: Verify that our solution (8) satisfies Lorentz gauge.

$$\frac{\partial \phi}{\partial t} = \int \frac{1}{4\pi\epsilon_0 R} \cdot \frac{\partial \tilde{\rho}(\mathbf{r}', t)}{\partial t} dV' = \mathcal{R} \left[\int \frac{1}{4\pi\epsilon_0 R} \cdot \frac{\partial \rho(\mathbf{r}', t)}{\partial t} dV' \right] \quad (15)$$

$$\nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\frac{\tilde{\mathbf{J}}(\mathbf{r}', t)}{R} \right) dV'. \quad (16)$$

Use the conservation of charge to make a connection of two expressions above. By (13),

$$-\mathcal{R} \left[\frac{\partial \rho(\mathbf{r}', t)}{\partial t} \right] = \mathcal{R}[\nabla' \cdot \mathbf{J}(\mathbf{r}', t)] = \nabla' \cdot \mathcal{R}[\mathbf{J}(\mathbf{r}', t)] + \nabla \cdot \mathcal{R}[\mathbf{J}(\mathbf{r}', t)]. \quad (17)$$

So,

$$\begin{aligned}
\frac{1}{c^2} \frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} &= \frac{\mu_0}{4\pi} \int \left[-\frac{1}{R} (\nabla' \cdot \tilde{\mathbf{J}} + \nabla \cdot \tilde{\mathbf{J}}) + \left(\frac{\nabla \cdot \tilde{\mathbf{J}}}{R} - \frac{\nabla R}{R^2} \cdot \tilde{\mathbf{J}} \right) \right] dV' \\
&= -\frac{\mu_0}{4\pi} \int \left[\frac{1}{R} (\nabla' \cdot \tilde{\mathbf{J}}) + \nabla \left(\frac{1}{R} \right) \cdot \tilde{\mathbf{J}} \right] dV' \\
&= -\frac{\mu_0}{4\pi} \int \left[\frac{\nabla' \cdot \tilde{\mathbf{J}}}{R} - \nabla' \cdot \left(\frac{1}{R} \right) \cdot \tilde{\mathbf{J}} \right] dV \\
&= -\frac{\mu_0}{4\pi} \int \nabla' \cdot \left(\frac{\tilde{\mathbf{J}}}{R} \right) dV' = -\frac{\mu_0}{4\pi} \oint_S \frac{\tilde{\mathbf{J}} \cdot d\mathbf{S}}{R} = 0.
\end{aligned} \tag{18}$$

3 Distribution of Electromagnetic Field

3.1 Precise Solution

By

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{19}$$

and (8)

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \left[\frac{\rho_{\text{ret}}(\mathbf{r}', t)}{R^2} \nabla R + \frac{\nabla R}{cR} \frac{\partial \rho_{\text{ret}}(\mathbf{r}', t)}{\partial t} - \frac{1}{c^2 R} \frac{\partial \mathbf{J}_{\text{ret}}}{\partial t} \right] dV', \tag{20}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\mathbf{J}_{\text{ret}} \times \frac{\nabla R}{R^2} - \frac{\nabla R}{cR} \times \frac{\partial \mathbf{J}_{\text{ret}}}{\partial t} \right] dV'. \tag{21}$$

3.2 Time-harmonic Solution

It is difficult to do the integral if there's some complicated retarded term, we try to solve some simple case. Suppose all the current evolve in the form of $\mathbf{j}_0 e^{-i\omega t}$, then

$$\mathbf{A} = \frac{\mu_0 e^{-i\omega t}}{4\pi} \int \frac{\mathbf{j}_0 e^{ikR}}{R} dV'. \tag{22}$$

This is also difficult to calculate if the distribution is not simple. We can use different approximations in the following according to the specific case.

- Near-field approximation: $kR \ll 1$, retardation effect can be neglected.
- Far-field approximation: $R \approx r \gg r'$, $\mathbf{A} \approx \frac{\mu_0}{4\pi r} e^{i(kr - \omega t)} \int \mathbf{j}_0 e^{-i\mathbf{k} \cdot \mathbf{r}'} dV'$.
- Long-wavelength approximation: In addition, $\mathbf{k} \cdot \mathbf{r}' \ll 1$, $e^{-i\mathbf{k} \cdot \mathbf{r}'} \approx 1 - i\mathbf{k} \cdot \mathbf{r}'$.

Example: A linear antenna of length d oscillates in a full-wave mode with angular frequency $\omega = \frac{2\pi c}{d}$. Find the radiated power per unit solid angle.

To calculate the energy flux, we only need to consider a large sphere, so we take the far-field approximation,

$$\begin{aligned}\mathbf{A} &= \frac{\mu_0 I_0}{4\pi r} e^{i(kr-\omega t)} \int_{-\frac{\lambda}{2}}^{+\frac{\lambda}{2}} \sin(kz') e^{-ikz' \cos \theta} dz' \hat{\mathbf{z}} \\ &= -\frac{i\mu_0 I_0}{2\pi kr} e^{i(kr-\omega t)} \frac{\sin\left(\frac{\pi}{2} \cos \theta\right)}{\sin^2 \theta} \hat{\mathbf{z}}\end{aligned}\quad (23)$$

Retain the leading term of $\frac{1}{r}$ in \mathbf{B} and \mathbf{E} , $\nabla \rightarrow i\mathbf{k}$,

$$\mathbf{B} = -\frac{\mu_0 I_0}{2\pi r} \frac{\sin\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} e^{i(kr-\omega t)} \hat{\boldsymbol{\phi}}, \quad (24)$$

$$\mathbf{E} = \frac{\mu_0 I_0 c}{2\pi r} \frac{\sin\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} e^{i(kr-\omega t)} \hat{\boldsymbol{\theta}}. \quad (25)$$

$$\bar{\mathbf{S}} = \frac{c\mu_0 I_0^2}{8\pi^2 r^2} \left[\frac{\sin\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \right]^2 \hat{\mathbf{r}}. \quad (26)$$

$$\frac{d\bar{P}}{d\Omega} = \frac{c\mu_0 I_0^2}{8\pi^2} \left[\frac{\sin\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \right]^2. \quad (27)$$

3.3 Multiple Expansion

If we have the condition of long-wavelength approximation, we can do the multiple expansion of the vector potential. Also, we retain the term of $\frac{1}{r}$.

$$\mathbf{A} \approx \frac{\mu_0}{4\pi r} \int \tilde{\mathbf{J}} dV'. \quad (28)$$

We first introduce two lemmas.

Lemma.

$$\int \mathbf{J} dV = \frac{d\mathbf{p}}{dt}. \quad (29)$$

Proof.

$$\mathbf{J} = \mathbf{J} \cdot \nabla \mathbf{r} = \nabla \cdot (\mathbf{J}\mathbf{r}) - (\nabla \cdot \mathbf{J})\mathbf{r} = \nabla \cdot (\mathbf{J}\mathbf{r}) + \frac{\partial \rho}{\partial t} \mathbf{r}. \quad (30)$$

The following is easy. □

Lemma.

Let

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r} \times \mathbf{J} dV, \quad \mathbf{D} = \int 3\rho \mathbf{r} \mathbf{r} dV, \quad (31)$$

then,

$$\mathbf{F} \cdot \int \mathbf{r} \mathbf{J} dV = \frac{1}{6} \mathbf{F} \cdot \dot{\mathbf{D}} - \mathbf{F} \times \mathbf{m}. \quad (32)$$

where \mathbf{F} is an arbitrary vector.

Proof. We have

$$\nabla \cdot (\mathbf{J} \mathbf{r} \mathbf{r}) = (\nabla \cdot \mathbf{J}) \mathbf{r} \mathbf{r} + \mathbf{J} \mathbf{r} + \mathbf{r} \mathbf{J}, \quad (33)$$

$$\mathbf{F} \cdot (\mathbf{J} \mathbf{r} - \mathbf{r} \mathbf{J}) = \mathbf{F} \times (\mathbf{r} \times \mathbf{J}), \quad (34)$$

So,

$$\mathbf{r} \mathbf{J} = \frac{\nabla \cdot (\mathbf{J} \mathbf{r} \mathbf{r}) + \frac{\partial \rho}{\partial t} \mathbf{r} \mathbf{r} + (\mathbf{r} \mathbf{J} - \mathbf{J} \mathbf{r})}{2}. \quad (35)$$

The following is easy. \square

Now we expand $\mathcal{R}[\mathbf{J}(\mathbf{r}', t)](\mathbf{r}'' = \mathbf{r}', \mathbf{r})$ around $\mathbf{r}'' = 0$,

$$\mathcal{R}[\mathbf{J}(\mathbf{r}', t)](\mathbf{r}) = \mathcal{R}[\mathbf{J}(\mathbf{r}', t)](0, \mathbf{r}) - \mathbf{r}' \cdot \nabla \mathcal{R}[\mathbf{J}(\mathbf{r}', t)](0, \mathbf{r}) + o(\mathbf{r}'). \quad (36)$$

The minus sign comes from exchanging the derivative of retardation positions. In this cases, it is convenient to use retard-time-transition operator \mathcal{T} defined as $(\mathcal{T}_a f)(t) = f(t - a)$. Then

$$\mathcal{R}[\mathbf{J}(\mathbf{r}', t)](\mathbf{r}) = \mathcal{T}_{\frac{r}{c}} \mathbf{J}(\mathbf{r}', t) - \mathbf{r}' \cdot \nabla (\mathcal{T}_{\frac{r}{c}} \mathbf{J}(\mathbf{r}', t)) + o(\mathbf{r}'). \quad (37)$$

By (12),

$$\mathcal{R}[\mathbf{J}] = \mathcal{T} \mathbf{J} + \mathbf{r}' \cdot \nabla r \frac{(\mathcal{T} \dot{\mathbf{J}})}{c} + o(\mathbf{r}'). \quad (38)$$

By two lemmas above, we have

$$\mathbf{A} \approx \frac{\mu_0}{4\pi r} \mathcal{T}_{\frac{r}{c}} \left[\dot{\mathbf{p}} - \frac{\hat{\mathbf{r}}}{c} \times \dot{\mathbf{m}} + \frac{1}{6} \frac{\hat{\mathbf{r}}}{c} \cdot \ddot{\mathbf{D}} \right]. \quad (39)$$

In the following part, we can use $\nabla \rightarrow -\frac{\hat{\mathbf{r}}}{c} \frac{\partial}{\partial t}$ to calculate \mathbf{E} and \mathbf{B} retaining the leading term of $\frac{1}{r}$. If we only consider the electric dipole term, the energy flow density is

$$\mathbf{S}(t) = \mathcal{T} \left[\frac{\mu_0 (\hat{\mathbf{r}} \times \ddot{\mathbf{p}})^2}{16\pi^2 r^2 c} \hat{\mathbf{r}} \right]. \quad (40)$$