GP1 HW3

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Problem 1 (Four springs harmonic oscillation)
Four fixed points are:

$$\vec{r}_1 = (l_0, 0)\,,\ \vec{r}_2 = (-l_0, 0)\,,\ \vec{r}_3 = (0, l_0)\,,\ \vec{r}_4 = (0, -l_0)\,.$$

The potential energy of the system is:

$$V = \sum_{i=1}^{4} \frac{1}{2} k \left[(\vec{r} - \vec{r}_i) \cdot (\vec{r} - \vec{r}_i) - l_0^2 \right] = 2kr^2. \tag{1.1}$$

$$m\ddot{\vec{r}} = -\nabla V = -4k\vec{r}. \tag{1.2}$$

So effective force constant is

$$\boxed{k' = 4k.} \tag{1.3}$$

Problem 2 (Driven harmonic oscillation)
Let the mass of the spar buoy be m and the distance moving away from the initial point be x, the height of the wave be

$$y = h \sin\left(\frac{2\pi t}{T}\right). \tag{2.1}$$

Then the extra force after wave come leads to

$$m\ddot{x} = k\,(y - x)\,. \tag{2.2}$$

where k is a constant that satisfying:

$$mg = kL. \tag{2.3}$$

Then, we can deduce that

$$\ddot{x} + \frac{g}{L} x = \frac{g}{L} h \sin\left(\frac{2\pi t}{T}\right). \tag{2.4}$$

$$x = \frac{h}{1 - \frac{L}{g}\left(\frac{2\pi}{T}\right)^2} \sin\left(\frac{2\pi t}{T}\right). \tag{2.5}$$

Problem 3 (Damped harmonic oscillation I)
Let $\omega_0 = \sqrt{\frac{k}{m}}$.

1. (a) For $t < 0$, $x(t) = 0$. For $t > 0$:

$$m\ddot{x} = -kx - m\gamma\dot{x} + F_0. \tag{3.1}$$

The solution is

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t} + \frac{F_0}{k}. \tag{3.2}$$

where

$$\lambda_1 = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} = 0,\ \lambda_2 = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}. \tag{3.3}$$

The initial conditions are:

$$x(0) = 0,\ \dot{x}(0) = 0. \tag{3.4}$$

Thus,

$$A = \frac{F_0}{k}\frac{-\lambda_2}{\lambda_2 - \lambda_1},\ B = \frac{F_0}{k}\frac{\lambda_1}{\lambda_2 - \lambda_1}. \tag{3.5}$$

$$\boxed{x(t) = \frac{F_0}{k}\left(\frac{-\lambda_2}{\lambda_2 - \lambda_1}e^{\lambda_1 t} + \frac{\lambda_1}{\lambda_2 - \lambda_1}e^{\lambda_2 t} + 1\right)} \tag{3.6}$$

(b)

$$m\ddot{x} = -kx - m\gamma\dot{x} + F_0\cos\left(\omega_0 t\right). \tag{3.7}$$

$$x(t) = \frac{F_0}{m\omega_0\gamma}\sin\left(\omega_0 t\right) + \frac{F_0}{m\gamma\left(\lambda_2 - \lambda_1\right)}\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right). \tag{3.8}$$

2. The amplitude of the oscillation is

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\gamma\omega\right)^2}}. \tag{3.9}$$

When

$$\boxed{\omega^2 = \omega_0^2 - \frac{\gamma^2}{2}},\ \text{if it's positive.} \tag{3.10}$$

The amplitude get maximum. If $0 > \omega_0^2 - \frac{\gamma^2}{2}$, there does not exist a maximum.

Problem 4 (Damped oscillation II)
We have known that after a long time, the motion of the oscillation is

$$x(t) = A\cos\left(\omega t + \phi\right). \tag{4.1}$$

where

$$A = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\beta\omega\right)^2}}, \tag{4.2}$$

and ϕ satisfying

$$\tan\left(\phi\right) = \frac{\beta\omega}{\omega^2 - \omega_0^2}. \tag{4.3}$$

1. $$P(t) = F_0 \cos\left(\omega t\right) \dot{x} = -F_0 A\omega \cos\left(\omega t\right) \sin\left(\omega t + \phi\right). \tag{4.4}$$

   The average rate is

   $$\langle P \rangle = \int_0^T P(t) \, \mathrm{d}t \,/T = -\frac{1}{2} F_0 A\omega \sin\left(\phi\right) = \frac{1}{2} m A^2 \omega^2 \beta. \tag{4.5}$$

2. $$\int_0^T \beta \dot{x}\dot{x} \, \mathrm{d}t \,/T = \frac{1}{2} m A^2 \omega^2 \beta. \tag{4.6}$$

3. $$\langle P \rangle = \frac{F_0^2 \beta}{2m} \frac{1}{\omega^2 + \frac{\omega_0^4}{\omega^2} + \beta^2 - 2\omega_0^2} \tag{4.7}$$

   iff. $\omega = \omega_0$, $\langle P \rangle$ get the maximum:

   $$\boxed{\langle P \rangle_{\max} = \frac{F_0^2}{2m\beta}.} \tag{4.8}$$