QM HW7

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October 26, 2025

Problem 1 (Hydrogen atom wavefunction)
(1)

$$\psi_{n_r,l,m} = R_{n_r,l}(r)Y_l^m(\theta,\phi), \tag{1.1}$$

where, Y_l^m is the spherical harmonics and

$$R_{n_R,l}(r) \sim \rho^l \mathrm{e}^{-\frac{\rho}{2}} L_{n_r}^{2l+1}(\rho), \tag{1.2}$$

$$\rho = \frac{2r}{na_0},\ a_0\text{is a constant.} \tag{1.3}$$

$$E_n = \frac{E_0}{n^2}. \tag{1.4}$$

Problem 2 (Gaussian orbital approximation)
(1)

$$\psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}\mathrm{e}^{-\frac{r}{a}},\ \psi_{1s}^G(\mathbf{r}) = \sqrt{\frac{2\sqrt{2}}{\pi^{\frac{3}{2}}\lambda^3}}\mathrm{e}^{-\frac{r^2}{\lambda^2}}. \tag{2.1}$$

$$\begin{aligned}\mathrm{Err}(\lambda) &= 4\pi\int_0^{+\infty}\left|\frac{1}{\sqrt{\pi a^3}}\mathrm{e}^{-\frac{r}{a}} - \sqrt{\frac{2\sqrt{2}}{\pi^{\frac{3}{2}}\lambda^3}}\mathrm{e}^{-\frac{r^2}{\lambda^2}}\right|^2 \mathrm{d}r \\ &= 2 - \sqrt{\frac{128\sqrt{2}}{\sqrt{\pi}a^3\lambda^3}}\mathrm{e}^{\left(\frac{\lambda}{2a}\right)^2}\int_0^{+\infty}\mathrm{e}^{-\left(\frac{r}{\lambda}+\frac{\lambda}{2a}\right)^2}\,\mathrm{d}r\,.\end{aligned} \tag{2.2}$$

Problem 3 (2D hydrogen atom)
Let $\psi = R(r)\mathrm{e}^{in\phi}$, then the Schröedinger equation is

$$R'' + \frac{1}{\rho}R' - \frac{n^2}{\rho^2}R + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)R = 0, \tag{3.1}$$

where,

$$\kappa^2 = -\frac{2mE}{\hbar^2},\ \rho = 2\kappa r,\ \lambda = \frac{me^2}{\hbar^2\kappa}. \tag{3.2}$$

Considering the tendency at $\rho \to \infty$ and $\rho \to 0$, we have the form of R as

$$R(\rho) = \rho^{|n|}e^{-\frac{\rho}{2}}w(\rho). \tag{3.3}$$

Then, $w(\rho)$ satisfies confluent hypergeometric equation:

$$\rho w'' + (2|n| + 1 - \rho)\, w' + \left(\lambda - |n| - \frac{1}{2}\right) w = 0. \tag{3.4}$$

When

$$\lambda = n_r + |n| + \frac{1}{2},\ \text{with } n_r \text{ a natural number} \tag{3.5}$$

the solution is a polynomial. So

$$E_n = -\frac{me^4}{2\hbar^2 \left(N + \frac{1}{2}\right)^2}. \tag{3.6}$$

where $N = n_r + |n|$. The degeneracy is $2N+1$. In 3D case, the energy is related to a integer with power of -2, and has a degeneracy of n^2.

Problem 4 (Edge spectrum of the edge state of QHE)

Problem 5 (Schwinger boson representation of angular momentum)
(1)

$$[J_\mu, J_\nu] = \frac{1}{4}\sigma^\mu_{\alpha\beta}\sigma^\nu_{\rho\lambda}[a^\dagger_\alpha a_\beta, a^\dagger_\rho a_\lambda]. \tag{5.1}$$

$$[a\alpha^\dagger a_\beta, a^\dagger_\rho a_\lambda] = a^\dagger_\alpha a_\lambda \delta_{\beta\rho} - a^\dagger_\beta a_\rho \delta_{\alpha\lambda}. \tag{5.2}$$

So,

$$[J_\mu, J_\nu] = \frac{1}{4} a^\dagger_\alpha a_\beta [\sigma^\mu, \sigma^\nu]_{\alpha\beta} = \mathrm{i}\epsilon_{\mu\nu\lambda}\frac{1}{2} a^\dagger_\alpha \sigma^\lambda_{\alpha\beta} a_\beta = \mathrm{i}\epsilon_{\mu\nu\lambda} J_\lambda. \tag{5.3}$$

(2)

$$\sigma^\mu_{\alpha\beta}\sigma^\mu_{\rho\lambda} = 2\delta_{\alpha\lambda}\delta_{\beta\rho} - \delta_{\alpha\beta}\delta_{\rho\lambda}. \tag{5.4}$$