

Spherical Functions

1 Legendre Polynomial

$$P_l(x) = \sum_{n=0}^l \frac{1}{(n!)^2} \frac{(l+n)!}{(l-n)!} \left(\frac{x-1}{2}\right)^n. \quad (1)$$

Where l is a natural number.

1.1 Rodrigues Formula:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l. \quad (2)$$

1.2 Differential Equation:

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_l(x)}{dx} \right] + l(l+1)P_l(x) = 0. \quad (3)$$

We can get

$$P_l(1) = 1 \quad (4)$$

via (1) and obtain

$$P_l(-x) = (-)^l P_l(x) \quad (5)$$

from (2).

1.3 Explicit Expression:

$$P_l(x) = \sum_{k=0}^{[l/2]} (-)^k \frac{(2l-2k)!}{2^k k! (l-k)! (l-2k)!} x^{l-2k}. \quad (6)$$

So,

$$P_{2l}(0) = (-)^l \frac{(2l)!}{(2^l l!)^2}, \quad P_{2l+1}(0) = 0. \quad (7)$$

1.4 Orthogonal Completeness

$$\int_{-1}^1 P_k(x) P_l(x) dx = \frac{2}{2l+1} \delta_{kl}. \quad (8)$$

1.5 Generating Function

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l, \quad |t| < \min \left| x \pm \sqrt{x^2 - 1} \right|. \quad (9)$$

1.6 Recursive Relation

Use (9) to get

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x), \quad (10)$$

and

$$P_l(x) = P'_{l+1}(x) - 2xP'_l(x) + P_{l-1}(x), \quad (11)$$

$$P'_{l+1}(x) = xP'_l(x) + (l+1)P_l(x), \quad (12)$$

2 Associated Legendre Function

2.1 Differential Equation

$$\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_l^m(x) \right] + \left[l(l+1) - \frac{m^2}{1-x^2} \right] P_l^m(x) = 0, \quad (13)$$

where, $m \leq l$ are natural numbers. We have the solution

$$P_l^m(x) = (-)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x), \quad (14)$$

$$P_l^{-m}(x) = (-)^m \frac{(l-m)!}{(l+m)!} P_l^m(x). \quad (15)$$

2.2 Orthogonal Relation

$$\int_{-1}^1 P_l^m(x) P_k^m(x) dx = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{lk}, \quad (16)$$

3 Spherical Harmonics

Special Harmonic functions are the normalized associated Legendre functions.

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}. \quad (17)$$

3.1 Orthogonal Relation

$$\int_0^{2\pi} d\phi \int_0^\pi Y_l^{m'}(\theta, \phi) Y_l^{m*}(\theta, \phi) \sin \theta d\theta = \delta_{ll'} \delta_{mm'}, \quad (18)$$

3.2 Parity

Under the transformation of $\mathbf{r} \rightarrow -\mathbf{r}$,

$$Y_l^m(\pi - \theta, \phi + \pi) = (-1)^m Y_l^m(\theta, \phi). \quad (19)$$

3.3 Addition Theorem

$$P_l(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}') = \frac{4\pi}{2l+1} \sum_{m=-l}^l Y_l^{m*}(\hat{\mathbf{n}}) Y_l^m(\hat{\mathbf{n}}'). \quad (20)$$