

FUNDAMENTAL ALGEBRA & ANALYSIS

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Chapter 1

Topology

1.1 Topological spaces

Proposition 1.1.1 Let X be a set and for any $x \in X$, let \mathcal{G}_x be a filter contained in the principal filter of $\{x\}$ ($\forall U \in \mathcal{G}_x, x \in U$). Denote by \mathcal{T} the set

$$\{U \in \wp(X) \mid \forall x \in U, U \in \mathcal{G}_x\}.$$

Then the following conditions are satisfied.

- (1) $\{\emptyset, X\} \subseteq \mathcal{T}$.
- (2) If $(U_1, U_2) \in \mathcal{T}^2$, then $U_1 \cap U_2 \in \mathcal{T}$.
- (3) If I is a set and $(U_i)_{i \in I} \in \mathcal{T}^I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

Moreover, $\forall x \in X$, $\mathcal{B}_x = \{U \in \mathcal{T} \mid x \in U\}$ is a filter basis contained in \mathcal{G}_x . It generates \mathcal{G}_x if the following condition is satisfied:

$$\forall U \in \mathcal{G}_x, \exists V \in \mathcal{G}_x, V \subseteq U \text{ and } \forall y \in V, V \in \mathcal{G}_y.$$

Proof

- (1) $\emptyset \in \mathcal{T}, X \in \bigcap_{x \in X} \mathcal{G}_x$.
- (2) $\forall x \in U_1 \cap U_2, U_1 \in \mathcal{G}_x, U_2 \in \mathcal{G}_x$, so $U_1 \cap U_2 \in \mathcal{G}_x$.
- (3) Let $U = \bigcup_{i \in I} U_i$. $\forall x \in U, \exists i \in I, x \in U_i$, so $U_i \in \mathcal{G}_x$. Since $U \supseteq U_i$, so $U \in \mathcal{G}_x$.

$$\mathcal{B}_x := \{U \in \mathcal{T} \mid x \in U\}.$$

If $U \in \mathcal{B}_x$, then $x \in U$, so $U \in \mathcal{G}_x$. Hence $\mathcal{B}_x \subseteq \mathcal{G}_x$. If $(U, V) \in \mathcal{B}_x^2$, then $U \cap V \in \mathcal{T}$, and $x \in U \cap V$. So $U \cap V \in \mathcal{B}_x$. So \mathcal{B}_x is a filter basis. Suppose the condition is satisfied. For any $U \in \mathcal{G}_x, \exists V \in \mathcal{G}_x \cap \mathcal{T}$, such that $V \subseteq U$. Note

that $V \in \mathcal{B}_x$, so \mathcal{G}_x is generated by \mathcal{B}_x . □

Definition 1.1.2 Let X be a set. We call **topology** on X any subset \mathcal{T} of $\wp(X)$ that satisfies the following conditions:

- (1) $\{\emptyset, X\} \subseteq \mathcal{T}$.
- (2) If $(U_1, U_2) \in \mathcal{T}^2$, then $U_1 \cap U_2 \in \mathcal{T}$.
- (3) For any set I , $\forall (U_i)_{i \in I} \in \mathcal{T}^I$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

(X, \mathcal{T}) is called a **topological space**.

If $\forall x \in X$, \mathcal{B}_x is a filter basis of X contained in the principal filter of $\{x\}$, then

$$\mathcal{T} = \{U \in \wp(X) \mid \forall x \in U, \exists V_x \in \mathcal{B}_x, V_x \subseteq U\}$$

is a topology on X , called the **topology generated by** $(\mathcal{B}_x)_{x \in X}$. More generally, if $\forall x \in X$, S_x is a subset of the principal filter of $\{x\}$ and \mathcal{G}_x is the filter generated by S_x , then we say that

$$\mathcal{T} = \{U \in \wp(X) \mid \forall x \in U, U \in \mathcal{G}_x\}$$

is the topology generated by $(S_x)_{x \in X}$.

Example 1.1.3

- (1) Let $\mathcal{G}_x = \{X\}$. The topology by $(\mathcal{G}_x)_{x \in X}$ is $\{\emptyset, X\}$, called the **trivial topology** on X .
- (2) Let $\mathcal{G}_x = \mathcal{F}_{\{x\}}$ be the principal filter. The topology generated by $(\mathcal{G}_x)_{x \in X}$ is $\wp(X)$. This topology is called the **discrete topology** on X .
- (3) Let (X, d) be a semimetric space.

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0}, d(x, y) = d(y, x), d(x, z) \leq d(x, y) + d(y, z), d(x, x) = 0.$$

$\forall \varepsilon > 0$, $\forall x \in X$, let $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$, $\{B(x, \varepsilon) \mid \varepsilon \in \mathbb{R}_{>0}\} =: \mathcal{B}_x$ is a filter basis on X , \mathcal{B}_x is contained in the principal filter of $\{x\}$. The topology

$$\mathcal{T} = \{U \in \wp(X) \mid \forall x \in U, \exists \varepsilon \in \mathbb{R}_{>0}, B(x, \varepsilon) \subseteq U\}$$

is called the **topology induced by the semimetric** d .

- (4) Let (G, \leq) be a totally ordered set $\forall x \in G$, let $S_x = \{G_{>a} \mid a < x\} \cup \{G_{<b} \mid x < b\}$

Proposition 1.1.4 $\forall x \in X, \forall \varepsilon \in \mathbb{R}_{>0}, B(x, \varepsilon) \in \mathcal{T}$.

Proof $\forall y \in B(x, \varepsilon), d(x, y) < \varepsilon$. Let $r = \varepsilon - d(x, y) > 0$, we claim that $B(y, r) \subseteq B(x, \varepsilon)$. Let $z \in B(y, r), d(y, z) < r$. Hence,

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r = d(x, y) + \varepsilon - d(x, y) = \varepsilon.$$

□

Remark 1.1.5 On \mathbb{R} , one has a metric

$$d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0},$$

$$(a, b) \longmapsto |a - b|.$$

$$B(x, \varepsilon) =]x - \varepsilon, x + \varepsilon[.$$

$$\mathcal{B}_x = \{]x - \varepsilon, x + \varepsilon[\mid \varepsilon \in \mathbb{R}_{>0}\}.$$

Let \mathcal{T}_d be the topology generated by $(\mathcal{B}_x)_{x \in \mathbb{R}}$. Let \mathcal{T} be the order topology generated by $(S_x)_{x \in \mathbb{R}}$, where

$$S_x := \{\mathbb{R}_{>a} \mid a < x\} \cup \{\mathbb{R}_{**0}**$$

Proposition 1.1.6 For any $x \in \mathbb{R}, \mathcal{F}(\mathcal{B}_x) = \mathcal{F}(S_x)$.

Proof $\forall \varepsilon > 0,]x - \varepsilon, x + \varepsilon[= \mathbb{R}_{. So $\mathcal{F}(\mathcal{B}_x) \subseteq \mathcal{F}(S_x)$.$

$\forall a \in \mathbb{R}, a < x, \mathbb{R}_{>a} \supseteq]a, 2x - a[=]x - (x - a), x + (x - a)[, \mathbb{R}_{>a} \in \mathcal{F}(\mathcal{B}_x)$.

$\forall b \in \mathbb{R}, b > x, \mathbb{R}_{**0}**$

So, $\mathbb{R}_{**0}** \subseteq \mathcal{F}(\mathcal{B}_x)$. Hence $S_x \subseteq \mathcal{F}(\mathcal{B}_x)$, which leads to $\mathcal{F}(S_x) \subseteq \mathcal{F}(\mathcal{B}_x)$. □

Definition 1.1.7 Let (X, \mathcal{T}) be a topological space. For any $x \in X$ and any $V \in \wp(X)$, if there exists $U \in \mathcal{T}$ such that $x \in U \subseteq V$, then we say that V is a **neighborhood of x** . We call **open subset** of X any subset of X that belongs to \mathcal{T} . If $U \in \mathcal{T}$, such that $x \in U$, we say that U is an **open neighborhood of x** . We denote by $\mathcal{V}_x(\mathcal{T})$ the set of all neighborhoods of x .

Proposition 1.1.8 $\mathcal{V}_x(\mathcal{T})$ is a filter on X contained in the principal filter of $\{x\}$. Moreover, the topology generated by $(\mathcal{V}_x(\mathcal{T}))_{x \in X}$ identifies with \mathcal{T} .

Proof

(1) If $(V_1, V_2) \in \mathcal{V}_x(\mathcal{T})^2$, $\exists(U_1, U_2) \in \mathcal{T}^2$, such that $x \in U_1 \subseteq V_1$, $x \in U_2 \subseteq V_2$. Hence, $x \in U_1 \cap U_2 \subseteq V_1 \cap V_2$, so $V_1 \cap V_2 \in \mathcal{V}_x(\mathcal{T})$.

(2) If $V \in \mathcal{V}_x(\mathcal{T})$, $W \in \wp(X)$, $V \subseteq W$. $\exists U \in \mathcal{T}$, $x \in U \subseteq V \subseteq W$, so $W \in \mathcal{V}_x(\mathcal{T})$. Let \mathcal{T}' be the topology generated by $(\mathcal{V}_x(\mathcal{T}))_{x \in X}$. By definition,

$$\mathcal{T}' = \{U \subseteq X \mid \forall x \in U, U \in \mathcal{V}_x(\mathcal{T})\}.$$

For any $U \in \mathcal{T}$, $\forall x \in U$, U is a open neighborhood of x , so $U \in \mathcal{T}'$. Let $U \in \mathcal{T}'$, $\forall x \in U$, $\exists V_x \in \mathcal{T}$, $x \in V_x \subseteq U$.

$$U = \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} V_x \subseteq U.$$

$$U = \bigcup_{x \in U} V_x \in \mathcal{T}.$$

□

Proposition 1.1.9 Let X be a set, $(\mathcal{T}_i)_{i \in I}$ be a family of topologies on X . Then

$$\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$$

is a topology on X .

Proof

(1) $\forall i \in I$, $\{\emptyset, X\} \subseteq \mathcal{T}_i$, so $\{\emptyset, X\} \subseteq \mathcal{T}$.

(2) If $(U_1, U_2) \in \mathcal{T}^2$, then for any $i \in I$, $U_1 \cap U_2 \in \mathcal{T}_i$, so $U_1 \cap U_2 \in \bigcap_{i \in I} \mathcal{T}_i$.

(3) For any set J and any $(U_j)_{j \in J} \in \mathcal{T}^J$, one has $\forall i \in I$, $\forall j \in J$, $U_j \in \mathcal{T}_i$, so

$$\bigcup_{j \in J} U_j \in \mathcal{T}_i.$$

Therefore,

$$\bigcup_{j \in J} U_j \in \bigcap_{i \in I} \mathcal{T}_i.$$

□

Definition 1.1.10 Let S be a subset of $\wp(X)$, we denote by \mathcal{T}_S the intersection of all topologies containing S , we call it the topology generated by S .

Definition 1.1.11

Let \mathcal{B} be a subset of $\wp(X)$, we say that \mathcal{B} is a **topological basis** if:

- (1) $X = \bigcup_{V \in \mathcal{B}} V$.
- (2) $\forall (U, V) \in \mathcal{B} \times \mathcal{B}, \forall x \in U \cap V, \exists W_x \in \mathcal{B}, x \in W_x \subseteq U \cap V$.

Definition 1.1.12 Let X be a set and \mathcal{T}_1 and \mathcal{T}_2 be two topologies on X . If $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that \mathcal{T}_1 is coarser than \mathcal{T}_2 and \mathcal{T}_2 is finer than \mathcal{T}_1 .

If $S \subseteq \wp(X)$, we denote by \mathcal{T}_S the intersection of all topology containing S . It is the coarsest topology containing S .

Proposition 1.1.13 Let S be a subset of $\wp(X)$. Let

$$\mathcal{B}_S := \{X\} \cup \left\{ \bigcap_{i=1}^n A_i \mid n \in \mathbb{N}_{\geq 1}, (A_1, \dots, A_n) \in S^n \right\},$$

then, \mathcal{B}_S is a topological basis on X . Moreover, $\mathcal{T}_S = \mathcal{T}_{\mathcal{B}_S}$.

Proof Since $X \in \mathcal{B}_S$, $\bigcup_{V \in \mathcal{B}_S} V = X$. Let $(U, V) \in \mathcal{B}_S \times \mathcal{B}_S$. If $U = X$, then $U \cap V = V \in \mathcal{B}_S$. Similarly, if $V = X$, then $U \cap V = U \in \mathcal{B}_S$. If $U = A_1 \cap \dots \cap A_n, V = B_1 \cap \dots \cap B_m$, then $\{A_1, \dots, A_n, B_1, \dots, B_m\} \subseteq \mathcal{B}_S$.

$$U \cap V = A_1 \cap \dots \cap A_n \cap B_1 \cap \dots \cap B_m \in \mathcal{B}_S.$$

Since $S \subseteq \mathcal{B}_S \subseteq \mathcal{T}_{\mathcal{B}_S}$, so $\mathcal{T}_S \subseteq \mathcal{T}_{\mathcal{B}_S}$, $X \in \mathcal{T}_S$. If $(A_1, \dots, A_n) \in S^n$, then $(A_1, \dots, A_n) \in \mathcal{T}_{\mathcal{B}_S}$. So $A_1 \cap \dots \cap A_n \in \mathcal{T}_S$. Hence $\mathcal{B}_S \subseteq \mathcal{T}_S$. Therefore, $\mathcal{T}_{\mathcal{B}_S} \subseteq \mathcal{T}_S$, so $\mathcal{T}_{\mathcal{B}_S} = \mathcal{T}_S$. \square

Proposition 1.1.14 Let \mathcal{B} be a topological basis on a set X . Then

$$\mathcal{T}_{\mathcal{B}} = \left\{ U \in \wp(X) \mid \exists \text{ a set } I \text{ and } (V_i)_{i \in I} \in \mathcal{B}^I, U = \bigcup_{i \in I} V_i \right\}.$$

Proof We denote by \mathcal{T} the set

$$\{U \in \wp(X) \mid U \text{ can be written as the union of a family sets in } \mathcal{B}\}.$$

By definition, $\mathcal{B} \subseteq \mathcal{T} \subseteq \mathcal{T}_{\mathcal{B}}$. It remains to check that \mathcal{T} is a topology.

By definition, $X \in \mathcal{T}$, $\emptyset \in \mathcal{T}$. Moreover, the union of a family of elements of \mathcal{T} remains in \mathcal{T} . Let $U = \bigcup_{i \in I} U_i$ and $V = \bigcup_{j \in J} V_j$ be elements of \mathcal{T} , where, $U_i \in \mathcal{B}, V_j \in \mathcal{B}$. Then

$$U \cap V = \bigcup_{(i,j) \in I \times J} (U_i \cap V_j).$$

For any $x \in U_i \cap V_j$, $\exists W_x^{(i,j)} \in \mathcal{B}$, $x \in W_x^{(i,j)} \subseteq U_i \cap V_j$. $U_i \cap V_j = \bigcup_{x \in U_i \cap V_j} W_x^{(i,j)}$,

so

$$U \cap V = \bigcup_{(i,j) \in I \times J} \bigcup_{x \in U_i \cap V_j} W_x^{(i,j)}.$$

□

1.2 Convergence

We fix a topology space (E, \mathcal{T}) , $l \in E$ and $S \subseteq \wp(E)$ that generates the filter $\mathcal{V}_l(\mathcal{T})$ of all neighborhood of l .

Definition 1.2.1 Let $f : X \rightarrow Y$ be a mapping. If \mathcal{F} is a filter on X , we denote by $f_*(\mathcal{F})$ the set $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{F}\}$.

Proposition 1.2.2 $f_*(\mathcal{F})$ is a filter on Y .

Proof Let $(B_1, B_2) \in f_*(\mathcal{F})$,

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \in \mathcal{F}.$$

Let $B \in f_*(\mathcal{F})$, $C \supseteq B$. $f^{-1}(C) \supseteq f^{-1}(B) \in \mathcal{F}$, so $f^{-1}(C) \in \mathcal{F}$. □

Proposition 1.2.3 Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$, be mappings. \mathcal{F} be a filter on X . Then

$$(g \circ f)_*(\mathcal{F}) = g_*(f_*(\mathcal{F})).$$

Proof

$$\begin{aligned}
(g \circ f)_*(\mathcal{F}) &= \{C \subseteq Z \mid (g \circ f)^{-1}(C) \in \mathcal{F}\} \\
&= \{C \subseteq Z \mid f^{-1}(g^{-1}(C)) \in \mathcal{F}\} \\
&= \{C \subseteq Z \mid g^{-1}(C) \in f_*(\mathcal{F})\} \\
&= g_*(f_*(\mathcal{F})).
\end{aligned}$$

□

Proposition 1.2.4 Let \mathcal{B} be a filter basis in X , $f : X \rightarrow Y$ be a mapping and \mathcal{F} be the filter generated by \mathcal{B} . Then $f(\mathcal{B}) : \{f(U) \mid U \in \mathcal{B}\}$ is a filter basis on Y and $f_*(\mathcal{F})$ is the filter generated by $f(\mathcal{B})$.

Proof Let U and V be elements of \mathcal{B} . Then $\exists W \in \mathcal{B}$, $W \subseteq U \cap V$. Hence $f(W) \subseteq f(U \cap V) \subseteq f(U) \cap f(V)$. Moreover, for any $U \in \mathcal{B}$, $U \subseteq f^{-1}(f(U))$. So $f^{-1}(f(U)) \in \mathcal{F}$. Therefore, $f(\mathcal{B}) \subseteq f_*(\mathcal{F})$. Let $A \in f_*(\mathcal{F})$. Then, $f^{-1}(A) \in \mathcal{F}$. So $\exists V \in \mathcal{B}$, $V \subseteq f^{-1}(A)$. Hence $f(V) \subseteq A$. Therefore, $f_*(\mathcal{F})$ is a filter basis generated by $f_*(\mathcal{B})$ □

Definition 1.2.5 Let $f : X \rightarrow E$ be a mapping, \mathcal{F} be a non-degenerate filter on X . If $f_*(\mathcal{F}) \supseteq \mathcal{V}_l(\mathcal{T})$, we say that f **converges** to l along \mathcal{F} .

$$\lim_{\mathcal{F}} f = l$$

denotes “ f converges to l along \mathcal{F} ”.

This condition is equivalent to

$$\forall V \in S_l, f^{-1}(V) \in \mathcal{F}.$$

If \mathcal{B} is a filter basis which generates \mathcal{F} . This condition is also

$$\forall V \in S_l, \exists U \in \mathcal{B}, f(U) \subseteq V.$$

Example 1.2.6

(1) Let $I \subseteq \mathbb{N}$ be an infinite subset and $x = (x_n)_{n \in I} \in \mathbb{N}^I$. Let \mathcal{F} be the Fréchet filter on I . If x converges to l along \mathcal{F} , or equivalently,

$$\forall V \in S_l, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, x_n \in V.$$

We say that the sequence $(x_n)_{n \in I}$ **converges to** l when n tends to the infinity,

denote as

$$\lim_{n \rightarrow +\infty} x_n = l.$$

(2) Let (X, \mathcal{T}_X) be a topological space. Let $Y \subseteq X$ be a subset of X , $p \in X$. Let

$$\mathcal{F} = \mathcal{V}_p(\mathcal{T}_X)|_Y = \{V \cap Y \mid V \in \mathcal{V}_p(\mathcal{T}_X)\}.$$

Assume that \mathcal{F} is non-degenerate. Let $f : X \rightarrow E$ be a mapping. If f converges to l along \mathcal{F} , we say that $f(x)$ converges to l when $x \in Y$ tends to p , denoted as

$$\lim_{x \in Y, x \rightarrow p} f(x) = l.$$

This condition is equivalent to:

$$\forall V \in S_l, \exists U \text{ a open neighborhood of } p, \forall x \in U \cap Y, f(x) \in V.$$

In general, if $g : Y \rightarrow E$ is a mapping such that

$$\forall V \in S_l, \exists U \text{ a open neighborhood of } p, \forall x \in U \cap Y, g(x) \in V,$$

then we say $g(x)$ converges to l when $x \in Y$ tends to p , denoted as

$$\lim_{x \in Y, x \rightarrow p} g(x) = l.$$

Remark 1.2.7 If (E, d) is a semimetric space, \mathcal{T} is the semimetric topology. Then condition in (1) becomes:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \varepsilon.$$

The conditions in (2) becomes:

$$\forall \varepsilon > 0, \exists U \text{ a open neighborhood of } p, \forall x \in U \cap Y, d(g(x), l) < \varepsilon.$$

If furthermore, (X, \mathcal{T}_X) is a semimetric space with semimetric d_X . The condition becomes

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in Y, d_X(x, p) < \delta \Rightarrow d(g(x), l) < \varepsilon.$$

Example 1.2.8

(3) Consider $X = \mathbb{R}$. Let $Y \subseteq X$. Consider the filter \mathcal{F} generated by $\{\mathbb{R}_{>M}, M \in \mathbb{R}\}$. Suppose that $\mathcal{F}|_Y$ is non-degenerate. Let $g : Y \rightarrow E$. If $\lim_{\mathcal{F}|_Y} g = l$, we say that $g(x)$ converges to l when x tends to $+\infty$, denoted as

$$\lim_{x \in Y, x \rightarrow +\infty} g(x) = l.$$

This condition is

$$\forall V \in S_l, \exists M \in \mathbb{R}_{>0}, \forall x \in Y, x > M \Rightarrow g(x) \in V.$$

If (E, d) is a metric space, it becomes:

$$\forall \varepsilon > 0, \exists M \in \mathbb{R}_{>0}, \forall x \in Y, x > M \Rightarrow d(g(x), l) < \varepsilon.$$

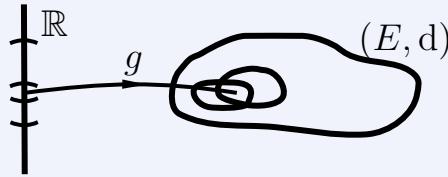


Figure 1.2.1: Example on \mathbb{R}

Example 1.2.9 Let (G, \leq) be a totally ordered set, \mathcal{F} be the ordered topology on G . It is generated by $\{G_{>a} \mid a \in G\} \cup \{G_{<b} \mid b \in G\}$. If $l \in G$, then $\mathcal{V}_x(\mathcal{F})$ is generated by

$$S_l := \{G_{>a} \mid a < l\} \cup \{G_{<b} \mid l < b\}.$$

Assume that (G, \leq) is order complete. Let $f : X \rightarrow G$ be a mapping and \mathcal{F} be a non-degenerate filter on X .

(1) Assume that f converges to l along \mathcal{F} . $\forall a < l, U_a := f^{-1}(G_{>a}) \in \mathcal{F}$. $\forall x \in U_a, f(x) > a$. So $\liminf_{\mathcal{F}} f \geq a$. If $\sup(G_{<l}) = l$, then $\liminf_{\mathcal{F}} f \geq l$. If $\sup(G_{<l}) < l$, we denote $a = \sup(G_{<l})$. $\forall a \in U_a, f(x) \geq l$. So $\liminf_{\mathcal{F}} f \geq l$. Similarly, $\liminf_{\mathcal{F}} f \leq l$. So f admits l as its limit.

(2) Assume that $\limsup_{\mathcal{F}} f = \liminf_{\mathcal{F}} f = l$.

$$\liminf_{\mathcal{F}} f = l \Rightarrow \sup_{U \in \mathcal{F}} f^i(U) = l, \forall a < l, \exists U \in \mathcal{F}, f^i(U) > a, f^{-1}(G_{>a}) \in \mathcal{F}.$$

$$\limsup_{\mathcal{F}} f = l \Rightarrow \forall b > l, f^{-1}(G_{<b}) \in \mathcal{F}.$$

Therefore, f converges to l along \mathcal{F} .

1.3 Continuity

Definition 1.3.1 Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, f be a function from X to Y , and $p \in \text{Dom}(f)$. If for any neighborhood U of $f(p)$, there exists a neighborhood V of p such that

$$f(V) \subseteq U,$$

then we say that the function f is **continuous** at the point p .

If f is continuous at any $p \in \text{Dom}(f)$, then we say that f is **continuous**.

Remark 1.3.2

(1) The continuity of f at p is equivalent to:

$$\lim_{\substack{x \in \text{Dom}(f) \\ x \rightarrow p}} f(x) = f(p),$$

namely, f converges to $f(p)$ when x tends to p .

(2) Let $\mathcal{V}_{f(p)}(\mathcal{T}_Y)$ and $\mathcal{V}_p(\mathcal{T}_X)$ be filters of neighborhoods of $f(p)$ and p respectively. Let \mathcal{B}_p be a filter basis that generates $\mathcal{V}_p(\mathcal{T}_X)$. Let $S_{f(p)}$ be a subset of $\mathcal{V}_{f(p)}(\mathcal{T}_Y)$ that generates $\mathcal{V}_{f(p)}(\mathcal{T}_Y)$. Then the continuity of f at p is equivalent to:

$$\forall U \in S_{f(p)}, \exists V \in \mathcal{B}_p, f(V) \subseteq U.$$

In the case where X and Y are metric spaces, this condition becomes:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in \text{Dom}(f), d(x, p) < \delta \Rightarrow d(f(x), f(p)) < \varepsilon.$$

Proposition 1.3.3 Let (X, \mathcal{T}_X) , (Y, \mathcal{T}_Y) , (Z, \mathcal{T}_Z) be topological spaces. $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be functions. $p \in \text{Dom}(g \circ f)$. Assume that f is continuous at p and g is continuous at $f(p)$. Then $g \circ f$ is continuous at p .

Proof Let U be a neighborhood of $g(f(p))$. Since g is continuous at $f(p)$, there exists a neighborhood V of $f(p)$ such that $g(V) \subseteq U$. Since f is continuous at p , there exists a neighborhood W of p , such that $f(W) \subseteq V$. Hence $g(f(W)) \subseteq g(V) \subseteq U$. So $g \circ f$ is continuous at p . \square

Example 1.3.4 Let (X, \mathcal{T}_X) be a topological space. Then Id_X and constant mapping are continuous.

Theorem 1.3.5 Let (X, \mathcal{T}_X) , (Y, \mathcal{T}_Y) be topological spaces, $f : X \rightarrow Y$ a function, $p \in \text{Dom}(f)$. Consider the following conditions:

- (1) f is continuous at p .
- (2) For any $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$, if $\lim_{n \rightarrow \infty} x_n = p$, then

$$\lim_{n \rightarrow \infty} f(x_n) = f(p).$$

One has (1) \Rightarrow (2). If p has a countable basis of neighborhoods, (2) \Rightarrow (1).

Proof Let $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$ such that $\lim_{n \rightarrow \infty} x_n = p$. f is continuous at p , so for any neighborhood U of $f(p)$, there exists a neighborhood V of p , such that $f(V) \subseteq U$. Since $\lim_{n \rightarrow \infty} x_n = p$, there exists $N \in \mathbb{N}$, so that for any $n \in \mathbb{N}_{>N}$, $x_n \in V$. Hence for any $n \in \mathbb{N}_{>N}$, $f(x_n) \in U$. Hence $\lim_{n \rightarrow \infty} f(x_n) = f(p)$. Assume that p has a countable basis of neighborhood. Let $(W_n)_{n \in \mathbb{N}}$ be a sequence of neighborhood of p , such that $\{W_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis. For $n \in \mathbb{N}$,

$$V_n := \bigcap_{i \in \mathbb{N}_{\leq n}} W_i.$$

V_n is a neighborhood of p . If f is not continuous at p , then there exists an neighborhood of p , U , such that

$$\forall n \in \mathbb{N}, f(V_n) \not\subseteq U.$$

We pick $x_n \in V_n$ but $f(x_n) \notin U$. For any neighborhood V of p , there exists $N \in \mathbb{N}$ such that $V_N \subseteq V$, so that $x_n \in V$ for any $n \in \mathbb{N}_{>N}$. Hence, x_n converges to p . But $f(x_n)$ cannot converge to $f(p)$. \square

Lemma 1.3.6 Let (X, \mathcal{T}_X) be a topological space. $V \subseteq X$. If $\forall p \in V$, V is a neighborhood of p , then $V \in \mathcal{T}_X$. In fact $\forall p \in V$, there exists $W_p \in \mathcal{T}_X$, $p \in W_p \subseteq V$. Hence

$$V = \bigcup_{p \in V} \{p\} \subseteq \bigcup_{p \in V} W_p \subseteq V.$$

Proposition 1.3.7 Let (X, \mathcal{T}_X) , (Y, \mathcal{T}_Y) be topological spaces, $\mathcal{S} \subseteq \mathcal{T}_Y$, \mathcal{S} generates \mathcal{T}_Y . The following statements are equivalent:

- (1) f is continuous.
- (2) For any $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.
- (3) For any $U \in \mathcal{S}$, $f^{-1}(U) \in \mathcal{T}_X$.

Proof

(1) \Rightarrow (2): For any $p \in f^{-1}(U)$, one has $f(p) \in U$. Hence, there is a neighborhood V_p of p , $f(V_p) \subseteq U$, or equivalently $V_p \subseteq f^{-1}(U)$. Therefore, $f^{-1}(U) \in \mathcal{T}_X$.

(3) \Rightarrow (2):

$$\mathcal{T}'_Y = \{U \in \wp(Y) \mid f^{-1}(U) \in \mathcal{T}_X\}$$

By definition, $\{\emptyset, Y\} \subseteq \mathcal{T}'_Y$. If $(U_1, U_2) \in \mathcal{T}'_Y \times \mathcal{T}'_Y$, then $f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap f^{-1}(U_2) \in \mathcal{T}_X$. So $U_1 \cap U_2 \in \mathcal{T}'_Y$. $(U_i)_{i \in I} \in (\mathcal{T}'_Y)^I$, then

$$f^{-1}\left(\bigcup_{i \in I} U_i\right) = \bigcup_{i \in I} f^{-1}(U_i) \in \mathcal{T}_X.$$

So \mathcal{T}'_Y is a topology, by (3), $\mathcal{S} \subseteq \mathcal{T}'_Y \Rightarrow \mathcal{T}_Y \subseteq \mathcal{T}'_Y$. \square

1.4 Initial Topology

Definition 1.4.1 Let X be a set, $((Y_i, \mathcal{T}_i))_{i \in I}$ a family of topological spaces, $(f_i : X \rightarrow Y_i)_{i \in I}$ a family of mappings. We call **initial topology** on X induced by $(f_i)_{i \in I}$ the topology generated by

$$\bigcup_{i \in I} \{f_i^{-1}(U_i) \mid U_i \in \mathcal{T}_i\}.$$

It is the coarsest topology on X making all f_i continuous.

Proposition 1.4.2 Let \mathcal{T} be the initial topology on X induced by $(f_i)_{i \in I}$.

(1)

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{T}_j \right\}$$

is topological basis that generates \mathcal{T} .

(2) Let (Z, \mathcal{T}_Z) be topological space, $h : Z \rightarrow X$ be a function and $p \in \text{Dom}(h)$. Then h is continuous at p if and only if $\forall i \in I$, $f_i \circ h$ is continuous at p .

Proof

(1) Let

$$S = \bigcup_{i \in I} \{f_i^{-1}(U_i) \mid U_i \in \mathcal{T}_i\},$$

\mathcal{B}' be the set of the intersections of all finitely elements of S (We have proved that \mathcal{B}' is a basis of \mathcal{T}). $\mathcal{B} \subseteq \mathcal{B}'$. Let i_1, \dots, i_n elements of I , $U_{i_k} \in \mathcal{T}_{i_k}$, $J = \{i_1, \dots, i_n\}$, $j \in J$, $A_j = \{k \in \{1, \dots, n\} \mid i_k = j\}$, $W_j = \bigcap_{k \in A_j} U_{i_k}$.

$$\bigcap_{k=1}^n f_{i_k}^{-1}(U_{i_k}) = \bigcap_{j \in J} f_j^{-1}(W_j) \in \mathcal{B}.$$

(2) Since f_i is continuous at p , if h is continuous then $\forall i \in I$, $f_i \circ h$ is continuous. Assume $\forall i \in I$, $f_i \circ h$ is continuous, then

$$\forall i \in I, \forall U_i \in \mathcal{T}_i, (f_i \circ h)^{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)).$$

Therefore, for any $V \in S$, $h^{-1}(V) \in \mathcal{T}_Z$. Hence h is continuous. \square

Example 1.4.3 Let (X_i, \mathcal{T}_i) be topological spaces, $X = \prod_{i \in I} X_i$, $\pi_i : X \rightarrow X_i$ be a projection. The initial topology on X induced by $(\pi_i)_{i \in I}$ is called the **product topology**.

1.5 Uniform Continuity

Definition 1.5.1 Let (X, d_X) , (Y, d_Y) be semimetric spaces, $f : X \rightarrow Y$ be a function, $\alpha \in \mathbb{R}_{\geq 0}$. If for any $(x_1, x_2) \in \text{Dom}(f)^2$, $d(f(x_1), f(x_2)) \leq \alpha \cdot d(x_1, x_2)$, then we say that f is α -Lipschitzian. If there exists $\alpha \in \mathbb{R}_{\geq 0}$ such that f is α -Lipschitzian, then we say that f is **Lipschitzian**.

If

$$\forall \varepsilon > 0, \exists \delta > 0, \forall (x_1, x_2) \in \text{Dom}(f)^2, d_X(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon,$$

then we say that f is **uniformly continuous**.

Proposition 1.5.2 Let (X, d_X) , (Y, d_Y) be semimetric spaces, $f : X \rightarrow Y$ be a function. If f is uniformly continuous, then f continuous.

Proof Let $p \in \text{Dom}(f)$. For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall (x_1, x_2) \in \text{Dom}(f)^2, d_X(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon.$$

In particular,

$$\forall x \in \text{Dom}(f), \exists \delta > 0, d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon.$$

□

Definition 1.5.3 Let K be a field, we call absolute value on K any mapping,

$$|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0},$$

- (1) $\forall a \in K, a = 0_K$ if and only if $|a| = 0$.
 - (2) $\forall (a, b) \in K \times K, |ab| = |a| |b|$.
 - (3) $\forall (a, b) \in K \times K, |a + b| \leq |a| + |b|$.
- The pair $(K, |\cdot|)$ is called a **valued field**.

Example 1.5.4 Let (K, \leq) be a totally ordered field, then $|a| = \max\{-a, a\}$ is an absolute value on K .

Example 1.5.5 Let p be a prime number. Any non-zero rational number α can be written in the form

$$\alpha = p^{\text{ord}_p(\alpha)} \cdot \frac{m}{n},$$

where, $\text{ord}_p(\alpha) \in \mathbb{Z}$, $p \nmid mn$. If $\alpha = 0$, we set (by convention) $\text{ord}_p(\alpha) = +\infty$.

Properties:

- (1) $\text{ord}_p(\alpha\beta) = \text{ord}_p(\alpha) + \text{ord}_p(\beta)$.
- (2) $\alpha = p^{\text{ord}_p(\alpha)} \frac{m}{n}$, $\beta = p^{\text{ord}_p(\beta)} \frac{u}{v}$, $\text{ord}_p(\alpha) > \text{ord}_p(\beta)$, $p \nmid nvu$.

$$\alpha + \beta = p^{\text{ord}_p(\beta)} \frac{p^{\text{ord}(\alpha) - \text{ord}(\beta)} mv + nu}{nv}.$$

- (3) If $\text{ord}(\alpha) = \text{ord}(\beta)$, then $\text{ord}_p(\alpha + \beta) \geq \text{ord}_p(\alpha) = \text{ord}_p(\beta)$.

$$\alpha + \beta = p^{\text{ord}_p(\alpha)} \frac{mv + nu}{nv}.$$

Proposition 1.5.6 The mapping

$$|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0},$$

$$\begin{cases} |\alpha|_p = p^{-\text{ord}(\alpha)}, & \text{if } \alpha \neq 0 \\ |\alpha|_p = 0, & \text{if } \alpha = 0 \end{cases}$$

is an absolute value on \mathbb{Q} .

Proof If $\alpha = 0$, then $|\alpha|_p > 0$. If $(\alpha, \beta) \in \mathbb{Q}^2$, when $0 \in \{\alpha, \beta\}$, then $\alpha\beta = 0$ and $0 = |\alpha\beta|_p = |\alpha|_p|\beta|_p$. When $0 \notin \{\alpha, \beta\}$,

$$|\alpha\beta|_p = p^{-\text{ord}_p(\alpha\beta)} = p^{-\text{ord}_p(\alpha)-\text{ord}_p(\beta)} = |\alpha|_p|\beta|_p.$$

If $\alpha = 0$, $|\alpha\beta|_p = |\beta|_p$. If $\beta = 0$, $|\alpha\beta|_p = |\alpha|_p$, if $0 \notin \{\alpha, \beta\}$,

$$|\alpha + \beta|_p = p^{-\text{ord}_p(\alpha+\beta)} \leq p^{\max\{\text{ord}_p(\alpha), \text{ord}_p(\beta)\}} \leq \max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p + |\beta|_p.$$

□

Remark 1.5.7 Let $(K, |\cdot|)$ be a valued field. If for any $(\alpha, \beta) \in K^2$ satisfies $|\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}$, we say that $(K, |\cdot|)$ is **non-archimedean**, otherwise, we say that $(K, |\cdot|)$ is **archimedean**. $(\mathbb{R}, |\cdot|)$ and $(\mathbb{Q}, |\cdot|)$ are archimedean.

Definition 1.5.8 Let $(K, |\cdot|)$ be a valued filed, V a vector spaced over K . We call **seminorm** on V any mapping

$$\|\cdot\| : V \longrightarrow \mathbb{R}_{\geq 0}$$

that satisfies the following conditions:

$$(1) \forall(a, x) \in K \times V, \|ax\| = |a| \cdot \|x\|.$$

$$(2) \forall(x, y) \in V \times V, \|x + y\| \leq \|x\| + \|y\|.$$

Note that (1) implies that $\|0_V\| = |0_K| \cdot \|0_V\| = 0$.

The pair $(V, \|\cdot\|)$ is called **seminormed vector space over** $(K, |\cdot|)$. If $\forall(x, y) \in V \times V, \|x + y\| \leq \max\{\|x\|, \|y\|\}$, then we say that $\|\cdot\|$ is **ultrametric**. If $\forall x \in V \setminus \{0\}, \|x\| > 0$, then we say that $\|\cdot\|$ is a **norm** and $(V, \|\cdot\|)$ is a **normed vector space over** $(K, |\cdot|)$.

Example 1.5.9 $d : V \times V \longrightarrow \mathbb{R}_{\geq 0}$, $d(x, y) := \|x - y\|$ is a semi-metric.

Example 1.5.10 Let $(K, |\cdot|)$ be a valued field.

(1) $(K, |\cdot|)$ is a normed vector space over $(K, |\cdot|)$. ($d(x, y) = |x - y|$ is a metric.)

(2) Let $(V_1, \|\cdot\|_1), \dots, (V_n, \|\cdot\|_n)$ be seminormed vector spaces over $(K, |\cdot|)$,

$$V = V_1 \oplus \cdots \oplus V_n.$$

$$\|\cdot\|_{l^\infty} : V \longrightarrow \mathbb{R}_{\geq 0}, (x_1, \dots, x_n) \longmapsto \max_{i \in \{1, \dots, n\}} \|x_i\|_i, x_i \in V_i, i \in \{1, \dots, n\},$$

$$\|\cdot\|_{l^1} : V \longrightarrow \mathbb{R}_{\geq 0}, (x_1, \dots, x_n) \longmapsto \sum_{i \in \{1, \dots, n\}} \|x_i\|_i, x_i \in V_i, i \in \{1, \dots, n\}.$$

$$\forall \lambda \in K, \forall (x_1, \dots, x_n) \in V,$$

$$\begin{aligned} \|\lambda(x_1, \dots, x_n)\|_{l^\infty} &= \|(\lambda \cdot x_1, \dots, \lambda x_n)\|_{l^\infty} \\ &= \max_{i \in \{1, \dots, n\}} \|\lambda x_i\|_i \\ &= \max_{i \in \{1, \dots, n\}} |\lambda| \|x_i\|_i \\ &= |\lambda| \max_{i \in \{1, \dots, n\}} \|x_i\|_i \\ &= |\lambda| \cdot \|(x_1, \dots, x_n)\|_{l^\infty}. \end{aligned}$$

$$\begin{aligned} \|\lambda(x_1, \dots, x_n)\|_{l^1} &= \|(\lambda \cdot x_1, \dots, \lambda x_n)\|_{l^1} \\ &= \sum_{i \in \{1, \dots, n\}} \|\lambda x_i\|_i \\ &= \sum_{i \in \{1, \dots, n\}} |\lambda| \|x_i\|_i \\ &= |\lambda| \sum_{i \in \{1, \dots, n\}} \|x_i\|_i \\ &= |\lambda| \cdot \|(x_1, \dots, x_n)\|_{l^1}. \end{aligned}$$

$$\forall x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V,$$

$$\begin{aligned} \|x + y\|_{l^\infty} &= \|(x_1 + y_1, \dots, x_n + y_n)\|_{l^\infty} = \max_{i \in \{1, \dots, n\}} \|x_i + y_i\|_i \\ &\leq \max_{i \in \{1, \dots, n\}} \|x_i\|_i + \|y_i\|_i \leq \|x\|_{l^\infty} + \|y\|_{l^\infty}. \end{aligned}$$

$$\begin{aligned} \|x + y\|_{l^1} &= \sum_{i \in \{1, \dots, n\}} \|x_i + y_i\|_i \\ &\leq \sum_{i \in \{1, \dots, n\}} \|x_i\|_i + \|y_i\|_i = \|x\|_{l^1} + \|y\|_{l^1}. \end{aligned}$$

(3) Let $(V, \|\cdot\|)$ be a seminormed vector space over K , $f : W \longrightarrow V$ be a K -linear mapping. We denote by $\|\cdot\|_f$ the mapping $W \longrightarrow \mathbb{R}_{\geq 0}$ define as

$$\forall x \in W, \|x\|_f := \|f(x)\|.$$

$\forall(\lambda, x) \in K \times W,$

$$\|\lambda x\|_f = \|f(\lambda x)\| = \|\lambda f(x)\| = |\lambda| \cdot \|f(x)\| = \lambda \|x\|_f.$$

$\forall(x, y) \in W \times W,$

$$\|x + y\|_f = \|f(x + y)\| = \|f(x)\| + \|f(y)\| \leq \|x\|_f + \|y\|_f.$$

Therefore, $\|\cdot\|_f$ is a seminorm on W , called the **seminorm induced by (the K -linear) mapping f** .

(4) Let $(V, |\cdot|)$ be a seminormed vector space over K , let $\pi : V \rightarrow E$ be a surjective K -linear mapping. We denote by $\|\cdot\|_\pi$ the mapping

$$E \rightarrow \mathbb{R}_{\geq 0},$$

$$\alpha \mapsto \inf_{x \in \pi^{-1}(\alpha)} \|x\|.$$

If $(\lambda, \alpha) \in K \times E$,

$$\|\lambda\alpha\|_\pi = \inf_{x \in \pi^{-1}(\lambda\alpha)} \|x\| = \inf_{x \in \pi^{-1}(\alpha)} |\lambda| \|x\| = |\lambda| \|\alpha\|_\pi.$$

If $(\alpha, \beta) \in E \times E$,

$$\begin{aligned} \|\alpha + \beta\|_\pi &= \inf_{z \in \pi^{-1}(\alpha + \beta)} \|z\| = \inf_{(x,y) \in \pi^{-1}(\alpha) \times \pi^{-1}(\beta)} \|x + y\| \\ &\leq \inf_{(x,y) \in \pi^{-1}(\alpha) \times \pi^{-1}(\beta)} \|x\| + \|y\| \\ &= \inf_{x \in \pi^{-1}(\alpha)} \|x\| + \inf_{y \in \pi^{-1}(\beta)} \|y\|. \end{aligned}$$

Hence $\|\cdot\|_\pi$ is a seminorm on E called the **quotient seminorm of $\|\cdot\|$ induced by π** .

Proposition 1.5.11 Let $(V, \|\cdot\|)$ be a seminormed vector space over a valued field $(K, |\cdot|)$.

- (1) For any $a \in V$, the mapping $\tau_a : V \rightarrow V$, $\tau_a(x) = x + a$ is 1-Lipschitzian.
- (2) For any $\lambda \in K$, the mapping $m_\lambda : V \rightarrow V$, $m_\lambda(x) := \lambda \cdot x$ is λ -Lipschitzian.
- (3) The mapping $\|\cdot\| : V \rightarrow \mathbb{R}$ is 1-Lipschitzian.

Proof

- (1) $\forall(x, y) \in V \times V$, $\|\tau_a(x) - \tau_a(y)\| = \|(x + a) - (y + a)\| = \|x - y\|$.

- (2) $\forall (x, y) \in V \times V, \|m_\lambda(x) - m_\lambda(y)\| = \|\lambda x - \lambda y\| = |\lambda| \|x - y\|.$
 (3) $\forall (x, y) \in V \times V, \|x\| = \|(x-y)+y\| \leq \|y\| + \|x-y\|. \text{ So } \|x\| - \|y\| \leq \|x-y\|.$ Similarly, $\|y\| - \|x\| \leq \|y-x\|.$ Hence,

$$\|\|x\| - \|y\|\| \leq \|x - y\|.$$

□

Definition 1.5.12 $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be two seminormed vector spaces over a valued field $(K, |\cdot|)$, and φ a K -linear mapping from E to F . We define $\|\varphi\| \in [0, +\infty]$ as

$$\|\varphi\| := \sup_{x \in E, \|x\|_E \neq 0} \frac{\|\varphi(x)\|_F}{\|x\|_E}.$$

In the case where $\|x_E\| = 0$, for any $x \in E$, by convention, $\|\varphi(x)\|$ is defined to be 0. If $\|\varphi\| < +\infty$, we say that φ is bounded. We denote by $\mathcal{L}(E, F)$ the set of all bounded K -linear mappings from E to F .

Remark 1.5.13 In the case when $(E, \|\cdot\|_E) = (K, \|\cdot\|)$,

$$\|\varphi\| = \sup_{x \in \mathcal{B}(0,1)} \|\varphi(x)\|_F.$$

Proposition 1.5.14

- (1) For any $\varphi \in \mathcal{L}(E, F)$ be the mapping φ is $\|\varphi\|$ -Lipschitzian. In particular, φ is continuous.
 (2) Suppose that there exists $\lambda \in K$, such that $|\lambda| > 1$. If $\varphi : E \rightarrow F$ is continuous at 0_E , then $\varphi \in \mathcal{L}(E, F)$.

Proof For any $(x, y) \in E \times E$:

- (1) $\|\varphi(x) - \varphi(y)\|_F = \|\varphi(x-y)\|_F \leq \|\varphi\| \|x-y\|_E.$
 (2) $\mathcal{B}(0_F, 1) := \{\alpha \in F \mid \|\alpha\|_F < 1\}$ is a neighborhood of 0_F . There exists $\varepsilon > 0$ such that

$$\varphi(\overline{\mathcal{B}}(0_E, \varepsilon)) \subseteq \mathcal{B}(0_F, 1)$$

where

$$\overline{\mathcal{B}}(0_E, \varepsilon) := \{x \in E \mid \|x\|_E < \varepsilon\}.$$

Let $x \in E \setminus \{0\}$, there exists $n \in \mathbb{Z}$, such that $\|\lambda^n x\|_E = |\lambda|^n \|x\|_E < \varepsilon$ and

$\|\lambda^{n+1}x\|_E = |\lambda|^{n+1} \|x\|_E \geq \varepsilon$. Thus,

$$\|\varphi(x)\|_F = \|\lambda^{-n}\varphi(\lambda^n x)\|_F = |\lambda|^{-n} \|\varphi(\lambda^n x)\|_F \leq |\lambda|^{-n} \leq \frac{|\lambda|}{\varepsilon} \|x\|_E.$$

Therefore, $\|\varphi\| \leq \frac{\lambda}{\varepsilon}$. □

Proposition 1.5.15 Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be two seminormed vector spaces over a valued field $(K, |\cdot|)$. Then $\mathcal{L}(E, F)$ is a vector subspace of F^E , and $\|\cdot\|$ is a seminorm on $\mathcal{L}(E, F)$, called the operator seminorm.

Proof Let φ, ψ be K -linear mappings from E to F . For any $x \in E$, such that $\|x\|_E \neq 0$.

$$\begin{aligned} \|(\varphi + \psi)(x)\|_F &= \|\varphi(x) + \psi(x)\|_F \leq \|\varphi(x)\|_F + \|\psi(x)\|_F \\ &\leq \|\varphi\| \|x\|_E + \|\psi\| \|x\|_E = (\|\varphi\| + \|\psi\|) \|x\|_E. \end{aligned}$$

$$\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|.$$

So, $\varphi, \psi \in \mathcal{L}(E, F) \Rightarrow \varphi + \psi \in \mathcal{L}(E, F)$. Let $\lambda \in K^\times$ and $\varphi \in \mathcal{L}(E, F)$, for any $x \in E$, $\|x\|_E \neq 0$. One has

$$\|(\lambda\varphi)(x)\|_F = \|\lambda \cdot \varphi(x)\|_F = |\lambda| \|\varphi(x)\|_F \leq |\lambda| \|\varphi\| \|x\|_E.$$

So, $\|\lambda\| \leq |\lambda| \|\varphi\|$, $\lambda\varphi \in \mathcal{L}(E, F)$. So $\mathcal{L}(E, F)$ is a vector subspace of F^E . Note that we can apply to λ^{-1} and $\lambda\varphi$ and get

$$\|\lambda^{-1} \cdot \lambda\varphi\| = \|\varphi\| \leq |\lambda^{-1}| \|\lambda\varphi\| = |\lambda|^{-1} \|\lambda\varphi\|, |\lambda| \leq \|\lambda\varphi\|.$$

Hence, $|\lambda| \|\varphi\| = \|\lambda\varphi\|$ and therefore $\|\cdot\|$ is a seminorm on $\mathcal{L}(E, F)$. □

Definition 1.5.16 Let E be a vector space over K , and $\|\cdot\|_1$ and $\|\cdot\|_2$ be seminorms on E . We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, if there exists $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\forall x \in E, c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1.$$

Proposition 1.5.17 Let E be a vector space over K , and $\|\cdot\|_1$ and $\|\cdot\|_2$ be seminorms on E . If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then they define the same topology on E .

Proof Let \mathcal{T}_1 and \mathcal{T}_2 to be the topologies defined by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then

$$\text{Id}_E : (E, \mathcal{T}_1) \longrightarrow (E, \mathcal{T}_2)$$

is bounded. So it is continuous, so $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Similarly, $\mathcal{T}_1 \subseteq \mathcal{T}_2$. \square

Proposition 1.5.18 Let $n \in \mathbb{N}_{\geq 1}$, and (X_i, d_i) , $i \in \{1, 2, \dots, n\}$ be n semi-normed vector spaces over K . Let $X = \prod_{i=1}^n X_i$ and

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0},$$

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) \longmapsto \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i).$$

Then d is a semimetric on X , and the topology induced by d is the product topology of \mathcal{T}_{d_i} (topology induced on X_i by d_i) $i \in \{1, \dots, n\}$.

Proof

$$(1) d((x_1, \dots, x_n), (x_1, \dots, x_n)) = \max_{i \in \{1, \dots, n\}} d_i(x_i, x_i) = 0.$$

(2)

$$\begin{aligned} d((x_1, \dots, x_n), (y_1, \dots, y_n)) &= \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) \\ &= \max_{i \in \{1, \dots, n\}} d_i(y_i, x_i) = d((y_1, \dots, y_n), (x_1, \dots, x_n)). \end{aligned}$$

(3)

$$\begin{aligned} &d((x_1, \dots, x_n), (z_1, \dots, z_n)) \\ &= \max_{i \in \{1, \dots, n\}} d_i(x_i, z_i) \\ &\leq \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) + \max_{i \in \{1, \dots, n\}} d_i(y_i, z_i) \\ &= d((x_1, \dots, x_n), (y_1, \dots, y_n)) + d((y_1, \dots, y_n), (z_1, \dots, z_n)). \end{aligned}$$

So d is a semimetric on X . For $i \in \{1, \dots, n\}$, $\mathcal{T}_i := \mathcal{T}_{d_i}$, where \mathcal{T}_d is the topology induced by d . Let $\pi_i : X \longrightarrow X_i$ be the project mapping (continuous with the product topology on X .) For any $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ in X ,

$$d(x, y) = \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) = \max_{i \in \{1, \dots, n\}} d_i(\pi_i(x), \pi_i(y)),$$

have $\forall i \in \{1, \dots, n\}$, $d_i(x_i, y_i) \leq d(x, y)$ which implies that

$$\text{Id}_X : (X, \mathcal{T}_d) \longrightarrow (X, \mathcal{T})$$

is continuous. So $\mathcal{T} \subseteq \mathcal{T}_d$.

$$\begin{aligned}\mathcal{B}((p_1, \dots, p_n), \varepsilon) &= \{(x_1, \dots, x_n) \mid d((p_1, \dots, p_n), (x_1, \dots, x_n)) < \varepsilon\} \\ &= \prod_{i=1}^n \mathcal{B}(p_i, \varepsilon) \in \mathcal{T}.\end{aligned}$$

□

1.6 Closed Subsets

Example 1.6.1 (Review of open subsets) In \mathbb{R} , an interval of the form $]a, b[$ is open, since $]a, b[= \mathcal{B}(\frac{a+b}{2}, \frac{b-a}{2})$. An interval of the form $]a, +\infty[$ is open, since $]a, +\infty[= \bigcup_{n \in \mathbb{N}_{\geq 1}}]a, a+n[$.

Definition 1.6.2 Let (X, \mathcal{T}) be a topological space. We say a subset Y of X is **closed** if $X \setminus Y$ is open.

Remark 1.6.3

- (1) \emptyset, X are closed.
- (2) If F_1, F_2 are closed subset of X , then $F_1 \cup F_2$ is closed.
- (3) If $(F_i)_{i \in I}$ is a non-empty family of closed subsets of X , then $\bigcap_{i \in I} F_i$ is closed.

Example 1.6.4 Let $(a, b) \in \mathbb{R}^2$, $a < b$, then $[a, b] \subseteq \mathbb{R}^2$ is closed. Moreover, $]-\infty, a]$ is closed.

Proposition 1.6.5 Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, and $f : X \rightarrow Y$ be a mapping, then the following statements are equivalent:

- (1) f is continuous.
- (2) For any closed subset F of Y , $f^{-1}(F)$ is a closed subset of X .

Proof

(1) \Leftrightarrow (2): f is continuous if and only if, for any open subset U of Y , $f^{-1}(U) \in \mathcal{T}_X$. Let $F \subseteq Y$ be closed, then $Y \setminus F$ is open. So $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is open, so $f^{-1}(F)$ is closed.

(2) \Leftrightarrow (1): Let $U \in \mathcal{T}_Y$, then $F = Y \setminus U$ is closed, so $f^{-1}(F) = X \setminus f^{-1}(U)$ is

closed. So $f^{-1}(U) \in \mathcal{F}_Y$. □

Example 1.6.6

In \mathbb{R}^2 , $\{(x, y) \in \mathbb{R}^2 \mid x \geq 1\}$ is closed. Since $\mathbb{R}^2 \setminus \{(x, y) \mid x \geq 1\} =]-\infty, 0[\times \mathbb{R}$. Since $f(x, y) = x + y$ is continuous, then $f^{-1}([0, +\infty]) = \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0\}$ is closed.

Example 1.6.7 Let (X, d) be a semimetric space. Let $Y \subseteq X$, $Y \neq \emptyset$. We define, for any $x \in X$,

$$d(x, Y) = \inf_{y \in Y} d(x, y) \in \mathbb{R}_{\geq 0}.$$

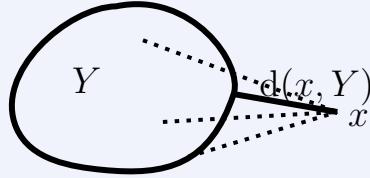


Figure 1.6.1: Definition of $d(\cdot, Y)$.

The mapping $d(\cdot, Y) : X \rightarrow \mathbb{R}$, $x \mapsto d(x, Y)$ is 1-Lipschitzian. Let $(x, x') \in X \times X$, $\forall y \in Y$,

$$d(x, Y) - d(x', Y) \leq d(x, y) - d(x', y) \leq d(x, x').$$

Taking the supremum, we get

$$d(x, Y) - d(x', Y) \leq d(x, x').$$

By symmetry between x and x' , $d(x', Y) - d(x, Y) \leq d(x, x')$. So

$$|d(x, Y) - d(x', Y)| \leq |d(x, x')|.$$

For any $r > 0$, define

$$B(Y, r) := \{x \in X \mid d(x, Y) < r\},$$

$$\overline{B}(Y, r) := \{x \in X \mid d(x, Y) \leq r\}$$

$B(Y, r)$ is open, $\overline{B}(Y, r)$ is closed. If $Y = \{y\}$ is a one point set, $B(Y, r)$ and $\overline{B}(Y, r)$ are defined as $B(y, r)$ and $\overline{B}(y, r)$ respectively.

Definition 1.6.8 Let (X, \mathcal{T}) be a topological space.

(1) Let \mathcal{F} be a non-degenerate filter, an element $p \in X$ is called **adherent point of \mathcal{F}** if $\mathcal{F} \cup \mathcal{V}_p(\mathcal{T})$ generates a non-degenerate filter. ($\forall U \in \mathcal{F}, \forall V \in \mathcal{V}_p(\mathcal{T}), U \cap V \neq \emptyset$.)

(2) Let $Y \subseteq X$. We say that $p \in X$ is an adherent point of Y if it is an adherent point of the principal filter $\mathcal{F}_Y = \{U \subseteq X \mid Y \subseteq U\}$. (For any neighborhood V of p , $Y \cap V \neq \emptyset$.) We denote by \overline{Y} the set of all adherent points of Y called the **closure** of Y . Clearly, $Y \subseteq \overline{Y}$.

Proposition 1.6.9 Let (X, \mathcal{T}_X) be a topological space, $Y \subseteq X$. Then \overline{Y} is the smallest closed subset containing Y . Namely,

$$\overline{Y} = \bigcap_{\substack{F \subseteq X \text{ closed,} \\ Y \subseteq F}} F.$$

Proof Let $p \in \overline{Y}$. If there exists a closed subset F containing Y such that $p \notin F$. So $p \in X \setminus F$. Hence $X \setminus F \in \mathcal{V}_p(\mathcal{T})$. So $\emptyset = (X \setminus Y) \cap Y \supseteq (X \setminus F) \cap Y \neq \emptyset$. Contradiction. Therefore,

$$\overline{Y} \subseteq \bigcap_{\substack{F \subseteq X \text{ closed,} \\ Y \subseteq F}} F.$$

Suppose that $x \in X \setminus \overline{Y}$. There exists an open neighborhood U of x such that $U \cap Y = \emptyset$. So $x \notin F := X \setminus U$.

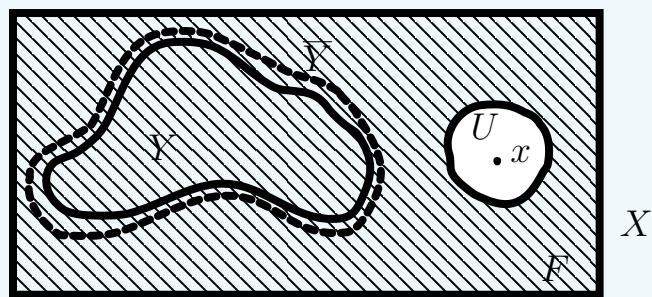


Figure 1.6.2: Closure

Note that F is closed and $F \supseteq Y$. Therefore,

$$X \setminus \overline{Y} \subseteq \bigcup_{\substack{F \subseteq X \text{ closed,} \\ Y \subseteq F}} (X \setminus F).$$

which leads to

$$\overline{Y} \supseteq X \setminus \left(\bigcup_{\substack{F \subseteq X \text{ closed,} \\ Y \subseteq F}} (X \setminus F) \right) = \bigcap_{\substack{F \subseteq X \text{ closed,} \\ Y \subseteq F}} F.$$

□

Definition 1.6.10 Let (X, \mathcal{T}) be a topological space and $Y \subseteq X$. We denote by Y° the set of $p \in Y$ such that Y is a neighborhood of p .

Proposition 1.6.11 Y° is the least^a open subset of X such that is contained in Y . Moreover,

$$X \setminus Y^\circ = \overline{X \setminus Y}.$$

^alargest

Proof $\forall y \in Y^\circ$, there exists $U_y \in \mathcal{T}$ such that $y \in U_y \subseteq Y$. Therefore, $\forall x \in U_y$, Y is a neighborhood of x , hence, $U_y \subseteq Y^\circ$. We thus obtain

$$Y^\circ = \bigcup_{y \in Y^\circ} \{y\} \subseteq \bigcup_{y \in Y^\circ} U_y \subseteq Y^\circ.$$

Hence Y° is open.

If $U \subseteq Y$ is open, then $\forall x \in U$, Y is a neighborhood of x . So $U \subseteq Y^\circ$. Therefore, Y° is the largest open subset that is contained in Y .

$$X \setminus Y^\circ = X \setminus \bigcup_{\substack{U \in \mathcal{T} \\ U \subseteq Y}} U = \bigcap_{\substack{U \in \mathcal{T} \\ U \subseteq Y}} X \setminus U \stackrel{F=X \setminus U}{=} \bigcap_{\substack{F \text{ closed} \\ X \setminus Y \subseteq F}} F = \overline{X \setminus Y}.$$

□

Definition 1.6.12 Let (X, \mathcal{T}) be a topological space. We equip $X \times X$ with the product topology, (a topological basis is given by $\{U \times V \mid (U, V) \in \mathcal{T}^2\}$) Let

$$\Delta_X := \{(x, x) \mid x \in X\} \subseteq X \times X.$$

If Δ_X is closed, we say that (X, \mathcal{T}) is a **Hausdorff space**. (Or (X, \mathcal{T}) is separated.)

Proposition 1.6.13 (X, \mathcal{T}) is a Hausdorff space if and only if $\forall (x, y) \in X \times X$, $x \neq y$, there exists $(U, V) \in \mathcal{V}_x(\mathcal{T}) \times \mathcal{V}_y(\mathcal{T})$, such that $U \cap V = \emptyset$.

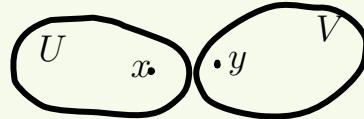


Figure 1.6.3: Hausdorff space

Proof

“ \Rightarrow ”: If $(x, y) \in X \times X$, $x \neq y$, then $(x, y) \in (X \times X) \setminus \Delta_X$. There exists $(U, V) \subseteq (X \times X) \setminus \Delta_X$, such that $(x, y) \in U \times V$, so $(U \times V) \cap \Delta_X = \emptyset$. Thus $U \cap V = \emptyset$. (If $p \in U \cap V$, then $(p, p) \in (U \times V) \cap \Delta_X$.)

“ \Leftarrow ”: For any $(x, y) \in X \times X$, $x \neq y$, $\exists U \in \mathcal{T}, \forall V \in \mathcal{T}$, $x \in U, y \in V, U \cap V = \emptyset$. Then $(x, y) \in U \times V$ and $(U \times V) \cap \Delta_X = \emptyset$. So Δ_X is closed. \square

Proposition 1.6.14 Let (X, \mathcal{T}) be a Hausdorff space. Let \mathcal{F} be a non-degenerate filter on X . If \mathcal{F} has a limit point^a, then its limit point is unique.

^aLet (X, \mathcal{T}) be a topological space and \mathcal{F} be a filter on X . If $p \in X$ is such that $\mathcal{V}_p(\mathcal{T}) \subseteq \mathcal{F}$, then we say that p is a limit point of \mathcal{F} .

Proof (By contradiction) Suppose that x and y are limit points of \mathcal{F} , $x \neq y$. Since X is Hausdorff, $\exists (U, V) \in \mathcal{T}^2$, $x \in U, y \in V, U \cap V = \emptyset$. Since x and y are limit points of \mathcal{F} , $U \in \mathcal{F}, V \in \mathcal{F}$. This contradicts the hypothesis that \mathcal{F} is non-degenerate. \square

Example 1.6.15 Any metric space is Hausdorff.

Let (X, d) be a metric space, $\forall (x, y) \in X \times X$, $x \neq y$, $d(x, y) > 0$. Let $\varepsilon = \frac{d(x, y)}{2}$. $B(x, \varepsilon) \cap B(y, \varepsilon) = \emptyset$. In fact, if $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$,

$$d(x, y) \leq d(x, z) + d(z, y) < 2\varepsilon = d(x, y).$$

Proposition 1.6.16 Let (X, \mathcal{T}) be a topological space, Y be a subset of X and $p \in X$.

(1) Let Z be a set and $f : Z \rightarrow X$ be a mapping. Let \mathcal{F} be a non-degenerate filter on Z . If p is a limit of f along \mathcal{F} , and if $f(Z) \subseteq Y$, then $p \in \overline{Y}$.

(2) Suppose that p has a countable neighborhood basis. If $p \in \overline{Y}$, then there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in Y that converges to p .

Proof

(1) p is a limit of f along \mathcal{F} if and only if $\mathcal{V}_p(\mathcal{T}) \subseteq f_*(\mathcal{F})$, or equivalently

$$\forall U \in \mathcal{V}_p(\mathcal{T}), f^{-1}(U) \in \mathcal{F}.$$

$f(f^{-1}(U)) \subseteq U \cap Y$, since $f(X) \subseteq Y$. Hence $U \cap Y \neq \emptyset$. So $p \in \overline{Y}$.

(2) Since p has a countable neighborhood basis, there exists a decreasing sequence $V_0 \supseteq V_1 \supseteq \dots$ of neighborhood of p such that $\{V_n \mid n \in \mathbb{N}\}$ forms a filter basis of $\mathcal{V}_p(\mathcal{T})$. For any $n \in \mathbb{N}$, $V_n \cap Y = \emptyset$, we take $y_n \in V_n \cap Y$. The sequence $(y_n)_{n \in \mathbb{N}}$ converges to p since $\forall n \in \mathbb{N}$, $\{y_k \mid k \in \mathbb{N}, k \geq n\} \subseteq V_n$. \square

Example 1.6.17 Let (X, d) be a semimetric space. $Y \subseteq X$, $\varepsilon > 0$. If $(y_n)_{n \in \mathbb{N}}$ is a sequence in $B(Y, \varepsilon)$, that converges to some $p \in X$, then

$$\lim_{n \rightarrow \infty} d(y_n, Y) = d(p, Y).$$

Therefore, $\overline{B(Y, \varepsilon)} \subseteq \overline{B}(Y, \varepsilon) := \{x \in X \mid d(x, Y) \leq \varepsilon\}$.

Proposition 1.6.18 Let (X, d) be a semimetric space, $Y \subseteq X$ be a closed subset. $\forall x \in X \setminus Y$, $d(x, Y) > 0$.

Proof $X \setminus Y$ is open, so $\exists \varepsilon > 0$ such that $B(X, \varepsilon) \subseteq X \setminus Y$. So $\forall y \in Y$, $d(x, y) \geq \varepsilon$. Hence, $d(x, y) \geq \varepsilon$. \square

Corollary 1.6.19 Let $(V, \|\cdot\|)$ be a semimetric space, W be a closed vector subspace of V . $Q = V/W$. Then the quotient seminorm

$$\|\cdot\|_Q : Q \longrightarrow R,$$

$$\alpha \longmapsto \inf_{x \in V, [x] = \alpha} \|x\|$$

is a norm.

Proof Let $\alpha \in Q \setminus \{0\}$ and $x \in V$ such that $\alpha = [x]$. Since $\alpha \neq 0$, $x \notin W$.

$$0 < d(x, W) := \inf_{y \in W} \|x - y\| = \inf_{\substack{x' \in V \\ [x'] = \alpha}} \|x'\| = \|\alpha\|_Q.$$

\square

Proposition 1.6.20 If (X, \mathcal{T}) is a Hausdorff space, then, $\forall x \in X$, $\{x\}$ is closed.

Proof $\forall y \in X \setminus \{x\}$, $y \neq x$. So $\exists (U, V) \in \mathcal{T} \times \mathcal{T}$, $x \in U$, $y \in V$. $U \cap V = \emptyset$. So $V \subseteq X \setminus \{x\}$. Hence $X \setminus \{x\}$ is a neighborhood of y . \square

Remark 1.6.21 Let $(V, \|\cdot\|)$ be a seminorm space. $W \subseteq V$, $Q = V/W$ and $\|\cdot\|_Q$ is the quotient seminorm. The mapping $\pi : V \rightarrow Q$, $x \mapsto [x]$ is continuous since $\|[x]\|_Q \leq \|x\|$. If $\|\cdot\|_Q$ is a norm then $\{0_Q\}$ is closed (since Q is Hausdorff). So $W = \pi^{-1}(\{0_Q\}) = \ker(\pi)$ is closed. This shows that $\|\cdot\|_Q$ is a norm $\Leftrightarrow W$ is closed.

1.7 Completeness

Definition 1.7.1 Let (X, d) be a semimetric space, $Y \subseteq X$, we define the diameter of Y as $\text{diam}(Y) := \sup_{(x,y) \in Y^2} d(x, y)$. If $\text{diam}(Y) < +\infty$, we say that Y is **bounded**.

Remark 1.7.2 Let (E, d) be a semimetric space.

(1) If A and B are subsets of E , then

$$A \subseteq B \Rightarrow \text{diam}(A) \leq \text{diam}(B).$$

(2) If $A \subseteq B \subseteq E$ and B is bounded, then A is bounded.

(3) If $A = \{x_1, \dots, x_n\} \subseteq E$, $n \in \mathbb{N}_{\geq 1}$. Then

$$\text{diam}(A) = \max_{(i,j) \in \{1, \dots, n\}^2} d(x_i, x_j) < +\infty.$$

So A is bounded.

(4) $\forall p \in X$, $\text{diam}(\bar{B}(p, r)) \leq 2r$, $\forall r \in \mathbb{R}_{>0}$. In fact, $\forall (x, y) \in \bar{B}(p, r)^2$, $d(x, y) \leq d(x, p) + d(p, y) \leq 2r$.

Proposition 1.7.3 Let (E, d) be a semimetric space, $A \subseteq E$. Suppose that A is bounded. Let $r = \text{diam}(A)$. For any $p \in A$, $A \subseteq \bar{B}(p, r)$.

Proof $\forall x \in A$, $d(p, x) \leq \text{diam}(A) = r$. \square

Proposition 1.7.4 Let (E, d) be a semimetric space, $A \subseteq E$, $B \subseteq E$ and $(x_0, y_0) \in A \times B$. Then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + \text{diam}(B) + d(x_0, y_0).$$

Proof Let $(x, y) \in (A \cup B)^2$.

Case 1. $\{x, y\} \in A$, $d(x, y) \leq \text{diam}(A)$.

Case 2. $\{x, y\} \in B$, $d(x, y) \leq \text{diam}(B)$.

Case 3. $x \in A$, $y \in B$,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}(A) + \text{diam}(B) + d(x_0, y_0).$$

Case 4. $x \in B$, $y \in A$. Same as case 3.

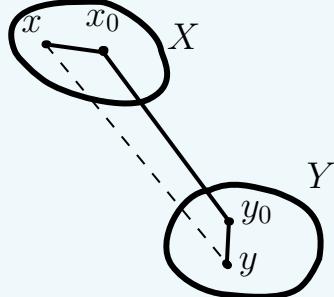


Figure 1.7.1: proposition. 1.7.4

□

Corollary 1.7.5 If $A \cap B \neq \emptyset$, then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + \text{diam}(B).$$

Definition 1.7.6 Let (E, d) be a semimetric space, and \mathcal{F} be a non-degenerate filter on E . If $\inf_{A \in \mathcal{F}} \text{diam}(A) = 0$, we say that \mathcal{F} is a **Cauchy filter**.

Example 1.7.7 $E =]0, 1]$. If \mathcal{F} is generated by $]0, \varepsilon[$, $\varepsilon \leq 1$, then \mathcal{F} is a Cauchy filter.

Proposition 1.7.8 Let (E, d) be a semimetric space and \mathcal{F} be a non-degenerate filter on E . If \mathcal{F} has a limit point p , then \mathcal{F} is a Cauchy filter.

Proof $\forall \varepsilon > 0, \bar{B}(p, \frac{\varepsilon}{2}) \in \mathcal{F}, \text{diam}(\bar{B}(p, \frac{\varepsilon}{2})) \leq \varepsilon.$ So $\inf_{A \in \mathcal{F}} \text{diam}(A) = 0.$ \square

Proposition 1.7.9 Let (E, d) be a semimetric space, and \mathcal{F} be a Cauchy filter on $E.$ Any adherent point of \mathcal{F} is a limit point of $\mathcal{F}.$

Proof Let $\varepsilon > 0.$ Let $A \in \mathcal{F}$ such that $\text{diam}(A) < \frac{\varepsilon}{2}.$ Note that $B(p, \frac{\varepsilon}{4}) \cap A \neq \emptyset.$ So

$$\text{diam}(B(p, \frac{\varepsilon}{4}) \cup A) \leq \text{diam}(B(p, \frac{\varepsilon}{4})) + \text{diam}(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

So $A \subseteq B(p, \varepsilon).$ \square

Definition 1.7.10 Let (E, d) be a semimetric space. $I \subseteq \mathbb{N}$ be an infinite subset, \mathcal{F} be the Fréchet filter on $I.$ We say that a sequence $x := (x_n)_{n \in I} \in E^I$ is a Cauchy sequence if $x_*(\mathcal{F}) := \{U \subseteq E \mid x^{-1}(U) \in \mathcal{F}\}$ is a Cauchy filter. Or equivalently,

$$\forall \varepsilon > 0, \exists N \in I, \forall n, m \in I_{\geq N}, d(x_n, x_m) \leq \varepsilon.$$

Proof “ \Rightarrow ” $\forall \varepsilon > 0, \exists U \in x_*(\mathcal{F}), \text{diam}(U) \leq \varepsilon.$ Since $x^{-1}(U) \in \mathcal{F}, \exists N \in \mathbb{N}$ such that $I_{\geq N} \subseteq x^{-1}(U).$ So $\{x_n \mid n \in I_{\geq N}\} \subseteq U,$ which leads to

$$\forall (n, m) \in I_{\geq N}^2, d(x_n, x_m) \leq \varepsilon.$$

“ \Leftarrow ” $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \varepsilon.$ \square

Proposition 1.7.11 Let (E, d) be a semimetric space, and $(x_n)_{n \in I}$ be a sequence in $E.$

- (1) If $(x_n)_{n \in I}$ is convergent, then it is a Cauchy sequence.
- (2) If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\{x_n \mid n \in I\}$ is bounded.
- (3) If $(x_n)_{n \in I}$ is a Cauchy sequence, then any of its subsequence is a Cauchy sequence.
- (4) If $(x_n)_{n \in I}$ is a Cauchy sequence, and if there exists a subsequence $(x_n)_{n \in J}$ that converges to some $p \in E,$ then $(x_n)_{n \in I}$ converges to $p.$

Proof Let \mathcal{F}_I be a Fréchet filter on $I.$

- (1) $x_*(\mathcal{F}_I)$ has a limit point. So it is a Cauchy filter.

(2) $\exists N \in \mathbb{N}$ such that $\text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq 1 < +\infty$. So $\text{diam}(\{x_n \mid n \in I\}) < +\infty$ since

$$\{x_n \mid n \in I\} = \{x_n \mid n \in I_{< N}\} \cup \{x_n \mid n \in I_{\geq N}\}.$$

(3) Let J be an infinite subset of I , $\lambda : J \rightarrow I$ be the inclusion mapping, and \mathcal{F}_J be the Fréchet filter. Then $\lambda_*(\mathcal{F}_J) \supseteq \mathcal{F}_I$.

$$(x \circ \lambda)_*(\mathcal{F}_J) = x_*(\lambda_*(\mathcal{F}_J)) \supseteq x_*(\mathcal{F}_I).$$

Since $x_*(\mathcal{F}_I)$ is a Cauchy filter, $(x \circ \lambda)_*(\mathcal{F}_J)$ is a Cauchy filter.

(4) We keep the notation introduced in the proof of (3). If p is a limit point of $x \circ \lambda_*(\mathcal{F}_J)$. Then p is an adherent point of $x_*(\mathcal{F}_I)$. Since $x_*(\mathcal{F}_I)$ is a Cauchy filter, p is a limit point of $x_*(\mathcal{F}_I)$. \square

Definition 1.7.12 Let (X, d) be a semimetric space. If any Cauchy filter on X has a limit point, then we say that (X, d) is complete.

Proposition 1.7.13 Let (X, d) be a metric space and Y be a subset of X . If (Y, d) is complete, then Y is a closed subset of X .

Proof Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in Y that converges in X to some $p \in X$. So $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in Y , thus it converges to some $q \in Y$. Since X is Hausdorff, $p = q$. Then $\bar{Y} = Y$ (Since \bar{Y} is the set of limits of all sequences in Y). So Y is closed. \square

Example 1.7.14 $(\mathbb{R}, |\cdot|)$ is complete.

Let $(x_n)_{n \in I}$ be a Cauchy sequence in \mathbb{R} . Let $M > 0$ such that $\forall n \in I, |x_n| \leq M$. Hence,

$$-M \leq \liminf_{n \rightarrow +\infty} x_n \leq \limsup_{n \rightarrow +\infty} x_n \leq M.$$

By Bolzano-Weierstrass, $\exists J \subseteq I$ infinite such that $(x_n)_{n \in J}$ converges to $\limsup_{n \rightarrow +\infty} x_n \in \mathbb{R}$. So $(x_n)_{n \in I}$ converges.

Proposition 1.7.15 Let (X, d) be a semimetric space, (X, d) is complete if and only if any Cauchy sequence in X is convergent.

Proof It suffices to prove “ \Leftarrow ”. Suppose that all Cauchy sequence in X converges. Let \mathcal{F} be a Cauchy filter on X . $\forall n \in \mathbb{N}$, let $A_n \in \mathcal{F}$, $\text{diam}(A) < \frac{1}{n+1}$. $\forall n \in \mathbb{N}$, let $B_n = A_0 \cap \dots \cap A_n \in \mathcal{F}$, $\text{diam}(B) \leq \frac{1}{n+1}$. Take $x_n \in B_n$. Then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Hence $(x_n)_{n \in \mathbb{N}}$ converges to $p \in X$.

We claim that p is a limit point of \mathcal{F} . If V is a neighborhood of p , then $\exists N \in \mathbb{N}$ such that $\{x_n \mid n \in \mathbb{N}_{\geq N}\} \subseteq V$. So $V \cap B_n \neq \emptyset$, $\forall n \in \mathbb{N}_{\geq N}$.

Let $A \in \mathcal{F}$, $A \cap B_n \neq \emptyset$. Take $y_n \in A \cap B_n$, $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. $d(y_n, x_n) \leq \frac{1}{1+n}$, so $(y_n)_{n \in \mathbb{N}}$ converges to p . Thus $V \cap A \neq \emptyset$. \square

Proposition 1.7.16 Let n be a positive integer. For any $i \in \{1, \dots, n\}$, let (X_i, d_i) be a semimetric space. Let $X = X_1 \times \dots \times X_n$ and

$$\pi_i : X \longrightarrow X_i$$

$$(x_1, \dots, x_n) \longmapsto x_i$$

be the projection mapping. We equip X with the product semimetric d

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) := \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i).$$

Let \mathcal{F} be a non-degenerate filter on X . For any $i \in \{1, \dots, n\}$, let

$$\mathcal{F}_i = (\pi_i)_*(\mathcal{F}).$$

- (1) The filter \mathcal{F} is a Cauchy filter if and only if \mathcal{F}_i is a Cauchy filter for any $i \in \{1, \dots, n\}$.
- (2) The filter \mathcal{F} has a limit point if and only if \mathcal{F}_i has a limit point for any $i \in \{1, \dots, n\}$. If p_i is the limit point of \mathcal{F}_i , then $p = (p_1, \dots, p_n)$ is a limit point of \mathcal{F} .
- (3) If $(X_1, d_1), \dots, (X_n, d_n)$ are complete, then (X, d) is complete.

Proof

(1) Since π_i are Lipschitzian, if \mathcal{F} is a Cauchy filter, then \mathcal{F}_i are all Cauchy filters. Conversely, let \mathcal{F} be a non-degenerate filter such that \mathcal{F}_i is a Cauchy filter for all $i \in \{1, \dots, n\}$. For any $\varepsilon > 0$, any $i \in \{1, \dots, n\}$, there exists $A_i \in \mathcal{F}_i$ such that $\text{diam}(A_i) < \varepsilon$. We define

$$A := A_1 \times \dots \times A_n = \bigcap_{i=1}^n \pi_i^{-1}(A_i) \in \mathcal{F}.$$

For any $(x_1, \dots, x_n), (y_1, \dots, y_n) \in A$, $d(x, y) = \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) < \varepsilon$.

(2) π_i is Lipschitzian so also continuous $i \in \{1, \dots, n\}$, therefore, if $p = (p_1, \dots, p_n)$ a limit point of \mathcal{F} , then $\pi_i(p) = p_i$ is a limit point of \mathcal{F}_i .

Conversely, Suppose that p_i is a limit point of \mathcal{F}_i . Let U be a neighborhood of p . There exists $U_i \in \mathcal{F}_i$ neighborhood of p_i such that $U_1 \times \dots \times U_n \subseteq U$.

$$U \supseteq \bigcap_{i=1}^n \pi_i^{-1}(U_i) \in \mathcal{F},$$

so $U \in \mathcal{F}$. □

Proposition 1.7.17 Let $(X, d_X), (Y, d_Y)$ be two semimetric spaces, $f : X \rightarrow Y$ be uniformly continuous. For any Cauchy filter \mathcal{F} on X , $f_*(\mathcal{F})$ is also a Cauchy filter on Y .

Proof $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\forall (x, y) \in X \times X, d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Let $A \in \mathcal{F}$ such that $\text{diam}(A) < \delta$, then $f(A) \in f_*(\mathcal{F})$ and $\text{diam}(f(A)) < \varepsilon$. □

Proposition 1.7.18 Let $(X, d), (X', d')$ be two semimetric spaces, $f : X \rightarrow X'$ injective mapping. Assume that there exists positive constants C_1 and C_2 such that $\forall (x, y) \in X \times X$, $C_1 \cdot d(x, y) \leq d'(f(x), f(y)) \leq C_2 \cdot d(x, y)$.

- (1) For any sequence $(x_i)_{i \in I}$ in X , the sequence $(x_i)_{i \in I}$ is a Cauchy sequence if and only if the sequence $(f(x_i))_{i \in I}$ is a Cauchy sequence.
- (2) For any $(x_i)_{i \in I}$ in X , $(x_i)_{i \in I}$ is a convergent if and only if $(f(x_i))_{i \in I}$ is convergent.
- (3) If (X, d) is complete, then $(f(X), d')$ is complete. $f(X)$ is closed in X' if in addition we assume that d' is a metric.

Proof Note that $f : X \rightarrow f(X)$, $f^{-1} : f(X) \rightarrow X$ are Lipschitzian. Apply the previous proposition. □

Theorem 1.7.19 Let $(K, |\cdot|)$ be a valued field such that $(K, |\cdot|)$ is complete. Let V be a finite dimensional vector space over K .

- (1) All possible norms on K are equivalent.
- (2) For any norm $\|\cdot\|$ on V , the normed space $(V, \|\cdot\|)$ is complete.

(3) If we equip V with the topology induced by an arbitrary norm $\|\cdot\|$. For any normed vector space $(V, \|\cdot\|)$, any K -linear mapping $f : V \rightarrow V'$ is bounded and $f(V)$ is closed in V' .

Proof Let $e := (e_i)_{i=1}^n$ be a basis of V . Then, the mapping

$$\|\cdot\|_e : V \rightarrow \mathbb{R}_{>0}, \|a_1e_1 + \dots + a_ne_n\| = \max_{i \in \{1, \dots, n\}} |a_i|$$

is a norm on V . For $f : V \rightarrow V'$,

$$\|f(a_1e_1 + \dots + a_ne_n)\|' = \|a_1f(e_1) + \dots + a_nf(e_n)\|' \quad (1.7.1)$$

$$\leq |a_1| \|f(e_1)\|' + \dots + |a_n| \|f(e_n)\|' \quad (1.7.2)$$

$$\leq \max_{i \in \{1, \dots, n\}} |a_i| \sum_{i=1}^n \|f(e_i)\|'. \quad (1.7.3)$$

Therefore the K -linear mapping $f : (V, \|\cdot\|_e) \rightarrow (V', \|\cdot\|')$ is bounded. In particular, for any $\|\cdot\|$ on V , $\text{Id} : (V, \|\cdot\|) \rightarrow (V, \|\cdot\|)$ is bounded. So there exists $C > 0$, $\|\cdot\| \leq C\|\cdot\|_e$.

To prove the theorem, we reason by induction with respect to $n = \dim(V)$.

In case when $n = 0$, $V = \{0\}$ has the unique norm $\|\cdot\|_0$ (constant mapping with the value $0 \in \mathbb{R}$). Any sequence in $\{0_V\}$ is constant, and hence convergent ($\{0\}, \|\cdot\|_0$ complete). If $f : (\{0_V\}, \|\cdot\|) \rightarrow (V, \|\cdot\|)$ is K -linear, then it is bounded. The unique $f(\{0_V\})$ is a one-point set, which is closed, since $(V, \|\cdot\|)$ is Hausdorff.

In case when $n = 1$, let e_1 be the basis of V , and $\|\cdot\|$ be an arbitrary norm on V .

$$\|a_1e_1\| = |a_1| \|e_1\| = \|a_1e_1\|_{e_1} \cdot \|e_1\|,$$

so $\|\cdot\|$ is equivalent to $\|\cdot\|_{e_1}$.

(2) Since $(K, |\cdot|)$ is complete, so $(V, \|\cdot\|_{e_1})$ is also complete, since

$$(K, |\cdot|) \rightarrow (V, \|\cdot\|_{e_1})$$

$$a \mapsto ae_1$$

is an isomorphism.

(3) We have seen that the mapping $f : (V, \|\cdot\|_{e_1}) \rightarrow (V', \|\cdot\|')$ is bounded. So $f : (V, \|\cdot\|) \rightarrow (V', \|\cdot\|')$ is also bounded. $f(V) = \{0\}$, or $(f(V), \|\cdot\|')$ is of dimension 1, so $(f(V), \|\cdot\|')$ is complete, so $f(V)$ is closed.

Suppose that the theorem holds for normed vector space of dimension $< n$. Then the case of dimension n :

Let $e = (e_i)_{i=1}^n$ be a basis of V . Let $W = \text{span}_K(\{e_1, \dots, e_n\})$. If $\|\cdot\|$ is a norm on V , then, by the induction hypothesis,

(i) $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent on W , that is, $\exists A > 0$ such that $\forall(a_1, \dots, a_{n-1})$,

$$\max\{|a_1|, \dots, |a_n|\} = \|a_1e_1 + \dots + a_{n-1}e_{n-1}\|_e \leq A\|a_1e_1 + \dots + a_{n-1}e_{n-1}\|.$$

(ii) $(W, \|\cdot\|)$ is complete.

(iii) W is a closed subset of V .

Let $Q = V/W$, and $\|\cdot\|_Q$ be the quotient norm on Q . Let $(b_1, \dots, b_n) \in K^n$,

$$s = b_1e_1 + \dots + b_ne_n \in V, t = b_1e_1 + \dots + b_ne_n + w \in V, \alpha = [s] = b_n[e_n] \in Q.$$

$$\|t\| = \|s - b_ne_n\| \leq \|s\| + |b_n|\|e_n\|.$$

$$\text{Take } B = \frac{\|e_n\|}{\|[e_n]\|_Q} \in \mathbb{R}, \|s\| \geq \|\alpha\|_Q = |b_n|\|[e_n]\|_Q = B^{-1}|b_n|\|e_n\|.$$

$$B^{-1}\|s\| \geq (\|t\| - |b_n|\cdot\|e_n\|) B^{-1}$$

$$\|s\| \geq B^{-1} \cdot |b_n| \cdot \|e_n\|.$$

$$(B^{-1} + 1)\|s\| \geq B^{-1}\|t\| \geq B^{-1}A^{-1} \max\{|b_1|, \dots, |b_n|\}$$

Take $C = \min\left\{\frac{B^{-1}A^{-1}}{B^{-1}+1}, B^{-1} \cdot \|e_n\|\right\}$. Then $\|s\| \geq C \max\{|b_1|, \dots, |b_n|\}$. So $\|\cdot\|$ is equivalent to $\|\cdot\|_e$.

(2) Since $(V, \|\cdot\|_e)$ is complete and $\|\cdot\|$ is equivalent to $\|\cdot\|_e$, we obtain that $(V, \|\cdot\|)$ is also complete.

(3) Since $f : (V, \|\cdot\|_e) \rightarrow (V', \|\cdot\|')$ is bounded, $f : (V, \|\cdot\|) \rightarrow (V', \|\cdot\|')$ is also bounded.

If f is not injective, $\dim(f(V)) < n$, so $(f(V), \|\cdot\|')$ is complete, $f(V)$ is thus closed.

If f is injective, then $\|f(\cdot)\|'$ and $\|\cdot\|$ are equivalent norms on V . So $(f(V), \|\cdot\|')$ is complete. Hence $f(V)$ is closed. □

Proposition 1.7.20 Let (X, d) be a complete semimetric space. Let $Y \subseteq X$ be a closed subset. Then (Y, d) is complete.

Proof Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in Y . It is also a Cauchy sequence in X , so converges to some $l \in X$. Since Y is closed, one has $l \in Y$. So (Y, d) is complete. □

1.8 Compactness

Definition 1.8.1 Let X be a set and \mathcal{F} be a non-degenerate filter on X . If there does not exist any non-degenerate filter \mathcal{G} on X such that $\mathcal{F} \subsetneq \mathcal{G}$, then we say that \mathcal{F} is an ultrafilter.

Proposition 1.8.2 For any non-degenerate filter \mathcal{F} on X , there exists an ultrafilter on X containing \mathcal{F} .

Proof Let Θ be a set of non-degenerate filters containing \mathcal{F} , equipped with \subseteq . Let Θ_0 be a non-empty totally ordered subset. Let $\mathcal{F}' = \bigcup_{\mathcal{H} \in \Theta_0} \mathcal{H}$.

We prove that \mathcal{F}' is a filter.

(1) Let $(V_1, V_2) \in \mathcal{F}' \times \mathcal{F}'$, $\exists \mathcal{H}_1, \mathcal{H}_2$ in Θ_0 , $V_1 \in \mathcal{H}_1$, $V_2 \in \mathcal{H}_2$. Since Θ_0 is totally ordered, either $\mathcal{H}_1 \subseteq \mathcal{H}_2$ or $\mathcal{H}_2 \subseteq \mathcal{H}_1$.

If $\mathcal{H}_1 \subseteq \mathcal{H}_2$, $V_1 \cap V_2 \in \mathcal{H}_2 \subseteq \mathcal{F}'$. If $\mathcal{H}_2 \subseteq \mathcal{H}_1$, $V_1 \cap V_2 \in \mathcal{H}_1 \subseteq \mathcal{F}'$.

(2) Let $V \in \mathcal{F}'$, let $\mathcal{H} \in \Theta_0$ such that $V \in \mathcal{H}$. $\forall U \in \wp(X)$, $U \supseteq V$, one has $U \in \mathcal{H} \subseteq \mathcal{F}'$. So $\mathcal{F}' \in \Theta$. It is an upper bound of Θ_0 . By Zorn's lemma, there exists maximal $\mathcal{G} \in \Theta$, it is an ultrafilter containing \mathcal{F} .

□

Proposition 1.8.3 Let X be a set and \mathcal{F} be a non-degenerate filter on X . The following conditions are equivalent.

- (1) \mathcal{F} is an ultrafilter.
- (2) $\forall A \in \wp(X)$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.
- (3) $\forall (A, B) \in \wp(X)^2$, if $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Proof

(1) \Rightarrow (2): Suppose that $A \in \wp(X)$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$. Let $B \in \mathcal{F}$. If $B \cap A = \emptyset$, then $B \subseteq X \setminus A$, $(X \setminus A) \in \mathcal{F}$. So $\mathcal{F} \cup \{A\}$ generates a non-degenerate filter $\mathcal{F}' \supsetneq \mathcal{F}$, contradiction.

(2) \Rightarrow (3): Suppose that $B \notin \mathcal{F}$. Then $X \setminus B \in \mathcal{F}$. So $(A \cup B) \cap (X \setminus B) \in \mathcal{F}$. So $A \in \mathcal{F}$.

(3) \Rightarrow (1): Let \mathcal{F}' be a non-degenerate filter such that $\mathcal{F} \subsetneq \mathcal{F}'$. Take $A \in \mathcal{F}' \setminus \mathcal{F}$. Then $X = A \cup (X \setminus A)$. Since $A \notin \mathcal{F}$, $X \setminus A \in \mathcal{F} \subseteq \mathcal{F}'$. So $\emptyset = A \cap (X \setminus A) \in \mathcal{F}'$. Contradiction. □

Corollary 1.8.4 Let $f : X \rightarrow Y$ be mapping of sets. If \mathcal{F} is an ultrafilter on X , then $f_*(\mathcal{F})$ is an ultrafilter on Y .

Proof Let A and B be subsets of Y such that

$$A \cup B \in f_*(\mathcal{F}) := \{C \subseteq Y \mid f^{-1}(C) \in \mathcal{F}\}$$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \in \mathcal{F}.$$

Since \mathcal{F} is an ultrafilter, $f^{-1}(A) \in \mathcal{F}$ or $f^{-1}(B) \in \mathcal{F}$. Namely, $A \in f_*(\mathcal{F})$ or $B \in f_*(\mathcal{F})$. \square

Definition 1.8.5 Let (X, \mathcal{T}) be a topological space, $Y \subseteq X$ and $(U_i)_{i \in I}$ be a family of subset of X .

- (1) If $Y \subseteq \bigcup_{i \in I} U_i$, we say that $(U_i)_{i \in I}$ is a **cover** of Y .
- (2) If $\exists J \in I$ such that $Y \subseteq \bigcup_{j \in J} U_j$, we say that $(U_j)_{j \in J}$ is a **subcover** of $(U_i)_{i \in I}$ of Y .
- (3) If $(U_i)_{i \in I} \in \mathcal{T}^I$ is a cover of Y , we say that $(U_i)_{i \in I}$ is an **open cover** of Y .
- (4) If I is a finite set and $(U_i)_{i \in I}$ is a cover of Y , we say that $(U_i)_{i \in I}$ is a **finite open cover**.

Proposition 1.8.6 Let (X, \mathcal{T}) be a topological space and $Y \subseteq X$. The following conditions are equivalent:

- (1) For any ultrafilter \mathcal{G} on X such that $Y \in \mathcal{G}$, \mathcal{G} has a limit point in Y .
- (2) For any non-degenerate filter \mathcal{F} on X , such that $Y \in \mathcal{F}$, \mathcal{F} has an adherent point in Y .
- (3) Any open cover of Y has a finite subcover.

Proof

(1) \Rightarrow (2) Let \mathcal{G} be an ultrafilter containing \mathcal{F} and $x \in Y$ be a limit point of \mathcal{G} . For any $U \in \mathcal{V}_x(\mathcal{T})$, $U \in \mathcal{G}$, so $\forall A \in \mathcal{F}$, $U \cap A \neq \emptyset$.

(2) \Rightarrow (1) Let \mathcal{G} be an ultrafilter, and let $x \in Y$ be an adherent point of \mathcal{G} . Then the filter \mathcal{G}' generated by $\mathcal{G} \cup \mathcal{V}_x(\mathcal{T})$ is non-degenerate and contains \mathcal{G} . So $\mathcal{G}' = \mathcal{G}$. This means $\mathcal{V}_x(\mathcal{T}) \subseteq \mathcal{G}$.

(2) \Rightarrow (3) Let $(U_i)_{i \in I}$ be an open cover of Y . Suppose that it does not have any finite subcover. For any $i \in I$, let $F_i = X \setminus U_i$. For any finite subset J of I ,

$Y \not\subseteq \bigcup_{j \in J} U_j$. So

$$Y \cap \left(X \setminus \bigcup_{j \in J} U_j \right) = Y \cap \left(\bigcap_{j \in J} F_j \right) \neq \emptyset.$$

So $\{F_i \mid i \in I\} \cup \{Y\}$ generates a non-degenerate filter \mathcal{F} . It has an adherent point of $x \in Y$. Since $Y \subseteq \bigcup_{i \in I} U_i$, $\exists i_0 \in I, x \in U_{i_0}$. So $U_{i_0} \in \mathcal{V}_x(\mathcal{T})$. This is impossible since $U_{i_0} \cap F_{i_0} = \emptyset$.

(3) \Rightarrow (2) Let \mathcal{F} be a non-degenerate filter such that $Y \in \mathcal{F}$. Suppose that \mathcal{F} does not have any adherent point in Y .

For any $y \in Y$, there exists open neighborhood U_y of y and $A_y \in \mathcal{F}$, such that $U_y \cap A_y = \emptyset$. Since $Y \subseteq \bigcup_{y \in Y} U_y$, $\exists \{y_1, \dots, y_n\} \subseteq Y$ such that $Y \subseteq \bigcup_{i=1}^n U_{y_i}$.

Take $A = \left(\bigcap_{i=1}^n A_{y_i} \right) \cap Y \in \mathcal{F}, A \neq \emptyset, A \subseteq Y$.

$$\begin{aligned} A &= A \cap Y \subseteq A \cap \bigcup_{i=1}^n U_{y_i} \\ &= \bigcup_{i=1}^n (A \cap U_{y_i}) \\ &\subseteq \bigcup_{i=1}^n (A_{y_i} \cap U_{y_i}) \\ &= \emptyset. \end{aligned}$$

Contradiction. □

Definition 1.8.7 Let (X, \mathcal{T}) be a topological space. If $Y \subseteq X$ satisfies the equivalent conditions described in the previous proposition, we say that Y is **compact**.

Proposition 1.8.8 Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be two topological spaces, $f : X \rightarrow Y$ be a continuous mapping. If $F \subseteq X$ is compact, then $f(F)$ is also compact.

Proof Let $(V_i)_{i \in I}$ be a open cover of $f(F)$. Then

$$F \subseteq f^{-1}(f(F)) \subseteq \bigcup_{i \in I} f^{-1}(V_i).$$

Since F is compact, $\exists J \subseteq I$ finite such that $F \subseteq \bigcup_{j \in J} f^{-1}(V_j)$. Hence $f(F) \subseteq \bigcup_{j \in J} V_j$. \square

Proposition 1.8.9 Let (X, \mathcal{T}) be a topological space, A be a compact subset of X , F is a closed subset of X , then $A \cap F$ is a compact subset of X .

Proof Let $(U_i)_{i \in I}$ be an open cover of $A \cap F$. Then

$$A \subseteq \left(\bigcup_{i \in I} U_i \right) \cup (X \setminus F).$$

So $\exists J \subseteq I$ finite, $A \subseteq (\bigcup_{j \in J} U_j) \cup (X \setminus F)$. Hence

$$A \cap F \subseteq \left(\bigcup_{j \in J} U_j \right) \cup ((X \setminus F) \cap F) = \bigcup_{j \in J} U_j.$$

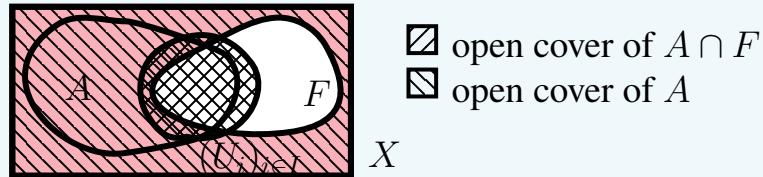


Figure 1.8.1: proposition 1.8.9

\square

Proposition 1.8.10 Let (X, \mathcal{T}) be a Hausdorff topological space, A be a compact subset of X .

- (1) For any $x \in X \setminus A$, there exists open subsets U and V of X , such that $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$.
- (2) A is closed.

Proof

- (1) $\forall y \in A, \exists U_y \in \mathcal{T}, V_y \in \mathcal{T}$ such that $y \in U_y, x \in V_y, U_y \cap V_y = \emptyset$. (Hausdorff) Since $A \subseteq \bigcup_{y \in A} U_y$ and A is compact. $\exists \{y_1, \dots, y_n\} \subseteq A, A \subseteq$

$\bigcup_{i=1}^n U_{y_i}$. Let $U = \bigcup_{i=1}^n U_{y_i}$, $V = \bigcap_{i=1}^n V_{y_i}$. These are open subsets of X , and $A \subseteq U$, $x \in V$.

$$U \cap V = \bigcup_{i=1}^n U_{y_i} \cap V \subseteq \bigcup_{i=1}^n U_{y_i} \cap V_{y_i} = \emptyset.$$

(1) \Rightarrow (2): $\forall x \in X \setminus A$, $\exists (U, V) \in \mathcal{T}^2$, $A \subseteq U$, $x \in V$, $U \cap V = \emptyset$. Hence, $V \subseteq X \setminus A$. So $X \setminus A$ is a neighborhood of x .

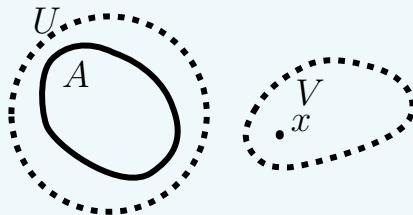


Figure 1.8.2: proposition 1.8.10

□

Corollary 1.8.11 Let (X, \mathcal{T}) be a Hausdorff topological space, and A and B be disjoint compact subset of X . There exist $(U, V) \in \mathcal{T}^2$, such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$.

Proof By (1) of the previous proposition.

$$\forall x \in B, \exists (U_x, V_x) \in \mathcal{T}^2, A \subseteq U_x, x \in V_x, U_x \cap V_x = \emptyset.$$

$B \subseteq \bigcup_{x \in B} V_x$. So, $\exists \{x_1, \dots, x_n\} \subseteq B$, such that $B \subseteq \bigcup_{i=1}^n V_{x_i}$. Let

$$U = \bigcap_{i=1}^n U_{x_i} \in \mathcal{T}, V = \bigcup_{i=1}^n V_{x_i} \in \mathcal{T}.$$

Then, $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

□

Theorem 1.8.12 Let (X, \mathcal{T}) be a topological space and A be a compact subset of X . Let (I, \leq) be a partially ordered set and $(F_i)_{i \in I}$ be a decreasing family of closed subsets of X . Assume that, any finite subset of I has an upper bound in I .

If $\left(\bigcap_{i \in I} F_i\right) \cap A = \emptyset$, then exists $i_0 \in I$ such that $F_{i_0} \cap A = \emptyset$.

Remark 1.8.13 In the particular case where $I = \mathbb{N}$, the theorem becomes: If $(A_n)_{n \in \mathbb{N}}$ is a sequence of compact non-empty subsets of X such that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$, then $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$.

Proof $\forall i \in I$, let $U_i = X \setminus F_i \in \mathcal{T}$. Since $\left(\bigcap_{i \in I} F_i\right) \cap A = \emptyset$. $A \subseteq X \setminus \bigcap_{i \in I} F_i = \bigcup_{i \in I} U_i$. Since A is compact, $\exists J \subseteq I$, finite such that $A \subseteq \bigcup_{j \in J} U_j$. Let i_0 be an upper bound of J . $\forall j \in J$, $j \leq i_0$. So $F_j \supseteq F_{i_0}$. $U_j \subseteq U_{i_0}$. Hence $A \subseteq U_{i_0}$, $A \cap F_{i_0} = \emptyset$. \square

Theorem 1.8.14 (Tychonoff) Let I be a non-empty set and (X_i, \mathcal{T}_i) , $i \in I$ be topological spaces. Let $X = \prod_{i \in I} X_i$, and \mathcal{T} be the product topology of $(\mathcal{T}_i)_{i \in I}$. For any $i \in I$, let A_i be a compact subset of X_i , let $A = \prod_{i \in I} A_i \subseteq X$. Then A is compact subset of X .

Proof For any $i \in I$, let $\pi_i : (x_j)_{j \in I} \mapsto x_i$ be the projection mapping.

$$A = \bigcap_{i \in I} \pi_i^{-1}(A_i).$$

Let \mathcal{F} be an ultrafilter on X such that $A \in \mathcal{F}$. Then $\forall i \in I$, $\pi_{i*}(\mathcal{F}) =: \mathcal{F}_i$ is an ultrafilter on X_i such that $A_i \in \mathcal{F}_i$. Since A_i is compact, \mathcal{F}_i has a limit point $x_i \in A_i$. Let $x = (x_i)_{i \in I}$. Let U be a neighborhood of x . There exists $J \subseteq I$ finite and $U_j \in \mathcal{F}_j$ ($j \in J$) such that $x_j \in U_j$ and $\bigcap_{j \in J} \pi_j^{-1}(U_j) \subseteq U$. $\forall j \in J$, $U_j \in \mathcal{F}_j$.

Since x_j is a limit point of \mathcal{F}_j . Hence $\pi_j^{-1}(U_j) \in \mathcal{F}$. Thus $\bigcap_{j \in J} \pi_j^{-1}(U_j) \in \mathcal{F}$, which implies $U \in \mathcal{F}$. \square

1.9 Compact Metric Spaces

Definition 1.9.1 Let (X, \mathcal{T}) be a topological space and $Y \subseteq X$. If any sequence in Y has a subsequence that converges to some element of X . We say that Y is **sequentially compact**.

Theorem 1.9.2 Let (X, \mathcal{T}) be a topological space and $Y \subseteq X$. If Y is compact and if each $y \in Y$ has a countable neighborhood basis, then Y is sequentially compact.

Proof Let $x = (x_n)_{n \in I}$ be a sequence in Y and \mathcal{F} be the Fréchet filter of I . Then $x_*(\mathcal{F})$ is a non-degenerate filter on X and $Y \in x_*(\mathcal{F})$. Since Y is compact, $x_*(\mathcal{F})$ has an adherent point $p \in Y$. Let $(U_k)_{k \in \mathbb{N}}$ be a decreasing sequence of neighborhood of p such that $\{U_k \mid k \in \mathbb{N}\}$ forms a neighborhood basis of p . Therefore we can construct in a recursive way a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ in I such that $x_{n_k} \in U_k$. Thus $(x_{n_k})_{k \in \mathbb{N}}$ converges to p . \square

Theorem 1.9.3 Let (X, d) be a metric space, and $Y \subseteq X$. The following conditions are equivalent:

- (1) Y is compact.
- (2) (Y, d) is complete and

$$\forall \varepsilon > 0, \exists A_\varepsilon \subseteq Y \text{ finite}, Y \subseteq \bigcup_{x \in A_\varepsilon} B(x, \varepsilon).$$

- (3) Y is sequentially compact.

Proof

(1) \Rightarrow (2) Let \mathcal{F} be a Cauchy filter on Y . Let $f : Y \rightarrow X$ be the inclusion mapping. Then $f_*(\mathcal{F})$ is a Cauchy filter on X and $Y \in f_*(\mathcal{F})$. Since Y is compact, $f_*(\mathcal{F})$ has an adherent point $l \in Y$. So l is a limit point of $f_*(\mathcal{F})$ (since $f_*(\mathcal{F})$ is a Cauchy filter.) For any $U \in V_l(\mathcal{T})$, $U \in f_*(\mathcal{F})$, namely, $f^{-1}(U) = U \cap Y \in \mathcal{F}$. Thus l is a limit point of \mathcal{F} . Since $Y \subseteq \bigcup_{y \in Y} B(y, \varepsilon)$. Since Y is compact, $\exists A_\varepsilon \subseteq Y$ finite, such that $Y \subseteq \bigcup_{x \in A_\varepsilon} B(x, \varepsilon)$.

(2) \Rightarrow (1) Let \mathcal{F} be an ultrafilter on X such that $Y \in \mathcal{F}$. For any $\varepsilon > 0$, $\bigcup_{x \in A_\varepsilon} B(x, \varepsilon) \in \mathcal{F}$. Hence \mathcal{F} is a Cauchy filter, which implies that $\mathcal{F}|_Y := \{Y \cap U \mid U \in \mathcal{F}\}$ is a Cauchy filter. Thus $\mathcal{F}|_Y$ has a limit point $l \in Y$, which is also a limit

point of \mathcal{F} .

(2) \Rightarrow (3) is already known.

(3) \Rightarrow (1) Let $(x_n)_{n \in I}$ be a Cauchy sequence in Y . Since Y is sequentially compact, $(x_n)_{n \in I}$ has a subsequence that converges to some $l \in Y$. Since $(x_n)_{n \in I}$ is a Cauchy sequence, it converges to l . We prove the second statement by contradiction.

Assume that $\varepsilon > 0$ is such that Y cannot be covered by finitely many balls centered in Y with radius ε . We construct recursively a sequence $(x_n)_{n \in \mathbb{N}}$ as follows. $x_0 \in Y$ is chosen arbitrarily. If x_0, \dots, x_n are chosen, we pick

$$x_{n+1} \in Y \setminus \bigcup_{i=0}^n B(x_i, \varepsilon).$$

Then $\forall (i, j) \in \mathbb{N}^2, i \neq j, d(x_i, x_j) < \varepsilon$. So $(x_n)_{n \in \mathbb{N}}$ does not have any Cauchy subsequence. It cannot have a convergent subsequence. \square

Example 1.9.4 Let $(a, b) \in \mathbb{R}^2, a < b$. $[a, b]$ is closed. $\forall \varepsilon, \exists N \in \mathbb{N}_{>0}. \frac{b-a}{N} < \varepsilon$. $\forall i \in \{0, \dots, N\}$, let $x_i = a + \frac{i}{N}(b-a)$. $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$.

$$[a, b] \subseteq \bigcup_{i=0}^N B(x_i, \varepsilon) = \bigcup_{i=0}^n [x_i - \varepsilon, x_i + \varepsilon].$$

So $[a, b]$ is compact.

Proposition 1.9.5 Let (X, d) be a metric space and $Y \subseteq X$. If Y is compact, then Y is bounded and closed.

Proof Since X is Hausdorff, Y is closed. $\exists A \subseteq Y$ finite, $Y \subseteq \bigcup_{x \in A} B(x, 1)$. Since each $B(x, 1)$ is bounded, so is Y . \square

Definition 1.9.6 Let (X, \mathcal{T}) be a topological space. If $\forall x \in X, x$ has a compact neighborhood, we say that X is **locally compact**.

Example 1.9.7 $(\mathbb{R}, |\cdot|)$ is locally compact.

$\forall x \in \mathbb{R}, [x-1, x+1]$ is compact neighborhood of x .

Proposition 1.9.8 Let $(K, |\cdot|)$ be a locally compact valued field. Then,

- (1) $(K, |\cdot|)$ is complete.
- (2) Assume that exists $a \in K$, $|a| > 1$. Let $(V, \|\cdot\|)$ be a finite-dimensional vector space over K . Then any bounded closed subset of V is compact.
- (3) $(V, \|\cdot\|)$ is locally compact.

Proof

(2) Assume that $V = K^n$, $\|(a_1, \dots, a_n)\| = \max\{|a_1|, \dots, |a_n|\}$. Let $A \subseteq K^n$, bounded, closed. Let $R > 0$, $A \subseteq \bar{B}(0_V, R)$.

Since $(K, |\cdot|)$ is locally compact, there exists a compact neighborhood U of $0_K \in K$. $\exists \varepsilon > 0$, $\bar{B}(0, \varepsilon) \subseteq U$. So $\bar{B}(0_K, \varepsilon)$ is compact. Take $n \in \mathbb{N}_{\geq 1}$ such that $|a|^n \varepsilon > R$.

$$f : K \longrightarrow K \text{ continuous}$$

$$b \longmapsto a^n b$$

$f(\bar{B}(0_K, \varepsilon)) = a^n \bar{B}(0_K, \varepsilon) \supseteq \bar{B}(0_K, R)$. is compact. By Tychonoff theorem, $\bar{B}(0_V, R)$ is compact. Since A is closed, A is compact.

(3) $\bar{B}(0_V, R)$ is compact for any $R > 0$.

(1) $B(0_K, R)$ is compact for any $R > 0$. For any Cauchy sequence $(a_n)_{n \in \mathbb{N}}$ in K , $\exists R > 0$. $\{a_n \mid n \in \mathbb{N}\} \subseteq \bar{B}(0_K, R)$, so $(a_n)_{n \in \mathbb{N}}$ converges. \square

Lemma 1.9.9 Let $A \subseteq R$ be a non-empty compact subset. Then A has a greatest and least element.

Proof Since A is non-empty and bounded, $\{\sup A, \inf A\} \subseteq \mathbb{R}$. Since $A = \bar{A}$, $\sup A \in A$, $\inf A \in A$. \square

Theorem 1.9.10 Let (X, \mathcal{T}) be a topological space and $f : X \longrightarrow \mathbb{R}$ be a continuous mapping. If $Y \subseteq X$ is compact, then $f|_Y$ has its maximum and minimum. ($\exists x_1 \in Y, f(x_1) \geq f(y), \forall y \in Y$. $\exists x_2 \in Y, \forall y \in Y, f(x_2) \leq f(y)$.)

Proof $f(Y) \subseteq \mathbb{R}$ is compact. \square

Theorem 1.9.11 Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \longrightarrow Y$ be a continuous mapping. If (X, d_X) is compact, then f is uniformly continuous.

Proof f is uniformly continuous if and only if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X, d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon.$$

Suppose by contradiction that f is not uniformly continuous. That is

$$\exists \varepsilon > 0, \forall \delta > 0, \exists (x_1, x_2) \in X^2, d_X(x_1, x_2) < \delta \text{ and } d_Y(f(x_1), f(x_2)) \geq \varepsilon.$$

So we can choose sequences $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ and $(y_n)_{n \in \mathbb{N}_{\geq 1}}$ in X such that

$$\forall n \in \mathbb{N}, d_X(x_n, y_n) \leq \frac{1}{n}, d_Y(f(x_n), f(y_n)) \geq \varepsilon.$$

By compactness of X , there exists a subsequence $I \subseteq \mathbb{N}_{\geq 1}$ finite, such that $(x_n)_{n \in I}$ converges to some $a \in X$, and $(y_n)_{n \in I}$ converges to some $b \in X$. Since $d_X : X \times X \rightarrow \mathbb{R}_{\geq 0}$ is continuous,

$$0 \leq d_X(a, b) = \lim_{n \in I, n \rightarrow +\infty} d_X(x_n, y_n) \leq \lim_{n \in I, n \rightarrow +\infty} \frac{1}{n}.$$

□

1.10 Path Connectedness

Definition 1.10.1 Let (X, \mathcal{T}) be a topological space and C be a subset of X . If for any $(x, y) \in C \times C$, there exists a continuous mapping $\varphi : [0, 1] \rightarrow C$, such that $\varphi(0) = x$ and $\varphi(1) = y$, we say that C is **path connected**.

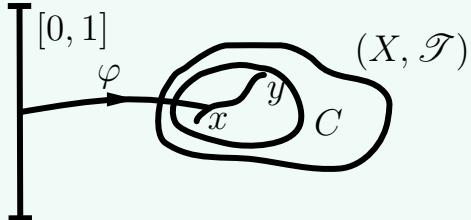


Figure 1.10.1: Path connected

Proposition 1.10.2 Let (X, \mathcal{T}_X) , (Y, \mathcal{T}_Y) be two topological spaces and $f : X \rightarrow Y$ be continuous mapping. If $C \subseteq X$ is path connected, then $f(C) \subseteq Y$ is path connected.

Proof Let $(a, b) \in C \times C$. There exist $(x, y) \in C \times C$ such that $f(x) = a$ and $f(y) = b$. Since C is path connected, so we have a continuous $\varphi : [0, 1] \rightarrow C$ such that $\varphi(0) = x$ and $\varphi(1) = y$. Then $f \circ \varphi : [0, 1] \rightarrow f(C)$ is continuous, and

$$(f \circ \varphi)(0) = f(x) = a, \quad (f \circ \varphi)(1) = f(y) = b.$$

Hence $f(C)$ is path connected. □

Theorem 1.10.3 Let I be a subset of \mathbb{R} . I is path connected if and only if I is an interval.

Proof

⇐ Assume that I is an interval. Let $(a, b) \in I \times I$ such that $a \leq b$. One has $[a, b] \subseteq I$. Consider $\varphi : [0, 1] \rightarrow I$ defined as:

$$\varphi(t) = \frac{a+b}{2} + \frac{b-a}{2}t, \quad t \in [0, 1].$$

This is the sum of a constant mapping and a linear mapping. Hence φ is continuous. So I is path connected.

⇒ Assume that I is path connected. Let $(a, b) \in I \times I$, $a \leq b$, $\varphi : [0, 1] \rightarrow I$ a continuous mapping such that $\varphi(0) = a$ and $\varphi(1) = b$. Let $c \in [a, b]$. It is

suffices to show that $c \in I$. Assume that $c \notin I$. We have $a < c < b$. We will define two sequences $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ in a recursive way: $x_0 = 0, y_0 = 1$. We want to have $\varphi(x_n) < \varphi(y_n)$. Let $(z_n) := \frac{x_n + y_n}{2}$.

$$\begin{cases} x_{n+1} = x_n, y_{n+1} = z_n, & \text{if } \varphi(z_n) > c, \\ x_{n+1} = z_n, y_{n+1} = y_n, & \text{if } \varphi(z_n) < c. \end{cases}$$

Then the sequence $(x_n)_{n \in \mathbb{N}}$ is increasing, $(y_n)_{n \in \mathbb{N}}$ is decreasing, and $0 \leq y_n - x_n \leq \frac{1}{2^n}$. So $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}$ converge to the same point $d \in [0, 1]$. Note that

$$\forall n \in \mathbb{N}, (c - \varphi(x_n))(c - \varphi(y_n)) < 0.$$

$$0 \leq (c - \varphi(d))^2 = \lim_{n \rightarrow \infty} (c - \varphi(x_n))(c - \varphi(y_n)) \leq 0.$$

So, $\varphi(d) = c$. Contradiction. □

Corollary 1.10.4 Let I be an interval in \mathbb{R} . If $f : I \rightarrow \mathbb{R}$ is a continuous mapping, then $f(I)$ is also an interval. In particular, if f has a positive value and a negative value, then it must have a zero.

Proof Since I is path connected, so $f(I)$ is path connected. Hence $f(I) \subseteq \mathbb{R}$ is an interval. □

Chapter 2

Differential Calculus

2.1 Landau symbol

In this section, we fix a complete valued field $(K, |\cdot|)$ and a normed vector space $(V, \|\cdot\|)$ over K .

Definition 2.1.1 Let X be a set, $f : X \rightarrow V$, $g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. Let $Y \subseteq X$ be a subset. We use the expression

$$f(x) = \mathcal{O}(g(x))$$

to denote the statement:

$$\exists C > 0, \forall x \in Y, \|f(x)\| \leq C \cdot g(x).$$

Let \mathcal{F} be a filter on X , we use the expression

$$f(x) = \mathcal{O}(g(x)) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists C > 0, \exists A \in \mathcal{F}, \|f(x)\| \leq C \cdot g(x), \forall x \in A.$$

We use the expression

$$f(x) = o(g(x)) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists \varepsilon : X \rightarrow \mathbb{R}_{\geq 0}, \exists A \in \mathcal{F}, \lim_{\mathcal{F}} \varepsilon = 0 \text{ and } \forall x \in A, \|f(x)\| \leq \varepsilon(x)g(x).$$

Proposition 2.1.2 Let X be a set and \mathcal{F} be a filter on X .

(1) Let $f : X \rightarrow V, g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f(x) = o(g(x))$ along \mathcal{F} , then $f(x) = \mathcal{O}(g(x))$ along \mathcal{F} .

(2)

1. Let $f_1 : X \rightarrow V, f_2 : X \rightarrow V$ and $g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f_1(x) = \mathcal{O}(g(x))$ and $f_2(x) = \mathcal{O}(g(x))$ along \mathcal{F} , then $f_1(x) + f_2(x) = \mathcal{O}(g(x))$ along \mathcal{F} .

2. Let $f_1 : X \rightarrow V, f_2 : X \rightarrow V$ and $g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f_1(x) = o(g(x))$ and $f_2(x) = o(g(x))$ along \mathcal{F} , then $f_1(x) + f_2(x) = o(g(x))$ along \mathcal{F} .

(3) Let $\lambda : X \rightarrow K, f : X \rightarrow V, g : X \rightarrow \mathbb{R}_{\geq 0}, h : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings.

1. If $\lambda(x) = \mathcal{O}(g(x))$ along $\mathcal{F}, f(x) = \mathcal{O}(h(x))$ along \mathcal{F} , then

$$(\lambda f)(x) = \lambda(x)f(x) = \mathcal{O}(g(x)h(x)).$$

2. If $\lambda(x) = \mathcal{O}(g(x))$ along $\mathcal{F}, f(x) = o(h(x))$ along \mathcal{F} , or if $\lambda(x) = o(g(x))$ along $\mathcal{F}, f(x) = \mathcal{O}(h(x))$ along \mathcal{F} , then

$$\lambda(x)f(x) = o(g(x)h(x)).$$

Proof

(1) We have $\varepsilon : X \rightarrow \mathbb{R}_{\geq 0}, A \in \mathcal{F}$ such that $\lim_{\mathcal{F}} \varepsilon = 0$ and $\forall x \in A, \|f(x)\| \leq \varepsilon(x)g(x)$. Since $\lim_{\mathcal{F}} \varepsilon = 0$, there exists $B \in \mathcal{T}$ such that $\forall x \in B, |\varepsilon(x)| < 1$, hence $\forall x \in A \cap B, \|f(x)\| \leq g(x)$.

(2)

1. $A_1, A_2 \in \mathcal{F}, C_1, C_2 > 0, \forall x \in A_1, \|f_1(x)\| \leq C_1g(x), \forall x \in A_2, \|f_2(x)\| \leq C_2g(x)$. So $f_1(x) + f_2(x) = \mathcal{O}(g(x))$

2. Let $\varepsilon_1 : X \rightarrow \mathbb{R}_{\geq 0}, \varepsilon_2 : X \rightarrow \mathbb{R}_{\geq 0}, A \in \mathcal{F}, \lim_{\mathcal{F}} \varepsilon_1 = \lim_{\mathcal{F}} \varepsilon_2 = 0$. $\forall x \in A_1, \|f_1(x)\| \leq \varepsilon_1(x) \cdot g(x), \forall x \in A_2, \|f_2(x)\| \leq \varepsilon_2(x)g(x)$. So $\lim_{\mathcal{F}} \varepsilon_1 + \varepsilon_2 = 0$.

$$\forall x \in A_1 \cap A_2, \|f_1(x) + f_2(x)\| \leq \|f_1(x)\| + \|f_2(x)\| \leq (\varepsilon_1(x) + \varepsilon_2(x))g(x).$$

(3)

1. There exists $(C_1, C_2) \in \mathbb{R}_{>0}^2$ and $(A_1, A_2) \in \mathcal{F}^2$ such that

$$\forall x \in A_1, |\lambda(x)| \leq C_1 g(x), \forall x \in A_2, \|f(x)\| \leq C_2 h(x).$$

Hence,

$$\forall x \in A_1 \cap A_2, \|(\lambda(x)f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C_1 C_2 g(x) h(x).$$

2. We assume that

$$\lambda(x) = \mathcal{O}(g(x)) \text{ along } \mathcal{F}, f(x) = o(h(x)) \text{ along } \mathcal{F}.$$

There exists $(A_1, A_2) \in \mathcal{F} \times \mathcal{F}, C \in \mathbb{R}_{\geq 0}$ and a mapping $\varepsilon : X \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\forall x \in A_1, |\lambda(x)| \leq C \cdot g(x), \forall x \in A_2, \|f(x)\| \leq \varepsilon(x) h(x).$$

Then one has

$$\lim_{\mathcal{F}} C\varepsilon(x) = 0$$

and

$$\forall x \in A_1 \cap A_2, \|(\lambda(x)f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C \cdot g(x) \cdot \varepsilon(x) h(x)$$

As required. □

Example 2.1.3

(1) Let $I \subseteq \mathbb{N}$ infinite. Let $(V, \|\cdot\|)$ be a normed vector space over complete valued field $(K, |\cdot|)$. Let \mathcal{F} be the filter on I . Let $(x_n)_{n \in I} \in V^I, (b_n)_{n \in I} \in \mathbb{R}_{\geq 0}^I$. We denote by

$$x_n = \mathcal{O}(b_n), n \in I, n \rightarrow +\infty$$

the statement $x_n = \mathcal{O}(b_n)$ along \mathcal{F} . Namely,

$$\exists N \in \mathbb{N}, \exists C > 0, \forall n \in I_{\geq N}, \|x_n\| \leq C \cdot b_n.$$

$$x_n = o(b_n), n \in I, n \rightarrow +\infty$$

denotes the statement $x_n = o(b_n)$ along \mathcal{F} . Namely,

$$\exists (\varepsilon_n)_{n \in I} \text{ such that } \lim_{n \rightarrow +\infty} \varepsilon_n = 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, \|x_n\| \leq \varepsilon_n \cdot b_n.$$

(2) Let (X, \mathcal{T}) be a topological space, $Y \subseteq X$, $y_0 \in \bar{Y}$. Let $f : Y \rightarrow V$ and $g : Y \rightarrow \mathbb{R}_{\geq 0}$ be mappings.

$$\mathcal{F} = \mathcal{V}_{y_0}(\mathcal{T})|_Y := \{U \cap Y \mid U \text{ is a neighborhood of } y_0\}$$

$f(y)\mathcal{O}(g(y))$, $y \in Y$, $y \rightarrow y_0$ denotes $f(y) = \mathcal{O}(g(y))$ along \mathcal{F} . Namely,

$$\exists C > 0, \exists U \in \mathcal{V}_{y_0}(\mathcal{T}), \forall y \in U \cap Y, \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)), y \in Y, y \rightarrow y_0$$

denotes $f(y) = o(g(y))$ along \mathcal{F} . Namely,

$$\exists \varepsilon : Y \rightarrow \mathbb{R}_{\geq 0}, \lim_{y \in Y, y \rightarrow y_0} \varepsilon(y) = 0, \exists U \in \mathcal{V}_{y_0}(\mathcal{T}),$$

$$\forall y \in U \cap Y, \|f(y)\| \leq \varepsilon(y)g(y).$$

(3) Let \mathcal{F} be a filter on \mathbb{R} generated by subsets of the form $[a, +\infty[$. Let $Y \subseteq \mathbb{R}$ not bounded from above. Let $f : Y \rightarrow V$ and $g : Y \rightarrow \mathbb{R}_{\geq 0}$ be mappings. Then

$$f(y) = \mathcal{O}(g(y)), y \in Y, y \rightarrow +\infty$$

denotes $f(y) = \mathcal{O}(g(y))$ along $\mathcal{F}|_Y$. Namely,

$$\exists C > 0, \exists a \in \mathbb{R}, \forall y \in Y_{\geq a}, \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)), y \in Y, y \rightarrow +\infty$$

denotes $f(y) = o(g(y))$ along $\mathcal{F}|_Y$. Namely,

$$\exists \varepsilon : Y \rightarrow \mathbb{R}_{\geq 0}, \lim_{y \rightarrow +\infty} \varepsilon(y) = 0, \exists a \in \mathbb{R}, \forall y \in Y_{\geq a}, \|f(y)\| \leq \varepsilon(y)g(y).$$

2.2 Differentiability

We fix a complete valued field $(K, |\cdot|)$. We suppose that there exists $a \in K^\times$, such that $|a| < 1$. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K .

$$\mathcal{L}(E, F) := \{\varphi \in \text{Hom}_K(E, F) \mid \|\varphi\| < +\infty\}.$$

$(\mathcal{L}(E, F), \|\cdot\|)$ is a normed vector space over K .

Definition 2.2.1 Let $U \subseteq E$ be subset and $p \in U^\circ$. We say that a mapping $f : U \rightarrow F$ is **differentiable** at p if there exists $\varphi \in \mathcal{L}(E, F)$ such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

If $U = U^\circ$ and f is differentiable at every point of U , we say that f is **differentiable** on U .

Proposition 2.2.2 Assume that $f : U \rightarrow F$ is differentiable at $p \in U^\circ$. There exists a unique $\varphi \in \mathcal{L}(E, F)$ such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

Lemma 2.2.3 $\forall \eta \in \mathcal{L}(E, F), \forall r > 0$.

$$\|\eta\| = \sup_{x \in E, 0 < \|x\|_E \leq r} \frac{\|\eta(x)\|_F}{\|x\|_E} = \sup_{x \in E, 0 < \|x\|_E < r} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

Proof (of Lemma) $\|\eta\| \geq \sup_{x \in E, 0 < \|x\|_E < r} \frac{\|\eta(x)\|_F}{\|x\|_E}$. $\forall y \in E \setminus \{0\}, \|a^N y\|_E = |a|^N \|y\|_E < r$.

$$\frac{\|\eta(a^N y)\|_F}{\|a^N y\|_E} = \frac{|a|^N \cdot \|\eta(y)\|_F}{|a|^N \cdot \|y\|_E} = \frac{\|\eta(y)\|_F}{\|y\|_E} \leq \sup_{x \in E, 0 < \|x\|_E < r} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

□

Proof (of Proposition) Suppose $\varphi, \psi \in \mathcal{L}(E, F)$ are such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E,$$

$$f(p + h) - f(p) - \psi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

Then

$$\varphi(h) - \psi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

$$\exists r > 0, \exists \varepsilon : \overline{B}(0_E, r) \rightarrow \mathbb{R}_{\geq 0} \text{ such that } \lim_{h \rightarrow 0_E} \varepsilon(h) = 0.$$

$$\forall h \in \overline{B}(0_E, r), \|(\varphi - \psi)(h)\|_F = \varepsilon(h)\|h\|_E.$$

$$\|\varphi - \psi\| = \sup_{x \in E, 0 < \|x\|_E < r'} \frac{\|\varphi(x) - \psi(x)\|_F}{\|x\|_E} \leq \sup_{0 < \|h\|_E < r'} \varepsilon(h).$$

Taking the limit when $r' \rightarrow 0$, by $\limsup_{h \rightarrow 0_E} \varepsilon(h) = 0$. We get $\|\varphi - \psi\| = 0$, hence $\varphi = \psi$. \square

Definition 2.2.4 Let $U \subseteq E$ and $f : U \rightarrow F$ be a mapping that is differentiable at $p \in U^\circ$. The unique $\varphi \in \mathcal{L}(E, F)$ such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E$$

is called the **differential** of f at p and is denoted as

$$D(f(p)).$$

Example 2.2.5

(1) $f : U \rightarrow F$, $f(x) \equiv c$, $c \in F$.

$$f(x + h) - f(x) = 0_E = o(\|h\|_E).$$

So f is differentiable at every point of U and $D(f(x)) = 0_F$.

(2) $\varphi \in \mathcal{L}(E, F)$.

$$\varphi(p + h) - \varphi(p) - \varphi(h) = 0_F = o(\|h\|_E).$$

So φ is differentiable at every point of E and $D(\varphi(p)) = \varphi$.

(3) Let $(F_i, \|\cdot\|_i)$ be normed vector spaces over K , $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$. Suppose that $F = F_1 \oplus \dots \oplus F_n$ and

$$\|(s_1, \dots, s_n)\|_F = \max\{\|s_1\|_1, \dots, \|s_n\|_n\}.$$

Let $U \subseteq E$ be an open subset, $f_i : U \rightarrow F_i$ be a mapping.

$$f : U \rightarrow F, \quad f(x) = (f_1(x), \dots, f_n(x)).$$

$$f(p + h) - f(p) = (f_1(p + h) - f_1(p), \dots, f_n(p + h) - f_n(p)).$$

Suppose that each f_i is differentiable

$$\begin{aligned} & f(p + h) - f(p) - (Df_1(p)(h), \dots, Df_n(p)(h))|_F \\ &= \max_{i \in \{1, \dots, n\}} \|f_i(p + h) - f_i(p) - Df_i(p)(h)\|_{F_i} \\ &= o(\|h\|_E). \end{aligned}$$

So f is differentiable at p and

$$Df(p)(h) = (Df_1(p)(h), \dots, Df_n(p)(h)).$$

(4) Suppose that $E = K$. If $U \subseteq K$ is open and $f : U \rightarrow F$ is differentiable at $p \in U$. We denote by $f'(p)$ the element $Df(p)(1) \in F$.

$$f(p+h) - f(p) - Df(p)(h) = o(\|h\|_E).$$

So

$$f(p+h) - f(p) - hf'(p) = o(\|h\|_E),$$

$$\frac{f(p+h) - f(p)}{h} - f'(p) = o(1).$$

That is,

$$\lim_{h \rightarrow 0} \frac{f(p+h) - f(p)}{h} = f'(p).$$

Theorem 2.2.6 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, $(G, \|\cdot\|_G)$ be normed vector spaces over a complete valued field $(K, |\cdot|)$. Let $U \subseteq E$ and $V \subseteq F$ be open subsets, $f : U \rightarrow F$ and $g : V \rightarrow G$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If f is differentiable at p and g is differentiable at $f(p)$, then $g \circ f : U \rightarrow G$ is differentiable at p and

$$D(g \circ f)(p)(h) = Dg(f(p))(Df(p)(h)).$$

Proof

$$f(p+h) - f(p) - Df(p)(h) = o(\|h\|_E),$$

so,

$$f(p+h) - f(p) = \mathcal{O}(\|h\|_E).$$

$$\begin{aligned} & g(f(p+h)) - g(f(p)) - Dg(f(p))(f(p+h) - f(p)) \\ &= o(\|f(p+h) - f(p)\|_F) = o(\mathcal{O}\|h\|_E) = o(\|h\|_E). \end{aligned}$$

$$\begin{aligned} & Dg(f(p))(f(p+h) - f(p)) - Dg(f(p))(Df(p)(h)) \\ &= Dg(f(p))(f(p+h) - f(p) - Df(p)(h)) \\ &= \mathcal{O}(o(\|h\|_E)) = o(\|h\|_E). \end{aligned}$$

So,

$$g(f(p+h)) - g(f(p)) - Dg(f(p))(Df(p)(h)) = o(\|h\|_E).$$

□

Remark 2.2.7 If $(E, \|\cdot\|_E) = (K, |\cdot|)$,

$$(g \circ f)'(p) = Dg(f(p))(f'(p)).$$

If $E = F = K$, $\|\cdot\|_E = \|\cdot\|_F = |\cdot|$.

$$(g \circ f)'(p) = g'(f(p)) \cdot f'(p).$$

Remark 2.2.8 Let $U \subseteq E$ be open. $f : U \rightarrow F_1 \times \cdots \times F_n$. If f is differentiable at $p \in U$, for any $i \in \{1, \dots, n\}$, the mapping

$$f_i := \pi_i \circ f : U \rightarrow F_i$$

is differentiable at p and

$$D(f_i)(p)(h) = D\pi_i(f(p))(Df(p)(h)) = \pi_i(Df(p)(h)).$$

2.3 Multilineal Mappings

Definition 2.3.1 Let K be a commutative unitary ring. Let $E_1, \dots, E_n; F$ be K -modules. We say that

$$\varphi : E_1 \times \cdots \times E_n \rightarrow F$$

is n -linear if for any $i \in \{1, \dots, n\}$ and any $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in E_1 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_n$, the mapping

$$E_i \rightarrow F, x_i \mapsto \varphi(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$$

is a homomorphism of K -modules. (K -linear mapping)

If $n = 1$, 1-linear is also called linear.

If $n = 2$, 2-linear is also called bilinear.

Example 2.3.2

- (1) $K \times K \rightarrow K$ $(a, b) \mapsto ab$ is bilinear.
- (2) $K^n \times K^n \rightarrow K$ $(x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$ is bilinear.
- (3) $K \times \cdots \times K \rightarrow K$ $(x_1, \dots, x_n) \mapsto x_1 \cdots x_n$ is n -linear.

Definition 2.3.3 We denote by $\text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$ the set of n -linear mappings from $E_1 \times \dots \times E_n$ to F .

Definition 2.3.4 Let $(K, |\cdot|)$ be a complete valued field.

Let $(E_1, \|\cdot\|_{E_1}), \dots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$ be normed vector spaces over K . For any $\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$, we define

$$\|\varphi\| := \sup_{x_i \in E_i \setminus \{0\}, i=1, \dots, n} \frac{\|\varphi(x_1, \dots, x_n)\|_F}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_n}}.$$

We denote by $\mathcal{L}(E_1 \times \dots \times E_n, F)$ the set

$$\{\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F) \mid \|\varphi\| < +\infty\}.$$

$\mathcal{L}^{(n)}(E_1 \times \dots \times E_n, F)$ is a normed vector space of $\text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$, and the norm is $\|\cdot\|$.

Theorem 2.3.5 Let $\varphi \in \mathcal{L}^{(n)}(E_1 \times \dots \times E_n, F)$. For any $p = (p_1, \dots, p_n) \in E_1 \times \dots \times E_n$, φ is differentiable at p and

$$D\varphi(p)(h_1, \dots, h_n) = \sum_{i=1}^n \varphi(p_1, \dots, p_{i-1}, h_i, p_{i+1}, \dots, p_n).$$

Proof

$$\begin{aligned} \varphi(p+h) - \varphi(p) &= \sum_{i=1}^n \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, p_i + h_i, p_{i+1}, \dots, p_n) \\ &\quad - \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, p_i, p_{i+1}, \dots, p_n) \\ &= \sum_{i=1}^n \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, h_i, p_{i+1}, \dots, p_n) \\ &= \sum_{i=1}^n \varphi(p_1, \dots, p_{i-1}, h_i, p_{i+1}, \dots, p_n) + o(\|h_i\|). \end{aligned}$$

□