Quantum Mechanics 2025 HW5

Due 10/28 in Class

October 14, 2025

Problem 1. Current, gauge transformation

Consider the Schrödinger equation of a charged particle in the electromagnetic fields. According to EM, $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$.

1)

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left(-\frac{1}{2m}(\mathbf{P} - \frac{q}{c}\mathbf{A})^2 + \phi(\mathbf{r},t)\right)\psi(\mathbf{r},t), \tag{1}$$

where $\mathbf{P} = -i\hbar\nabla$. Define the probability density $\rho(\mathbf{r},t) = \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t)$. In order to have the probability conservation, i.e., $\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0$, how should we define the probability current $\mathbf{j}(\mathbf{r},t)$?

2) Consider under the gauge transformation,

$$\begin{aligned} \mathbf{A}'(\mathbf{r},t) &= \mathbf{A}'(\mathbf{r},t) + \nabla f(\mathbf{r},t) \\ \phi'(\mathbf{r},t) &= \phi(\mathbf{r},t) - \frac{1}{c}\frac{\partial}{\partial t}f(\mathbf{r},t), \end{aligned} \tag{2}$$

Please show that $\mathbf{B}$ and $\mathbf{E}$ are invariant.

3) Define $\psi'(\mathbf{r},t) = e^{i\varphi(\mathbf{r},t)}\psi(\mathbf{r},t)$. Please figure out how to chose $\varphi(\mathbf{r},t)$, such that $\psi'(\mathbf{r},t)$ satisfies the Schrödinger equation under $\mathbf{A}', \phi'$,

$$i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r},t) = \left(-\frac{1}{2m}(\mathbf{P} - \frac{q}{c}\mathbf{A}')^2 + \phi'(\mathbf{r},t)\right)\psi'(\mathbf{r},t). \tag{3}$$

4) Prove that $\rho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ you obtained in 1) are invariant under the gauge transformation defined in 2).

Problem 2. Landau gauge

In the Landau gauge for a uniform magnetic field $A_x = By$, and $A_y = 0$. Consider a special case that the impurity potential $V_{imp}(y)$ only depends on y, such that the Hamiltonian is

$$H = \frac{(P_x - \frac{q}{c}A_x)^2}{2m} + \frac{P_y^2}{2m} + V(y). \tag{4}$$

1) By plugging in $\psi(x, y) = \frac{1}{\sqrt{L_x}} e^{ik_x x}\phi_{k_x}(y)$, reduce the problem into a 1D problem in the y-direction with the following k_x-dependent Hamiltonian $H_y(k_x)$.

$$H_y(k_x)\phi_{n,k_x}(y) = E_n(k_x)\phi_{n,k_x}(y) \tag{5}$$

2) You may use the Hellman-Feynman theorem to show that $I_x(n, k_x) = \frac{q}{L_x}\frac{\partial E_n}{\hbar\partial k}$.

3) Prove that the Hall conductance is quantized $\sigma_{xy} = \frac{q^2}{h}\nu$, where ν is the integer filling number. Why σ_{xy} is insensitive to the concrete form of $V_{imp}(y)$.