Quantum Mechanics 2025 HW11

No need to turn in

December 22, 2025

Problem 1. Resonance scattering

Consider a 3D attractive spherical δ-potential well with $\frac{2m}{\hbar^2}V(r) = \gamma\delta(r-R)$ where $\gamma < 0$ and R is the radius of the shell of the δ-potential. In the following, we only consider the s-wave channel, i.e., $l = 0$, and define $u(r) = rR_0(r)$ where $R_0(r)$ is the radial solution in the s-wave channel.

1) Prove that there exists a value $|\gamma_c|$, such that, when $|\gamma| > |\gamma_c|$ there is one bound state, and when $|\gamma| < |\gamma_c|$ there is no bound state. Determine the value of the dimensionless parameter $|\gamma_c|R$, and the localization length of the bound state in the limit of $|\gamma| \to |\gamma_c|$.

2) Find the equation of phase shift that $\tan\delta_0(k)$ satisfies, and check that this expression is consistent with the fact $\delta_0(k)$ is an odd function of k.

It is not required here, but if you can justify why $\delta_0(k)$ is an odd function, you can get some extra credits.

Expand the expression $k\cot\delta_0(k)$ around $k = 0$ and keep the first two leading terms. Show that it can be expanded as

$$k\cot\delta_0(k) = -\tfrac{1}{a_0} + r_0k^2, \tag{1}$$

and determine the values of scattering length a_0 and the interaction range r_0. Show that in the case that $|\gamma| \to |\gamma_c| + 0^+$, a_0 is the same as the localization length of the bound state at the leading order. In this case, between $|a_0|$ and r_0, which can be much larger than R, and which is at the same order of R?

Now let us consider the locations of resonance scattering.

3) Show that at $|\gamma| < |\gamma_c|$, there is no resonance. Sketch the plots of $\tan\delta_0(k)$ v.s. kR, and $\delta_0(k)$ v.s. kR. What is the value of $\delta_0(k=0)$? If right at $|\gamma| = |\gamma_c|$, again sketch the plots of $\tan\delta_0(k)$ v.s. kR, and $\delta_0(k)$ v.s. kR at $|\gamma| = |\gamma_c|$. What is the value of $\delta_0(k=0)$ in this case?

4) Let us consider the case that $|\gamma|$ slightly larger than $|\gamma_c|$. Show that there is only one solution for $\tan\delta_0(k) = \pm\infty$ at $0 < kR < \frac{\pi}{2}$. Again sketch the plots of $\tan\delta_0(k)$ v.s. kR, and $\delta_0(k)$ v.s. kR. What is the value of $\delta_0(k=0)$ in this case?

If $|\gamma|$ further goes large will $\delta_0(k=0)$ change or not? You further check the case at which there are three solutions $\tan\delta_0(k) = \pm\infty$, and use plots to support yourarguments.

Combining all the cases in 4) and 5), can you relate $\delta_0(k=0)$ to the number of bound states?

5) Consider the limit of $|\gamma| \to +\infty$, show that the resonance solutions, i.e., $\tan\delta_0(k) = \pm\infty$, occur at $kR \to (n+\frac{1}{2})\pi + 0^-$, and at $kR \to n\pi + 0^+$. For which case, the radial

wavefunction behaves like a meta-stable quasi-bound state, i.e., the magnitude of $u(r)$ is much stronger at $r < R$ than that at $r > R$?