QM HW5

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Problem 1 (Coherent states)
Let $l = \sqrt{\frac{\hbar}{m\omega}}$, then

$$x = \frac{l}{\sqrt{2}}\left(a + a^\dagger\right),\ p = \frac{\hbar}{il}\frac{a - a^\dagger}{\sqrt{2}}. \tag{1.1}$$

$$\langle x\rangle = \frac{l}{\sqrt{2}}\langle\alpha|a + a^\dagger|\alpha\rangle = \frac{l}{\sqrt{2}}\left(\alpha + \alpha^*\right). \tag{1.2}$$

$$\langle p\rangle = \frac{\hbar}{il}\frac{\langle\alpha|a - a^\dagger|\alpha\rangle}{\sqrt{2}} = \frac{\hbar}{il}\frac{\alpha - \alpha^*}{\sqrt{2}}. \tag{1.3}$$

$$x^2 = \frac{l^2}{2}\left[a^2 + {a^\dagger}^2 + \{a, a^\dagger\}\right] = \frac{l^2}{2}\left[a^2 + {a^\dagger}^2 + 2a^\dagger a + 1\right]. \tag{1.4}$$

$$p^2 = \frac{\hbar^2}{2l^2}[2a^\dagger a + 1 - (a^2 + {a^\dagger}^2)]. \tag{1.5}$$

$$\left\langle x^2\right\rangle = \frac{l^2}{2}\left[(\alpha + \alpha^*)^2 + 1\right]. \tag{1.6}$$

$$\left\langle p^2\right\rangle = \frac{\hbar^2}{2l^2}\left[-(\alpha - \alpha^*)^2 + 1\right]. \tag{1.7}$$

Thus,

$$\sqrt{\overline{\Delta x^2}}\sqrt{\overline{\Delta p^2}} = \frac{\hbar}{2}. \tag{1.8}$$

It reaches the minimum uncertainty, so we call it the most classical quantum state.

Problem 2 (Wavefunctions of Harmonic Oscillator)

$$0 = \langle x|\, a\, |0\rangle = \langle x|\, \frac{1}{\sqrt{2}}\left[\frac{x}{l} + \frac{ipl}{\hbar}\right]|0\rangle = \frac{1}{\sqrt{2}}\left(\frac{x}{l}\psi_0 + l\frac{\mathrm{d}\psi_0}{\mathrm{d}x}\right). \tag{2.1}$$

Hence,

$$\psi_0(x) = Ae^{-\frac{1}{2}\frac{x^2}{l^2}}. \tag{2.2}$$

Normalize,

$$|A|^2 \int_{-\infty}^{+\infty} e^{-\frac{x^2}{l^2}}\,\mathrm{d}x = |A|^2\, l\Gamma\left(\frac{1}{2}\right) = 1. \tag{2.3}$$

Therefore,

$$\boxed{\psi_0(x) = \frac{e^{-\frac{x^2}{2l^2}}}{l^{1/2}\pi^{1/4}}.} \tag{2.4}$$

By $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$,

$$\psi_{n+1}(x) = \frac{1}{\sqrt{2(n+1)}}\left(\frac{x}{l} - l\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi_n(x). \tag{2.5}$$

So,

$$\boxed{\psi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{x}{l} - l\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \psi_0(x).} \tag{2.6}$$

Problem 3 (High dimensional Oscillator)
(1) We have $[x_i, p_j] = i\hbar\delta_{ij}$, so

$$[a^i, a_j^\dagger] = \delta^i{}_j,\ \ [a^i, a_i^\dagger] = D. \tag{3.1}$$

$$x^i x_i = \frac{l^2}{2}\left(a^i a_i + a^{i\dagger} a_i^\dagger + 2a_i^\dagger a^i + [a^i, a_i^\dagger]\right). \tag{3.2}$$

$$p^i p_i = -\frac{\hbar^2}{2l^2}\left(a^i a_i + a^{i\dagger} a_i^\dagger - 2a_i^\dagger a^i - [a^i, a_i^\dagger]\right). \tag{3.3}$$

Let $N = a_i^\dagger a_i$, then

$$\boxed{H = \hbar\omega(N + D/2).} \tag{3.4}$$

$$a' = Ua,\ \ a'^\dagger = a^\dagger U^\dagger,\ \ N' = a^\dagger\, U^\dagger U a = N. \tag{3.5}$$

Hence, H is invariant under the transformation.
(2) We only need to check $[Q_{ij}, N] = 0$. We have

$$[a_i, a_k^\dagger a^k] = a^k\delta_{ik} = a_i,\ \ [a_i^\dagger, a_k^\dagger a^k] = -a^k\delta_{ik} = -a_i^\dagger. \tag{3.6}$$

Since A_{ij} is a number, $[A_{ij}, N] = 0$, thus

$$\begin{aligned}[a_i^\dagger A_{ij} a_j, N] &= a_i^\dagger A_{ij}[a_j, N] + [a_i^\dagger, N]A_{ij}a_j \\ &= a_i^\dagger A_{ij} a_j - a_i^\dagger A_{ij} a_j \\ &= 0.\end{aligned} \tag{3.7}$$

(I'm confused about why we need a A_{ij}. It is just a number.)
(3) Anything commutable with H is a conservation. $a_i a_j^\dagger$ is conserved, which means the angular momentum is conserved.

Problem 4 (Quantum Virial Theorem)
(1) We use the Schrödinger picture. Let $|\alpha, t\rangle$ be a state[1]. The Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\alpha\rangle = H|\alpha\rangle. \tag{4.1}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\alpha|\,\mathbf{x}\cdot\mathbf{p}\,|\alpha\rangle = \frac{1}{i\hbar}\langle\alpha|[\mathbf{x}\cdot\mathbf{p}, H]|\alpha\rangle. \tag{4.2}$$

$$[x^i p_i, \frac{p^j p_j}{2m} + V(\mathbf{x})] = [x^i, \frac{p^j p_j}{2m}]p_i + x^i[p_i, V(\mathbf{x})] = i\hbar\left(\frac{p^i p_i}{m} - x^i\partial_i V\right). \tag{4.3}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\alpha|\,\mathbf{x}\cdot\mathbf{p}\,|\alpha\rangle = \langle\alpha|\frac{p^2}{m} - \mathbf{x}\cdot\nabla V|\alpha\rangle. \tag{4.4}$$

Therefore,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\alpha|\,\mathbf{x}\cdot\mathbf{p}\,|\alpha\rangle = \left\langle\frac{p^2}{m}\right\rangle - \langle \mathrm{x}\cdot\nabla V\rangle.} \tag{4.5}$$

(2) Since H is Hermitian and E is real, we have

$$(H-E)|n,\lambda\rangle = 0,\ \langle n,\lambda|(H-E) = 0. \tag{4.6}$$

Thus,

$$\left(\frac{\partial}{\partial\lambda}\langle n,\lambda|\right)(H-E)|n,\lambda\rangle = 0,\ \langle n,\lambda|(H-E)\left(\frac{\partial}{\partial\lambda}|n,\lambda\rangle\right) = 0. \tag{4.7}$$

Let $\frac{\partial}{\partial\lambda}$ act on the following equation,

$$\langle n,\lambda|(H-E)|n,\lambda\rangle = 0, \tag{4.8}$$

we obtain,

$$\langle n,\lambda|\frac{\partial}{\partial\lambda}(H-E)|n,\lambda\rangle. \tag{4.9}$$

Exactly,

$$\boxed{\frac{\partial E}{\partial\lambda} = \langle n,\lambda|\frac{\partial H}{\partial\lambda}|n,\lambda\rangle}. \tag{4.10}$$

(3)
(4) For harmonic oscillator, $V = \frac{1}{2}m\omega^2 x^2$, $\mathbf{x}\cdot\nabla V = m\omega^2 x^2$.

$$\mathbf{x}\cdot\mathbf{p} = \frac{\hbar}{2}\left(a^2 - a^{\dagger 2} - 3\right). \tag{4.11}$$

So,

$$\langle\mathbf{x}\cdot\mathbf{p}\rangle = -\frac{3\hbar}{2} \tag{4.12}$$

[1] Shorten as $|\alpha\rangle$

$$\langle n|p^2|n\rangle = \left(n+\frac{3}{2}\right)\frac{\hbar m\omega}{2}, \;\; \langle n|x^2|n\rangle = \left(n+\frac{3}{2}\right)\frac{\hbar}{2m\omega}. \tag{4.13}$$

Quantum Viral theorem holds. We can find that $\langle \mathbf{x}\cdot\mathbf{p}\rangle$ is not dependent on n.

Problem 5 (Operator normal product)
Since $|0\rangle$ is the ground state and $H = \omega a^\dagger a$, $a\,|0\rangle = 0$.

$$e^{\alpha a}\,|0\rangle = \sum_{k=1}^{\infty}\frac{\alpha^k}{k!}a^k\,|0\rangle = |0\rangle\,. \tag{5.1}$$

So,

$$\langle 0|\!:e^A:\!|0\rangle = \langle 0|e^{\alpha' a^\dagger}e^{\alpha a}|0\rangle = 1. \tag{5.2}$$

$$\langle 0|AB|0\rangle = \alpha\beta'. \tag{5.3}$$

By Baker-Hausdorff lemma, we can deduce that: If $[[A,B],A] = [[A,B],B] = 0$, then

$$\exp(AB) = \exp(BA)\exp([A,B]). \tag{5.4}$$

Thus,

$$e^{\alpha a}e^{\beta' a^\dagger} = e^{\beta' a^\dagger}e^{\alpha a}e^{\alpha\beta'}. \tag{5.5}$$

$$e^{\alpha' a}e^{\alpha a}e^{\beta' a^\dagger}e^{\beta a} = e^{(\alpha'+\beta')a^\dagger}e^{(\alpha+\beta)a}e^{\alpha\beta'}. \tag{5.6}$$

That is

$$:e^A::e^B: = :e^{A+B}:e^{\langle 0|AB|0\rangle}. \tag{5.7}$$

$$e^A = :e^A:e^{\frac{1}{2}\alpha\alpha'}. \tag{5.8}$$

Let $A = B$ in (5.3), we get $\alpha\alpha' = \langle 0|A^2|0\rangle$. Then Plug (5.8) into (5.7), we obtain

$$e^A e^B = :e^{A+B}:e^{\langle 0|AB+\frac{A^2}{2}+\frac{B^2}{2}|0\rangle}. \tag{5.9}$$

Therefore,

$$\boxed{\langle 0|e^A e^B|0\rangle = \langle 0|\!:e^{A+B}:\!|0\rangle\, e^{\langle 0|AB+\frac{A^2}{2}+\frac{B^2}{2}|0\rangle} = e^{\langle 0|AB+\frac{A^2}{2}+\frac{B^2}{2}|0\rangle}.} \tag{5.10}$$