QM HW3

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Problem 1 (A quantum particle in an infinitely deep potential well)
(1) When $|x| > \frac{L}{2}$, $\psi(x) = 0$. Now devoted exclusively to the case where $|x| < \frac{L}{2}$.By Schrödinger equation,

$$E\psi + \frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = 0. \tag{1.1}$$

Let $k = \sqrt{2mE/\hbar^2}$,

$$\psi(x) = A\sin(kx) + B\cos(kx). \tag{1.2}$$

For odd parity, $B = 0$. For even parity, $A = 0$. Take the boundary condition

$$\psi(\pm\frac{L}{2}) = 0. \tag{1.3}$$

$$\cos\left(\frac{k^+L}{2}\right) = 0, \sin\left(\frac{k^-L}{2}\right) = 0. \tag{1.4}$$

So, we have

$$k_n^+ = \frac{(2n-1)\,\pi}{L},\ k_n^- = \frac{2n\pi}{L}. \tag{1.5}$$

That is

$$\boxed{E_n^+ = \frac{\hbar^2\pi^2(2n-1)^2}{2mL^2},\ E_n^- = \frac{\hbar^2\pi^2(2n)^2}{2mL^2}.} \tag{1.6}$$

$$\psi_n^+ = A_n^+\cos\left(\frac{(2n-1)\,\pi x}{L}\right),\ \psi_n^- = A_n^-\sin\left(\frac{2n\pi x}{L}\right). \tag{1.7}$$

Normalize the wave functions,

$$\boxed{\psi_n^+ = \sqrt{\frac{2}{L}}\cos\left(\frac{(2n-1)\,\pi x}{L}\right),\ \psi_n^- = \sqrt{\frac{2}{L}}\sin\left(\frac{2n\pi x}{L}\right)}. \tag{1.8}$$

(2)

$$\Psi(x,t) = \sqrt{\frac{1}{L}}\left[e^{-iE_1^+t}\cos\left(\frac{\pi x}{L}\right) + e^{-iE_1^-t}\sin\left(\frac{2\pi x}{L}\right)\right] \tag{1.9}$$

$$= \boxed{\sqrt{\frac{1}{L}}\left[e^{-i\frac{\hbar^2\pi^2}{2mL^2}t}\cos\left(\frac{\pi x}{L}\right) + e^{-i\frac{2\hbar^2\pi^2}{mL^2}t}\sin\left(\frac{2\pi x}{L}\right)\right]}. \tag{1.10}$$

(3) By symmetry,

$$\langle x \rangle = 0,\ \langle p \rangle = 0. \tag{1.11}$$

$$\left\langle x^2 \right\rangle = \int_{-\frac{L}{2}}^{\frac{L}{2}} \psi^* x^2 \psi \ \mathrm{d}x = \frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}. \tag{1.12}$$

$$\left\langle p^2 \right\rangle = {}^1 2mE = \frac{n^2\pi^2\hbar^2}{L^2}. \tag{1.13}$$

So,

$$\boxed{\sqrt{\Delta x^2} = L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}}, \tag{1.14}$$

$$\boxed{\sqrt{\Delta p^2} = \frac{n\pi\hbar}{L}}, \tag{1.15}$$

$$\boxed{\sqrt{\Delta x^2}\sqrt{\Delta p^2} = \hbar\sqrt{\frac{n^2\pi^2}{12} - \frac{1}{2}} \geq \hbar\sqrt{\frac{\pi^2}{12} - \frac{1}{2}} > \frac{\hbar}{2}}. \tag{1.16}$$

Problem 2 (δ-function)
(1) Easy to check that

$$\lim_{a\to 0} \frac{1}{\pi} \frac{a}{x^2 + a^2} = \lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} = \begin{cases} 0 & , x \neq 0 \\ +\infty & , x = 0 \end{cases}. \tag{2.1}$$

$$\int_{-\epsilon}^{\epsilon} \frac{a}{x^2 + a^2} \ \mathrm{d}x = 2\arctan\left(\frac{\epsilon}{a}\right). \tag{2.2}$$

$$\lim_{a\to 0} \frac{2}{\pi} \arctan\left(\frac{\epsilon}{a}\right) = 1. \tag{2.3}$$

$$\lim_{a\to 0} \frac{1}{a\sqrt{\pi}} \int_{-\epsilon}^{\epsilon} e^{-\frac{x^2}{a^2}} \ \mathrm{d}x = \frac{\Gamma(\frac{1}{2})}{\sqrt{\pi}} = 1. \tag{2.4}$$

(2)

$$\frac{1}{x \pm i\epsilon} = \frac{x}{x^2 + \epsilon^2} \mp \frac{i\epsilon}{x^2 + \epsilon^2}. \tag{2.5}$$

$\frac{x}{x^2+\epsilon^2}$ is an odd function, in the interval where $|x| < \epsilon$, it contributes $O(\epsilon)$. Once $\lim_{\epsilon\to 0} \frac{x}{x^2+\epsilon^2} = \frac{1}{x}$ and by (1), we obtain

$$\boxed{\frac{1}{x \pm i\epsilon} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x).} \tag{2.6}$$

(3) Let $A := \{x_n \mid g(x_n) = 0, g'(x_n) \neq 0\}$. Around $x = x_n$,

$$g(x) = g(x_n) + g'(x_n)(x - x_n) = g'(x_n)(x - x_n). \tag{2.7}$$

[1] By Schrödinger equation: $\frac{p^2}{2m} |\psi\rangle = E |\psi\rangle$, then $\left\langle p^2 \right\rangle = \langle \psi | p^2 | \psi \rangle = 2mE$.

Thus,

$$\delta[g(x)] = \delta(\sum_{x_n \in A} g'(x_n)(x - x_n)) = \sum_{x_n \in A} \delta[g'(x_n)(x - x_n)]. \tag{2.8}$$

By $\delta(ax) = \frac{\delta(x)}{|a|}$ $(a \neq 0)$,

$$\boxed{\delta[g(x)] = \sum_{x_n \in A} \frac{\delta(x - x_n)}{|g'(x_n)|}.} \tag{2.9}$$

In particular,

$$\boxed{\delta(x^2 - a^2) = \frac{\delta(x - a) + \delta(x + a)}{2a}.} \tag{2.10}$$

(4)

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} [\delta(x - a)] (xf(x)) \,\mathrm{d}x \tag{2.11}$$

$$= \delta(x - a)xf(x)|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x - a) [f(x) + xf'(x)] \,\mathrm{d}x \tag{2.12}$$

$$= f(a) + af'(a). \tag{2.13}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}^2}{\mathrm{d}x^2} [\delta(x - a)] \left(x^2 f(x)\right) \mathrm{d}x \tag{2.14}$$

$$= \int_{-\infty}^{\infty} -\frac{\mathrm{d}}{\mathrm{d}x} [\delta(x - a)] \frac{\mathrm{d}}{\mathrm{d}x} \left[x^2 f(x)\right] \mathrm{d}x \tag{2.15}$$

$$= \int_{-\infty}^{\infty} \delta(x - a) \frac{\mathrm{d}^2}{\mathrm{d}x^2} \left[x^2 f(x)\right] \mathrm{d}x \tag{2.16}$$

$$= a^2 f''(a) + 4af'(a) + 2f(a). \tag{2.17}$$

Problem 3 (Momentum representation)
(1)

$$\langle x| \hat{p}^2 |\psi\rangle = \langle x| \hat{p} (\hat{p} |\psi\rangle) = -i\hbar \langle x| \hat{p} |\psi\rangle = -\hbar^2 \psi(x). \tag{3.1}$$

(2)

$$\langle x| \hat{H} |\psi\rangle = \langle x| \hat{p}^2/2m + \hat{V}(x) |\psi\rangle = \left(-\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2} m\omega^2\right) \psi(x). \tag{3.2}$$

(3)

$$\langle p| \hat{x} |\psi\rangle = \int \mathrm{d}p' \langle p| \hat{x} |p'\rangle \langle p'|\psi\rangle \tag{3.3}$$

$$\langle p| \hat{x} |p'\rangle = i\hbar \frac{\partial}{\partial p} \delta(p - p') = -i\hbar \frac{\partial}{\partial p'} \delta(p - p'). \tag{3.4}$$

Similar to Problem 2-(4),

$$\langle p| \hat{x} |\psi\rangle = i\hbar \frac{\partial}{\partial p}\psi(p). \tag{3.5}$$

(4)

$$\begin{aligned}\langle p| H |\psi\rangle &= \int \mathrm{d}p' \langle p| \frac{\hat{p^2}}{2m} |p'\rangle \langle p'|\psi\rangle + \frac{1}{2}m\omega^2 \langle p| \hat{x}^2 |\psi\rangle \\ &= \int \mathrm{d}p' \frac{p^2}{2m}\delta(p-p')\psi(p) - \frac{1}{2}m\omega^2\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}p^2}\psi(p) \\ &= \left[\frac{p^2}{2m} - \frac{1}{2}m\omega^2\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}p^2}\right]\psi(p). \end{aligned}\tag{3.6}$$

Problem 4 (Orbital angular momentum)
(1)

$$\begin{aligned}[L_i, L_j] =&[\epsilon_{imn}x_m p_n, \epsilon_{jkl}x_k p_l] \\ =&\epsilon_{imn}\epsilon_{jkl}\left(x_m x_k[p_n, p_l] + x_m[p_n, x_k]p_l + x_k[x_m, p_l]p_n + [x_m, x_k]p_l p_n\right) \\ =&i\hbar\epsilon_{imn}\epsilon_{jln}(x_m p_l - x_l p_m) \\ =&i\hbar(x_i p_j - x_j p_i) \\ =&i\hbar\epsilon_{ijk}\epsilon_{kmn}x_m p_n \\ =&i\hbar\epsilon_{ijk}L_k. \end{aligned}\tag{4.1}$$

$$[L_j L_j, L_i] = \{L_j, [L_j, L_i]\} = i\hbar\epsilon_{jik}\{L_j, L_k\} = 0. \tag{4.2}$$

(2)

$$[L_i, x_j] = \epsilon_{imn}\left(x_m[p_n, x_j] + [x_m, x_j]p_n\right) = \epsilon_{imn}\left(-i\hbar x_m\delta_{nj}\right) = i\hbar\epsilon_{ijk}x_k. \tag{4.3}$$

$$[L_i, p_j] = \epsilon_{imn}\left(x_m[p_n, p_j] + [x_m, p_j]p_n\right) = \epsilon_{imn}\left(i\hbar p_n\delta_{mj}\right) = i\hbar\epsilon_{ijk}p_k. \tag{4.4}$$

(3)

$$[L_i, x_j x_j] = \{x_j, [L_i, x_j]\} = i\hbar\epsilon_{ijk}\{x_j, x_k\} = 0. \tag{4.5}$$

The last equality holds since ϵ_{ijk} is antisymmetric and $\{x_j, x_k\}$ is symmetric. Similarly,

$$[L_i, p_j p_j] = \{p_j, [L_i, p_j]\} = i\hbar\epsilon_{ijk}\{p_j, p_k\} = 0. \tag{4.6}$$

(4) In coordinate representation,

$$L_i = i\hbar\epsilon_{ijk}x_j\frac{\mathrm{d}}{\mathrm{d}x_k}. \tag{4.7}$$

In momentum representation,

$$L_i = i\hbar\epsilon_{ijk}p_k\frac{\mathrm{d}}{\mathrm{d}p_j}. \tag{4.8}$$

Problem 5 (Complete set of Mechanical variables)
We have the fact that

$$[A, f(B)] = \frac{\partial f(B)}{\partial B}[A, B]. \tag{5.1}$$

if $[A, B]$ is communicative with any operator.

$$H = \frac{p^2}{2m} - \frac{e^2}{r} \tag{5.2}$$

By Problem 4, we have

$$[L_i, H] = \frac{[L_i, p^2]}{2m} - e^2 \left[L_i, \frac{1}{\sqrt{r^2}}\right] = 0. \tag{5.3}$$

$$\begin{aligned}[L^2, H] =& \frac{[L^2, p^2]}{2m} - e^2 \left[L^2, \frac{1}{\sqrt{r^2}}\right] \\ =& \frac{\{L_i, [L_i, p^2]\}}{2m} - e^2\{L_i, [L_i, \frac{1}{\sqrt{r^2}}]\} \\ =& 0. \end{aligned} \tag{5.4}$$

Therefore L_i, L^2 are compatible with H.

Problem 6 (Gaussian and uncertainty principle)
Uniform probability distribution:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, \psi^*(x)\psi(x) = AA^* \frac{\Gamma\left(\frac{1}{2}\right)}{(l^{-2})^{\frac{1}{2}}} = 1. \tag{6.1}$$

So,

$$\boxed{A = \sqrt{\frac{1}{\sqrt{\pi} l}},} \tag{6.2}$$

if we choose a positive real number as the coefficient.
(1) By the symmetry, easy to find

$$\langle x \rangle = \langle p \rangle = 0. \tag{6.3}$$

Then,

$$\begin{aligned}\left\langle (\Delta x)^2 \right\rangle = \left\langle x^2 \right\rangle =& \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi^*(x) x^2 \psi(x) \\ =& \int_{-\infty}^{+\infty} A^2 e^{-\frac{x^2}{l^2}} x^2\, \mathrm{d}x \\ =& A^2 l^3 \Gamma\left(\frac{3}{2}\right) \\ =& \boxed{\frac{1}{2} l^2}. \end{aligned} \tag{6.4}$$

$$\begin{aligned}\left\langle (\Delta p)^2 \right\rangle = \left\langle p^2 \right\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi^*(x) p^2 \psi(x) \\ &= \int_{-\infty}^{+\infty} A^2 e^{-\frac{x^2}{2l^2}} \left(-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) e^{-\frac{x^2}{2l^2}}\, \mathrm{d}x \\ &= \frac{A^2\hbar^2}{l} \Gamma\left(\frac{3}{2}\right) \\ &= \boxed{\frac{\hbar^2}{2l^2}}.\end{aligned} \tag{6.5}$$

Hence,

$$\boxed{\sqrt{\left\langle (\Delta x)^2 \right\rangle} \sqrt{\left\langle (\Delta p)^2 \right\rangle} = \frac{\hbar}{2}.} \tag{6.6}$$

It reaches the lower bound of the uncertainty principle.
(2)

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \mathrm{d}x\, \psi(x) e^{-i\frac{px}{\hbar}} = \frac{Al}{\sqrt{\hbar}} e^{-\frac{p^2 l^2}{2\hbar^2}}. \tag{6.7}$$

$$\begin{aligned}\left\langle (\Delta p)^2 \right\rangle = \left\langle p^2 \right\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}p\, \psi^*(p) p^2 \psi(p) \\ &= \frac{A^2 l^2}{\hbar} \frac{\hbar^3}{l^3} \Gamma\left(\frac{3}{2}\right) \\ &= \boxed{\frac{\hbar^2}{2l^2}}.\end{aligned} \tag{6.8}$$

$$\begin{aligned}\left\langle (\Delta x)^2 \right\rangle = \left\langle x^2 \right\rangle &= \int_{-\infty}^{+\infty} \mathrm{d}p\, \psi^*(p) x^2 \psi(p) \\ &= \int_{-\infty}^{+\infty} \frac{A^2 l^2}{\hbar} e^{-\frac{p^2 l^2}{2\hbar^2}} \left(-\hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}p^2}\right) e^{-\frac{p^2 l^2}{2\hbar^2}}\, \mathrm{d}p \\ &= \boxed{\frac{1}{2} l^2}.\end{aligned} \tag{6.9}$$

Still,

$$\boxed{\sqrt{\left\langle (\Delta x)^2 \right\rangle} \sqrt{\left\langle (\Delta p)^2 \right\rangle} = \frac{\hbar}{2}.} \tag{6.10}$$