Westlake University
Fundamental algebra and analysis I

Final exam of December 30th 2024, 12:00-18:00

The use of any electronic devices is strictly prohibited. The statement of a question, even if not justified, is permitted to be used in the answers to subsequent questions. The degree of clarity in the writing is an important evaluation factor in this exam.

Part I: Power series

Preamble. In this part, we fix a field K equipped with an absolute value $|\cdot|$. We assume that K is complete under the metric

$$\big((a,b) \in K \times K\big) \longmapsto |a-b|.$$

By convention, for any $a \in K$, the expression a^0 denotes the multiplicative neutral element $1 \in K$.

We also fix a normed vector space $(V, \|\cdot\|)$ over K, which is again assumed to be complete under the metric

$$\big((x,y) \in V \times V\big) \longmapsto \|x-y\|.$$

In other words, $(V, \|\cdot\|)$ is a Banach space over K.

If $(x_n)_{n\in\mathbb{N}}$ is a sequence in V, then the expression $\sum_{n\in\mathbb{N}} x_n$ denotes the sequence

$$\Big(\sum_{j\in\{0,\ldots,n\}} x_j\Big)_{n\in\mathbb{N}}$$

and is called the series associated with the sequence $(x_n)_{n\in\mathbb{N}}$. If this sequence converges in V, we say that the series $\sum_{n\in\mathbb{N}} x_n$ *converges.* If the series of real numbers $\sum_{n\in\mathbb{N}} \|x_n\|$ converges, we say that the series $\sum_{n\in\mathbb{N}} x_n$ *converges absolutely.*

Since the normed vector space $(V, \|\cdot\|)$ is assumed to be complete, we have proved in the course that, if a series $\sum_{n\in\mathbb{N}} x_n$ in V converges absolutely, then it converges.

Let X be set and $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of mappings from X to V.

(1) We say that the series $\sum_{n\in\mathbb{N}} \varphi_n$ *converges pointwisely* if, for any $x \in X$, the series $\sum_{n\in\mathbb{N}} \varphi_n(x)$ converges in V.

(2) We say that the series $\sum_{n\in\mathbb{N}} \varphi_n$ *converges normally* if

$$\sum_{n\in\mathbb{N}} \|\varphi_n\|_{\sup} < +\infty,$$

where

$$\|\varphi_n\|_{\sup} := \sup_{x\in X} \|\varphi_n(x)\|.$$

Since $(V, \|\cdot\|)$ is assumed to be complete, if the series $\sum_{n\in\mathbb{N}} \varphi_n$ converges normally, then it converges pointwisely to a certain mapping $\varphi : X \to V$. Moreover, $(\varphi_n)_{n\in\mathbb{N}}$ also converges uniformly to φ.

Questions.

1. Let ε be an element of $[0, 1[$. Prove that

$$\sum_{n\in\mathbb{N}} n\varepsilon^n \leqslant \frac{\varepsilon}{(1-\varepsilon)^2}.$$

Hint: One can write $n\varepsilon^n$ *as* $\sum_{k=1}^{n} \varepsilon^n$ *and then switch the order of summations in*

$$\sum_{n=1}^{N} \sum_{k=1}^{n} \varepsilon^n$$

to get

$$\sum_{n=1}^{N} n\varepsilon^n = \sum_{k=1}^{N} \sum_{n=k}^{N} \varepsilon^n.$$

2. Let ε be an element of $[0, 1[$. Prove that

$$\sum_{n\in\mathbb{N},\, n\geqslant 2} \binom{n}{2} \varepsilon^n \leqslant \frac{\varepsilon^2}{(1-\varepsilon)^3}.$$

Hint: Write $\binom{n}{2}$ *as* $1 + 2 + \cdots + (n-1)$*, and then perform a swith of summation order as in the previous question.*

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3. Let α and β be elements of K and $n \in \mathbb{N}$. Prove the following equality

$$\beta^n - \alpha^n - n\alpha^{n-1}(\beta - \alpha) = (\beta - \alpha)^2 \sum_{\ell=0}^{n-2} (n - \ell - 1)\beta^\ell \alpha^{n-\ell-2}.$$

4. Let $n \in \mathbb{N}$ and $a \in K$. Prove that the mapping

$$g_n : K \longrightarrow K, \quad b \longmapsto (b - a)^n$$

is differentiable and determine the derivative g_n'.

5. Prove that the mapping

$$K \times V \longrightarrow V, \quad (b, x) \longmapsto bx$$

is differentiable and determine its differential.
(*Hint: One can use the fact that it is a bounded bilinear mapping.*)

6. Let x be an element of V, $n \in \mathbb{N}$ and $a \in K$. Let f_n be the mapping from K to V that sends $b \in K$ to $(b - a)^n x$. Prove that $f_n : K \to V$ is differentiable and determine f_n'.

7. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in V, $a \in K$ and R be a positive real number. Assume that, for any $r \in [0, R[$, the sequence $(r^n \|x_n\|)_{n \in \mathbb{N}}$ is bounded. For any $n \in \mathbb{N}$, let $p_n : K \to V$ be the mapping that sends $b \in K$ to $(b - a)^n x_n$.

(1) Prove that, for any positive real number r such that $r < R$, the series of mappings $\sum_{n \in \mathbb{N}} p_n$ converges normally on

$$\overline{B}(a, r) = \{b \in K \mid |b - a| \leqslant r\}.$$

(2) Deduce that the series $\sum_{n \in \mathbb{N}} p_n$ converges simply on

$$B(a, R) = \{b \in K \mid |b - a| < R\}.$$

We denote by $f : B(a, R) \to V$ the pointwise limit of the series $\sum_{n \in \mathbb{N}} p_n$ on $B(a, R)$.

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(3) Prove that, for any positive real number r such that $r < R$, the series of mappings

$$\sum_{n\in\mathbb{N}} p'_n$$

converges normally on $\overline{B}(a,r)$. Deduce that the series converges simply on $B(a,R)$. We denote by $g : B(a,R) \to V$ the pointwise limit of the series

$$\sum_{n\in\mathbb{N}} p'_n$$

on $B(a,R)$.
(*Hint: use Question* **1.**)

(4) Prove that f is differentiable on $B(a,R)$ and, for any $b \in B(a,R)$, $f'(b) = g(b)$.

(5) Deduce that, for any $k \in \mathbb{N}$, the mapping f is of class C^k.
(*Hint: reason by induction on* k.)

8. Let M be a positive real number. Prove that

$$\lim_{n\to+\infty} M^{\frac{1}{n}} = 1.$$

(*Hint: Let* ε *be an arbitrary positive real number. Compare* $(1+\varepsilon)^n$ *with* M *and* M^{-1}.)

9. Let $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$ be a sequence in V. We denote by

$$R(\boldsymbol{x}) = \sup\{r \in \mathbb{R}_{\geqslant 0} \,|\, (r^n\|x_n\|)_{n\in\mathbb{N}} \text{ is bounded}\} \in [0,+\infty].$$

We call $R(\boldsymbol{x})$ the *radius of convergence* of the power series

$$\sum_{n\in\mathbb{N}} b^n x_n, \quad b \in K.$$

(1) Prove that the set

$$I(\boldsymbol{x}) := \{r \in \mathbb{R}_{\geqslant 0} \,|\, (r^n\|x_n\|)_{n\in\mathbb{N}} \text{ is bounded}\}$$

is an interval.

(2) Deduce that, for any non-negative real number r such that $r < R(\boldsymbol{x})$, the sequence $(r^n\|x_n\|)_{n\in\mathbb{N}}$ is bounded.

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(3) Prove the following equality (here by convention $0^{-1} = +\infty$)

$$R(\boldsymbol{x})^{-1} = \limsup_{n\to+\infty} \|x_n\|^{\frac{1}{n}}.$$

(4) Suppose that $x_n \neq \mathbf{0}$ for any $n \in \mathbb{N}$. Prove that

$$\liminf_{n\to+\infty} \frac{\|x_n\|}{\|x_{n+1}\|} \leqslant R(\boldsymbol{x}) \leqslant \limsup_{n\to+\infty} \frac{\|x_n\|}{\|x_{n+1}\|}$$

Deduce that, if the sequence

$$\Big(\frac{\|x_n\|}{\|x_{n+1}\|}\Big)_{n\in\mathbb{N}}$$

has a limit ℓ, then $R(\boldsymbol{x}) = \ell$.
Hint: For natural numbers n and N such that $n < N$, express $\|x_N\|$ as

$$\|x_n\| \prod_{j=n}^{N-1} \frac{\|x_{j+1}\|}{\|x_j\|}.$$

Then, use

$$s_n = \sup_{j\in\mathbb{N},\, j\geqslant n} \frac{\|x_j\|}{\|x_{j+1}\|}, \quad \ell_n := \inf_{j\in\mathbb{N},\, j\geqslant n} \frac{\|x_j\|}{\|x_{j+1}\|}$$

to estimate $\|x_N\|$ and obtain $\ell_n \leqslant R(\boldsymbol{x}) \leqslant s_n$.

10. In this question, we suppose that $(K, |\cdot|)$ is $\mathbb{R}$ equipped with the usual absolute value. Let U be an open subset of $\mathbb{R}$, $a \in U$ and $f : U \to V$ be a mapping which is of class C^∞. Let R be a positive real number such that $]a-R, a+R[\, \subseteq U$. Suppose that, for any $r \in [0, R[$, one has

$$\lim_{n\to+\infty} \frac{r^n}{n!} \sup_{b\in[a-r,a+r]} \|f^{(n)}(b)\| = 0.$$

Prove that the series of functions

$$\sum_{n\in\mathbb{N}} \frac{(b-a)^n}{n!} f^{(n)}(a), \quad b \in \mathbb{R}$$

converges pointwisely on $]a-R, a+R[$ to f.
(*Hint: Apply Taylor-Lagrange inequality.*)

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Part II: Exponential in a Banach algebra

Preamble. As in Part I, we fix a complete valued field $(K, |\cdot|)$. We call a *Banach algebra* over K any complete normed vector space $(A, \|\cdot\|)$ over K equipped with an K-bilinear mapping (called the *multiplication*)

$$A \times A \longrightarrow A, \quad (x, y) \longmapsto xy$$

such that

(a) A equipped with the multiplication forms a monoid.

(b) for any $(x, y) \in A \times A$,

$$\|xy\| \leqslant \|x\| \cdot \|y\|.$$

Questions.

11. Let $(V, \|\cdot\|_V)$ be a Banach space over K. Let $\mathscr{L}(V)$ the K-vector space of all bounded K-linear mappings from V to itself. We equipped $\mathscr{L}(V)$ with the operator norm $\|\cdot\|$ defined as

$$\forall\, \varphi \in \mathscr{L}(V), \quad \|\varphi\| := \sup_{x \in V \setminus \{0\}} \frac{\|\varphi(x)\|_V}{\|x\|_V}.$$

Prove that $\mathscr{L}(V)$ equipped with the following composition law

$$\Big((\psi, \varphi) \in \mathscr{L}(V) \times \mathscr{L}(V)\Big) \longmapsto \psi \circ \varphi,$$

forms a Banach algebra over K.

Hint: By a result of the course, $\mathscr{L}(V)$ equipped with the operator norm forms a Banach space (you need not reprove this result in the exam).

12. We equip $\mathbb{R}$ with the usual absolute value. Let $\mathbb{C}$ be the field of complex numbers and $|\cdot| : \mathbb{C} \to \mathbb{R}$ be the mapping defined as

$$\forall\, (a, b) \in \mathbb{R}^2, \quad |a + bi| = \sqrt{a^2 + b^2}.$$

Check that, $(\mathbb{C}, |\cdot|)$ equipped with the usual multiplication of complex numbers forms a Banach algebra over $\mathbb{R}$.

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Hint: One can use without justification the fact that the mapping of usual multiplication

$$\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}, \quad (z, w) \longmapsto zw$$

is $\mathbb{R}$-bilinear and defines a structure of commutative monoid on $\mathbb{C}$.

In the rest of Part II, we assume that $(K, |\cdot|)$ is $\mathbb{R}$ equipped with the usual absolute value.

We fix a Banach algebra $(A, \|\cdot\|)$ over K. For any $\varphi \in A$, we denote by $\boldsymbol{x}_\varphi$ the sequence

$$\left(\frac{1}{n!}\varphi^n\right)_{n\in\mathbb{N}},$$

where by convention $0! = 1$ and φ^0 denotes the multiplicative neutral element of A.

13. Prove that $R(\boldsymbol{x}_\varphi) = +\infty$. For any $t \in \mathbb{R}$, denote by $\exp(t\varphi)$ the limit of the series

$$\sum_{n\in\mathbb{N}} \frac{t^n}{n!}\varphi^n.$$

14. Let φ be an element of A. Prove that the mappings $A \to A$, $\psi \mapsto \varphi\psi$ and $A \to A$, $\psi \mapsto \psi\varphi$ are both continuous.

15. Prove that, for any $t \in \mathbb{R}$, $\exp(t\varphi)\varphi = \varphi\exp(t\varphi)$.
Hint: Use the previous question and the fact that $\exp(t\varphi)$ *is the limit of the sequence*

$$\left(\sum_{k=0}^{n} \frac{t^k}{k!}\varphi^k\right)_{n\in\mathbb{N}}.$$

16. Prove that the mapping $t \mapsto \exp(t\varphi)$ is differentiable and determine its derivative.

17. Prove that the mapping $(t \in \mathbb{R}) \mapsto \exp(t\varphi)$ is a morphism of monoids from $(\mathbb{R}, +)$ to $(A, \cdot)$.
Hint: For fixed $s \in \mathbb{R}$, consider the derivative of the mapping

$$\Phi_s : (t \in \mathbb{R}) \longmapsto \exp((t+s)\varphi)\exp(-t\varphi).$$

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18. The purpose of this exercice is to discuss properties of the exponential and logarithmic functions. We consider the mapping $\exp(\cdot) : \mathbb{R} \to \mathbb{R}$ that sends $t \in \mathbb{R}$ to the limit of

$$\sum_{n\in\mathbb{N}} \frac{t^n}{n!}.$$

(1) Prove that, for any $t \in \mathbb{R}$, $\exp(t) > 0$.

(2) Prove that the mapping $\exp(\cdot) : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

(3) Prove that the image of the mapping $\exp(\cdot)$ identifies with $\mathbb{R}_{>0}$.

(4) Denote by $\ln(\cdot) : \mathbb{R}_{>0} \to \mathbb{R}$ the inverse of the mapping $\exp(\cdot)$. Prove that $\ln(\cdot)$ is of class C^∞.

(5) For any $n \in \mathbb{N}$, compute the n-th derivative of $\ln(\cdot)$.

(6) Prove that the Taylor series of the mapping $\ln(\cdot)$ at 1 is given by

$$\sum_{n\in\mathbb{N}_{\geqslant 1}} (t-1)^n \frac{(-1)^{n-1}}{n}, \quad t \in \mathbb{R}.$$

Prove that this series converges pointwisely to $\ln(\cdot)$ on $]0, 2[$.

Hint: Prove that the series

$$\sum_{n\in\mathbb{N}_{\geqslant 0}} (-1)^n (t-1)^n$$

converges pointwisely on $]0, 2[$ *to* $1/t$, *and the convergence is uniform on any compact interval contained in* $]0, 2[$.

(7) Let α be a real number. We consider the mapping

$$f_\alpha : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}, \quad t \longmapsto \exp(\alpha \ln(t)).$$

Prove that f_α is a morphism of multiplicative groups and determine its Taylor series at 1.

19. In this question, we consider the mapping

$$\mathbb{R} \longrightarrow \mathbb{C}, \quad t \longmapsto \exp(ti),$$

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where $i \in \mathbb{C}$ denotes the imaginary unit. For $t \in \mathbb{R}$, we denote by $\cos(t)$ and $\sin(t)$ the real and imaginary parts of $\exp(ti)$ respectively. In other words, one has

$$\forall\, t \in \mathbb{R}, \quad \exp(ti) = \cos(t) + \sin(t)i$$

(1) Prove that, for any $t \in \mathbb{R}$

$$\exp(-ti) = \overline{\exp(ti)}.$$

Deduce that

$$\forall\, t \in \mathbb{R}, \quad |\exp(ti)| = 1.$$

(2) Prove that $\cos(\cdot)$ and $\sin(\cdot)$ are of class C^∞. Determine their derivatives.

(3) Let α be a non-zero real number. Prove that, for any $x \in \mathbb{R}$, there exists $n \in \mathbb{Z}$ such that

$$|x - n\alpha| < |\alpha|.$$

(4) Let G be a subgroup of $(\mathbb{R}, +)$. Prove that, either $\overline{G} = \mathbb{R}$, or G is discrete, namely for any $x \in G$, x has a neighbourhood V such that $V \cap G = \{x\}$.
Hint: Prove that G is discrete if and only if there is a neighbourhood V of 0 in $\mathbb{R}$ such that $V \cap G = \{0\}$.

(5) Prove that the kernel N of the morphism of groups

$$(\mathbb{R}, +) \longrightarrow (\mathbb{C} \setminus \{0\}, \times), \quad (t \in \mathbb{R}) \longmapsto \exp(ti)$$

is discrete.
Hint: Prove that N closed, and then deduce that, if N is not discrete, then it is equal to $\mathbb{R}$. Establish a contradiction by considering the derivative tof the mapping $t \mapsto \exp(ti)$.

(6) Prove that there exists a unique real number $\pi > 0$ such that

$$N = \{2\pi n \,|\, n \in \mathbb{Z}\}.$$

(7) Prove that $\exp(\pi i) = -1$.

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(8) Prove that $\sin(x) > 0$ when $x \in]0, \pi[$.
Hint: First prove that $\sin(x) \neq 0$ *for any* $x \in]0, \pi[$. *Then prove that there exists* $x_0 \in]0, \pi[$ *such that* $\sin(x_0) > 0$. *One can examine the derivative of* $\sin(\cdot)$ *at* 0. *Finally prove that* $\sin(\cdot)$ *cannot take negative values on* $]0, \pi[$ *by using the theorem of intermediate values.*

(9) Prove that the images of $\cos(\cdot)$ and $\sin(\cdot)$ are both equal to $[-1, 1]$.

(10) Prove that $\sin(\frac{\pi}{2}) = 1$.
Hint: Let $z = \cos(\frac{\pi}{2}) + \sin(\frac{\pi}{2})i$ *and determine* z^2.

(11) Prove that $\cos(\cdot)$ is strictly decreasing on $[0, \frac{\pi}{2}]$ and $\sin(\cdot)$ is strictly increasing on $[0, \frac{\pi}{2}]$.

(12) Prove that the image of $(t \in \mathbb{R}) \mapsto \exp(ti)$ is equal to

$$\{z \in \mathbb{C} \,|\, |z| = 1\}.$$

20. Let $(V, \|\cdot\|)$ be a Banach space over $\mathbb{R}$ and $A = \mathscr{L}(V)$ equipped with the composition of mappings and the operator norm (see Question **11.**).

(1) Let x_0 be an element of X and $\Phi : \mathbb{R} \to V$ be the mapping that sends $t \in \mathbb{R}$ to $\exp(t\varphi)(x_0)$. Prove that Φ is differentiable and

$$\Phi'(t) = \varphi(\Phi(t)).$$

(2) Resolve the following ordinary differential equation in $\mathbb{R}^2$:

$$\begin{cases} x'(t) = y(t), \\ y'(t) = -x(t), \end{cases} \qquad (x(0), y(0)) = (0, 1).$$

Part III: Summation as an integral

Preamble. We denote by $\mathbb{R}^{\oplus \mathbb{N}}$ the set of mappings $f : \mathbb{N} \to \mathbb{R}$ such that

$$\{n \in \mathbb{N} \,|\, f(n) \neq 0\}$$

is finite. It is a vector subspace of $\mathbb{R}^{\mathbb{N}}$.

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Questions.

21. Check that $\mathbb{R}^{\oplus\mathbb{N}}$ is a Riesz space over $\mathbb{N}$. Namely, for any

$$(f,g) \in \mathbb{R}^{\oplus\mathbb{N}} \times \mathbb{R}^{\oplus\mathbb{N}}, \quad \Big((n \in \mathbb{N}) \longmapsto \min\{f(n), g(n)\}\Big) \in \mathbb{R}^{\oplus\mathbb{N}}.$$

22. Denote by $\Sigma : \mathbb{R}^{\oplus\mathbb{N}} \to \mathbb{R}$ the mapping which sends $f \in \mathbb{R}^{\oplus\mathbb{N}}$ to

$$\sum_{n\in\mathbb{N},\, f(n)\neq 0} f(n).$$

Prove that Σ is an integral operator, namely Σ is an $\mathbb{R}$-linear mapping, $\Sigma(f) \geqslant 0$ when $f(n) \geqslant 0$ for any $n \in \mathbb{N}$, and, for any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ that converges pointwisely to 0, one has

$$\lim_{n\to+\infty} \Sigma(f_n) = 0.$$

23. Let g be an element of $\mathbb{R}^{\mathbb{N}}_{\geqslant 0}$. By using the theorem of Beppo Levi, prove that g is Σ-integrable in the sense of Daniell if and only if the sequence

$$\sum_{k=0}^{n} g(k), \quad n \in \mathbb{N}$$

converges in $\mathbb{R}$.

24. Let f be an element of $\mathbb{R}^{\mathbb{N}}$. Prove that f is Σ-integrable in the sense of Daniell if and only if the series

$$\sum_{n\in\mathbb{N}} f(n)$$

converges absolutely.

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Part IV: Stieltjes integral

Preamble. We denote by S the $\mathbb{R}$-vector subspace of $\mathbb{R}^{\mathbb{R}}$ that is composed of the mappings of the form

$$\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i,b_i]},$$

where $n \in \mathbb{N}$ and for any $i \in \{1,\ldots,n\}$, (a_i,b_i) is an element of $\mathbb{R}\times\mathbb{R}$ such that $a_i < b_i$. We denote by $S^{\uparrow}$ the set of elements in $]-\infty,+\infty]^{\mathbb{R}}$ that can be written as a pointwise limit of an increasing sequence in S.

For any $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the integral part of x, namely

$$\lfloor x \rfloor := \sup\{n \in \mathbb{Z} \mid n \leqslant x\}.$$

We denote by $\langle x \rangle$ the decimal part of x, which is defined as

$$\langle x \rangle := x - \lfloor x \rfloor.$$

25. Prove that, for any $x \in \mathbb{R}$,

$$0 \leqslant \langle x \rangle < 1.$$

26. Denote by $\varphi : \mathbb{R} \to \mathbb{R}$ the mapping sending $x \in \mathbb{R}$ to $\lfloor x \rfloor$. Prove that φ is increasing and right-continuous. We denote by $I_\varphi : S \to \mathbb{R}$ the integral operator defined as follows

$$I_\varphi\left(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i,b_i]}\right) := \sum_{i=1}^{n} \lambda_i(\varphi(b_i) - \varphi(a_i)).$$

27. Let (a,b) be an element of $\mathbb{R}\times\mathbb{R}$ such that $a<b$. Prove that, for any $\lambda \in [0,+\infty]$, one has $\lambda \mathbb{1}_{]a,b[} \in S^{\uparrow}$, where by convention $(+\infty)0 := 0$.
Hint: Consider the sequence

$$\left(\lambda \mathbb{1}_{]a,b-(b-a)/n]}\right)_{n\in\mathbb{N}_{\geqslant 1}}.$$

28. Let n be an integer. Determine $I_\varphi(\mathbb{1}_{]n,n+1[})$.

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29. Let $f \in \mathbb{R}^{\mathbb{R}}$. Prove that, for any $(a,b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, the mapping $\mathbb{1}_{]a,b]} f$ is I_φ-integrable, and

$$\int_{]a,b]} f(t)\,\mathrm{d}\lfloor t \rfloor = \sum_{n \in \mathbb{Z},\, a < n \leqslant b} f(n).$$

In the questions **30.–32.**, let (a,b) be an element of $\mathbb{R} \times \mathbb{R}$ such that $a < b$, and U be an open subset of $\mathbb{R}$ containing $[a,b]$. Let $\psi : U \to \mathbb{R}$ be a mapping of class C^1.

30. Prove that ψ' is bounded on the interval $[a,b]$.

31. Let $M = \sup_{t \in [a,b]} |\psi'(t)|$. Prove that the mapping

$$\psi_M : [a,b] \longrightarrow \mathbb{R}, \quad \psi_M(t) = \psi(t) + Mt$$

is increasing.

32. Prove that

$$\int_{]a,b]} \psi(t)\,\mathrm{d}\lfloor t \rfloor = \psi(b)\lfloor b \rfloor - \psi(a)\lfloor a \rfloor - \int_a^b \lfloor t \rfloor \psi'(t)\,\mathrm{d}t.$$

Hint: For any $n \in \mathbb{Z}$*,*

$$\int_{]n,n+1]} \psi(t)\,\mathrm{d}\lfloor t \rfloor = \psi(n+1), \quad \int_n^{n+1} \lfloor t \rfloor \psi'(t)\,\mathrm{d}t = n(\psi(n+1) - \psi(n))$$

Part V: Stirling's formula

For any $n \in \mathbb{N}$, let

$$w_n = \int_0^{\frac{\pi}{2}} \sin(t)^n\,\mathrm{d}t.$$

33. Prove that $(n+2)w_{n+2} = (n+1)w_n$ for any $n \in \mathbb{N}$.

34. Prove that the sequence $(w_n)_{n \in \mathbb{N}}$ is positive and decreasing.

35. Prove that the sequence

$$(n+1)w_n w_{n+1}, \quad n \in \mathbb{N}$$

is constant.

36. Prove that

$$\frac{n+1}{n+2} \leqslant \frac{w_{n+1}}{w_n} \leqslant 1.$$

Deduce that

$$\lim_{n\to+\infty} \frac{w_{n+1}}{w_n} = 1.$$

37. Prove that, for any $n \in \mathbb{N}$,

$$w_{2n} = \frac{(2n)!}{2^{2n}(n!)^2} \cdot \frac{\pi}{2}, \quad w_{2n+1} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

38. Let x be a real number such that $x \geqslant 1$. Prove that

$$\sum_{n\in\mathbb{N},\, 1\leqslant n\leqslant x} \frac{1}{n} = \frac{\lfloor x \rfloor}{x} + \int_1^t \frac{\lfloor t \rfloor}{t^2}\,\mathrm{d}t.$$

Hint: apply Question **32.**.

39. Prove that the mapping

$$(t \in \mathbb{R}) \longrightarrow \frac{\langle t \rangle}{t^2}$$

is Lebesgue integrable on $]1, +\infty[$. We denote by γ the value

$$1 - \int_1^{+\infty} \frac{\langle t \rangle}{t^2}\,\mathrm{d}t.$$

40. Prove that

$$\sum_{n\in\mathbb{N},\, 1\leqslant n\leqslant x} \frac{1}{n} = \ln(x) + \gamma + O\Big(\frac{1}{x}\Big)$$

when $x \to +\infty$.

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41. Prove that,

$$\sum_{n\in\mathbb{N},\,1\leqslant n\leqslant x} \ln(n) = x\ln(x) - x + O(\ln(x))$$

when $x \to +\infty$.

42. Let $b_1 : \mathbb{R} \to \mathbb{R}$ and $b_2 : \mathbb{R} \to \mathbb{R}$ be the mappings defined as follows:

$$\forall\, t \in \mathbb{R}, \quad b_1(t) := \langle t\rangle - \frac{1}{2}, \quad b_2(t) := \frac{1}{2}\langle t\rangle^2 - \frac{1}{2}\langle t\rangle.$$

Prove that b_1 is locally Lebesgue integrable and b_2 is its primitive function.

43. Let x be a real number such that $x \geqslant 1$. Prove that

$$\int_1^x \ln(t)\,\mathrm{d}t = x\ln(x) - x + 1.$$

44. Deduce that the sequence

$$(n!e^n n^{-n-\frac{1}{2}})_{n\in\mathbb{N},\,n\geqslant 1}$$

converges to a positive real number.

45. Prove that

$$\lim_{n\to+\infty} \frac{n!e^n}{n^n\sqrt{n}} = \sqrt{2\pi}.$$

The end