Pauli Matrix

We define

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

We define $\sigma^0 = I_{2\times2}(\mathbb{C})$ just for convenience, it is used for balancing the index, you should not replace I any where. In the following part, we will use i, j, k to represent $1, 2, 3$ and other Latin letters to represent $0, 1, 2, 3$. Also, we will use Greek letters to represent $1, 2$, which are the components of matrices. So there are something crazy:

$$\delta^\mu{}_\mu = 2, \ \delta^i{}_i = 3, \ \delta^a{}_a = 4. \tag{2}$$

Property 1.

$$\sigma^i \sigma^j = \delta^{ij} I + \mathrm{i} \varepsilon^{ijk} \sigma_k. \tag{3}$$

If we define a general[1] ϵ^{abc}:

$$\epsilon_{abc} = \begin{cases} \varepsilon^{abc} & , 0 \notin \{a, b, c\} \\ 0 & , 0 \in \{a, b, c\} \end{cases} \tag{4}$$

Replace ε^{ijk} with ϵ^{abc}, the equality is still valid:

$$\boxed{\sigma^a \sigma^b = \delta^{ab} \sigma^0 + \mathrm{i} \epsilon^{abc} \sigma_c.} \tag{5}$$

Property 2. For any $i \in \{1, 2, 3\}$,

$$\boxed{\det(\sigma^i) = -1, \ \mathrm{tr}(\sigma^i) = 0, \ (\sigma^i)^\dagger = \sigma^i.} \tag{6}$$

Property 3 (Completeness).
Any $M \in M_{2\times2}(\mathbb{C})$ can be written as a linear combination of $\sigma^0, \sigma^1, \sigma^2, \sigma^3$. Since,

$$\sigma^0 + \sigma^3 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \ \sigma^0 - \sigma^3 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \ \sigma^1 + \mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \ \sigma^1 - \mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \tag{7}$$

Let $M = \beta_a \sigma^a$, then

$$M \sigma^b = \beta_a \sigma^a \sigma^b = \beta_a \left(\delta^{ab} I + \mathrm{i} \epsilon^{abc} \sigma_c \right). \tag{8}$$

We take its trace, the second term at right hand side is zero, so

$$\boxed{\beta^b = \frac{1}{2} \mathrm{tr}(M \sigma^b).} \tag{9}$$

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[1]It is also anti-symmetric: $\forall a, b, c \in \{0, 1, 2, 3\}, \ (a - b)(b - c)(c - a) = 0 \Rightarrow \epsilon^{abc} = 0.$

That means

$$M^{\lambda}{}_{\rho}\delta^{\mu}{}_{\lambda}\delta^{\rho}{}_{\nu} = M^{\mu}{}_{\nu} = \frac{1}{2}M^{\lambda}{}_{\rho}\left(\sigma_a\right)^{\rho}{}_{\lambda}\left(\sigma^a\right)^{\mu}{}_{\nu}.\tag{10}$$

Since $M^{\lambda}{}_{\rho}$ is arbitrary, we can deduce that

$$\boxed{\left(\sigma_a\right)^{\rho}{}_{\lambda}\left(\sigma^a\right)^{\mu}{}_{\nu} = 2\delta^{\mu}{}_{\lambda}\delta^{\rho}{}_{\nu}.}\tag{11}$$

In particular,

$$\left(\sigma_i\right)^{\rho}{}_{\lambda}\left(\sigma^i\right)^{\mu}{}_{\nu} = 2\delta^{\mu}{}_{\lambda}\delta^{\rho}{}_{\nu} - \delta^{\rho}{}_{\lambda}\delta^{\mu}{}_{\nu}.\tag{12}$$

Property 4 (Relation with cross product of vectors).
Let u^i, v^i be two vectors, then

$$\boxed{\left(u_i\sigma^i\right)\left(v_j\sigma^j\right) = \mathrm{i}\left(u^i v^j \varepsilon_{ijk}\right)\sigma^k + u_k v^k I.}\tag{13}$$

In particular,

$$\left(u_i\sigma^i\right)^2 = \left(u^i u_i\right)^2 I.\tag{14}$$

Property 5 (Exponential form).
Let **n** be a unitary vector, then we can deduce that

$$\boxed{\exp\left[\mathrm{i}\left(n_i\sigma^i\right)\frac{\theta}{2}\right] = I\cos\frac{\theta}{2} + \mathrm{i}\left(n_i\sigma^i\right)\sin\frac{\theta}{2}.}\tag{15}$$