

# FUNDAMENTAL ALGEBRA & ANALYSIS

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# Chapter 1

# Differential Calculus

## 1.1 Landau symbol

In this section, we fix a complete valued field  $(K, |\cdot|)$  and a normed vector space  $(V, \|\cdot\|)$  over  $K$ .

**Definition 1.1.1** Let  $X$  be a set,  $f : X \rightarrow V$ ,  $g : X \rightarrow \mathbb{R}_{\geq 0}$  be mappings. Let  $Y \subseteq X$  be a subset. We use the expression

$$f(x) = \mathcal{O}(g(x))$$

to denote the statement:

$$\exists C > 0, \forall x \in Y, \|f(x)\| \leq C \cdot g(x).$$

Let  $\mathcal{F}$  be a filter on  $X$ , we use the expression

$$f(x) = \mathcal{O}(g(x)) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists C > 0, \exists A \in \mathcal{F}, \|f(x)\| \leq C \cdot g(x), \forall x \in A.$$

We use the expression

$$f(x) = o(g(x)) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists \varepsilon : X \rightarrow \mathbb{R}_{\geq 0}, \exists A \in \mathcal{F}, \lim_{\mathcal{F}} \varepsilon = 0 \text{ and } \forall x \in A, \|f(x)\| \leq \varepsilon(x)g(x).$$

**Proposition 1.1.2** Let  $X$  be a set and  $\mathcal{F}$  be a filter on  $X$ .

(1) Let  $f : X \rightarrow V, g : X \rightarrow \mathbb{R}_{\geq 0}$  be mappings. If  $f(x) = o(g(x))$  along  $\mathcal{F}$ , then  $f(x) = \mathcal{O}(g(x))$  along  $\mathcal{F}$ .

(2)

1. Let  $f_1 : X \rightarrow V, f_2 : X \rightarrow V$  and  $g : X \rightarrow \mathbb{R}_{\geq 0}$  be mappings. If  $f_1(x) = \mathcal{O}(g(x))$  and  $f_2(x) = \mathcal{O}(g(x))$  along  $\mathcal{F}$ , then  $f_1(x) + f_2(x) = \mathcal{O}(g(x))$  along  $\mathcal{F}$ .

2. Let  $f_1 : X \rightarrow V, f_2 : X \rightarrow V$  and  $g : X \rightarrow \mathbb{R}_{\geq 0}$  be mappings. If  $f_1(x) = o(g(x))$  and  $f_2(x) = o(g(x))$  along  $\mathcal{F}$ , then  $f_1(x) + f_2(x) = o(g(x))$  along  $\mathcal{F}$ .

(3) Let  $\lambda : X \rightarrow K, f : X \rightarrow V, g : X \rightarrow \mathbb{R}_{\geq 0}, h : X \rightarrow \mathbb{R}_{\geq 0}$  be mappings.

1. If  $\lambda(x) = \mathcal{O}(g(x))$  along  $\mathcal{F}, f(x) = \mathcal{O}(h(x))$  along  $\mathcal{F}$ , then

$$(\lambda f)(x) = \lambda(x)f(x) = \mathcal{O}(g(x)h(x)).$$

2. If  $\lambda(x) = \mathcal{O}(g(x))$  along  $\mathcal{F}, f(x) = o(h(x))$  along  $\mathcal{F}$ , or if  $\lambda(x) = o(g(x))$  along  $\mathcal{F}, f(x) = \mathcal{O}(h(x))$  along  $\mathcal{F}$ , then

$$\lambda(x)f(x) = o(g(x)h(x)).$$

### Proof

(1) We have  $\varepsilon : X \rightarrow \mathbb{R}_{\geq 0}, A \in \mathcal{F}$  such that  $\lim_{\mathcal{F}} \varepsilon = 0$  and  $\forall x \in A, \|f(x)\| \leq \varepsilon(x)g(x)$ . Since  $\lim_{\mathcal{F}} \varepsilon = 0$ , there exists  $B \in \mathcal{T}$  such that  $\forall x \in B, |\varepsilon(x)| < 1$ , hence  $\forall x \in A \cap B, \|f(x)\| \leq g(x)$ .

(2)

1.  $A_1, A_2 \in \mathcal{F}, C_1, C_2 > 0, \forall x \in A_1, \|f_1(x)\| \leq C_1g(x), \forall x \in A_2, \|f_2(x)\| \leq C_2g(x)$ . So  $f_1(x) + f_2(x) = \mathcal{O}(g(x))$

2. Let  $\varepsilon_1 : X \rightarrow \mathbb{R}_{\geq 0}, \varepsilon_2 : X \rightarrow \mathbb{R}_{\geq 0}, A \in \mathcal{F}, \lim_{\mathcal{F}} \varepsilon_1 = \lim_{\mathcal{F}} \varepsilon_2 = 0$ .  $\forall x \in A_1, \|f_1(x)\| \leq \varepsilon_1(x) \cdot g(x), \forall x \in A_2, \|f_2(x)\| \leq \varepsilon_2(x)g(x)$ . So  $\lim_{\mathcal{F}} \varepsilon_1 + \varepsilon_2 = 0$ .

$$\forall x \in A_1 \cap A_2, \|f_1(x) + f_2(x)\| \leq \|f_1(x)\| + \|f_2(x)\| \leq (\varepsilon_1(x) + \varepsilon_2(x))g(x).$$

(3)

1. There exists  $(C_1, C_2) \in \mathbb{R}_{>0}^2$  and  $(A_1, A_2) \in \mathcal{F}^2$  such that

$$\forall x \in A_1, |\lambda(x)| \leq C_1 g(x), \forall x \in A_2, \|f(x)\| \leq C_2 h(x).$$

Hence,

$$\forall x \in A_1 \cap A_2, \|(\lambda(x)f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C_1 C_2 g(x) h(x).$$

2. We assume that

$$\lambda(x) = \mathcal{O}(g(x)) \text{ along } \mathcal{F}, f(x) = o(h(x)) \text{ along } \mathcal{F}.$$

There exists  $(A_1, A_2) \in \mathcal{F} \times \mathcal{F}, C \in \mathbb{R}_{\geq 0}$  and a mapping  $\varepsilon : X \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\forall x \in A_1, |\lambda(x)| \leq C \cdot g(x), \forall x \in A_2, \|f(x)\| \leq \varepsilon(x) h(x).$$

Then one has

$$\lim_{\mathcal{F}} C\varepsilon(x) = 0$$

and

$$\forall x \in A_1 \cap A_2, \|(\lambda(x)f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C \cdot g(x) \cdot \varepsilon(x) h(x)$$

As required. □

### Example 1.1.3

(1) Let  $I \subseteq \mathbb{N}$  infinite. Let  $(V, \|\cdot\|)$  be a normed vector space over complete valued field  $(K, |\cdot|)$ . Let  $\mathcal{F}$  be the filter on  $I$ . Let  $(x_n)_{n \in I} \in V^I, (b_n)_{n \in I} \in \mathbb{R}_{\geq 0}^I$ . We denote by

$$x_n = \mathcal{O}(b_n), n \in I, n \rightarrow +\infty$$

the statement  $x_n = \mathcal{O}(b_n)$  along  $\mathcal{F}$ . Namely,

$$\exists N \in \mathbb{N}, \exists C > 0, \forall n \in I_{\geq N}, \|x_n\| \leq C \cdot b_n.$$

$$x_n = o(b_n), n \in I, n \rightarrow +\infty$$

denotes the statement  $x_n = o(b_n)$  along  $\mathcal{F}$ . Namely,

$$\exists (\varepsilon_n)_{n \in I} \text{ such that } \lim_{n \rightarrow +\infty} \varepsilon_n = 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, \|x_n\| \leq \varepsilon_n \cdot b_n.$$

(2) Let  $(X, \mathcal{T})$  be a topological space,  $Y \subseteq X$ ,  $y_0 \in \bar{Y}$ . Let  $f : Y \rightarrow V$  and  $g : Y \rightarrow \mathbb{R}_{\geq 0}$  be mappings.

$$\mathcal{F} = \mathcal{V}_{y_0}(\mathcal{T})|_Y := \{U \cap Y \mid U \text{ is a neighborhood of } y_0\}$$

$f(y)\mathcal{O}(g(y))$ ,  $y \in Y$ ,  $y \rightarrow y_0$  denotes  $f(y) = \mathcal{O}(g(y))$  along  $\mathcal{F}$ . Namely,

$$\exists C > 0, \exists U \in \mathcal{V}_{y_0}(\mathcal{T}), \forall y \in U \cap Y, \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)), y \in Y, y \rightarrow y_0$$

denotes  $f(y) = o(g(y))$  along  $\mathcal{F}$ . Namely,

$$\exists \varepsilon : Y \rightarrow \mathbb{R}_{\geq 0}, \lim_{y \in Y, y \rightarrow y_0} \varepsilon(y) = 0, \exists U \in \mathcal{V}_{y_0}(\mathcal{T}),$$

$$\forall y \in U \cap Y, \|f(y)\| \leq \varepsilon(y)g(y).$$

(3) Let  $\mathcal{F}$  be a filter on  $\mathbb{R}$  generated by subsets of the form  $[a, +\infty[$ . Let  $Y \subseteq \mathbb{R}$  not bounded from above. Let  $f : Y \rightarrow V$  and  $g : Y \rightarrow \mathbb{R}_{\geq 0}$  be mappings. Then

$$f(y) = \mathcal{O}(g(y)), y \in Y, y \rightarrow +\infty$$

denotes  $f(y) = \mathcal{O}(g(y))$  along  $\mathcal{F}|_Y$ . Namely,

$$\exists C > 0, \exists a \in \mathbb{R}, \forall y \in Y_{\geq a}, \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)), y \in Y, y \rightarrow +\infty$$

denotes  $f(y) = o(g(y))$  along  $\mathcal{F}|_Y$ . Namely,

$$\exists \varepsilon : Y \rightarrow \mathbb{R}_{\geq 0}, \lim_{y \rightarrow +\infty} \varepsilon(y) = 0, \exists a \in \mathbb{R}, \forall y \in Y_{\geq a}, \|f(y)\| \leq \varepsilon(y)g(y).$$

## 1.2 Differentiability

We fix a complete valued field  $(K, |\cdot|)$ . We suppose that there exists  $a \in K^\times$ , such that  $|a| < 1$ . Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces over  $K$ .

$$\mathcal{L}(E, F) := \{\varphi \in \text{Hom}_K(E, F) \mid \|\varphi\| < +\infty\}.$$

$(\mathcal{L}(E, F), \|\cdot\|)$  is a normed vector space over  $K$ .

**Definition 1.2.1** Let  $U \subseteq E$  be subset and  $p \in U^\circ$ . We say that a mapping  $f : U \rightarrow F$  is **differentiable** at  $p$  if there exists  $\varphi \in \mathcal{L}(E, F)$  such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

If  $U = U^\circ$  and  $f$  is differentiable at every point of  $U$ , we say that  $f$  is **differentiable** on  $U$ .

**Proposition 1.2.2** Assume that  $f : U \rightarrow F$  is differentiable at  $p \in U^\circ$ . There exists a unique  $\varphi \in \mathcal{L}(E, F)$  such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

**Lemma 1.2.3**  $\forall \eta \in \mathcal{L}(E, F), \forall r > 0$ .

$$\|\eta\| = \sup_{x \in E, 0 < \|x\|_E \leq r} \frac{\|\eta(x)\|_F}{\|x\|_E} = \sup_{\substack{x \in E \\ 0 < \|x\|_E < r}} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

**Proof (of Lemma)**  $\|\eta\| \geq \sup_{x \in E, 0 < \|x\|_E < r} \frac{\|\eta(x)\|_F}{\|x\|_E}$ .  $\forall y \in E \setminus \{0\}, \|a^N y\|_E = |a|^N \|y\|_E < r$ .

$$\frac{\|\eta(a^N y)\|_F}{\|a^N y\|_E} = \frac{|a|^N \cdot \|\eta(y)\|_F}{|a|^N \cdot \|y\|_E} = \frac{\|\eta(y)\|_F}{\|y\|_E} \leq \sup_{\substack{x \in E \\ 0 < \|x\|_E < r}} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

□

**Proof (of Proposition)** Suppose  $\varphi, \psi \in \mathcal{L}(E, F)$  are such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E,$$

$$f(p + h) - f(p) - \psi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

Then

$$\varphi(h) - \psi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

$$\exists r > 0, \exists \varepsilon : \overline{B}(0_E, r) \rightarrow \mathbb{R}_{\geq 0} \text{ such that } \lim_{h \rightarrow 0_E} \varepsilon(h) = 0.$$

$$\forall h \in \overline{B}(0_E, r), \|(\varphi - \psi)(h)\|_F = \varepsilon(h)\|h\|_E.$$

$$\|\varphi - \psi\| = \sup_{\substack{x \in E \\ 0 < \|h\|_E < r'}} \frac{\|\varphi(h) - \psi(h)\|_F}{\|h\|_E} \leq \sup_{0 < \|h\|_E < r'} \varepsilon(h).$$

Taking the limit when  $r' \rightarrow 0$ , by  $\limsup_{h \rightarrow 0_E} \varepsilon(h) = 0$ . We get  $\|\varphi - \psi\| = 0$ , hence  $\varphi = \psi$ .  $\square$

**Definition 1.2.4** Let  $U \subseteq E$  and  $f : U \rightarrow F$  be a mapping that is differentiable at  $p \in U^\circ$ . The unique  $\varphi \in \mathcal{L}(E, F)$  such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E$$

is called the **differential** of  $f$  at  $p$  and is denoted as

$$D(f(p)).$$

### Example 1.2.5

(1)  $f : U \rightarrow F$ ,  $f(x) \equiv c$ ,  $c \in F$ .

$$f(x + h) - f(x) = 0_E = o(\|h\|_E).$$

So  $f$  is differentiable at every point of  $U$  and  $D(f(x)) = 0_F$ .

(2)  $\varphi \in \mathcal{L}(E, F)$ .

$$\varphi(p + h) - \varphi(p) - \varphi(h) = 0_F = o(\|h\|_E).$$

So  $\varphi$  is differentiable at every point of  $E$  and  $D(\varphi(p)) = \varphi$ .

(3) Let  $(F_i, \|\cdot\|_i)$  be normed vector spaces over  $K$ ,  $i \in \{1, \dots, n\}$ ,  $n \in \mathbb{N}$ . Suppose that  $F = F_1 \oplus \dots \oplus F_n$  and

$$\|(s_1, \dots, s_n)\|_F = \max\{\|s_1\|_1, \dots, \|s_n\|_n\}.$$

Let  $U \subseteq E$  be an open subset,  $f_i : U \rightarrow F_i$  be a mapping.

$$f : U \rightarrow F, \quad f(x) = (f_1(x), \dots, f_n(x)).$$

$$f(p + h) - f(p) = (f_1(p + h) - f_1(p), \dots, f_n(p + h) - f_n(p)).$$

Suppose that each  $f_i$  is differentiable

$$\begin{aligned} & f(p + h) - f(p) - (Df_1(p)(h), \dots, Df_n(p)(h))|_F \\ &= \max_{i \in \{1, \dots, n\}} \|f_i(p + h) - f_i(p) - Df_i(p)(h)\|_{F_i} \\ &= o(\|h\|_E). \end{aligned}$$

So  $f$  is differentiable at  $p$  and

$$Df(p)(h) = (Df_1(p)(h), \dots, Df_n(p)(h)).$$

(4) Suppose that  $E = K$ . If  $U \subseteq K$  is open and  $f : U \rightarrow F$  is differentiable at  $p \in U$ . We denote by  $f'(p)$  the element  $Df(p)(1) \in F$ .

$$f(p+h) - f(p) - Df(p)(h) = o(\|h\|_E).$$

So

$$\begin{aligned} f(p+h) - f(p) - hf'(p) &= o(\|h\|_E), \\ \frac{f(p+h) - f(p)}{h} - f'(p) &= o(1). \end{aligned}$$

That is,

$$\lim_{h \rightarrow 0} \frac{f(p+h) - f(p)}{h} = f'(p).$$

**Theorem 1.2.6** Let  $(E, \|\cdot\|_E)$ ,  $(F, \|\cdot\|_F)$ ,  $(G, \|\cdot\|_G)$  be normed vector spaces over a complete valued field  $(K, |\cdot|)$ . Let  $U \subseteq E$  and  $V \subseteq F$  be open subsets,  $f : U \rightarrow F$  and  $g : V \rightarrow G$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . If  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $f(p)$ , then  $g \circ f : U \rightarrow G$  is differentiable at  $p$  and

$$D(g \circ f)(p)(h) = Dg(f(p))(Df(p)(h)).$$

### Proof

$$f(p+h) - f(p) - Df(p)(h) = o(\|h\|_E),$$

so,

$$f(p+h) - f(p) = \mathcal{O}(\|h\|_E).$$

$$\begin{aligned} &g(f(p+h)) - g(f(p)) - Dg(f(p))(f(p+h) - f(p)) \\ &= o(\|f(p+h) - f(p)\|_F) = o(\mathcal{O}\|h\|_E) = o(\|h\|_E). \end{aligned}$$

$$\begin{aligned} &Dg(f(p))(f(p+h) - f(p)) - Dg(f(p))(Df(p)(h)) \\ &= Dg(f(p))(f(p+h) - f(p) - Df(p)(h)) \\ &= \mathcal{O}(o(\|h\|_E)) = o(\|h\|_E). \end{aligned}$$

So,

$$g(f(p+h)) - g(f(p)) - Dg(f(p))(Df(p)(h)) = o(\|h\|_E).$$

□

**Remark 1.2.7** If  $(E, \|\cdot\|_E) = (K, |\cdot|)$ ,

$$(g \circ f)'(p) = Dg(f(p))(f'(p)).$$

If  $E = F = K$ ,  $\|\cdot\|_E = \|\cdot\|_F = |\cdot|$ .

$$(g \circ f)'(p) = g'(f(p)) \cdot f'(p).$$

**Remark 1.2.8** Let  $U \subseteq E$  be open.  $f : U \rightarrow F_1 \times \cdots \times F_n$ . If  $f$  is differentiable at  $p \in U$ , for any  $i \in \{1, \dots, n\}$ , the mapping

$$f_i := \pi_i \circ f : U \rightarrow F_i$$

is differentiable at  $p$  and

$$D(f_i)(p)(h) = D\pi_i(f(p))(Df(p)(h)) = \pi_i(Df(p)(h)).$$

### 1.3 Multilineal Mappings

**Definition 1.3.1** Let  $K$  be a commutative unitary ring. Let  $E_1, \dots, E_n; F$  be  $K$ -modules. We say that

$$\varphi : E_1 \times \dots \times E_n \rightarrow F$$

is  $n$ -linear if for any  $i \in \{1, \dots, n\}$  and any  $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in E_1 \times \dots \times E_{i-1} \times E_{i+1} \times \dots \times E_n$ , the mapping

$$E_i \rightarrow F, x_i \mapsto \varphi(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$$

is a homomorphism of  $K$ -modules. ( $K$ -linear mapping)

If  $n = 1$ , 1-linear is also called linear.

If  $n = 2$ , 2-linear is also called bilinear.

#### Example 1.3.2

- (1)  $K \times K \rightarrow K$   $(a, b) \mapsto ab$  is bilinear.
- (2)  $K^n \times K^n \rightarrow K$   $(x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$  is bilinear.
- (3)  $K \times \dots \times K \rightarrow K$   $(x_1, \dots, x_n) \mapsto x_1 \cdots x_n$  is  $n$ -linear.

**Definition 1.3.3** We denote by  $\text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$  the set of  $n$ -linear mappings from  $E_1 \times \dots \times E_n$  to  $F$ .

**Definition 1.3.4** Let  $(K, |\cdot|)$  be a complete valued field.

Let  $(E_1, \|\cdot\|_{E_1}), \dots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$  be normed vector spaces over  $K$ . For any  $\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$ , we define

$$\|\varphi\| := \sup_{\substack{x_i \in E_i \setminus \{0\} \\ i \in \{1, \dots, n\}}} \frac{\|\varphi(x_1, \dots, x_n)\|_F}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_n}}.$$

We denote by  $\mathcal{L}(E_1 \times \dots \times E_n, F)$  the set

$$\{\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F) \mid \|\varphi\| < +\infty\}.$$

$\mathcal{L}^{(n)}(E_1 \times \dots \times E_n, F)$  is a normed vector space of  $\text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$ , and the norm is  $\|\cdot\|$ .

**Theorem 1.3.5** Let  $(E_1, \|\cdot\|_{E_1}), \dots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$  be normed vector spaces over  $K$ . Let  $\varphi \in \mathcal{L}^{(n)}(E_1 \times \dots \times E_n, F)$ . For any  $p = (p_1, \dots, p_n) \in E_1 \times \dots \times E_n$ ,  $\varphi$  is differentiable at  $p$  and

$$D\varphi(p)(h_1, \dots, h_n) = \sum_{i=1}^n \varphi(p_1, \dots, p_{i-1}, h_i, p_{i+1}, \dots, p_n).$$

### Proof

$$\begin{aligned} \varphi(p+h) - \varphi(p) &= \sum_{i=1}^n \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, p_i + h_i, p_{i+1}, \dots, p_n) \\ &\quad - \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, p_i, p_{i+1}, \dots, p_n) \end{aligned}$$

$$\begin{aligned} &\varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, h_i, p_{i+1}, \dots, p_n) \\ &\quad - \varphi(p_1, \dots, p_{i-1}, h_i, p_{i+1}, \dots, p_n) \\ &= \sum_{j=1}^{i-1} \varphi(p_1 + h_1, \dots, p_{j-1} + h_{j-1}, h_j, p_{j+1}, \dots, h_i, \dots, p_n). \end{aligned}$$

$$\begin{aligned}
& \|\varphi(p_1 + h_1, \dots, p_{j-1} + h_{j-1}, h_j, p_{j+1}, \dots, h_i, \dots, p_n)\|_F \\
& \leq \|\varphi\| \cdot \prod_{k=1}^{j-1} \|p_k + h_k\|_{E_k} \cdot \|h_j\|_{E_j} \cdot \prod_{k=j+1}^{i-1} \|p_k\|_{E_k} \cdot \|h_i\|_{E_i} \cdot \prod_{k=i+1}^n \|p_k\|_{E_k} \\
& = \mathcal{O}(\|h\|^2) = o(h), \quad h \rightarrow 0.
\end{aligned}$$

□

**Definition 1.3.6** Let  $K$  be a commutative unitary ring.  $n \in \mathbb{N}_{\geq 1}$ ,  $E$  and  $F$  be  $K$ -modules. We say that

$$\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$$

is **symmetric** if

$$\forall \sigma \in \mathfrak{S}_{\{1, \dots, n\}}, \quad \forall (x_1, \dots, x_n) \in E^n, \quad \varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varphi(x_1, \dots, x_n).$$

Let  $P : E \rightarrow F$  be a mapping. If there exists a symmetric  $\varphi \in \text{Hom}_K^{(n)}(E \times \dots \times E, F)$  such that

$$\forall x \in E, \quad P(x) = \varphi(x, \dots, x),$$

we say that  $P$  is a **homogeneous polynomial mapping of degree  $n$** .

If  $F = K$ ,  $P$  is called a **homogeneous polynomial** on  $E$ . The symmetric polynomial mapping  $\varphi$  is called the **polarization** of  $P$ .

**Proposition 1.3.7** Let  $(K, |\cdot|)$  be a complete valued field that is non-trivial. Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces over  $K$ . Assume that  $P : E \rightarrow F$  is a homogeneous polynomial mapping of degree  $n$ . Which admits a bounded polarization  $\varphi$ . Then  $P$  is differentiable on  $E$  and,

$$\forall (x, h) \in E \times E, \quad DP(x)(h) = n\varphi(x, \dots, x, h).$$

**Proof** Let

$$\begin{aligned}
\Delta : E & \longrightarrow E^n, \\
x & \longmapsto (x, \dots, x).
\end{aligned}$$

Then  $P = \varphi \circ \Delta$ . Since  $\varphi$  and  $\Delta$  are differentiable, so it is  $P$ .

Moreover,

$$\begin{aligned}
 DP(x)(h) &= D\varphi(\Delta(x))(D\Delta(x)(h)) \\
 &= D\varphi(x, \dots, x)(h, \dots, h) \\
 &= \sum_{i=1}^n \varphi(x, \dots, x, h, x, \dots, x) \\
 &= n\varphi(x, \dots, x, h).
 \end{aligned}$$

□

**Remark 1.3.8** Assume that  $E = K$ . Let  $P : K \rightarrow F$  be a homogeneous polynomial mapping of degree  $n$  of form  $P(x) = x^n s$ , where  $s \in F$ . Its polarization is of the form

$$\varphi(a_1, \dots, a_n) = a_1 \cdots a_n s.$$

$$P'(x) = DP(x)(1) = n\varphi(x, \dots, x, 1) = nx^{n-1}s.$$

**Proposition 1.3.9** Let  $n$  be a positive integer  $n \geq 2$ . Let  $(E_1, \|\cdot\|_1), \dots, (E_n, \|\cdot\|_n), (F, \|\cdot\|_F)$  be normed vector spaces. For any  $i \in \{1, \dots, n\}$ , the mapping

$$\mathcal{L}^{(n)}(E_1, \dots, E_n; F) \xrightarrow{f} \mathcal{L}^{(i)}(E_1, \dots, E_i, \mathcal{L}^{(n-i)}(E_{i+1}, \dots, E_n; F))$$

$$\varphi \longmapsto \left( \begin{array}{c} E_1 \times \dots \times E_n \xrightarrow{\mathcal{L}^{(i)}(E_{i+1}, \dots, E_n; F)} \\ (x_1, \dots, x_i) \longmapsto \left( \begin{array}{c} (x_{i+1}, \dots, x_n) \longmapsto \varphi(x_1, \dots, x_n) \\ E_{i+1} \times \dots \times E_n \in F \end{array} \right) \end{array} \right)$$

is an isomorphism of vector spaces over  $K$ , and in the same time an isometry, ( $\|f(\varphi)\| = \|\varphi\|$ ).

**Remark 1.3.10**

$$\forall (x_1, \dots, x_n) \in E_1 \times \dots \times E_n, f(\varphi)(x_1, \dots, x_i)(x_{i+1}, \dots, x_n) = \varphi(x_1, \dots, x_n)$$

**Proof**  $\forall (x_1, \dots, x_n) \in E_1 \times \dots \times E_n$ ,

$$\begin{aligned}
 \varphi(x_1, \dots, x_n) : E_{i+1} \times \dots \times E_n &\longrightarrow F \text{ is bounded} \\
 (x_{i+1}, \dots, x_n) &\longmapsto \varphi(x_1, \dots, x_n)
 \end{aligned}$$

Since

$$\|\varphi(x_1, \dots, x_n)\|_F \leq (\|\varphi\| \cdot \|x_1\| \dots \|x_i\|) \|x_{i+1}\| \dots \|x_n\|.$$

$$\begin{aligned}
\|f(\varphi)\| &= \sup_{x_j \in E_j \setminus \{0\}, j=1, \dots, i} \frac{\|\varphi(x_1, \dots, x_i, \cdot)\|}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_i}} \\
&= \sup_{x_j \in E_j \setminus \{0\}, j=1, \dots, i} \sup_{x_k \in E_k \setminus \{0\}, k=i+1, \dots, n} \frac{\|\varphi(x_1, \dots, x_n)\|_F}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_n}} \\
&= \|\varphi\|.
\end{aligned}$$

Hence  $f$  is injective. ( $\ker(f) = \{0\}$ )

For any  $\psi \in \mathcal{L}^{(i)}(E_1, \dots, E_i, \mathcal{L}^{(n-i)}(E_{i+1}, \dots, E_n))$ ,

$$\begin{aligned}
\varphi : E_1 \times \dots \times E_n &\longrightarrow F \\
(x_1, \dots, x_n) &\longmapsto \psi(x_1, \dots, x_i)(x_{i+1}, \dots, x_n)
\end{aligned}$$

belongs to  $\mathcal{L}^{(n)}(E_1, \dots, E_n; F)$  and  $f(\varphi) = \psi$ . So  $f$  is surjective.  $\square$

**Corollary 1.3.11** If  $E_1, \dots, E_n$  are all finite dimensional, then

$$\mathcal{L}^{(n)}(E_1, \dots, E_n; F) = \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F).$$

**Proof** If  $n = 1$ ,  $\mathcal{L}(E_1, F) = \text{Hom}_K(E_1, F)$ .

$$\begin{aligned}
\mathcal{L}^{(n)}(E_1, \dots, E_n; F) &\cong \mathcal{L}(E_1, \mathcal{L}^{(n-1)}(E_2, \dots, E_n; F)) \\
&= \text{Hom}_K(E_1, \text{Hom}_K^{(n-1)}(E_2 \times \dots \times E_n, F)) \\
&\cong \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F).
\end{aligned}$$

$\square$

Let  $(K, |\cdot|)$  be a complete nontrivial valued field. Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces over  $K$ .

**Definition 1.3.12** Let  $U \subseteq E$  be an open subset of  $E$ ,  $f : U \rightarrow F$  be a mapping.

If  $f$  is continuous on  $U$ , we say that  $f$  is **of class  $\mathcal{C}^0$**  and we denote by

$$\text{D}^0 f$$

the mapping  $f : U \rightarrow F$ . Denote by

$$\mathcal{C}^0(U, F)$$

the set of mappings from  $U$  to  $F$ .

$$U \xrightarrow{(f,g)} K \times K \xrightarrow{\times} K$$

$$p \longmapsto (f(p), g(p)) \longmapsto f(p) \times g(p)$$

Let  $p \in U$ . If  $f$  is differentiable on an open neighborhood  $V$  of  $p$  such that  $V \subseteq U$ . Then

$$\begin{aligned} Df : V &\longrightarrow \mathcal{L}(E, F) \\ x &\longmapsto Df(x) \end{aligned}$$

is a mapping. If  $Df$  is  $(n-1)$ -times differentiable at  $p$ , we say that  $f$  is **of class  $\mathcal{C}^n$**  at  $p$ . If  $f$  is of class  $\mathcal{C}^n$  at every point of  $U$ , we say that  $f$  is **n-times differentiable** at  $p$ . We denote by

$$D^n f(p) \in \mathcal{L}^{(n)}(E, \dots, E, F)$$

the  $n$ -linear mapping that sends  $(h_1, \dots, h_n) \in E^n$  to

$$D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n) \in F.$$

### Remark 1.3.13

$$D^n f(p)(h_1, \dots, h_n) = D^i(D^{n-i} f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n).$$

## 1.4 Convexity

**Definition 1.4.1** Let  $E$  be a vector space over a field  $K$ .  $S \subseteq E$  be a non-empty subset.

We call affine combination of elements of  $S$  any element of  $E$  of the form

$$a_1s_1 + a_2s_2 + \cdots + a_ns_n,$$

where  $n \in \mathbb{N}_{\geq 1}$ ,  $s_1, \dots, s_n \in S$ ,  $a_1, \dots, a_n \in K$  such that

$$a_1 + a_2 + \cdots + a_n = 1.$$

We denote by  $\text{Aff}(S)$  the set of all affine combinations of elements of  $S$ . One has  $S \subseteq \text{Aff}(S)$ .  $\text{Aff}(S)$  is called the affine hull of  $S$ .

If  $S = \text{Aff}(S)$ , we say that  $S$  is an affine subspace of  $E$ .

### Proposition 1.4.2

(1) If  $F$  is a vector subspace of  $E$ ,  $\forall p \in E$ ,

$$p + F = \{p + x \mid x \in F\}$$

is an affine subspace of  $E$ .

(2) If  $A \subseteq E$  is an affine subspace of  $E$ . For any  $p \in A$ ,

$$A - p := \{x - p \mid x \in A\}$$

is a vector subspace of  $E$ , which is not dependent on the choice of  $p$ . We call it the vector space **associated** with  $A$ .

### Proof

(1) Let  $(x_1, \dots, x_n) \in F^n$ ,  $(a_1, \dots, a_n) \in K^n$ , such that  $\sum_{i=1}^n a_i = 1$ . Then

$$\begin{aligned} \sum_{i=1}^n a_i(p + x_i) &= p \cdot \sum_{i=1}^n a_i + \sum_{i=1}^n a_i x_i \\ &= p + \sum_{i=1}^n a_i x_i \in p + F. \end{aligned}$$

(2) Let  $(x_1, \dots, x_n) \in A^n, (b_1, \dots, b_n) \in K^n$ .

$$\begin{aligned} \sum_{i=1}^n b_i(x_i - p) &= \sum_{i=1}^n b_i x_i - \left( \sum_{i=1}^n b_i \right) p \\ &= \left( \sum_{i=1}^n b_i x_i + \left( 1 - \sum_{i=1}^n b_i \right) p \right) - p \\ &\in A - p. \end{aligned}$$

Let  $q \in A$ ,  $\forall x \in A$ ,  $x - p = (x - q) + (q - p) \in A - q$ . So  $A - p \subseteq A - q$ . By symmetry,  $A - q \subseteq A - p$ . Hence  $A - p = A - q$ .  $\square$

**Example 1.4.3** Let  $A$  be an  $m$  by  $p$  matrix with coefficients in  $\mathbb{R}$ . Let  $(b_1, \dots, b_n) \in E^m$ . Consider the linear equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

The solution set is

$$S := \{(x_1, \dots, x_p) \in E^p \mid A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\}.$$

Claim:  $S$  is an affine subspace of  $E^p$ .

**Proof** Let  $\underline{x}^{(1)}, \dots, \underline{x}^{(n)}$  be elements of  $S$ , where  $\underline{x}^{(i)} = (x_1^{(i)}, \dots, x_p^{(i)})$ . Let  $(a_1, \dots, a_n) \in \mathbb{R}^n$ ,  $\underline{x} = a_1 \underline{x}^{(1)} + \dots + a_n \underline{x}^{(n)}$ .

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = A \left( a_1 \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_p^{(1)} \end{pmatrix} + \dots + a_n \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_p^{(n)} \end{pmatrix} \right).$$

$$a_1 A \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_p^{(1)} \end{pmatrix} + \dots + a_n A \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_p^{(n)} \end{pmatrix} = (a_1 + \dots + a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

$$x_j = a_1 x_j^{(1)} + \dots + a_n x_j^{(n)}.$$

$\square$

**Proposition 1.4.4** Let  $S \subseteq E$ . Then  $\text{Aff}(S)$  is the smallest affine subspace of  $E$  containing  $S$ .

### Proof

Let  $A \subseteq E$  be an affine subspace containing  $S$ .  $\forall n \in \mathbb{N}_{\geq 1}, \forall (x_1, \dots, x_n) \in S^n \subseteq A^n, (a_1, \dots, a_n) \in \mathbb{R}$ ,  $a_1 + \dots + a_n = 1$ , one has

$$\sum_{i=1}^n a_i x_i \in A.$$

So  $\text{Aff}(S) \subseteq A$ .

To show that  $\text{Aff}(S)$  is an affine subspace containing  $S$ , it is sufficient to check that  $\text{Aff}(S)$  is an affine subspace.

If  $S = \emptyset$ , then  $\text{Aff}(S) = \emptyset$ . It is an affine subspace.

Suppose that  $S \neq \emptyset, p \in S$ . We prove that  $\text{Aff}(S) - p$  is equal to  $\text{Span}_{\mathbb{R}}(S - p)$ . Let  $y = a_1 x_1 + \dots + a_n x_n \in \text{Aff}(S)$ .

$$y - p = a_1(x_1 - p) + \dots + a_n(x_n - p) \in \text{Span}_{\mathbb{R}}(S - p).$$

Let  $(x_1, \dots, x_n) \in S^n, (b_1, \dots, b_n) \in \mathbb{R}^n$ .

$$\sum_{i=1}^n b_i(x_i - p) = \left( \sum_{i=1}^n b_i x_i + \left(1 - \sum_{i=1}^n b_i\right)p \right) - p \in \text{Aff}(S) - p.$$

□

**Definition 1.4.5** Let  $S \subseteq E$ . We call **convex combination** of elements of  $S$  any element of  $E$  of the form

$$a_1 s_1 + a_2 s_2 + \dots + a_n s_n,$$

where  $n \in \mathbb{N}_{\geq 1}, s_1, \dots, s_n \in S, a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$  such that

$$a_1 + a_2 + \dots + a_n = 1.$$

We denote by  $\text{Conv}(S)$  the set of all convex combinations of elements of  $S$ .  $\text{Conv}(S)$  is called the **convex hull** of  $S$ . One has  $S \subseteq \text{Conv}(S) \subseteq \text{Aff}(S)$ .

**Proposition 1.4.6** Let  $E$  be a vector space over  $\mathbb{R}$  and  $C \subseteq E$ . Then  $C$  is convex

if and only if

$$\forall(x, y) \in C^2, \forall\lambda \in [0, 1], \lambda x + (1 - \lambda)y \in C.$$

**Proof** It is sufficient to check “ $\Leftarrow$ ”. We prove by induction on  $n$  that

$$\forall n \in \mathbb{N}_{\geq 1}, \forall(x_1, \dots, x_n) \in C^n, \forall(a_1, \dots, a_n) \in \mathbb{R}_{\geq 0}^n, \sum_{i=1}^n a_i = 1, \sum_{i=1}^n a_i x_i \in C.$$

The case where  $n = 1$  is trivial. The case where  $n = 2$  comes from the hypothesis. Suppose  $n \geq 3$  in assuming that the statement holds for any integer less than  $n$ . If  $a_n = 1$ , then  $a_1 = \dots = a_{n-1} = 0$ , so  $\sum_{i=1}^n a_i x_i = x_n \in C$ . If  $a_n < 1$ , we have  $a_1 + \dots + a_{n-1} = 1 - a_n > 0$ . By the induction hypothesis,

$$x := \sum_{i=0}^{n-1} \frac{a_i}{1 - a_n} x_i \in C.$$

Taking  $y = x_n, t = 1 - a_n$ ,

$$C \ni tx + (1 - t)y = \sum_{i=1}^n a_i x_i.$$

□

## 1.5 Mean Value Theorems

**Theorem 1.5.1** (Mean Value Inequality) Let  $(F, \|\cdot\|_F)$  be normed vector spaces over  $\mathbb{R}$ . Let  $(a, b) \in \mathbb{R}^2$  such that  $a < b$ . Let  $f : [a, b] \rightarrow F$  be a continuous mapping that is differentiable on  $]a, b[$ . Then

$$\|f(b) - f(a)\|_F \leq (b - a) \cdot \sup_{t \in ]a, b[} \|f'(t)\|_F.$$

**Proof** We may suppose that  $\sup_{t \in ]a, b[} \|f'(t)\|_F < +\infty$ . Take

$$M > \sup_{t \in ]a, b[} \|f'(t)\|_F.$$

Let  $m = \frac{a+b}{2}$ . Let

$$J = \{x \in [m, b] \mid \forall t \in [m, x], \|f(t) - f(m)\|_F \leq M(t - m)\}.$$

It is an interval containing  $m$ . So it is of the form

$$[m, c[ \text{ or } [m, c]$$

$$\forall t \in [m, c[, \|f(t) - f(m)\|_F \leq M(t - m).$$

Taking the limit  $t < c, t \rightarrow c$ , we get  $c \in J$ . So  $J = [m, c]$ . We then check  $c = b$ .

If  $c \neq b$ , then  $c \in ]a, b[$ , so  $f$  is differentiable at  $c$ . That is

$$\|f(c + h) - f(c)\|_F = \|f'(c)h + o(\|h\|)\|_F \leq \|f'(c)\|_F h + o(\|h\|), h \rightarrow 0.$$

Since  $M > \|f'(c)\|_F$ ,  $\exists h_0 > 0$  such that

$$\forall h \in ]0, h_0], \|f(c + h) - f(c)\|_F \leq Mh.$$

$$\begin{aligned} \|f(c + h) - f(m)\| &\leq \|f(c + h) - f(c)\| + \|f(c) - f(m)\| \\ &\leq Mh + M(c - m) = M(c + h - m). \end{aligned}$$

So  $[m, c + h_0] \subseteq J$ , contradiction. Thus  $b = c$ .  $\|f(b) - f(m)\|_F \leq M(b - m)$ .

By the same reason,  $\|f(m) - f(a)\|_F \leq M(m - a)$ . So

$$\|f(b) - f(a)\|_F \leq \|f(b) - f(m)\|_F + \|f(m) - f(a)\|_F \leq M(b - a).$$

Taking the limit when  $M \rightarrow \sup_{t \in ]a, b[} \|f'(t)\|_F$ , we get the announced result.  $\square$

**Corollary 1.5.2** Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces over  $\mathbb{R}$ .  $U \subseteq E$  be an open subset, and  $(x, y) \in U^2$  such that

$$[x, y] = \{tx + (1 - t)y \mid t \in [0, 1]\} \subseteq U.$$

Let  $f : U \rightarrow F$  be a differentiable mapping. Then

$$\|f(x) - f(y)\|_F \leq \left( \sup_{z \in ]x, y[} \|\mathrm{D}f(z)\| \right) \cdot \|x - y\|_E.$$

**Proof** Let

$$\begin{aligned} g : [0, 1] &\longrightarrow U \\ t &\longmapsto tx + (1 - t)y. \end{aligned}$$

$$g(0) = x, g(1) = y, g'(t) = x - y.$$

Then,

$$(f \circ g)'(t) = Df(g(t))(x - y),$$

$$D(f \circ g)(t)(1) = Df(g(t))(Dg(t)(1)).$$

By the theorem,

$$\begin{aligned} \|f(x) - f(y)\|_F &= \|f(g(1)) - f(g(0))\|_F \\ &\leq \sup_{t \in ]0,1[} \|Df(g(t))(x - y)\|_F \\ &\leq \sup_{t \in ]0,1[} |Df(g(t))| \cdot \|x - y\|_E \\ &= \sup_{z \in [x,y]} \|Df(z)\| \cdot \|x - y\|_E. \end{aligned}$$

□

**Definition 1.5.3** Let  $(X, \mathcal{T})$  be a topological space,  $p \in X$ . Let  $U$  be a neighborhood of  $p$  and  $f : U \rightarrow \mathbb{R}$  be a mapping. If there exists a neighborhood  $V$  of  $p$  such that  $p \in V \subseteq U$  and

$$\forall x \in V, f(p) \geq f(x),$$

we say that  $p$  is a **local maximum point** of  $f$  on  $U$ .

If  $p$  is a local maximum point or a local minimum point, we say that  $p$  is a **local extremum** of  $f$  on  $U$ .

If  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  are normed vector spaces.  $U \subseteq E$  open,  $f : U \rightarrow F$  is differentiable. If  $p \in U$  is such that

$$Df(p) = 0 \in \mathcal{L}(E, F),$$

we say that  $p$  is a **critical point** of  $f$ .

**Theorem 1.5.4** Let  $(E, \|\cdot\|)$  be a normed vector space over  $\mathbb{R}$ .  $U \subseteq E$  be an open subset,  $f : U \rightarrow \mathbb{R}$  be a differentiable mapping. If  $p \in U$  is a local extremum point of  $f$ , then it is a critical point ( $Df(p) = 0$ ).

**Proof** There exists  $r > 0$  such that  $p + B(0, r) \subseteq U$  and

$$(h \in B(0, r)) \mapsto f(p + h) - f(p) \in \mathbb{R}$$

does not change the sign.

$\forall h \in B(0, r), \forall \in [0, 1],$

$$(f(p + th) - f(p))(f(p - th) - f(p)) \geq 0.$$

Taking the limit when  $t \rightarrow 0, -Df(p)(h)^2 \geq 0$ . So  $Df(p)(h) = 0$ .  $\square$

**Theorem 1.5.5** (Rolle) Let  $(a, b) \in \mathbb{R}^2, a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping that is differentiable on  $]a, b[$ . If  $f(a) = f(b)$ , then

$$\exists t \in ]a, b[, f'(t) = 0.$$

**Proof** If there exists  $t$  which is in  $]a, b[$  and is an extremum point of  $f$ , then  $f'(t) = 0$ . Since  $[a, b]$  is compact and  $f$  is continuous, so  $f$  attains its maximum and minimum.

If the extremum points of  $f$  are in  $\{a, b\}$ . Since  $f(a) = f(b)$ ,  $f$  is compact, so  $f'(t) = 0$  on  $]a, b[$ .  $\square$

**Theorem 1.5.6** (Gronwall inequality) Let  $(F, \|\cdot\|)$  be a normed vector space over  $\mathbb{R}$ ,  $(a, b) \in \mathbb{R}^2, a < b$ . Let  $f : [a, b] \rightarrow F$  and  $g : [a, b] \rightarrow \mathbb{R}$  be differentiable mappings on  $]a, b[$ . If  $\forall t \in ]a, b[, \|f'(t)\| \leq g'(t)$ , then

$$\|f(b) - f(a)\|_F \leq g(b) - g(a).$$

**Proof** Let  $m \in ]a, b[$ . Let  $\varepsilon > 0$ ,

$$J := \{t \in [m, b] \mid \forall s \in [m, t], \|f(s) - f(m)\|_F \leq g(s) - g(m) + \varepsilon(s - m)\}.$$

Since  $f$  and  $g$  are continuous,  $J$  is a closed interval of the form  $[m, c]$ .

If  $c < b$ ,

$$\begin{aligned} f(c + h) &= f(c) + hf'(c) + o(h), \\ g(c + h) &= g(c) + hg'(c) + o(h), \quad h > 0, h \rightarrow 0. \end{aligned}$$

$\exists \delta > 0$ , such that  $[c, c + \delta] \subseteq [c, b]$  and  $\forall h \in [0, \delta]$ ,

$$\|f(c + h) - f(c)\| \leq h\|f'(c)\| + \frac{\varepsilon}{2}h.$$

$$g(c + h) - g(c) \geq hg'(c) - \frac{\varepsilon}{2}h.$$

So,

$$\|f(c + h) - f(c)\| \leq g(c + h) - g(c) + \varepsilon h.$$

By the triangle inequality,

$$\|f(c+h) - f(m)\| \leq g(c+h) - g(m) + \varepsilon(c+h-m).$$

So  $J \supseteq [m, c+\delta]$ , contradiction.

Therefore  $c = b$ .

$$\|f(b) - f(m)\| \leq g(b) - g(m) + \varepsilon(b-m).$$

A similar argument shows that

$$\|f(m) - f(a)\| \leq g(m) - g(a) + \varepsilon(m-a).$$

Hence,

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \varepsilon(b-a).$$

$$\|f(c+h) - f(c) + hf'(c)\| \leq \varphi(h)h, \lim_{h \rightarrow 0} \varphi(h) = 0.$$

$$\exists \delta > 0, \forall h > 0, 0 \leq h < \delta \Rightarrow |\varphi(h)| \leq \frac{\varepsilon}{2}.$$

□

**Theorem 1.5.7** (Mean value theorem of Lagrange) Let  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping that is differentiable on  $]a, b[$ . Then

$$\exists \xi \in ]a, b[, f(b) - f(a) = f'(\xi)(b-a).$$

**Proof** Let  $g : [a, b] \rightarrow \mathbb{R}$ .

$$g(t) := f(b) - f(t) + C(b-t), \text{ where } C = -\frac{f(b) - f(a)}{b-a}.$$

Then  $g(a) = g(b) = 0$ ,  $g'(t) = -f'(t) - C$ .

$$\exists \xi \in ]a, b[, g'(\xi) = 0, f'(\xi) = -C = \frac{f(b) - f(a)}{b-a}.$$

□

**Theorem 1.5.8** (Darboux) Let  $I$  be an open interval in  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a differentiable mapping. Then  $f'(I)$  is an interval.

**Proof** Let  $a, b$  be two elements in  $I$  such that  $a < b$ . Let

$$\begin{aligned} g : [a, b] &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \frac{f(t) - f(a)}{t - a}, & t \neq a \\ f'(a), & t = a \end{cases} \end{aligned}$$

$g$  is continuous, and  $g([a, b])$  is an interval. By the mean value theorem of Lagrange,  $g([a, b]) \subseteq f'(I)$ .

Let

$$\begin{aligned} h : [a, b] &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \frac{f(t) - f(b)}{t - b}, & t \neq b \\ f'(b), & t = b \end{cases} \end{aligned}$$

$h([a, b])$  is an interval contained in  $f'(I)$ .

$h([a, b]) \cup g([a, b])$  is an interval since

$$\frac{f(b) - f(a)}{b - a} \in h([a, b]) \cap g([a, b]),$$

$$\{f'(a), f'(b)\} \subseteq h([a, b]) \cup g([a, b]).$$

So the interval linking  $f'(a), f'(b)$  is contained in  $f'(I)$ . Hence,  $f'(I)$  is an interval.

□

## 1.6 Higher Differential

We fix a complete non-trivially valued field  $(K, |\cdot|)$ . Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces over  $K$ .

**Definition 1.6.1** Let  $U \subseteq E$  be an open subset,  $f : U \rightarrow F$  be a mapping,  $p \in U$ .

(1) If  $f$  is continuous at  $p$ , we say that  $f$  is 0-time differentiable at  $p$ , and we let

$$D^0 f(p) := f(p).$$

(2) If  $f$  is differentiable at  $p$ , we say that  $f$  is 1-time differentiable at  $p$ , and we let

$$D^1 f(p) := Df(p).$$

(3) Let  $n \geq 2$ . If exists open neighborhood  $V$  of  $p$  such that  $V \subseteq U$  and  $f$  is differentiable on  $V$  and  $Df$  is  $n - 1$ -time differentiable on  $V$ , we say that  $f$  is

$n$ -time differentiable at  $p$ , and we let

$$\mathrm{D}^n f(p) \in \mathcal{L}(E, \dots, E, F)$$

be the multilinear mapping sending  $(h_1, \dots, h_n) \in E^n$  to

$$\mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \dots, h_{n-1})(h_n).$$

If  $E = K$ ,  $\mathrm{D}^n f(p)(1, \dots, 1)$  is denoted as  $f^{(n)}(p) \in F$ .  $f^{(0)}(p)$  is often denoted as  $f(p)$ .

**Remark 1.6.2**  $\forall i \in \{1, \dots, n\}$ ,

$$\mathrm{D}^n f(p)(h_1, \dots, h_n) = \mathrm{D}^i(\mathrm{D}^{n-i} f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n).$$

If  $E = K$ ,

$$f^{(n)}(p)(h_1, \dots, h_n) = \mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \dots, h_{n-1})(h_n).$$

**Definition 1.6.3** Let  $X$  be a set, we denote by  $\mathfrak{S}_X$  the element of all bijection from  $X$  to  $X$ .  $(\mathfrak{S}_X, \circ)$  forms a group. The identity mapping  $\mathrm{Id}_X$  is the neutral element of  $(\mathfrak{S}_X, \circ)$ .  $(\mathfrak{S}_X, \circ)$  is called the symmetric group of  $X$ . The elements of  $(\mathfrak{S}_X, \circ)$  are called permutations of  $X$ .

Let  $n \in \mathbb{N}_{\geq 2}$ ,  $x_1, \dots, x_n$  be distinct elements of  $X$ . We denote by  $(x_1 x_2 \cdots x_n)$  the element of  $\mathfrak{S}_X$  that sends  $x_i$  to  $x_{i+1}$ ,  $(i \in \{1, \dots, n-1\})$ ,  $x_n$  to  $x_1$ ,  $y \in X \setminus \{x_1, \dots, x_n\}$  to  $y$  itself. This element is called an  $n$ -cycle. A 2-cycle is also called a transposition.

**Remark 1.6.4**  $\mathfrak{S}_X$  acts on  $X$ .

$$\begin{aligned} \mathfrak{S}_X \times X &\longrightarrow X \\ (\sigma, x) &\longmapsto \sigma(x). \end{aligned}$$

If  $\sigma \in \mathfrak{S}_X$ ,  $x \in X$ , we denote by  $\mathrm{orb}_\sigma(x)$  the set  $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$ .

$$\langle \sigma \rangle := \{\sigma^n \mid n \in \mathbb{Z}\} \subseteq \mathfrak{S}_X$$

is a group.  $\mathrm{orb}_\sigma(x)$  is the orbit of  $x$  under the action of  $\langle \sigma \rangle$ .

**Proposition 1.6.5** If  $\text{orb}_\sigma(x)$  is finite of  $d$  elements, then  $\sigma^d(x) = x$ , and  $\text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{d-1}(x)\}$ . Moreover, the restriction of  $\sigma$  to  $\text{orb}_\sigma(x)$  identifies to the restriction of the cycle  $(x, \sigma(x), \dots, \sigma^{d-1}(x))$ .

**Proof** Since  $\text{orb}_\sigma(x)$  is finite,

$$\{(n, m) \in \mathbb{Z}^2 \mid n < m, \sigma^n(x) = \sigma^m(x)\}$$

Let

$$l = \min\{m - n \mid (n, m) \in \mathbb{Z}^2, n < m, \sigma^n(x) = \sigma^m(x)\}.$$

Then  $x, \sigma(x), \dots, \sigma^{l-1}(x)$  are distinct, and  $\sigma^l(x) = x$ .  $\forall n \in \mathbb{Z}$ , then  $n$  can be written as  $n = lp + r$ , where  $p \in \mathbb{Z}, r \in \{0, \dots, l-1\}$ .

$$\sigma^n(x) = \sigma^r(\sigma^{lp}(x)) = \sigma^r((\sigma^l \circ \dots \circ \sigma^l)(x)) = \sigma^r(x).$$

So,  $\text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{l-1}(x)\}$ , ( $l = d$ ). □

**Remark 1.6.6** If  $X$  is finite, then  $X$  can be written as a distinct union of orbits (under the action of  $\langle \sigma \rangle$ ). Let  $d_i = \#(\text{orb}_\sigma(x_i)), i = 1, \dots, n$ , then

$$\sigma|_{\text{orb}_\sigma(x^{(i)})} = (x^{(i)}, \sigma(x^{(i)}), \dots, \sigma^{d_i-1}(x^{(i)}))|_{\text{orb}_\sigma(x^{(i)})}.$$

So  $\sigma = \tau_1 \circ \dots \circ \tau_n$ , where  $\tau_i := (x^{(i)}, \sigma(x^{(i)}), \dots, \sigma^{d_i-1}(x^{(i)}))$ .

**Corollary 1.6.7** Suppose that  $X$  is finite. Any  $\sigma \in \mathfrak{S}_X$  can be written as a composition of transpositions.

**Proof**

$$(x_1 \dots x_n) = (x_1 x_2) \circ (x_2 \dots x_n),$$

So,

$$(x_1 \dots x_n) = (x_1 x_2) \circ (x_2 x_3) \circ \dots \circ (x_{n-1} x_n). □$$

**Definition 1.6.8** Denote by  $\mathfrak{S}_n$  the symmetric group  $\mathfrak{S}_{\{1, \dots, n\}}$ . A composition of the form  $(i \ i+1)$ ,  $i \in \{1, \dots, n-1\}$  is called an adjacent transposition.

**Corollary 1.6.9** Any  $\sigma \in \mathfrak{S}_n$  can be written as a composition of adjacent transpositions.

**Proof** Let  $(j, k) \in \{1, \dots, n\}^2$ ,  $j < k$ ,

$$(j-1 \ j) \circ (j \ k) \circ (j-1 \ j) = (j-1 \ k).$$

$$(j \ k) = (j \ j+1) \circ (j+1 \ j+2) \circ \dots \circ (k-1 \ k) \circ \dots (j \ j+1).$$

□

**Theorem 1.6.10** (Schwarz) Let  $U \subseteq E$  be an open subset,  $f : U \rightarrow F$  be a mapping.  $n \in \mathbb{N}_{\geq 1}$ ,  $p \in U$ . Assume that  $f$  is  $n$ -times differentiable at  $p$ . Then  $\forall \sigma \in \mathfrak{S}_n, \forall (h_1, \dots, h_n) \in E^n$ ,

$$D^n f(p)(h_1, \dots, h_n) = D^n f(p)(h_{\sigma(1)}, \dots, h_{\sigma(n)}).$$

**Proof (By induction)** The case where  $n = 1$  is trivial. Case  $n = 2$ : Exists  $V$  open,  $p \in V \subseteq U$ .  $f$  is differentiable on  $V$  and  $Df$  is differentiable at  $p$ .

$$Df(p+h)(\cdot) - Df(p)(\cdot) - D^2 f(p)(h, \cdot) = o(\|h\|_E).$$

Let  $\varepsilon > 0, \exists \delta > 0, \forall h \in E, \|h\|_E \leq 2\delta \Rightarrow p + h \in V$  and

$$\|Df(p+h)(\cdot) - Df(p)(\cdot) - D^2 f(p)(h, \cdot)\| \leq \varepsilon \|h\|_E.$$

Let  $h \in E$  such that  $\|h\|_E \leq \delta$ . Define  $g_h : B(0, \delta) \rightarrow F$  as

$$g_h(k) = f(p+h+k) - f(p+h) - f(p+k) + f(p) - D^2 f(p)(h, k).$$

Then,

$$\begin{aligned} Dg_h(k)(\cdot) &= Df(p+h+k)(\cdot) - Df(p+k)(\cdot) - D^2 f(p)(h, \cdot) \\ &= Df(p+h+k)(\cdot) - Df(p)(\cdot) - D^2 f(p)(h+k, \cdot) \\ &\quad - (Df(p+k)(\cdot) - Df(p)(\cdot) - D^2 f(p)(k, \cdot)) \end{aligned}$$

$$\|Dg_h(k)(\cdot)\| \leq \varepsilon \|h+k\|_E + \varepsilon \|k\|_E \leq 3\varepsilon \max\{\|k\|_E, \|h\|_E\}.$$

$g_h(0) = 0$ . Therefore,  $\|g_h(k)\| \leq 3\varepsilon \max\{\|k\|_E, \|h\|_E\}^2$  (mean value inequality).

$$\|g_h(k) - g_h(0)\| \leq \left( \sup_{t \in ]0,1[} \|Dg_h(tk)\| \right) \cdot \|k\|.$$

Therefore,

$$f(p+h+k) - f(p+h) - f(p+k) + f(p) - D^2 f(p)(h, k) = o(\max\{\|k\|_E, \|h\|_E\}^2).$$

By symmetry,

$$f(p+h+k) - f(p+h) - f(p+k) + f(p) - D^2 f(p)(k, h) = o(\max\{\|k\|_E, \|h\|_E\}^2).$$

$$D^2 f(p)(h, k) - D^2 f(p)(k, h) = o(\max\{\|h\|_E, \|k\|_h\}^2).$$

$$D^2 f(p)(th, tk) - D^2 f(p)(tk, th) = o(|t|^2), \quad t \rightarrow 0.$$

$$D^2 f(p)(h, k) - D^2 f(p)(k, h) = o(1), \quad t \rightarrow 0.$$

Suppose  $n \geq 3$ .

$$D^n f(p)(h_1, \dots, h_n) = D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n).$$

If  $\sigma = (j \ j+1)$ ,  $j \leq 2$ ,

$$D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n) = D^{n-1}(Df)(p)(h_{\sigma(1)}, \dots, h_{\sigma(n-1)})(h_n)$$

by the induction hypothesis, if  $\sigma = (n-1 \ n)$ ,

$$D^n f(p)(h_1, \dots, h_n) = D^2 \left( (D^{n-2} f)(h_1, \dots, h_{n-2})(h_{n-1}, h_n) \right)$$

$$\begin{aligned} D^n f(p)(h_{\sigma(1)}, \dots, h_{\sigma(n)}) &= D^n f(p)(h_1, \dots, h_{n-1}) \\ &= D^2 \left( (D^{n-2} f)(h_1, \dots, h_{n-2})(h_n, h_{n-1}) \right) \\ &= D^n f(p)(h_1, \dots, h_n). \end{aligned}$$

□

## 1.7 Taylor's Formula

**Theorem 1.7.1** (Toylor-Young) Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces over  $\mathbb{R}$ ,  $U \subseteq E$  open,  $n \in \mathbb{N}$ ,  $f : U \rightarrow F$  be a mapping,  $p \in U$ . Suppose that  $f$  is  $n$ -times differentiable at  $p$ . Then

$$f(x) = f(p) + \sum_{k=1}^n \frac{1}{k!} D^k f(p)(x - p, \dots, x - p) + o(\|x - p\|^n), \quad x \rightarrow p.$$

**Proof (By induction on  $n$ )**

$n = 0$ ,  $f(x) = f(p) + o(1)$  follows by continuity of  $f$ ;  $n = 1$  follows by the differentiability of  $f$ .

From  $n - 1$  to  $n$ . Let  $g : U \rightarrow F$

$$g(x) = f(x) - f(p) - \sum_{k=1}^n \frac{1}{k!} D^k f(p)(x - p, \dots, x - p).$$

$g$  is differentiable on an open neighborhood of  $p$ ,

$$Dg(x)(h) = Df(x)(h) - \sum_{k=1}^n \frac{1}{k!} k D^k f(p)(x - p, \dots, x - p, h)$$

$$Dg(x) = Df(x) - \sum_{l=0}^{n-1} \frac{1}{l!} D^l(Df)(x - p, \dots, x - p) \stackrel{\text{hyp.}}{=} o(\|x - p\|^{n-1}), \quad x \rightarrow p.$$

So  $g(x) = o(\|x - p\|^n)$ .

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(p, \delta), \|Dg(x)\| \leq \varepsilon \|x - p\|^{n-1}.$$

$g(p) = 0$ , so

$$\|g(x) - g(p)\| \leq \varepsilon \|x - p\|^{n-1} \cdot \|x - p\| = \varepsilon \|x - p\|^n.$$

□

**Theorem 1.7.2** (Taylor-Lagrange) Let  $(a, b) \in \mathbb{R}^2$ ,  $a < b$ .  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping. Suppose that  $f$  is  $(n + 1)$ -times differentiable on  $]a, b[$  and  $\forall k \in \{3, \dots, n\}, f^{(k)} : ]a, b[ \rightarrow \mathbb{R}$  tends to a continuous mapping  $[a, b] \rightarrow \mathbb{R}$ .

Then

$$\exists \xi \in ]a, b[, f(b) - \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) = \frac{f^{(n+1)}(\xi)(b-a)^{n+1}}{(n+1)!}.$$

**Proof** Let  $g : [a, b] \rightarrow \mathbb{R}$ .

$$g(t) := \sum_{k=0}^n \frac{(b-t)^k}{k!} f^{(k)}(t) - C \frac{(b-t)^{n+1}}{(n+1)!}.$$

$$\text{Then } g(b) = f(b), g(a) = \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) - C \frac{(b-a)^{n+1}}{(n+1)!}.$$

$$\begin{aligned} g'(t) &= \sum_{k=0}^n \frac{(b-t)^k}{k!} f^{(k+1)}(t) - \sum_{k=1}^n \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(t) + C \frac{(b-t)^n}{n!} \\ &= \frac{(b-t)^n}{n!} f^{(n+1)}(t) + C \frac{(b-t)^n}{n!}. \end{aligned}$$

Take  $C$  such that  $g(a) = g(b)$ . Then by Rolle's theorem,  $\exists \xi \in ]a, b[, g'(\xi) = 0$ ,  $C = -f^{(n+1)}(\xi)$ . Then,

$$g(a) = \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{f^{(n+1)}(\xi)}{(n+1)!} + \frac{f^{(n+1)}(\xi)}{(n+1)!} (b-a)^{n+1} = f(b) = g(b).$$

□

**Theorem 1.7.3** Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be two normed vector spaces over  $\mathbb{R}$ ,  $U \subseteq E$  be an open subset, and  $f : U \rightarrow F$  be a mapping that is  $(n+1)$ -times differentiable, where  $n \in \mathbb{N}$ . Let  $p \in U$ ,  $h \in E$  such that  $\forall t \in [0, 1], p + th \in U$ . Let

$$M = \sup_{t \in [0, 1]} \|D^{n+1}f(p + th)\|.$$

Then,

$$\|f(p+h) - \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h)\|_F \leq \frac{M}{(n+1)!} \|h\|_E^{n+1}.$$

**Proof** We define  $\phi : [0, 1] \rightarrow F$

$$\phi(t) = f(p + th) + \sum_{k=1}^n \frac{(1-t)^k}{k!} D^k f(p + th)(h, \dots, h).$$

$$\phi(0) = \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h), \quad \phi(1) = f(p + h).$$

$$\begin{aligned} \phi'(t) &= Df(p + h)(h) + \sum_{k=1}^n \frac{(1-t)^k}{k!} D^{k+1} f(p + th)(h, \dots, h) \\ &\quad - \sum_{k=1}^n \frac{(1-t)^{k-1}}{(k-1)!} D^k f(p + th)(h, \dots, h) \\ &= \sum_{k=0}^n \frac{(1-t)^k}{k!} D^{k+1} f(p + th)(h, \dots, h) \\ &\quad - \sum_{l=0}^{n-1} \frac{(1-t)^l}{(l)!} D^{l+1} f(p + th)(h, \dots, h) \\ &= \frac{(1-t)^n}{n!} D^{n+1} f(p + th)(h, \dots, h). \end{aligned}$$

So,

$$\|\phi'(t)\| \leq M \|h\|_E^{n+1} \frac{(1-t)^n}{n!}, \quad t \in [0, 1].$$

By Gronwall's inequality,

$$\|\phi(1) - \phi(0)\|_F \leq M \cdot \|h\|^{n+1} \frac{1}{(n+1)!}.$$

□

## 1.8 Banach Space

**Proposition 1.8.1** Let  $(X, d)$  be a metric space and  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . If

$$\sum_{n \in NN} d(x_n, x_{n+1}) < +\infty,$$

then  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence.

**Proof** Let  $N \in \mathbb{N}$ . If  $(n, m) \in \mathbb{N}^2$ ,  $n > m$ , by the triangle inequality,

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k \geq N} d(x_k, x_{k+1}).$$

So,

$$0 \leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \leq \sum_{k \geq N} d(x_k, x_{k+1}).$$

Taking the limit when  $N \rightarrow +\infty$ , we get

$$\lim_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0.$$

Let  $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ . If  $\left(\sum_{k=0}^n a_k\right)_{n \in \mathbb{N}}$  converges to some  $l$  in  $\mathbb{R}$ . Then,  $l - \sum_{k=0}^{N-1} a_k$  converges to 0. If  $a_k \leq 0$  for any  $k \in \mathbb{N}$ ,  $l - \sum_{k=0}^{N-1} a_k = \sum_{k=N}^{+\infty} a_k$ .

$$l - \sum_{k=0}^{N-1} a_k = \lim_{n \rightarrow +\infty} \left( \sum_{k=0}^n a_k - \sum_{k=0}^{N-1} a_k \right) = \lim_{n \rightarrow +\infty} \sum_{k=N}^n a_k.$$

□

**Definition 1.8.2** Let  $(K, |\cdot|)$  be a complete valued field and  $(E, \|\cdot\|)$  be a normed vector space over  $K$ . If  $E$  equipped with the metric

$$\begin{aligned} E \times E &\longrightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\longmapsto \|x - y\|_E. \end{aligned}$$

is complete, we say that  $(E, \|\cdot\|)$  is a **Banach space**.

Let  $(E, \|\cdot\|)$  be a Banach space. If  $(x_n)_{n \in \mathbb{N}}$  is a sequence in  $E$  such that  $\sum_{n \in \mathbb{N}} \|x_n\| < +\infty$ , we say that  $\sum_{n \in \mathbb{N}} x_n$  **converges absolutely**.

**Remark 1.8.3** Suppose that  $\sum_{n \in \mathbb{N}} x_n$  converges absolutely. Then  $\left(\sum_{k=0}^n x_k\right)_{n \in \mathbb{N}}$  is a Cauchy sequence, since

$$\|x_n\| = \left\| \sum_{k=0}^n x_k - \sum_{k=0}^{n-1} x_k \right\|.$$

So,  $\sum_{n \in \mathbb{N}} x_n$  converges.

**Theorem 1.8.4** (Root test of Cauchy) Let  $(E, \|\cdot\|)$  be a Banach space and  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ . Let

$$r = \limsup_{n \rightarrow \infty} \|x_n\|^{\frac{1}{n}} \in [0, +\infty]$$

If  $r < 1$ , then  $\sum_{n \in \mathbb{N}} x_n$  converges absolutely.

If  $r > 1$ , then  $\sum_{n \in \mathbb{N}} x_n$  diverges.

**Lemma 1.8.5** If a series  $\sum_{n \in \mathbb{N}} x_n$  converges, then  $\lim_{n \rightarrow +\infty} \|x_n\| = 0$ .

**Proof (of lemma)**

$$\|x_n\| = \left\| \sum_{k=0}^n x_k - \sum_{k=0}^{n-1} x_k \right\|.$$

Since  $\sum_k^n x_k$  converges to some  $l \in E$ .

$$\lim_{n \rightarrow +\infty} \|x_n\| = \lim_{n \rightarrow +\infty} \left\| \sum_{k=0}^n x_k - \sum_{k=0}^{n-1} x_k \right\| = \|l - l\| = 0.$$

□

**Proof (of theorem)** If  $r > 1$ ,  $\exists \beta > 1$  such that  $r > \beta$ . Since  $r = \limsup_{n \rightarrow +\infty} \|x_n\|^{\frac{1}{n}}$ ,  $\exists I \subseteq \mathbb{N}$  infinite such that  $\lim_{n \in I, n \rightarrow +\infty} \|x_n\|^{\frac{1}{n}} = r$  (Bolzano-Weierstrass).

$$\exists N \in \mathbb{N}, \forall n \in I_{\geq N}, \|x_n\|^{\frac{1}{n}} \geq \beta.$$

So,  $\|x_n\| \geq \beta^n \geq 1$ . So  $\sum_{n \in \mathbb{N}} x_n$  diverges.

If  $r < 1$ ,  $\exists \alpha \in ]0, 1[$ ,  $r < \alpha$ . Since  $r = \limsup_{n \rightarrow +\infty} \|x_n\|^{\frac{1}{n}}$ ,

$$\exists N \in \mathbb{N}, \forall n \geq N, \|x_n\|^{\frac{1}{n}} \leq \alpha, \|x_n\| \leq \alpha^n.$$

So,

$$\sum_{n \geq N} \|x_n\| \leq \sum n \geq N\alpha^n = \frac{\alpha^N}{1 - \alpha} < +\infty.$$

Therefore,  $\sum_{n \in \mathbb{N}} x_n$  converges absolutely. □

**Theorem 1.8.6** (Ratio test of D'Alembert) Let  $(E, \|\cdot\|)$  be a Banach space and  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ .

(1) If

$$\limsup_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < 1,$$

then  $\sum_{n \in \mathbb{N}} x_n$  converges absolutely.

(2) If

$$\limsup_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} > 1,$$

then  $\sum_{n \in \mathbb{N}} x_n$  diverges.

### Proof

(1) Let  $0 < \alpha < 1$  such that

$$\limsup_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < \alpha.$$

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}_N, \|x_{n+1}\| \alpha \|x_n\| \leq \alpha^{n+1-N} \|x_N\|.$$

Thus,

$$\sum_{n \geq N} \|x_n\| \leq \sum_{n \geq N} \|x_N\| \alpha^{n-N} = \|x_N\| \frac{1}{1-\alpha} < +\infty.$$

(2) Let  $\beta > 1$  such that

$$\liminf_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} > \beta.$$

$$\exists N \in \mathbb{N}, x_N \neq 0, \text{ and } \forall n \in \mathbb{N}_{\geq N}, \|x_{n+1}\| \geq \beta \|x_n\|$$

$$\forall n \geq N, \|x_n\| \geq \beta^{n-N} \|x_N\| \rightarrow +\infty (n \rightarrow +\infty)$$

So  $\sum_{n \in \mathbb{N}} x_n$  diverges. □

Let  $z \in \mathbb{C}$ . The series  $\sum_{n \in \mathbb{N}} \frac{z^n}{n!}$  converges absolutely since

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \rightarrow 0 (n \rightarrow +\infty).$$

We denote by  $e^z$  this limit.

## 1.9 Local inversion

**Definition 1.9.1** Let  $X$  be a topological space and  $Y \subseteq X$ . If  $\overline{Y} = X$ , we say that  $Y$  is dense.

**Theorem 1.9.2** (Baire) Let  $(X, d)$  be a complete metric space. Let  $(\Omega_n)_{n \in \mathbb{N}}$  be a sequence of dense open subset of  $X$ . Let  $\Omega = \bigcap_{n \in \mathbb{N}} \Omega_n$ , then  $\Omega$  is dense in  $X$ .

**Proof** Suppose that  $\Omega$  is not dense. Let  $x_0 \in X \setminus \overline{\Omega}$ , exists  $\varepsilon > 0$  such that  $B(x_0, \varepsilon) \subseteq X \setminus \overline{\Omega}$ .

Let  $r_0 = \varepsilon$ . We construct in a recursive way sequence  $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$  and  $(r_n)_{n \in \mathbb{N}} \in \mathbb{R}_{\geq 0}^n$  as follows.

Suppose that  $(x_n, r_n)$  is chosen.  $B(x_n, r_n) \cap \Omega_n \neq \emptyset$ . We pick  $x_{n+1} \in X$  and  $r_{n+1} \leq \frac{x_n}{2}$  such that  $B(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n) \cap \Omega_n$ ,  $d(x_{n+1}, x_n) < r_n$ .  $\sum_{n \in \mathbb{N}} r_n < +\infty$  (ratio test).

Then the sequence converges to some  $l$ . For any  $n \in \mathbb{N}$ ,  $x_n \in B(x_0, \varepsilon)$ . So  $l \in \overline{B}(x_0, \varepsilon)$ .

Moreover,  $\forall n \in \mathbb{N}$ ,  $l \in \overline{B}(x_{n+1}, r_{n+1}) \subseteq B_{x_n, r_n} \cap \Omega_n$ . Thus  $l \in \bigcap_{n \in \mathbb{N}} \Omega_n = \Omega$ . Contradiction.  $\square$

**Corollary 1.9.3** Let  $(X, d)$  be a non-empty complete metric space and  $(Y_n)_{n \in \mathbb{N}}$  be a family of closed subsets of  $X$  such that  $X = \bigcup_{n \in \mathbb{N}} Y_n$ . Then exists  $n \in \mathbb{N}$  such that  $Y_n^\circ \neq \emptyset$ .

**Proof** Let  $\Omega_n = X \setminus Y_n$ . Suppose that  $\forall n \in \mathbb{N}$ ,  $Y_n^\circ = \emptyset$ . Then  $\overline{\Omega}_n = X \setminus Y_n^\circ = X$ . Thus  $\Omega := \bigcap_{n \in \mathbb{N}} \Omega_n$  is dense in  $X$ . Namely,  $X = \Omega$ . So

$$\emptyset = X \setminus \overline{\Omega} = (X \setminus \Omega)^\circ = (X \setminus \bigcap_{n \in \mathbb{N}} \Omega_n)^\circ = \left( \bigcup_{n \in \mathbb{N}} Y_n \right)^\circ = X^\circ = X.$$

Contradiction.  $\square$

**Theorem 1.9.4** (Banach) Let  $(K, |\cdot|)$  be a complete non-trivially valued field, and  $E$  be a vector space over  $K$ . Let  $\|\cdot\|_1, \|\cdot\|_2$  be two norms on  $E$  such that  $(E, \|\cdot\|_1)$  and  $(E, \|\cdot\|_2)$  are both Banach spaces.

If  $\exists C > 0$  such that  $\|\cdot\|_2 \leq C\|\cdot\|_1$ . Then  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent. ( $\exists C' > 0$ ,  $\|\cdot\|_1 \leq C'\|\cdot\|_2$ )

**Proof** For  $x \in E$  and  $r > 0$ . Let

$$B_i(x, r) := \{y \in E \mid \|y - x\|_i < r\}, \quad i = 1, 2$$

$\forall y \subseteq E$ , let  $\overline{Y}^{\|\cdot\|_2}$  be the closure of  $Y$  in  $(E, \|\cdot\|_2)$ .

$$E = \bigcup_{n \geq 1} B_1(0, n) = \bigcup_{n \geq 1} \overline{B_1(0, n)}^{\|\cdot\|_2}.$$

Hence,  $\exists n_0 \geq 1, p \in E, r_0 > 0$  such that

$$B_2(p, r_0) \subseteq \overline{B_1(0, n_0)}^{\|\cdot\|_2}$$

or equivalently,

$$B_2(0, r_0) \subseteq \overline{B_1(-p, n_0)}^{\|\cdot\|_2} \subseteq \overline{B_1(0, n_0 + \|p\|_1)}^{\|\cdot\|_2}$$

since  $\forall x \in B_1(-p, n_0)$

$$\|x\|_1 = \|x - p + p\|_1 \leq \|x - p\| + \|p\|_1 < n_0 + \|p\|_1.$$

Let  $r_1 = n_0 + \|p\|_1$ ,

$$B_2(0, r_0) \subseteq \overline{B_1(0, r_1)}^{\|\cdot\|_2} \subseteq B_1(0, r_1) + B_2(0, r_0|a|)$$

where  $a \in K, 0 < |a| < \frac{1}{2}$ .

In fact,  $\forall x \in \overline{B_1(0, r_0)}^{\|\cdot\|_2}$ , exists sequence  $(x_n)_{n \in \mathbb{N}} \in B_1(0, r_1)^{\mathbb{N}}$ , such that  $x_n \rightarrow x$  in  $(E, \|\cdot\|_2)$ ,  $\exists n \in \mathbb{N}, \|x_n - x\|_2 < r_0|a|$

$$B_2(0, r_0|a|^n) \subseteq B_1(0, r_1|a|^n) + B_2(0, r_0|a|^{n+1})$$

Let  $y \in B_2(0, r_0)$ , we choose  $(x_0, y_0) \in B_1(0, r_1) \times B_2(0, r_0|a|)$  such that  $y = x_0 + y_0$ . When  $(x_n, y_n)$  si chosen, let  $(x_{n+1}, y_{[n+1]}) \in B_1(0, r_0|a|^{n+1}) \times B_2(0, r_0|a|^{n+2})$ ,  $y_n = x_{n+1} + y_{n+1}$ ,  $y = y_n + \sum_{k=0}^n x_k$ . So  $\sum_{n \in \mathbb{N}} x_n$  converges to  $y$ .

Moreover,  $\sum_{n \in \mathbb{N}} \|x_n\| < +\infty$ , so it converges in  $(E, \|\cdot\|_1)$  to some  $x$ . Therefore,  $x = y$  since  $\|\cdot\|_2 \leq C\|\cdot\|_1$ . So  $\|y\|_1\|x\|_1 \leq \sum_{n \in \mathbb{N}} \|x_n\|_1 \leq \frac{r_1}{1-|a|}$ .

Therefore  $B_2(0, r_0) \subseteq B_1(0, \frac{r_1}{1-|a|})$ . So  $\|\cdot\|_1$  is bounded by a constant times  $\|\cdot\|_2$ .  $\square$