GP1 HW4

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Problem 1 (The hyperbolic orbit)
(1)

$$\frac{\mathrm{d}\vec{S}}{\mathrm{d}t} = \frac{1}{2}\vec{v} \times \vec{r} = \frac{\vec{L}}{2m} = \text{const.} \tag{1.1}$$

(2)

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{1.2}$$

Let $\vec{v} = \vec{u} + \vec{C}$, where $\vec{C}$ is a constant vector and $|\vec{u}| = \frac{GM}{L}$ Then,

$$\left|\vec{v} - \frac{GM}{L}\vec{e}\right| = \frac{GM}{L}. \tag{1.3}$$

$\vec{v}$ can only be a part of circle, so $E > 0$.

Problem 2 (A parabolic orbit)
By Problem 1 we can know that when $E = 0$, the orbit is a parabola.

Problem 3 (Repulsive inverse-square force field)
(1) Same as Problem 1.
(2)Let

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{k}{mr^2}\hat{r},\ k > 0. \tag{3.1}$$

We have

$$\dot{\theta} = \frac{L}{mr^2}. \tag{3.2}$$

So

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}\theta} = \frac{k}{L}\hat{r} = -\frac{k}{L}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}\theta}. \tag{3.3}$$

Thus,

$$\vec{v} = \vec{v_0} - \frac{k}{L}\vec{\theta}. \tag{3.4}$$

$E = \frac{1}{2}mv^2 + \frac{k}{r} > 0$, so the velocity circle does not contain the origin. So the orbit is a hyperbola.

Problem 4 (Proof of Kepler's First Law)
We start from Binet's equation:

$$h^2u^2\left(\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u\right)=-\frac{F}{m}. \tag{4.1}$$

Where $h=r^2\dot{\theta}$ is a constant, $u=\frac{1}{r}$, $F=-\frac{GMm}{r^2}=-GMmu^2$. Hence,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}+u=\frac{Gm}{h^2}. \tag{4.2}$$

$$u=A\cos\left(\theta+\theta_0\right)+\frac{GM}{h^2}. \tag{4.3}$$

$$r=\frac{1}{A\cos\left(\theta+\theta_0\right)+\frac{GM}{h^2}}. \tag{4.4}$$

We can use the initial condition to determine A and θ_0. Anyway, we know it is a quadratic curve.