Dirac Delta Function

1 One-Dimentiinal Case

Definition 1.
$\delta(x)$ is a generalized function satisfies:

$$\delta(x) := \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}, \tag{1}$$

$$\int_{-\epsilon}^{+\epsilon} \delta(x) \, \mathrm{d}x = 1, \quad \epsilon > 0. \tag{2}$$

All the equalities should be understand under integration.

Property 1.

$$\delta(-x) = \delta(x). \tag{3}$$

Proof.

$$\delta(-x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}, \tag{4}$$

$$\int_{-\epsilon}^{+\epsilon} \delta(-x) \, \mathrm{d}x = -\int_{+\epsilon}^{-\epsilon} \delta(-x) \, \mathrm{d}(-x) = \int_{-\epsilon}^{+\epsilon} \delta(x) \, \mathrm{d}x, \quad \epsilon > 0. \tag{5}$$

Property 2.

$$\delta(ax) = \frac{1}{|a|}\delta(x) \quad (a \neq 0). \tag{6}$$

Proof.

$$\delta(ax) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases} = \frac{\delta(x)}{|a|}, \tag{7}$$

$$\int_{-\epsilon}^{+\epsilon} \delta(ax) \, \mathrm{d}x = \begin{cases} \frac{1}{a} \int_{-\epsilon}^{+\epsilon} \delta(ax) \, \mathrm{d}(ax) \,, \, a > 0 \\ \frac{1}{a} \int_{+\epsilon}^{-\epsilon} \delta(ax) \, \mathrm{d}(ax) \,, \, a < 0 \end{cases} = \frac{1}{|a|} \int_{-\epsilon}^{+\epsilon} \delta(x) \, \mathrm{d}x. \tag{8}$$

Property 3.

$$f(x)\delta(x-a) = f(a)\delta(x-a). \tag{9}$$

Property 4.

$$\int \delta(x-y)\delta(y-a) \, \mathrm{d}y = \delta(x-a). \tag{10}$$

Property 5.

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} \, \mathrm{d}k. \tag{11}$$

Proof.
Add a convergence factor to soft cutoff the divergent integration.

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} e^{-\epsilon k^2} \, \mathrm{d}k = \frac{e^{-\frac{x^2}{4\epsilon}}}{2\sqrt{\pi\epsilon}}. \tag{12}$$

Then check it like above proof.

Property 6.

$$\delta[g(x)] = \sum_n \frac{\delta(x - x_n)}{|g'(x_n)|}, \; g(x_n) = 0, g'(x_n) \neq 0. \tag{13}$$

Proof.
Around $x = x_n$,

$$g(x) = g(x_n) + g'(x_n)(x - x_n) = g'(x_n)(x - x_n) \tag{14}$$

Thus,

$$\delta[g(x)] = \delta(\sum_n g'(x_n)(x - x_n)) = \sum_n \delta[g'(x_n)(x - x_n)]. \tag{15}$$

By Property 2,

$$\delta[g(x)] = \sum_n \frac{1}{|g'(x_n)|} \delta(x - x_n), \; g(x_n) = 0, g'(x_n) \neq 0. \tag{16}$$

For those $g(x_n) = 0, g'(x_n) = 0$, take $2 \leq m = \min\left(\{m \in \mathbb{N} \mid g^m(x_n) \neq 0\}\right)$. Then we induct on m that

$$\delta[g^m(x_n)(x - x_n)^m] = 0. \tag{17}$$

For $m = 2$,

$$\delta[g''(x_n)(x - x_n)^2] = \tag{18}$$

2 High-Dimentiinal Case

Definition 2.

$$\delta^n(\mathbf{x} - \mathbf{x_0}) = \prod_{i=1}^{n} \delta(x_i - x_{0i}) \tag{19}$$

Property 7 (Normalization)**.**

$$\int_{\mathbb{R}^n} \delta^n(\mathbf{x}-\mathbf{x_0})\,\mathrm{d}x = \prod_{i=1}^{n} \int_{-\infty}^{+\infty} \delta(x_i - x_{0i})\,\mathrm{d}x = 1. \tag{20}$$

Property 8 (Sifting Property)**.**

$$\int_{\mathbb{R}^n} f(\mathbf{x})\delta^n(\mathbf{x}-\mathbf{x_0})\,\mathrm{d}x = f(\mathbf{x_0}). \tag{21}$$

Property 9 (Coordinate Transformations)**.**
For transformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$ with Jacobian $J = \left|\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right|$,

$$\delta^n(\mathbf{x}-\mathbf{x_0}) = \frac{1}{|J|}\delta^n(\mathbf{y}(\mathbf{x}) - \mathbf{y}(\mathbf{x_0})). \tag{22}$$

Easy to check the integration.

Property 10 (Integration by Parts Generalization)**.**

$$\int f(\mathbf{x})\nabla\delta(\mathbf{x}-\mathbf{x_0})\mathrm{d}^n\mathbf{x} = -\nabla f(\mathbf{x_0}). \tag{23}$$

Property 11 (Relationship with Laplacian Operator)**.**

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x_0}|^{n-2}}\right) = -(n-2)S_n\delta^n(\mathbf{x}-\mathbf{x_0}),\ n \neq 2, \tag{24}$$

where

$$S_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{25}$$

is the surface area of the n-dimensional unit sphere. In particular, $n = 3$:

$$\nabla^2\left(\frac{1}{|\mathbf{r}-\mathbf{r_0}|^2}\right) = -4\pi\delta^3(\mathbf{r}-\mathbf{r_0}). \tag{26}$$

Property 12 (Scaling Property)**.**

$$\delta^n(\alpha\mathbf{x}) = \frac{1}{|\alpha|^n}\delta^n(\mathbf{x}). \tag{27}$$

Property 13 (Composition with Functions)**.**
For a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with simple zeros at $\mathbf{x}_k$:

$$\delta(f(\mathbf{x})) = \sum_k \frac{\delta(\mathbf{x}-\mathbf{x}_k)}{|\nabla f(\mathbf{x}_k)|}. \tag{28}$$