Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 9–1 : Differentiability : fundamental notions

1. State whether the following statements are true or false.

(1) $o(g(x)) \pm o(g(x)) = o(g(x))$, $x \to x_0$;
(2) $O(g_1(x)) \cdot o(g_2(x)) = o(g_1(x) \cdot g_2(x))$, $x \to x_0$;
(3) $o(O(g(x))) = o(g(x))$, $x \to x_0$;
(4) $O(o(g(x)) = o(g(x))$, $x \to x_0$;
(5) $o(x) = O(x)$, $o(x^2)/x = o(x)$, $x \to x_0$;
(6) $O(g_1(x)) \cdot O(g_2(x)) = O(g_1(x) \cdot g_2(x))$, $x \in I$;
(7) $o(g_1(x)) \cdot o(g_2(x)) = o(g_1(x) \cdot g_2(x))$, $x \to x_0$;
(8) $o(o(g(x))) = o(g(x))$, $x \to x_0$;
(9) $O(O(g(x))) = O(g(x))$, $x \to x_0$.

2. For all $\alpha > 0$, prove that $\exp\left(\frac{\ln B}{\ln \ln B}\right) = O(B^\alpha)$ when $B \geqslant 3$.

3. In this exercise, we denote by $\mathbb{C}[\![T]\!]$ the set of formal power series with coefficients in $\mathbb{C}$. Remind that, as a set, $\mathbb{C}[\![T]\!]$ identifies with $\mathbb{C}^{\mathbb{N}}$, the set of all sequences in $\mathbb{C}$ parametrized by $\mathbb{N}$. Moreover, a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}^{\mathbb{N}}$, which is viewed as a formal power series, is written formally as

$$\sum_{n\in\mathbb{N}} a_n T^n.$$

We equip $\mathbb{C}[\![T]\!]$ with an additive composition law and a multiplicative composition law as follows :

$$\Big(\sum_{n\in\mathbb{N}} a_n T^n\Big) + \Big(\sum_{n\in\mathbb{N}} b_n T^n\Big) = \sum_{n\in\mathbb{N}} (a_n + b_n) T^n,$$
$$\Big(\sum_{n\in\mathbb{N}} a_n T^n\Big)\Big(\sum_{n\in\mathbb{N}} b_n T^n\Big) = \sum_{n\in\mathbb{N}} \Big(\sum_{\substack{(\ell,m)\in\mathbb{N}^2\\ \ell+m=n}} a_\ell b_m\Big) T^n.$$

Recall that $\mathbb{C}[\![T]\!]$ forms a commutative unitary ring under these composition laws. Moreover, the mapping

$$\mathbb{C} \times \mathbb{C}[\![T]\!] \longrightarrow \mathbb{C}[\![T]\!], \qquad \Big(\lambda, \sum_{n\in\mathbb{N}} a_n T^n\Big) \longmapsto \sum_{n\in\mathbb{N}} (\lambda a_n) T^n$$

equip $\mathbb{C}[\![T]\!]$ with a structure of vector space over $\mathbb{R}$. We denote by $D : \mathbb{C}[\![T]\!] \to \mathbb{C}[\![T]\!]$ the formal derivative mapping defined as

$$D\Big(\sum_{n\in\mathbb{N}} a_n T^n\Big) = \sum_{n\in\mathbb{N}} (n+1)a_{n+1}T^n.$$

For any $R \in \,]0, +\infty]$, let $\mathbb{U}_R$ be the set of formal power series

$$\sum_{n\in\mathbb{N}} a_n T^n$$

such that

$$\forall\, r \in [0, R[, \quad \sum_{n\in\mathbb{N}} |a_n| r^n < +\infty.$$

In the remaining of the exercice, we fix a positive real number R.

(1) Show that D is an $\mathbb{C}$-linear mapping, which satisfy the following Lebniz rule :

$$\forall\, (\varphi, \psi) \in \mathbb{C}[\![T]\!], \quad D(\varphi\psi) = \varphi D(\psi) + D(\varphi)\psi.$$

(2) Prove that $\mathbb{U}_R$ is an $\mathbb{C}$-vector subspace and a subring of $\mathbb{C}[\![T]\!]$.

(3) Prove that, for any $\varphi \in \mathbb{U}_R$, one has $D(\varphi) \in \mathbb{U}_R$.

(4) Let

$$\varphi = \sum_{n\in\mathbb{N}} a_n T^n \in \mathbb{U}_R.$$

Prove that, for any $t \in \,]-R, R[$, the series

$$\sum_{n\in\mathbb{R}} a_n t^n$$

converges absolutely. We denote by $f_\varphi(t)$ the limit of this series.

(5) Let $\varphi \in \mathbb{U}_R$. Prove that $f_\varphi : \,]-R, R[\to \mathbb{C}$ is a continuous mapping.

(6) Let $\varphi \in \mathbb{U}_R$. Prove that the mapping $f_\varphi(\cdot)$ is differentiable on any point of $]-R, R[$ and the following equality holds

$$f'_\varphi = f_{D(\varphi)}.$$

(7) For any $t \in \mathbb{R}$, let $\exp(t)$ be the limit of the following series

$$\sum_{n\in\mathbb{N}} \frac{t^n}{n!}.$$

Prove that the mapping $\exp(\cdot)$ is differentiable on $\mathbb{R}$ and that $\exp' = \exp$.

(8) Recall that we have shown that $\exp(\cdot)$ is a strictly increasing mapping from $\mathbb{R}$ to $\mathbb{R}_{>0}$. Let $\ln : \mathbb{R}_{>0} \to \mathbb{R}$ be the inverse mapping of exp. Show that ln is differentiable and that

$$\forall\, x \in \mathbb{R}_{>0}, \quad \ln'(x) = \frac{1}{x}.$$

(9) Prove that the function

$$\mathbb{R}^{\times} \longrightarrow \mathbb{R}, \quad x \longmapsto \ln|x|$$

is differentiable at any point of $\mathbb{R}^{\times}$ and its derivative is $x \mapsto 1/x$.

(10) Let $a \in \mathbb{R}$. Denote by $f_a : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ the mapping sending $t \in \mathbb{R}_{>0}$ to $t^a := \exp(a\ln(t))$. Show that f_a is differentiable and

$$f_a'(t) = at^{a-1}.$$

(11) Let $b > 0$. Denote by $g_b : \mathbb{R} \to \mathbb{R}_{>0}$ the mapping sending $t \in \mathbb{R}$ to $b^t := \exp(t\ln(b))$. Prove that g_b is differentiable on $\mathbb{R}$ and

$$g_b'(t) = b^t \ln(b).$$

(12) Let ω be a complex number. Prove that the mapping from $\mathbb{R}$ to $\mathbb{C}$ sending t to $\exp(\omega t)$ is differentiable and determine its derivative. One can show that the mapping from $\mathbb{R}$ to $\mathbb{C}$ sending $t \in \mathbb{R}$ to $\omega t \in \mathbb{C}$ is a bounded $\mathbb{R}$-linear mapping.

(13) Prove that, for any $t \in \mathbb{R}$, one has

$$\cos(t) = \frac{\exp(it) + \exp(-it)}{2}, \quad \sin(t) = \frac{\exp(it) - \exp(-it)}{2i}.$$

(14) Prove that the mappings $\sin : \mathbb{R} \to \mathbb{R}$ and $\cos : \mathbb{R} \to \mathbb{R}$ are differentiable. Moreover, the following equality holds :

$$\sin' = \cos, \quad \cos' = -\sin.$$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable mapping. We suppose that there exists $\ell \in \mathbb{R}$ such that

$$\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = \ell.$$

(1) Prove that there exists $(a, b, c) \in \mathbb{R}^3$ such that $a < b < c$ and that

$$(f(b) - f(a))(f(b) - f(c)) \geqslant 0.$$

(2) Deduce that there exists $\xi \in \mathbb{R}$ such that $f'(\xi) = 0$.

5. Let $(a,b) \in \mathbb{R}^2$ such that $a < b$. Let $f, g : [a,b] \to \mathbb{R}$ be continuous mapping that is differentiable on $]a,b[$. We assume that $g'(x) \neq 0$ for any $x \in\,]a,b[$.

(1) Denote by α the real number

$$\frac{f(b) - f(a)}{g(b) - g(a)}.$$

Prove that $h = f - \alpha g$ is continuous on $[a,b]$ and differentiable on $]a,b[$. Verify that $h(b) = h(a)$.

(2) Prove that there exists $\xi \in\,]a,b[$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

6. We denote by $\iota : \mathbb{R}^\times \to \mathbb{R}^\times$ the mapping sending $x \in \mathbb{R}^\times$ to x^{-1}.

(1) Let $p \in \mathbb{R}^\times$. Prove that

$$\iota(x) - \iota(p) = -\frac{\iota(x)}{p}(x - p) = -\frac{1}{p^2}(x - p) + \frac{\iota(x)}{p^2}(x - p)^2.$$

(2) Prove that ι is differentiable on $\mathbb{R}^\times$ and

$$\iota'(x) = -\frac{1}{x^2}.$$

(3) Let I be an open interval and $f : I \to \mathbb{R}^\times$. Prove that, if f is differentiable at some point $x \in I$, then $1/f$ is also differentiable at x, and its derivative at x is equal to

$$-\frac{f'(x)}{f(x)^2}.$$

(4) Let n be an integer and $f_n : \mathbb{R}^\times \to \mathbb{R}^\times$ be the mapping sending $x \in \mathbb{R}^\times$ to x^n. Show that

$$f_n'(x) = nx^{n-1}.$$

(5) For any $t \in \mathbb{R} \setminus \{(n + \frac{1}{2})\pi \,|\, n \in \mathbb{Z}\}$, let

$$\tan(t) := \frac{\sin(t)}{\cos(t)}.$$

Prove that the mapping $\tan(\cdot)$ is differentiable and determine its derivative.

(6) For any $t \in \mathbb{R} \setminus \{n\pi \mid n \in \mathbb{Z}\}$, let

$$\cot(t) = \frac{\cos(t)}{\sin(t)}.$$

Show that the mapping $\cot(\cdot)$ is differentiable and determine its derivative.

7. (1) Prove that the restriction of the mapping sin to $[-\frac{\pi}{2}, \frac{\pi}{2}]$ is strictly increasing and the image of $[-\frac{\pi}{2}, \frac{\pi}{2}]$ by sin is $[-1, 1]$.

(2) Let $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ be the inverse mapping of the restriction of sin to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Prove that arcsin is continuous on $[-1, 1]$ and is differentiable on $]-1, 1[$. Determine its derivative.

(3) Prove that the restriction of the mapping cos to $[0, \pi]$ is strictly decreasing and the image of $[0, \pi]$ by cos is $[-1, 1]$.

(4) Let $\arccos : [-1, 1] \to [0, \pi]$ be the inverse mapping of the restriction of cos to $[0, \pi]$. Prove that arccos is continuous on $[-1, 1]$ and is differentiable on $]-1, 1[$. Determine its derivative.

(5) Prove that the restriction of tan to $]-\frac{\pi}{2}, \frac{\pi}{2}[$ is strictly increasing and its image is $\mathbb{R}$.

(6) Let $\arctan : \mathbb{R} \to]-\frac{\pi}{2}, \frac{\pi}{2}[$ be the inverse of the restriction of tan to $]-\frac{\pi}{2}, \frac{\pi}{2}[$. Prove that the mapping arctan is differentiable and determine its derivative.

8. Let a and b be positive real numbers such that $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$.

(1) Prove that there exists $\xi \in]a, b[$ such that

$$\frac{f(b) - f(a)}{\ln(b) - \ln(a)} = \xi f'(\xi).$$

(2) Let x be a real number such that $x > 1$. Prove that, for any $n \in \mathbb{N}_{\geqslant 1}$, there exists $\xi_n \in]1, x[$ such that

$$x^{1/n} - 1 = \frac{1}{n}\xi_n^{1/n} \ln(x).$$

(3) Prove that

$$\lim_{n \to +\infty} n(x^{1/n} - 1) = \ln(x).$$

9. For any $n \in \mathbb{N}_{\geqslant 1}$, let $f_n : [0, 1] \to [0, 1]$ be the mapping sending $t \in [0, 1]$ to t^n.

(1) Show that the sequence $(f_n)_{n\in\mathbb{N}_{\geqslant 1}}$ converges simply to some mapping. Determine the limit mapping.
(2) Is the above convergence uniform?

10. Let X be a topological space and $(E, \|\cdot\|)$ be a Banach space. Let I be an infinite subset of $\mathbb{N}$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence of mappings from X to E. For any $n \in \mathbb{N}$, let

$$\|f_n\|_{\sup} := \sup_{x\in X} \|f_n(x)\| \in [0, +\infty].$$

We say that the series $\sum_{n\in I} f_n$ *converges normally* if

$$\sum_{n\in I} \|f_n\|_{\sup} < +\infty.$$

(1) Suppose that the series $\sum_{n\in I} f_n$ converges normally. Prove that $\sum_{n\in I} f_n$ converges uniformly.
(2) For any $n \in \mathbb{N}_{\geqslant 1}$ and any $x \in \mathbb{R}$, let

$$f_n(x) = \frac{nx}{1 + n^5x^2}.$$

Prove that the series $\sum_{n\in\mathbb{N}_{\geqslant 1}} f_n$ converges normally.

11. (1) Let a be a positive real number. Prove that

$$\frac{1}{a+1} < \ln(a+1) - \ln(a) < \frac{1}{a}.$$

(2) Prove that the function $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined as

$$f(x) = \left(1 + \frac{1}{x}\right)^x$$

is increasing.
(3) Prove that the function $g : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ defined as

$$g(x) = \left(1 + \frac{1}{x}\right)^{x+1}$$

is decreasing.
(4) Determine

$$\lim_{x\to+\infty} f(x) \quad \text{and} \quad \lim_{x\to+\infty} g(x)$$