Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 9–3 : Differentiability : functions of several variables

1. Find the following limits if they exist.

(1) $\displaystyle\lim_{x\to\infty, y\to\infty} \frac{|x|+|y|}{x^2+y^2}$

(2) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$

(3) $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)^{x^2y^2}$

(4) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^6y^8}{(x^2+y^4)^5}$

(5) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

(6) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3}$

(7) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^3+y^6)^2}$

(8) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\arctan(x^3+y^3)}{x^2+y^2}$

(9) $\displaystyle\lim_{(x,y)\to(0,0)} \left(x\sin\frac{1}{y}+y\sin\frac{1}{x}\right)$

(10) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x}$

(11) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\log(x+e^y)}{\sqrt{x^2+y^2}}$

(12) $\displaystyle\lim_{(x,y)\to(0,0)} (x+y)\sin\frac{1}{x}\sin\frac{1}{y}$

(13) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}$

(14) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy(x^2-y^2)}{x^2+y^2}$

(15) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sqrt{xy}}{(x^2+y^2)^p},\ p>0$

(16) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{\sqrt{x^2+y^2}}$

2. Let $U\subset\mathbb{R}^d$ be a non-empty open convex set. We say $f: U\to\mathbb{R}$ is convex if and only if for all $x,y\in U$ and all $\lambda\in[0,1]$ we have

$$f((1-\lambda)x+\lambda y)\le(1-\lambda)f(x)+\lambda f(y).$$

(We say that f is strictly convex if the strictly inequality holds.)

(1) Suppose f is differentiable. Show that f is convex if and only if for all $x=(x_1,\dots,x_d), y=(y_1,\dots,y_d)\in U$ we have

$$f(y)-f(x)\ge\sum_{i=1}^{d}\frac{\partial f}{\partial x_i}(x)(y_i-x_i).$$

(Show that f is strictly convex if the strict inequality holds.)

(2) Suppose that f is convex on $\mathbb{R}^d$ and $f(0) = 0$. Show that there exist α, β such that for all $x \in \mathbb{R}^d$ we have

$$f(x) \geq \alpha \|x\| + \beta.$$

3. Let $f : U \to \mathbb{R}$ be a convex function, where $U \subset \mathbb{R}^d$ is a non-empty open convex set.

(1) Show that any local minimal point of f is also a global minimal point.

(2) Show that the global minimal points of f form a convex set.

(3) Show that if f is strictly convex then it has at most one global minimal point.

(4) Suppose that $f \in C^1$ and $x^* \in U$ is a global minimal point. Then we have $\frac{\partial f}{\partial x_i}(x^*) = 0$ for all $i = 1, 2, \cdots, d$.

4. The mapping $f : \mathbb{R}^d \to \mathbb{R}$ is given by $f(x_1, \cdots, x_d) = x_1 x_2 \cdots x_d$. Find the extremal points of f on the set

$$\Big\{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} \prod_{j \neq i} x_j = 1\Big\}.$$

5. Suppose $f(x, y, z)$ is continuous on the cube $[a, b] \times [a, b] \times [a, b]$. Show that

$$g(x, y) = \max_{a \leq z \leq b} f(x, y, z)$$

is continuous in the square $[a, b] \times [a, b]$.

6. Let $\Omega \subset \mathbb{R}^d$ be an open set. Suppose $f : \Omega \to \mathbb{R}$ has bounded partial derivatives (that is, there exists $M > 0$ such that $\left|\frac{\partial f}{\partial x_i}(x)\right| \leq M$ for all $i = 1, 2, \cdots, d$ and all $x \in \Omega$). Show that f is continuous on Ω. Is f differentiable on Ω?

7. Suppose that $f(x, y)$ is separately continuous. Show that $f(x, y)$ is continuous provided that one of the following conditions hold.

(1) The continuity of $f(x, y)$ as a function of x is uniform with respect to y, that is, for any x and $\varepsilon > 0$, there exists δ depending on x, ε but not y such that for all x' such that $|x' - x| \leq \delta$ and all y we have

$$|f(x, y) - f(x', y)| \leq \varepsilon.$$

(2) The continuity of $f(x, y)$ as a function of y is uniform with respect to x,

(3) Particularly, $f(x,y)$ is λ-Lipschitz with respect to y, that is, for all y, y' and x, we have

$$|f(x,y) - f(x,y')| \le \lambda |y - y'|.$$

(4) $f(x,y)$ is λ-Lipschitz with respect to x.

8. Let $V \subset \mathbb{R}^d$ be a non-empty open set. Let $K \subset V$ be a bounded closed set. Show that there exists a continuous function $f : \mathbb{R}^d \to [0,1]$ such that

$$f(x) = \begin{cases} 1, & \text{if } x \in K \\ 0, & \text{if } x \notin V. \end{cases}$$

9. Suppose that $f(x)$ is continuous in a bounded open set $D \subset \mathbb{R}^d$. Show that f is uniformly continuous in D if and only if for any $x_0 \in \partial D$, $\lim_{x \in D, x \to x_0} f(x)$ exists. Here, ∂D means the boundary of D.

10. Show that

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$ but not differentiable at $(0,0)$.

11. Define

$$f(x,y) = \begin{cases} xy \sin \frac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Prove that

(1) $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$ exist.

(2) $\frac{\partial f}{\partial x}(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are not continuous at $(0,0)$.

(3) $f(x,y)$ is differentiable at $(0,0)$.

12. Let $\alpha, \beta > 0$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = \begin{cases} 0, & (x,y) = (0,0) \\ \sin\left(\frac{|x|^\alpha + |y|^\beta}{x^2+y^2}\right), & (x,y) \neq (0,0). \end{cases}$$

When is $f \in C^1(\mathbb{R}^2)$?

13. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x,y) = \begin{cases} 1, & (x,y) = (0,0) \\ \frac{\sin^2 x + \sin^2 y + \sin^2(x+y)}{x \sin x + y \sin y + (x+y)\sin(x+y)}, & (x,y) \neq (0,0). \end{cases}$$

Do the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist ? Is $f \in C^1(\mathbb{R}^2)$?

14. Suppose that $\frac{\partial f}{\partial x}(x, y)$ exists at $(0, 0)$ and that $\frac{\partial f}{\partial y}(x, y)$ is continuous at $(0, 0)$. Show that $f(x, y)$ is differentiable at $(0, 0)$.

15. The function φ satisfies $\varphi(0) = 0$ and $|\varphi(t)| \le t^2$ in a small neighbourhood of 0. Show that $f(x, y) := \varphi(|xy|)$ is differentiable at $(0, 0)$.

16. Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $f : \Omega \to \mathbb{R}$ be a differentiable function. Prove the following statements.

(1) If f achieves the maximal value at x_0, then $df(x_0) = 0$.

(2) Suppose that Ω is open and convex. If $df(x) = 0$ for all $x \in \Omega$, then f is a constant.

(3) Suppose that Ω is open and connected. (Connectivity means that $\emptyset$ and Ω are the only subsets of Ω that are both open and closed). If $df(x) = 0$ for all $x \in \Omega$, then f is a constant.