Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 10 : Integration calculus

In this exercise sheet, we will always dentoe by a, b two real numbers with $a \leq b$, unless explicitly stated otherwise. We will denote by $R([a,\, b])$, $P([a,\, b])$ and $C([a,\, b])$, respectively, the set of integrable functions on $[a,\, b]$, the set of piecewise uniformly continuous functions on $[a,\, b]$ and the set of continuous functions on $[a,\, b]$.

1. In this exercise, let Ω be a non-empty set, S be a vector subspace of $\mathbb{R}^{\Omega}$ which is stable by $\wedge$ and $I : S \to \mathbb{R}$ be an integral operator. Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of elements of the vector space $\widetilde{L}^1(I)$ and let $f : \Omega \to \,]-\infty, +\infty]$ be the pointwise limit of the sequence $(f_n)_{n\in\mathbb{N}}$. We denote by α the limit of the sequence $(I(f_n))_{n\in\mathbb{N}}$.

(1) Let $\varepsilon \in \mathbb{R}_{>0}$. Prove that, for any $n \in \mathbb{N}_{\geqslant 1}$ there exists $u_n \in S^{\uparrow}$ such that $f_n - f_{n-1} \leqslant u_n$ and that

$$I(f_n - f_{n-1}) \geqslant I(u_n) - \frac{\varepsilon}{2^n}.$$

(2) Prove that, for any $n \in \mathbb{N}_{\geqslant 1}$,

$$f_n \leqslant u_1 + \cdots + u_n$$

and

$$I(f_n) \geqslant I(u_1) + \cdots + I(u_n) - \varepsilon.$$

(3) Let u be the pointwise limit of the series $\sum_{n\in\mathbb{N}_{\geqslant 1}} u_n$. Prove that $u \in S^{\uparrow}$ and

$$I(u) = \sum_{n\in\mathbb{N}_{\geqslant 1}} I(u_n).$$

(4) Prove that $u \geqslant f$ and

$$\alpha \geqslant I(u) - \varepsilon \geqslant \overline{I}(f) - \varepsilon.$$

(5) Prove that, for any $n \in \mathbb{N}$, there exists $\ell_n \in S^{\downarrow}$ such that $\ell_n \leqslant f_n$ and

$$I(\ell_n) \geqslant I(f_n) - \varepsilon.$$

Deduce that

$$\underline{I}(f) \geqslant I(f_n) - \varepsilon.$$

(6) Prove that $\underline{I}(f) \geqslant \alpha - \varepsilon$.

(7) Suppose that $\alpha \in \mathbb{R}$. Prove that f belongs to $\widetilde{L}^1(I)$ and $I(f) = \alpha$.

(8) Deduce that $I : \widetilde{L}^1(I) \to \mathbb{R}$ is an integral operator.

(9) Suppose that $\alpha = +\infty$. Prove that $I(f) = +\infty$.

(10) Prove that, if $(g_n)_{n\in\mathbb{N}}$ is a decreasing sequence in in $\widetilde{L}^1(I)$ such that
$$\lim_{n\to+\infty} I(g_n) \in \mathbb{R},$$
then the sequence $(g_n)_{n\in\mathbb{N}}$ converges pointwisely to an element $g \in \widetilde{L}^1(I)$ and
$$I(g) = \lim_{n\to+\infty} I(g_n).$$

2. In this exercice, let Ω be a non-empty set, S be a vector subspace of $\mathbb{R}^\Omega$ which is stable by $\wedge$ and $I : S \to \mathbb{R}$ be an integral operator. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\widetilde{L}^1(I)$. Assume that there exists $g \in \widetilde{L}^1(I)$ such that $f_n \geqslant g$ for any $n \in \mathbb{N}$.

(1) For any $n \in \mathbb{N}$, let
$$g_n = \inf_{m\in\mathbb{N}_{\geqslant n}} f_m.$$
Prove that g_n is the pointwise limit of the decreasing sequence
$$(f_n \wedge \cdots \wedge f_{n+k})_{k\in\mathbb{N}}.$$

(2) Prove that, for any $n \in \mathbb{N}$, $g_n \in \widetilde{L}^1(I)$ and $I(g_n) \leqslant I(f_n)$.

(3) Prove that
$$\lim_{n\to+\infty} I(g_n) \leqslant \liminf_{n\to+\infty} I(f_n).$$

(4) Prove that
$$\liminf_{n\to+\infty} f_n \in \widetilde{L}^1(I)^{\uparrow}$$
and
$$I\Big(\liminf_{n\to+\infty} f_n\Big) \leqslant \liminf_{n\to+\infty} I(f_n).$$

3. In this exercice, let Ω be a non-empty set, S be a vector subspace of $\mathbb{R}^\Omega$ which is stable by $\wedge$ and $I : S \to \mathbb{R}$ be an integral operator. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\widetilde{L}^1(I)$, which converges simply to a mapping $f : \Omega \to \mathbb{R}$. Assume that there exists $g \in \widetilde{L}^1(I)$ such that $|f_n| \leqslant g$ for any $n \in \mathbb{N}$.

(1) Prove that $f \in \widetilde{L}^1(I)$.

(2) Prove that

$$I(f) \leqslant \liminf_{n \to +\infty} I(f_n), \quad I(-f) \leqslant \liminf_{n \to +\infty} I(-f_n).$$

(3) Deduce that $(I(f_n))_{n \in \mathbb{N}}$ converges to $I(f)$.

4. In this exercice, we let S be the set of mappings in $\mathbb{R}^{\mathbb{R}}$ which can be written in the form of

$$\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i, b_i]},$$

where $n \in \mathbb{N}$, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, and for any $i \in \{1, \ldots, n\}$,

$$(a_i, b_i) \in \mathbb{R}^2, \quad a_i < b_i.$$

Denote by $I : S \to \mathbb{R}$ the integral operator defined as

$$I(\lambda_1 \mathbb{1}_{]a_1, b_1]} + \cdots + \lambda_n \mathbb{1}_{]a_n, b_n]}) = \sum_{i=1}^{n} \lambda_i (b_i - a_i).$$

If A is a subset of $\mathbb{R}$ and f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that the mapping

$$f_A : \mathbb{R} \to \mathbb{R}, \quad t \longmapsto \begin{cases} f(t), & \text{if } t \in A \cap \mathrm{Dom}(f), \\ 0, & \text{else} \end{cases}$$

belongs to $\widetilde{L}^1(I)^{\uparrow} \cup \widetilde{L}^1(I)^{\downarrow}$, then we denote by

$$\int_A f(t) \, \mathrm{d}t$$

the value $I(f_A)$. If $f_A \in \widetilde{L}^1(I)$, then we say that f is *integrable* on A.

(1) Let a and b be two real numbers such that $a \leqslant b$. Prove that, for any $f \in \widetilde{L}^1(I)$, the mappings

$$\mathbb{1}_{]a,b]} f, \quad \mathbb{1}_{[a,b[} f, \quad \mathbb{1}_{]a,b[} f, \quad \mathbb{1}_{[a,b]} f$$

are all elements of $\widetilde{L}^1(I)$, and

$$\int_{]a,b]} f(t) \, \mathrm{d}t = \int_{[a,b[} f(t) \, \mathrm{d}t = \int_{]a,b[} f(t) \, \mathrm{d}t = \int_{[a,b]} f(t) \, \mathrm{d}t.$$

In what follows, we denote by

$$\int_a^b f(t) \, \mathrm{d}t$$

this value.

(2) Let a and b be real numbers such that $a < b$ and $f : [a, b] \to \mathbb{R}$ be a mapping. We assume that f is integrable on $[a, b]$.

(a) Prove that, for any $x \in [a, b]$, the function f is integrable on $[a, x]$, and the mapping

$$(x \in [a, b]) \longrightarrow \int_a^x f(t)\,\mathrm{d}t$$

is continuous.

(b) Suppose that f is continuous. Prove that the mapping

$$x \longmapsto \int_a^x f(t)\,\mathrm{d}t$$

is differentiable on $]a, b[$ and its derivative coincides with the restriction of f to $]a, b[$.

(c) Let $F : [a, b] \to \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$ and satisfies $F'(x) = f(x)$ for any $x \in\ ]a, b[$. Prove that

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

(3) Let a and b be real numbers such that $a < b$. Determine

$$\int_a^b t^n\,\mathrm{d}t, \quad \int_a^b \exp(t)\,\mathrm{d}t, \quad \int_a^b \cos(t)\,\mathrm{d}t, \quad \int_a^b \sin(t)\,\mathrm{d}t.$$

(4) Prove that the mapping

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \frac{1}{1+t^2}$$

is integrable on $\mathbb{R}$ and determine its integral.

(5) Prove that the mapping

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \exp(-|t|)$$

is integrable on $\mathbb{R}$ and determine its integral.

(6) Prove that the mapping

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \exp(-t^2)$$

is integrable on $\mathbb{R}$.

5. In this exercice, we let S be the set of mappings in $\mathbb{R}^{\mathbb{R}}$ which can be written in the form of

$$\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i,b_i]},$$

where $n \in \mathbb{N}$, $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, and for any $i \in \{1, \ldots, n\}$,

$$(a_i, b_i) \in \mathbb{R}^2, \quad a_i < b_i.$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous mapping. Denote by $I_\varphi : S \to \mathbb{R}$ the integral operator defined as

$$I_\varphi(\lambda_1 \mathbb{1}_{]a_1,b_1]} + \cdots + \lambda_n \mathbb{1}_{]a_n,b_n]}) = \sum_{i=1}^{n} \lambda_i(\varphi(b_i) - \varphi(a_i)).$$

If A is a subset of $\mathbb{R}$ and f is a function from $\mathbb{R}$ to $\mathbb{R}$ such that the mapping

$$f_A : \mathbb{R} \to \mathbb{R}, \quad t \longmapsto \begin{cases} f(t), & \text{if } t \in A \cap \mathrm{Dom}(f), \\ 0, & \text{else} \end{cases}$$

belongs to $\widetilde{L}^1(I)^{\uparrow} \cup \widetilde{L}^1(I)^{\downarrow}$, then we denote by

$$\int_A f(t) \,\mathrm{d}\varphi(t)$$

the value $I(f_A)$. If $f_A \in \widetilde{L}^1(I)$, then we say that f is $\mathrm{d}\varphi$-*integrable* on A. For any $a \in \mathbb{R}$, we denote by $\Delta\varphi(a)$ the difference $\varphi(a) - \varphi_-(a)$, where

$$\varphi_-(a) := \lim_{t>0,\, t\to 0} \varphi(a - t).$$

(1) Let a be a real number, show that $\mathbb{1}_{\{a\}}$ is $\mathrm{d}\varphi$-integrable on $\mathbb{R}$. Show that

$$\int_{\mathbb{R}} \mathbb{1}_{\{a\}}(t) \,\mathrm{d}\varphi(t) = \Delta\varphi(a).$$

(2) Let a and b be two real numbers such that $a \leqslant b$. Prove that, for any $f \in \widetilde{L}^1(I_\varphi)$, the mappings

$$\mathbb{1}_{]a,b]} f, \quad \mathbb{1}_{[a,b[} f, \quad \mathbb{1}_{]a,b[} f, \quad \mathbb{1}_{[a,b]} f$$

are all elements of $\widetilde{L}^1(I)$, and

$$\int_{]a,b]} f(t)\,\mathrm{d}\varphi(t) = \int_{]a,b[} f(t)\,\mathrm{d}\varphi(t) + f(b)\Delta\varphi(b),$$
$$\int_{[a,b[} f(t)\,\mathrm{d}\varphi(t) = \int_{]a,b[} f(t)\,\mathrm{d}\varphi(t) + f(a)\Delta\varphi(a),$$
$$\int_{[a,b]} f(t)\,\mathrm{d}\varphi(t) = \int_{]a,b[} f(t)\,\mathrm{d}\varphi(t) + f(a)\Delta\varphi(a) + f(b)\Delta\varphi(b).$$

(3) Let a and b be real numbers such that $a < b$. Assume that ψ is an integrable function on $[a,b]$ and that

$$\forall\, x \in [a,b], \quad \varphi(x) = \varphi(a) + \int_a^x \psi(t)\,\mathrm{d}t.$$

Prove that, for any function f which is $\mathrm{d}\varphi$-integrable on $]a,b]$, the function $f\psi$ is integrable on $]a,b]$, and

$$\int_{]a,b]} f(t)\,\mathrm{d}\varphi(t) = \int_a^b f(t)\psi(t)\,\mathrm{d}t.$$

(4) We assume that φ is continuous. Let a and b be real numbers such that $a < b$. Prove that, for any integrable function f on $]\varphi(a),\varphi(b)]$, the function $f\circ\varphi$ is $\mathrm{d}\varphi$-integrable on $]a,b]$, and one has

$$\int_{]\varphi(a),\varphi(b)]} f(x)\,\mathrm{d}x = \int_{]a,b]} f(\varphi(t))\,\mathrm{d}\varphi(t).$$

(5) Compute

$$\int_{\mathbb{R}} x\exp(-x^2)\,\mathrm{d}x.$$

6. Find the primitive functions of the following mappings.

(1) $(x \in \mathbb{R}_{>0}) \mapsto x^\alpha$, where $\alpha \in \mathbb{R}$,
(2) $(x \in \mathbb{R}\setminus\{0\}) \mapsto x^{-1}$,
(3) $(x \in \mathbb{R}) \mapsto a^x$, where $a \in \mathbb{R}_{>0}$,
(4) $(x \in \mathbb{R}) \mapsto \cos(x)$,
(5) $(x \in \mathbb{R}) \mapsto \sin(x)$,
(6) $(x \in \,]-\pi/2,\pi/2[) \mapsto 1/\cos(x)^2$,
(7) $(x \in \,]-1,1[) \mapsto 1/\sqrt{1-x^2}$,
(8) $(x \in \,]-1,1[) \mapsto 1/(1+x^2)$.

7. Determine the primitive functions of the following functions.

(1) $f(x) = x^3 - x + 1$,
(2) $f(x) = \cos(x) - \sin(x)$,
(3) $f(x) = 1 - e^x + x$,
(4) $f(x) = \sqrt{x} + \frac{1}{x} + \frac{2}{x^2}$,
(5) $f(x) = (x+1)/x^2$.

8. Using the method of change of variables, determine the primitive functions of the following functions.

(1) $f(x) = x/(1+x^2)$,
(2) $f(x) = \cos(x)\sin(x)^2$,
(3) $f(x) = \ln(x)/x$,
(4) $f(x) = 1/(x\ln(x))$,
(5) $f(x) = e^x/(1+e^x)$,
(6) $f(x) = x\sqrt{1+x^2}$.

9. Compute the following integrals.

$$\int_0^{\pi} (1 - \cos(3x))\,\mathrm{d}x, \quad \int_0^{\sqrt{\pi}} x\cos(x^2)\,\mathrm{d}x, \quad \int_1^2 \frac{\ln(x)}{x}\,\mathrm{d}x.$$

10. Using the method of integration by parts, compute the following integrals.

$$\int_0^1 xe^x\,\mathrm{d}x, \quad \int_1^e x^2\ln(x)\,\mathrm{d}x.$$

11. Using the method of integration by parts, determine the primitive function of the following functions.

(1) $x \mapsto \arctan(x)$,
(2) $x \mapsto \ln(x)^2$,
(3) $x \mapsto \sin(\ln(x))$,
(4) $x \mapsto \ln(x + \sqrt{1+x^2})$,
(5) $x \mapsto e^{2x}(\tan(x)+1)^2$.

12. The purpose of this exercise is to determine the primitive function of the mapping $x \mapsto \ln(x)^n$. We denote by F_n a primitive function of $x \mapsto \ln(x)^n$ such that $F_n(1) = (-1)^n n!$.

(1) Prove that, for any positive integer n, one has

$$F_n(x) + nF_{n-1}(x) = x\ln(x)^n.$$

(2) Prove that

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \frac{n!}{(n-k)!} x \ln(x)^{n-k}.$$

13. Let $(a, b, m, n) \in \mathbb{R}^2 \times \mathbb{N}^2$ such that $a < b$. We denote by $A_{m,n}$ the integral

$$\int_a^b (t-a)^m (t-b)^n \, \mathrm{d}t.$$

(1) Prove that

$$A_{m,n} = -\frac{m}{n+1} A_{m-1,n+1}.$$

(2) Determine the value of $A_{0,n}$ for $n \in \mathbb{N}$.

(3) Prove that, for any $n \in \mathbb{N}$,

$$A_{n,n} = \frac{(-1)^{n+1}(a-b)^{2n+1}}{(2n+1)\binom{2n}{n}}.$$

14. Using a change of variables, determine the following integrals.

(1) $\displaystyle\int_1^4 \frac{1-\sqrt{t}}{\sqrt{t}} \, \mathrm{d}t,$

(2) $\displaystyle\int_0^{\pi} \frac{\sin(t)}{1+\cos(t)^2} \, \mathrm{d}t,$

(3) $\displaystyle\int_1^{e} \frac{1}{(2t\ln(t)+t)} \, \mathrm{d}t.$

(4) $\displaystyle\int_0^1 \frac{1}{1+e^t} \, \mathrm{d}t.$

(5) $\displaystyle\int_1^3 \frac{\sqrt{t}}{t+1} \, \mathrm{d}t,$

(6) $\displaystyle\int_{-1}^1 \sqrt{1-t^2} \, \mathrm{d}t.$

(7) $\displaystyle\int_0^{\pi/6} \frac{1}{\cos(t)} \, \mathrm{d}t$ (by the change of variables $t \mapsto \sin(t)$),

(8) $\displaystyle\int_1^2 \frac{2t}{\sqrt{1+t}} \, \mathrm{d}t$ (by the change of variables $t \mapsto \sqrt{1+t}$),

(9) $\displaystyle\int_0^3 \frac{1}{\sqrt{1+\sqrt{1+t}}} \, \mathrm{d}t$ (by the change of variables $t \mapsto \sqrt{1+t}$).

15. Determine the primitive function of the following functions.

(1) $x \mapsto \dfrac{1}{x^3-1}$,
(2) $x \mapsto \dfrac{x^3+2x}{x^2+x+1}$,
(3) $x \mapsto \dfrac{1}{x^3-7x+6}$,
(4) $x \mapsto \dfrac{4x^2}{x^4-1}$.

16. For any $n \in \mathbb{N}_{\geqslant 1}$, let

$$A_n = \int_0^1 \frac{1}{(x^2+1)^n}\,\mathrm{d}x.$$

(1) Determine A_1.
(2) Prove that, for any $n \in \mathbb{N}$ such that $n \geqslant 1$, one has

$$A_{n+1} = \frac{2n-1}{2n}A_n + \frac{1}{n2^{n+1}}.$$

(3) Determine A_3.

17. Prove that the function $\ln(\cdot)$ is Lebesgue integrable on $]0,1]$ and determine its integral.

18. Prove that the function $t \mapsto e^{-|t|}$ is Lebesgue integrable on $\mathbb{R}$ and determine its integral.

19. Prove that the function $t \mapsto e^{-t^2}$ is Lebesgue integrable on $\mathbb{R}$.

20. The purpose of this exercise is to study the Lebesgue integrability of the mapping

$$f_\alpha : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}, \quad x \longmapsto \frac{1}{x^\alpha}.$$

(1) Find a primitive function F_α of the mapping f_α.
(2) Prove that the mapping f_α is Lebesgue integrable on $]0,1]$ if and only if $\alpha < 1$.
(3) Prove tha the mapping f_α is Lebesgue integrable on $[1,+\infty[$ is Lebesgue integrable on $[1,+\infty[$ if and only if $\alpha > 1$.

21. Let α and β be real numbers. The purpose of this exercise is to determine the Lebesgue integrability of the function

$$f_{\alpha,\beta}(x) = \frac{1}{x^\alpha \ln(x)^\beta}$$

on $[e,+\infty[$.

(1) Suppose that $\alpha > 1$. Prove that $f_{\alpha,\beta}$ is Lebesgue integrable on $[e, +\infty[$.
(2) Determine a primitive function of $f_{1,\beta}$.
(3) Prove that $f_{1,\beta}$ is Lebesgue integrable on $[e, +\infty[$ if and only if $\beta > 1$.
(4) Suppose that $\alpha < 1$. Prove that $f_{\alpha,\beta}$ is not Lebesgue integrable on $[e, +\infty[$

22. Let $f : [0, +\infty[$ be a continuous mapping which is Lebesgue integrable on $[0, +\infty[$.

(1) Prove that, if the limit
$$\lim_{x\to+\infty} f(x)$$
exists, then it is necessarily 0.
(2) Suppose that f is uniformly continuous. Prove that
$$\lim_{x\to+\infty} f(x) = 0.$$
(3) Construct a continuous and integrable function f on $[0, +\infty[$ which does not converge when $x \to +\infty$.

23. For any $x \in \mathbb{R}$, denote by f_x the mapping
$$(t \in]0, +\infty[) \longmapsto \frac{\sin(xt)}{te^t}$$

(1) Show that, for any $x \in \mathbb{R}$, the mapping f_x is Lebesgue integrable on $]0, +\infty[$.
(2) For any $x \in \mathbb{R}$, let
$$F(x) = \int_0^{+\infty} \frac{\sin(xt)}{te^{-t}}\,\mathrm{d}t.$$
Prove that the mapping F is of class C^1 and determine F'.
(3) Prove that $F(x) = \arctan(x)$.

24. In this exercice, we consider the following integral
$$F(x) = \int_0^{+\infty} \frac{e^{-xt}}{1+t^2}\,\mathrm{d}t, \quad x \in \mathbb{R}_{>0}.$$

(1) Prove that, for any $x \in \mathbb{R}_{>0}$, $F(x)$ is well defined and takes value in $\mathbb{R}_{>0}$.

(2) Prove that
$$\lim_{x\to+\infty} F(x) = 0.$$
(3) Prove that F is of class C^2.
(4) Prove that F verifies the following equation
$$F''(x) + F(x) = \frac{1}{x}.$$

25. The purpose of this exercise is to compute the value of the following integral
$$A = \int_0^{+\infty} e^{-t^2}\,\mathrm{d}t.$$
For any $x \in \mathbb{R}_{\geqslant 0}$, let
$$f(x) = \int_0^x e^{-t^2}\,\mathrm{d}t, \quad g(x) = \int_0^1 \frac{e^{-(t^2+1)x^2}}{t^2+1}\,\mathrm{d}t.$$
(1) Prove that, for any $x \in \mathbb{R}_{\geqslant 0}$, one has
$$g(x) + f(x)^2 = \frac{\pi}{4}.$$
(2) Determine the limits of $f(x)$ and $g(x)$ when $x \to +\infty$.
(3) Deduce that $A = \sqrt{\pi}/2$.

26. Let $f : [0,1] \to \mathbb{R}_{>0}$ be a continuous mapping. For any $\alpha \in \mathbb{R}_{\geqslant 0}$, let
$$F(\alpha) = \int_0^1 f(t)^\alpha\,\mathrm{d}t.$$
(1) Prove that F is differentiable on $\mathbb{R}_{>0}$ and that $F'(\alpha)$ has a limit when $\alpha \to 0$.
(2) Prove that
$$\lim_{\alpha\to 0} F(\alpha)^{1/\alpha} = \int_0^1 \ln f(t)\,\mathrm{d}t.$$

27. Prove that the following limits exist and find the limits.
(a) $\displaystyle\lim_{n\to+\infty} \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$;
(b) $\displaystyle\lim_{n\to+\infty} \frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}$;
(c) $\displaystyle\lim_{n\to+\infty} \frac{1}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots \sin\frac{n\pi}{n}\right)$;

(d) $\lim\limits_{n\to+\infty}\left(\frac{1^\alpha}{n^{\alpha+1}}+\frac{2^\alpha}{n^{\alpha+1}}+\cdots\frac{n^\alpha}{n^{\alpha+1}}\right)$;

(e) $\lim\limits_{n\to+\infty}\sin\frac{\pi}{n}\cdot\sum_{k=0}^{n-1}\frac{1}{2+\cos\frac{k\pi}{n}}$.

28. Find the derivatives of the following functions.

(a) $f(x)=\int_0^{x^3}\sin^3 t dt$;

(b) $f(x)=\int_1^x\frac{1}{1+t^2+\sin^2 t}dt$;

(c) $f(x)=\sin\left(\int_0^x\cos\left(\int_0^y\sin^2 tdt\right)dy\right)$;

(d) $f^{-1}(x)$, where $f(x)=\int_1^x\frac{1}{t}dt$;

(e) $f^{-1}(x)$, where $f(x)=\int_0^x\frac{1}{\sqrt{1-t^2}}dt,\quad x\in(0,1)$.

29. (a) Let $I,K\subseteq\mathbb{R}$ denote intervals. Suppose that $f:I\to\mathbb{R}$ is a continuous function and $\alpha,\beta:K\to I$ are differentiable functions. Define a function $F:K\to\mathbb{R}$ by

$$F(x)=\int_{\alpha(x)}^{\beta(x)}f(t)dt.$$

Prove that F is differentiable on K and in particular,

$$F'(x)=f(\beta(x))\beta'(x)-f(\alpha(x))\alpha'(x),\quad\forall x\in K.$$

(b) Find the derivative for the function $F:(1,\infty)\to\mathbb{R}$ defined by

$$F(x)=\int_{x^a}^{x^b}\frac{1}{\log t}dt,$$

where $a,b>0$.

30. (Oscillation and Lebesgue's theorem). We say that a subset $E\subset\mathbb{R}$ has (Lebesgue) *measure zero* if for any $\varepsilon>0$ there exists a countable family of open intervals (I_n) such that $E\subset\cup_n I_n$ and $\sum_n l(I_n)\le\varepsilon$, where $l(I_n)$ denotes the length of I_n, i.e., $l(I_n)=b_n-a_n$ if I_n has endpoints a_n and b_n.

(a) Prove that a countable union of sets of measure zero has again measure zero. In particular, any countable set has measure zero.

In the following consider a bounded function $f : [a,b] \to \mathbb{R}$. The *oscillation* of f on a subset $A \subset [a,b]$ is the quantity

$$\omega(f;A) = \sup_{x_1,x_2\in A} |f(x_1) - f(x_2)|$$

and the *oscillation* of f at a point $x \in [a,b]$ is defined to be

$$\omega(f;x) = \lim_{\delta\to 0^+} \omega(f;[x-\delta, x+\delta] \cap [a,b]).$$

Denote by E the set of discontinuities of f.

(b) Prove that $E = \{x \mid \omega(f;x) > 0\}$.

(c) Prove that if f is Riemann integrable, then $\{x \mid \omega(f;x) > 1/n\}$ has measure zero for all $n \in \mathbb{N} \setminus \{0\}$. (Hint : use Darboux sums)

(d) (*Lebesgue's theorem*) Prove that f is Riemann integrable if and only if E has measure zero.

31. (Second mean value theorem for integrals) Let a and b be two real numbers such that $a < b$, f a continuous and monotone function on the closed interval $[a,b]$, and g a continuous function on $[a,b]$. Let G be the function defined by

$$G(x) = \int_a^x g(t)\,\mathrm{d}t$$

for every x in $[a,b]$, and consider the integral :

$$I = \int_a^b f(t)g(t)\,\mathrm{d}t = \int_a^b f(t)G'(t)\,\mathrm{d}t.$$

(a) Prove that for every real number $\varepsilon > 0$, one can find a finite family of elements a_i (for $i = 0, 1, \ldots, n$) in $[a,b]$ such that

$$\left| I - \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f(a_i)G'(t)\,\mathrm{d}t \right| < \varepsilon.$$

(b) Establish the equality :

$$\sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f(a_i)G'(t)\,\mathrm{d}t = f(b)G(b) - \sum_{i=2}^{n} G(a_{i-1})[f(a_i) - f(a_{i-1})].$$

(c) Let

$$\mathcal{I}_\varepsilon = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} f(a_i)G'(t)\,dt.$$

Prove that the number $\mathcal{I}_\varepsilon - f(b)G(b)$ lies within the interval bounded by the numbers

$$[f(a_1) - f(b)][\inf_{x\in[a,b]} G(x)] \quad \text{and} \quad [f(a_1) - f(b)][\sup_{x\in[a,b]} G(x)].$$

Deduce that the number $I - f(b)G(b)$ lies within the interval bounded by the numbers

$$[f(a) - f(b)][\inf_{x\in[a,b]} G(x)] \quad \text{and} \quad [f(a) - f(b)][\sup_{x\in[a,b]} G(x)].$$

(d) Consider the continuous function H defined on the interval $[a,b]$ by $H(x) = [f(a) - f(b)]G(x)$; prove that there exists a number c in the interval $[a,b]$ such that $H(c) = I - f(b)G(b)$. Deduce the following equality :

$$I = f(a) \int_a^c g(t)\,\mathrm{d}t + f(b) \int_c^b g(t)\,\mathrm{d}t.$$

(This equality is called the *second mean value theorem for integrals.*)

32. (Cauchy-Schwarz, Hölder and Minkowski inequalities)

(a) Show that if f is Riemann integrable on $[a,b]$, then so is $|f|^p$ for $p \geq 0$.
In the following write

$$\|f\|_p = \left(\int_a^b |f|^p(x)\mathrm{d}x\right)^{1/p}, \quad p > 1.$$

(b) Starting from Hölder's inequality for sums, obtain *Hölder's inequality* for integrals (*Cauchy-Schwarz inequality* for $p = 2$) :

$$\left|\int_a^b (f\cdot g)(x)\mathrm{d}x\right| \leq \|f\|_p \|g\|_q,$$

if f, g are Riemann-integrable functions on $[a,b]$, $p > 1, q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.
Note that the equality occurs when $|f|^p/\|f\|_p^p = |g|^q/\|g\|_q^q$. Deduce that

$$\|f\|_p = \sup\left\{\int_a^b (f\cdot g)(x)\mathrm{d}x \mid \|g\|_q \leq 1, g \text{ is Riemann integrable}\right\}$$

(c) Deduce *Minkowski's inequality* for integrals :

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

if f, g is Riemann integrable on $[a, b]$ and $p \geq 1$.

(d) Show that the previous inequality reverses direction if $0 < p < 1$.

(e) Verify that if f is a continuous convex function on $\mathbb{R}$ and φ an arbitrary continuous function on $\mathbb{R}$, then Jensen's inequality

$$f\left(\frac{1}{c}\int_0^c \varphi(t)\mathrm{d}t\right) \leq \frac{1}{c}\int_0^c f(\varphi(t))\mathrm{d}t$$

holds for $c \neq 0$.

33. Let f be a function of class C^2 on $\mathbb{R}$ and x a point in $\mathbb{R}$ such that $f''(x) \neq 0$.

(a) Show that there exists $\eta > 0$ such that for all $h \in [-\eta, \eta] \setminus \{0\}$, there exists a unique number $\theta \in (0, 1)$ with

$$f(x+h) = f(x) + hf'(x+\theta h).$$

The function defined on $[-\eta, \eta] \setminus \{0\}$ that associates h with this unique θ is denoted by θ_x.

(b) Show that

$$\lim_{h\to 0} \theta_x(h) = \frac{1}{2}.$$

34. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Use the fundamental theorem of calculus and Darboux theorem to prove the intermediate value theorem.

35. Let f be a continuous function on $[-1, 1]$. Prove that $\lim\limits_{h\to 0^+} \displaystyle\int_{-1}^{1} \frac{h}{h^2+x^2} f(x)\,\mathrm{d}x = \pi f(0)$. (Hint : consider the integrals on a carefully-chosen small neighborhood of 0 dependent of h and on its complement separately, and use the mean value theorem.)

36. Let $a, b > 0$. Calculate the interior area of the ellipse, i.e., the region enclosed by the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

37. We propose to calculate the area of the region delimited by the two curves :

$$f(x) = \frac{x^2}{2} \quad \text{and} \quad g(x) = \frac{1}{1+x^2}.$$

(1) Show that the two curves intersect at the points of abscissa $x = 1$ and $x = -1$.

(2) Show that $f(x) < g(x)$ for all $x \in (-1,\, 1)$.

(3) Deduce that the bounded region is located above the graph of f and below the graph of g between the two vertical lines of equation $(x = 1)$ and $(x = -1)$.

(4) Deduce that the desired area A is expressed :

$$A = \int_{-1}^{1} g(x)\, dx - \int_{-1}^{1} f(x)\, dx.$$

(5) Conclude that $A = \dfrac{\pi}{2} - \dfrac{1}{3}$.

38. Find the following limits

(1) $\displaystyle\lim_{n\to\infty} \frac{\sqrt[n]{(n+1)\cdots(n+n)}}{n}$ (2) $\displaystyle\lim_{n\to\infty} \prod_{k=1}^{n} \left(2+\cos\left(\frac{k\pi}{n}\right)\right)^{\pi/n}$

(3) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k}$ (4) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{2n+2k-1}$

39. We partition the interval $[1, 2]$ into n subintervals by the points $\{1 + k/n\}_{k=0}^{n}$. We denote by U_n and L_n the corresponding upper and lower Darboux sums, respectively. Show that

$$U_1 = 1,\quad L_1 = 1 - \frac{1}{2},\quad U_2 = 1 - \frac{1}{2} + \frac{1}{3},\quad L_2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}, \cdots\cdots$$

and generally the sequence $U_1, L_1, U_2, L_2 \cdots, U_n, L_n \cdots$ is identical with the sequence of the partial sums of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} + \cdots$$

Then show that

$$\lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}\right) = \log 2.$$

40. For any pair of reals $\{a, b\}$ with $a \le b$ assume that a map $I_{a,b} : R([a,\, b]) \to \mathbb{R}$ is given and satisfies the following conditions :

(1) $I_{a,b}$ is a linear functional on $R([a,\, b])$, i.e.

$$I_{a,b}(\lambda f + g) = \lambda I_{a,b}(f) + I_{a,b}(g), \quad \forall\, f, g \in R([a,\, b]),\ \lambda \in \mathbb{R};$$

(2) $I_{a,b}(1) = b - a\,;$
(3) $f \geq 0 \;\Rightarrow\; I_{a,b}(f) \geq 0\,;$
(4) $I_{a,b}(f) = I_{a,c}(f) + I_{c,b}(f)$ for any $f \in R([a,\, b])$ and $c \in [a,\, b]$.

Our aim is to show that $I_{a,b}(f) = \int_a^b f(x)dx$.

(1) Show that $f \leq g \;\Rightarrow\; I_{a,b}(f) \leq I_{a,b}(g)$ and $|I_{a,b}(f)| \leq I_{a,b}(|f|)$.
(2) Show that if $f, g \in R([a,\, b])$ differ only at a finite number of points $x_1, \cdots, x_n$ in $[a,\, b]$, then $I_{a,b}(f) = I_{a,b}(g)$.
(3) Show that $I_{a,b}(f) = \int_a^b f(x)dx$ first for any step function, then for any $f \in R([a,\, b])$.

41. Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Show that for any $\varepsilon > 0$ there exist continuous functions $g, h : [a, b] \to \mathbb{R}$ such that $g(x) \leq f(x) \leq h(x)$ for any $x \in [a, b]$ and

$$\int_a^b (h(x) - g(x))dx \leq \varepsilon.$$

Particularly, we have

$$\int_a^b (h(x) - f(x))dx \leq \varepsilon, \quad \int_a^b (f(x) - g(x))dx \leq \varepsilon.$$

42. Let $f \in R([a,\, b])$ and define $F(x) = \int_a^x f(t)dt$ for $x \in [a,\, b]$.

(1) Show that F is continuous on $[a,\, b]$.
(2) Assume that f admits a right (resp. left) limit at $x_0 \in [a,\, b]$. Show that F is right (resp. left) differentiable at x_0 and its right (resp. left) derivative $F'(x_0+)$ (resp. $F'(x_0-)$ is equal to $f(x_0+)$ (resp. $f(x_0-)$).
(3) How about the converse to the previous assertion? Namely, if F is right (resp. left) differentiable at x_0, does f admit a right (resp. left) limit at x_0? Justify your answer by a proof or by a counter-example.
(4) Assume now that f is continuous on $[a,\, b]$ and that $u : I \to [a, b]$ is a differentiable function, where I is an interval. Show that the function G defined by

$$G(x) = \int_a^{u(x)} f(t)dt$$

is differentiable on I.

43. We propose to show the **second mean value theorem** : Let f : $[a,\, b] \to \mathbb{R}_+$ be decreasing and $g : [a,\, b] \to \mathbb{R}$ be integrable. Then there exists $c \in [a,\, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a+)\int_a^c g(x)dx,$$

where $f(a+) = \lim_{x\searrow a} f(x)$. In the following we assume that $f(a+) = f(a) > f(b)$, otherwise, there is nothing to prove. Let

$$G(x) = \int_a^x g(t)dt.$$

(1) Let $m = \min_{x\in[a,b]} G(x)$ and $M = \max_{x\in[a,b]} G(x)$. Show that the desired statement is equivalent to

$$(*) \qquad mf(a) \le \int_a^b f(x)g(x)dx \le Mf(a).$$

Further, replacing g by $-g$, it suffices to show the second inequality above.

(2) Show the second inequality of $(*)$ if f is a step function.

(3) Show that for any $\varepsilon > 0$ there exists a decreasing step function φ such that

$$0 \le \varphi \le f \quad \text{and} \quad \int_a^b (f(x) - \varphi(x))dx < \varepsilon.$$

(4) Deduce the second inequality of $(*)$.

(5) Show that if $f : [a,\, b] \to \mathbb{R}$ is monotone, then there exists $c \in [a,\, b]$ such that

$$\int_a^b f(x)g(x)dx = f(a+)\int_a^c g(x)dx + f(b-)\int_c^a g(x)dx.$$

44. Suppose that $F(x)$ is differentiable on $[a, b]$ and $F'(x) = f(x)$. Show that $f(x)$ is integrable if and only if there exists an integrable function $g(x)$ such that for all $x \in [a, b]$ it holds that

$$F(x) = F(a) + \int_a^x g(t)dt.$$

45. Let f be a decreasing nonnegative continuous function on $(0, +\infty)$. For any integer $n \ge 1$, we define

$$u_n = \sum_{k=1}^{n} f(k) \quad \text{and} \quad I_n = \int_1^n f(x)dx.$$

(1) Show that

$$\int_{n}^{n+1} f(x)dx \le f(n) \le \int_{n-1}^{n} f(x)dx, \quad n \ge 2.$$

(2) Show that there exist $\alpha, \beta > 0$ such that $I_{n+1} - \alpha \le u_n - \beta \le I_n$ for all $n \ge 2$.

(3) Deduce that $\{u_n\}_{n=1}^{\infty}$ and $\{I_n\}_{n=1}^{\infty}$ either converge or diverge at the same time.

46. Let $(x_n)_{n\ge 1} \subset [a,\, b]$ be a sequence converging to x_0. Let f, g be two bounded functions on $[a,\, b]$ that differ only at x_n for $n \ge 0$. Show that if one of them is integrable, so the other does. In this case, we have

$$\int_a^b f(x)dx = \int_a^b g(x)dx.$$

47. Let $T > 0$ and f be a continuous function on $\mathbb{R}$. Show that the following assertions are equivalent :

(1) f is T-periodic, i.e. $f(x+T) = f(x)$ for any $x \in \mathbb{R}$;

(2) the function $F(x) = \int_x^{x+T} f(t)dt$ is constant on $\mathbb{R}$.

48. Let (f_n) be a sequence in $C([a,\, b])$ converging uniformly to f on $[a,\, b]$. Let (c_n) be a sequence in $[a,\, b]$ converging uniformly to c. Define

$$F_n(x) = \int_{c_n}^{x} f_n(t)dt \;\text{ and }\; F(x) = \int_{c}^{x} f_n(t)dt, \quad x \in [a,\, b].$$

(1) Show that $\lim\limits_{n\to\infty} \displaystyle\int_c^{c_n} \big(f_n(t) - f(t)\big)dt = 0$.

(2) Show that (F_n) uniformly converges to F on $[a,\, b]$.

49. Let $\{u_i\}_{i=1}^{N} \subset [0,1]$ be a sequence. For any $(a,b) \subset [0,1]$, let $N(a,b)$ be the number of u_i's in (a,b). Show that the following statements are equivalent :

(a) For any $(a,b) \subset [0,1]$, it holds that $\lim_{N\to\infty} \frac{N(a,b)}{N} = b - a$.

(b) For every continuous function $f : [0,1] \to \mathbb{R}$, it holds that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(u_k) = \int_0^1 f(x)dx.$$