Hermite Ploynomial

1 Explicit Expression

$$H_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!(n-2k)!}(2z)^{n-2k}. \tag{1}$$

2 Generating Function

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}. \tag{2}$$

3 Hermite Equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{3}$$

This is a Sturm-Liouville type equation with weight function $w(x) = e^{-x^2}$.
Proof.

$$\frac{\partial G}{\partial s} = \sum_{n=0}^{\infty} \frac{1}{n!} H_{n+1}(z)s^n. \tag{4}$$

$$2sG = \sum_{n=1}^{\infty} 2n\frac{1}{n!} H_{n-1}s^n. \tag{5}$$

Compare the coefficients of s^n,

$$\boxed{H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0.} \tag{6}$$

4 Recurrence Relations

$$\frac{\partial G}{\partial z} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\mathrm{d}}{\mathrm{d}z} H_n(z)s^n. \tag{7}$$

Hence,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}z} H_n = 2nH_{n-1}.} \tag{8}$$

5 Rodrigues Formula

$$e^{-(s-z)^2} = \sum_{n=0}^{\infty} \frac{H_n(z)e^{-z^2}}{n!} s^n. \tag{9}$$

$$H_n(z)e^{-z^2} = \left. \frac{\mathrm{d}^n}{\mathrm{d}s^n} e^{-(s-z)^2} \right|_{s=0}. \tag{10}$$

$\mathrm{d}s = -\,\mathrm{d}(z-s)$, hence

$$\boxed{H_n(z) = (-)^n e^{z^2} \frac{\mathrm{d}^n}{\mathrm{d}z^n} e^{-z^2}.} \tag{11}$$

6 Parity Property

$$H_n(-x) = (-)^n H_n(x). \tag{12}$$

7 Special Values

$$H_{2m}(0) = (-)^m \frac{(2m)!}{m!}. \tag{13}$$

$$H_{2m+1}(0) = 0. \tag{14}$$

8 Orthogonality Relation

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}\,dx = \sqrt{\pi}2^n n!\delta_{mn} \tag{15}$$

Proof.

$$\int_{-\infty}^{+\infty} G_1(s,z)G_2(t,z)\,\mathrm{d}z = e^{-(z-(s+t))^2} e^{2st} = \Gamma(\frac{1}{2})e^{2st}. \tag{16}$$

Hence,

$$\int_{-\infty}^{+\infty} G_1(s,z)G_2(t,z)\,\mathrm{d}z = \sqrt{\pi}e^{2st}. \tag{17}$$

$$G_1(s,z)G_2(t,z) = \sum_{(n,m)\in\mathbb{N}^2} \frac{1}{n!m!} H_n H_m s^n t^m. \tag{18}$$

$$e^{2st} = \sum_{n=0}^{+\infty} \frac{(2st)^n}{n!}. \tag{19}$$

Therefore,

$$\boxed{\int_{-\infty}^{+\infty} H_n(z)H_m(z)e^{-z^2}\,\mathrm{d}z = \delta_{nm}2^n n!\sqrt{\pi}.} \tag{20}$$