Quantum Mechanics 2025 HW4

Due 10/09 in Class

September 30, 2025

Problem 1. Probability current

Consider the Schrödinger equaiton

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r},t)\right)\psi(\mathbf{r},t). \tag{1}$$

Define the probability density $\rho(x,t) = \psi^\dagger(x,t)\psi(x,t)$. In order to have the probability conservation, i.e., $\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0$, how should we define the probability current $\mathbf{j}(\mathbf{r},t)$?

Problem 2. time-evolution

Consider a physical observable O, and its expectation value $\bar{O}(t) = \langle\psi(t)|O|\psi(t)\rangle$. The Hamiltonian of the system is time-independent.

Prove the following statements:

1) If $[O,H] = 0$, then $\frac{d}{dt}\bar{O}(t) = 0$.

2) If $|\psi(t)\rangle$ itself is the time-evolution of an eigen-state of H, then $\frac{d}{dt}\bar{O}(t) = 0$.

Problem 3. The f-sum rule

Consider a particle in 1D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \tag{2}$$

By calculating $[[H,x],x]$ prove

$$\sum_l |\langle l|x|n\rangle|^2(E_l - E_n) = \hbar^2/2m, \tag{3}$$

where $|n\rangle$ and $|k\rangle$ are energy eigenstates of H with the eigenvalues E_n and E_k, respectively. This is essentially the quantization condition that Heisenberg used to establish quantum mechanics.

Problem 4. The double δ-potential

A particle of mass m moves in 1D x under the potential $V(x)$.

$$V(x) = \gamma\left(\delta(x + \tfrac{a}{2}) + \delta(x - \tfrac{a}{2})\right). \tag{4}$$

1) Consider the scattering states: At $x < -\frac{a}{2}$, it is a superposition of an incident wave superposed with a reflection wave $\psi(x) = e^{ikx} + Re^{-ikx}$. At $x > \frac{a}{2}$, it is the transmission wave $\psi(x) = Se^{ikx}$. Find the expression of the scattering amplitude R and S.

2) Please find the conditions for the perfect transmission, i.e., $|S| = 1$.

3) Please find the bound state solutions at $\gamma < 0$ Express R and S in terms of E. Please show that the bound state information shows up as the pole of the scattering amplitude.

Problem 5. Bound states

Consider a 1D potential $V(x) \leq 0$ for $+\infty < x < +\infty$. As $|x| \to +\infty$, $V(x) \to 0$, and $V(x = 0) \neq 0$. Please prove that for the Hamiltonian

$$H = -\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2} + V(x), \tag{5}$$

there must exist at least one bound state $\psi_b(x)$ whose eigen-energy $E_b < 0$.

(Hint: Consider the following problem. Suppose two potential functions $V_{1,2}(x)$ satisfying $V_1(x) < V_2(x)$ for $+\infty < x < +\infty$, then prove that the ground state energy E_{g1} for the system with potential V_1 is smaller than that of E_{g2} for the system with potential V_2.)

Problem 6. Hermite Polynomial

In class, we derived the differential equation

$$\tfrac{d^2}{dz^2}u_n(z) - 2z\tfrac{d}{dz}u_n(z) + (\lambda_n - 1)u_n(z) = 0. \tag{6}$$

1) Try the solution that $u_n(z) = \sum_k a_k z^k$. Figure out the recursion relation of the coefficient a_k. Show that only when $\lambda_n = 2n+1$ for $n = 0, 1, 2, ...$, the expression of $u_n(z)$ can be truncated into a polynomial. The corresponding energy level is $E_n = (n+\frac{1}{2})\hbar\omega$. Then up to a normalization factor $u_n = H_n(z)$ with $H_n(z)$ the Hermite polynomial.

2) According to the generation function $G(z,s) = e^{-s^2+2zs} = \sum_0^{+\infty} \frac{1}{n!}H_n(z)s^n$, prove that

$$H_n(z) = (-)^n e^{z^2} \tfrac{d^n}{dz^n} e^{-z^2}. \tag{7}$$

3) According to $\frac{\partial G(z,s)}{\partial s} = -2(s-z)G(z,s)$, prove the recursion relation

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0. \tag{8}$$

According to $\frac{\partial G(z,s)}{\partial z} = 2sG(z,s)$, prove that $\frac{d}{dz}H_n = 2nH_{n-1}$.

4) Set $G_1(z,s) = e^{-s^2+2zs} = \sum_0^{+\infty}\frac{1}{n!}H_n(z)s^n$ and $G_2(z,t) = e^{-t^2+2tz} = \sum_0^{+\infty}\frac{1}{n!}H_n(z)t^n$. Prove that

$$\int_{-\infty}^{+\infty} dz G_1(z,s)G_2(z,t)e^{-z^2} = \sqrt{\pi}e^{2st}. \tag{9}$$

According to this result, prove that

$$\int_{-\infty}^{+\infty} dz H_n(z)H_m(z)e^{-z^2} = \delta_{nm}2^n n!\sqrt{\pi}. \tag{10}$$