Nabla Opertor

This article aim to give a proof[1] of the expression[2] of nabla operator in different coordinate system and explain its “meaning”. And this will be a cheat sheet.

1 The Nabla Operator ∇

Nabla operator has three action modules: “association”, “inner product”, “cross product”[3]. Or we can say we denote three different actions as a same notation . They have different meanings, and so have the different expressions.

2 Physical Component and Orthogonal Basis

$\mathbf{g}^i, \mathbf{g}_i$ as the natural basis have different units. That is inconvenient for physics calculation. So we construct another covariant basis:

$$\mathbf{g}_{(i)} = \frac{\mathbf{g}_i}{\sqrt{g_{\underline{ii}}}} = \beta^j_{(i)}\mathbf{g}_j, \tag{1}$$

where, underline means do not take summation, and

$$\beta^j_{(i)} = \frac{\delta^j_i}{\sqrt{\mathbf{g}_{\underline{i}} \cdot \mathbf{g}_{\underline{i}}}}. \tag{2}$$

Then the physical component can be written as

$$v^{(i)} = \sqrt{g_{\underline{ii}}} v^i, \quad v_{(i)} = \frac{1}{\sqrt{g_{\underline{ii}}}} v_i. \tag{3}$$

If $\mathbf{g}_i$ is orthogonal, let

$$|\mathbf{g}_i| = A_i, \tag{4}$$

which are called the **Lamé coefficient**, then

$$g_{ij} = \begin{cases} 0, & i \neq j \\ A_i^2, & i = j. \end{cases} \tag{5}$$

Let

$$\mathbf{e}_i = \frac{\mathbf{g}_i}{A_i}, \quad \mathbf{e}^i = A_i \mathbf{g}^i, \tag{6}$$

then,

$$\mathbf{e}_i = \mathbf{e}^i = \mathbf{e}(i). \tag{7}$$

$$\Gamma^k_{ij} = 0, (i \neq j \neq k), \quad \Gamma^i_{ij} = \frac{1}{A_i}\frac{\partial A_i}{\partial x^j}, \quad \Gamma^j_{ii} = -\frac{A_i}{A_j^2}\frac{\partial A_i}{\partial x^j}, (i \neq j). \tag{8}$$

[1] I will use tensor notation.
[2] Mainly for 3D vectors.
[3] In my words.

3 Gradient

Let $\mathbf{T}$ be a tensor, then we define the gradient of the tensor as

$$\nabla \mathbf{T} = \mathbf{g}^i \frac{\partial \mathbf{T}}{\partial x^i} = \frac{\mathbf{e}^i}{A_i} \frac{\partial \mathbf{T}}{\partial x^i}. \tag{9}$$

4 Divergence

Note that

$$\frac{\partial \sqrt{g}}{\partial x^i} = \Gamma^j_{ji} \sqrt{g}, \tag{10}$$

we have

$$\nabla \cdot \mathbf{F} = \partial_i F^i + F^m \Gamma^i_{im} = \partial_i F^i + F^m \frac{1}{\sqrt{g}} \partial_m \sqrt{g} = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} F^i \right). \tag{11}$$

In orthogonal coordinate system, we have

$$\nabla \cdot \mathbf{F} = \sum_{i=1}^{3} \frac{1}{A_1 A_2 A_3} \partial_i \left(\frac{A_1 A_3 A_3}{A_i} F(i) \right). \tag{12}$$

5 Curl

$$\begin{aligned} \nabla \times \mathbf{F} &= \epsilon^{ijk} \nabla_i F_j \mathbf{g}_k = \epsilon^{ijk} \left(\partial_i F_j - F_m \Gamma^m_{ij} \right) \mathbf{g}^k = \epsilon^{ijk} \partial_i F_j \mathbf{g}^k & (13) \\ &= \frac{1}{\sqrt{g}} \begin{vmatrix} \mathbf{g}_1 & \mathbf{g}_1 & \mathbf{g}_1 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix}. & (14) \end{aligned}$$

In orthogonal coordinate system, we have

$$\nabla \times \mathbf{F} = \frac{1}{A_1 A_2 A_3} \begin{vmatrix} A_1 \mathbf{e}_1 & A_2 \mathbf{e}_2 & A_3 \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ A_1 F(1) & A_2 F(2) & A_3 F(3) \end{vmatrix}. \tag{15}$$

6 Laplacian

For a scalar function f, we have

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{A_1 A_2 A_3} \sum_{i=1}^{3} \partial_i \left(\frac{A_1 A_2 A_3}{A_i^2} \partial_i f \right). \tag{16}$$

7 Special 3D Coordinates

7.1 Rectangular coordinates

$$A_1 = A_2 = A_3 = 1. \tag{17}$$

7.2 Cylindrical Coordinates

$$A_1 = A_3 = 1, \quad A_2 = \rho. \tag{18}$$

7.3 Spherical Coordinates

$$A_1 = 1, \quad A_2 = r, \quad A_3 = r \sin\theta. \tag{19}$$