Quantum Mechanics 2025 HW4

Due 10/09 in Class

September 23, 2025

Problem 1. Probability current

Consider the Schrödinger equaiton

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r}, t)\right)\psi(\mathbf{r}, t). \tag{1}$$

Define the probability density $\rho(x, t) = \psi^\dagger(x, t)\psi(x, t)$. In order to jsbr the probability conservation, i.e., $\frac{\partial}{\partial t}\rho + \nabla \cdot \mathbf{j} = 0$, how should we define the probability current $\mathbf{j}(\mathbf{r}, t)$?

Problem 2. time-evolution

Consider a physical observable O, and its expectation value $\bar{O}(t) = \langle\psi(t)|O|\psi(t)\rangle$. The Hamiltonian of the system is time-independent.

Prove the following statements:

1) If $[O, H] = 0$, then $\frac{d}{dt}\bar{O}(t) = 0$.

2) If $|\psi(t)\rangle$ itself is the time-evolution of an eigen-state of H, then $\frac{d}{dt}\bar{O}(t) = 0$.

Problem 3. f-sum rule

Consider a particle in 1D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \tag{2}$$

By calculating $[[H, x], x]$ prove

$$\sum_{n,l} |\langle n|x|l\rangle| = \hbar^2/2m, \tag{3}$$

where $|n\rangle$ is an energy eigenstate of H with the eigenvalue E_n.

Problem 4. The double δ-potential

A particle of mass m moves in 1D x under the potential $V(x)$.

$$V(x) = \gamma \left(\delta(x + \tfrac{a}{2}) + \delta(x - \tfrac{a}{2}) \right). \tag{4}$$

1) Consider the scattering states: At $x < -\frac{a}{2}$, it is a superposition of an incident wave superposed with a reflection wave $\psi(x) = e^{ikx} + Re^{-ikx}$. At $x > \frac{a}{2}$, it is the transmission wave $\psi(x) = Se^{ikx}$. Find the expression of the scattering amplitude R and S.

2) Please find the bound state solutions. Express R and S in terms of E. Please show that the bound state information shows up as the pole of the scattering amplitude.

Problem 5. Quantum Viral theorem

1) Consider a particle in the 3D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \tag{5}$$

By calculating $[\mathbf{x} \cdot \mathbf{p}, H]$, prove that

$$\tfrac{d}{dt} \langle \mathbf{x} \cdot \mathbf{p} \rangle = \langle p^2/m \rangle - \langle \mathbf{x} \cdot \nabla V \rangle. \tag{6}$$

To identify the relation with the classic Viral theorem, we need the left-hand-side vanish. Under what condition does it vanish?