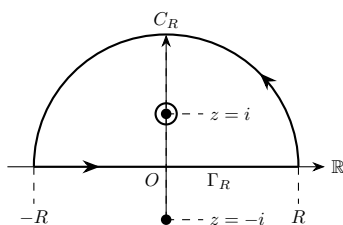# QM HW10

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Problem 1

$$H'(t) = -\frac{F_0\tau/\omega}{\tau^2 + t^2}x = -\frac{F_0\tau/\omega}{\tau^2 + t^2}\frac{a + a^\dagger}{\sqrt{2}}\sqrt{\frac{\hbar}{m\omega}}. \tag{1.1}$$

$$c^{(1)} = \frac{-\mathrm{i}}{\hbar}\int_{-\infty}^{+\infty} \mathrm{d}t \exp(\mathrm{i}\omega t)\langle 1|\,H'\,|0\rangle = \frac{\mathrm{i}F_0}{\sqrt{2\hbar m\omega^3}}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega\tau z}\frac{\mathrm{d}z}{1+z^2}. \tag{1.2}$$

By residue theorem,

$$\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\omega\tau z}\frac{\mathrm{d}z}{1+z^2} = \pi\mathrm{e}^{-\omega\tau}. \tag{1.3}$$

So,

$$P = |c^{(1)}|^2 = \frac{\pi^2 F_0^2\mathrm{e}^{-2\omega\tau}}{2\hbar m\omega^3}. \tag{1.4}$$

$\tau \gg 1/\omega$ means low frequency of transition, it is proper.

Problem 2

$$H'(t) = eE_0 z\exp(-t/\tau),\ t > 0. \tag{2.1}$$

$$c_f^{(1)}(+\infty) = \frac{-\mathrm{i}}{\hbar}\int_0^{+\infty}\mathrm{d}t'\,\langle f|\,H'(t)\,|i\rangle\,\mathrm{e}^{\mathrm{i}\omega_{fi}t'}. \tag{2.2}$$

$$\langle f|\,z\,|i\rangle = \sqrt{\frac{4\pi}{3}}\,\langle R_{nl}|\,r\,|R_{10}\rangle \cdot \langle Y_l^m|\,Y_1^0\,|Y_0^0\rangle. \tag{2.3}$$

Since Y_1^0 is odd and by Wigner-Eckart theorem, only $l = 1$ and $m = 0$, the element can not be zero.

$$\langle Y_l^m | Y_1^0 | Y_0^0 \rangle = \frac{1}{\sqrt{4\pi}} \int |Y_1^0|^2 \, \mathrm{d}\Omega = \frac{1}{\sqrt{4\pi}}. \tag{2.4}$$

$$\int_0^{+\infty} e^{-t/\tau + \mathrm{i}\omega_{fi}t} \mathrm{d}t = \frac{1}{1/\tau - \mathrm{i}\omega_{fi}}. \tag{2.5}$$

So,

$$P_{210} = \frac{e^2 E_0^2 |\langle R_{nl} | r | R_{10} \rangle|^2 \tau^2}{2\hbar^2 \left(1 + \omega^2 \tau^2\right)}. \tag{2.6}$$

where E_0 is the energy of ground state, and $\omega = \frac{3}{4}\frac{E_0}{\hbar}$.

Problem 3

(a) Let

$$\Delta\omega = \omega - \omega_{21}. \tag{3.1}$$

Define new variables:

$$\tilde{c}_1(t) = c_1(t)e^{-i\Delta\omega\,t/2}, \qquad \tilde{c}_2(t) = c_2(t)e^{i\Delta\omega\,t/2}. \tag{3.2}$$

Then,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \Delta\omega & 2\gamma/\hbar \\ 2\gamma/\hbar & -\Delta\omega \end{pmatrix}\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix}. \tag{3.3}$$

Thus the effective Hamiltonian in the rotated frame is

$$H_{\text{eff}} = \frac{\hbar}{2}\begin{pmatrix} \Delta\omega & 2\gamma/\hbar \\ 2\gamma/\hbar & -\Delta\omega \end{pmatrix} = \frac{\hbar}{2}\left(\Delta\omega\,\sigma_z + \frac{2\gamma}{\hbar}\,\sigma_x\right). \tag{3.4}$$

$$U(t) = \exp\left(-\frac{i}{\hbar}H_{\text{eff}}\,t\right) = \exp\left[-it\begin{pmatrix} \Delta\omega/2 & \gamma/\hbar \\ \gamma/\hbar & -\Delta\omega/2 \end{pmatrix}\right]. \tag{3.5}$$

The eigenvalues of the matrix are $\pm\Omega_R$ with

$$\Omega_R = \sqrt{\left(\frac{\gamma}{\hbar}\right)^2 + \left(\frac{\Delta\omega}{2}\right)^2}. \tag{3.6}$$

Using the identity $\exp(-i\boldsymbol{n}\cdot\boldsymbol{\sigma}\,\theta) = \cos\theta\,I - i\sin\theta\,(\boldsymbol{n}\cdot\boldsymbol{\sigma})$, we obtain

$$U(t) = \cos(\Omega_R t)\,I - i\sin(\Omega_R t)\frac{1}{\Omega_R}\begin{pmatrix} \Delta\omega/2 & \gamma/\hbar \\ \gamma/\hbar & -\Delta\omega/2 \end{pmatrix}. \tag{3.7}$$

With $\tilde{c}_1(0) = 1$, $\tilde{c}_2(0) = 0$, equation (3.7) gives

$$\tilde{c}_2(t) = -i\,\frac{\gamma}{\hbar\Omega_R}\sin(\Omega_R t). \tag{3.8}$$

Returning to the original variables,

$$c_2(t) = \tilde{c}_2(t)e^{-i\Delta\omega\, t/2} = -i\,\frac{\gamma}{\hbar\Omega_R}e^{-i\Delta\omega\, t/2}\sin(\Omega_R t). \tag{3.9}$$

Hence the occupation probability of the excited state is

$$\boxed{|c_2(t)|^2 = \frac{\gamma^2/\hbar^2}{\gamma^2/\hbar^2 + (\omega-\omega_{21})^2/4}\sin^2\!\left[\sqrt{\frac{\gamma^2}{\hbar^2}+\frac{(\omega-\omega_{21})^2}{4}}\,t\right]} \tag{3.10}$$

and, by normalization,

$$|c_1(t)|^2 = 1 - |c_2(t)|^2. \tag{3.11}$$

(b)

$$c_2^{(1)}(t) = -\frac{i}{\hbar}\gamma\int_0^t e^{-i\Delta\omega t'}\,dt' = \frac{\gamma}{\hbar\Delta\omega}(1 - e^{-i\Delta t}). \tag{3.12}$$

So,

$$P_{1\to 2}^{(1)}(t) = \frac{4\gamma^2}{\hbar^2\Delta\omega^2}\sin^2\!\left(\frac{\Delta\omega t}{2}\right). \tag{3.13}$$

For the cases where $\Delta\omega \gg \frac{\gamma}{\hbar}$, the approximation is valid. But for the cases where $\Delta\omega \ll \frac{\gamma}{\hbar}$, the approximation is not valid. This is because this resonance case.

Problem 4

$$H'(t) = eE_0(x\cos(\omega t) - y\sin(\omega t)),\ t > 0. \tag{4.1}$$

$$x\cos(\omega t) - y\sin(\omega t) \sim (Y_1^{+1}+Y_1^{-1})\cos(\omega t) + i(Y_1^{+1}-Y_1^{-1})\sin(\omega t). \tag{4.2}$$

That is

$$H'(t) \sim Y_1^{+1}e^{i\omega t} + Y_1^{-1}e^{-i\omega t}. \tag{4.3}$$

Since x, y has odd parity and by Wigner-Eckart theorem, only when $\Delta m = \pm 1$, $\Delta l = \pm 1$, the element can not be zero.

Problem 5
(a) We have shown that the possible final states should be $l' = 1$ and $m' = 0/\pm 1$. By symmetry, three banch have the same ratio.
(b) $l' = l \pm 1$, $m' = m$ or $m \pm 1$. By Wigner-Eckart theorem,

$$P(m') = |\langle l,m;1,(m'-m)|l',m'\rangle|^2 \tag{5.1}$$

(1) For a fixed l', the ratio is

$$|\langle l,m;1,-1|l',m-1\rangle|^2 : |\langle l,m;1,0|l',m\rangle|^2 : |\langle l,m;1,1|l',m+1\rangle|^2. \tag{5.2}$$

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