Chapter 1

Basic logic

The purpose of this chapter is to introduce the mathematical logic language used throughout the book, including the fundamentals of propositional logic and predicate logic. By studying this chapter, students will master the basic methods of mathematical reasoning, laying a solid foundation for learning algebra and analysis.

1.1 Mathematical statements

1.1.1 Definition. We call *statement* a declarative sentence without free variable in a given mathematical theory, whose truth value (that is, *true* or *false*) can in principle be determined. In a consistent mathematical theory, any statement is either true or false, but cannot be true and false in the same time.

1.1.2 Example. "2 is an even number", "$1 > 2$" are both statements. The former one is true, the latter is false.

1.1.3 Example. "$1 + 2$" is a computational formula, it is *not* a statement. The inequality "$2x + 1 > 0$" contains a free variable x, so judging its truth value is meaningless. Therefore, it is *not* a statement. In this book, we use the symbol $:=$ to interpret the notation on the left hand side of the symbol as the expression on the right hand side. For example, "$a := 3 + 5$" means that we denote by a the value of $3 + 5$. This expression is *not* a statement.

1.1.4 Definition. In a mathematical theory, statements admitted as true without justification are called the *axioms*. All statements that can be rigorously deduced from axioms are true. A statement that is confirmed to be true through rigorous mathematical proof is called a *theorem*. A statement that can be deduced from a theorem via straightforward reasoning is usually referred to as a *corollary* of that *theorem*.

In mathematical literature, a statement is often called a *proposition*. In this book, the term "*proposition*" is rather used to label theorems that are relatively simple or not repeatedly applied in the book.

1.1.5 Remark. Some mathematical conjectures have neither been proved nor disproved. However, these conjectures have similar forms with the statements the truth values of which have been determined. In principle the truth values of these conjectures could be determined in the future. They should be classified as statements. Furthermore, there exist declarative sentences that are neither provable nor disprovable within a certain mathematical theory. However, in an enriched axiomatic mathematical theory, their truth values can be in principle determined. Such declarative sentences should also be recognised as statements in the enriched mathematical theory.

1.2 Negation

Starting from given statements, compound statements can be constructed through the syntax of construction. This section introduces the negation of statements.

1.2.1 Definition. Let P be a statement. Then the sentence "*not* P" is also a statement, called the *negation* of P. It has the opposite true value of that of P. Sometimes we denote the statement "*not* P" as $\neg P$.

1.2.2 Notation. When a statement is expressed by a linking verb of judgement, its negation can be expressed by negating the linking verb. For example, the negation of the statement "2 *is an even number*" can be expressed as "2 *is not an even number*".

If a statement is expressed as a relation linked by a relation symbol, its negation could be expressed by overlaying the relation symbol with a diagonal line from the upper right to the lower left. For example, the negation of "$1 > 2$" can be expressed as "$1 \not> 2$".

1.2.3 Remark. The double negation of a statement P has the same truth value as that of the statement P. This point can be seen by the table of truth values as follows, where T stands for "true" and F stands for "false".

P	$\neg P$	$\neg\neg P$
T	F	T
F	T	F

1.3 Conjunction and disjunction

1.3.1 Definition. Let P and Q be statements. Then "P *and* Q" is a statement, called the *conjunction* of P and Q, often denoted as $P \wedge Q$. When both P and Q are true, the statement $P \wedge Q$ is true, otherwise it is false.

Similarly, the sentence "P *or* Q" is a statement, called the *disjunction* of P and Q, often denoted as $P \vee Q$. When both P and Q are false, the statement $P \vee Q$ is false, otherwise it is true.

1.3.2 Remark. We describe the truth values of conjunction and disjunction in the following table.

P	Q	$P \wedge Q$	$P \vee Q$	$Q \wedge P$	$Q \vee P$
T	T	T	T	T	T
T	F	F	T	F	T
F	T	F	T	F	T
F	F	F	F	F	F

We observe from the table that $P \wedge Q$ and $Q \wedge P$ have the same truth value, and $P \vee Q$ and $Q \vee P$ have the same truth value.

1.3.3 Proposition. *Let P and Q be statements. The statements $\neg(P \wedge Q)$ and $(\neg P) \vee (\neg Q)$ have the same truth value, and $\neg(P \vee Q)$ and $(\neg P) \wedge (\neg Q)$ have the same truth value.*

Proof. This can be observed from the following tables.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$(\neg P) \wedge (\neg Q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

$\square$

1.4 Conditional statement

1.4.1 Definition. Let P and Q be statements. The sentence "*if* P, *then* Q" is a statement, often denoted as $P \Rightarrow Q$. It has the same truth value as that of $(\neg P) \vee Q$. A statement of this form is called a *conditional statement*. We describe its true value in the following table.

P	Q	$\neg P$	$P \Rightarrow Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

1.4.2 Remark. We observe from the table that, only in the case where P is true and Q is false, the statement $P \Rightarrow Q$ is false, otherwise it is true. Therefore, if one can prove Q under the assumption that P is true, then the statement $P \Rightarrow Q$ is true. However, the statement $P \Rightarrow Q$ itself does not signify that we are considering a proof of the statement Q from P.

We also observe from the table that, in the case where both P and $P \Rightarrow Q$ are true, the statement Q must be true. This type of reasonings often appear in a mathematical proof.

1.4.3 Proposition. *Let P, Q and R be statements. If both $P \Rightarrow Q$ and $Q \Rightarrow R$ are true, then $P \Rightarrow R$ is true.*

Proof. It suffices to treat the case where P is true, since otherwise $P \Rightarrow R$ is automatically true. In the case where P is true, since $P \Rightarrow Q$ is true, we deduce that Q is true. Furthermore, since $Q \Rightarrow R$ is true, we deduce that R is true. Therefore, $P \Rightarrow R$ is true. $\square$

1.4.4 Proposition. *Let P and Q be statements. The statements $P \Rightarrow Q$ and $(\neg Q) \Rightarrow (\neg P)$ have the same truth value.*

Proof. We could conclude from the following table.

P	Q	$\neg P$	$\neg Q$	$(\neg Q) \Rightarrow (\neg P)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

$\square$

1.4.5 Definition. Let P and Q be two statements. The statement $(\neg Q) \Rightarrow (\neg P)$ is called the *contrapositive* of the statement $P \Rightarrow Q$. If one proves the contrapositive statement $(\neg Q) \Rightarrow (\neg P)$, then, by Proposition 1.4.4, we obtain that the statement $P \Rightarrow Q$ is also true. This method is called *proof by contraposition.*

1.4.6 Example. Let n be an integer. We prove by contraposition that, if n^2 is an even number, then n is an even number. Note that the contrapositive of this statement says that, if n is not an even number, then n^2 is not an even number. Since n is an integer, if n is not an even number, it must be an odd number, namely it is of the form $2k + 1$, with k being an integer. Therefore,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1$$

is an odd number. Thus we have proved the contrapositive statement. Hence the initial statement is also true.

1.5 Biconditional statement

1.5.1 Definition. Let P and Q be statements. Then "P *if and only if* Q" is also a statement, often denoted as $P \Leftrightarrow Q$. When P and Q have the same truth value, the statement $P \Leftrightarrow Q$ is true, otherwise $P \Leftrightarrow Q$ is false. By definition, the statements $P \Leftrightarrow Q$ and $Q \Leftrightarrow P$ have the same true value.

1.5.2 Proposition. *Let P and Q be statements. Then $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ and $P \Leftrightarrow Q$ have the same truth value.*

Proof. We could conclude from the following table.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

□

1.5.3 Remark. The above proposition shows that, given two statements P and Q, to prove that they have the same truth value, it suffices to prove conditional statements of both directions $P \Rightarrow Q$ and $Q \Rightarrow P$. The statement $Q \Rightarrow P$ is called the *converse* of $P \Rightarrow Q$. The contraposition of $Q \Rightarrow P$, namely $(\neg P) \Rightarrow (\neg Q)$, is called the *inverse* of $P \Rightarrow Q$. Proposition 1.4.4 shows that the converse and the inverse of a conditional statement have the same truth value. Therefore, to justify the biconditional statement $P \Leftrightarrow Q$, it also suffices to prove the conditional statement $P \Rightarrow Q$ and its inverse $(\neg P) \Rightarrow (\neg Q)$.

1.5.4 Example. Let n be an integer. Then n^2 is an even number if and only if n is an even number. In fact, we have proved in Example 1.4.6 that, if n^2 is an even number, then n is an even number. It remains to check that, if n is an even number, then n^2 is also an even number. Assume that n is of the form $2k$, where k is an integer. Then $n^2 = 4k^2$ is divisible by 2. Hence n^2 is an even number.

1.6 Proof by contradiction

1.6.1 Definition. In a consistent mathematical theory, a statement cannot be true and false at the same time. Let P be a statement. If we assume that $\neg P$ is true and deduce that a certain statement is both true and false, then we say that a *contradiction* happens and the assumption $\neg P$ is false. Thus the statement P is true. Such a reasoning is called *proof by contradiction*.

1.6.2 Example. We prove by contradiction that the equation $x^2 = 2$ does not have any rational solution. Suppose by contradiction that p/q is a solution of the equation $x^2 = 2$, where p is an integer and q is a positive integer, which do not have common prime divisor. By definition, one has $p^2 = 2q^2$. Hence p^2 is an even number. By Example 1.5.4, we obtain that p is an even number. Hence there exists $p_1 \in \mathbb{Z}$ such that $p = 2p_1$. Hence we deduce from $p^2 = 2q^2$ that $2q_1^2 = q^2$. This shows that q^2 is an even number and hence q is an even number. Thus p and q has 2 as a common prime divisor, which leads to a contradiction.

Exercises

1. Let P and Q be statements. Use truth tables to determine the truth values of the following statements according to the truth values of P and Q:

$$P \wedge \neg P,\ P \vee \neg P,\ (P \vee Q) \Rightarrow (P \wedge Q),\ (P \Rightarrow Q) \Rightarrow (Q \Rightarrow P)$$

2. Let P and Q be statements.

(1) Show that $P \Rightarrow (Q \wedge \neg Q)$ has the same truth value as $\neg P$.

(2) Show that $(P \wedge \neg Q) \Rightarrow Q$ has the same truth value as $P \Rightarrow Q$.

3. Consider the following statements:

$$
\begin{aligned}
P &:= \text{“Little Bear is happy”},\\
Q &:= \text{“Little Bear has done her math homework”},\\
R &:= \text{“Little Rabbit is happy”}.
\end{aligned}
$$

Express the following statements using P, Q, and R, along with logical connectives:

(1) If Little Bear is happy and has done her math homework, then Little Rabbit is happy.

(2) If Little Bear has done her math homework, then she is happy.

(3) Little Bear is happy only if she has done her math homework.

4. Does the following reasoning hold? Justify your answer.

— It is known that Little Bear is both smart and lazy, or Little Bear is not smart.

— It is also known that Little Bear is smart.

— Therefore, Little Bear is lazy.

5. Does the following reasoning hold? Justify your answer.

— It is known that at least one of the lion or the tiger is guilty.

— It is also known that either the lion is lying or the tiger is innocent.

— Therefore, the lion is either lying or guilty.

6. An explorer arrives at a cave with three closed doors, numbered 1, 2, and 3. Exactly one door hides treasure, while the other two conceal deadly traps.

— Door 1 states: “*The treasure is not here*”;

— Door 2 states: “*The treasure is not here*”;

— Door 3 states: “*The treasure is behind Door 2*”.

Only one of these statements is true. Which door should the explorer open to find the treasure?

7. The Kingdom of Truth sent an envoy to the capital of the Kingdom of Lies. Upon entering the border, the envoy encountered a fork with three paths: dirt, stone, and concrete. Each path had a signpost:

— The concrete path's sign: "*This path leads to the capital, and if the dirt path leads to the capital, then the stone path also does.*"

— The stone path's sign: "*Neither the concrete nor the dirt path leads to the capital.*"

— The dirt path's sign: "*The concrete path leads to the capital, but the stone path does not.*" All signposts lie. Which path should the envoy take?

8. Let a and b be real numbers. Prove that, if $a \neq -1$ and $b \neq -1$, then $ab + a + b \neq -1$.

9. Let a, b, and c be positive real numbers such that $abc > 1$ and

$$a + b + c < \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

Prove the following:

(1) None of a, b, or c equals 1.

(2) At least one of a, b, or c is greater than 1.

(3) At least one of a, b, or c is less than 1.

10. Let $a \neq 0$ and b be real numbers. For real numbers x and y, prove that if $x \neq y$, then $ax + b \neq ay + b$.

11. Let $n \geq 2$ be an integer. Prove that if n is composite, then there exists a prime number p dividing n such that $p \leq \sqrt{n}$.

12. Let n be an integer. Prove that either 4 divides n^2 or 4 divides $n^2 - 1$.

13. Let n be an integer. Prove that 12 divides $n^2(n^2 - 1)$.

14. Prove that any integer divisible by 4 can be written as the difference of two perfect squares.

15. Let x and y be non-zero integers. Prove that $x^2 - y^2 \neq 1$.

16. A plane has 300 seats and is fully booked. The first passenger ignores their assigned seat and chooses randomly. Subsequent passengers take their assigned seat if available; otherwise, they choose randomly. What is the probability that the last passenger sits in their assigned seat?

17. Little Bear, Little Goat, and Little Rabbit are all wearing hats. A parrot prepared four red feathers and four blue feathers to decorate their hats. The parrot selected two feathers for each hat-wearing animal to place on their hats. Each animal cannot see the feathers on their own hat but can see the feathers on the other animals' hats. Here is their conversation:

— Little Bear: *I don't know what color the feathers on my hat are, but I know the other animals also don't know what color the feathers on their hats are.*"

— Little Goat: *Haha, now even without looking at Little Bear's hat, I know what color the feathers on my hat are.*"

— Little Rabbit: *Now I know what color the feathers on my hat are.*"

— Little Bear: *Hmm, now I also know what color the feathers on my hat are.*"

Question: What color are the feathers on Little Goat's hat?

18. The Sphinx tells the truth on one fixed weekday and lies on the other six. Cleopatra visits The Sphinx for three consecutive days:

— Day 1: The Sphinx declared, "*I lie on Monday and Tuesday.*"

— Day 2: The Sphinx declared, "*Today is either Thursday, or Saturday, or Sunday.*"

— Day 3: The Sphinx declared, "*I lie on Wednesday and Friday.*"

On which day does the Sphinx tell the truth? On which days of the week did Cleopatra visit the Sphinx?

Chapter 2

Sets

This book presents the fundamentals of algebra and analysis based on set theory. Set theory, proposed initially by Cantor, is the cornerstone of modern mathematics. Before the emergence of set theory, the progress of mathematics was often built upon intuition and visual imagery. Moreover, due to the limitations of available tools, the scope of mathematical research was relatively narrow—for instance, analysis was often confined to studying functions with analytic expressions. Cantor's set theory introduced a new language and new tools to mathematics, greatly advancing the development of modern mathematics.

Naive set theory regards a set as a collection of distinct objects that are clearly defined. However, treating any such collection as a set without restriction leads to paradoxes. For example, Russell's paradox considers the collection of all sets that do not contain themselves. If this collection is treated as a set, one arrives at a contradiction regardless of whether the set contains itself or not. Determining which types of collections should be considered as sets is a subtle issue that lacks universal agreement. Axiomatic set theory treats set theory itself as a structure governed by a system of axioms. These axioms ensure the existence of constructions needed in mathematics while avoiding known paradoxes, thus providing a sound foundation for mathematics.

The goal of this chapter is to introduce some fundamental concepts of set theory and basic structures to prepare for the chapters that follow. Since our aim is not to give a systematic introduction to mathematical logic, we will explain ideas as much as possible using natural language rather than formal language.

2.1 Roster notation

2.1.1 Definition. In naive set theory, a set refers to a certain collection of *distinct* objects. The objects in a set are called *elements* of it. Two sets A and B are said to be *equal* if they have the same elements. We denote by $A = B$ the statement "*A and B are equal*".

If A is a set and a is an object, we denote by $a \in A$ the statement

"a is an element of A";

we denote by $a \notin A$ the statement

"a is not an element of A".

If a is an element of A, we also say that *a belongs to A*.

2.1.2 Notation. The *roster method* of representing a set refers to a notation where all elements of the set are explicitly enumerated in a single row within a pair of curly braces. For example the set consisting of the elements 1, 2 and 3 can be denoted as $\{1, 2, 3\}$. The represented set is independent of the enumeration order or element repetition. For example, one has

$$\{1, 2, 3\} = \{3, 2, 1\} = \{1, 1, 2, 3\}.$$

When representing a set using the roster method, an ellipsis $(\ldots)$ may be used to indicate elements with an obvious pattern. For example, the set of positive integers less than 100 can be written as:

$$\{1, 2, \ldots, 100\}.$$

Similarly, if n is a natural number the set of natural numbers not exceeding n can be written as

$$\{0, 1, \ldots, n\}.$$

When no ambiguity arises, an ellipsis may also be used to omit infinitely many elements with a clear pattern. For example, the set of all even integers can be expressed in roster notation as:

$$\{..., -6, -4, -2, 0, 2, 4, 6, ...\}.$$

When representing sets using the roster method, if the elements are indexed, an ellipsis can be used to summarise elements corresponding to indices with obvious patterns. For example, if n is a positive integer and for each $i \in \{1, \ldots, n\}$, a mathematical object x_i is given, then the set of these objects can be written as:

$$\{x_1, \ldots, x_n\}.$$

Similarly, if, for every natural number n, a mathematical object y_n, then the set of these objects can be written as:

$$\{y_0, y_1, y_2, \ldots\}.$$

More generally, if I is a set and if, for any $i \in I$, a mathematical object x_i is given, then the set of these objects can be written as

$$\{x_i \mid i \in I\},$$

where I is called the *index set* of this roster writing. For example,

$$\{2n \mid n \in \mathbb{Z}\}$$

denotes the set of all even numbers,

$$\{2n + 1 \mid n \in \mathbb{Z}\}$$

denotes the set of all odd numbers.

2.1.3 Definition. Let A and B be sets. We denote by $A \times B$ the following set of ordered pairs

$$\{(x, y) \mid x \in A,\ y \in B\},$$

and call it the *Cartesian product* of sets A and B.

More generally, if n is a positive integer and $A_1, \ldots, A_n$ be sets, we denote by

$$A_1 \times \cdots \times A_n$$

the set of all n-tuples $(x_1, \ldots, x_n)$, where $x_1 \in A_1, \ldots, x_n \in A_n$.

2.2 Subsets and powersets

2.2.1 Definition. Let A and B be sets. If any element of A is an element of B, we say that A is a *subset* of B. We denote by $A \subseteq B$ or by $B \supseteq A$ the statement

"*A is a subset of B*".

If A is a subset of B and A is *not* equal to B, we say that A is a *proper subset* of B, denoted by $A \subset B$ or by $B \supset A$.

If A is a subset (resp. proper subset) of B, we also say that A is *contained* (resp. *strictly contained*) in B, or that B *contains* (resp. *contains strictly*) A.

2.2.2 Remark. By definition, any set A is a subset of itself. Moreover, for any sets A and B, if $A \subseteq B$ and $B \subseteq A$, then the sets A and B have the same elements, namely $A = B$.

2.2.3 Definition. We denote by $\varnothing$ the set that does not contain any element, and we call it the *empty set*. By definition, the empty set is a subset of any set.

2.2.4 Proposition. *Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

Proof. Let x be an element of A. Since $A \subseteq B$, x is an element of B. Since $B \subseteq C$, x is an element of C. Therefore, one has $A \subseteq C$. □

2.2.5 Definition. Let X be a set. We denote by $\mathscr{P}(X)$ the set of all subsets of X, called the *power set* of X. Note that $\{\varnothing, X\} \subseteq \mathscr{P}(X)$.

2.2.6 Example. The only subset of the empty set is itself, namely $\mathscr{P}(\varnothing) = \{\varnothing\}$. Moreover, $\mathscr{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\}$.

2.3 Set-builder notation

2.3.1 Definition. Let A be a set. If for any $x \in A$, we fix a statement $P(x)$, then we say that $P(\cdot)$ is a *condition* on A. For example, "*x is an odd number*" is a condition on $\mathbb{Z}$. If $P(x)$ is true, then we say that x *satisfies* the condition $P(\cdot)$.

Let A be a set and $P(\cdot)$ be a condition on A. We denote by

$$\{x \in A \,|\, P(x)\}$$

the set of elements $x \in A$ such that $P(x)$ is true. This is a subset of A. Such representation of a new set by a condition on a given set is called *set-builder notation*.

2.3.2 Notation. Sometimes we combine the roster notation and the set-builder notation to describe a set. Let I be a set. For any $i \in I$, let x_i be a mathematical object. Let $P(\cdot)$ be a condition on I, and I_P be the set $\{i \in I \,|\, P(i)\}$. The we use the expression

$$\{x_i \,|\, i \in I,\ P(i)\}$$

to denote the set

$$\{x_i \,|\, i \in I_P\}.$$

2.3.3 Example. In the set-builder notation, it is important to fix an environmental set on which we consider a condition. If, for any set A, we consider a statement $P(A)$ (having a determined truth value), then the collection of all sets A such that $P(A)$ is true does not necessarily form a set.

For example, Russell's paradox considers the collection of all sets A such that $A \notin A$. If we consider this collection as a set, either it belongs to itself or not will lead to a contradiction.

Moreover, the collection of all sets should not be consider as a set. In fact, if the collection $\mathscr{S}$ of all sets is a set, then by the set-builder notation, the construction in Russell's paradox should also be a set, which would be expressed as

$$\{A \in \mathscr{S} \,|\, A \notin A\}.$$

This leads to a contradiction.

2.4 Set difference

2.4.1 Definition. Let A and B be sets. Then the set

$$\{x \in B \mid x \notin A\}$$

is called the *set difference* of B and A. In the case where A is a subset of B, we also call $B \setminus A$ as the *complement* of A in B.

2.4.2 Example. Let A be a set, $P(\cdot)$ be a condition on A. Then the following equality holds:

$$\{x \in A \mid \neg P(x)\} = A \setminus \{x \in A \mid P(x)\}. \tag{2.1}$$

2.4.3 Proposition. *Let A and B be sets. Then $B \setminus A = \varnothing$ if and only if $B \subseteq A$. In particular, in the case where A is a subset of B, the set $B \setminus A$ is empty if and only if $A = B$.*

Proof. Suppose that $B \setminus A = \varnothing$. For any element x of B, one has $x \in A$ since otherwise $x \in B \setminus A$, which leads to a contradiction. Hence $B \subseteq A$.

Suppose that $B \setminus A \neq \varnothing$. Let x be an arbitrary element of $B \setminus A$. It is hence an element of B that does not belong to A. Therefore B is not included in A.

Suppose that $A \subseteq B$. If $B \setminus A = \varnothing$, by the first conclusion we obtain $B \subseteq A$, hence we deduce $A = B$ by the condition $A \subseteq B$. Conversely, if $A = B$, by definition one has $B \setminus A = \varnothing$. □

2.4.4 Proposition. *Let A and B be sets. Then*

$$B \setminus (B \setminus A) = \{x \in B \mid x \in A\}.$$

When $A \subseteq B$, one has

$$B \setminus (B \setminus A) = A.$$

In particular, $B \setminus \varnothing = B$.

Proof. By definition,

$$B \setminus (B \setminus A) = \{x \in B \mid x \notin B \setminus A\}.$$

Let x be an element of $B \setminus (B \setminus A)$. By definition, $x \in B$ and $x \notin B \setminus A$. If x does not belong to A, then $x \in B \setminus A$, which leads to a contradiction. Hence we obtain $x \in A$.

Conversely, let x be a common element of A and B. By definition, $x \in B$, and x is not an element of $B \setminus A$ since $x \in A$. Hence x belongs to $B \setminus (B \setminus A)$.

In the case where $A \subseteq B$, by definition one has

$$\{x \in B \mid x \in A\} = A.$$

Hence

$$B \setminus (B \setminus A) = A.$$

Apply this equality to the case where $A = B$, by Proposition 2.4.3 we get

$$B \setminus \varnothing = B \setminus (B \setminus B) = B.$$

□

2.5 Quantifiers

2.5.1 Definition. Let A be a set and $P(\cdot)$ be a condition on A.

We use the expression

$$\forall\, x \in A,\ P(x)$$

to denote the statement

$$\{x \in A \,|\, P(x)\} = A.$$

We often read it as

"*for any $x \in A$, x satisfies the condition $P(\cdot)$*".

We use the expression

$$\exists\, x \in A,\ P(x)$$

to denote the statement

$$\{x \in A \,|\, P(x)\} \neq \varnothing.$$

We often read it as

"*there exists $x \in A$ which satisfies the condition $P(\cdot)$*".

2.5.2 Example. Let $P(\cdot)$ be any condition on the empty set $\varnothing$. Note that

$$\{x \in \varnothing \,|\, P(x)\}$$

is a subset of $\varnothing$. Hence it is equal to $\varnothing$. Therefore,

$$\forall\, x \in \varnothing,\ \mathbb{P}(x)$$

is true, and

$$\exists\, x \in \varnothing,\ \mathbb{P}(x)$$

is false.

2.5.3 Proposition. (1) *The statements*

$$\text{“}\exists\, x \in A,\ \neg P(x)\text{” } \textit{and} \text{ “}\forall\, x \in A,\ P(x)\text{”}$$

have opposite truth values.

(2) *The statements*

$$\text{“}\forall\, x \in A,\ \neg P(x)\text{” } \textit{and} \text{ “}\exists\, x \in A,\ P(x)\text{”}$$

have opposite truth values.

Proof. (1) Note that

$$\{x \in A \,|\, \neg P(x)\} = A \setminus \{x \in A \,|\, P(x)\}.$$

By Proposition 2.4.3, $\{x \in A \,|\, P(x)\} = A$ if and only if $\{x \in A \,|\, \neg P(x)\} = \varnothing$. Hence the statements

$$\text{“}\exists\, x \in A,\ \neg P(x)\text{” and “}\forall\, x \in A,\ P(x)\text{”}$$

have opposite truth values.

(2) We apply (1) to the condition $\neg P(\cdot)$ to obtain that

$$\text{“}\forall\, x \in A,\ \neg P(x)\text{” and “}\exists\, x \in A,\ \neg\neg P(x)\text{”}$$

have opposite truth values. Since, for any $x \in A$, $\neg\neg P(x)$ and $P(x)$ have the same truth value, we obtain that “$\exists\, x \in A$, $\neg\neg P(x)$” and “$\exists\, x \in A$, $P(x)$” have the same truth value. The statement is thus proved. □

2.6 Sufficient and necessary conditions

2.6.1 Definition. Let A be a set, $P(\cdot)$ and $Q(\cdot)$ be conditions on A. If

$$\{x \in A \,|\, P(x)\} \subseteq \{x \in A \,|\, Q(x)\},$$

we say that $P(\cdot)$ is a *sufficient condition* of $Q(\cdot)$, and $Q(\cdot)$ is a *necessary condition* of $P(\cdot)$. If

$$\{x \in A \,|\, P(x)\} = \{x \in A \,|\, Q(x)\},$$

we say that $P(\cdot)$ is a *necessary and sufficient condition* of $Q(\cdot)$, or that the conditions $P(\cdot)$ and $Q(\cdot)$ are *equivalent*.

2.6.2 Proposition. *Let A be a set, $P(\cdot)$ and $Q(\cdot)$ be conditions on A.*

(1) *$P(\cdot)$ is a sufficient condition of $Q(\cdot)$ if and only if*

$$\forall\, x \in A,\ P(x) \Rightarrow Q(x).$$

(2) *$P(\cdot)$ is a necessary condition of $Q(\cdot)$ if and only if*

$$\forall\, x \in A,\ Q(x) \Rightarrow P(x).$$

(3) *$P(\cdot)$ is a necessary and sufficient condition of $Q(\cdot)$ if and only if*

$$\forall\, x \in A,\ P(x) \Leftrightarrow Q(x).$$

Proof. (1) By 2.4.3, we obtain that $P(\cdot)$ is a sufficient condition of $Q(\cdot)$ if and only if

$$\{x \in A \,|\, P(x)\} \setminus \{x \in A \,|\, Q(x)\} = \varnothing.$$

Moreover,

$$\begin{aligned}\{x \in A \,|\, P(x)\} \setminus \{x \in A \,|\, Q(x)\} &= \{x \in A \,|\, P(x) \wedge (\neg Q(x))\} \\ = A \setminus \{x \in A \,|\, (\neg P(x)) \vee Q(x)\} &= A \setminus \{x \in A \,|\, P(x) \Rightarrow Q(x), \}\end{aligned}$$

where the second equality comes from (2.1) and Proposition 1.3.3. By Proposition 2.4.3 ,

$$\{x \in A \,|\, P(x)\} \setminus \{x \in A \,|\, Q(x)\} = \varnothing.$$

if and only if

$$\{x \in A \,|\, P(x) \Rightarrow Q(x)\} = A,$$

namely

$$\forall\, x \in A,\ P(x) \Rightarrow Q(x).$$

(2) follows from (1) by switching $P(\cdot)$ and $Q(\cdot)$.

(3) follows from (1), (2), and Proposition 1.5.2. □

2.7 Union

2.7.1 Definition. Let I be a set. For any $i \in I$, let A_i be a set. We say that $(A_i)_{i\in I}$ is a *family of sets* parametrised by I.

We denote by $\bigcup_{i\in I} A_i$ the set consisting of all elements of all A_i. It is called the *union* of the sets A_i, $i \in I$. By definition, a mathematical object x belongs to $\bigcup_{i\in I} A_i$ if and only if

$$\exists\, i \in I,\ x \in A_i.$$

In particular, $\bigcup_{i\in I} A_i$ is empty when $I = \varnothing$.

2.7.2 Notation. Let n be a positive integer, $A_1, \ldots, A_n$ be sets. We denote $\bigcup_{i\in\{1,\ldots,n\}} A_i$ as

$$A_1 \cup \ldots \cup A_n.$$

Note that it does not depend on the order of $A_1, \ldots, A_n$.

2.7.3 Proposition. *Let I be a set and $(A_i)_{i\in I}$ be a family of sets parametrised by I. Let B be a set. Then $\bigcup_{i\in I} A_i \subseteq B$ if and only if*

$$\forall\, i \in I,\ A_i \subseteq B.$$

Proof. Let $A := \bigcup_{i\in I} A_i$.

For any $i \in I$ one has $A_i \subseteq A$. If $A \subseteq B$, then by Proposition 2.2.4 we deduce that $A_i \subseteq B$.

Conversely, we suppose that, for any $i \in I$, one has $A_i \subseteq B$. Let x be an element of A. By definition, there exists $i \in I$ such that $x \in A_i$. Since $A_i \subseteq B$, one has $x \in B$. Therefore, $A \subseteq B$. □

2.7.4 Corollary. *Let B and I be sets. For any $i \in I$, let $P_i(\cdot)$ be a condition on B. Then*

$$\{x \in B \mid \exists\, i \in I,\ P_i(x)\} = \bigcup_{i\in I} \{x \in B \mid P_i(x)\}.$$

Proof. Let

$$A := \{x \in B \mid \exists\, i \in I,\ P_i(x)\}.$$

For any $i \in I$, let

$$A_i := \{x \in B \mid P_i(x)\}.$$

For any $x \in A$, there exists $i \in I$ such that $P_i(x)$ is true. Hence $x \in \bigcup_{i\in I} A_i$.

Conversely, by definition, for any $i \in I$, one has $A_i \subseteq A$. Hence, by Proposition 2.7.3, one has

$$\bigcup_{i\in I} A_i \subseteq A.$$

□

2.7.5 Proposition. *Let $(A_i)_{i\in I}$ be a family of sets, and B be a set. Then*

$$\left(\bigcup_{i\in I} A_i\right) \setminus B = \bigcup_{i\in I} (A_i \setminus B).$$

Proof. Let $A := \bigcup_{i\in I} A_i$.

For any $i \in I$, one has $A_i \subseteq A$. Hence $A_i \setminus B \subseteq A \setminus B$. By Proposition 2.7.3, we obtain $\bigcup_{i\in I} (A_i \setminus B) \subseteq A \setminus B$.

Conversely, if $x \in A \setminus B$, then $x \in A$ and $x \notin B$. By definition, there exists $i \in I$ such that $x \in A_i$, and hence $x \in A_i \setminus B$. This leads to $A \setminus B \subseteq \bigcup_{i\in I} (A_i \setminus B)$. □

2.8 Intersection

2.8.1 Definition. Let I be a non-empty set and $(A_i)_{i\in I}$ be a family of sets parametrised by I. We denote by $\bigcap_{i\in I} A_i$ the set of all common elements of A_i, $i \in I$. This set is called the *intersection* of A_i, $i \in I$. Note that, if i_0 is an arbitrary element of I, the set-builder notation ensures that

$$\{x \in A_{i_0} \,|\, \forall\, i \in I,\ x \in A_i\}$$

is a set. This set is the intersection of $(A_i)_{i\in I}$.

By definition, an mathematical object x belongs to $\bigcap_{i\in I} A_i$ if and only if

$$\forall\, i \in I,\ x \in A_i.$$

2.8.2 Notation. Let n be a positive integer, $A_1, \ldots, A_n$ be sets. We denote $\bigcap_{i\in\{1,\ldots,n\}} A_i$ as

$$A_1 \cap \ldots \cap A_n.$$

Note that it does not depend on the order of $A_1, \ldots, A_n$. In particular, if A and B are two sets, then

$$\{x \in B \,|\, x \in A\} = B \cap A.$$

Therefore, the first statement of Proposition 2.4.4 becomes

for any sets A and B, one has $B \setminus (B \setminus A) = B \cap A$.

2.8.3 Remark. In set theory, it does not make sense to consider the intersection of an empty family of sets. In fact, if such an intersection existed as a set, for any mathematical object x, since the statement

$$\forall\, i \in \varnothing,\ x \in A_i$$

is true (see Example 2.5.2), we would obtain that x belongs to $\bigcap_{i\in\varnothing} A_i$. By Russell's paradox, this is impossible.

2.8.4 Proposition. *Let I be a non-empty set and $(A_i)_{i\in I}$ be a set parametrised by I. Let B be a set. Then $B \subseteq \bigcap_{i\in I} A_i$ if and only if*

$$\forall\, i \in I,\ B \subseteq A_i.$$

Proof. Let $A = \bigcap_{i\in I} A_i$.

Suppose that $B \subseteq A$. For any $x \in B$, one has $x \in A$, and hence

$$\forall\, i \in I,\ x \in A_i.$$

Therefore, for any $i \in I$, B is contained in A_i.

Suppose that, for any $i \in I$, $B \subseteq A_i$. Then, for any $x \in B$ and any $i \in I$, one has $x \in A_i$. Hence, for any $x \in B$, one has $x \in A$. Therefore, $B \subseteq A$. □

2.8.5 Corollary. *Let B be a set, I be a non-empty set. For any $i \in I$, let $P_i(\cdot)$ be a condition on B. Then*

$$\{x \in B \,|\, \forall\, i \in I,\ P_i(x)\} = \bigcap_{i \in I} \{x \in B \,|\, P_i(x)\}.$$

Proof. Let

$$A := \{x \in B \,|\, \forall\, i \in I,\ P_i(x)\}.$$

For any $i \in I$, let

$$A_i := \{x \in B \,|\, P_i(x)\}.$$

For any $x \in A$ and any $i \in I$, $P_i(x)$ is true. Hence $A \subseteq A_i$. By Proposition 2.8.4, we obtain

$$A \subseteq \bigcap_{i \in I} A_i.$$

Conversely, if $x \in \bigcap_{i \in I} A_i$, then for any $i \in I$, one has $x \in A_i$. Hence $x \in B$, and for any $i \in I$, $P_i(x)$ is true. Thus $x \in A$. □

2.8.6 Proposition. *Let B be a set, $(A_i)_{i \in I}$ be a family of sets. The following equality holds*

$$\left(\bigcap_{i \in I} A_i\right) \setminus B = \bigcap_{i \in I} (A_i \setminus B).$$

Proof. Let $A := \bigcap_{i \in I} A_i$. For any $i \in I$, one has $A \subseteq A_i$. Hence

$$A \setminus B = \{x \in A \,|\, x \notin B\} \subseteq \{x \in A_i \,|\, x \notin B\}.$$

By Proposition 2.8.4, we get

$$A \setminus B \subseteq \bigcap_{i \in I} (A_i \setminus B).$$

Conversely, if $x \in \bigcap_{i \in I} (A_i \setminus B)$, then, for any $i \in I$, one has $x \in A_i \setminus B$, namely $x \in A_i$ and $x \notin B$. Thus $x \in \bigcap_{i \in I} A_i$ and $x \notin B$. Therefore $x \in A \setminus B$.

□

2.8.7 Proposition. *Let I be a set and $(A_i)_{i \in I}$ be a family of sets parametrised by I. For any set B, the following statements hold.*

(1) $B \cap (\bigcup_{i \in I} A_i) = \bigcup_{i \in I} (B \cap A_i)$.

(2) *If* $I \neq \varnothing$, $B \cup (\bigcap_{i \in I} A_i) = \bigcap_{i \in I} (B \cup A_i)$,

(3) *If* $I \neq \varnothing$, $B \setminus \bigcup_{i \in I} A_i = \bigcap_{i \in I} (B \setminus A_i)$,

(4) *If* $I \neq \varnothing$, $B \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I}(B \setminus A_i)$.

Proof. (1) By Corollary 2.7.4, we obtain

$$B \cap (\bigcup_{i \in I} A_i) = \{x \in B \mid \exists\, i \in I,\ x \in A_i\} = \bigcup_{i \in I}\{x \in B \mid x \in A_i\} = \bigcup_{i \in I}(B \cap A_i).$$

(2) Let $A := \bigcap_{i \in I} A_i$. By definition, for any $i \in I$, one has $A \subseteq A_i$ and hence $B \cup A \subseteq B \cup A_i$. Thus, by Proposition 2.8.4, we obtain

$$B \cup \left(\bigcap_{i \in I} A_i\right) \subseteq \bigcap_{i \in I}(B \cup A_i).$$

Conversely, let $x \in \bigcap_{i \in I}(B \cup A_i)$. For any $i \in I$, one has $x \in B \cup A_i$. If $x \in B$, then $x \in B \cup (\bigcap_{i \in I} A_i)$; otherwise one has

$$\forall\, i \in I,\ x \in A_i,$$

and we still get $x \in B \cup (\bigcap_{i \in I} A_i)$.

(3) By Proposition 2.5.3,

$$B \setminus \bigcup_{i \in I} A_i = \{x \in B \mid \neg(\exists\, i \in I,\ x \in A_i)\} = \{x \in B \mid \forall\, i \in I,\ x \notin A_i\}.$$

By Corollary 2.8.5, this is equal to

$$\bigcap_{i \in I}\{x \in B \mid x \notin A_i\} = \bigcap_{i \in I}(B \setminus A_i).$$

(4) By Proposition 2.5.3,

$$B \setminus \bigcap_{i \in I} A_i = \{x \in B \mid \neg(\forall\, i \in I,\ x \in A_i)\} = \{x \in B \mid \exists\, i \in I,\ x \notin A_i\}.$$

By Corollary 2.7.4, this is equal to

$$\bigcup_{i \in I}\{x \in B \mid x \notin A_i\} = \bigcup_{i \in I}(B \setminus A_i).$$

□

Exercices

1. Let $A = \{1, 2, 3, \text{Pikachu}\}$, $B = \{a, b, c, \text{Pikachu}\}$. Determine $A \cup B$ and $A \cap B$.

2. For any $k \in \mathbb{N}$, let
$$k\mathbb{N} = \{kn \mid n \in \mathbb{N}\}.$$

(1) Determine $2\mathbb{N} \cap 3\mathbb{N}$.

(2) Does the equality $2\mathbb{N} \cup 3\mathbb{N} = \mathbb{N}$ hold?

(3) Determine
$$\bigcap_{k \in \mathbb{N}} k\mathbb{N}.$$

3. Are the following sets equal:
$$A = \{(x, y) \in \mathbb{R}^2 \mid 4x - y = 1\}, \quad B = \{(t + 1, 4t + 3) \mid t \in \mathbb{R}\}?$$

Prove your conclusion.

4. Is the statement $0 \in \{\{0\}\}$ true?

5. Let A and B be two sets such that $A \subseteq B$. Prove that $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

6. Let A, B and C be sets. Prove the following equalities:
$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$$

7. Let I and J be non-empty sets, $(A_{i,j})_{(i,j) \in I \times J}$ be a family of sets. The goal of this exercise is to compare the following sets:
$$M = \bigcup_{i \in I} \bigcap_{j \in J} A_{i,j}, \quad N = \bigcap_{j \in J} \bigcup_{i \in I} A_{i,j}.$$

For any $i \in I$, let
$$M_i = \bigcap_{j \in J} A_{i,j}.$$

For any $j \in J$, let
$$N_j = \bigcup_{i \in I} A_{i,j}.$$

Thus,
$$M = \bigcup_{i \in I} M_i, \quad N = \bigcap_{j \in J} N_j.$$

(1) Let $(i, j) \in I \times J$. Prove that $M_i \subseteq A_{i,j} \subseteq N_j$.

(2) Prove that $M \subseteq N$.

(3) Consider the case $I = J = \mathbb{N}$. For any $(i, j) \in \mathbb{N}\times\mathbb{N}$, let $A_{i,j} = \{|i-j|\}$. Determine the sets M and N. Are they equal?

8. Let X and Y be sets.

(1) Prove that $\mathscr{P}(X \cap Y) = \mathscr{P}(X) \cap \mathscr{P}(Y)$.

(2) Prove that $\mathscr{P}(X) \cup \mathscr{P}(Y) \subseteq \mathscr{P}(X \cup Y)$.

(3) Construct two sets X and Y such that the equality

$$\mathscr{P}(X) \cup \mathscr{P}(Y) = \mathscr{P}(X \cup Y)$$

does not hold.

9. Let A and B be sets. Prove that

$$B \setminus A = (B \cup A) \setminus A = B \setminus (A \cap B).$$

10. Let A, B and C be sets. Prove that

$$C \setminus (B \setminus A) = (C \cap A) \cup (C \setminus B).$$

11. If X and Y are two sets, we denote by $X\Delta Y$ the symmetric difference

$$(X \setminus Y) \cup (Y \setminus X).$$

(1) Prove that, for any sets X and Y,

$$X\Delta Y = (X \cup Y) \setminus (X \cap Y).$$

(2) Prove that, if $X \supseteq Y$, then $X\Delta Y = X \setminus Y$.

12. Let A, B and C be sets.

(1) Compute $A\Delta A$ and $A\Delta\varnothing$.

(2) Prove that $(A\Delta B) \cap C = (A \cap C)\Delta(B \cap C)$.

13. For each of the following statements, determine a statement of the opposite truth value, where no quantifier is preceded by a negation symbol. Determine their truth values.

(1) $\exists\, x \in \mathbb{R},\, \forall\, y \in \mathbb{R},\, x + y \geq 0;$

(2) $\exists\, x \in \mathbb{R},\, \exists\, y \in \mathbb{R},\, x + y \geq 0;$

(3) $\forall\, x \in \mathbb{R},\, \forall\, y \in \mathbb{R},\, x + y \geq 0;$

(4) $\forall\, x \in \mathbb{R},\, \exists\, y \in \mathbb{R},\, x + y \geq 0;$

14. By using quantifiers, rewrite the following statements.

(1) Any natural number has a real square root.

(2) Any natural number is strictly smaller than a certain real number.

(3) There is a real number that is smaller or equal to any natural number.

15. Consider the condition on $\mathbb{R}$:

$$P(x) := \text{“}\forall\, y \in [0,1],\, x \geq y \Rightarrow x \geq 2y\text{”}, \quad x \in \mathbb{R}.$$

Determine all real numbers satisfying $P(\cdot)$.

16. For each of the following conditions on $\mathbb{R}$, describe the sets of $x \in \mathbb{R}$ that satisfy the condition.

(1) $((x > 0) \wedge (x < 1)) \vee (x = 0)$,

(2) $(x > 3) \wedge (x < 5) \wedge (x \neq 4)$,

(3) $((x \leq 0) \wedge (x > 1)) \vee (x = 4)$,

(4) $(x \geq 0) \Rightarrow (x \geq 2)$.

17. Consider the following statement:

$$\exists\, a \in \mathbb{R}_{<0},\ \forall\, x \in \mathbb{R}_{\geqslant 1},\ x > a \Rightarrow \Big((x^2 > \tfrac{a^2}{4}) \vee (x \leq 0)\Big).$$

Determine a statement of the opposite truth value, where no quantifier is preceded by a negation symbol. Is this statement true?

18. Recall that a prime number is by definition a natural number that is ≥ 2, the only divisors of which are 1 and itself. Let $\mathbb{P}$ be the set of all prime numbers. Express the following statements as formulas by using quantifiers.

(1) If a prime number divides the product of two integers, then it divides at least one between them.

(2) Any integer that is greater or equal to 2 is divisible by a prime number.

(3) The natural number 2 is the only prime number that is even.

(4) Any primer number that is ≥ 5 is congruent to 1 or -1 modulo 6.

(5) Any even number that is > 2 can be written as the sum of two prime numbers.

(6) Any prime number that is congruent to 1 modulo 4 can be written as the sum of two squares.

(7) For any natural number n such that $n \geqslant 2$, there is always a primer number that lies between n and $2n$.

(8) There exist infinitely many prime numbers.

(9) Let p be an integer such that $p \geq 2$. Then p is a prime number if and only if p divides $(p-1)! + 1$.

19. Let x, y, x', and y' be mathematical objects. Prove that

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$$

if and only if $x = x'$ and $y = y'$.

Chapter 3

Correspondences

3.1 Correspondence and its inverse

3.1.1 Definition. We call a *correspondence* any triplet of the form

$$f = \big(\mathscr{D}_f, \mathscr{A}_f, \Gamma_f\big),$$

where $\mathscr{D}_f$, $\mathscr{A}_f$ are two sets, called respectively the *departure set* and the *arrival set* of f, and Γ_f is a subset of $\mathscr{D}_f \times \mathscr{A}_f$, called the *graph* of f.

If X and Y are two sets and f is a correspondence of the form $\big(X, Y, \Gamma_f\big)$, we say that f is a correspondence *from* X *to* Y.

3.1.2 Definition. Let f be a correspondence. We denote by f^{-1} the correspondence defined as follows:

$$\mathscr{D}_{f^{-1}} := \mathscr{A}_f, \quad \mathscr{A}_{f^{-1}} := \mathscr{D}_f,$$
$$\Gamma_{f^{-1}} := \{(y, x) \in \mathscr{A}_f \times \mathscr{D}_f \,|\, (x, y) \in \Gamma_f\}.$$

The correspondence f^{-1} is called the *inverse correspondence* of f. Clearly one has

$$(f^{-1})^{-1} = f, \tag{3.1}$$

namely f is the inverse correspondence of f^{-1}.

3.1.3 Example. Let X be a set. Denote by Δ_X the following subset of $X \times X$:

$$\{(x, x) \,|\, x \in X\},$$

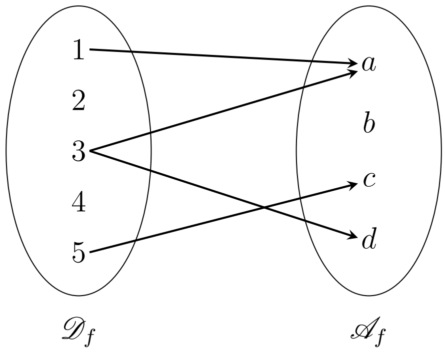
called the *diagonal subset* of $X \times X$. The correspondence (X, X, Δ_X) is called the *identity correspondence* of X, denoted as Id_X. By definition, one has $\mathrm{Id}_X^{-1} = \mathrm{Id}_X$.

3.1.4 Example. Let X and Y be two sets. There is a correspondence from X to Y whose graph is the empty set. This correspondence is called the *empty correspondence* from X to Y. Note that the inverse correspondence of the empty correspondence from X to Y is the empty correspondence from Y to X.

3.2 Illustration of a correspondence

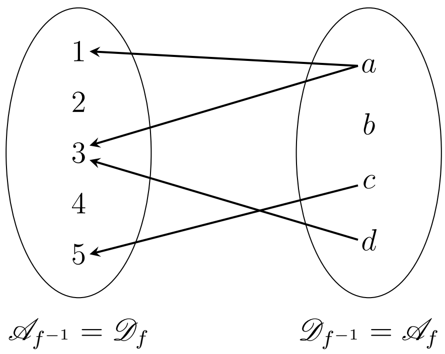
3.2.1 Remark. A correspondence can be viewed as a oriented graph. The elements of $\mathscr{D}_f$ and $\mathscr{A}_f$ are illustrated by two groups of vertices. For any ordered pair in the graph Γ_f, we link the corresponding vertices by an arrow. In the following figure, we illustrate a correspondence from $\{1, 2, 3, 4, 5\}$ to $\{a, b, c, d\}$.

Figure 3.1: Visualization of a correspondance f



The inverse correspondence f^{-1} can be visualised by inverting the direction of the arrows in the above figure.

Figure 3.2: Visualization of the inverse correspondance f^{-1}



3.2.2 Remark. We can also represent a correspondence f by a table, whose rows are labelled by elements of $\mathscr{D}_f$ and whose columns are labelled by elements of $\mathscr{A}_f$. For each pair in Γ_f we mark the cell of corresponding coordinates by a ✓. For example, the correspondence described by Figure 3.3 can be represented by the following table.

Table 3.1: Table representation of the correspondence f

	a	b	c	d
1	✓			
2				
3	✓			✓
4				
5			✓	

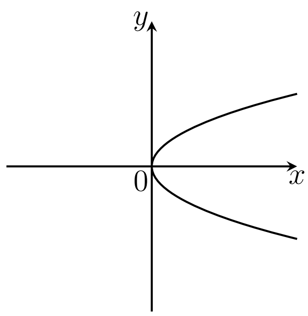
Its inverse correspondence can be represented by the transposed table.

Table 3.2: Table representation of the inverse correspondence f^{-1}

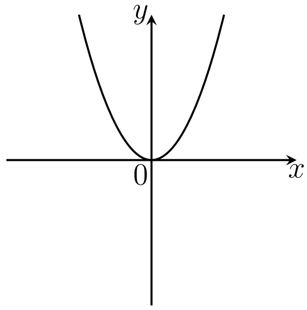
	1	2	3	4	5
a	✓		✓		
b					
c					✓
d			✓		

3.2.3 Remark. A correspondence from $\mathbb{R}$ to $\mathbb{R}$ can be illustrated by its graph in the coordinate plane. Consider for example such a correspondence f, with

$$\Gamma_f := \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x = y^2\}.$$



The inverse correspondence is illustrated by a parabola that opens upward.



3.3 Image and preimage

3.3.1 Definition. Let X and Y be sets, and f be a correspondence from X to Y. If (x, y) is an element of Γ_f, we say that x is a *preimage* of y under f, and y is an *image* of x under f.

If A is a set, we denote by $f(A)$ the set

$$\{y \in \mathscr{A}_f \mid \exists\, x \in A \text{ such that } (x, y) \in \Gamma_f\},$$

called the *image* of A by the correspondence f.

If B is a set, the set $f^{-1}(B)$ is called the *preimage* of B by the correspondence f. Note that it is by definition the image of B by the inverse correspondence f^{-1}.

3.3.2 Definition. Let f be a correspondence. The set $f(\mathscr{D}_f)$ is called the *range* of f, denoted as $\mathrm{Im}(f)$. The set $f^{-1}(\mathscr{A}_f)$ is called the *domaine of definition* of f, denoted as $\mathrm{Dom}(f)$. Note that the domain of definition of a correspondence f is the projection of the graph Γ_f to the departure set $\mathscr{D}_f$, and the range of a correspondence f is the projection of the graph Γ_f to the arrival set $\mathscr{A}_f$.

For any sets A and B,

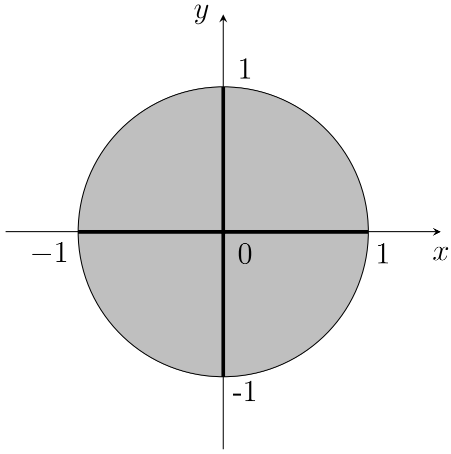
$$f(A) \subseteq \mathrm{Im}(f), \quad f^{-1}(B) \subseteq \mathrm{Dom}(f).$$

Moreover,

$$\mathrm{Dom}(f) = \mathrm{Im}(f^{-1}), \quad \mathrm{Im}(f) = \mathrm{Dom}(f^{-1}).$$

3.3.3 Example. The following figure illustrates a correspondence from $\mathbb{R}$ to $\mathbb{R}$, the image of which is

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$



The domain of definition and the range of this correspondence are both $[-1, 1]$.

3.3.4 Proposition. *Let f be a correspondence.*

(1) *If A and A' are two sets such that $A' \subseteq A$, then one has $f(A') \subseteq f(A)$.*

(2) *If B and B' are two sets such that $B' \subseteq B$, then one has $f^{-1}(B') \subseteq f^{-1}(B)$.*

Proof. It suffices to prove the first statement. Let y be an element of $f(A')$. By definition there exists $x \in A'$ such that $(x, y) \in \Gamma_f$. Since $A' \subseteq A$, one has $x \in A$ and hence $y \in f(A)$. $\square$

3.3.5 Proposition. *Let f be a correspondence. The following equalities hold:*

$$\mathrm{Im}(f) = f(\mathrm{Dom}(f)), \quad \mathrm{Dom}(f) = f^{-1}(\mathrm{Im}(f)).$$

Proof. Since $\mathrm{Dom}(f) \subseteq \mathscr{D}_f$, by Proposition 3.3.4, one has

$$f(\mathrm{Dom}(f)) \subseteq f(\mathscr{D}_f) = \mathrm{Im}(f).$$

Let y be an element of $\mathrm{Im}(f)$. There exists $x \in \mathscr{D}_f$ such that $(x, y) \in \Gamma_f$. By definition one has $x \in \mathrm{Dom}(f)$ and hence $y \in f(\mathrm{Dom}(f))$. Therefore the equality $\mathrm{Im}(f) = f(\mathrm{Dom}(f))$ is true. Applying this equality to f^{-1}, we obtain the second equality. $\square$

3.3.6 Proposition. *Let f be a correspondence.*

(1) *Let A be a set and y be an mathematical object. Then y belongs to $f(A)$ if and only if $A \cap f^{-1}(\{y\}) \neq \varnothing$.*

(2) *Let B be a set and x be a mathematical object. Then x belongs to $f^{-1}(B)$ if and only if $B \cap f(\{x\}) \neq \varnothing$.*

Proof. (1) By definition, $y \in f(A)$ if and only if there exists $x \in A$ such that $(x, y) \in \Gamma_f$, or equivalently $x \in A \cap f^{-1}(\{y\})$

Applying (1) to f^{-1}, we obtain (2). □

3.3.7 Proposition. *Let f be a correspondence.*

(1) *Let I be a set and $(A_i)_{i\in I}$ be a family of sets parametrised by I. Then the equality*

$$f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$$

holds. Moreover, if I is not empty, then

$$f\left(\bigcap_{i\in I} A_i\right) \subseteq \bigcap_{i\in I} f(A_i)$$

(2) *Let I be a set and $(B_i)_{i\in I}$ be a family of sets parametrised by I. Then the equality*

$$f^{-1}\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^{-1}(B_i)$$

holds. Moreover, if I is not empty, then

$$f^{-1}\left(\bigcap_{i\in I} B_i\right) \subseteq \bigcap_{i\in I} f^{-1}(B_i).$$

Proof. (1) By Propositions 3.3.6 and 2.8.7, we obtain

$$\begin{aligned}
f\left(\bigcup_{i\in I} A_i\right) &= \left\{ y \in Y \,\middle|\, \left(\bigcup_{i\in I} A_i\right) \cap f^{-1}(y) \neq \varnothing \right\} \\
&= \left\{ y \in Y \,\middle|\, \bigcup_{i\in I} (A_i \cap f^{-1}(\{y\})) \neq \varnothing \right\} \\
&= \left\{ y \in Y \,\middle|\, \exists\, i \in I,\ A_i \cap f^{-1}(\{y\}) \neq \varnothing \right\} = \bigcup_{i\in I} f(A_i).
\end{aligned}$$

Let $A = \bigcap_{i\in I} A_i$. For any $i \in I$, one has $A \subseteq A_i$ and hence, by Proposition 3.3.4, one has $f(A) \subseteq f(A_i)$. By Proposition 2.8.4, we get

$$f(A) \subseteq \bigcap_{i\in I} f(A_i).$$

Applying (1) to f^{-1}, we obtain (2). □

3.4 Composition

3.4.1 Definition. Let f and g be correspondences. We define the *composite* of g and f as the correspondence $g \circ f$ from $\mathscr{D}_f$ to $\mathscr{A}_g$ whose graph $\Gamma_{g\circ f}$ is composed of the elements (x,z) of $\mathscr{D}_f \times \mathscr{A}_g$ such that there exists some object y satisfying $(x,y) \in \Gamma_f$ and $(y,z) \in \Gamma_g$ (note that the object y should be an element of $\mathscr{A}_f \cap \mathscr{D}_g$). In other words,

$$\Gamma_{g\circ f} = \{(x,z) \in \mathscr{D}_f \times \mathscr{A}_g \,|\, \exists\, y \in \mathscr{A}_f \cap \mathscr{D}_g,\ (x,y) \in \Gamma_f \text{ and } (y,z) \in \Gamma_g\}.$$

3.4.2 Proposition. *Let f and g be correspondences. The following equality holds:*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{3.2}$$

Proof. Let $x \in \mathscr{A}_{f^{-1}} = \mathscr{D}_f$, $z \in \mathscr{D}_{g^{-1}} = \mathscr{A}_g$. Then $(z,x) \in \Gamma_{(g\circ f)^{-1}}$ if and only if $(x,z) \in \Gamma_{g\circ f}$, namely there exists y such that $(x,y) \in \Gamma_f$ and $(y,z) \in \Gamma_g$. This is also equivalent to the existence of y such that $(z,y) \in \Gamma_{g^{-1}}$ and $(y,x) \in \Gamma_{f^{-1}}$, namely $(z,x) \in \Gamma_{f^{-1}\circ g^{-1}}$. □

3.4.3 Proposition. *Let f, g and h be correspondences. The following equality holds:*

$$h \circ (g \circ f) = (h \circ g) \circ f. \tag{3.3}$$

Proof. It suffices to verify that $h \circ (g \circ f)$ and $(h \circ g) \circ f$ have the same graph. By definition, an element $(x,w) \in \mathscr{D}_f \times \mathscr{A}_h$ belongs to the graph of $h \circ (g \circ f)$ if and only if there exists z such that $(x,z) \in \Gamma_{g\circ f}$ and $(z,w) \in \Gamma_h$, which is equivalent to the existence of y and z such that $(x,y) \in \Gamma_f$, $(y,z) \in \Gamma_g$ and $(z,w) \in \Gamma_h$. A similar argument shows that this condition is also equivalent to $(x,w) \in \Gamma_{(h\circ g)\circ f}$. Hence the equality (3.3) holds. □

3.4.4 Proposition. *Let X and Y be sets, f be a correspondence from X to Y. Then the following equalities hold:*

$$f \circ \mathrm{Id}_X = f = \mathrm{Id}_Y \circ f.$$

Proof. Let $(x,y) \in X \times Y$. By definition, (x,y) belongs to the graph of $\mathrm{Id}_X \circ f$ if and only if there exists x' such that $(x,x') \in \Delta_X$ and $(x',y) \in \Gamma_f$, that is, $x = x'$ and $(x,y) \in \Gamma_f$. Therefore $f \circ \mathrm{Id}_X = f$. Applying this equality to f^{-1}, we obtain

$$f^{-1} \circ \mathrm{Id}_Y = f^{-1}.$$

Taking the inverse correspondences, by (3.1) and (3.2) we deduce

$$\mathrm{Id}_Y \circ f = f.$$

□

3.4.5 Proposition. *Let f and g be correspondences.*

(1) *For any set A, one has*

$$(g \circ f)(A) = g(f(A)).$$

In particular,

$$\mathrm{Im}(g \circ f) = g(\mathrm{Im}(f)) \subseteq \mathrm{Im}(g).$$

If in addition $\mathrm{Dom}(g) \subseteq \mathrm{Im}(f)$*, then the equality* $\mathrm{Im}(g \circ f) = \mathrm{Im}(g)$ *holds.*

(2) *For any set B, one has*

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)).$$

In particular,

$$\mathrm{Dom}(g \circ f) = f^{-1}(\mathrm{Dom}(g)) \subseteq \mathrm{Dom}(f).$$

If in addition $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$*, then the equality* $\mathrm{Dom}(g \circ f) = \mathrm{Dom}(f)$ *holds.*

Proof. (1) By definition

$$\begin{aligned}(g \circ f)(A) &= \{z \in \mathscr{A}_g \mid \exists\, x \in A,\ (x, z) \in \Gamma_{g \circ f}\}\\ &= \{z \in \mathscr{A}_g \mid \exists\, x \in A,\ \exists\, y \in \mathscr{A}_f,\ (x, y) \in \Gamma_f,\ (y, z) \in \Gamma_g\}\\ &= \{z \in \mathscr{A}_g \mid \exists\, y \in f(A),\ (y, z) \in \Gamma_g\} = g(f(A)).\end{aligned}$$

Applying this equality to the case where $A = \mathscr{D}_f$, we obtain

$$\mathrm{Im}(g \circ f) = (g \circ f)(\mathscr{D}_f) = g(f(\mathscr{D}_f)) = g(\mathrm{Im}(f)) \subseteq \mathrm{Im}(g).$$

In the case where $\mathrm{Dom}(g) \subseteq \mathrm{Im}(f)$, by Propositions 3.3.5 and 3.3.4 we obtain

$$\mathrm{Im}(g) = g(\mathrm{Dom}(g)) \subseteq g(\mathrm{Im}(f)) = \mathrm{Im}(g \circ f).$$

Hence the equality $\mathrm{Im}(g \circ f) = \mathrm{Im}(g)$ holds.

Applying (1) to g^{-1} and f^{-1}, by (3.2) we obtain (2). □

3.5 Surjectivity

3.5.1 Definition. Let f be a correspondence. If $\mathscr{A}_f = \mathrm{Im}(f)$, we say that f is *surjective*. If f^{-1} is surjective, or equivalently $\mathrm{Dom}(f) = \mathscr{D}_f$, we say that f is a *multivalued mapping*.

3.5.2 Proposition. *Let f be a correspondence.*

(1) *Assume that f is surjective. Then, for any subset B of $\mathscr{A}_f$, one has $B \subseteq f(f^{-1}(B))$.*

(2) *Assume that f is a multivalued mapping. Then, for any subset A of $\mathscr{D}_f$, one has $A \subseteq f^{-1}(f(A))$.*

Proof. (1) Let y be an element of B. Since f is surjective, there exists $x \in \mathscr{D}_f$ such that $(x, y) \in \Gamma_f$. Therefore, $x \in f^{-1}(B)$ and hence $y \in f(f^{-1}(B))$.

Applying (1) to f^{-1}, we obtain (2). □

3.5.3 Proposition. *Let f and g be correspondences.*

(1) *If $g \circ f$ is surjective, so is g.*

(2) *If $g \circ f$ is a multivalued mapping, so is f.*

Proof. (1) By (1) of Proposition 3.4.5, one has

$$\mathrm{Im}(g \circ f) \subseteq \mathrm{Im}(g) \subseteq \mathscr{A}_g = \mathscr{A}_{g \circ f}. \tag{3.4}$$

If $g \circ f$ is surjective, namely $\mathrm{Im}(g \circ f) = \mathscr{A}_{g \circ f}$, then we deduce from (3.4) that $\mathrm{Im}(g) = \mathscr{A}_g$, namely g is surjective.

Applying (1) to g^{-1} and f^{-1}, we obtain (2). □

3.5.4 Proposition. *Let f et g be correspondences.*

(1) *If g is surjective and $\mathrm{Dom}(g) \subseteq \mathrm{Im}(f)$, then $g \circ f$ is also surjective.*

(2) *If f is a multivalued mapping and $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$, then $g \circ f$ is a multivalued mapping.*

Proof. (1) Since $\mathrm{Dom}(g) \subseteq \mathrm{Im}(f)$, by (1) of Proposition 3.4.5 we obtain

$$\mathrm{Im}(g \circ f) = \mathrm{Im}(g).$$

Since g is surjective,

$$\mathrm{Im}(g) = \mathscr{A}_g = \mathscr{A}_{g \circ f}.$$

Hence $g \circ f$ is also surjective.

Applying (1) to g^{-1} and f^{-1}, we obtain (2). □

3.6 Injectivity

3.6.1 Definition. Let f be a correspondence. If each element of $\mathscr{D}_f$ has at most one image under f, we say that f is a *function*. If f^{-1} is a function, or equivalently, each element of $\mathscr{A}_f$ has at most one preimage under f, we say that f is *injective*.

3.6.2 Notation. Functions form a special case of correspondences. The defining feature of functions is that corresponding to each element in the domain of definition, is a unique element in the arrival set of the function.

Let f be a function, and let $x \in \mathrm{Dom}(f)$. We denote the unique image of x under f as $f(x)$, and we say that f *sends* $x \in \mathrm{Dom}(f)$ to $f(x)$ or $f(x)$ is the *value* of f at x. We can also use the notation

$$x \longmapsto f(x)$$

to indicate the correspondence of x to its image under f.

3.6.3 Proposition. *Let f be a correspondence.*

(1) *Assume that f is injective. For any set A one has $f^{-1}(f(A)) \subseteq A$.*

(2) *Assume that f is a function. For any set B one has $f(f^{-1}(B)) \subseteq B$.*

Proof. (1) Let x be an element of $f^{-1}(f(A))$. By definition, there exists $y \in f(A)$ such that $(x, y) \in \Gamma_f$. Since $y \in f(A)$ there exist $x' \in A$ such that $(x', y) \in \Gamma_f$. Since y admits at most one preimage, we obtain $x = x'$. Hence $x \in A$.

Applying (1) to f^{-1}, we obtain (2). □

3.6.4 Proposition. *Let f and g be correspondences.*

(1) *If f and g are functions, so is $g \circ f$. Moreover, for any $x \in \mathrm{Dom}(g \circ f)$, one has $(g \circ f)(x) = g(f(x))$.*

(2) *If f and g are injective, so is $g \circ f$.*

Proof. Let x be an element of $\mathrm{Dom}(g \circ f)$. Assume that z and z' are images of x under $g \circ f$. Let y and y' be such that

$$(x, y) \in \Gamma_f, \quad (y, z) \in \Gamma_g, \quad (x, y') \in \Gamma_f, \quad (y', z') \in \Gamma_g.$$

Since f is a function, one has $y = y' = f(x)$. Since g is a function, we deduce that $z = z' = g(f(x))$. Therefore $g \circ f$ is a function, and the equality $(g \circ f)(x) = g(f(x))$ holds for any $x \in \mathrm{Dom}(g \circ f)$.

Applying (1) to g^{-1} and f^{-1}, we obtain (2). □

3.6.5 Proposition. *Let f and g be correspondences.*

(1) *If $g \circ f$ is injective and $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$, then f is also injective.*

(2) *If $g \circ f$ is a function and $\mathrm{Dom}(g) \subseteq \mathrm{Im}(f)$, then g is also a function.*

Proof. (1) Let y be an element of the image of f. Let x and x' be preimages of y under f. Since $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$, one has $y \in \mathrm{Dom}(g)$. Hence there exists $z \in \mathscr{A}_g$ such that $(y, z) \in \Gamma_g$. We then deduce that (x, z) and (x', z) are elements of $\Gamma_{g\circ f}$. Since $g \circ f$ is injective, we obtain $x = x'$. Therefore, f is injective.

Applying (1) to g^{-1} and f^{-1}, we obtain (2). □

3.6.6 Proposition. *Let f be a correspondence, and I be a non-empty set.*

(1) *Suppose that f is a function. For any family $(B_i)_{i\in I}$ of sets parametrised by I, one has*
$$f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i).$$

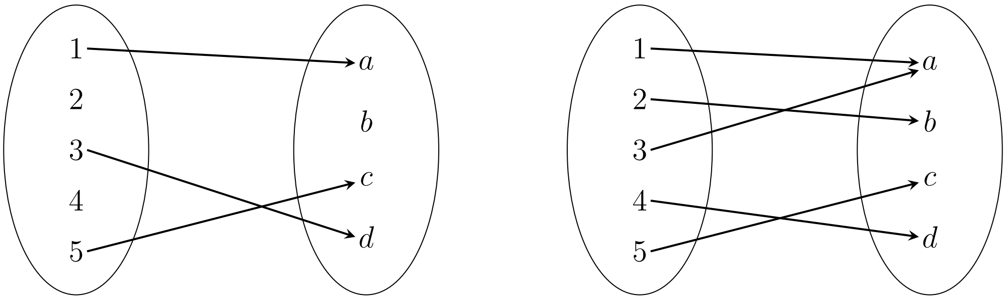
(2) *Suppose that f is injective. For any family $(A_i)_{i\in I}$ of sets parametrised by I, one has*
$$f\Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I} f(A_i).$$

Proof. (1) Let x be an element of $\bigcap_{i\in I} f^{-1}(B_i)$. For any $i \in I$, one has $f(x) \in B_i$. Hence $x \in f^{-1}\big(\bigcap_{i\in I} B_i\big)$. Therefore we obtain
$$f^{-1}\Big(\bigcap_{i\in I} B_i\Big) \supseteq \bigcap_{i\in I} f^{-1}(B_i).$$
Combing with (2) of Proposition 3.3.7, we obtain the equality
$$f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i).$$

Applying (1) to f^{-1}, we obtain (2). □

3.6.7 Example. In the following we illustrate two functions.

Figure 3.3: Visualization of two functions



The function illustrated on the left hand side of the figure is injective and not surjective, that illustrated on the right hand side is surjective and not injective.

3.7 Mappings

3.7.1 Definition. A correspondence f is said to be a *mapping* if any element of $\mathscr{D}_f$ has a unique image, or equivalently, f is a function and $\mathscr{D}_f = \mathrm{Dom}(f)$. Note that f is a mapping if and only if f^{-1} is both injective and surjective.

3.7.2 Notation. Let X and Y be sets. We denote by Y^X the set of all mappings from X to Y. An element $u \in Y^X$ is often written in the form of a family of elements of Y parametrised by X as follows

$$(u(x))_{x\in X}.$$

In the case where $X = \{1, \ldots, n\}$, where n is a positive integer, the set $Y^{\{1,\ldots,n\}}$ is also denoted as Y^n. An element u of Y^n is often written as an n-tuple

$$(u(1), \ldots, u(n)).$$

3.7.3 Example. (1) Let X be a set. The identity correspondence Id_X is a mapping. It is also called the *identity mapping* of X.

(2) Let X and Y be sets and y be an element of Y. The mapping from X to Y sending any $x \in X$ to y is called the *constant mapping with value y*.

3.7.4 Remark. Let $f : X \to Y$ be a mapping, I be a set.

(1) By (1) of Proposition 3.3.7, for any family of sets $(A_i)_{i\in I}$, one has

$$f\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} f(A_i).$$

By (2) of Proposition 3.3.7, for any family of sets $(B_i)_{i\in I}$, one has

$$f^{-1}\Big(\bigcup_{i\in I} B_i\Big) = \bigcup_{i\in I} f^{-1}(B_i).$$

(2) Assume that I is not empty. By (1) of Proposition 3.3.7, for any family of sets $(A_i)_{i\in I}$, one has

$$f\Big(\bigcap_{i\in I} A_i\Big) \subseteq \bigcap_{i\in I} f(A_i).$$

By (1) of Proposition 3.6.6, for any family of sets $(B_i)_{i\in I}$, one has

$$f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i).$$

(3) By (2) of Proposition 3.6.3, for any set B, one has $f(f^{-1}(B)) \subseteq B$. Since f is a function and f^{-1} is injective, by (1) of Proposition 3.6.3 and (2) of Proposition 3.5.2, for any subset A of X one has $f^{-1}(f(A)) = A$.

3.7.5 Proposition. *Let f and g be mappings. Suppose that $\mathrm{Im}(f) \subseteq \mathscr{D}_g$. Then $g\circ f$ is also a mapping. Moreover, for any $x \in \mathscr{D}_f = \mathscr{D}_{g\circ f}$ one has*

$$(g\circ f)(x) = g(f(x)).$$

Proof. Note that $\mathscr{D}_g = \mathrm{Dom}(g)$ since g is a mapping. Hence the statement is a direct consequence of Propositions 3.6.4 and 3.5.4. □

3.7.6 Remark. Let $f : X \to Y$ and $g : Y \to Z$ be mappings.

(1) By Proposition 3.5.4, if f and g are both surjective, so is $g\circ f$. By Proposition 3.5.3, if $g\circ f$ is surjective, so is g.

(2) By Proposition 3.6.4, if f and g are both injective, so is $g\circ f$. By Proposition 3.6.5, if $g \circ f$ is injective, so is f.

3.8 Bijection

3.8.1 Definition. Let f be a mapping, that is, a correspondence such that f^{-1} is injective and surjective. If f is injective and surjective, we say that f is a *bijection*, or a *one-to-one correspondence*. Note that a correspondence is a bijection if and only if its inverse is a bijection.

3.8.2 Proposition. *Let X and Y be sets, f be a correspondence from X to Y. If f is a bijection, then $f^{-1}\circ f=\mathrm{Id}_X$ and $f\circ f^{-1}=\mathrm{Id}_Y$. Conversely, if there exists a correspondence g such that $g\circ f=\mathrm{Id}_X$ and $f\circ g=\mathrm{Id}_Y$, then f is a bijection and $g=f^{-1}$.*

Proof. If f is a bijection, then f and f^{-1} are both mappings. By Proposition 3.7.5, one has

$$\begin{aligned}\forall\, x\in X,\quad (f^{-1}\circ f)(x)=f^{-1}(f(x))=x,\\ \forall\, y\in Y,\quad (f\circ f^{-1})(y)=f(f^{-1}(y))=y.\end{aligned}$$

Hence $f^{-1}\circ f=\mathrm{Id}_X$ and $f\circ f^{-1}=\mathrm{Id}_Y$.

Assume that g is a correspondence such that $g\circ f=\mathrm{Id}_X$ and $f\circ g=\mathrm{Id}_Y$. Since identity correspondences are surjective mappings, by Proposition 3.5.3, we deduce from the equality $g\circ f=\mathrm{Id}_X$ (which can also be written as $f^{-1}\circ g^{-1}=\mathrm{Id}_X$) that g and f^{-1} are surjective. In particular, $\mathrm{Dom}(f)=X=\mathrm{Im}(g)$.

Similarly, we deduce from the equality $f\circ g=\mathrm{Id}_Y$ (which can also be written as $g^{-1}\circ f^{-1}=\mathrm{Id}_Y$) that f and g^{-1} are surjective. In particular, $\mathrm{Dom}(g)=Y=\mathrm{Im}(f)$.

Since identity correspondences are injective, by Proposition 3.6.5, we deduce from $g\circ f=\mathrm{Id}_X$ that f is injective. Similarly, we deduce from $f\circ g=\mathrm{Id}_Y$ that f is a function. Therefore, f is a mapping which is injective and surjective, namely a bijection.

Finally, by Propositions 3.4.4 and 3.4.3, we obtain

$$g=g\circ\mathrm{Id}_Y=g\circ(f\circ f^{-1})=(g\circ f)\circ f^{-1}=\mathrm{Id}_X\circ f^{-1}=f^{-1}.$$

□

3.8.3 Proposition. *Let $f:X\to Y$ and $g:Y\to Z$ be bijections. Then the composite correspondence $g\circ f$ is also a bijection.*

Proof. This is a direct consequence of Propositions 3.7.5, 3.6.4 and 3.5.4. □

3.8.4 Proposition. *Let X and Y be sets, f be a correspondence from X to Y, and g be a correspondence from Y to X. If $f\circ g$ and $g\circ f$ are bijections, then f and g are both bijections.*

Proof. By Proposition 3.5.3, f and g are surjective and are multivalued mappings. In particular,

$$\mathrm{Dom}(f) = X, \quad \mathrm{Im}(f) = Y, \quad \mathrm{Dom}(g) = Y, \quad \mathrm{Im}(g) = X.$$

Therefore, by Proposition 3.6.5, we deduce that f and g are injective and are functions. Hence f and g are both bijections. □

3.9 Direct product

3.9.1 Definition. Let I be a set and $(A_i)_{i\in I}$ be a family of sets parametrised by I. We denote by

$$\prod_{i\in I} A_i$$

the set of all mappings from I to $\bigcup_{i\in I} A_i$ which send any $i \in I$ to an element of A_i. This set is called the *direct product* of $(A_i)_{i\in I}$. Using Notation 3.7.2, we often write an element of the direct product in the form of a family $x := (x_i)_{i\in I}$ parametrised by I, where each x_i is an element of A_i, called the i-th *coordinate* of x. In the case where I is the empty set, the union $\bigcup_{i\in I} A_i$ is empty. Therefore, the direct product contains a unique element (identity mapping of $\varnothing$).

For each $j \in I$, we denote by

$$\mathrm{pr}_j : \prod_{i\in I} A_i \longrightarrow A_j$$

the mapping which sends each element $(a_i)_{i\in I}$ of the direct product to its j-th coordinate a_j. This mapping is called the *projection to the j-th coordinate.*

3.9.2 Notation. Let n be a non-zero natural number. If $(A_i)_{i\in\{1,\ldots,n\}}$ is a family of sets parametrised by $\{1,\ldots,n\}$, then the set

$$\prod_{i\in\{1,\ldots,n\}} A_i$$

is often denoted as

$$A_1 \times \cdots \times A_n.$$

3.9.3 Axioms (Axiom of choice)**.** In this book, we adopt the following axiom. If I is a non-empty set and if $(A_i)_{i\in I}$ is a family of non-empty sets, then the direct product $\prod_{i\in I} A_i$ is not empty.

3.9.4 Proposition. *Let I be a set and $(A_i)_{i\in I}$ be a family of sets parametrised by I. For any set X, the mapping*

$$\Big(\prod_{i\in I} A_i\Big)^X \longrightarrow \prod_{i\in I} A_i^X, \tag{3.5}$$

which sends f to $(\mathrm{pr}_i \circ f)_{i\in I}$, is a bijection.

Proof. Let $(f_i)_{i\in I}$ be an element of

$$\prod_{i\in I} A_i^X,$$

where each f_i is a mapping from X to A_i. Let $f : X \to \prod_{i\in I} A_i$ be the mapping which sends $x \in X$ to $(f_i(x))_{i\in I}$. By definition, for any $i \in I$ one has

$$\forall\, x \in X, \quad \mathrm{pr}_i(f(x)) = f_i(x).$$

Therefore the mapping (3.5) is surjective.

If f and g are two mappings from X to $\prod_{i\in I} A_i$ such that $\mathrm{pr}_i \circ f = \mathrm{pr}_i \circ g$ for any $i \in I$, then, for any $x \in X$ one has

$$\forall\, i \in I, \quad \mathrm{pr}_i(f(x)) = \mathrm{pr}_i(g(x)).$$

Hence $f(x) = g(x)$ for any $x \in X$, namely $f = g$. Therefore the mapping (3.5) is injective. □

3.9.5 Notation. Let I be a set, $(A_i)_{i\in I}$ be a family of sets parametrised by I.

Let X be a set. For any $i \in I$, let $f_i : X \to A_i$ be a mapping from X to A_i. By Proposition 3.9.4, there exists a unique mapping $f : X \to \prod_{i\in I} A_i$ such that $\mathrm{pr}_i \circ f = f_i$ for any $i \in I$. By abuse of notation, we denote by $(f_i)_{i\in I}$ this mapping.

Let $(B_i)_{i\in I}$ be a family of sets parametrised by I. For any $i \in I$, let $g_i : B_i \to A_i$ be a mapping from B_i to A_i. We denote by

$$\prod_{i\in I} g_i : \prod_{i\in I} B_i \longrightarrow \prod_{i\in I} A_i$$

the mapping which sends $(b_i)_{i\in I}$ to $(g_i(b_i))_{i\in I}$. In the case where $I = \{1, \ldots, n\}$, where n is a non-zero natural number, the mapping $\prod_{i\in\{1,\ldots,n\}} g_i$ is also denoted as

$$g_1 \times \cdots \times g_n.$$

3.9.6 Proposition. *Let $f : X \to Y$ be a mapping.*

(1) *If f is surjective, then there exists an injective mapping $g : Y \to X$ such that $f \circ g = \mathrm{Id}_Y$.*

(2) *If f is injective and X is not empty, then there exists a surjective mapping $h : Y \to X$ such that $h \circ f = \mathrm{Id}_X$.*

Proof. (1) The case where $Y = \varnothing$ is trivial since in this case $X = \varnothing$ and f is the identity mapping of $\varnothing$. In the following, we assume that Y is not empty. Since f is surjective, for any $y \in Y$, the set $f^{-1}(\{y\})$ is not empty. Hence the direct product

$$\prod_{y \in Y} f^{-1}(\{y\})$$

is not empty. In other words, there exists a mapping g from Y to X such that $f(g(y)) = y$ for any $y \in Y$, that is $f \circ g = \mathrm{Id}_Y$. By (2) of Remark 3.7.6, g is injective.

(2) Let x_0 be an element of X. We define a mapping $h : Y \to X$ as follows:

$$h(y) := \begin{cases} f^{-1}(y), & \text{if } y \in \mathrm{Im}(f), \\ x_0, & \text{else.} \end{cases}$$

Then, by construction one has $h \circ f = \mathrm{Id}_X$. By (1) of Remark 3.7.6, h is surjective. □

3.10 Restriction and extension

3.10.1 Definition. Let f and g be correspondences. If $\Gamma_f \subseteq \Gamma_g$, we say that f is a *restriction* of g and that g is an *extension* of f. By definition, f is a restriction of g if and only if f^{-1} is a restriction of g^{-1}.

Let X and Y be sets, h be a correspondence from X to Y, and A be a subset of X. Denote by $h|_A$ the correspondence from A to Y such that

$$\Gamma_{h|_A} = \Gamma_h \cap (A \times Y).$$

We call it the *restriction of h to A.*

3.10.2 Proposition. *Let g be a correspondence.*

(1) *If g is a function, then all its restrictions are functions.*

(2) *If g is injective, the all its restrictions are injective.*

(3) *If g is a function and A is a subset of* $\mathrm{Dom}(g)$*, then $g|_A$ is a mapping.*

Proof. (1) Let f be a restriction of g. Then one has $\mathrm{Dom}(f) \subseteq \mathrm{Dom}(g)$. For any $x \in \mathrm{Dom}(f)$, if $(x, y) \in \Gamma_f$, then $(x, y) \in \Gamma_g$. Since g is a function, x admits at most one image under g. Therefore, it also admits at most one image under f. Hence f is a function.

Applying (1) to g^{-1}, we obtain (2).

(3) By (1), $g|_A$ is a function. Moreover, for any $x \in A$,

$$(x, g(x)) \in (A \times \mathscr{A}_g) \cap \Gamma_g = \Gamma_{g|_A},$$

which implies that $x \in \mathrm{Dom}(g|_A)$. Hence $g|_A$ is a mapping. □

3.10.3 Definition. Let X be a set and A be a subset of X. The restriction of the identity mapping Id_X to A is called the *inclusion mapping* of A into X, denoted by $j_A : A \to X$. Note that, if h is a correspondence from X to a set Y, then $h|_A$ identifies with $h \circ j_A$, the composite correspondence of h with the inclusion mapping $j_A : A \to X$.

3.11 Equivalence relation

3.11.1 Definition. Let X be a set. We call *binary relation* on X any correspondence from X to itself. If R is a binary relation on X, for any $(x, y) \in X \times X$, we denote by $x \, R \, y$ the statement "$(x, y) \in \Gamma_R$", and denotes by $x \not R \, y$ the statement "$(x, y) \notin \Gamma_R$". If Y is a subset of X, then there is a unique binary relation on Y, the graph of which is $\Gamma_R \cap (Y \times Y)$. We call it the *restriction of R to Y* as a binary relation. We emphasise that it is different from the restriction of R to Y as a correspondence.

3.11.2 Notation. Let X be a set, n be a positive integer, $R_1, \ldots, R_n$ be binary relations on X. If $x_0, \ldots, x_n$ are elements of X, then

$$x_0 \, R_1 \, x_1 R_2 \ldots R_{n-1} x_{n-1} \, R_n \, x_n$$

denotes

$$\forall \, i \in \{0, \ldots, n-1\}, \quad x_i \, R_i \, x_{i+1}.$$

3.11.3 Definition. Let X be a set and R be a binary relation on X.

(a) If for any $x \in X$, one has $x \, R \, x$, then we say that the binary relation R is *reflexive*. Note that R is reflexive if and only if $\Delta_X \subseteq \Gamma_R$.

(b) If for any $(x, y) \in X \times X$, $x \, R \, y$ implies $y \, R \, x$, then we say that the binary relation R is *symmetric*. Note that R is symmetric if and only if $R = R^{-1}$.

(c) If for all elements x, y and z of X, $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$, then we say that the binary relation R is *transitive*. Note that R is transitive if and only if $\Gamma_{R\circ R} \subseteq \Gamma_R$.

(d) If R is reflexive, symmetric and transitive, then we say that R is an *equivalence relation*.

3.11.4 Lemma. *Let X be a set and $\sim$ be an equivalence relation on X. For any $x \in X$, denote by $[x]$ the subset*

$$\{y \in X \,:\, y \sim x\}.$$

(1) *If x_1 and x_2 are elements of X such that $x_1 \sim x_2$, then one has $[x_1] = [x_2]$.*

(2) *If x_1 and x_2 are elements of X such that $x_1 \not\sim x_2$, then one has $[x_1] \cap [x_2] = \varnothing$.*

Proof. (1) Let y be an element of $[x_1]$. By definition, $y \sim x_1$. Since $x_1 \sim x_2$, by the transitivity we obtain $y \sim x_2$, namely $y \in [x_2]$. Therefore, we obtain $[x_1] \subseteq [x_2]$. Moreover, by the symmetry one has $x_2 \sim x_1$, and hence, by what has been proved above, we obtain $[x_2] \subseteq [x_1]$. Therefore $[x_1] = [x_2]$.

(2) We reason by contraposition. Assume that z is a common element of $[x_1]$ and $[x_2]$. By definition, one has $z \sim x_1$ and $z \sim x_2$. Hence, by (1), we deduce $[x_1] = [z] = [x_2]$. □

3.11.5 Definition. Let X be a set, $\sim$ be an equivalence relation on X. A subset of X of the form

$$[x] := \{y \in X \,:\, y \sim x\}, \text{ where } x \in X$$

is called an *equivalence class* under the equivalence relation $\sim$, and the element x is called a *representative* of the equivalence class. We denote by $X/{\sim}$ the set of all equivalence classes under $\sim$, called the *quotient set* of X by the equivalence relation $\sim$. This is a subset of the power set $\mathscr{P}(X)$. The mapping from X to $X/{\sim}$ which sends $x \in X$ to the equivalence class $[x]$ represented by x is called the *projection mapping*. Note that the projection mapping is surjective.

3.11.6 Proposition. *Let X be a set and $\sim$ be an equivalence relation on X. Then the elements of $X/{\sim}$ are pairwise disjoint sets and one has*

$$X = \bigcup_{A \in X/\sim} A. \tag{3.6}$$

In other words, X is the disjoint union of the equivalence classes under $\sim$.

Proof. By Lemma 3.11.4, two equivalence classes under $\sim$ are either disjoint or equal, which shows that the sets in $X/\sim$ are pairwise disjoint. Moreover, by the reflexivity of $\sim$, for any $x \in X$, one has $x \in [x]$. Hence

$$X \subseteq \bigcup_{A \in X/\sim} A.$$

Conversely, for any $A \in X/\sim$, one has $A \subseteq X$. Hence, by Proposition 2.7.3, one has

$$\bigcup_{A \in X/\sim} A \subseteq X.$$

Hence the equality (3.6) holds. $\square$

3.11.7 Example. Let p be a natural number. Consider the binary relation $\sim_p$ on $\mathbb{Z}$ defined as follows:

$$x \sim_p y \text{ if and only if there exists } n \in \mathbb{Z} \text{ such that } x - y = pn.$$

One can check that $\sim_p$ is an equivalence relation on $\mathbb{Z}$. We call it the *relation of congruence* modulo p. For any $a \in \mathbb{Z}$, we denote by $a + p\mathbb{Z}$ the equivalence class of a under $\sim_p$.

3.11.8 Proposition. *Let X be a set, $\sim$ be an equivalence relation on X, and $f : X \to Y$ be a mapping. Denote by $\pi : X \to X/\sim$ the projection mapping, Assume that, for any $(x, x') \in X \times X$ such that $x \sim x'$, one has $f(x) = f(x')$. There exists a unique mapping $\widetilde{f} : X/\sim \to Y$ such that $\widetilde{f} \circ \pi = f$ (namely, such that $\widetilde{f}([x]) = f(x)$ for any $x \in X$). Moreover, the equality $\mathrm{Im}(f) = \mathrm{Im}(\widetilde{f})$ holds.*

Proof. By (1) of Proposition 3.9.6, since π is surjective, there exists an injective mapping $g : X/\sim \to X$ such that $\pi \circ g = \mathrm{Id}_{X/\sim}$. Therefore, if $\widetilde{f} : X/\sim \to Y$ is a mapping such that $\widetilde{f} \circ \pi = f$, then one has

$$f \circ g = (\widetilde{f} \circ \pi) \circ g = \widetilde{f} \circ (\pi \circ g) = \widetilde{f} \circ \mathrm{Id}_{X/\sim} = \widetilde{f}.$$

This shows the uniqueness of $\widetilde{f}$. Moreover, since Since $f = \widetilde{f} \circ \pi$ and since π is surjective, by (1) of Proposition 3.4.5, we obtain $\mathrm{Im}(\widetilde{f}) = \mathrm{Im}(f)$.

It remains to check that, if we define $\widetilde{f}$ as $f \circ g$, then the mapping $\widetilde{f}$ satisfies the equality $\widetilde{f} \circ \pi = f$. Note that, for any $\alpha \in X/\sim$, one has $\pi(g(\alpha)) = \alpha$, namely $g(\alpha)$ is a representative of α. Therefore, for any $x \in X$, one has $g(\pi(x)) \sim x$ and hence

$$\widetilde{f}(\pi(x)) = f(g(\pi(x))) = f(x).$$

$\square$

3.11.9 Definition. Under the notation and the hypothesis of Proposition 3.11.8, the mapping $\widetilde{f} : X/{\sim} \to Y$ is said to be *induced by f by passing to quotient.*

3.11.10 Corollary. *Let $f : X \to Y$ be a mapping of sets. The binary relation R_f on X defined by*

$$x \, R_f \, x' \text{ if and only if } f(x) = f(x')$$

is an equivalence relation. Moreover, f induces by passing to quotient an injective mapping $\widetilde{f} : X/R_f \to Y$.

Proof. For any $x \in X$ one has $f(x) = f(x)$ and hence $x \, R_f \, x$. If x and x' are elements of X such that $x \, R_f \, x'$, then one has $f(x') = f(x)$ and hence $x' \, R_f \, x$. If x, x' and x'' are elements of X such that $x \, R_f \, x'$ and $x' \, R_f \, x''$, then the equalities $f(x) = f(x') = f(x'')$ hold, and hence $x \, R_f \, x''$.

By Proposition 3.11.8, the mapping f induces by passing to quotient a mapping $\widetilde{f} : X/R_f \to Y$. If α and β are two elements of X/R_f, which are represented by elements x and y of X respectively, and such that $\widetilde{f}(\alpha) = \widetilde{f}(\beta)$, then one has $f(x) = f(y)$ and hence $x \, R_f \, y$. This leads to $\alpha = \beta$. Therefore $\widetilde{f}$ is injective. □

3.11.11 Corollary. *Let I be a set and $(X_i)_{i \in I}$ be a family of sets parametrised by I. For any $i \in I$, let R_i be an equivalence relation on X_i and $\pi_i : X_i \to X_i/R_i$ be the projection mapping. Let $X = \prod_{i \in I} X_i$ and let $\sim$ be the binary relation on X defined as follows*

$$(x_i)_{i \in I} \sim (y_i)_{i \in I} \quad \text{if and only if } \ \forall \, i \in I, \ x_i \, R_i \, y_i.$$

(1) *The binary relation $\sim$ is an equivalence relation on X.*

(2) *The mapping*

$$\prod_{i \in I} \pi_i : X \longrightarrow \prod_{i \in I} (X_i/R_i)$$

induces by passing to quotient a bijection from $X/{\sim}$ to $\prod_{i \in I}(X_i/R_i)$.

Proof. Note that, for any elements $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ of X, $(x_i)_{i \in I} \sim (y_i)_{i \in I}$ if and only if $(\pi_i(x_i))_{i \in I} = (\pi_i(y_i))_{i \in I}$. Therefore, by Corollary 3.11.10, the binary relation $\sim$ is an equivalence relation, and the mapping $\prod_{i \in I} \pi_i$ induces by passing to quotient an injective mapping from $X/{\sim}$ to $\prod_{i \in I}(X_i/R_i)$. Moreover, by Proposition 3.11.8 this mapping have the same image as that of $\prod_{i \in I} \pi_i$. Since $\prod_{i \in I} \pi_i$ is surjective, we obtain that the induced mapping is also surjective. □

Exercises

1. Here is a table that shows courses chosen by 6 students, followed by a list of the assigned instructors.

	Analysis	Physics	Chemistry	Programming	Calculus	Phys. Lab	Chem. Lab
Charles	$\surd$	$\surd$		$\surd$			
Charlie		$\surd$		$\surd$	$\surd$		
Carlo			$\surd$		$\surd$		$\surd$
Karl	$\surd$	$\surd$				$\surd$	
Charuzu		$\surd$	$\surd$		$\surd$		
Charlot	$\surd$	$\surd$	$\surd$				

- Analysis: Maicon,
- Physics: Mikkel,
- Chemistry: Michelle,
- Programming: Mikhail,
- Calculus: Micheal,
- Physics Lab: Mitchell,
- Chemistry Lab: Mihai.

Let X denote the set that consists of these 6 students, Y denote the set of all the courses, Z denote the set of all the instructors, f denote the correspondence from X to Y based on the above table, g denote the correspondence from Y to Z based on the assignment of the course instructors.

(1) Draw a table of g.

(2) Is the correspondence f injective? surjective? a function? a multi-valued mapping? What about g?

(3) Draw a table of the composite $g \circ f$. What can we conclude from it?

(4) Draw a table of the composite $f^{-1} \circ f$. What can we conclude from it?

2. For each of the following mappings, determine if it is injective, surjective. Justify your answer.

(1) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 3x + 4$.

(2) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = 3x + 4$.

(3) $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 4$.

(4) $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x + y, x - y)$.

3. For each of the following functions from $\mathbb{R}$ to $\mathbb{R}$, determine its domain of definition and its range.

(1) $\cos \circ \sin$.

(2) $\exp \circ \ln$.

(3) $\ln \circ \exp$.

(4) ι defined as $\iota(x) = x^{-1}$.

(5) $\iota \circ \iota$.

4. Consider the correspondence φ from $\mathbb{R}$ to $\mathbb{R}$ whose graph is given by

$$\{(x, y) \,:\, x^2 + y^2 \leq 1\}.$$

(1) Draw the graph of the correspondance φ.

(2) Determine the domain of definition and the range of φ.

(3) We view φ as a binary relation on $\mathbb{R}$. Is this binary relation reflexive, symmetric, transitive?

(4) Draw the graph of $\varphi \circ \varphi$.

5. Consider the mapping $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x^2$. Let

$$A = \{x \in \mathbb{R} \mid -1 \leqslant x \leqslant 4\}.$$

Determine $f(A)$ and $f^{-1}(A)$.

6. Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined as

$$f(x) = 3x + 1, \quad g(x) = x^2 - 1.$$

Determine the functions $f \circ g$ and $g \circ f$. Are they equal?

7. For each of the following function h from $\mathbb{R}$ to $\mathbb{R}$. Determine two functions f and g such that $h = f \circ g$.

(1) $h(x) = \sqrt{3x - 1}$,

(2) $h(x) = \sin(x + \pi/2)$,

(3) $h(x) = 1/(x + 1)$.

8. Let f and g be mappings from $\mathbb{N}$ to $\mathbb{N}$ defined as

$$f(n) = 2n, \quad g(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Determine $g \circ f$ and $f \circ g$. Is the mapping f injective, surjective, a bijection? Is the mapping g injective, surjective, a bijection?

9. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the mapping sending $(n, p) \in \mathbb{N} \times \mathbb{N}$ to $2^n(2p + 1)$.

(1) Show that f is injective.

(2) Determine the range of f.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping. For each of the following statements, construct a statement of the opposite truth value, where no quantifier is preceded by a negation symbol.

(1) $\forall\, x \in \mathbb{R},\ f(x) \neq 0,$

(2) $\forall\, M \in \mathbb{R}_{>0},\ \exists\, A \in \mathbb{R}_{>0},\ \forall\, x \geq A,\ f(x) > M.$

(3) $\forall\, x \in \mathbb{R},\ f(x) > 0 \Rightarrow x \leq 0.$

(4) $\forall \varepsilon \in \mathbb{R}_{>0},\ \exists\, \eta \in \mathbb{R}_{>0},\ \forall\, (x, y) \in \mathbb{R} \times \mathbb{R},\ |x - y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon.$

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping. By using the quantifiers, express the following statements:

(1) f is constant.

(2) 0 lies in the range of f.

(3) f only takes non-negative values.

12. We consider the following condition $P(\cdot)$ on the set $\mathbb{R}^{\mathbb{R}}$ of mappings from $\mathbb{R}$ to $\mathbb{R}$

$$P(f) := \text{“}\exists\, M \in \mathbb{R},\ \forall\, x \in \mathbb{R},\ f(x) \leq M\text{”}.$$

(1) Construct a condition $Q(\cdot)$ on $\mathbb{R}^{\mathbb{R}}$ without quantifier proceeded by a negation symbol, such that, for any $f \in \mathbb{R}^{\mathbb{R}}$, $Q(f)$ and $\neg P(f)$ have the same truth value.

(2) Construct an example of mapping in $\mathbb{R}^{\mathbb{R}}$ that satisfies the condition $P(\cdot)$.

(3) Construct an example of mapping in $\mathbb{R}^{\mathbb{R}}$ that does not satisfies the condition $P(\cdot)$.

(4) For each of the following conditions, determine if it is satisfied by all mappings in $\mathbb{R}^{\mathbb{R}}$, if it is not satisfied by none of the mappings in $\mathbb{R}^{\mathbb{R}}$, or if it is satisfied by some but not all of the mappings in $\mathbb{R}^{\mathbb{R}}$.

a) $\exists\, x \in \mathbb{R}, \forall\, M \in \mathbb{R}, f(M) \leq x$.

b) $\exists\, x \in \mathbb{R}, \forall\, M \in \mathbb{R}, f(x) \leq M$.

c) $\forall\, M \in \mathbb{R}, \exists\, x \in \mathbb{R}, f(x) \leq M$.

d) $\forall\, M \in \mathbb{R}, \exists\, x \in \mathbb{R}, f(M) \leq x$.

13. Let n and p be two natural numbers, $X = \{1, \ldots, n\}$ and $Y = \{1, \ldots, p\}$.

(1) How many correspondences are there from X to Y?

(2) How many injective correspondences are there from X to Y?

(3) How many surjective correspondences are there from X to Y?

(4) How many functions are there from X to Y?

(5) How many mappings are there from X to Y?

(6) How many bijections are there from X to Y.

14. For any natural number n, let E_n be the set $\{1, \ldots, n\}$ (by convention E_0 denotes the empty set). For any $(n,p) \in \mathbb{N}^2$, let $S_{n,p}$ be the number of surjective mappings from E_n to E_p.

(1) Determine $S_{n,p}$ for $(n,p) \in \mathbb{N}^2$ such that $p > n$.

(2) For any $n \in \mathbb{N}$, determine $S_{n,0}$.

(3) Show that, for any non-zero natural number p, one has

$$\sum_{i=0}^{p} (-1)^i \binom{p}{i} = 0,$$

where

$$\binom{n}{p} := \frac{n!}{p!(n-p)!}$$

(4) Let i, q and p be natural numbers such that $i \leq q \leq p$. Show that

$$\binom{p}{q}\binom{q}{i} = \binom{p}{i}\binom{p-i}{q-i}.$$

(5) Show that, for any $(i,p) \in \mathbb{N}^2$ such that $i < p$, one has

$$\sum_{q=i}^{p} (-1)^q \binom{p}{q}\binom{q}{i} = 0.$$

(6) Let p and q be natural numbers such that $q \leq p$. Show that the number of mappings from E_n to E_p whose range has exactly q elements is equal to $S_{n,q}\binom{p}{q}$.

(7) Let p and n be natural numbers such that $p \leq n$ and $n \geq 1$. Show that

$$p^n = \sum_{q=1}^{p} \binom{p}{q} S_{n,q}.$$

(8) Let p and n be natural numbers such that $p \leq n$ and $n \geq 1$. Show that

$$(-1)^p S_{n,p} = \sum_{i=1}^{p} (-1)^i \binom{p}{i} i^n.$$

(9) Let p and n be natural numbers such that $1 \leq p < n$. Show that

$$S_{n,p} = p(S_{n-1,p} + S_{n-1,p-1}).$$

(10) Let $p \in \mathbb{N}$. Determine $S_{p,p}$ and $S_{p+1,p}$.

(11) Let p and n be natural numbers such that $p \leq n$. Show that the number of surjective functions from E_n to E_p is equal to

$$\sum_{j=p}^{n} \binom{n}{j} S_{j,p}.$$

(12) Prove the equality

$$\sum_{j=p}^{n} \binom{n}{j} S_{j,p} = \frac{1}{p+1} S_{n+1,p+1}.$$

15. For any natural number n, let E_n be the set $\{1, \ldots, n\}$ (by convention E_0 denotes the empty set).

(1) Let n and p be natural numbers such that $n \leq p$. Show that the number of injective mappings from E_n to E_p is equal to $p!/(p-n)!$.

(2) Show that the number of injective functions from E_n to E_p is

$$\sum_{k=0}^{\min\{n,p\}} \binom{n}{k} \binom{p}{k} k!.$$

16. Let X be a set and $\mathscr{P}(X)$ be the power set of X, namely $\mathscr{P}(X)$ is the set of all subsets of X. The purpose of this exercise is to prove that there does not exist any injective mapping from $\mathscr{P}(X)$ to X. We reason by contradiction assuming that $\Phi : \mathscr{P}(X) \to X$ is an injective mapping.

(1) Show that Φ^{-1} is a function from X to $\mathscr{P}(X)$.

(2) Let A be the set of $n \in \mathrm{Im}(\Phi)$ such that $n \notin \Phi^{-1}(n)$. Show that $\Phi^{-1}(\Phi(A)) = A$.

(3) Examine the truth value of the statement $\Phi(A) \in A$ and obtain a contradiction.