QM HW6

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Problem 1 (Current, gauge transformation)
(1) Use the substitution:

$$-i\hbar\nabla \longrightarrow -i\hbar\nabla - \frac{q\vec{A}}{c}, \tag{1.1}$$

the probability current will be:

$$\vec{j} = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right) \longrightarrow \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right) - \frac{q}{mc}\vec{A}\left|\psi\right|^2. \tag{1.2}$$

Hence,

$$\boxed{\vec{j} = \frac{\hbar}{m}\mathrm{Im}\left(\psi^*\nabla\psi\right) - \frac{q}{mc}\vec{A}\left|\psi\right|^2.} \tag{1.3}$$

(2)

$$A'_\mu = A_\mu + \partial_\mu f. \tag{1.4}$$

Since the commutator is antisymmetric and the derivative is commutative,

$$\partial_{[\mu}\partial_{\nu]}f = 0. \tag{1.5}$$

Thus,

$$F'_{\mu\nu} = \partial_{[\mu}A'_{\nu]} = \partial_{[\mu}A_{\nu]} + \partial_{[\mu}\partial_{\nu]}f = F_{\mu\nu}. \tag{1.6}$$

So,

$$\mathbf{E}' = \mathbf{E},\ \mathbf{B}' = \mathbf{B}. \tag{1.7}$$

(3)

$$\frac{\partial\psi'}{\partial t} = i\frac{\partial\varphi}{\partial t}\psi' + e^{i\varphi}\frac{\partial\psi}{\partial t} \tag{1.8}$$

Cancelate $e^{i\varphi}$, and plug in Schrödinger equation, we should take

$$\boxed{\varphi = \frac{q}{\hbar c}f.} \tag{1.9}$$

(4)

$$\rho' = e^{i\varphi}\psi e^{-i\varphi}\psi^* = \psi\psi^* = \rho. \tag{1.10}$$

If we let

$$\psi = \sqrt{\rho} e^{\frac{iS}{\hbar}}, \tag{1.11}$$

then,

$$\mathbf{j} = \frac{\rho}{m}\left(\nabla S - \frac{q\mathbf{A}}{c}\right). \tag{1.12}$$

$$S' = S + \hbar\varphi, \quad \nabla S' - \frac{q\mathbf{A}'}{c} = \nabla S - \frac{q\mathbf{A}}{c}. \tag{1.13}$$

So the probability current is invariant under the gauge transformation.

Problem 2 (Landau gauge)
(1)

$$\left(p_x - \frac{qB}{c}y\right)^2 e^{ik_x x} = \left(\hbar k_x - \frac{qB}{c}y\right)^2 e^{ik_x x}. \tag{2.1}$$

$$\left[\frac{\left(\hbar k_x - \frac{qBy}{c}\right)^2}{2m} + (V(y) - E)\right]\phi_{k_x}(y) = \frac{\hbar^2}{2m}\phi''_{k_x}(y). \tag{2.2}$$

We can define $H_y(k_x)$ as

$$H_y(k_x) = \frac{\left(\hbar k_x - \frac{qBy}{c}\right)^2}{2m} + \frac{p_y^2}{2m} + V(y). \tag{2.3}$$

(2) Note that

$$\frac{\partial H_y(k_x)}{\partial k_x} = \frac{\hbar k_x - qBy/c}{m}. \tag{2.4}$$

$$I_x(x, k_x) = qJ = \frac{q\rho}{m}\left(\hbar k_x - \frac{qBy}{c}\right) = \frac{q}{L_x}\frac{\partial E_n}{\hbar \partial k_x}. \tag{2.5}$$

(3)

$$I_{x,n} = \frac{1}{2\pi}\int \mathrm{d}k_x\, I_x(n, k_x) = \frac{q^2}{h}\Delta V / L_x. \tag{2.6}$$

$$\sigma_{xy} = \frac{I_x}{E_y} = \frac{q^2}{h}\nu. \tag{2.7}$$

This result is not related to V_{imp}.