Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 8–2: Topology (2)

1. We now define three topologies on the real line $\mathbb{R}$, all of which are of interest.

If $\mathscr{B}$ is the collection of all open intervals in the real line,

$$]a, b[= \{x \mid a < x < b\},$$

the topology generated by $\mathscr{B}$ is called the **standard topology** (or **Euclidean topology)** on the real line. Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

If $\mathscr{B}'$ is the collection of all half-open intervals of the form

$$[a, b[= \{x \mid a \leqslant x < b\},$$

where $a < b$, the topology generated by $\mathscr{B}'$ is called the **lower limit topology** on $\mathscr{R}$. When $\mathscr{R}$ is given the lower limit topology, we denote it by $\mathbb{R}_\ell$.

Finally let K denote the set of all numbers of the form $\frac{1}{n}$, for $n \in \mathbb{N}^+$, and let $\mathscr{B}''$ be the collection of all open intervals (a, b), along with all sets of the form $(a, b) \smallsetminus K$. The topology generated by $\mathscr{B}''$ will be called the K-**topology** on $\mathbb{R}$. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_K$.

(1) Prove that all three of these collections are bases.

(2) Prove that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than the standard topology on $\mathbb{R}$, but are not comparable with one another.

2. Check that $\mathcal{T} = \{(a, +\infty) \mid a \in \mathbb{R}\}$ is a topology on $\mathbb{R}$, find $\mathcal{V}_p(\mathcal{T})$ for $p \in \mathbb{R}$.

3. Show that

$$\lim_{\mathcal{F}} f$$

does not need to be unique.

4. What happens with $\lim_{\mathcal{F}} f$ for the degenerate $\mathcal{F}$? Do we really need it?

Assume that $\mathcal{V}_p(\mathcal{T}_x)/\operatorname{Dom}(f)$ is non-degenerate. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be two topological spaces, $f : X \to Y$ be a continuous map. We say that $f(X)$ converges to ℓ when $x \in \operatorname{Dom}(f)$ tends to p and write

$$\lim_{x \in \operatorname{Dom}(f), x \to p} f(x) = \ell$$

when

$$\lim_{\mathcal{V}_p(\mathcal{T}_X)/\operatorname{Dom}(f)} f = \ell.$$

5. (a) Prove that f is continuous at p if and only if

$$\lim_{x \in \operatorname{Dom}(f), x \to p} f(x) = f(p),$$

i.e., $f(x)$ converges to $f(p)$ when $x \in \operatorname{Dom}(f)$ tends to p.

(b) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Prove that

$$\mathrm{id} : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$$

is continuous if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

(c) Prove that

$$\begin{array}{rccc} x_0 : & (X, \mathcal{T}_1) & \longrightarrow & (X, \mathcal{T}_2) \\ & x & \mapsto & x_0 \end{array}$$

is locally continuous.

6. Let $(L, |\cdot|)$ be a valued field. Show that $|1_K| = |-1_K| = 1$.

7. Let $(E, \|\cdot\|)$ be a semi-normed vector space. Show that

$$\begin{array}{ccc} E \times E & \longrightarrow & \mathbb{R} \\ (x, y) & \mapsto & \|x - y\| \end{array}$$

is a semi-norm.

8. Let (X, d) be a pseudo-metric space.

(1) For $x, y \in X$, check that $x \sim y \Leftrightarrow d(x, y) = 0$ is an equivalent relation.

(2) Let $X^* = X/\sim$ be the quotient space. Show that

$$\begin{array}{rccc} d^* & X^* \times X^* & \longrightarrow & \mathbb{R}_{\geqslant 0} \\ & ([x], [y]) & \mapsto & d(x, y) \end{array}$$

is a metric.

9. Let $(K, |\cdot|)$ be a valued field, $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be two seminormed spaces over K, and $f : E \to F$ be a morphism. Prove that the following statements are equivalent.

(1) f is continuous.

(2) $\exists\, p \in E$, such that f is continuous at p.

(3) f is continuous at $0 \in E$.

(4) f is bounded in some neighborhood of 0.

(5) $\exists\, M \geqslant 0$, such that $\forall\, x \in E$, $\|x\| \leqslant 1 \Rightarrow \|f(x)\| \leqslant M$.

(6) $\exists\, M \geqslant 0$, such that $\forall\, x \in E$, $\|f(x)\| \leqslant M\|x\|$.

(7) f is uniformly continuous.

10. Let $\mathcal{L}(E, F)$ be the set of linear maps from E to F, where $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are two normed vector space over a valued field $(K, |\cdot|)$. For a $f \in \mathcal{L}(E, F)$, we define the **operator norm** of f as

$$\|f\| \colon = \sup_{x \in E, \|x\| \neq 0} \frac{\|f(x)\|_F}{\|x\|_E}.$$

Prove the following statements.

(1) $\|f\| = \sup\{\|f(x)\|_F \mid x \in E,\ \|x\|_E \leqslant 1\}$.

(2) $\|f\| = \inf\{M \geqslant 0 \mid \forall\, x \in E,\ \|f(x)\|_F \leqslant M\|x\|_E\}$.

(3) If $E \neq \{0\}$, then $\|f\| = \sup\{\|f(x)\|_F \mid x \in E,\ \|x\|_E = 1\}$.

11. Let X be a topological space. We say that X is **irreducible** if $X = X_1 \cup X_2$ with the closed subsets X_1 and X_2, then we have $X = X_1$ or $X = X_2$.

(1) Let $\mathbb{R}$ be the topological space equipped with the standard topology. Prove that if I is an irreducible subspace of $\mathbb{R}$, then I consists of a single point.

(2) Let X be an irreducible topological space. Prove that the intersection of two non-empty open subsets of X is non-empty.

(3) Let X be a topological space. Prove that the set of irreducible subspaces of X admits maximal elements for the inclusion relation, and such a maximal element is a closed subset.

Let X be a topological space. We call these maximal elements determined in (3) the **irreducible components** of X.

(4) Prove that the union of all irreducible components of a topological space X is just X.

(5) Let X be irreducible. Prove that any non-empty open subset of X is dense in X and is irreducible.

(6) Let U be an open subset of X. Prove that the irreducible components of U are the $\{X_i \cap U\}_i$, where the X_i are the irreducible components of X which meet U.

(7) Suppose that X is a finite union of irreducible closed subsets $(Z_j)_{j=1}^n$. Prove that every irreducible component Z of X is equal to one of the Z_j, $j = 1, \ldots, n$. If, moreover, there is no inclusion relation between the $(Z_j)_{j=1}^n$, then prove that the $(Z_j)_{j=1}^n$ are exactly the irreducible components of X.

12. Let $\mathbb{A}^n_{\mathbb{C}} = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{C}\}$, and $A = \mathbb{C}[T_1, \ldots, T_n]$. For a $P = (a_1, \ldots, a_n) \in \mathbb{A}^n_{\mathbb{C}}$ and an $f \in A$, we define $f(P) = f(a_1, \ldots, a_n)$ for simplicity. We define

$$Z(f) = \{P \in \mathbb{A}^n_{\mathbb{C}} \mid f(P) = 0\}$$

as the set of zeros of f. Let $T \subseteq A$, we define

$$Z(T) = \{P \in \mathbb{A}^n_{\mathbb{C}} \mid f(P) = 0, \ \forall\, f \in T\}$$

as the common zeros of the elements in T.

Let $Y \subseteq \mathbb{A}^n_{\mathbb{C}}$. We say that Y is an **algebraic set** if there exists a $T \subseteq A$, such that $Y = Z(T)$.

(1) Prove that the union of two algebraic sets is an algebraic set, the intersection of any family of algebraic sets is an algebraic set, and the empty set and the whole space are algebraic sets.

(2) We take the open subsets of $\mathbb{A}^n_{\mathbb{C}}$ as the complements of the algebraic sets. Prove that this operation defines a topology on $\mathbb{A}^n_{\mathbb{C}}$.
We call it the **Zariski topology** on $\mathbb{A}^n_{\mathbb{C}}$.

(3) We consider $\mathbb{A}^1_{\mathbb{C}}$. Describe the algebraic set of $\mathbb{A}^1_{\mathbb{C}}$. Is the Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ Hausdorff or not?

(4) Prove that the Zariski topology on $\mathbb{A}^1_{\mathbb{C}}$ is irreducible (see the previous exercise for the definition of irreducible topology).

(5) Let $Y \subseteq \mathbb{A}^n_{\mathbb{C}}$. We define $I(Y) = \{f \in A \mid f(P) = 0, \ \forall\, P \in Y\}$. Prove the following results.

i. If $T_1 \subseteq T_2$ are subsets of A, then $Z(T_1) \supseteq Z(T_2)$.

ii. If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n_{\mathbb{C}}$, then $I(Y_1) \supseteq I(Y_2)$.

iii. For any two subsets Y_1, Y_2 of $\mathbb{A}^n_{\mathbb{C}}$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.