QM HW9

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Problem 1
(1) The energy of three lowest-lying states are

$$\begin{aligned} E_0 &= \hbar\omega \\ E_1 &= \frac{3\hbar\omega}{2} \\ E_2 &= 2\hbar\omega. \end{aligned} \tag{1.1}$$

E_0 is not degenerate, while E_1 and E_2 have degeneracy of $2, 3$ respectively.
(2) E_0 is non-degenerate, $\langle 0|V|0\rangle = 0$, so $\Delta_0^{(1)} = 0$.

$$V = \delta m\omega^2 l^2 \frac{(a_1 + a_1^\dagger)(a_2 + a_2^\dagger)}{2}. \tag{1.2}$$

$$\frac{\delta m\omega^2 l^2}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \langle\psi|1_1\rangle \\ \langle\psi|1_2\rangle \end{pmatrix} = \Delta_1^{(1)} \begin{pmatrix} \langle\psi|1_1\rangle \\ \langle\psi|1_2\rangle \end{pmatrix}. \tag{1.3}$$

So,

$$\Delta_1^{(1)} = \pm\frac{\delta m\omega^2 l^2}{2}, \tag{1.4}$$

with respect to eigen-kets $\frac{|1,0\rangle + |0,1\rangle}{\sqrt{2}}$ and $\frac{|1,0\rangle - |0,1\rangle}{\sqrt{2}}$. Similarly,

$$\Delta_2^{(1)} = \pm\frac{\delta m\omega^2 l^2}{\sqrt{2}} \text{ or } 0. \tag{1.5}$$

(3)

Problem 2
Let

$$l = \sqrt{\frac{\hbar}{m\omega}},\ a = \frac{1}{\sqrt{2}}\left(\frac{x}{l} + \frac{\mathrm{i}lp}{\hbar}\right),\ N = a^\dagger a. \tag{2.1}$$

Then,

$$H_0 = \hbar\omega\left(N + \frac{1}{2}\right),\ H' = \frac{\epsilon}{4}\hbar\omega\left((a^\dagger)^2 + a^2 + 2N + 1\right). \tag{2.2}$$

Perturbation:

$$\Delta_0^{(1)} = \langle 0|H'|0\rangle = \frac{\epsilon}{4}\hbar\omega. \tag{2.3}$$

$$\Delta_0^{(2)} = \sum_{k=1}^{+\infty} \frac{|\langle k| H' |0\rangle|^2}{E_0 - E_k} = -\frac{\epsilon^2}{16}\hbar\omega. \tag{2.4}$$

So,

$$\tilde{E_0} \approx \hbar\omega \left(\frac{1}{2} + \frac{\epsilon}{4} - \frac{\epsilon^2}{16}\right). \tag{2.5}$$

$$|\tilde{0}^{(1)}\rangle = \sum_{k=1}^{+\infty} \frac{|k\rangle \langle k| H' |0\rangle}{E_0 - E_k} = -\frac{\epsilon\sqrt{2}}{8} |2\rangle. \tag{2.6}$$

Thus,

$$\tilde{\psi}(x) \approx \langle x| \left(|0\rangle - \frac{\epsilon\sqrt{2}}{8} |2\rangle\right) \sim \mathrm{e}^{-\frac{x^2}{2l^2}} \left(H_0\left(\frac{x}{l}\right) - \frac{\epsilon\sqrt{2}}{8} H_2\left(\frac{x}{l}\right)\right) \tag{2.7}$$

Exact:

Let

$$m' = m\sqrt{1+\epsilon},\ l' = \sqrt{\frac{\hbar}{m'\omega}},\ a' = \frac{1}{\sqrt{2}}\left(\frac{x}{l'} + \frac{\mathrm{i}l'p}{\hbar}\right). \tag{2.8}$$

Then,

$$H = \sqrt{1+\epsilon}\hbar\omega\left(N' + \frac{1}{2}\right). \tag{2.9}$$

$$\tilde{E_0} = \frac{1}{2}\sqrt{1+\epsilon}\hbar\omega \approx \hbar\omega\left(\frac{1}{2} + \frac{\epsilon}{4} - \frac{\epsilon^2}{16}\right). \tag{2.10}$$

$$\tilde{\psi}(x) \sim \mathrm{e}^{-\frac{x^2}{2l^2}} \left[1 - \frac{\epsilon}{8} - \frac{m\omega}{4\hbar}\epsilon x^2\right]. \tag{2.11}$$

Problem 3

(1)

$$V = \lambda r^2 \cdot 2\sqrt{\frac{2\pi}{15}}\left(Y_2^2 + Y_2^{-2}\right). \tag{3.1}$$