Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 8-1 : Topology (1)

1. We equip $\mathbb{R}$ with the topology defined by the usual metric $(a, b) \mapsto |a - b|$.

(1) Let a and b be two real numbers such that $a < b$. Express the interval $]a, b[$ as an open ball. Determine its center and its radius.

(2) Let a be a real number. Show that the intervals $]a, +\infty[$ and $]-\infty, a[$ are open subsets of $\mathbb{R}$.

(3) Let a be a real number. Show that the intervals $[a, +\infty[$ and $]-\infty, a]$ are closed subsets of $\mathbb{R}$.

(4) Let a and b be real numbers such that $a \leqslant b$. Show that $[a, b]$ is a closed subset of $\mathbb{R}$.

2. (1) If $\{\mathcal{T}_i\}_{i\in I}$ is a family of topologies on X, show that $\bigcap_{i\in I} \mathcal{T}_i$ is a topology on X. Is $\bigcup_{i\in I} \mathcal{T}_i$ a topology on X?

(2) Let $\{\mathcal{T}_i\}_{i\in I}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathcal{T}_i$, and a unique largest topology contained in all $\mathcal{T}_i$.

(3) If $X = \{a, b, c\}$, let

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ and } \mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Find the smallest topology containing $\mathcal{T}_1$ and $\mathcal{T}_2$, and the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

3. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

(1) Prove that such a collection $\mathscr{B}$ is a topological basis.

(2) Give an example to show that the collection $\mathscr{B}$ is not a topology on $X \times Y$.

(3) A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis $\mathcal{S}$ is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$. In other words, we have

$$\mathcal{T} = \left\{ \bigcup_{B\in\mathscr{B}'} B \mid \mathscr{B}' \subset \mathscr{B} \right\},$$

where $\mathscr{B} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S},\ i = 1, \ldots, n,\ n \in \mathbb{N}^+\}$.

Prove that the above definition is well-defined, which means you need to prove that $\mathcal{T}$ is a topology. **Hint :** you only need to prove that all the finite intersections of elements in $\mathcal{S}$ forms a topological basis.

(4) Let $\mathscr{B}$ be a basis for the topology of X, and $\mathscr{C}$ be a basis for the topology of Y. Prove that the collection

$$\mathscr{D} = \{B \times C \mid B \in \mathscr{B}, C \in \mathscr{C}\}$$

is a basis for the topology of $X \times Y$.

(5) Let X, Y be two topological spaces,

$$\begin{array}{rccc} \pi_1 : & X \times Y & \longrightarrow & X \\ & (x, y) & \mapsto & x \end{array},$$

and

$$\begin{array}{rccc} \pi_2 : & X \times Y & \longrightarrow & X \\ & (x, y) & \mapsto & y \end{array}$$

be the projections. Prove that

$$\mathcal{S} = \{\pi_1^{-1}(U) \mid U \text{ open in } X\} \cup \{\pi_2^{-1}(V) \mid V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

(6) Describe the product topology of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where each $\mathbb{R}$ is equipped with the standard topology.

4. Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, we define the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}.$$

(1) Prove that $\mathcal{T}_Y$ is a topology on Y, which is called the **subspace topology**. With this topology, Y is called a subspace of X, where its open sets consist of all intersections of open sets of X with Y.

(2) Let $\mathscr{B}$ be a basis for the topology of X. Prove that the collection

$$\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.

(3) With all the above notations, give an example to show that an open subset of Y with respect to $\mathcal{T}_Y$ do not necessarily to be an open subset of X with respect to $\mathcal{T}$.

(4) Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

(5) Let A be a subspace of X and B be a subspace of Y. Prove that the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

5. We equip $\mathbb{R}$ with the topology defined by the usual metric. Let I be an open subset of $\mathbb{R}$. We assume that I is not empty. We define a binary relation $\sim$ on I as follows : for any $(a, b) \in I \times I$, $a \sim b$ if and only if the closed interval with extremities a and b is contained in I, that is, $[a, b] \subseteq I$ when $a \leqslant b$, and $[b, a] \subseteq I$ when $b \leqslant a$.

(1) Show that $\sim$ is an equivalence relation on I.

(2) Let J be an equivalence class of I under the equivalence relation $\sim$. Show that J is an interval.

(3) Let J be an interval in $\mathbb{R}$ which is contained in I. Show that J is contained in some equivalence class of I under the equivalence relation $\sim$.

(4) Show that any equivalence class of I under the equivalence relation $\sim$ is an interval of the form $]a, b[$ with $(a, b) \in [-\infty, +\infty]^2$, $a < b$.

(5) Show that I is a disjoint union of countably many open intervals.

6. Let n be a positive integer and (X_i, d_i), $i \in \{1, \ldots, n\}$ be a family of metric spaces. Let X be the product set $X_1 \times \cdots \times X_n$. Recall that the elements of X are n-tuples $x = (x_1, \ldots, x_n)$ with $x_1 \in X_1, \ldots, x_n \in X_n$. We define a function $d : X \times X \to \mathbb{R}_{\geqslant 0}$ as follows : for any elements $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of X,

$$d(x, y) := \max_{i \in \{1, \ldots, n\}} d_i(x_i, y_i).$$

(1) Show that d is a metric on X. We call it the *product metric* of $d_1, \ldots, d_n$.

(2) Let $x = (x_1, \ldots, x_n)$ be an element of X and $r > 0$. Show that

$$B_d(x, r) = B_{d_1}(x_1, r) \times \cdots \times B_{d_n}(x_n, r).$$

(3) Let $(x^{(k)})_{k \in \mathbb{N}}$ be an element of $X^{\mathbb{N}}$, and $x = (x_1, \ldots, x_n) \in X$. We write each $x^{(k)} \in X$ in the form of $(x_1^{(k)}, \ldots, x_n^{(k)})$, where $x_i^{(k)} \in X_i$ for any $i \in \{1, \ldots, n\}$.

(3.a) Show that $(x^{(k)})_{k \in \mathbb{N}}$ is a Cauchy sequence if and only if each $(x_i^{(k)})_{k \in \mathbb{N}}$ is a Cauchy sequence, where $i \in \{1, \ldots, n\}$.

(3.b) Show that the sequence $(x^{(k)})_{k\in\mathbb{N}}$ converges in X to x if and only if, for each $i \in \{1, \ldots, n\}$, the sequence $(x_i^{(k)})_{k\in\mathbb{N}}$ in X converges to x_i.

(4) Show that, if all metric spaces $(X_1, d_1), \ldots, (X_n, d_n)$ are complete, then (X, d) is also complete.

(5) Let Y be a topological space. For any $i \in \{1, \ldots, n\}$, let $f_i : Y \to X_i$ be a mapping. Let $f : Y \to X_1 \times \cdots \times X_n$ be the mapping sending $y \in Y$ to $(f_1(y), \ldots, f_n(y))$. Show that the mapping f is continuous at some $y_0 \in Y$ if and only if each mapping f_i is continuous at y_0, where $i \in \{1, \ldots, n\}$.

7. Let V be a vector space over $\mathbb{R}$. We call *norm on V* any mapping $\|\cdot\| : V \to \mathbb{R}_{\geqslant 0}$ that satisfies the following conditions :

(i) for any $x \in V \setminus \{\mathbf{0}_V\}$, $\|x\| > 0$,

(ii) for any $(a, x) \in \mathbb{R} \times V$, $\|ax\| = |a| \cdot \|x\|$,

(iii) for any $(x, y) \times V \times V$, $\|x + y\| \leqslant \|x\| + \|y\|$.

The couple $(V, \|\cdot\|)$ is called a normed vector space over $\mathbb{R}$.

(1) Show that the function

$$d : V \times V \longrightarrow \mathbb{R}_{\geqslant 0}, \quad (x, y) \in V \times V \longmapsto \|x - y\|$$

is a metric on V. We call this metric the *metric associated with* the norm $\|\cdot\|$.

(2) Let n be a positive integer. Show that the mapping

$$\|\cdot\|_{\ell^\infty} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}, \quad \|(x_1, \ldots, x_n)\|_{\ell^\infty} := \max_{i\in\{1,\ldots,n\}} |x_i|.$$

Show that $\|\cdot\|_{\ell^\infty}$ is a norm on $\mathbb{R}^n$.

(3) Let $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be the metric associated with the norm $\|\cdot\|_{\ell^\infty}$. Show that the metric space $(\mathbb{R}^n, d_\infty)$ is complete. One can express d_∞ as a product metric.

(4) Show that the mapping

$$\|\cdot\|_{\ell^1} : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geqslant 0}, \quad \|(x_1, \ldots, x_n)\|_{\ell^1} := \sum_{i=1}^n |x_i|$$

is a norm on $\mathbb{R}^n$.

(5) Show that, for any $x \in \mathbb{R}^n$, one has

$$\|x\|_{\ell^\infty} \leqslant \|x\|_{\ell^1} \leqslant n\|x\|_{\ell^\infty}.$$

(6) Let $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ the metric associated with the norm $\|\cdot\|_{\ell^1}$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}^n$ and x be an element of $\mathbb{R}^n$.

(6.a) Show that $(x^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in (X, d_∞) if and only if it is a Cauchy sequence in (X, d_1).

(6.b) Show that $(x^{(k)})_{k\in\mathbb{N}}$ converges to x in (X, d_∞) if and only if it converges to x in (X, d_1).

(7) Deduce that the metric space (X, d_1) is complete.

8. Let X and Y be metric spaces and α be a positive constant. We say that a mapping $f : X \to Y$ is *α-Lipschitzian* if

$$\forall\,(a,b) \in X \times X, \quad d(f(a), f(b)) \leqslant \alpha d(a,b).$$

(1) Let X and Y be metric spaces and $f : X \to Y$ be a mapping which is α-Lipschitzian for some positive constant α. Show that f is uniformly continuous.

(2) Let (X, d) be a metric space and x_0 be an element of X. Show that the mapping $f : X \to \mathbb{R}$ sending $x \in X$ to $d(x_0, x)$ is 1-Lipschitzian.

(3) Let (X, d) be a metric space and x_0 be an element of X. Show that, for any $r > 0$, the set

$$\overline{B}(x_0, r) := \{x \in X \mid d(x_0, x) \leqslant r\}$$

is a closed subset of X.

(4) Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. Show that the mapping $\|\cdot\| : V \to \mathbb{R}$ is 1-Lipschitzian.

9. Let X and Y be metric spaces, and $f : X \to Y$ be a mapping, and x_0 be an element of X, we say that f is locally Lipschitzian at x_0 if there exists $r > 0$ and $\alpha > 0$ such that

$$\forall (a,b) \in B(x_0, r) \times B(x_0, r), \quad d(f(a), f(b)) \leqslant \alpha d(a,b).$$

Prove that, if f is locally Lipschitzian at x_0, then it is continuous at x_0.

10. We equip $\mathbb{R}$ with the usual metric and $\mathbb{R}^2$ with the product metric.

(1) Show that the mapping

$$+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad (a,b) \longmapsto a + b$$

is 2-Lipschitzian. Deduce that this mapping is continuous.

(2) Let Y be a topological space and $f : Y \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be mappings. Assume that both f and g are continuous at some $y_0 \in Y$. Show that the mapping

$$f + g : Y \longrightarrow \mathbb{R}, \quad (y \in Y) \longmapsto f(y) + g(y)$$

is continuous at y_0.

(3) Let a be a real number. Show that the mapping

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto at$$

is $|a|$-Lipschitzian. Deduce that the mapping is continuous.

(4) Let Y be a topological space and $f : Y \to \mathbb{R}$ be a mapping which is continuous at some $y_0 \in Y$. Show that, for any $a \in \mathbb{R}$, the mapping

$$af : Y \longrightarrow \mathbb{R}, \quad (y \in Y) \longmapsto af(y)$$

is continuous at y_0.

(5) Show that the mapping

$$\mathbb{R} \times \mathbb{R}, \quad (a, b) \longmapsto ab$$

is locally Lipschitzian at any point $(a_0, b_0) \in \mathbb{R} \times \mathbb{R}$.

(6) Let Y be a topological space and $f : Y \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be mappings. Assume that both f and g are continuous at some $y_0 \in Y$. Show that the mapping

$$fg : Y \longrightarrow \mathbb{R}, \quad (y \in Y) \longmapsto f(y)g(y)$$

is continuous at y_0.

(7) Let P be an element of the polynomial ring $\mathbb{R}[T]$. Show that the mapping

$$\mathbb{R} \longrightarrow \mathbb{R}, \quad x \longmapsto P(x)$$

is continuous.

(8) Let n be a positive integer. We equip $\mathbb{R}^n$ with the product metric of the usual metric on $\mathbb{R}$.

(8.a) Show that, for any $i \in \{1, \ldots, n\}$, the mapping

$$p_i : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (x_1, \ldots, x_n) \longmapsto x_i$$

is 1-Lipschitzian.

(8.b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be an $\mathbb{R}$-linear mapping. Show that there exists $(a_1, \ldots, a_n) \in \mathbb{R}^n$ such that

$$\forall\,(x_1, \ldots, x_n) \in \mathbb{R}^n, \quad f(x_1, \ldots, x_n) = ax_1 + \cdots + ax_n.$$

(8.c) Deduce that any $\mathbb{R}$-linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$ is continuous.

(8.d) Let m be another positive integer. We equip $\mathbb{R}^m$ with the product metric of the usual metric on $\mathbb{R}$. Show that any $\mathbb{R}$-linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous.

11. Prove that the following subsets of $\mathbb{R}^2$ (equipped with the product metric of the usual metric) are open. Draw the picture of each of these subsets.

(1) $\{(x, y) \in \mathbb{R}^2 \,|\, x < y\}$.
(2) $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\}$.
(3) $\{(x, y) \in \mathbb{R}^2 \,|\, xy > 1\}$.
(4) $\{(x, y) \in \mathbb{R}^2 \,|\, x \neq 0\}$
(5) $\{(x, y) \in \mathbb{R}^2 \,|\, |x| < |y|\}$.

12. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let R be a postive real number such that the sequence $(a_n R^n)_{n\in\mathbb{N}}$ is bounded.

(1) Let t be an element of $\mathbb{R}$ such that $|t| < R$. Show that the series

$$\sum_{n\in\mathbb{N}} a_n t^n$$

converges absolutely. We denote by $f(t)$ the limit of this series.

(2) For any $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the mapping sending $t \in \mathbb{R}$ to

$$\sum_{k=0}^{n} a_k t^k.$$

Let r be a positive real number such that $r < R$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $B(0, r)$.

(3) Deduce that the mapping $f : B(0, R) \to \mathbb{R}$ is continuous.

(4) Prove that the function

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \sum_{n\in\mathbb{N}} \frac{t^n}{n!}$$

is well-defined and is continuous.

(5) Prove that the function

$$B(0,1) \longrightarrow \mathbb{R}, \quad t \longmapsto \frac{(-1)^n}{n} t^n$$

is well-defined and is continuous.

13. Let $P \in \mathbb{R}[T]$ be a polynomial of odd degree. Show that there exists $t \in \mathbb{R}$ such that $P(t) = 0$.

14. In this exercise, we consider the mapping $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^3 - 3x - 3.$$

(1) Show that the mapping f is strictly increasing on $[1, +\infty[$.

(2) Show that the equation $x^3 - 3x - 3 = 0$ has a unique solution in the interval $[2, 2 + \frac{1}{9}]$.

(3) Use the algorithm of dichotomy to give an upper and a lower bound of the solution with an error $< 100^{-1}$.

15. Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a monotone mapping.

(1) Show that, for any interval J in $\mathbb{R}$, $f^{-1}(J)$ is an interval.

(2) Assume that $f(I)$ is an interval. Show that f is continuous.

(3) Prove that, if f is strictly monotone and continuous, then $f^{-1} : f(I) \to \mathbb{R}$ is a continuous mapping.

(4) Denote by $\ln :]0, +\infty[\to \mathbb{R}$ the inverse mapping of exp. Show that ln is a continuous mapping.