Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 7 - 1 : Limits – series (I)

1. State whether the following sequences $(x_n)_{n\in\mathbb{N}}$ converge or not. If yes, please find the sequential limit and if not, please explain why.

(1) $x_n = (-1)^n$;

(2) $x_n = (-1)^n \frac{1}{n}$;

(3) $x_n = \sin n$;

(4) $x_n = r^n$, where $|r| < 1$;

(5) $x_n = \displaystyle\sum_{k=1}^{n} r^k$, where $|r| < 1$.

2. Let $a > 0$, $b > 1$ be two real numbers.

(1) Prove $\lim\limits_{n\to\infty} \frac{1}{b^n} = 0$.

(2) Prove that there exists an $n_0 \in \mathbb{N}$, such that for all $n > n_0$, $\frac{a}{n} \leqslant \frac{1}{2}$.

(3) Prove $\lim\limits_{n\to\infty} \frac{a^n}{n!} = 0$ by the facts above.

3. Prove the following statements about limits.

(1) $\displaystyle\lim_{n\to+\infty} \frac{n!}{n^n} = 0$;

(2) $\displaystyle\lim_{n\to+\infty} \sqrt[n]{a} = 1$ $(a > 0)$;

(3) $\displaystyle\lim_{n\to+\infty} \frac{n^k}{a^n} = 0$ $(a > 0,\ k \in \mathbb{N})$;

(4) $\displaystyle\lim_{n\to+\infty} \sqrt[n]{n^k} = 1$ $(k \in \mathbb{N})$;

(5) $\displaystyle\lim_{n\to+\infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n)^2}\right) = 0$;

(6) $\displaystyle\lim_{n\to+\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}\right) = \frac{1}{2}$;

(7) $\displaystyle\lim_{n\to+\infty} \left(\frac{1}{2} + \frac{2}{2^2} + \cdots + \frac{n}{2^n}\right) = 2$;

(8) $\displaystyle\lim_{n\to+\infty} \frac{3}{2} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{2^{2^n}+1}{2^{2^n}} = 2$;

(9) $\displaystyle\lim_{n\to+\infty} \left(\frac{1}{2} + \frac{3}{2^2} + \cdots + \frac{2n-1}{2^n}\right) = 3$.

(10) $\lim\limits_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0.$

(11) $\lim\limits_{n\to\infty} ((n+1)^a - n^a) = 0$, where $a < 1$.

4. Prove the limit by following steps.

(1) For $n \in \mathbb{N}^*$, prove $\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$.

(2) Prove $\lim\limits_{n\to\infty} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = 0$.

5. Find or prove the following limits.

(1) Prove $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Hint : Prove $\sqrt[n]{n} \leqslant 1 + \frac{2}{\sqrt{n}}$.

(2) Find $\lim\limits_{n\to\infty} \sqrt[n]{1 + \frac{1}{2} + \cdots + \frac{1}{n}}$.

6. Calculate the following limits.

(1) $\lim\limits_{n\to\infty} \frac{a_m n^m + \cdots + a_0}{b_l n^l + \cdots + b_0}$, where $a_0, \ldots, a_m, b_0, \ldots, b_l \in \mathbb{R}$. Discuss for the cases of $m > l$, $m = l$ and $m < l$.

Hint : Begin this subject from the study of $\lim\limits_{n\to\infty} \frac{1}{n}$.

(2) $\lim\limits_{n\to+\infty} (\sqrt{n+1} - \sqrt{n})\sqrt{n}$;

(3) $\lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{n^2+1}}{n+2}$;

(4) $\lim\limits_{n\to+\infty} \left(\dfrac{1}{1\cdot 3} + \dfrac{1}{2\cdot 4} + \cdots + \dfrac{1}{n(n+2)}\right)$;

(5) $\lim\limits_{n\to+\infty} \sum\limits_{k=1}^{n} \dfrac{1}{k(k+1)(k+2)}$.

7. Prove the following statements about limits using the squeeze theorem.

(1) $\lim\limits_{n\to+\infty} \left[\dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \cdots + \dfrac{1}{(2n)^2}\right] = 0$;

(2) $\lim\limits_{n\to+\infty} \sqrt[n]{a_1^n + a_2^n + \cdots + a_k^n} = \max\limits_{1\leqslant i\leqslant k}\{a_i\}, (a_i \geqslant 0,\ i = 1, 2, \ldots, k)$;

(3) $\lim\limits_{n\to+\infty} \sqrt[n]{(\sqrt[n]{n} - 1)} = 1$;

(4) $\lim\limits_{n\to+\infty} \sqrt[n]{a_1 a_2 \cdots a_n} = a,\ (a_n > 0,\ \lim\limits_{n\to\infty} a_n = a)$.

8. Suppose that (x_n) is a sequence of real numbers and $\lim\limits_{n\to+\infty} x_n = a$. Prove that

(1) $\lim\limits_{n\to+\infty} \dfrac{x_1 + 2x_2 + \cdots + nx_n}{n^2} = \dfrac{a}{2}$.

(2) Suppose that $0 < r < 1$. Use the conclusion in (1) to calculate the following limits :

i. $\lim\limits_{n\to+\infty} \dfrac{1}{n^2}(r + 2^2r^2 + \cdots + n^2r^n)$;

ii. $\lim_{n\to+\infty} \frac{1}{n^2}(1+2^{\frac{3}{2}}+3^{\frac{4}{3}}+\cdots+n^{\frac{n+1}{n}})$.

9. Suppose that $\varphi : \mathbb{N} \to \mathbb{N}$ such that φ is strictly monotonically increasing, and $\varphi \neq \mathrm{id}_{\mathbb{N}}$.

(1) Prove that there exists $N \in \mathbb{N}$, such that for any $n \in \mathbb{N}$ and $n \geq N$, $\varphi(n) > n$.

(2) Let $\varphi^0 := id_{\mathbb{N}}$, and inductively define

$$\varphi^{n+1} := \varphi \circ \varphi^n, \quad \forall n \in \mathbb{N} \cup \{0\}.$$

Prove that for any $k \in \mathbb{N}$ and $n \in \mathbb{N}$ with $n \geq N$, $\varphi^k(n) \geq n+k$.

(3) Suppose that (x_n) is a sequence of real numbers satisfying $x_{\varphi(n)} = x_n$ for any $n \in \mathbb{N}$. Prove that if (x_n) is convergent, then there exists $M \in \mathbb{N}$, such that for any $n \in \mathbb{N}$ with $n \geq M$, $x_n = x_M$.

10. Suppose that (x_n) is a sequence of real numbers satisfying $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. Prove that if

$$\lim_{n\to+\infty} \frac{x_1+x_2+\ldots x_n}{n} = a,$$

then $\lim_{n\to+\infty} x_n = a$.

(1) Suppose that (a_n) is a sequence of real numbers satisfying

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lambda,$$

where $|\lambda| < 1$. Prove that $\lim_{n\to+\infty} a_n = 0$.

(2) Use the conclusion in (a) to prove the following statements about limits :

i. $\lim_{n\to+\infty} \frac{a^n}{n!} = 0 \ (a \in \mathbb{R})$;

ii. $\lim_{n\to+\infty} \frac{1^2 \cdot 3^2 \cdot (2n-1)^2 a^n}{2^2 \cdot 4^2 \cdots (2n)^2} = 0 \ (|a| < 1)$.

11. (1) Suppose that (a_n) is a sequence of real numbers satisfying

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \lambda,$$

where $\lambda > 1$. Prove that $\lim_{n\to+\infty} a_n = +\infty$.

(2) Use the conclusion in (a) to prove the following statements about limits :

i. $\lim_{n\to+\infty} \frac{3 \cdot 6 \cdot 9 \cdots 3n}{7 \cdot 10 \cdot 13 \cdots (3n+4)} a^n = +\infty \ (a > 1)$;

ii. $\lim\limits_{n\to+\infty} \dfrac{1\cdot 3\cdot 5\cdots(2n-1)a^{2n}}{2\cdot 4\cdot 6\cdots(2n-2)\cdot(2n)} = +\infty \ (|a|>1)$.

12. Prove the following statements about superior and inferior limits.

(1) Prove that $\overline{\lim\limits_{n\to+\infty}}(-1)^n(1+\dfrac{1}{n}) = 1$ and $\underline{\lim\limits_{n\to+\infty}}(-1)^n(1+\dfrac{1}{n}) = -1$.

(2) Prove that $\underline{\lim\limits_{n\to+\infty}}\left[(1+(-1)^n)n+\dfrac{1}{n}\right] = 0$ but it has no superior limit.

(3) Prove that $\overline{\lim\limits_{n\to+\infty}}\left[((-1)^n-1)n+\dfrac{1}{n}\right] = 0$ but it has no inferior limit.

(4) Prove that (n^2) and $(-n)$ have neither superior limit nor inferior limit.

13. (1) Let (u_n) be a sequence that has both superior limit and inferior limit. Prove that $\underline{\lim\limits_{n\to+\infty}} u_n \leq \overline{\lim\limits_{n\to+\infty}} u_n$.

(2) Prove that u_n converges to a real number l if and only if $\underline{\lim\limits_{n\to+\infty}} u_n = \overline{\lim\limits_{n\to+\infty}} u_n = l$.

14. We consider the sequence (x_n) and (y_n) defined by :

$$\begin{cases} x_{2n} = 1+\frac{1}{2n+1} \\ x_{2n+1} = 0 \end{cases} \qquad \begin{cases} y_{2n} = 0 \\ y_{2n+1} = \frac{1}{2n+2}+1. \end{cases}$$

(1) Calculate $\overline{\lim} x_n$, $\underline{\lim} x_n$ and $\overline{\lim}(x_n+y_n)$.

(2) Calculate $\underline{\lim} x_n$, $\overline{\lim}(-x_n)$ and $\underline{\lim}(-x_n)$.

(3) Prove that if (u_n) is a bounded sequence and if $\alpha > 0$, then $\overline{\lim}\alpha x_n = \alpha\overline{\lim} u_n$ and $\underline{\lim}\alpha x_n = \alpha\underline{\lim} u_n$.

15. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real numbers defined by $r_n := \sum\limits_{k=1}^{n} a_k$, where $a_k \in \mathbb{R}$ for all $1 \leq k \leq n$. We suppose that the sequence $(\sqrt[k]{|a_k|})_{k\in\mathbb{N}}$ has a superior limit λ.

(1) Suppose that $\lambda < 1$ and μ is a real number with $\lambda < \mu < 1$. If there exists an integer K such that $\forall\ k \in \mathbb{N}$ with $k \geq K$, $\sqrt[k]{|a_k|} < \mu$, prove that the sequence (r_n) is convergent.

(2) Suppose that $\lambda > 1$ and μ is a real number with $\lambda > \mu > 1$. If for any $k \in \mathbb{N}$, there exists an $p \in \mathbb{N}$ such that $p \geqslant k$ and $\sqrt[k]{|a_k|} > \mu$, prove that the sequence (r_n) is divergent.

(3) State the convergence or divergence of the sequence (r_n), where $r_n := \sum_{k=1}^{n} a_k$ with a_k defined below :

i. $a_{2j} = \left(\frac{j}{2j+1}\right)^{2j}, a_{2j+1} \left(\frac{1}{2j+1}\right)^{2j+1}$;

ii. $a_{2j} = \left(\frac{3j-1}{2j+1}\right)^{2j}, a_{2j+1} \left(\frac{1}{2j+1}\right)^{2j+1}$.

16. Let (r_n) be a sequence of real numbers defined by $r_n := \sum_{k=1}^{n} a_k$, where $a_k \in \mathbb{R}$ and $a_k > 0$ for all $1 \le k \le n$.

(1) Prove that if the sequence $\left(\frac{a_{k+1}}{a_k}\right)$ has an superior limit strictly smaller than 1, then the sequence (r_n) is convergent.

(2) Prove that if $\left(\frac{a_{k+1}}{a_k}\right)$ has a inferior limit strictly larger than 1, then the sequence (r_n) is divergent.

(3) State the convergence or divergence of the sequence (r_n), where $r_n := \sum_{k=1}^{n} a_k$ with a_k defined below :

i. $a_{2j} = \frac{j!}{1\cdot 9\cdot 25 \cdots (2j+1)^2}, a_{2j+1} = \frac{j!}{1\cdot 9\cdot 25 \cdots (2j+1)^2(2j+3)}$;

ii. $a_{2j+1} = \frac{\cos 1 \cdots \cos \frac{1}{2j+1}}{2^{j+1}}, a_{2j+2} = \frac{\cos 1 \cdots \cos \frac{1}{2j+1}}{2^j}$;

iii. $a_{2j+1} = \frac{\exp\left(-1+\frac{1}{2}+\cdots+\frac{1}{j}\right)}{2j+1}, a_{2j+2} = \frac{\exp\left(1+\frac{1}{2}+\cdots+\frac{1}{j+1}\right)}{2j+2}$.