GP1 HW2

Jiete XUE*

September 18, 2025

Problem 1
We have

$$\dot{v} = \frac{1}{2}\frac{\mathrm{d}(v^2)}{\mathrm{d}x}, \tag{1.1}$$

Plug in Newton's second law, we have

$$F = m\dot{v} = \frac{1}{2}m\frac{\mathrm{d}(v^2)}{\mathrm{d}x}. \tag{1.2}$$

Then we can do the integration both sides.

$$v^2 = v_0^2 + \frac{2}{m}\int_{x_0}^{x} F\,\mathrm{d}x. \tag{1.3}$$

In particular, if F is a constant, we have

$$v^2 = v_0^2 + \frac{2}{m}F(x - x_0). \tag{1.4}$$

Problem 2
(1)

$$m\dot{v}_x = qv_yB \tag{2.1}$$
$$m\dot{v}_y = -qv_xB + qE \tag{2.2}$$
$$m\dot{v}_z = 0 \tag{2.3}$$

Since $v_{z0} = 0$ and by (2.3), we have

$$v_z(t) = 0. \tag{2.4}$$

So the motion remains in $z = 0$ plane.
(2) We need $\dot{v}_y = 0$ to satisfy particle moving unreflected through the field. Then, by (2.3),

$$v_x = \frac{E}{B} \tag{2.5}$$

*SID: 20253121013

(3) Let $r = x + iy$, where i is the imaginary unit. Rewrite the equation in 1. as:

$$m\ddot{r} = iq(E - \dot{r}B). \tag{2.6}$$

Then we can deduce that

$$r = \frac{i}{2m}qEt^2 + Ae^{-iqBt/m} + C. \tag{2.7}$$

By the initial condition,

$$r(0) = 0,\ \dot{r}(0) = v_{x0}, \tag{2.8}$$

we have

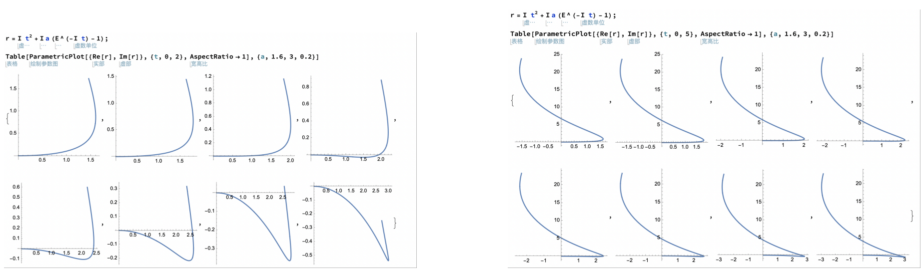
$$r(t) = \frac{i}{2m}qEt^2 + \frac{iv_{x0}m}{qB}\left(e^{-iqBt/m} - 1\right), \tag{2.9}$$

$$\dot{r}(t) = i\frac{qE}{m}t + v_{x0}\left(e^{-iqBt/m} - 1\right). \tag{2.10}$$

(4) Reparameterize r to

$$r = i\eta t^2 + i\epsilon\left(e^{-i\omega t} - 1\right), \tag{2.11}$$

where η and ϵ are both real numbers.



(a) In the initial stage

(b) In the following stage

Figure 1: The motion of a charged particle in a electromagnetic field.

Problem 3

We just need to consider only one direction, and the others will be the same. Assume that

$$v_{\rho 0} = v_0\cos\theta,\ v_{z0} = v_0\sin\theta. \tag{3.1}$$

Then,

$$z = -\frac{1}{2}g\left(\frac{\rho}{v_0}\right)^2(1 + \tan^2\theta) + \rho\tan\theta. \tag{3.2}$$

We want to know the boundary when θ have taken all the values it can reach.

$$\frac{\partial z}{\partial(\tan\theta)} = -g\left(\frac{\rho}{v_0}\right)^2\tan\theta + \rho = 0. \tag{3.3}$$

Plug in (3.2), we obtain:

$$z = \frac{v_0^2}{2g} - \frac{1}{2} g \frac{\rho^2}{v_0^2}. \tag{3.4}$$

That's exactly the boundary.

Problem 4 (Fall wtih resistance)
(1) At this scale and speed, we suppose quadratic drag force take the main role. Set down straight as the positive direction of axis z.

$$mg - \frac{\kappa\pi}{4} D^2 \rho_a \dot{z}^2 = m\ddot{z}. \tag{4.1}$$

Let $z = z^{(0)} + z^{(1)}, z^{(1)} << z^{(0)}$,

$$z^{(0)} = \frac{1}{2} g t^2. \tag{4.2}$$

$$-\frac{\kappa\pi}{4} D^2 \rho_a (\dot{z}^{(0)})^2 \approx m\ddot{z}^{(1)}. \tag{4.3}$$

Thus,

$$z^{(1)}(t) \approx -\frac{\kappa\pi}{480} \frac{D^2 g^2 \rho_a}{m} t^6 = -\epsilon t^6. \tag{4.4}$$

Let $t = t^{(0)} + t^{(1)}, t^{(0)} << t^{(1)}$,

$$t^{(0)} = \sqrt{\frac{2z}{g}} \approx 3.35s, \tag{4.5}$$

$$g t^{(0)} t^{(1)} - \epsilon \left(t^{(0)} \right)^6 \approx 0. \tag{4.6}$$

Hence,

$$t^{(1)} \approx \frac{\epsilon (t^{(0)})^5}{g}. \tag{4.7}$$

$$\Delta t \approx \frac{\Delta\epsilon (t^{(0)})^5}{g} \approx 0.42s \tag{4.8}$$

It is difficult to tell difference at the top of the tower. But the people at the bottom can find they didn't hit the ground at the same time.
(2) I think this problem makes no sense, and I have no reason to believe that the formula you give still holds in such high speed. So I refuse to answer.

Problem 5 (The Longest Fall)
(1)

$$\Delta m_1 = \sigma \Delta\Omega r_1^2,\ \Delta m_2 = \sigma\Delta\Omega r_2^2. \tag{5.1}$$

$$F_1 = \frac{G\Delta m_1 m}{r_1^2} = \frac{G\Delta m_2 m}{r_2^2} = F_2. \tag{5.2}$$

That means the force generate by the opposite spheres cancel each other.
(2) Suppose the density of the spheres is ρ, then

$$F(r) = -\frac{G\rho\left(\frac{4\pi}{3}r^3\right)m}{r^2} = -\frac{4\pi\rho G m}{3}r. \tag{5.3}$$

By Newton's second law, we have

$$m\ddot{r} = -\frac{4\pi\rho G m}{3}r. \tag{5.4}$$

Its solution is:

$$r(t) = R\cos(\omega t + \phi). \tag{5.5}$$

Where R is the amplitude, $\omega = \sqrt{\frac{4\pi\rho G}{3}}$ is the angular frequency and ϕ is the phase. It is a harmonic oscillation, and its period is

$$T = 2\pi\sqrt{\frac{3}{4\pi\rho G}}. \tag{5.6}$$

(3) For satellite,

$$m\dot{\theta}^2 R = \frac{Gm\rho\frac{4\pi}{3}R^3}{R^2}. \tag{5.7}$$

Hence,

$$\dot{\theta} = \sqrt{\frac{3}{4\pi\rho G}} \tag{5.8}$$

$$proj = R\cos\theta = R\cos\left(\sqrt{\frac{4\pi\rho G}{3}}t + \phi\right) = r(t). \tag{5.9}$$

(4)

$$\vec{F}(\vec{r}) = -\frac{4\pi\rho G m}{3}\vec{r}. \tag{5.10}$$

Let $\vec{\tau}$ be the direction of the tunnel. Then, we have the projection of $\vec{F}$ in the direction of $\vec{\tau}$ and the motion equation:

$$\vec{F}(\vec{\tau}\cdot\vec{r}) = \vec{\tau}\cdot\vec{F}(\vec{r}) = \vec{\tau}\cdot\ddot{\vec{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(\vec{\tau}\cdot\vec{r}). \tag{5.11}$$

Thus,

$$\vec{\tau}\cdot\vec{r} = R\cos\left(\sqrt{\frac{4\pi\rho G}{3}}t + \phi\right). \tag{5.12}$$

It is still a harmonic oscillation with the same period.