QM HW2

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Problem 1 (Harmonic osicllator)
(1) We've known that $x_{nm} \neq 0 \Rightarrow m = n \pm 1$

$$x_{n,n}^2 = \sum_k x_{n,k} x_{k,n} = x_{n,n-1} x_{n-1,n} + x_{n,n+1} x_{n+1,n} \tag{1.1}$$

Since

$$x_{n+1,n} = \sqrt{\frac{\hbar}{2m\omega_0}(n+1)} \tag{1.2}$$

We have

$$\boxed{x_{n,n}^2 = \frac{\hbar}{m\omega_0}\left(n + \frac{1}{2}\right)} \tag{1.3}$$

$$(p^2)_{n,n} = m^2(\dot{x})_{n,n}^2 = \hbar\omega_0 m(n + 1/2) \tag{1.4}$$

$$\boxed{\sqrt{(x^2)_{n,n}(p^2)_{n,n}} = \hbar\left(n + \frac{1}{2}\right)} \tag{1.5}$$

When $n = 0$,

$$\sqrt{(x^2)_{n,n}(p^2)_{n,n}} = \frac{\hbar}{2} \tag{1.6}$$

(2)

$$\boxed{x_{n,m}(t) = \sqrt{\frac{\hbar}{2m\omega_0}}\left(\sqrt{n}e^{i\omega_0 t}\delta_{m,n-1} + \sqrt{n+1}e^{-i\omega_0 t}\delta_{m,n+1}\right)} \tag{1.7}$$

$$\boxed{p_{n,m}(t) = i\sqrt{\frac{\hbar m\omega_0}{2}}\left(\sqrt{n}e^{i\omega_0 t}\delta_{m,n-1} - \sqrt{n+1}e^{-i\omega_0 t}\delta_{m,n+1}\right)} \tag{1.8}$$

$$\begin{aligned}
[x,p]_{n,m}(t) &= \sum_k \left(x_{n,k}p_{k,m}e^{i\omega_{n,m}t} - p_{m,k}x_{k,n}e^{i\omega_{m,n}t}\right) && (1.9)\\
&= [x_{n,n-1}p_{n-1,m} + x_{n,n+1}p_{n+1,m} \\
&\quad - p_{m,n-1}x_{n-1,n} - p_{m,n+1}x_{n+1,n}\ ]\,\delta_{nm} && (1.10)\\
&= \boxed{i\hbar\delta_{nm}} && (1.11)
\end{aligned}$$

Problem 2 (Quantum equation of motion)
(1) By associative law, $\forall H_2(x)$,

$$[x, H_2(x)] = 0 \tag{2.1}$$

We expand $H_1(p)$ in form of

$$H_1(p) = \sum_{n=0}^{\infty} a_n p^n \tag{2.2}$$

We have

$$[x, p^n] = \sum_{i=0}^{n-1} p^i [x, p] p^{n-i-1} = in\hbar p^{n-1} = i\hbar \frac{\partial(p^n)}{\partial p} \tag{2.3}$$

Hence,

$$\boxed{\frac{\partial H}{\partial p} = \frac{1}{i\hbar}[x, H]} \tag{2.4}$$

Similarly,

$$\boxed{-\frac{\partial H}{\partial x} = \frac{1}{i\hbar}[p, H]} \tag{2.5}$$

Therefore,

$$\dot{x} = \frac{i}{\hbar}[x, H], \quad \dot{p} = \frac{i}{\hbar}[p, H] \tag{2.6}$$

(2) $\forall (n, m) \in \mathbb{N}^2$

$$[x, x^n p^m] = x x^n p^m - x^n p^m x = x^n (x p^m - p^m x) = x^n [x, p] \tag{2.7}$$

So,

$$\boxed{[x, x^n p^m] = i\hbar x^n \frac{\partial(p^m)}{\partial p}} \tag{2.8}$$

Hence (2.4) and (2.5) still hold. Let $x = x, y,\ p = p_x, p_y$.

$$\begin{aligned} \dot{x} &= \frac{1}{i\hbar}[x, H], \quad \dot{p}_x = \frac{1}{i\hbar}[p_x, H] \\ \dot{y} &= \frac{1}{i\hbar}[y, H], \quad \dot{p}_y = \frac{1}{i\hbar}[p_y, H] \end{aligned} \tag{2.9}$$

Problem 3 (De Broglie wave)
By relativity,

$$p = mv_g \ , \ E = mc^2 \tag{3.1}$$

$$v_g = \frac{pc^2}{E} \tag{3.2}$$

In wave case,

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \tag{3.3}$$

From

$$E^2 = p^2c^2 + m_0^2c^4 \tag{3.4}$$

we deduce that

$$\frac{pc^2}{E} = \frac{\mathrm{d}E}{\mathrm{d}p} = \hbar\frac{\mathrm{d}\omega}{\mathrm{d}p} \tag{3.5}$$

Thus,

$$\mathrm{d}p = \hbar \mathrm{d}k$$

$$\boxed{p = \hbar k} \tag{3.6}$$

Problem 4 (Schrödinger equation)
Let

$$\Psi(r,t) = e^{iS/k} = e^{i[W(r)-Et]/k} = \psi(r)e^{-iEt/k} \tag{4.1}$$

Considering $E = \hbar\omega$, take $k = \hbar$. Then

$$S = -i\hbar \ln\psi(r) - Et \tag{4.2}$$

Plug in

$$(\nabla S)^2 = 2m\left(E - V(r)\right) \tag{4.3}$$

we obtain

$$\frac{\hbar^2}{2m}\left(\nabla\psi\right)^2 + \left(E - V(r)\right)\psi^2 = 0 \tag{4.4}$$

Now, we want to linearize the equation. Let ∇ act on (4.4) and eliminate the common factor

$$\frac{\hbar^2}{2m}\nabla^2\psi + \left(E - V(r)\right)\psi + \frac{\nabla(E - V(r))\psi^2}{2\nabla\psi} = 0 \tag{4.5}$$

God believes S, E, V are smooth at the $\hbar$ scale. We ignore the last term.

$$\boxed{\frac{\hbar^2}{2m}\nabla^2\psi + \left(E - V(r)\right)\psi = 0} \tag{4.6}$$

Problem 5 (Quantum Potential)
Plug

$$\frac{\partial \Psi}{\partial t} = \frac{i}{\hbar}\frac{\partial S}{\partial t} e^{iS/\hbar} \tag{5.1}$$

$$\nabla \Psi = \frac{i}{\hbar} \nabla S e^{iS/\hbar} \tag{5.2}$$

$$\nabla^2 \Psi = \frac{1}{\hbar^2}\left[i\hbar \nabla^2 S - (\nabla S)^2\right] e^{iS/\hbar} \tag{5.3}$$

into Schrödinger equation,

$$\frac{\partial S}{\partial t} = \frac{1}{2m}\left[i\hbar \nabla^2 S - (\nabla S)^2\right] - V \tag{5.4}$$

When taking the limit $\hbar \to 0$,

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V = 0 \tag{5.5}$$

It is exactly the Hamilton-Jacobi equation. The quantum potential is

$$\boxed{V_{\text{quantum}} = -\frac{i}{2m\hbar}\nabla^2 S} \tag{5.6}$$