习题课 9.5. Warm Up, Jensen's Inequality, Ordinals.

1. Q: Compare $7/13$ and $6/11$

Solution:
$7/13 < 6/11 \iff (7/13) \times 11 \times 13 < 6/11 \times 11 \times 13$

$\iff 7\times 11 < 6 \times 13 \iff 77 < 78$. This is true

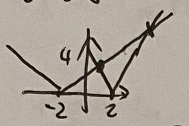
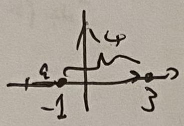
~~from then~~ Hence $7/13 < 6/11$

3. Q: Compare
$a = (10^9+2)^2 \quad b = (10^9-4)\times(10^9+8)$

Solution: Let $t = 10^9+2$

Then $a = t^2 \quad b = (t-6)\cdot(t+6)$
$= t^2-36$

$\Rightarrow b < a$.

5. Resolve in $\mathbb{R}$

① $|x+3| \le 5$ ② $|2x-4| \le |x+2|$ ③ $|x+1| + |x-3| \le 6$ ---(*)

Since $|a| \le b \iff \begin{cases} a \le b \\ -a \le b \end{cases}$ The original question $\iff$

① $\begin{cases} x+3 \le 5 \\ -(x+3) \le 5 \end{cases} \iff x \in [-8, 2]$

②. $|2x-4| \le |x+2| \iff (2x-4)^2 \le (x+2)^2$

$\iff 4x^2 - 16x + 16 \le x^2 + 4x + 4$

$\iff 3x^2 - 20x + 12 \le 0$

$\iff x^2 - \frac{20}{3}x + 4 \le 0$

$(x-\frac{10}{3})^2 - \frac{64}{9} \le 0$

$\iff (x-\frac{10}{3})^2 \le \frac{64}{9}$

$\iff |x - \frac{10}{3}| \le \frac{8}{3}$

$\iff \begin{cases} x - \frac{10}{3} \le \frac{8}{3} \\ -(x-\frac{10}{3}) \le \frac{8}{3} \end{cases}$

$\iff x \in [\frac{2}{3}, 6]$

③. split into 3 cases, $x \in]-\infty, 1[$, $[1,3]$ and $]3, \infty[$

Case I : $x \in]-\infty, 1[$. The ori

$(*) \iff -x-1-x+3 \le 6$

$\iff -2x \le 4 \iff x \ge -2$

In this case $x \in$ ~~$[-2,1[$~~ the solution set is $x \in [-2, 1[$

Case II : $x \in [1,3]$

$(*) \iff x+1-x+3 \le 6 \iff 4 \le 6$ which is true

In this case $\forall x \in [1,3]$ ~~all belongs~~ is a solution

Case III : $x \in]3, +\infty[$

$(*) \iff x+1+x-3 \le 6 \iff x \le 4$.

~~with~~ the solution set : $]3, 4]$

Combining these cases, the solution set is

$[-2, 1[\,\cup\, [1,3] \,\cup\,]3, 4]$

$= [-2, 4]$.

9. Determine the infimum of $A = \{\frac{a}{b} + \frac{b}{a} \mid (a,b) \in \mathbb{R}^2_{>0}\}$

Solution: 2

Proof: Let x ~~be the infimum of~~ be $\inf A$

when $a = b = 1$

① $\frac{a}{b} + \frac{b}{a} = 2 \in A$.

$\Rightarrow x \le 2$

②. ~~$\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} \ge 2$~~

Since $a^2+b^2 \ge 2ab$

$\frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} \ge 2$ when $a, b > 0$.

$\Rightarrow 2$ is a lower bound of A

$\Rightarrow x \ge 2$.

11. Let X be a set.

(1). Prove that the relation of inclusion $\subseteq$ is an order relation on $\mathcal{P}(X)$.

(2). Let I be a set and $(A_i)_{i\in I}$ be a family of elements of $\mathcal{P}(X)$. Prove that $\sup_{i\in I} A_i$, $\inf_{i\in I} A_i$ exists.

Proof: (1). It suffices to check $\subseteq$ is

①. reflexive. $\forall A \in \mathcal{P}(X)$, $A \subseteq A$

②. ~~antisymmetric~~ transitive $\forall A, B, C \in \mathcal{P}(X)$ $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

③. antisymmetric $\forall A, B \in \mathcal{P}(X)$ $A \subseteq B \wedge B \subseteq A \Rightarrow A = B$.

Thus $\subseteq$ is an order relation.

13. Prove that $\forall n\in\mathbb{N}$,

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1).$$

Pf: We proceed by induction on n

① $n=0$. $0=\frac{1}{6}\times 0\times 1\times 0=0$ holds.

② ~~not n=1 We~~ Suppose we have proved

$$\sum_{k=1}^{n-1} k^2 = \frac{1}{6}(n-1)\cdot n\cdot(2n-1) \text{ for some } n\geq 1.$$

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n-1} k^2 + n^2 = \frac{1}{6}(n-1)\cdot n\cdot(2n-1)+n^2$$

$$= \frac{1}{6}\cdot n(2n^2-3n+1)+\frac{1}{6}\cdot n\cdot 6n$$

$$= \frac{1}{6}\cdot n(2n^2+3n+1) = \frac{1}{6}n\cdot(n+1)(2n+1)$$

By induction, $\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)$ $\forall n\in\mathbb{N}$.

15. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ defined as

$u_0=1$, $u_{n+1}=\sqrt{2+u_n}$. Prove that, for any $n\in\mathbb{N}$, $0\leq u_n<2$.

Pf: By induction on n.

① $n=0$ ~~$u_0=1$~~ $0<u_0<2$ holds.

② Suppose we have $0<u_n<2$

~~$u_{n+1}=\sqrt{2+u_n} \Rightarrow 0<u_{n+1}$~~

$$2<2+u_n<4$$

$$0<\sqrt{2}<\underbrace{\sqrt{2+u_n}}_{u_{n+1}=}<2$$

~~By ind~~ Thus $0<u_{n+1}<2$ holds for every $n\in\mathbb{N}$.

17 $2^{n-1}\leq n!\leq n^n$.

Prove that, for any positive integer n,

Pf: ~~By induction on~~ n. We proceed by induction on n.

① $n=1$. ~~$2^{1-1}=1\leq 1!$~~ $2^{1-1}=1!=1^1=1$.

② Suppose we have $2^{n-1}\leq n!\leq n^n$. for some $n\geq 1$

$$2^n=2\times 2^{n-1}\leq 2\times n!\leq (n+1)\times n! = (n+1)!$$

$$(n+1)!=(n+1)\cdot n!\leq (n+1)\cdot n^n\leq (n+1)\cdot(n+1)^n=(n+1)^{n+1}.$$

□.

19. Let $(u_n)_{n\in\mathbb{N}}$ be the sequence in $\mathbb{R}$ defined as

$u_0=2$, $u_1=3$. $u_{n+2}=3u_{n+1}-2u_n$ for $n\in\mathbb{N}$.

Prove that, for any $n\in\mathbb{N}$, $u_n=2^n+1$

Pf: We proceed by induction on n.

① $n=0$ and 1. $u_0=2=2^0+1=2$

$u_1=3=2^1+1$

② ~~We~~ suppose we have $\forall n\leq k-1$ $u_n=2^n+1$ for $n=k$. $(k\geq 2)$

$$u_k=3u_{k-1}-2u_{k-2}$$

$$=3\cdot(2^{k-1}+1)-2(2^{k-2}+1)$$

$$=3\cdot 2^{k-1}+3-2^{k-1}-2$$

$$=2\cdot 2^{k-1}+1$$

$$=2^k+1.$$

Hence $u_n=2^n+1$ for every $n\in\mathbb{N}$.

21. Recall that $\sqrt{2}$ is irrational. Show that $\cos(1°)$ is irrational.

Pf: We ~~can~~ prove by contradiction.

Assume that $\cos 1°$ is rational.

Then we prove by induction, all $\cos(n°)$ $n\in\mathbb{N}$ are rational.

Upshot: $\cos n° + \cos(n-2)° = 2\cos(n-1)°\cos 1°$

3|

Extra: (Jensen's inequality.)

Def (Convex function on $\mathbb{R}$). Let $X \subseteq \mathbb{R}$ be an interval.

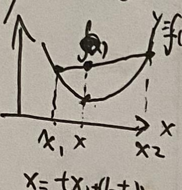
- A function $f: X \to \mathbb{R}$ is called convex ~~if~~ (resp. strictly convex) for all $x_1, x_2 \in X$, and $t \in [0,1]$,

$$f(tx_1+(1-t)x_2) \le t f(x_1)+(1-t)f(x_2)$$

(~~resp. f(tx1+(1-t~~)

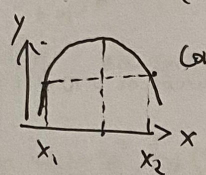
- A function $f: X \to \mathbb{R}$ is called concave if for all $x_1, x_2 \in X$ and $t \in [0,1]$,

$$f(tx_1+(1-t)x_2) \ge t f(x_1)+(1-t)f(x_2)$$



convex

$x = tx_1+(1-t)x_2$



concave

Fact: If ~~&~~ $f: X \to \mathbb{R}$ is twice differentiable, then it is convex if and only if ~~its Hessian~~ ~~$\nabla^2 f(x)$~~ is ~~semi-positive definite~~ $f''(x) \ge 0$ (resp. $f''(x) \le 0$) (resp. concave)

(1). Show that $f(x) = \log x$ is concave. Let X be $]0,+\infty[$,

(2). Let X be $]0,+\infty[$. p a real number ≥ 1. Show that $f(x) = x^p$ is convex.

(3). ~~Use induction~~ (Jensen's inequality) Let f be a convex function, $0 \le \alpha_i \le 1$ for $i=1,\dots,n$. ~~Show that~~ Such that $\sum_{i=1}^n \alpha_i = 1$. Show that for every $x_1,\dots,x_n \in X$, we have

$$f\left(\sum_{i=1}^n \alpha_i x_i\right) \le \sum_{i=1}^n \alpha_i f(x_i)$$

(4) Show that for all real numbers $x_1,\dots,x_n$ $\left(\sum_{i=1}^n x_i\right)^2 \le n \cdot \sum_{i=1}^n x_i^2$

(5) (Young's inequality) ~~use (1) to show that~~ For $p,q>1$ s.t. $\frac{1}{p}+\frac{1}{q}=1$, and $x,y \in]0,+\infty[$. Show that ~~$\log(\frac{1}{p}x^p+\frac{1}{q}y^q) \ge \frac{1}{p}\log(x^p)$~~ $xy \le \frac{1}{p}x^p + \frac{1}{q}y^p$. (Hint: ~~you~~ use (1))

(6). ~~Show~~ (Hölder's inequality) For $p,q>1$ s.t. $\frac{1}{p}+\frac{1}{q}=1$, $x_i, y_i > 0$ $i=1,\dots,n$. Show that

$$\sum_{i=1}^n x_i y_i \le \left(\sum_{j=1}^n x_j^p\right)^{1/p} \left(\sum_{j=1}^n y_j^q\right)^{1/q}$$

In particular, when $p=q=2$, this is the Cauchy-Schwarz inequality

$$\sum_{i=1}^n x_i y_i \le \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n y_i^2\right)^{1/2}.$$

~~(Hint: Show that [illegible])~~

Pf: (1) ~~By fact, it suffices to show $f''(x) \le 0$~~. $f''(x) = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} \le 0$ $\forall x > 0$. Using fact, we know $f(x) = \log x$ is concave.

(2). $f''(x) = (p x^{p-1})' = p \cdot (p-1) \cdot x^{p-2} \ge 0$ $\forall x > 0$ when $p > 1$. Using fact, we know $f(x) = x^p$ is convex.

(3). We prove by induction on n.

① case $n=1$. trivial.

~~Case n=2. definition of~~

② suppose for $n = K-1$, the statement holds. Let $\alpha_1, \dots, \alpha_K$ be K real numbers. $0 \le \alpha_i \le 1$, $1 \le i \le K$. Then at least one of α_i is not 1. $(K \ge 2)$ Assume $\alpha_1 < 1$. Then ~~by inducti~~

$$f\left(\sum_{i=1}^K \alpha_i x_i\right) = f\left(\alpha_1 x + (1-\alpha_1)\sum_{i=2}^K \frac{\alpha_i}{1-\alpha_1} x_i\right)$$

$$\le \alpha_1 f(x_1) + (1-\alpha_1) f\left(\sum_{i=2}^K \frac{\alpha_i}{1-\alpha_1} x_i\right)$$

$$\le \alpha_1 f(x_1) + (1-\alpha_1)\sum_{i=2}^K \frac{\alpha_i}{1-\alpha_1} f(x_i)$$

(by induction hyp. $\sum_{i=2}^k \frac{\alpha_i}{1-\alpha_1} = 1$)

$$= \sum_{i=1}^K \alpha_i f(x_i)$$

(4). The function $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ is convex since $f''(x) = 2 \ge 0$. Thus we ~~have~~ can apply (3)

$$\left(\frac{1}{n}\sum_{i=1}^n x_i\right)^2 = f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \le \sum_{i=1}^n \frac{1}{n} f(x_i) = \frac{1}{n}\sum_{i=1}^n x_i^2.$$

(f convex)

(5). $\log(\frac{1}{p}x^p+\frac{1}{q}y^p) \geq \frac{1}{p}\log(x^p)+\frac{1}{q}\log(y^q)$
$= \log xy.$

Since exp is monotone, we get.

$$\frac{1}{p}x^p+\frac{1}{q}y^p \geq xy$$

(6) we first prove the hint.

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} x_i y_i^{1-q} \cdot y_i^{q} \frac{\sum_{j=1}^{n} y_j^{q}}{\sum_{k=1}^{n} y_k^{q}}$$

$$= \sum_{j=1}^{n} y_j^{q} \cdot \sum_{i=1}^{n} x_i \cdot y_i^{1-q}\left(\frac{y_i^{q}}{\sum_{k=1}^{n} y_k^{q}}\right) \quad ---- (*)$$

Since $\sum_{i=1}^{n} \frac{y_i^{q}}{\sum_{k=1}^{n} y_k^{q}} = 1$, we can apply Jensen's inequality (3) for $f(x)=x^p$, by (2)

$$\cancel{(**) \leq}\left(\sum_{i=1}^{n} x_i y_i^{1-q}\left(\frac{y_i^{q}}{\sum_{k=1}^{n} y_k^{q}}\right)\right)^p \leq \sum_{i=1}^{n} (x_i y_i^{1-q})^p \cdot \left(\frac{y_i^{q}}{\sum_{k=1}^{n} y_k^{q}}\right)$$

$$= \sum_{i=1}^{n} x_i^p y_i^{-q} \cdot \frac{y_i^{q}}{\sum_{k=1}^{n} y_k^{q}} = \frac{\sum_{i=1}^{n} x_i^p}{\sum_{k=1}^{n} y_k^{q}} \quad \cancel{p(1-q)+q=0}$$

Substitute $(**)^{\frac{1}{p}}$ into $(*)$

$$(*) \leq \sum_{j=1}^{n} y_j^{q} \left(\frac{\sum_{i=1}^{n} x_i^p}{\sum_{k=1}^{n} y_k^{q}}\right)^{\frac{1}{p}}$$

$$= \left(\sum_{j=1}^{n} y_j^{q}\right)^{(1-\frac{1}{p})} \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$$

$$= \left(\sum_{j=1}^{n} y_j^{q}\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}$$ as desired □.

---

23. If x and y are two sets, we denote by $x \underline{\in} y$ the statement "$x=y$ or $x \in y$". We say that α is an ordinal if elements of α are all subsets of α and $(\alpha, \in)$ forms a well-ordered set.

(1). Let x, y be sets, prove that $x \underline{\in} y$ if and only if $x \in y \cup \{y\}$

(2). Prove that $\emptyset$ is an ordinal

(3). Let α be a set. Assume that its elements are all sets and $(\alpha \cup \{\alpha\}, \underline{\in})$ forms a well ordered set, then α is an ordinal.

(4). Let α be an ordinal. Prove that $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha = \bigcup_{x \in \alpha \cup \{\alpha\}} x$.

In the following question, if α is an ordinal, we denote by $\alpha+1$ the ordinal $\alpha \cup \{\alpha\}$.

(5). Let α, β be ordinals. Prove that if $\alpha+1=\beta+1$, then $\alpha=\beta$

(6) Let α be an ordinal, and x, y be two elements of α. Prove that one and only one of the following holds: $x \in y$, $x=y$, $y \in x$.

(7). Let α be a ordinal. Prove that all elements of α are ordinals

(8) Let α, β be ordinals, Prove that if $\beta \subseteq \alpha$, then β is an initial segment of α $\quad X_{<\alpha} \subseteq I \ \forall \alpha \in I$ then I is initial seg

(9). Let α be an ordinal (recall: $\beta = X_{<i}$ for some $i \in \alpha$) $= \{x \in \alpha \mid x < i\}$ prove that $\alpha+1$ identifies with the set of all initial segments of α. Deduce that all initial segments of α are ordinals.

(10) Let α, β be ordinals. Prove that the following conditions are equivalent.

(a) $\beta \in \alpha$. (b). β is an initial segment of α and $\beta \neq \alpha$.

(c). $\beta \subsetneq \alpha$.

(11). Let I be a non-empty set and $(\alpha_i)_{i \in I}$ be a family of ordinals. Prove that $\bigcap_{i \in I} \alpha_i$ is an ordinal.

(12) Let α, β be ordinals, ~~prove that~~ prove that either $\alpha \subseteq \beta$ or $\beta \subsetneq \alpha$.

(13). Let A be a family of ordinals. Prove that $(A, \leq)$ is a well-ordered set, and ~~the~~ "$\subseteq$" the binary relation "$\leq$" and $\underline{\in}$ coincide on A.

(14) Let I be a set and $(\alpha_i)_{i \in I}$ be a family of ordinals. Prove that $\bigcup_{i \in I} \alpha_i$ is an ordinal.

Pf (1). Since $x = y \iff x \in \{y\}$, $x \subseteq y \iff x \in y$ or $x = y$

~~$x \subseteq y$ if and only if~~ $\iff x \in y$ or $x \in \{y\} \iff x \in y \cup \{y\}$

(2). It suffices to check

① elements of ϕ are subsets of ϕ. Trivially true since there is no element of ϕ.

② $(\phi, \in)$ forms a well-ordered set, i.e. every nonempty subset of ϕ has a least element. Trivially true b/c there is no nonempty subset of ϕ.

(3). We check that

① ~~elements of α are subsets of α~~ $x \in \alpha \Rightarrow x \subseteq \alpha$. For every $x \in \alpha$, we know that $x \in \alpha \cup \{\alpha\}$. since $\alpha \cup \{\alpha\}$ is an ordinal. ~~x is subset~~ $x \subseteq \alpha \cup \{\alpha\}$

It suffice to show $\alpha \notin x$.

If $\alpha \in x$ and $x \in \alpha$ then $x = \alpha$ by anti symmetry of order $\in$. $\alpha \notin \alpha$.

Axiom: Recall the axiom of foundation in set theory: $\forall S, S \neq \phi \Rightarrow (\exists x \in S, S \cap x = \phi)$. Every nonempty set has a $\in$-minimal element.

~~If $\alpha \in x$~~ (Assume, by contradiction, $\alpha \in x$.)

Then $S = \{\alpha, x\}$ violates axiom of foundation. since $x \in \alpha \cap S$, $\alpha \in x \cap S$. This is a contradiction.

Hence $\alpha \notin x$.

② WTS $(\alpha, \in)$ is well-ordered. since it is a subset of well-ordered set $(\alpha \cup \{\alpha\}, \in)$. (Prop 4.6.2)

~~Any subset ... again well-ordered (Prop 4.6.2).~~

(4). We check that

① $\forall x \in \alpha \cup \{\alpha\}$: $x \in \alpha \cup \{\alpha\} \Rightarrow x \subseteq \alpha \cup \{\alpha\}$. either $x \in \alpha$ or $x = \alpha$.

a. If $x \in \alpha$. since α is ordinal, $x \subseteq \alpha \subseteq \alpha \cup \{\alpha\}$ $\Rightarrow x \subseteq \alpha \cup \{\alpha\}$

b. If $x = \alpha$, we have $x \subseteq \alpha \cup \{\alpha\}$

②. Every nonempty subset A of $\alpha \cup \{\alpha\}$ has a least element

a. If $A = \{\alpha\}$ then A has least element α.

b. If $A \neq \{\alpha\}$. then $A \cap \alpha$ is nonempty.

$A \wedge \alpha$ has a least element $x \in A \wedge \alpha$ since $(\alpha, \in)$ is well-ordered. This x is also least element of A. To see this ~~since~~ $x \subseteq \alpha$. trivial. if ~~$A = A \wedge \alpha$~~ $\alpha \notin A$ ①

$x \in \alpha \Rightarrow x \subseteq \alpha$ if ~~$A = (A \wedge \alpha) \cup \{\alpha\}$~~ $\alpha \in A$ ②

(5).

~~... $\{0, 1, 2, \dots\}$ ... $\{0, 1, 2, \dots, \omega, \omega+1, \dots\}$ ...~~

If $\alpha \cup \{\alpha\} = \beta \cup \{\beta\}$.

Then $\alpha \in \beta \cup \{\beta\}$, $\beta \in \alpha \cup \{\alpha\}$

a. If $\alpha \in \{\beta\}$, then $\alpha = \beta$. All done.

~~If $\alpha \in \beta$. then~~

~~b. It remains to deal the case $\alpha \in \beta$.~~

or b. If $\beta \in \{\alpha\}$. then $\alpha = \beta$. All done

It remains to deal with case $\alpha \in \beta$ and $\beta \in \alpha$. But this violates axiom of foundation, by taking $S = \{\alpha, \beta\}$.

(6). ~~$\alpha+1$ is also ordinal~~ Since ~~$x, y \in \alpha+1$~~. $\in$ is well-order, thus total order on α.

$\Rightarrow x \subseteq y$ or $y \subseteq x$.

$\Rightarrow x \in y$ or $x = y$ or $y \in x$.

(7). ① $\forall$ element $x \in \alpha$ $x \subseteq \alpha$.

Thus $(x, \in)$ as restriction of $(\alpha, \in)$ is also a well-order set well ordered

②. $\forall y \in x$. WTS $y \subseteq x$.

since $x \subseteq \alpha$, we know $y \in \alpha$ thus $y \subseteq \alpha$. ~~If~~

~~By (6) $x \in y$ or $x = y$ or $y \in x$~~

$\forall z \in y$, since $y \subseteq \alpha \Rightarrow$ we know $z \in \alpha$.

by (6) either $z \in x$ or $z = x$ or $x \in z$.

if $z = x$ then $x \in y \wedge y \in x$, contradiction! $x = y$ $x \in x$ $(S = \{x, y\})$.

if $x \in z$. then $x \in z \wedge z \in y \wedge y \in x$, contradiction! $x = y = z$ $\Rightarrow x \in x$.

thus $z \in x$

we have shown that $\forall z \in y \Rightarrow z \in x$. thus $y \subseteq x$

This finishes the proof. □

(8). prove if $\beta \subseteq \alpha$. β also an ordinal,

Claim

First. See next page

Let $X = \{x \in \alpha+1 \mid \forall t \in \beta, t \in x\} = \beta^u$ (upper bound in $\alpha+1$)

Since α is well-ordered. β^u has a least element l (β^u is nonempty since $\alpha \in \beta^u$)

WTS: $\beta = \alpha_{<l}$.

① $\beta \subseteq \alpha_{<l}$ since l is a upper bound of β.

②. $\forall x \in \alpha_{<l}$ ~~$x \in \beta$ since either~~ x is not a upper bound of β. $\exists y \in \beta$, s.t. $x \subseteq y$.

β is ordinal $\Rightarrow$ $y \subseteq \beta$. $\Rightarrow x = y \in \beta$ or $x \in y \subseteq \beta$.

either way. $x\in\beta$. Thus $(\alpha+1)_{<\ell} \subseteq \beta$

Combining ①②. $\beta = (\alpha+1)_{<\ell}$

(4). We further show that $\ell=\beta$ in the proof of (8)

$\beta = (\alpha+1)_{<\ell}$.

$\forall x\in\beta \subseteq\alpha\subseteq\alpha+1$ $x\le\ell$ and $x\ne\ell \Rightarrow x\in\ell$

$\forall x\in\ell$, $x\le \ell$ and $x\ne\ell \Rightarrow x\in\alpha_{<\ell}=\beta$

$\Rightarrow \beta=\ell$.

Thus $\beta = (\alpha+1)_{<\beta}$ and $\beta\in\alpha+1=\alpha\cup\{\alpha\}$

~~Now we define~~ now if $\beta=\alpha$. Then β is an initial segment of α.

If $\beta\in\alpha$. Then $\beta=(\alpha+1)_{<\beta} = \alpha_{<\beta}$ since the only ~~differ~~ element in $\alpha+1$ not in α is α and ~~$\alpha<\beta$ does not hold~~, $\alpha\notin(\alpha+1)_{<\beta}$.

$\Rightarrow\beta$ is initial segment of α

(9). ~~$\beta=(\alpha+1)_{<\beta}$ from the previous part~~

~~$\forall\beta\in\alpha+1$ know~~

~~From (8) we already have a map from $\alpha+1$~~

to $\forall\beta\in\alpha+1$, by (7) β is an ordinal and $\beta\subseteq\alpha$. By (8) we know β is an initial segment of α

It remains to show all initial segments I ~~have this form~~ are ordinals and $I\in\alpha+1$.

①. well ordered ✓

②. ~~elements of~~ Let $x\in I$. WTS $x\subseteq I$.

$x\in I\subseteq\alpha$. x is an ordinal by (7).

$x=(\alpha+1)_{<x}=\alpha_{<x}$ by (4) the proof of (8). $(x\ne\alpha)$

$\alpha_{<x}\subseteq I$ since I is initial segment

$\Rightarrow x\subseteq I$.

①+② $\Rightarrow I$ is ~~initial segment~~ ordinal.

by Claim: I is ordinal and $I\subseteq\alpha$

$\Rightarrow I\in\alpha+1$.

(8) We prove a stronger claim:

Claim: If β is an ordinal and $\beta\subseteq\alpha$

Then $\beta=(\alpha+1)_{<\beta}$. $(=\{x\in\alpha+1 \mid x\in\beta\})$ and $\beta\in\alpha+1$.

Claim ⇒ (8) If $\beta=\alpha$, If $\beta\in\alpha$

pf of Claim: ...

(10). (a)⇒(b) If $\beta\in\alpha$. By (7), β is an ordinal. Then $\beta\subseteq\alpha$ since α is ordinal

By (8), β is an initial segment.

We have $\beta\ne\alpha$ because otherwise, we have $\alpha\in\alpha$. Contradicting.

(b)⇒(c) trivial.

(c)⇒(a) by Claim: $\beta\in\underbrace{\alpha\cup\{\alpha\}}_{\alpha+1}$

$\beta\ne\alpha\Rightarrow\beta\in\alpha$.

(11) ~~② $\forall$ ... we~~

Let $\alpha=\bigcap_{i\in I}\alpha_i$. We ~~want to~~ show

①. $\forall x\in\alpha$, WTS: $x\subseteq\alpha$.

$x\in\alpha\Rightarrow\forall i\ x\in\alpha_i\Rightarrow\forall i, x\subseteq\alpha_i$

$\Rightarrow x\subseteq\bigcap_{i\in I}\alpha_i = \alpha$

② $\bigcap\alpha_i$ is a subset of well-ordered set α_{i_0}, thus also well ordered.
Fix some $i_0\in I$.

①+② ⇒ ~~ordinal~~ α is ordinal.

(12). Let γ be the ordinal $\alpha\cap\beta$ as in (11).

~~Assume~~ We prove by contradiction.

Assume neither $\alpha\subseteq\beta$ nor $\beta\subsetneq\alpha$ holds.

Then $\gamma\subsetneq\alpha$ and $\gamma\subsetneq\beta$

By (10). $\gamma\in\alpha$ and $\gamma\in\beta$.

$\Rightarrow\gamma\in\alpha\cap\beta\Rightarrow\gamma\in\gamma$, contradiction!

Thus either $\alpha\subseteq\beta$ or $\beta\subsetneq\alpha$ holds.

(13). The binary relation $\subsetneq$ and $\in$ coincide by (10) (c)⇔(a)

Now fix any nonempty subset $X \subseteq A$.

~~Wa~~ It suffice to find a least element.

~~Let $\alpha_X = \bigcap X$ (i.e. $\bigwedge_{\alpha \in X} \alpha$)~~

By ~~(11). α_X is an ordinal. it suffice to~~ show

$\alpha_X \in X$.

Fix some $\alpha \in X$. $\forall \beta \in X$, by (12) either $\beta \le \alpha$ or $\alpha \le \beta$

~~$\alpha \le \beta$ or $\beta \le \alpha$. $\beta \le \alpha$ or $\alpha \le \beta$.~~

~~i.e. ($\beta \in \alpha$ or $\beta = \alpha$) or $\alpha \le \beta$~~

~~Take $X_{\le \alpha}$ as a subset~~ ~~i.e. $\beta \in \alpha+1$ or $\alpha \le \beta$.~~

~~$\Rightarrow X_{\le \alpha}$ is a subset of $\alpha+1$.~~

$\beta \le \alpha \iff \beta \in \alpha$ or $\beta = \alpha \iff \beta \in \alpha+1$

Thus $X_{\le \alpha}$ is a subset of $\alpha+1$.

In particular, it is well-ordered. (and nonempty $\alpha \in X_{\le \alpha}$)

It has a least element γ

It is also a least element of X. ~~Since~~

($\forall \beta \notin X_{\le \alpha}$, ~~...~~ $\beta \not\le \alpha \Rightarrow \alpha \le \beta$) by (12)

(14) ①. ~~Let~~

Let $\alpha = \bigcup_{i \in I} \alpha_i$

if $x \in \alpha$. then $\exists i.\ x \in \alpha_i$

since α_i is ordinal, $x \subseteq \alpha_i$.

$\Rightarrow x \subseteq \alpha$.

②. $\forall$ nonempty subset $X \subseteq \alpha = \bigcup_{i \in I} \alpha_i$. Fix some $\beta \in X$

by (13) it has a least element γ. ~~$\exists$~~ $\exists i, \beta \in \alpha_i$ Thus well-ordered.

~~Fix some $x \in X$~~

~~$\exists i,\ x \in \alpha_i$.~~

~~Consider $X_{\le x}$ it is a subset of α_i~~

~~($\forall y \in X_{\le x}$, $y \le x$, $y \le x$)~~

~~$y \in x$ or $y = x$~~

~~$x \subseteq \alpha_i \Rightarrow y \in \alpha_i$ eitherway.) Thus well-ordered~~

~~$X_{\le x}$ is nonempty. $\Rightarrow X_{\le x}$ has a least element γ.~~

~~γ is also the least element of X~~

$0, 1, 2, 3, \dots .\ \omega.\ \omega+1, \omega+2, \dots .\ \omega \cdot 2, \omega \cdot 2+1 \dots$

$\omega \cdot 3 . \dots .\ \omega \cdot \omega .\ \omega \cdot \omega+1 .\ \omega \cdot \omega+2 \dots$

$\omega \cdot \omega \cdot \omega \cdot \omega . \dots \quad \omega^{\omega}$

$\omega^{\omega^{\omega}}$

$\dots$

~~Knaster-Tarski fixpoint theorem~~

Let X