QM HW9

Jiete XUE

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Problem 1 (2D Coupled Harmonic Oscillator)
(1) The energy of three lowest-lying states are

$$\begin{aligned} E_0 &= \hbar\omega \\ E_1 &= \frac{3\hbar\omega}{2} \\ E_2 &= 2\hbar\omega. \end{aligned} \tag{1.1}$$

E_0 is not degenerate, while E_1 and E_2 have degeneracy of $2, 3$ respectively.
(2) E_0 is non-degenerate, $\langle 0|V|0\rangle = 0$, so $\Delta_0^{(1)} = 0$.

$$V = \delta m\omega^2 l^2 \frac{(a_1 + a_1^\dagger)(a_2 + a_2^\dagger)}{2}. \tag{1.2}$$

$$\frac{\delta m\omega^2 l^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle\psi|1_1\rangle \\ \langle\psi|1_2\rangle \end{pmatrix} = \Delta_1^{(1)} \begin{pmatrix} \langle\psi|1_1\rangle \\ \langle\psi|1_2\rangle \end{pmatrix}. \tag{1.3}$$

So,

$$\Delta_1^{(1)} = \pm\frac{\delta m\omega^2 l^2}{2}, \tag{1.4}$$

with respect to eigen-kets $\frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}}$ and $\frac{|1,0\rangle - |0,1\rangle}{\sqrt{2}}$. Similarly,

$$\Delta_2^{(1)} = \pm\delta m\omega^2 l^2 \text{ or } 0. \tag{1.5}$$

(3) Let $u = \frac{x+y}{\sqrt{2}}$, $v = \frac{u-v}{\sqrt{2}}$, then,

$$H = \frac{p_u^2}{2m} + \frac{1}{2}m\omega^2(1+\delta)u^2 + \frac{p_u^2}{2m} + \frac{1}{2}m\omega^2(1-\delta)v^2. \tag{1.6}$$

Let,

$$\omega_u = \omega\sqrt{1+\delta},\ \omega_v = \omega\sqrt{1-\delta}. \tag{1.7}$$

Then,

$$E_{n_u,n_v} = \hbar\omega_u(n_u + 1/2) + \hbar\omega_v(n_v + 1/2) \approx \hbar\omega\left(n_u + n_v + 1 + \frac{\delta}{2}(n_u - n_v)\right). \tag{1.8}$$

We get the same result as perturbation theory.

Problem 2 (Quadratic Perturbation)
Let

$$l=\sqrt{\frac{\hbar}{m\omega}},\ a=\frac{1}{\sqrt{2}}\left(\frac{x}{l}+\frac{\mathrm{i}lp}{\hbar}\right),\ N=a^\dagger a. \tag{2.1}$$

Then,

$$H_0=\hbar\omega\left(N+\frac{1}{2}\right),\ H'=\frac{\epsilon}{4}\hbar\omega\left((a^\dagger)^2+a^2+2N+1\right). \tag{2.2}$$

Perturbation:

$$\Delta_0^{(1)}=\langle 0|H'|0\rangle=\frac{\epsilon}{4}\hbar\omega. \tag{2.3}$$

$$\Delta_0^{(2)}=\sum_{k=1}^{+\infty}\frac{|\langle k|H'|0\rangle|^2}{E_0-E_k}=-\frac{\epsilon^2}{16}\hbar\omega. \tag{2.4}$$

So,

$$\tilde{E_0}\approx\hbar\omega\left(\frac{1}{2}+\frac{\epsilon}{4}-\frac{\epsilon^2}{16}\right). \tag{2.5}$$

$$|\tilde{0}^{(1)}\rangle=\sum_{k=1}^{+\infty}\frac{|k\rangle\langle k|H'|0\rangle}{E_0-E_k}=-\frac{\epsilon\sqrt{2}}{8}|2\rangle. \tag{2.6}$$

Thus,

$$\tilde{\psi}(x)\approx\langle x|\left(|0\rangle-\frac{\epsilon\sqrt{2}}{8}|2\rangle\right)\sim \mathrm{e}^{-\frac{x^2}{2l^2}}\left(H_0\left(\frac{x}{l}\right)-\frac{\epsilon\sqrt{2}}{8}H_2\left(\frac{x}{l}\right)\right) \tag{2.7}$$

Exact:
Let

$$m'=m\sqrt{1+\epsilon},\ l'=\sqrt{\frac{\hbar}{m'\omega}},\ a'=\frac{1}{\sqrt{2}}\left(\frac{x}{l'}+\frac{\mathrm{i}l'p}{\hbar}\right). \tag{2.8}$$

Then,

$$H=\sqrt{1+\epsilon}\hbar\omega\left(N'+\frac{1}{2}\right). \tag{2.9}$$

$$\tilde{E_0}=\frac{1}{2}\sqrt{1+\epsilon}\hbar\omega\approx\hbar\omega\left(\frac{1}{2}+\frac{\epsilon}{4}-\frac{\epsilon^2}{16}\right). \tag{2.10}$$

$$\tilde{\psi}(x)\sim \mathrm{e}^{-\frac{x^2}{2l^2}}\left[1-\frac{\epsilon}{8}-\frac{m\omega}{4\hbar}\epsilon x^2\right]. \tag{2.11}$$

Problem 3 (Quadrupole Perturbation)
(1)

$$V=\lambda r^2\cdot 2\sqrt{\frac{2\pi}{15}}\left(Y_2^2+Y_2^{-2}\right). \tag{3.1}$$

By Wigner-Eckart Theorem,

$$V\doteq A\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}, \tag{3.2}$$

where A is a constant. Eigen-value of V is $E_1 = 0$, E_2, $E_3 = 0$. And eigen-kets are

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |-1\rangle\right), |-\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle - |-1\rangle\right), |0\rangle = |m=0\rangle \tag{3.3}$$

respectively. Threefold degeneracy has been removed completely.
(2)

Problem 4 (Second-order lifting of degeneracy)
(1) **Nondegenerate perturbation theory (Wrong)**
Unperturbed states: $|1\rangle, |2\rangle$ with energy E_1, $|3\rangle$ with energy E_2.
Using $E_n = E_n^{(0)} + \langle n| V |n\rangle + \sum_{k\neq n} \frac{|\langle k|V|n\rangle|^2}{E_n^{(0)} - E_k^{(0)}}$:
For $|1\rangle$: $\langle 1| V |1\rangle = 0$, only coupling to $|3\rangle$: $\langle 3| V |1\rangle = a^*$, so

$$E_1^{(2)} = \frac{|a|^2}{E_1 - E_2}$$

For $|2\rangle$:

$$E_2^{(2)} = \frac{|b|^2}{E_1 - E_2}$$

For $|3\rangle$: couplings to $|1\rangle, |2\rangle$:

$$E_3^{(2)} = \frac{|a|^2 + |b|^2}{E_2 - E_1}$$

Thus:

$$E_A \approx E_1 + \frac{|a|^2}{E_1 - E_2}, \quad E_B \approx E_1 + \frac{|b|^2}{E_1 - E_2}, \quad E_C \approx E_2 + \frac{|a|^2 + |b|^2}{E_2 - E_1}$$

(2) Exact
Characteristic equation:

$$\det \begin{pmatrix} E_1 - E & 0 & a \\ 0 & E_1 - E & b \\ a^* & b^* & E_2 - E \end{pmatrix} = 0$$

Expansion gives:

$$(E_1 - E)\left[(E_1 - E)(E_2 - E) - |b|^2\right] - |a|^2(E_1 - E) = 0$$

$$(E_1 - E)\left[(E_1 - E)(E_2 - E) - (|a|^2 + |b|^2)\right] = 0$$

So one root is $E = E_1$, others satisfy:

$$(E_1 - E)(E_2 - E) - S = 0, \quad S = |a|^2 + |b|^2$$

Solving:

$$E = \frac{E_1 + E_2 \pm \sqrt{(E_2 - E_1)^2 + 4S}}{2}$$

For small S:

$$E \approx E_1 - \frac{S}{\Delta}, \quad E_1, \quad E_2 + \frac{S}{\Delta}$$

(3) Degenerate perturbation theory

Degenerate subspace: $\{|1\rangle, |2\rangle\}$, unperturbed energy E_1.

Effective Hamiltonian:

$$(H_{\text{eff}})_{ij} = E_1 \delta_{ij} + V_{ij} + \sum_{k \notin \text{deg}} \frac{V_{ik} V_{kj}}{E_1 - E_k}$$

Only $k = 3$ contributes, with $E_3^{(0)} = E_2$:

$$H_{\text{eff}} = E_1 - \frac{1}{\Delta} \begin{pmatrix} |a|^2 & ab^* \\ a^* b & |b|^2 \end{pmatrix}$$

Eigenvalues:

$$\lambda = E_1, \quad E_1 - \frac{S}{\Delta}$$

Third eigenvalue from nondegenerate PT on $|3\rangle$:

$$E_3 \approx E_2 + \frac{S}{\Delta}$$

(4) Comparison

- Nondegenerate PT (wrong):
  $E_1 + \frac{|a|^2}{E_1 - E_2}$, $E_1 + \frac{|b|^2}{E_1 - E_2}$, $E_2 + \frac{S}{E_2 - E_1}$
- Exact
  E_1, $E_1 - \frac{S}{\Delta}$, $E_2 + \frac{S}{\Delta}$
- Degenerate PT (correct):
  Same as exact to second order

The nondegenerate treatment fails because it doesn't account for degeneracy lifting between $|1\rangle$ and $|2\rangle$.