Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 6 - 2 : linear algebra : vector and matrix

1. In $\mathbb{R}^3$, let

$$\mathbf{u} = (1, -2, 1), \ \mathbf{v} = (-1, 1, 0).$$

(1) For $\mathbf{w} = (-6, 3, 3)$, find $a, b \in \mathbb{R}$, such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(2) For $\mathbf{w}' = (6, 3, 3)$, prove $\mathbf{w}' \notin \mathrm{Span}(\mathbf{u}, \mathbf{v})$.

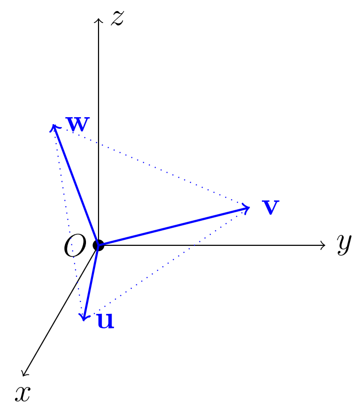
(3) Give the equation of coordinate of $\mathrm{Span}(\mathbf{u}, \mathbf{v})$.

2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three non-zero vectors in $\mathbb{R}^3$, which do not lie in a same plane.

(1) Prove $\mathrm{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbb{R}^3$.

(2) Prove that if $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0$, then $a = b = c = 0$.

(3) Consider the pyramid below and study the following problems.



(a) Is $\frac{1}{2}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ inside or outside the pyramid ?

(b) Prove : for an arbitrary point P in the pyramid, if we consider it as the vector $\overrightarrow{OP} = \mathbf{p}$, then there exist $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$ and $a + b + c \leqslant 1$, such that

$$\mathbf{p} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}.$$

(4) Write the surface $\Delta(O\mathbf{uv})$ of the pyramid as a set of the linear combinations of $\{\mathbf{u}, \mathbf{v}\}$.

(5) Write the surface $\Delta(\mathbf{uvw})$ of the pyramid as a set of the linear combinations of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

3. Let

$$\mathbf{u} = (1, 2, -5, 3), \ \mathbf{v} = (2, -1, 4, 7).$$

For $a, b \in \mathbb{R}$, when $(a, b, -37, -3) \in \mathrm{Span}(\mathbf{u}, \mathbf{v})$?

4. We consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \ B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 1 & -1 \end{pmatrix}, \ D = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
$$E = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 3 \end{pmatrix}, \ F = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

(1) In A, B, C, D, E, F, for which pairs the addition is well-defined ?

(2) In A, B, C, D, E, F, for which pairs the multiplication is well-defined ? If so, give the number of rows and columns of the product.

(3) Give AB, BA, CE, EF, FE.

(4) Give the transport of A, B, C, D, E, F.

5. (1) For $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, find a non-zero $B \in M_2(\mathbb{R})$, such that

$$AB = 0_{2,2},$$

and deduce that A is not invertible.

(2) For the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, give a non-zero matrix $B \in M_{3,2}$, such that

$$AB = 0_{2,2}\,.$$

(3) Let A be the same as in the previous exercise, give a non-zero matrix $C \in M_{3,2}(\mathbb{R})$, such that

$$AC = I_2.$$

6. Find $A \in M_2(\mathbb{R})$, such that :

(a) $A \neq \pm I_2$, but $A^2 = I_2$.

(b) $A^2 = -I_2$.

(c) $A \neq I_2$, but $A^3 = I_2$.

7. We consider the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

(1) Verify

$$(1,0)R(\theta) = (\cos\theta, \sin\theta),\ (0,1)R(\theta) = (-\sin\theta, \cos\theta). \tag{1}$$

Explain $R(\theta)$ is the linear transformation with the angle θ.

(2) Verify $\forall\, \theta_1, \theta_2 \in \mathbb{R}$,

$$R(\theta_1)R(\theta_2) = R(\theta_1 + \theta_2).$$

(3) Find $R(\theta)^k$, $k \in \mathbb{N}$.

(4) Find $R(\theta)^{-1}$.

8. Given $a, b, c, d \in \mathbb{R}$, and we consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(1) Verify $AB = BA = (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(2) Verify : when $\Delta := (ad - bc) \neq 0$, A is invertible, and

$$A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(3) When $\Delta = 0$, prove that A is not invertible. (**Hint :** verify $AX = 0$ has a non-zero solution.)

9. For $(a, b) \in \mathbb{R}^2$, we consider

$$M(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(1) By the fact

$$M(a, b) = aI_2 + bJ,\ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

prove :

$$M(a, b)M(c, d) = M(ac - bd, bc + ad).$$

(2) Give the explicit representation of $M(a, b)^k$ ($k \in \mathbb{N}$).

(3) Prove : $M(a,b)$ is invertible if and only if $(a,b) \neq (0,0)$, and in this time we have

$$M(a,b)^{-1} = M\left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right).$$

10. Suppose

$$A = \begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}.$$

Give the explicit representation of A^n, where $n \in \mathbb{N}$.

11. Suppose

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

(1) Verify $A^2 - 3A + 2I_3 = 0$.

(2) Determine $a_n, b_n \in \mathbb{R}$, such that

$$x^n = (x^2 - 3x + 2)Q(x) + (a_n x + b_n),$$

where $Q(x)$ is a polynomial.

(3) Give the explicit representation of A^n.

12. It is easy to see that λI_n $(\lambda \in \mathbb{R})$ is commutative with any matrix B of order n. Prove its converse : Suppose $A \in M_n(\mathbb{R})$ is commutative with any $B \in M_n(\mathbb{R})$, then $A = \lambda I_n$, where $\lambda \in \mathbb{R}$.

13. Solve the following linear systems of equations.

(1) $\begin{cases} x - 2y + 3z = 5, \\ 2x - 4y + z = 5, \\ 3x - 5y + 2z = 8. \end{cases}$

(2) $\begin{cases} -5x + y - 3z - 2t = 2, \\ x + 2y - 4z - t = 3, \\ -3x + 5y - 11z - 4t = 8, \\ -9x + 4y - 10z - 5t = 7, \end{cases}$ where t is a parameter.

(3) $\begin{cases} 2x_1 + x_2 + x_3 + x_4 = a, \\ x_1 + 2x_2 + x_3 + x_4 = b, \\ x_1 + x_2 + 2x_3 + x_4 = c, \\ x_1 + x_2 + x_3 + 2x_4 = d, \end{cases}$ where $a, b, c, d \in \mathbb{R}$ are fixed.

(4) $\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_2 + x_3 + x_4 = 0 \\ \quad \ddots \quad \ddots \quad \ddots \quad \vdots \\ x_{n-2} + x_{n-1} + x_n = 0 \\ x_{n-1} + x_n = 0 \end{cases}$, where $n \geqslant 3$.

14. For different choices of the parameter m, solve the following linear systems of equations.

(1) $\begin{cases} x + y + (m-1)z = m, \\ mz = m, \\ -x + (m^2 - m - 1)y + (m-1)z = m. \end{cases}$.

(2) $\begin{cases} x - my + m^2 z = m, \\ mx - m^2 y + mz = 1, \\ mx + y - m^3 z = -1. \end{cases}$

(3) $\begin{cases} x + y + z = m + 1, \\ mx + y + (m+1)z = m - 1, \\ x + my + mz = m + 2. \end{cases}$

15. Find the elementary matrices P_k, such that :

(a) $(x, y)P_1 = (y, x), \quad \forall\ (x, y) \in \mathbb{R}^2$.

(b) $(x, y)P_2 = (x, x + y), \quad \forall\ (x, y) \in \mathbb{R}^2$.

(c) $(x, y)P_3 = (-x, y), \quad \forall\ (x, y) \in \mathbb{R}^2$.

(d) $(x, y, z)P_4 = (x, z, y), \quad \forall\ (x, y, z) \in \mathbb{R}^3$.

(e) $(x, y, z)P_5 = (z, y, x), \quad \forall\ (x, y, z) \in \mathbb{R}^3$.

(f) $(x, y, z)P_6 = (x, y, z + x), \quad \forall\ (x, y, z) \in \mathbb{R}^3$.

16. Give the products of the following matrices.

(1) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

(2) $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(3) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

(4) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

17. Find the inverses of the matrices below.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}.$$

18. If $A \in M_n(\mathbb{R})$ satisfies

$$A^2 = I_n,$$

then we say that A is a *reflection*.

(1) Give A^{-1}.

(2) Prove : $\forall\ \lambda \in \mathbb{R}$, $|\lambda| \neq 1$, $A_\lambda = A - \lambda I_n$ is invertible, and give the inverse of A_λ. (**Hint :** calculate A_λ^2, and write it as the form of $bA_\lambda + cI_n$.)

19. Suppose $A \in M_n(\mathbb{R})$. If $A^2 = A$, then we say that A is a *projection*.

(1) If A is a projection, prove that so is $B = I_n - A$, and

$$AB = BA = 0;$$

give the inverse of $I_n + A$.

(2) Prove that for a projection A, it is invertible if and only if $A = I_n$.

20. For $\mathbf{u} = (u_1, \ldots, u_n)^\top, \mathbf{v} = (v_1, \ldots, v_n)^\top \in \mathbb{R}^n$, suppose $m_{ij} = x_i y_j - x_j y_i$ $(i, j \in [\![1, n]\!])$, $M = (m_{ij})_{1 \leqslant i,j \leqslant n}$. Consider the matrix $A = I_n + aM$, where $a \in \mathbb{R}$ is a parameter.

(a) Prove that if $AX = 0$, then there exist $\lambda, \mu \in \mathbb{R}$ such that $X = \lambda\mathbf{u} + \mu\mathbf{v}$.

(b) Prove that A is invertible.

21. Let $n \geqslant 2$, and we consider the matrix

$$J = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

of order n

(1) Calculate J^k, $k = 2, \ldots, n$, and deduce $J^n = 0_{n,n}$.

(2) Prove that $I + J$ is invertible, and give its inverse matrix.

(3) For a polynomial $P(x) = \sum\limits_{k=0}^{n-1} a_k x^k$, where $a_0 = 1$, give the representation of $P(J)$. Prove that $P(J)$ is invertible.

(4) Suppose

$$A = \begin{pmatrix} 1 & \binom{n}{1} & \cdots & \binom{n}{n-1} \\ & \ddots & \ddots & \vdots \\ & & 1 & \binom{n}{1} \\ & & & 1 \end{pmatrix}.$$

Give A^{-1}.

22. (1) Let $A \in M_n(\mathbb{R})$ whose elements in the diagonal are all odd numbers, and elements outside the diagonal are all even numbers. Prove that A is invertible.

(2) Suppose a goatherd has 2025 goats. For every goat, we can always divide the other 2024 ones into two part, each part have 1012 goats and they have the same weight. Prove that all the 2025 goats have the same weight.

23. All the vectors considered are in $\mathbb{R}^n$. Determine the assertions below are true or false.

(1) If $\mathbf{v}$ is not the linear combination of $\mathbf{u}$, then they are linearly independent.

(2) If $\mathbf{v}$ is the linear combination of $\mathbf{u}$, then they are linearly dependent.

(3) If a family of vectors contains the zero vector, then they are linearly dependent.

(4) If a family of vectors does not contain the zero vector, then they are linearly independent.

(5) If $p > n$, then $\mathbf{u}_1, \dots, \mathbf{u}_p$ are linearly dependent.

(6) If $p < n$, then $\mathbf{u}_1, \dots, \mathbf{u}_p$ cannot generate $\mathbb{R}^n$.

(7) If $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ are linearly independent, then $\mathrm{Span}(\mathbf{u}_1, \dots, \mathbf{u}_n) = \mathbb{R}^n$.

(8) If $\mathrm{Span}(\mathbf{u}_1, \dots, \mathbf{u}_n) = \mathbb{R}^n$, then $(\mathbf{u}_1, \dots, \mathbf{u}_n)$ are linearly independent.

(9) If $(\mathbf{u}_1, \dots, \mathbf{u}_p)$ are linearly dependent, $A \in M_{m,n}(\mathbb{R})$ is an arbitrary matrix, then $(A\mathbf{u}_1, \dots, A\mathbf{u}_p)$ are linearly dependent.

(10) If $(\mathbf{u}_1, \dots, \mathbf{u}_p)$ are linearly independent, $A \in M_{m,n}(\mathbb{R})$ is an arbitrary matrix, then $(A\mathbf{u}_1, \dots, A\mathbf{u}_p)$ are linearly independent.

24. Determine whether the families of vectors below are linearly independent.

(1) $\mathbf{u} = (2, 1), \quad \mathbf{v} = (1, 2).$

(2) $\mathbf{u} = (1, 2, 3), \quad \mathbf{v} = (2, 1, 3), \quad \mathbf{w} = (3, 2, 1).$

(3) $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (1, 1, -2)$, $\mathbf{w} = (1, a, -3)$, where $a \neq \frac{2}{3}$.
(4) In (3), choose $a = \frac{2}{3}$.
(5) $\mathbf{u} = (3, 1, 2, -1)$, $\mathbf{v} = (1, -5, 6, -11)$, $\mathbf{w} = (-1, 2, -3, 5)$.
(6) $\mathbf{u}_1 = (2, 1, 3, 9)$, $\mathbf{u}_2 = (1, 5, 4, 7)$, $\mathbf{u}_3 = (5, 3, 7, 15)$, $\mathbf{u}_4 = (-3, 2, -1, 4)$.

25. Let

$$\mathbf{u} = (1, 2, 1),\ \mathbf{v} = (2, 3, 3),\ \mathbf{w} = (3, 7, 1).$$

(1) Prove that $\mathcal{B} = (\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a basis of $\mathbb{R}^3$.
(2) Give the coordinate of (x, y, z) with respect to $\mathcal{B}$.

26. Extend the following families of vectors as a basis of the total space.

(1) $\mathbf{u}_1 = (1, 1, 2)$, $\mathbf{u}_2 = (2, 1, 2)$.
(2) $\mathbf{u}_1 = (1, 0, 1, 0)$, $\mathbf{u}_2 = (0, 1, 0, 1)$.
(3) $\mathbf{u}_1 = (1, 1, 1, 1)$, $\mathbf{u}_2 = (1, 2, 3, 4)$, $\mathbf{u}_3 = (1, 3, 5, 10)$.

27. Let K be a field and V be a finite dimensional vector space over K. Prove that, for any $n \in \mathbb{N}$,

$$\dim(V^n) = n \dim(V).$$

28. Let K be a field and V and W be finite dimensional vector spaces over K. Prove that $\mathrm{Hom}_{K\text{-Mod}}(V, W)$ is naturally equipped with a structure of vector space over K and

$$\dim(\mathrm{Hom}_{K\text{-Mod}}(V, W)) = \dim(V) \dim(W).$$

29. Let $u : \mathbb{R}^3 \to \mathbb{R}^4$ be the linear mapping defined as

$$u(x, y, z) = (-x + y, x - y, -x + z, -y + z).$$

(1) Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$ and $\{f_1, f_2, f_3, f_4\}$ be the canonical basis of $\mathbb{R}^4$. Express $u(e_1), u(e_2), u(e_3)$ in terms of f_1, f_2, f_3, f_4.
(2) Write the matrix of u with respect to the canonical basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.
(3) Prove that $\{f_1, f_2, u(e_1), u(e_2)\}$ forms a basis of $\mathbb{R}^4$.
(4) Write the matrix of u with respect to $\{e_1, e_2, e_3\}$ to $\{f_1, f_2, u(e_1), u(e_2)\}$.

30. Let F, G be two vector subspaces of $\mathbb{R}^5$, $\dim(F) = \dim(G) = 3$. Prove that $F \cap G \neq \{0\}$.

31. Suppose that F is a vector subspace of $\mathbb{R}^n$ of dimension p, $\mathbf{v}_1, \ldots, \mathbf{v}_p \in F$. Prove that the following statements are equivalent.

(a) $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ is a basis of F ;
(b) $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ are linearly independent ;
(c) $(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ generate F : $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p) = F$.

32. Determine whether the following f_k are linear transformation. If so, find the matrix A_k such that $f_k(X) = f_{A_k}(X) = XA_k$.

(a)

$$\begin{array}{rrcl} f_1 : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & (x, y) & \longmapsto & (x + y, x - y, x + 2y). \end{array}$$

(b)

$$\begin{array}{rrcl} f_2 : & \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ & (x, y) & \longmapsto & (x + y, x - y, 1). \end{array}$$

(c)

$$\begin{array}{rrcl} f_3 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (x, y, z) & \longmapsto & (x + y + z + a, x - y - z + b). \end{array}$$

(Discuss for different $(a, b) \in \mathbb{R}^2$.)

(d)

$$\begin{array}{rrcl} f_4 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & ax + by + cz. \end{array}$$

(e)

$$\begin{array}{rrcl} f_5 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R} \\ & (x, y, z) & \longmapsto & x^2 + y^2 + z^2. \end{array}$$

(f)

$$\begin{array}{rrcl} f_6 : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x, y, z) & \longmapsto & (x + 2y + 3z, x + 3y + 5z, x + 5y + 9z). \end{array}$$

33. In $\mathbb{R}^2$, consider

$$\mathbf{u} = (1, 1), \quad \mathbf{v} = (2, -1).$$

(1) Prove that $(\mathbf{u}, \mathbf{v})$ forms a basis of $\mathbb{R}^2$.
(2) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, such that

$$f(\mathbf{u}) = (2, -1), f(\mathbf{v}) = (-4, 2).$$

Find the representation matrix of f.
(3) Find the conditions of the parameter $a \in \mathbb{R}$, such that

$$(5, a) \in \mathrm{Im}(f).$$

34. Consider

$$\begin{array}{rrcl} f: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longmapsto & (-3x-y+z, 8x+3y-2z, -4x-y+z). \end{array}$$

(1) Determine the representation matrix of f.

(2) Determine a basis and the dimension of $\ker(f)$.

(3) Determine a basis and the dimension of $\mathrm{Im}(f)$.

35. Determine the rank of the following matrices.

$$A = \begin{pmatrix} 0 & -7 & 0 & 2 \\ -1 & -7 & 4 & -2 \\ 2 & 2 & -6 & 0 \\ -2 & -8 & 6 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 & -1 & 1 \\ 7 & 1 & 2 & 4 \\ 2 & 1 & 1 & 1 \\ 5 & 5 & 4 & 2 \end{pmatrix}.$$

36. According to $\lambda \in \mathbb{R}$, determine a basis and the dimension of the matrices below, and find their ranks.

(1) $\begin{pmatrix} -1-\lambda & 2 & 1 \\ 4 & 1-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{pmatrix}$.

(2) $\begin{pmatrix} 12-\lambda & -6 & 3 \\ -9 & -5-\lambda & 3 \\ -12 & -8 & 9-\lambda \end{pmatrix}$.

37. Let $A, B \in M_n(\mathbb{R})$. By the fact

$$(A - I_n)(B - I_n) = (AB - I_n) - (A - I_n) - (B - I_n),$$

prove : $\mathrm{rk}(AB - I_n) \leqslant \mathrm{rk}(A - I_n) + \mathrm{rk}(B - I_n)$.

38. Suppose $A \in M_{m,n}(\mathbb{R})$. Prove $\mathrm{rk}(AA^{\mathsf{T}}) = \mathrm{rk}(A)$, and $\mathrm{Im}(AA^{\mathsf{T}}) = \mathrm{Im}(A)$.

39. Let $A \in M_{m,n}(\mathbb{R})$.

(1) If $\mathrm{rk}(A) = n$, $B \in M_{n,p}(\mathbb{R})$. Prove $\mathrm{rk}(AB) = \mathrm{rk}(B)$.

(2) If $\mathrm{rk}(A) = m$, $C \in M_{p,m}(\mathbb{R})$. Prove $\mathrm{rk}(CA) = \mathrm{rk}(C)$.

40. Suppose $A \in M_{m,n}(\mathbb{R})$, whose rank is $r \geqslant 1$. Prove that there exist r matrices of rank 1 denoted by $A_1, \ldots, A_r$, such that

$$A_1 + \cdots + A_r = A.$$

41. Prove that there exists a matrix $A \in M_n(\mathbb{R})$, such that $\ker(A) = \mathrm{Im}(A)$ if and only if n is an even number. (**Hint :** consider the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.)

42. Suppose $A \in M_{3,2}(\mathbb{R})$, $B \in M_{2,3}(\mathbb{R})$, such that

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Prove : $BA = I_2$.

43. Let $A, B \in M_n(\mathbb{R})$.

(1) Prove

$$\ker(A) \subset \ker(BA), \quad \mathrm{Im}(BA) \subset \mathrm{Im}(B).$$

(2) Prove that $\mathbb{R}^n = \mathrm{Im}(A) + \ker(B)$ is valid if and only if $\mathrm{Im}(BA) = \mathrm{Im}(B)$.

44. Suppose $J \in M_n(\mathbb{R})$ is a nilpotent matrix, which means $\exists\, p \in \mathbb{N}^*$, such that $J^p = 0_{n,n}$. Prove $J^n = 0_{n,n}$.

45. Suppose $\mathbb{V}(\mathbb{R}^n)$ is the set of all vector subspaces of $\mathbb{R}^n$, $d : \mathbb{V}(\mathbb{R}^n) \to \mathbb{N}$ is a function which satisfies the following two properties :

(a) IF $F, G \in \mathbb{V}(\mathbb{R}^n)$, $F \cap G = \{0\} \Longrightarrow d(F+G) = d(F) + d(G)$.

(b) $d(\mathbb{R}^n) = n$.

Prove :

(1) If $\dim(F) = 1$, then $d(F) = 1$.

(2) d is the dimension : $\forall\, F \in \mathbb{V}(\mathbb{R}^n)$, $d(F) = \dim(F)$.