QM HW12

Jiete XUE

December 22, 2025

Problem 1 (Resonance scattering)
We have potential:

$$V(r) = \frac{\gamma\hbar^2}{2m}\delta(r-R). \tag{1.1}$$

Since we only consider the case where $l = 0$, the Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\cdot\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR_0) + (V(r)-E)R_0 = 0. \tag{1.2}$$

R_0 is the radial contribution.
Let $u(r) = rR_0(r)$, then,

$$-\frac{\hbar^2}{2m}u'' + (V-E)u = 0. \tag{1.3}$$

We suppose the solution at $r < R$ and $r > R$ are u_1 and u_2 respectively. Both of them are of the form

$$u_i(r) = A_i\mathrm{e}^{kr} + B_i\mathrm{e}^{-kr},\ i\in\{1,2\},\ k = \sqrt{-\frac{2mE}{\hbar^2}} > 0. \tag{1.4}$$

(1) We suppose the bound state exists, then $E < 0$, and $A_2 = 0$, or the integral of wavefunction will diverge. The derivative continuous at $r = 0$, thus equals zero, it require $A_1 + B_1 = 0$. Then the connection condition becomes

$$u(R) = A_1\left(\mathrm{e}^{kR} - \mathrm{e}^{-kR}\right) = B_2\mathrm{e}^{-kR}, \tag{1.5}$$

$$-\frac{\hbar^2}{2m}\left[\left(-B_2k\mathrm{e}^{-kR}\right) - A_1k\left(\mathrm{e}^{kR}+\mathrm{e}^{-kR}\right)\right] + \frac{\gamma\hbar^2}{2m}u(R) = 0. \tag{1.6}$$

Then we can deduce that

$$\frac{2kR}{\gamma R} = \left(\mathrm{e}^{-2kR} - 1\right). \tag{1.7}$$

The derivative of function $\mathrm{e}^{-x} - 1$ is monotone, and is -1 at $x = 0$, so $|\gamma_c| = R^{-1}$.

(2) Suppose $k=\sqrt{\frac{2mE}{\hbar^2}}$ and

$$u(r)=\begin{cases}A\sin(kr), & r<R,\\ C\sin(kr+\delta_0), & r>R.\end{cases} \tag{1.8}$$

The connection conditions give

$$A\sin(kR)=C[\sin(kR)\cos\delta_0+\cos(kR)\sin\delta_0], \tag{1.9}$$

$$Ak\cos(kR)=Ck[\cos(kR)\cos\delta_0-\sin(kR)\sin\delta_0]+\gamma A\sin(kR). \tag{1.10}$$

So,

$$\tan\delta_0=\frac{\gamma\sin^2(kR)}{k-\gamma\sin(kR)\cos(kR)}=\frac{1-\cos(2kR)}{2k-\gamma\sin(2kR)}. \tag{1.11}$$

It is odd function of k.

$$2k-\gamma\sin(2kR)=(1-\gamma R)(2k)+\frac{\gamma}{6}(2kR)^3+\mathcal{O}(k^5). \tag{1.12}$$

$$1-\cos(2kR)=\frac{1}{2}(2kR)^2-\frac{1}{24}(2kR)^4+\mathcal{O}(k^6). \tag{1.13}$$

So,

$$a_0=\left(\frac{1}{R}-\frac{1}{\gamma R^2}\right)^{-1},\ r_0=\frac{1}{3}\left(R+\frac{1}{\gamma}\right). \tag{1.14}$$