FUNDAMENTAL ALGEBRA & ANALYSIS

Contents

Chapter 1

Limit

1.1 Filters

Definition 1.1.1 Let X be a set. We call **filter** on X any non-empty subset $\mathcal{F}$ of $\wp(X)$ this satisfies:
(1) $\forall (V_1, V_2) \in \mathcal{F}^2, V_1 \cap V_2 \in \mathcal{F}$.
(2) $\forall V \in \mathcal{F}, \forall W \in \wp(X)$, if $V \subseteq W$, then $W \in \mathcal{F}$.

Remark 1.1.2
If $\varnothing \in \mathcal{F}$, then $\mathcal{F} = \wp(X)$, we say that $\mathcal{F}$ is degenerate.

Example 1.1.3 If $Y \subseteq X$, then

$$\mathcal{F}_Y := \{V \in \wp(X) \mid Y \subseteq V\}$$

is a filter, called the principal filter of Y.

If $\mathcal{F}$ is a non-degenerate filter such that, for any non-degenerate filter $\mathcal{G}$, one has $\mathcal{F} \nsubseteq \mathcal{G}$. We say that $\mathcal{F}$ is an **ultrafilter**.

Proposition 1.1.4 Let I be a non-empty set and $(\mathcal{F}_i)_{i\in I}$ is a family of filters on X, then $\mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i$ is also a filter on X.

Proof
(1) $\forall (V_1, V_2) \in \mathcal{F}^2$, one has

$$\forall i \in I, (V_1, V_2) \in \mathcal{F}_i^2,$$

so $V_1 \cap V_2 \in \mathcal{F}_i$. This leads to $V_1 \cap V_2 \in \mathcal{F}$.
(2) $\forall V \in \mathcal{F}$, one has $\forall i \in I, V \in \mathcal{F}_i$. If $W \in \wp(X), W \supseteq V$, then $\forall i \in I, W \in \mathcal{F}_i$. □

Definition 1.1.5 Let S be a subset of $\wp(X)$. We denote by $\mathcal{F}_S$ the intersection of all filters containing S. It is thus the least filter containing S. We call it the filter generated by S.

Remark 1.1.6 If $Y \subseteq X$, then the principal filter $\mathcal{F}_Y$ is generated by $\{Y\}$.

Proposition 1.1.7 Let X be a set and S be a non-empty subset of $\wp(X)$, then

$$\mathcal{F}_S := \{U \in \wp(X) \mid \exists n \in \mathbb{N}_{\geq 1}, \exists (A_1, \ldots, A_n) \in S^n, A_1 \cap \cdots \cap A_n \subseteq U\}.$$

Proof Denote by $\mathcal{F}'_S$ the set on the right hand side of the equality. One has $\mathcal{F}'_S \subseteq \mathcal{F}_S$. It remains to check that $\mathcal{F}'_S$ is a filter containing S. By definition, $S \subseteq \mathcal{F}'_S$. If $(U, V) \in \mathcal{F}'^2_S, \exists A_1, \ldots, A_n, B_1, \ldots, B_n \in S, A_1 \cap \cdots \cap A_n \subseteq U, B_1 \cap \cdots \cap B_n \subseteq V$, so $A_1 \cap \cdots \cap A_n \cap B_1 \cap \cdots \cap B_n \subseteq U \cap V$. If $W \supseteq U$, then $A_1 \cap \cdots \cap A_n \subseteq W$, so $W \in \mathcal{F}'_S$. □

Definition 1.1.8 We say that a subset S of $\wp(X)$ is a **filter basis** if, for any $(A, B) \in S \times S$, there exists $C \in S$, such that $C \subseteq A \cap B$.[a]

[a] If $n \in \mathbb{N}_{\geq 1}$ and $(A_1, \ldots, A_n) \in S^n$, $\exists C \in S$ such that $C \subseteq A_1 \cap \cdots \cap A_n$.

Remark 1.1.9 If S is a filter basis, then

$$\mathcal{F}_S = \{U \in \wp(X) \mid \exists A \in S, A \subseteq U\}.$$

If S is a subset of $\wp(X)$, then

$$\mathcal{B}_S := \{A_1 \cap \cdots \cap A_n \mid n \in \mathbb{N},\ (A_1, \ldots, A_n) \in S^n\}$$

is a filter basis containing S. Moreover, $\mathcal{F}_S = \mathcal{F}_{\mathcal{B}_S}$.

Proposition 1.1.10 Let X be a set. Then

$$\mathcal{F} = \{U \in \wp(X) \mid X \backslash U \text{ is finite}\}$$

is a filter on X. We call it the **Fréchet filter** of X.

Proof
If $(U, V) \in \mathcal{F}^2, X\backslash(U \cap V) = (X\backslash U) \cup (X\backslash V)$, is finite.
If $U \in \mathcal{F}, W \in \wp(X), U \subseteq W$, then $(X\backslash W) \subseteq (X\backslash U)$ is finite. □

Example 1.1.11 Let $I \subseteq \mathbb{N}$ be an infinite set. Let $J \subseteq \mathbb{N}$ be infinite, then $\{I_{\geq j} \mid j \in J\}$ is a filter basis that generates the Fréchet filter of I. $\{I_{\geq j} \mid j \in J\}$ is a totally ordered subset of $\wp(I)$, so it is a filter basis. For any $j \in J, I\backslash I_{\geq j} = I_{<j}$ is finite. Let $U \in$ Fréchet filter of I, $I\backslash U$ is finite. There exists $j \in J$ such that $\forall i \in I\backslash U, i < j$. So $I\backslash U \subseteq I_{<j}, U \supseteq I\backslash I_{j<} = I_{\geq j}$.

Example 1.1.12 Let X be a set. We call **pseudometric** on X any mapping

$$\mathrm{d} : X \times X \to \mathbb{R}_{\geq 0}.$$

such that,
(1) $\forall x \in X, \mathrm{d}(x, x) = 0.$
(2) $\forall (x, y) \in X^2, \mathrm{d}(x, y) = \mathrm{d}(y, x).$
(3) (Triangle inequality) $\forall (x, y, z) \in X^3, \mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z).$
(X, d) is called the **pseudometric space**. If

$$\forall (x, y) \in X^2, x \neq y \Rightarrow \mathrm{d}(x, y) > 0,$$

then (X, d) is called a **metric space**.
Let (X, d) be a pseudometric space. For any $x \in X$, and $\varepsilon \in \mathbb{R}_{\geq 0}$, we denote by $B(x, \varepsilon)$ the set

$$\{y \in X \mid \mathrm{d}(x, y) < \varepsilon\},$$

called the **open ball** *center at x of radius ε.*
Then

$$\mathcal{V}_x := \{U \in \wp(X) \mid \exists \varepsilon \in \mathbb{R}_{>0}, B(x, \varepsilon) \subseteq U\}$$

is a filter, called the **filter of neighborhood** of x.

Proposition 1.1.13 Let $J \subseteq \mathbb{R}_{>0}$ be a non-empty subset such that $\inf J = 0$. Then $\mathcal{B}_J = \{B(x, \varepsilon) \mid \varepsilon \in J\}$ is a filter basis such that $\mathcal{F}_{\mathcal{B}_J} = \mathcal{V}_x$.

Proof $\forall U \in \mathcal{V}_x, \exists \varepsilon \in J, \varepsilon < \delta,$

$$B(x, \varepsilon) \subseteq B(x, \delta) \subseteq U.$$

□

1.2 Order Limit

We fix a partially ordered set $(G, \leq)$ assumed to be order complete.

Example 1.2.1
(1) $\mathbb{R} \cup \{-\infty, +\infty\},\ \forall x \in \mathbb{R}, -\infty < x < +\infty$.
(2) $[0, +\infty]$.
(3) $(\wp(\Omega), \subseteq)$.

Definition 1.2.2 Let X be a set and $f : X \longrightarrow G$ be a mapping. For any $U \in \wp(X)$, we define

$$f^s(U) := \sup_{x \in U} f(x) = \sup f(U).$$

$$f^i(U) := \inf_{x \in U} f(x) = \inf f(U).$$

If $U \neq \varnothing$, $f^s(U) \geq f^i(U)$. Let $\mathcal{F}$ be a filter on X. We define

$$\limsup_{\mathcal{F}} f := \inf_{U \in \mathcal{F}} f^s(U).$$

$$\liminf_{\mathcal{F}} f := \sup_{U \in \mathcal{F}} f^i(U).$$

They are called the **superior limit** and the **inferior limit** of f along $\mathcal{F}$. If

$$\liminf_{\mathcal{F}} f = \limsup_{\mathcal{F}} f,$$

we say that f has a limit along $\mathcal{F}$, and we denote $\lim\limits_{\mathcal{F}} f$ this value.

Notation 1.2.3 Let $I \subseteq \mathbb{N}$ be an infinite subset. We call sequence in G parametrized by I any element of $G^I = \{(a_n)_{n \in I} \mid \forall n \in I, a_n \in G\}$. If $\mathcal{F}$ is the Fréchet filter on I, then for any $f = (a_n)_{n \in I} \in G^I$, $\limsup_{\mathcal{F}} f$ is denote as $\limsup\limits_{n \in I, n \to +\infty} a_n$ or as $\limsup\limits_{n \to +\infty} a_n$. Resp. $\liminf$.

Proposition 1.2.4 Let $f : X \longrightarrow G$ be a mapping and $\mathcal{F}$ be a non-degenerate filter. Then

$$\forall (U, V) \in \mathcal{F} \times \mathcal{F}, f^s(U) \geq f^i(V).$$

In particular

$$\limsup_{\mathcal{F}} f \geq \liminf_{\mathcal{F}} f.$$

Proof

$$f^s(U) \geq f^s(U \cap V) \geq f^i(U \cap V) \geq f^i(V).$$

Taking $\inf_{U \in \mathcal{F}}$, we get $\forall V \in \mathcal{F}$, $\limsup_{\mathcal{F}} f \geq f^i(V)$. Taking $\sup_{V \in \mathcal{F}}$, we get $\limsup_{\mathcal{F}} f \geq \liminf_{\mathcal{F}} f$. $\square$

Proposition 1.2.5 Let $f : X \longrightarrow G$ be a mapping, $\mathcal{B}$ be a filter basis on X and $\mathcal{F}$ be the filter generated by $\mathcal{B}$. Then

$$\limsup_{\mathcal{F}} f = \inf_{B \in \mathcal{B}} f^s(B), \ \liminf_{\mathcal{F}} f = \sup_{B \in \mathcal{B}} f^i(B).$$

Proof Since $\mathcal{B} \subseteq \mathcal{F}$, one has

$$\limsup_{\mathcal{F}} f = \inf_{U \in \mathcal{F}} f^s(U) \leq \inf_{B \in \mathcal{B}} f^s(B).$$

For any $U \in \mathcal{F}, \exists A \in \mathcal{B}$ such that $U \supseteq A$. One has

$$f^s(U) \geq f^s(A) \geq \inf_{B \in \mathcal{B}} f^s(B).$$

Taking $\inf_{U \in \mathcal{F}}$, we get

$$\limsup_{\mathcal{F}} f \geq \inf_{B \in \mathcal{B}} f^s(B).$$

$\square$

Consequence: If $I \subseteq \mathbb{N}$ is an infinite subset, $J \subseteq \mathbb{N}$ is another infinite subset, $\forall (a_n)_{n \in I} \in G^I$,

$$\limsup_{n \in I, n \to +\infty} a_n = \inf_{j \in J} \sup_{n \in I_{\geq j}} a_n,$$

$$\liminf_{n \in I, n \to +\infty} a_n = \sup_{j \in J} \inf_{n \in I_{\geq j}} a_n.$$

Example 1.2.6 $a_n = (-1)^n, (a_n)_{n \in \mathbb{N}} \in [-\infty, +\infty]^N$,

$$\limsup_{n \to +\infty} (-1)^n = \inf_{j \in 2\mathbb{N}} \sup_{n \geq j} (-1)^n = \inf_{j \in 2\mathbb{N}} 1 = 1.$$

$$\liminf_{n\to+\infty}(-1)^n = -1.$$

Example 1.2.7 $\left(\frac{1}{n}\right)_{n\in\mathbb{N}_{\geq 1}}$,

$$\limsup_{n\to+\infty}\frac{1}{n} = \inf_{j\in\mathbb{N}_{\geq 1}}\sup_{n\geq j}\frac{1}{n} = \inf_{j\in\mathbb{N}_{\geq 1}}\frac{1}{j} = 0,$$

$$\liminf_{n\to+\infty}\frac{1}{n} = \sup_{j\in\mathbb{N}_{\geq 1}}\inf_{n\geq j}\frac{1}{n} = \sup_{j\in\mathbb{N}_{\geq 1}} 0 = 0.$$

Proposition 1.2.8 Let $f, g : X \longrightarrow G$ be mappings and $\mathcal{F}$ be a filter on X. Suppose that there exists $A \in \mathcal{F}$ such that

$$\forall x \in A, f(x) \leq g(x).$$

Then,

$$\limsup_{\mathcal{F}} f \leq \limsup_{\mathcal{F}} g,\ \liminf_{\mathcal{F}} f \leq \liminf_{\mathcal{F}} g.$$

Proof Let

$$\mathcal{B} = \{U \in \mathcal{F} \mid U \subseteq A\}.$$

$\mathcal{B}$ is a filter basis, and $\mathcal{B} \in \mathcal{F}$. For any $V \in \mathcal{F}$, one has $V \cap A \in \mathcal{B}$ and $V \supseteq V \cap A$. So $\mathcal{F}$ is generated by $\mathcal{B}$. For any $B \in \mathcal{B}$, one has $B \subseteq A$ and hence

$$f^s(B) \leq g^s(B),\ f^i(B) \leq g^i(B).$$

So

$$\inf_{B\in\mathcal{B}} f^s(B) \leq \inf_{B\in\mathcal{B}} g^s(B),\ \sup_{B\in\mathcal{B}} f^i(B) \leq \sup_{B\in\mathcal{B}} g^i(B).$$

□

Theorem 1.2.9 (Squeeze Theorem) Let X be a set and $\mathcal{F}$ be a non-degenerate filter on X. Let f, g, h be elements of G^X. Assume that there exists $A \in \mathcal{F}$ such that

$$\forall x \in A, f(x) \leq g(x) \leq h(x).$$

If f and h have limits along $\mathcal{F}$, and

$$\lim_{\mathcal{F}} f = \lim_{\mathcal{F}} h,$$

then, g also has a limit along $\mathcal{F}$, and

$$\lim_{\mathcal{F}} f = \lim_{\mathcal{F}} g = \lim_{\mathcal{F}} h.$$

Proof

$$\lim_{\mathcal{F}} f = \limsup_{\mathcal{F}} f \leq \limsup_{\mathcal{F}} g \leq \limsup_{\mathcal{F}} h = \lim_{\mathcal{F}} h.$$

So

$$\limsup_{\mathcal{F}} g = \lim_{\mathcal{F}} f = \lim_{\mathcal{F}} h.$$

$$\lim_{\mathcal{F}} f = \liminf_{\mathcal{F}} f \leq \liminf_{\mathcal{F}} g \leq \liminf_{\mathcal{F}} h = \lim_{\mathcal{F}} h.$$

So

$$\liminf_{\mathcal{F}} g = \lim_{\mathcal{F}} f = \lim_{\mathcal{F}} h.$$

$\square$

Example 1.2.10 Let $a > 1$. Consider the sequence $\left(\frac{a^n}{n!}\right)_{n\in\mathbb{N}}$. If $n \geq N \geq 2a$, $a \leq \frac{N}{2}$, then

$$0 \leq \frac{a^n}{n!} \leq \frac{a^N}{N!} \cdot \frac{a^{n-N}}{(N+1)\dots n} \leq \frac{a^N}{N!}\frac{1}{2^{n-N}}.$$

For any $n \geq N$, $0 \leq \frac{a^n}{n!} \leq \frac{(2a)^N}{N!} \cdot \frac{1}{2^n}$. So by squeeze theorem, $\lim\limits_{n\to+\infty} \dfrac{a^n}{n!} = 0$.

Theorem 1.2.11 (Monotone Convergence Theorem) Let I be an infinite subset of $\mathbb{N}$ and $(a_n)_{n\in I} \in G^I$.
(1) If $(a_n)_{n\in I}$ is increasing, then $(a_n)_{n\in I}$ admits $\sup\limits_{n\in I} a_n$ as its limit.
(2) If $(a_n)_{n\in I}$ is decreasing, then $(a_n)_{n\in I}$ admits $\inf\limits_{n\in I} a_n$ as its limit.

Proof
(1) Let $l = \sup_{n\in I} a_n$, $\forall n \in \mathbb{N},\ a_n \leq l$. So

$$\limsup_{n\to+\infty} a_n \leq \limsup_{n\to+\infty} l = l.$$

$$\forall j \in I,\ \inf_{n\in I_{\geq j}} a_n = a_j,$$

so

$$\liminf_{n\to+\infty} a_n = \sup_{j\in I} \inf_{n\in I_{\geq j}} a_n = \sup_{j\in I} a_j = l.$$

Hence,

$$l = \liminf_{n\to+\infty} a_n \le \limsup_{n\to+\infty} a_n \le l.$$

Which means

$$\lim_{n\to+\infty} a_n = l.$$

□

Proposition 1.2.12 Let X be a set and $Y \subseteq X$.
(1) If $\mathcal{F}$ is a filter on X, then

$$\mathcal{F}|_Y := \{U \cap Y \mid U \in \mathcal{F}\}$$

is a filter on Y.
(2) If $\mathcal{B}$ is a filter basis on X, and $\mathcal{F}$ is the filter generated by $\mathcal{B}$, then

$$\mathcal{B}|_Y := \{B \cap Y \mid B \in \mathcal{B}\}$$

is a filter basis generates $\mathcal{F}|_Y$.

Proof
(1) Let U and V be elements of $\mathcal{F}$, one has

$$(U \cap Y) \cap (V \cap Y) = (U \cap V) \cap Y \in \mathcal{F}|_Y .$$

Let $U \in \mathcal{F}, W \subseteq Y, U \cap Y \subseteq W$. Let $V = U \cup W \in \mathcal{F}$.

$$Y \cap V = (U \cap Y) \cup (W \cap Y) = W.$$

Hence $W \in \mathcal{F}|_Y$.
(2) Let B_1, B_2 be elements of $\mathcal{B}$, then $\exists A \in B, A \subseteq B_1 \cap B_2$. Thus

$$A \cap Y \subseteq (B_1 \cap Y) \cap (B_2 \cap Y).$$

So $\mathcal{B}|_Y$ is a filter basis. Moreover, $\mathcal{B}|_Y \subseteq \mathcal{F}|_Y$. Let $U \in \mathcal{F}, \exists B \in \mathcal{B}$ such that $B \subseteq U$. Thus

$$B \cap Y \subseteq U \cap Y.$$

So $U \cap Y$ contains an element of $\mathcal{B}|_Y$. □

Example 1.2.13 Let $I \subseteq \mathbb{N}$ be an infinite subset, and $(a_n)_{n\in I} \in G^I$. If $J \subseteq I$ is an infinite subset, $\mathcal{F}$ be the filter on I, then $\mathcal{F}|_J$ is the Fréchet filter on J. $(a_n)_{n\in J}$ is called a subsequence of $(a_n)_{n\in I}$.

Proposition 1.2.14 Let $f : X \longrightarrow G$ be a mapping, $\mathcal{F}$ be a filter on X, $Y \subseteq X$. Then
(1)

$$\limsup_{\mathcal{F}|_Y} f|_Y \leq \limsup_{\mathcal{F}} f,$$

$$\liminf_{\mathcal{F}|_Y} f|_Y \geq \liminf_{\mathcal{F}} f.$$

(2) Suppose that $\mathcal{F}|_Y$ is non-degenerate and f has a limit along $\mathcal{F}$, then $f|_Y$ has a limit along $\mathcal{F}|_Y$ and

$$\lim_{\mathcal{F}} f = \lim_{\mathcal{F}|_Y} f|_Y.$$

(3) If $Y \in \mathcal{F}$, then

$$\limsup_{\mathcal{F}|_Y} = \limsup_{\mathcal{F}} f,$$

$$\liminf_{\mathcal{F}|_Y} = \liminf_{\mathcal{F}} f.$$

Proof
$\forall U \in \mathcal{F}, f^s(U \cap Y) \leq f^s(U)$. So

$$\limsup_{\mathcal{F}|_Y} f|_Y = \inf_{U\in\mathcal{F}} f^s(U \cap Y) \leq \inf_{U\in\mathcal{F}} f^s(U) = \limsup_{\mathcal{F}} f.$$

(2)

$$\lim_{\mathcal{F}} f = \limsup_{\mathcal{F}} f \geq \limsup_{\mathcal{F}|_Y} f|_Y \geq \liminf_{\mathcal{F}|_Y} f|_Y \geq \liminf_{\mathcal{F}} f = \lim_{\mathcal{F}} f.$$

(3) $\mathcal{F}|_Y$ is a filter basis that generates $\mathcal{F}$ if $Y \in \mathcal{F}$,

$$\limsup_{\mathcal{F}|_Y} f|_Y = \inf_{V\in\mathcal{F}|_Y} f^s(U) = \inf_{U\in\mathcal{F}} f^s(U) = \limsup_{\mathcal{F}} f.$$

□

Theorem 1.2.15 (Bolzano-Weierstrass) Suppose that G is totally ordered. Let $I \subseteq \mathbb{N}$ be an infinite subset and $(a_n)_{n\in I}$ be a sequence in G.

(1) There exists an infinite subset J_1 such that $(a_n)_{n\in I}$ is monotone and admits $\limsup_{n\in I, n\to+\infty} a_n$ as its limit.
(2) There exists an infinite subset J_2 such that $(a_n)_{n\in I}$ is monotone and admits $\liminf_{n\in I, n\to+\infty} a_n$ as its limit.

Proof
(1) Let

$$J = \{n \in I \mid \forall m \in I_{\geq n}, a_m \leq a_n\}.$$

If J is infinite, $(a_n)_{n\in J}$ is decreasing. Hence it admits

$$\alpha := \inf_{n\in J} a_n$$

as its limit. For any $n \in J$, $\sup_{m\in I_{\geq n}} a_m = a_n$. So

$$\limsup_{n\in I, n\to+\infty} a_n = \inf_{n\in J} \sup_{m\in I_{\geq n}} a_m = \alpha.$$

Suppose that J is finite. Pick $n_0 \in I$ such that $\forall j \in J, j < n_0$. We construct in a recursive way a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in I as follows: Suppose $n_0 < n_1 < \cdots < n_k$ have been chosen. Since G is totally ordered, there exists $i \in I$ such that $n_0 \leq i \leq n_k$ and

$$a_i = \max\{a_j \mid j \in I, n_0 \leq j \leq n_k\}.$$

Since $i \notin J$, there exists $n_{k+1} \in I, n_{k+1} > i$ such that

$$a_{n_{k+1}} > a_i.$$

Note that $n_{k+1} > n_k$. Let

$$J_1 = \{n_k \mid k \in \mathbb{N}\},$$

$(a_n)_{n\in J_1}$ is increasing, hence it admits

$$\beta := \sup_{n\in J} a_n$$

as its limit. For any $j \in I$ such that $j \geq n_0$, there exists $k \in \mathbb{N}$ such that $j \leq n_k$. Thus $a_j \leq a_{n_{k+1}} \leq \beta$. So $\limsup_{n\in I, n\to+\infty} a_n \leq \beta$. Moreover, since $J_1 \subseteq I$,

$$\beta = \lim_{n\in J_1, n\to+\infty} a_n = \limsup_{n\in J_1, n\to+\infty} a_n \leq \limsup_{n\in I, n\to+\infty} a_n.$$

Therefore,

$$\beta = \limsup_{n\in I, n\to+\infty} a_n.$$

$\square$

1.3 Partially Ordered Groups

Definition 1.3.1 Let $(G, *)$ be a group, and $\leq$ be a partial order on G. If

$$\forall (a, b, c) \in G^3, a < b \Rightarrow a * c < b * c \text{ and } c * a < c * b,$$

we say that $(G, *, \leq)$ is a **partially ordered group**. If in addition $\leq$ is a total order, we say that $(G, *, \leq)$ is a **totally ordered group**. (Resp. semigroup, monoid.)

Example 1.3.2 (1) $(\mathbb{R}, +, \leq)$. (2) $(\mathbb{R}_{>0}, \cdot, \leq)$. (3) $(\mathbb{N}\backslash\{0\}, \cdot, |)$.

Remark 1.3.3
(1) If $(G, *)$ is a partially ordered group, then

$$\forall (a, b, c) \in G^3, a \leq b \Rightarrow a * c \leq b * c,\ c * a \leq c * b.$$

(2) $(G, \hat{*}, \leq)$ is a partially ordered group.
Resp. semigroup, monoid.

Proposition 1.3.4
Let $(G, *, \leq)$ be a partially ordered semigroup. Let $(a_1, a_2, b_1, b_2) \in G^4$.
(1) If $a_1 \leq a_2,\ b_1 \leq b_2$, then $a_1 * b_1 \leq a_2 * b_2$.
(2) If $a_1 < a_2,\ b_1 \leq b_2$, then $a_1 * b_1 < a_2 * b_2$.
(3) If $a_1 \leq a_2,\ b_1 < b_2$, then $a_1 * b_1 < a_2 * b_2$.

Proof
(1) $a_1 * b_1 \leq a_2 * b_1 \leq a_2 * b_2$
(2),(3) At least one of the above inequality is strict. □

Proposition 1.3.5 Let $(G, *, \leq)$ be a partially ordered semigroup, $(x, y, a) \in G^3$. Assume that, either $\leq$ is a total order, or $(G, *)$ is a monoid and $a \in G^\times$. Then the following conditions are equivalent:
(1) $x \leq y$.
(2) $x * a \leq y * a$.
(3) $a * x \leq a * y$.

Proof By definition, $(1) \Rightarrow (2)$, $(1) \Rightarrow (3)$. Assume that $x * a \leq y * a$. If $(G, *)$

is a monoid and $a \in G^\times$, then

$$x = (x * a) * \iota(a) \leq (y * a) * \iota(a) = y.$$

Suppose that $\leq$ is a total order. If $x \not\leq y$, then $x > y$ and $x * a > y * a$, contradiction. $\square$

Corollary 1.3.6 A totally ordered semigroup satisfies the left and right cancellation laws.

Proof Let $(G, *, \leq)$ be a totally ordered semigroup. Let $(x, y, a) \in G^3$ such that $x * a = y * a$. Then

$$x * a \leq y * a, y * a \leq x * a.$$

Hence $x \leq y$ and $y \leq x$. $\square$

Proposition 1.3.7 Let $(G, *, \leq)$ be a partially ordered monoid. Then,

$$\iota : G^\times \longrightarrow G^\times$$

is strictly decreasing.

Proof Let $(x, y) \in G^\times \times G^\times$ such that $x < y$. Then

$$e = \iota(x) * x < \iota(x) * y,$$

where e is the neutual element of $(G, *)$. Thus

$$e * \iota(y) < \iota(x) * y * \iota(y).$$

That is $\iota(y) < \iota(x)$. $\square$

Proposition 1.3.8 Let $(G, *, \leq)$ be a totally ordered group, and e be the neutual element of $(G, *)$. If $G \neq \{e\}$, then G has neither a greatest element nor a least element.

Proof Suppose that $(G, \leq)$ has a greastest element β. We first show by contradiction that $\beta \neq e$. Suppose that $e = \max G$. Pick $x \in G, x \neq e$. Then $x < e$.

Thus $\iota(x) > \iota(e)$. Contradiction. If $\beta > e$, $\beta * \beta > e * \beta = \beta$, contradiction, too. $\square$

1.4 Enhancement

Definition 1.4.1 Let $(S, *, \leq)$ be a partially ordered semigroup. Suppose that $(S, \leq)$ has no greastest element and has no least element. Let $\perp$ and $\top$ be formal elements and let

$$\bar{S} = S \cup \{\perp, \top\}.$$

We extend $\leq$ to $\bar{S}$ by letting $\perp < x < \top, \forall x \in S$. We extend $*$ to a mapping

$$\left(\bar{S} \times \bar{S}\right) \setminus \{(\perp, \top), (\top, \perp)\} \longrightarrow \bar{S},$$

such that

$$\forall x \in S \cup \{\top\}, x * \top = \top * x = \top.$$

$$\forall x \in S \cup \{\perp\}, x * \perp = \perp * x = \perp.$$

$\top * \perp$ and $\perp * \top$ are NOT DEFINED. $(\bar{S}, *, \leq)$ is called the **enhancement** of $(S, *, \leq)$. If A and B are subset of $\bar{S}$, we denote by $A * B$ the set

$$\{x * y \mid (x, y) \in A \times B \, , \{x, y\} \neq \{\perp, \top\}\}.$$

Example 1.4.2
(1) $(\mathbb{R}, +, \leq), \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.
(2) $(\mathbb{R}_{>0}, \cdot, \leq), \bar{\mathbb{R}}_{>0} = \mathbb{R}_{>0} \cup \{0, +\infty\}$.

Remark 1.4.3
(1) $\forall (a, b) \in \bar{S} \times \bar{S}, a * b$ is defined if and only if $b * a$ is defined.
(2) If $*$ is commutative, and $a * b$ is defined, then $a * b = b * a$.

Definition 1.4.4 Let $a_0, \ldots, a_n$ be elements of $\bar{S}$, if $a_0 * \cdots * a_n$ is defined, and $(a_0, \ldots, a_{n-1}) * a_n$ is also defined, then we let $a_0 * \cdots * a_n = (a_0 * \cdots * a_{n-1}) * a_n$.

Proposition 1.4.5 Let $a_0, \ldots, a_n$ be elements of $\bar{S}$. For any $i \in \{0, \ldots, n\}$, $a_0 * \cdots * a_{i-1} * (a_i * a_{i+1}) * \cdots * a_n$ is defined if and only if $a_0 * \cdots * a_n$ is defined. Moreover, $a_0 * \cdots * a_n = a_0 * \ldots (a_{i-1} * a_i) * \cdots * a_n$.

Proof Both terms are defined is and only if

$$\{\top, \perp\} \not\subseteq \{a_0, \ldots, a_n\}.$$

If $\perp \in \{a_0, \ldots, a_n\}$, then both terms are equal to $\perp$. If $\top \in \{a_0, \ldots, a_n\}$, then both terms are equal to $\top$. $\square$

Proposition 1.4.6 Let $(S, *, \leq)$ be a partially ordered semigroup. Let $(a, b) \in \bar{S} \times \bar{S}$. If $a < b$, then for any $c \in S$, $a * c < b * c, c * a < c * b$.

Proof If $\{a, b\} \subseteq S$, this follows from the definition of a partially ordered semigroup. If $a = \perp$, then $b > \perp$. $a * c = c * a = \perp$. $b * c \neq \perp, c * b \neq \perp$. So $a*c < b*c, c*a < c*b$. If $\{a, b\} \subseteq S$ and $a \neq \perp$, then $b = \top$, $b*c = c*b = \top$. $a * c \neq \top, c * a \neq \top$. So $a * c < b * c$, $c * a < c * b$. $\square$

Proposition 1.4.7
Let $(S, *, \leq)$ be a partially ordered semigroup and $(x, y, a, b) \in \bar{S}^4$.
(1) If $x < a$ and $y < b$, then $x * y$ and $a * b$ are defined, and $x * y < a * b$.
(2) If $x \leq a, y \leq b$ and $x * y$ and $a * b$ are defined, $x * y \leq a * b$.

Proof
(1) Since $x < a, y < b, \top \notin \{x, y\}, \perp \notin \{a, b\}$. So $x * y$ and $a * b$ are defined. If $\top \in \{a, b\}$, then $a * b = \top$. Since $\top \notin \{x, y\}, x * y \neq \top$, so $x * y < a * b$. If $\perp \in \{x, y\}$, then $x * y = \perp$. Since $\perp \notin \{a, b\}$, $a * b \neq \perp$. So $x * y < a * b$. If $\top \notin \{a, b\}, \perp \notin \{x, y\}$,, then $\{x, y, a, b\} \subseteq S$. So $x * y < x * b < a * b$.
(2) If $\top \in \{x, y\}$, then $\top \in \{a, b\}$, so $x * y = \top = a * b$. If $\perp \in \{x, y\}$, then $\perp \in \{a, b\}$, so $x * y = \perp = a * b$. If $\top \in \{x, y\}$, then $x * y = \perp \leq a * b$. If $\perp \in \{x, y\}$, then $x*y \geq \top = a*b$. If $\{x, y, a, b\} \subseteq S$, then $x*y \leq a*y \leq a*b$.
$\square$

Proposition 1.4.8 Let $(S, *, \leq)$ be a partially ordered monoid and $e \in S$ be the neutual element. Let $(a, b) \in \bar{S} \times \bar{S}$, with $a \in S^{\times} \cup \{\perp, \top\}$. Then the following conditions are equivalent.
(1) $a < b$.
(2) $\iota(a) * b$ is defined and $e < \iota(a) * b$. (Where $\iota(\perp) = \top, \iota(\top) = \perp$.)
(3) $b * \iota(a)$ is defined and $e < b * \iota(a)$.

Proof Suppose that $a < b$. Then a and b cannot be both $\top$ or be both $\bot$. Hence $\{\iota(a), b\} \neq \{\bot, \top\}$. Therefore, $\iota(a) * b$ and

$$b * \iota(a)$$

are defined. If $\{a, b\} \subseteq S$, then $e = \iota(a) * a < \iota(a) * b$, $e = a * \iota(a) < b * \iota(a)$. If $a = \bot$, then $\iota(a) = \top$ and $\iota(a) * b = b * \iota(a) = \top$. So $e < \iota(a) * b, e < b * \iota(a)$. If $b = \top$, then $\iota(a) * b = b * \iota(a) = \top > e$.
Assume (2). $\iota(a) * b$ is defined and $e < \iota(a) * b$. If $a \in S$,

$$a = a * e < a * (\iota(a) * b) = (a * \iota(a)) * b = e * b = b.$$

If $a = \bot, \iota(a) = \top, b \neq \bot$, so $a < b$. If $a = \top, \iota(a) = \bot, \iota(a) * b = \bot < e$, contradiction. □

Corollary 1.4.9 Let $(S, *, \leq)$ be a partially ordered monoid and $(a, b) \in (S^\times \cup \{\bot, \top\})^2$. Then $a < b$ if and only if $\iota(a) > \iota(b)$.

Proof If $a < b$,then $\iota(a) * b$ is defined and

$$e < \iota(a) * b = \iota(a) * \iota(\iota(b)).$$

So $\iota(b) < \iota(a)$. □

Lemma 1.4.10 Let $(S, *, \leq)$ be a partially ordered monoid. Let $A \subseteq \bar{S}, b \in S^\times \cup \{\bot, \top\}$. Then the following statements hold:
(1) If $\sup(A) * b$ is defined, then $A * \{b\}$ has a supremum in $\bar{S}$, and

$$\sup(A * \{b\}) = \sup(A) * b.$$

(2) If $\inf(A) * b$ is defined, then $A * \{b\}$ has a infimum in $\bar{S}$, and

$$\inf(A * \{b\}) = \inf(A) * b.$$

(3) If $b * \sup(A)$ is defined, then $A * \{b\}$ has a supremum in $\bar{S}$, and

$$\sup(\{b\} * A) = b * \sup(A).$$

(4) If $b * \inf(A)$ is defined, then $A * \{b\}$ has a infimum in $\bar{S}$, and

$$\inf(\{b\} * A) = b * \inf(A).$$

Proof
(1) Suppose that $b = \perp$, then $\sup(A) \neq \top$, $\sup(A) * b = \perp$. $A * \{b\} \subseteq \{\perp\}$, so $\sup(A * \{b\}) = \perp$. Suppose that $b = \top$, then $\sup(A) \neq \perp$, $A \neq \varnothing$, $\sup(A) * b = \top$. So $\sup(A * \{b\}) = \top = \sup(A) * \top$. We suppose that $b \in S^{\times}$. $\forall a \in A,\ A \leq \sup(A)$, so $a * b \leq \sup(A) * b$. This means that $\sup(A) * b$ is an upper bound of $A * \{b\}$. Let M be an upper bound of $A * \{b\}$. For any $a \in A$, $a \times b \in A * \{b\}$, so $a * b \leq M$. Hence,

$$a = (a * b) * \iota(b) \leq M * \iota(b).$$

We then deduce $\sup(A) \leq M * \iota(b)$. Hence $\sup(A) * b \leq M * \iota(b) * b$. Therefore, $\sup(A) * b$ is the supremum of $A * \{b\}$. □

Remark 1.4.11 Consider

$$S = \{0\} \cup [2, 3[\cup [4, +\infty[\subseteq \mathbb{R}.$$

$A = [2, 3[, \sup(A) = 4, A + \{2\} = [4, 5[, \sup(A + \{2\}) = 5, \sup(A) + 2 = 6.$

Theorem 1.4.12 Let $(S, *, \leq)$ be a partially ordered group. Let A and B be subsets of $\bar{S}$.
(1) If $\sup(A) * \sup(B)$ is defined, then $A * B$ has a supremum in $\bar{S}$ and

$$\sup(A * B) = \sup(A) * \sup(B).$$

(2) If $\inf(A) * \inf(B)$ is defined, then $A * B$ has a infimum in $\bar{S}$ and

$$\inf(A * B) = \inf(A) * \inf(B).$$

Proof For any $(a, b) \in A \times B$, if $a * b$ id defined, then

$$a * b \leq \sup(A) * \sup(B).$$

So $\sup(A) * \sup(B)$ is an upper bound of $A * B$. If $\perp \in \{\sup(A), \sup(B)\}$, then $A * B$ has $\perp$ as an upper bound. So $\sup(A * B) = \perp = \sup(A) * \sup(B)$. We suppose that $\perp \notin \{\sup(A) * \sup(B)\}$. Thus $A \backslash \{\perp\} \neq \varnothing$, $B \backslash \{\perp\} \neq \varnothing$. Suppose that $\sup(A) = \top$. Take $b \in B \backslash \{\perp\}$.

$$\sup(A * B) \geq \sup(A) * \{b\} = \sup(A) * b = \top.$$

So $\sup(A * B) = \top = \sup(A) * \sup(B)$. Similarly, if $\sup(B) = \top$, then

$$\sup(A * B) = \top = \sup(A) * \sup(B).$$

Suppose that $\top \notin \{\sup(A), \sup(B)\}$. For any $b \in B$, $\sup(A) * b$ is defined since $\sup(A) \in S$. Hence

$$\sup(A) * \{b\} = \sup(A) * b.$$

$$A * B = \bigcup_{b \in B} A * \{b\},$$

$$\{\sup(A) * b \mid b \in B\} = \{\sup(A)\} * B.$$

By the lemma, $\{\sup(A)\} * B$ has a supremum, which is $\sup(A) * \sup(B)$. So $\sup(A * B)$ exists, and is equal to

$$\{\sup(A * \{b\}) \mid b \in B\} = \sup(A) * \sup(B).$$

$\square$

Corollary 1.4.13 Let $(S, *, \leq)$ be a partially ordered group. Let $f, g : X \longrightarrow \bar{S}$ be two mappings. Let

$$Y = \{x \in X \mid f(x) * g(x) \text{ is defined }\}.$$

Let

$$f * g : Y \longrightarrow \bar{S},$$

$$y \longmapsto f(y) * g(y).$$

(1) If $(\sup f) * (\sup g)$ is defined, and $f * g$ has a supremum, then

$$\sup(f * g) \leq \sup(f) * \sup(g).$$

(2) If $(\inf f) * (\inf g)$ is defined, and $f * g$ has a infimum, then

$$\inf(f * g) \geq \inf(f) * \inf(g).$$

Proof Let $A = f(X), B = g(X)$. By the theorem, $A * B$ has a supremum, and

$$\sup(A * B) = \sup(A) * \sup(B).$$

Let

$$C = (f * g)(Y) = \{f(y) * g(y) \mid y \in Y\}.$$

One has

$$C \subseteq A * B = \{f(x) * g(y) \mid (x, y) \in X \times X,\ f(x) * g(y) \text{ is defined }\}.$$

So $\sup(C) \leq \sup(A * B)$. $\square$

Theorem 1.4.14 Let $(S, *, \leq)$ be a partially ordered group. We suppose that $\bar{S}$ is order complete. Let X be a set and $f, g : X \longrightarrow \bar{S}$ be mappings. Let $\mathcal{F}$ be a filter on X that is non-degenerate. Suppose that $\forall x \in X$, $f(x) * g(x)$ is defined. Then

$$\limsup_{\mathcal{F}} (f * g) \leq \limsup_{\mathcal{F}} f * \limsup_{\mathcal{F}} g,$$

$$\limsup_{\mathcal{F}} (f * g) \geq \limsup_{\mathcal{F}} f * \liminf_{\mathcal{F}} g,$$

$$\liminf_{\mathcal{F}} (f * g) \geq \liminf_{\mathcal{F}} f * \liminf_{\mathcal{F}} g,$$

$$\liminf_{\mathcal{F}} (f * g) \leq \liminf_{\mathcal{F}} f * \limsup_{\mathcal{F}} g.$$

Provided that the term on the right hand side is defined.

Proof
(1) $\forall U \in \mathcal{F}$,

$$(f * g)^s(U) \leq f^s(U) * g^s(U).$$

Provided that $f^s(U) * g^s(U)$ is defined. If $\limsup_{\mathcal{F}} f * \limsup_{\mathcal{F}} g$ is defined, then

$$\limsup_{\mathcal{F}} (f * g) = \inf_{U \in \mathcal{F}} (f * g)^s(U) \leq \inf_{U \in \mathcal{F}} [f^s(U) * g^s(U)] = l.$$

$$\begin{aligned}\limsup_{\mathcal{F}} f * \limsup_{\mathcal{F}} g &= \left(\inf_{U \in \mathcal{F}} f^s(U)\right) * \left(\inf_{V \in \mathcal{F}} g^s(V)\right) \\ &= \inf_{(U,V) \in \mathcal{F} \times \mathcal{F},\ \text{defined}} (f^s(U) * g^s(V))\end{aligned}$$

If $(U, V) \in \mathcal{F} \times \mathcal{F}$ is such that $f^s(U) * g^s(V)$ is defined, then

$$f^s(U) * g^s(V) \geq f^s(U \cap V) * g^s(U \cap V) \geq l$$

provided that $f^s(U \cap V) * g^s(U \cap V)$ is defined. If $f^s(U \cap V) * g^s(U \cap V)$ is not defined, then $\top \in \{f^s(U), g^s(V)\}$, so that

$$f^s(U) * g^s(V) = \top \geq l.$$

Therefore,

$$\limsup_{\mathcal{F}} f * \limsup_{\mathcal{F}} g \geq l \geq \limsup_{\mathcal{F}} f * g.$$

(2)

$$\limsup_{\mathcal{F}} f * g \geq \limsup_{\mathcal{F}} f * \liminf_{\mathcal{F}} g.$$

Let $U \in \mathcal{F}$. Suppose that $(f * g)^s (U) \neq \top$. $\forall V \in \mathcal{F}$, one has $\forall x \in U \cap V$,

$$(f * g)^s (U \cap V) \geq f(x) * g(x) \geq f(x) * g^i(U \cap V) \geq f(x) * g^i(V).$$

So

$$(f * g)^s (U) \geq f^s(U \cap V) * g^i(V),$$

provided that $f^s(U \cap V) * g^i(V)$ is defined. Taking the infimum which respect to U, we obtain

$$\limsup_{\mathcal{F}} f * g \geq \limsup_{\mathcal{F}} f * g^i(V),$$

provided that $\limsup_{\mathcal{F}} f * g^i(V)$ is defined. Taking the supremum with respect to V, we obtain

$$\limsup_{\mathcal{F}} f * g \geq \limsup_{\mathcal{F}} f * \liminf_{\mathcal{F}} g.$$

$\square$

Corollary 1.4.15 Let $(S, *, \leq)$ be a partially ordered group, such that $\bar{S}$ is order complete. Let $f, g : X \longrightarrow \bar{S}$ be mappings such that $\forall x \in X, f(x) * g(x)$ is defined. Let $\mathcal{F}$ be a non-degenerate filter on X. Assume that g has a limit along $\mathcal{F}$.

(1)

$$\limsup_{\mathcal{F}} f * g = \limsup_{\mathcal{F}} f * \lim_{\mathcal{F}} g.$$

(2)

$$\liminf_{\mathcal{F}} f * g = \liminf_{\mathcal{F}} f * \lim_{\mathcal{F}} g.$$

Provided that the term on the right hand side is defined.

Proof

$$\limsup_{\mathcal{F}} f * g \leq \limsup_{\mathcal{F}} f * \limsup_{\mathcal{F}} g = \limsup_{\mathcal{F}} f * \lim_{\mathcal{F}} g.$$

$$\limsup_{\mathcal{F}} f * g \geq \limsup_{\mathcal{F}} f * \liminf_{\mathcal{F}} g = \limsup_{\mathcal{F}} f * \lim_{\mathcal{F}} g.$$

$\square$

Example 1.4.16
(1)

$$\limsup_{n\to+\infty}\left((-1)^n+\frac{1}{n}\right)=\limsup_{n\to+\infty}(-1)^n=1.$$

(2) For $a>1$, let $a=1+b$. $a^n=(1+b)^n\geq nb$. so

$$0\leq\frac{\sqrt{n}}{a^n}\leq\frac{1}{b\sqrt{n}},$$

$$\lim_{n\to+\infty}\frac{\sqrt{n}}{a^n}.$$

$$\forall k\in\mathbb{N}_{\geq 1},\ \lim_{n\to+\infty}\left(\frac{\sqrt{n}}{a^n}\right)^k.$$

Proposition 1.4.17 Let $(G,\leq)$ be a totally ordered set that is order complete. Let X be a set and $f,g:X\longrightarrow G$ be mappings. We denote by

$$f\wedge g:X\longrightarrow G,x\longmapsto\min\{f(x),g(x)\},$$

$$f\vee g:X\longrightarrow G,x\longmapsto\max\{f(x),g(x)\}.$$

Let $\mathcal{F}$ be a filter on X that is non-degenerate. Then

$$\limsup_{\mathcal{F}}(f\vee g)=\max\{\limsup_{\mathcal{F}}f,\limsup_{\mathcal{F}}g\},$$

$$\liminf_{\mathcal{F}}(f\wedge g)=\min\{\liminf_{\mathcal{F}}f,\liminf_{\mathcal{F}}g\}.$$

If f and g have limit along $\mathcal{F}$, so do $f\vee g$ and $f\wedge g$ and

$$\limsup_{\mathcal{F}}(f\vee g),\limsup_{\mathcal{F}}(f\wedge g).$$

Proof Let

$$\mathcal{B}_1=\{A\in\mathcal{F}\mid f^s(A)\geq g^s(A)\},$$

$$\mathcal{B}_2=\mathcal{F}\backslash\mathcal{B}_1=\{A\in\mathcal{F}\mid f^s(A)<g^s(A)\}.$$

Case 1. $\forall U\in\mathcal{F},\exists A\in\mathcal{B}_1,A\subseteq U$.

$$\limsup_{\mathcal{F}}(f\vee g)=\inf_{\mathcal{F}}(f\vee g)^s(U)=\inf_{A\in\mathcal{B}_1}(f\vee g)^s(A)=\inf_{A\in\mathcal{B}_1}f^s(A)=\limsup_{\mathcal{F}}f.$$

Case 2. $\exists W \in \mathcal{F}, \forall A \in \mathcal{B}_1, A \nsubseteq W$. If $\mathcal{B}_1 = \varnothing, \mathcal{B}_2 = \mathcal{F}$. $\quad (U \in \mathcal{F}, U \in \mathcal{B}_2)$ If $A \in \mathcal{B}_1, \forall U \in \mathcal{F}, U \subseteq W, U \cap A \notin \mathcal{B}_1$, so $U \cap A \in \mathcal{B}_2$ and $U \cap A \subseteq U$. (U contains an elements in $\mathcal{B}_2$.)

$$\limsup_{\mathcal{F}}(f \vee g) = \inf_{U \in \mathcal{F}|_W} (f \vee g)^s(U) = \inf_{A \in \mathcal{B}_2} (f \vee g)^s(A) = \inf_{U \in \mathcal{B}_2} g^s(A) = \liminf_{\mathcal{F}} g.$$

Hence,

$$\limsup_{\mathcal{F}}(f \vee g) \leq \max\{\limsup_{\mathcal{F}} f, \limsup_{\mathcal{F}} g\}.$$

Moreover, $f \vee g \geq f, f \vee g \geq g$, so

$$\limsup_{\mathcal{F}}(f \vee g) \geq \max\{\limsup_{\mathcal{F}} f, \limsup_{\mathcal{F}} g\}.$$

If $\lim_{\mathcal{F}} f$ and $\lim_{\mathcal{F}} g$ exists, then

$$\begin{aligned}\liminf_{\mathcal{F}}(f \wedge g) &\geq \min\{\liminf_{\mathcal{F}} f, \liminf_{\mathcal{F}} g\} \\ &= \max\{\lim_{\mathcal{F}} f, \lim_{\mathcal{F}} g\} = \limsup_{\mathcal{F}}(f \vee g).\end{aligned}$$

□

Proposition 1.4.18 Let $(S, *, \leq)$ be a partially orderes group. for any $A \subseteq S$,let

$$\iota(A) := \{\iota(a) \mid a \in A\}.$$

If A has an supremum in $\bar{S}$ then $\iota(A)$ has an infimum in $\bar{S}$ and

$$\inf(\iota(A)) = \iota(\sup(A)).$$

Resp. infimum, supremum.

Proof

$\forall a \in A, a \leq \sup(A)$, so $\iota(a) \geq \iota(\sup(A))$. Hence $\iota(\sup(A))$ is a lower bound of $\iota(A)$. Let m be a lower bound of $\iota(A)$. For any $a \in A$, $\iota(a) \geq m, \iota(m) \geq a$. Hence $\iota(m) \geq \sup(A), \iota(\sup(A)) \geq m$. □

Corollary 1.4.19 Let $(S, *, \leq)$ be a partially orderes group. $f : X \longrightarrow \bar{S}$ be a mapping, and $\mathcal{F}$ be a filter on X. Let

$$\iota(f) : X \longrightarrow \bar{S},\ x \longmapsto \iota(f(x)).$$

Assume that $\bar{S}$ is order complete. Then

$$\limsup_{\mathcal{F}} \iota(f) = \iota(\liminf_{\mathcal{F}} f),$$
$$\liminf_{\mathcal{F}} \iota(f) = \iota(\limsup_{\mathcal{F}} f).$$

Proof

$$\limsup_{\mathcal{F}} \iota(f) = \inf_{U\in\mathcal{F}} = \inf_{U\in\mathcal{F}} \iota\left(\inf_{x\in u} f(x)\right) = \iota\left(\sup_{U\in\mathcal{F}} \inf_{x\in U} f(x)\right) = \iota\left(\liminf_{\mathcal{F}} f\right).$$

$\square$

1.5 Absolute Values

We fix a totolly ordered abelian group $(R, +, \leq)$.

Remark 1.5.1 $\forall a \in R$, either $a = 0$, or $a > 0$, or $a < 0$. We add two final elements $-\infty, +\infty$ to R to construct the enhancement of $(R, +, \leq)$.

Definition 1.5.2 For any $x \in \bar{R}$, we let

$$|x| := \max\{x, -x\}.$$

By definition, $|-x| = |x|$.

Proposition 1.5.3 For any $(a, b) \in \bar{R} \times \bar{R}$, if $a + b$ is defined, then

$$|a + b| \leq |a| + |b|.$$

Proof If $\{-\infty, +\infty\} \cap \{a, b\} \neq \varnothing$, then

$$|a| + |b| = +\infty \leq |a + b|$$

if $a + b$ is defined. Suppose that $\{a, b\} \subseteq R$,

$$a + b \leq |a| + |b|,$$

$$-(a + b) = (-a) + (-b) \leq |a| + |b|,$$

so,

$$|a+b| \leq |a| + |b|\,.$$

$\square$

Corollary 1.5.4 For any $(a, b) \in \bar{R} \times \bar{R}$,

$$||a| - |b|| \leq |a-b|\,, ||a| - |b|| \leq |a+b|\,,$$

provided that the terms on both sides are defined.

Proof If $\{-\infty, +\infty\} \cap \{a, b\} \neq \varnothing, |a-b| = +\infty$. Suppose that $\{a, b\} \subseteq R$.

$$|a| - |b| = |(a-b) + b| - |b| \leq |a-b| + |b| - |b| = |a-b|\,.$$

Similarly,

$$|b| - |a| \leq |b-a| = |a-b|\,.$$

So,

$$||a| - |b|| \leq |a-b|\,.$$

$\square$

Theorem 1.5.5 Let X be a set, $\mathcal{F}$ be a non-degenerate filter on X, $f : X \longrightarrow R$ be a mapping and $l \in R$. The following statements are equivalent.
(1) f admits l along $\mathcal{F}$.
(2) $\limsup_{\mathcal{F}} |f - l| = 0$, where $f - l : X \longrightarrow R, x \longmapsto f(x) - l$. Moreover, there conditions imply.
(3) $\forall \varepsilon \in R_{>0}, \exists U \in \mathcal{F}, \forall x \in U, |f(x) - l| < \varepsilon$. The converse is true when $\inf R_{>0} = 0$.

Proof Note that

$$\limsup_{\mathcal{F}} (f - l) = \left(\limsup_{\mathcal{F}} f\right) - l,$$

$$\liminf_{\mathcal{F}} (f - l) = \left(\liminf_{\mathcal{F}} f\right) - l.$$

(1)⇒(2): (1) gives $\limsup_{\mathcal{F}} f = \liminf_{\mathcal{F}} f = l$. So

$$\limsup_{\mathcal{F}} (f - l) = \liminf_{\mathcal{F}} (f - l) = 0.$$

So,

$$\begin{aligned}\limsup_{\mathcal{F}} |f-l| &= \limsup_{\mathcal{F}} (f-l)\vee(l-f)\\ &= \max\{\limsup_{\mathcal{F}}(f-l), \limsup_{\mathcal{F}}(l-f)\}\\ &= 0.\end{aligned}$$

(2)⇒(1): $\max\{\limsup_{\mathcal{F}}(f-l), \limsup_{\mathcal{F}}(l-f)\} = 0$. So

$$\left(\limsup_{\mathcal{F}} f\right) - l \le 0, \quad l - \liminf_{\mathcal{F}} f \le 0.$$

Hence,

$$l \le \liminf_{\mathcal{F}} f \le \limsup_{\mathcal{F}} f \le l.$$

(2)⇒(3): One has $\inf_{U\in\mathcal{F}} \sup_{x\in U} |f(x)-l| = 0$. For any $\varepsilon \in R_{>0}$, ε is not a lower bound of

$$\left\{\sup_{x\in U} |f(x)-l| \mid U \in \mathcal{F}\right\}.$$

So, $\exists U \in \mathcal{F}, \sup_{x\in U} |f(x)-l| < \varepsilon$. So $\forall x \in U, |f(x)-l| < \varepsilon$.
(3)⇒(2): Under the hypothesis, $\inf R_{>0} = 0$.

$$\forall \varepsilon \in R_{>0}, \exists U \in \mathcal{F}, \forall x \in U, |f(x)-l| < \varepsilon.$$

$$0 \le \limsup_{\mathcal{F}} |f-l| \le \varepsilon.$$

Taking the infimum with respect to $\varepsilon \in R_{>0}$, we obtain $\limsup_{\mathcal{F}} |f-l| = 0$. □

Definition 1.5.6 Let $A \subseteq R$. If $\exists M \in R_{>0}$ such that $\forall a \in A, |a| \le M$,we say that A is **bounded**. If $f : X \longrightarrow R$ is such that $f(X)$ is bounded, we say that f is bounded.

Corollary 1.5.7 Let X be a set, $\mathcal{F}$ be a non-degenerate filter on X, and $f : X \longrightarrow R$ be a mapping. If f has a limit along $\mathcal{F}$ and $\lim_{\mathcal{F}} f \in R$, then there exists $U \in \mathcal{F}$ such that $f|_U$ is bounded.

Proof If $R = \{0\}$, then f is constant, so bounded. Suppose that there exists

$\varepsilon \in R_{>0}$,

$$\exists U \in \mathcal{F}, \forall x \in U, |f(x) - l| < \varepsilon,$$

where $l = \lim f$. So $|f(x)| \leq |f(x) - l| + |l| \leq \varepsilon + |l|$, on U. $\square$

1.6 Ordered Rings

Definition 1.6.1 Let $(R, +, \cdot)$ be a non-zero unitary ring and $\leq$ be a partially ordered group and

$$\forall (a, b) \in R_{>0} \times R_{>0}, ab > 0,$$

then we say that $(R, +, \cdot, \leq)$ is an **partially ordered ring**.
Let $\bar{R} = R \cup \{-\infty, +\infty\}$ be an enhancement of R. We extend partially the multiplication on $\bar{R}$ as follows

$$\forall x \in \bar{R}_{>0},\ x(+\infty) = (+\infty)x = +\infty,\ x(-\infty) = (-\infty)x = -\infty.$$

$$\forall x \in \bar{R}_{<0},\ x(+\infty) = (+\infty)x = (-\infty),\ x(-\infty) = (-\infty)x = +\infty.$$

$$\cancel{0 \cdot \infty}$$

are NOT defined.

$$(+\infty)^{-1} = (-\infty)^{-1} = 0.$$

We fix a partially ordered ring.

(1) If $a \geq 0, b \geq 0$, then $ab \geq 0$.
(2) $a < b \Rightarrow -b < -a$.
(3) $a \leq b \Rightarrow -b \leq -a$.
(4) If $\lambda > 0,\ a > b$ then $\lambda a > \lambda b,\ a\lambda > b\lambda,\ (-\lambda)a < (-\lambda)b,\ a(-\lambda) < b(-\lambda)$.
(5) If $\lambda > 0,\ a \geq b$, then $\lambda a \geq \lambda b,\ a\lambda \geq b\lambda,\ (-\lambda)a \leq (-\lambda)b,\ a(-\lambda) \leq b(-\lambda)$.

Proposition 1.6.2 Let $(R, +, \cdot, \leq)$ be a totally ordered unitary ring.
(1) $\forall a \in R,\ a \neq 0 \Rightarrow a^2 > 0$.
(2) $\forall a \in R^{\times},\ a > 0 \Rightarrow a^{-1} > 0, a < 0 \Rightarrow a^{-1} < 0$.

Proof
(1) Since $\leq$ is a total order, either $a > 0,\ a^2 = a \cdot a > 0$, or $a < 0,\ a^2 = (-a) \cdot (-a) > 0$.

(2) $a^2 > 0$. If $a^{-1} < 0$, then $a = a^{-1} \cdot a^2 < 0$. If $a^{-1} > 0,\ a = a^{-1} \cdot a^2 > 0$. $\square$