Pauli Matrix

We define

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

We define $\sigma^0 = I_{2\times 2}(\mathbb{C})$ just for convenience, it is used for balancing the index, you should not replace I any where. In the following part, we will use i, j, k to represent $1, 2, 3$ and other Latin letters to represent $0, 1, 2, 3$. Also, we will use Greek letters to represent $1, 2$, which are the components of matrices. So there are something crazy:

$$\delta^{\mu}{}_{\mu} = 2,\ \delta^{i}{}_{i} = 3,\ \delta^{a}{}_{a} = 4. \tag{2}$$

Property 1.

$$\sigma^i \sigma^j = \delta^{ij} I + \mathrm{i}\varepsilon^{ijk}\sigma_k. \tag{3}$$

If we define a general[1] ϵ^{abc}:

$$\epsilon_{abc} = \begin{cases} \varepsilon^{abc} & ,0 \notin \{a,b,c\} \\ 0 & ,0 \in \{a,b,c\} \end{cases} \tag{4}$$

Replace ε^{ijk} with ϵ^{abc}, the equality is still valid:

$$\boxed{\sigma^a \sigma^b = \delta^{ab}\sigma^0 + \mathrm{i}\epsilon^{abc}\sigma_c.} \tag{5}$$

Property 2. For any $i \in \{1, 2, 3\}$,

$$\boxed{\det\left(\sigma^i\right) = -1,\ \mathrm{tr}(\sigma^i) = 0,\ (\sigma^i)^\dagger = \sigma^i.} \tag{6}$$

Property 3 (Completeness)**.**
Any $M \in M_{2\times 2}(\mathbb{C})$ can be written as a linear combination of $\sigma^0, \sigma^1, \sigma^2, \sigma^3$. Since,

$$\sigma^0+\sigma^3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},\ \sigma^0-\sigma^3 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},\ \sigma^1+\mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},\ \sigma^1-\mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{7}$$

Let $M = \beta_a \sigma^a$, then

$$M\sigma^b = \beta_a \sigma^a \sigma^b = \beta_a \left(\delta^{ab} I + \mathrm{i}\epsilon^{abc}\sigma_c\right). \tag{8}$$

We take its trace, the second term at right hand side is zero, so

$$\boxed{\beta^b = \frac{1}{2}\mathrm{tr}(M\sigma^b).} \tag{9}$$

[1] It is also anti-symmetric: $\forall a, b, c \in \{0, 1, 2, 3\},\ (a-b)(b-c)(c-a) = 0 \Rightarrow \epsilon^{abc} = 0$.

That means

$$M^{\lambda}{}_{\rho}\delta^{\mu}{}_{\lambda}\delta^{\rho}{}_{\nu} = M^{\mu}{}_{\nu} = \frac{1}{2}M^{\lambda}{}_{\rho}\left(\sigma_a\right)^{\rho}{}_{\lambda}\left(\sigma^a\right)^{\mu}{}_{\nu}. \tag{10}$$

Since $M^{\lambda}{}_{\rho}$ is arbitrary, we can deduce that

$$\boxed{\left(\sigma_a\right)^{\rho}{}_{\lambda}\left(\sigma^a\right)^{\mu}{}_{\nu} = 2\delta^{\mu}{}_{\lambda}\delta^{\rho}{}_{\nu}.} \tag{11}$$

In particular,

$$\left(\sigma_i\right)^{\rho}{}_{\lambda}\left(\sigma^i\right)^{\mu}{}_{\nu} = 2\delta^{\mu}{}_{\lambda}\delta^{\rho}{}_{\nu} - \delta^{\rho}{}_{\lambda}\delta^{\mu}{}_{\nu}. \tag{12}$$

Property 4 (Relation with cross product of vectors)**.**
Let u^i, v^i be two vectors, then

$$\boxed{\left(u_i\sigma^i\right)\left(v_j\sigma^j\right) = \mathrm{i}\left(u^iv^j\varepsilon_{ijk}\right)\sigma^k + u_kv^kI.} \tag{13}$$

In particular,

$$\left(u_i\sigma^i\right)^2 = \left(u^iu_i\right)^2 I. \tag{14}$$

Property 5 (Exponential form)**.**
Let $\mathbf{n}$ be a unitary vector, then we can deduce that

$$\boxed{\exp\left[\mathrm{i}\left(n_i\sigma^i\right)\frac{\theta}{2}\right] = I\cos\frac{\theta}{2} + \mathrm{i}\left(n_i\sigma^i\right)\sin\frac{\theta}{2}.} \tag{15}$$

Property 6 (Projection Operators)**.** Consider

$$P_{\pm} = \frac{1}{2}\left(1 + \sigma^in_i\right), \tag{16}$$

we have

$$\boxed{P_{\pm}^2 = P_{\pm}}, \quad \boxed{\left(\sigma^in_i\right)P_{\pm} = \pm P_{\pm}}, \quad \boxed{P_+P_- = P_-P_+ = 0}. \tag{17}$$

It can used to construct the eigenstates of the Hamiltonian of the form (σ^in_i).