

QM HW2

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Problem 1 (Harmonic oscillator)

(1) We've known that $x_{nm} \neq 0 \Rightarrow m = n \pm 1$

$$x_{n,n}^2 = \sum_k x_{n,k} x_{k,n} = x_{n,n-1} x_{n-1,n} + x_{n,n+1} x_{n+1,n} \quad (1.1)$$

Since

$$x_{n+1,n} = \sqrt{\frac{\hbar}{2m\omega_0}}(n+1) \quad (1.2)$$

We have

$$\boxed{x_{n,n}^2 = \frac{\hbar}{m\omega_0} \left(n + \frac{1}{2}\right)} \quad (1.3)$$

$$(p^2)_{n,n} = m^2 (\dot{x})_{n,n}^2 = \hbar\omega_0 m(n + 1/2) \quad (1.4)$$

$$\boxed{\sqrt{(x^2)_{n,n}(p^2)_{n,n}} = \hbar \left(n + \frac{1}{2}\right)} \quad (1.5)$$

When $n = 0$,

$$\sqrt{(x^2)_{n,n}(p^2)_{n,n}} = \frac{\hbar}{2} \quad (1.6)$$

(2)

$$\boxed{x_{n,m}(t) = \sqrt{\frac{\hbar}{2m\omega_0}} (\sqrt{n}e^{i\omega_0 t}\delta_{m,n-1} + \sqrt{n+1}e^{-i\omega_0 t}\delta_{m,n+1})} \quad (1.7)$$

$$\boxed{p_{n,m}(t) = i\sqrt{\frac{\hbar m\omega_0}{2}} (\sqrt{n}e^{i\omega_0 t}\delta_{m,n-1} - \sqrt{n+1}e^{-i\omega_0 t}\delta_{m,n+1})} \quad (1.8)$$

$$[x, p]_{n,m}(t) = \sum_k (x_{n,k} p_{k,m} e^{i\omega_{n,m} t} - p_{m,k} x_{k,n} e^{i\omega_{m,n} t}) \quad (1.9)$$

$$= [x_{n,n-1} p_{n-1,m} + x_{n,n+1} p_{n+1,m} - p_{m,n-1} x_{n-1,n} - p_{m,n+1} x_{n+1,n}] \delta_{nm} \quad (1.10)$$

$$= \boxed{i\hbar\delta_{nm}} \quad (1.11)$$

Problem 2 (Quantum equation of motion)(1) By associative law, $\forall H_2(x)$,

$$[x, H_2(x)] = 0 \quad (2.1)$$

We expand $H_1(p)$ in form of

$$H_1(p) = \sum_{n=0}^{\infty} a_n p^n \quad (2.2)$$

We have

$$[x, p^n] = \sum_{i=0}^{n-1} p^i [x, p] p^{n-i-1} = i\hbar p^{n-1} = i\hbar \frac{\partial(p^n)}{\partial p} \quad (2.3)$$

Hence,

$$\boxed{\frac{\partial H}{\partial p} = \frac{1}{i\hbar} [x, H]} \quad (2.4)$$

Similarly,

$$\boxed{-\frac{\partial H}{\partial x} = \frac{1}{i\hbar} [p, H]} \quad (2.5)$$

Therefore,

$$\dot{x} = \frac{i}{\hbar} [x, H], \quad \dot{p} = \frac{i}{\hbar} [p, H] \quad (2.6)$$

(2) $\forall (n, m) \in \mathbb{N}^2$

$$[x, x^n p^m] = x x^n p^m - x^n p^m x = x^n (x p^m - p^m x) = x^n [x, p] \quad (2.7)$$

So,

$$\boxed{[x, x^n p^m] = i\hbar x^n \frac{\partial(p^m)}{\partial p}} \quad (2.8)$$

Hence (2.4) and (2.5) still hold. Let $x = x, y, p = p_x, p_y$.

$$\begin{aligned} \dot{x} &= \frac{1}{i\hbar} [x, H], & \dot{p}_x &= \frac{1}{i\hbar} [p_x, H] \\ \dot{y} &= \frac{1}{i\hbar} [y, H], & \dot{p}_y &= \frac{1}{i\hbar} [p_y, H] \end{aligned} \quad (2.9)$$

Problem 3 (De Broglie wave)

By relativity,

$$p = mv_g, \quad E = mc^2 \quad (3.1)$$

$$v_g = \frac{pc^2}{E} \quad (3.2)$$

In wave case,

$$v_g = \frac{d\omega}{dk} \quad (3.3)$$

From

$$E^2 = p^2 c^2 + m_0^2 c^4 \quad (3.4)$$

we deduce that

$$\frac{pc^2}{E} = \frac{dE}{dp} = \hbar \frac{d\omega}{dp} \quad (3.5)$$

Thus,

$$dp = \hbar dk$$

$p = \hbar k$

(3.6)

Problem 4 (Schrödinger equation)

Let

$$\Psi(r, t) = e^{iS/\hbar} = e^{i[W(r) - Et]/\hbar} = \psi(r) e^{-iEt/\hbar} \quad (4.1)$$

Considering $E = \hbar\omega$, take $k = \hbar$. Then

$$S = -i\hbar \ln \psi(r) - Et \quad (4.2)$$

Plug in

$$(\nabla S)^2 = 2m(E - V(r)) \quad (4.3)$$

we obtain

$$\frac{\hbar^2}{2m} (\nabla \psi)^2 + (E - V(r)) \psi^2 = 0 \quad (4.4)$$

Now, we want to linearize the equation. Let ∇ act on (4.4) and eliminate the common factor

$$\frac{\hbar^2}{2m} \nabla^2 \psi + (E - V(r)) \psi + \frac{\nabla(E - V(r)) \psi^2}{2\nabla \psi} = 0 \quad (4.5)$$

God believes S, E, V are smooth at the \hbar scale. We ignore the last term.

$\frac{\hbar^2}{2m} \nabla^2 \psi + (E - V(r)) \psi = 0$

(4.6)

Problem 5 (Quantum Potential)

Plug

$$\frac{\partial \Psi}{\partial t} = \frac{i}{\hbar} \frac{\partial S}{\partial t} e^{iS/\hbar} \quad (5.1)$$

$$\nabla \Psi = \frac{i}{\hbar} \nabla S e^{iS/\hbar} \quad (5.2)$$

$$\nabla^2 \Psi = \frac{1}{\hbar^2} \left[i\hbar \nabla^2 S - (\nabla S)^2 \right] e^{iS/\hbar} \quad (5.3)$$

into Schrödinger equation,

$$\frac{\partial S}{\partial t} = \frac{1}{2m} \left[i\hbar \nabla^2 S - (\nabla S)^2 \right] - V \quad (5.4)$$

When taking the limit $\hbar \rightarrow 0$,

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V = 0 \quad (5.5)$$

It is exactly the Hamilton-Jacobi equation. The quantum potential is

$$\boxed{V_{\text{quantum}} = -\frac{i\hbar}{2m} \nabla^2 S} \quad (5.6)$$

It vanishes as $\hbar \rightarrow 0$.