Westlake University
Fundamental algebra and analysis I

Final exam of January 7th 2026, 12:00-18:00 (E13-206)

The use of any electronic devices is strictly prohibited. The statement of a question, even if not justified, is permitted to be used in the answers to subsequent questions. The degree of clarity in the writing is an important evaluation factor in this exam.

Throughout the examination, we denote by $\mathbb{R}$ the field of real numbers and we equip it with the usual absolute value $|\cdot|$ on $\mathbb{R}$, which is given by

$$(a \in \mathbb{R}) \longmapsto \max\{a, -a\}.$$

Part I: Exponential of endomorphisms

Preamble. In this part, we fix a finite-dimensional vector space V over $\mathbb{R}$, equipped with a norm $\|\cdot\|_V$. We call *endomorphism* of V any $\mathbb{R}$-linear mapping from V to itself. We denote by $\mathrm{End}(V)$ the set of all endomorphisms of V. It is an $\mathbb{R}$-vector subspace of V^V. We equip it with the operator norm $\|\cdot\|$, which is defined as

$$\forall\, f \in \mathrm{End}(V), \quad \|f\| := \sup_{x \in V \setminus \{0\}} \frac{\|f(x)\|_V}{\|x\|_V}$$

For any $f \in \mathrm{End}(V)$ and any positive integer n, we denote by f^n the mapping

$$\underbrace{f \circ \cdots \circ f}_{n \text{ copies}}.$$

Moreover, by convention f^0 denotes the identity mapping Id_V.

For $n \in \mathbb{N}$, let

$$P_n : \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$$

the mapping that sends $f \in \mathrm{End}(V)$ to f^n.

Questions.

1. Prove that P_0 is differentiable and determine its differential.

2. Prove that P_1 is differentiable and determine its differential.

3. Prove that P_2 is differentiable and determine its differential.
(*Hint: Find a bilinear mapping*

$$B : \mathrm{End}(V) \times \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$$

such that $P_2(f) = B(f, f)$ *for any* $f \in \mathrm{End}(V)$. *In general* B *is not symmetric, so care is needed when computing its differential.*)

4. Let $f \in \mathrm{End}(V)$. Prove that the series

$$\sum_{n \in \mathbb{N}} \frac{1}{n!} f^n$$

converges absolutely. We denote by $\exp(f)$ its limit.

5. Let $\mathbf{0} : V \to V$ be the constant mapping sending $x \in V$ to the neutral element 0_V of V. Determine $\exp(\mathbf{0})$.

6. Let t be a real number. Determine $\exp(t\,\mathrm{Id}_V)$.

7. Prove that, for any $R > 0$, the series of functions

$$(f \in \mathrm{End}(V)) \longmapsto \sum_{k=0}^{n} \frac{1}{k!} f^k$$

converges normally on

$$\{f \in \mathrm{End}(V) \mid \|f\| \leq R\}$$

Deduce that $\exp(\cdot)$ is a continuous mapping.

8. Let f and g be two elements of $\mathrm{End}(V)$ such that $f \circ g = g \circ f$. Prove that

$$\exp(f + g) = \exp(f) \circ \exp(g).$$

9. Prove that $\exp(\cdot) : \mathrm{End}(V) \to \mathrm{End}(V)$ is differentiable at Id_V. Determine its differential at Id_V.

10. Prove that $P_n : \mathrm{End}(V) \to \mathrm{End}(V)$ is differentiable for any $n \in \mathbb{N}$. Deduce that $\exp(\cdot) : \mathrm{End}(V) \to \mathrm{End}(V)$ is differentiable.

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Part II: Urysohn's lemma

Preamble. In this part, we fix a Hausdorff topological space X. We assume that X is *locally compact*, namely any element $x \in X$ has a compact neighbourhood in X. The purpose of this part is to prove the following statement: for any compact subset K and any open subset U of X such that $K \subseteq U$, there exists a continuous mapping $f : X \to [0,1]$ such that

(1) for any $y \in K$, $f(y) = 1$,

(2) the *support* of f, which is defined as
$$\mathrm{Supp}(f) := \overline{\{x \in X \mid f(x) \neq 0\}},$$
is compact,

(3) $\mathrm{Supp}(f) \subseteq U$.

Questions.

11. Let x be an element of X and F be a closed subset of X such that $x \notin F$. Prove that there exists a compact neighbourhood C of x such that $C \cap F = \varnothing$.
(*Hint: Let N be a compact neighbourhood of x. Prove that $N \cap F$ is compact. Then use the Hausdorff property of X and the compactness of $N \cap F$ to separate x from $N \cap F$.*)

12. Let x be an element of X and V_x be an open neighbourhood of x. Prove that there exists an open neighbourhood W_x of x such that $\overline{W}_x$ is compact and $\overline{W}_x \subseteq V_x$.

13. Let Y and V be subsets of X such that Y is compact, V is open and $Y \subseteq V$. Prove that there exists an open subset W_Y of X such that $\overline{W}_Y$ is compact and
$$Y \subseteq W_Y \subseteq \overline{W}_Y \subseteq V.$$

14. The previous question allows us to construct an open subset W of X such that $\overline{W}$ is compact and
$$K \subseteq W \subseteq \overline{W} \subseteq U.$$
Prove that there exists a family $(U_r)_{r \in \mathbb{Q} \cap]0,1]}$ of open subsets of X which satisfies the following conditions:

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(1) $K \subseteq U_1$,

(2) for any $r \in \mathbb{Q} \cap]0,1]$, $\overline{U}_r \subseteq W$,

(3) for any $(r,s) \in (\mathbb{Q} \cap]0,1])^2$ such that $r < s$, one has $\overline{U}_s \subseteq U_r$.

(*Hint: one can write the elements of* $\mathbb{Q} \cap]0,1]$ *into a sequence* $(r_n)_{n\in\mathbb{N}}$ *such that* $r_0 = 1$ *and reason by induction.*)

15. For $x \in X$, let

$$f(x) = \sup\{r \in \mathbb{Q} \cap]0,1] \mid x \in U_r\},$$

where the supremum is taken in $[0,1]$, so that $\sup \varnothing$ is equal to 0. Prove that f is continuous.
(*Hint: for any* $t \in [0,1]$, *verify that*

$$\{x \in X \mid f(x) > t\} \text{ and } \{x \in X \mid f(x) < t\}$$

are open subsets of X.)

16. Check that $f|_K$ is constant of value 1 and $\mathrm{Supp}(f) \subseteq W$. Deduce that the support of f is compact.

17. Let $C_c(X)$ be the set of all continuous mappings from X to $\mathbb{R}$ which have a compact support. Prove that the topology of X identifies with the coarsest topology on X making all mappings in $C_c(X)$ continuous.

Part III: Radon measures

Preamble. In this part, we fix a locally compact Hausdorff space X. We denote by $\mathcal{A}$ the Borel σ-algebra on X. We suppose that there exists an increasing family $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X such that $K_n \subseteq K_{n+1}^\circ$ for any $n \in \mathbb{N}$ and that

$$X = \bigcup_{n\in\mathbb{N}} K_n.$$

We denote by $C_c(X)$ the set of all continuous mappings from X to $\mathbb{R}$ which have a compact support. Note that $C_c(X)$ is a vector subspace of $\mathbb{R}^X$. We call *positive linear functional* on $C_c(X)$ any $\mathbb{R}$-linear mapping I from $C_c(X)$ to $\mathbb{R}$ such that, for any $f \in C_c(X)$ satisfying

$$\forall\, x \in X,\ f(x) \geq 0,$$

one has $I(f) \geq 0$.

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Questions.

18. Prove that $C_c(X)$ is a Riesz space on X.

19. Prove that $\mathbb{1}_X \in C_c(X)^{\uparrow}$, where $C_c(X)^{\uparrow}$ denotes the set of mappings from X to $\mathbb{R} \cup \{+\infty\}$ that can be written as the pointwise limit of an increasing sequence in $C_c(X)$.
(*Hint: Use the results of Part II to prove that there exists, for any* $n \in \mathbb{N}$, *a continuous mapping* $g_n : X \to [0, 1]$ *such that* $g_n(x) = 1$ *for any* $x \in K_n$ *and that* $\operatorname{Supp}(g_n) \subseteq K_{n+1}$.)

20. Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $C_c(X)$ that converges pointwisely to the zero mapping. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to the zero mapping.

21. Let $I : C_c(X) \to \mathbb{R}$ be a positive linear functional. Prove that I is an integral operator on $C_c(X)$.
(*Hint: Let* $(f_n)_{n\in\mathbb{N}}$ *be a decreasing sequence in* $C_c(X)$ *that converges pointwise to* 0. *Let* K *be the support of* f_0. *By using the results of Part II, prove that there exists* $g : X \to [0, 1]$ *in* $C_c(X)$ *such that* $g(x) = 1$ *for any* $x \in K$. *Then prove that* $f_n \leq \|f_n\|_{\sup}\, g$, *where* $\|f_n\|_{\sup}$ *denotes* $\sup_{x\in X} f_n(x)$.)

22. Deduce that there exists a unique σ-finite measure μ_I on $(X, \mathcal{A})$ such that, any element f of $C_c(X)$ is integrable with respect μ and that

$$\int_X f \, \mathrm{d}\mu_I = I(f).$$

We call μ_I the Radon measure associated with the positive linear functional I.

23. For $p \in X$, let $I_p : C_c(X) \to \mathbb{R}$ be the mapping that sends $f \in C_c(X)$ to $f(p)$.

(a) Prove that I_p is a positive linear functional. We denote by δ_p the Radon measure associated with I_p and call it the *Dirac measure* at p.

(b) Prove that, for any $\mathcal{A}$-measurable mapping from X to $\mathbb{R}_{\geq 0}$, one has

$$\int_X f \, \mathrm{d}\delta_p = f(p).$$

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(*Hint: Apply a monotone class theorem.*)

24. We say that a subset Y of X is discrete if, for any $y \in Y$, there exists a neighbourhood V_y of y in X such that $V_y \cap Y = \{y\}$. Prove that, if Y is a discrete and closed subset of X, then, for any compact subset F of X, the intersection $Y \cap F$ is a finite set.

25. Let Y be a discrete and closed subset of X and $\lambda : Y \to \mathbb{R}_{\geq 0}$ be a mapping. Let $I_{Y,\lambda} : C_c(X) \to \mathbb{R}$ be the mapping that sends $f \in C_c(X)$ to

$$\sum_{y \in Y,\, f(y) \neq 0} \lambda(y) f(y).$$

(a) Prove that the mapping $I_{Y,\lambda}$ is well defined, and defines a positive linear functional on $C_c(X)$.

(b) We denote by $\delta_{Y,\lambda}$ the Radon measure on $(\Omega, \mathcal{A})$ associated with $I_{Y,\lambda}$. Prove, that, for any measurable mapping $f : X \to \mathbb{R}_{\geq 0}$, one has

$$\int_X f \,\mathrm{d}\delta_{Y,\lambda} = \sum_{y \in Y} \lambda(y) f(y) := \sup_{\substack{Z \subseteq Y \\ Z \text{ is finite}}} \sum_{y \in Z} \lambda(z) f(z).$$

Part IV: Series as an integral

Preamble. In this part, we denote by $\mathbb{N}_*$ the set of positive integers. We fix a mapping $\lambda : \mathbb{N}_* \to \mathbb{R}_{\geq 0}$. For any $x \in \mathbb{R}_{>0}$, let

$$S(x) = \sum_{n \in \mathbb{N}_*,\, n \leq x} \lambda(n).$$

Questions.

26. Prove that $\mathbb{N}_*$ is a discrete and closed subset of $\mathbb{R}_{>0}$.

27. Prove that the mapping S is increasing and right continuous.

28. Prove that the Radon measure $\delta_{\mathbb{N}_*,\lambda}$ identifies with the Lebesgue-Stieltjes measure on $\mathbb{R}_{>0}$ associated with S, namely, for any Borel measurable mapping $f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$, one has

$$\int_{\mathbb{R}_{>0}} f(x)\,\mathrm{d}S(x) = \int_{\mathbb{R}_{>0}} f\,\mathrm{d}\delta_{\mathbb{N}_*,\lambda} = \sum_{n\in\mathbb{N}_*} \lambda(n)f(n).$$

(*Hint: First check the case where f is of the form $\mathbb{1}_{]a,b]}$, where $(a,b) \in \mathbb{R}^2_>$, $a<b$. Then conclude by using the monotone class theorem.*)

29. Let $\varphi : \mathbb{R}_{>0} \to \mathbb{R}$ be a mapping which could be written as the difference of two increasing and right continuous mappings from $\mathbb{R}_{>0}$ to $\mathbb{R}$. Prove that, for any real number $x > 0$, one has

$$\sum_{n\in\mathbb{N}_*,\, n\leq x} \lambda(n)\varphi(n) = \varphi(x)S(x) - \int_1^x S(t)\,\mathrm{d}\varphi(t) + \sum_{n\in\mathbb{N}_*,\, n\leq x} \lambda(n)\Delta\varphi(n),$$

where

$$\Delta\varphi(n) := \varphi(n) - \lim_{\varepsilon>0,\,\varepsilon\to 0} \varphi(n-\varepsilon).$$

30. Let $(a,b) \in \mathbb{R}\times\mathbb{R}$ such that $0 < a < b$. Let $f : [a,b] \to \mathbb{R}$ be a continuous mapping which is continuously differentiable on $]a,b[$. Assume that f' extends to a continuous mapping form $[a,b]$ to $\mathbb{R}$.

(a) Prove that f can be written as the difference of two continuous and increasing mappings.
Hint: consider the mapping

$$x \longmapsto \int_a^x |f(t)|\,\mathrm{d}t.$$

(b) Prove that

$$\sum_{n\in\mathbb{N}_*,\, a<n\leq b} f(n) = \int_a^b f(t)\,\mathrm{d}t + \int_a^b \langle t\rangle f'(t)\,\mathrm{d}t - f(b)\langle b\rangle + f(a)\langle a\rangle,$$

where for any real number t, $\langle t\rangle$ denotes the decimal part of t, namely

$$\langle t\rangle := t - \lfloor t\rfloor,$$

with $\lfloor t\rfloor$ being the greatest integer bounded from above by t.

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31. Let $\varphi : \mathbb{R}_{>0} \to \mathbb{R}$ be a decreasing and right continuous mapping such that

$$\lim_{x \to +\infty} \varphi(x) = 0.$$

Prove that, for any $x \in \mathbb{R}_{>1}$, one has

$$\int_1^x \varphi(t)\,\mathrm{d}t \le \sum_{n \in \mathbb{N}_*,\, n \le x} \varphi(n) \le \varphi(1) + \int_1^x \varphi(t)\,\mathrm{d}t.$$

Deduce that the series

$$\sum_{n \in \mathbb{N}_*} \varphi(n) < +\infty$$

if and only if

$$\int_1^x \varphi(t)\,\mathrm{d}t < +\infty.$$

32. Let s be a real number. Prove that the series

$$\sum_{n \in \mathbb{N}_*} \frac{1}{n^s}$$

converges when $s > 1$, and it diverges when $s \le 1$.

33. Let α and β be real numbers. Prove that the series

$$\sum_{n \in \mathbb{N}_{\ge 2}} \frac{1}{n^\alpha \ln(n)^\beta}$$

diverges when $\alpha < 1$ or ($\alpha = 1$ and $\beta \le 1$). It converges when $\alpha > 1$ or ($\alpha = 1$ and $\beta > 1$).

34. Prove that there is a constant $\gamma > 0$ (called Euler's constant) such that

$$\sum_{n \in \mathbb{N}_*,\, n \le x} \frac{1}{n} = \ln(x) + \gamma + O\Big(\frac{1}{x}\Big), \quad x \to +\infty.$$

35. Let s be a real number such that $s > 1$. prove that

$$\sum_{n \in \mathbb{N}_*,\, n \le x} \frac{1}{n^s} = \zeta(s) + \frac{x^{1-s}}{1-s} + O(x^{-s}),\ x \to +\infty,$$

where

$$\zeta(s) := \sum_{n \in \mathbb{N}_*} \frac{1}{n^s}.$$

36. Let s be a real number such that $0 < s < 1$.

(a) Prove that
$$\left(\sum_{n\in\mathbb{N}_*,\, n\leq x} \frac{1}{n^s}\right) - \frac{x^{1-s}}{1-s}$$
has a limit when $x \to +\infty$. We denote by $Z(s)$ this limit.

(b) Prove that
$$\left(\sum_{n\in\mathbb{N}_*,\, n\leq x} \frac{1}{n^s}\right) - \frac{x^{1-s}}{1-s} - Z(s) = O(x^{-s}), \quad x \to +\infty.$$

(c) Let b be a real number such that $b \geq 0$. Prove that
$$\sum_{n\in\mathbb{N}_*,\, n\leq x} n^b = \frac{x^{b+1}}{b+1} + O(x^b), \quad x \to +\infty.$$

Part V: The average order of $d(n)$

Preamble. For any positive integer n, we denote by $d(n)$ the set of divisors of n, namely
$$d(n) = \sum_{k\in\mathbb{N}_*,\, k|n} 1.$$

37. Let x be a positive real number, prove the equality
$$\sum_{n\in\mathbb{N}_*,\, n\leq x} d(n) = \sum_{\substack{(k,\ell)\in\mathbb{N}_*^2,\\ k\ell\leq x}} 1.$$

38. Let x be a positive real number. Prove that
$$\sum_{n\in\mathbb{N}_*,\, n\leq x} d(n) = \sum_{\substack{k\in\mathbb{N}_*\\ k\leq x}} \sum_{\substack{\ell\in\mathbb{N}_*\\ \ell\leq x/k}} 1.$$

Deduce that
$$\sum_{n\in\mathbb{N}_*,\, n\leq x} d(n) = x\ln(x) + O(x), \quad x \to +\infty.$$

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39. Let x be a positive real number. Prove that

$$\sum_{n\in\mathbb{N}_*,\, n\leq x} d(n) = 2\Bigg(\sum_{\substack{k\in\mathbb{N}_* \\ k\leq\sqrt{x}}} \sum_{\substack{\ell\in\mathbb{N}_* \\ \ell\leq x/k}} 1 \Bigg) - \lfloor\sqrt{x}\rfloor^2.$$

40. Deduce that

$$\sum_{n\in\mathbb{N}_*,\, n\leq x} d(n) = x\ln(x) + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is Euler's constant introduced in Question **34.**

The end