QM HW7

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Problem 1 (Hydrogen atom wavefunction)
(1)

$$\psi_{n_r,l,m} = R_{n_r,l}(r) Y_l^m(\theta,\phi), \tag{1.1}$$

where, Y_l^m is the spherical harmonics and

$$R_{n_R,l}(r) \sim \rho^l \mathrm{e}^{-\frac{\rho}{2}} L_{n_r}^{2l+1}(\rho), \tag{1.2}$$

$$\rho = \frac{2r}{na_0},\ a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2}\text{is a constant.} \tag{1.3}$$

$$E_n = \frac{E_0}{n^2}. \tag{1.4}$$

(2)

$$\psi_{100}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0} \tag{1.5}$$

$$\psi_{200}(r,\theta,\phi) = \frac{1}{\sqrt{32\pi a_0^3}} \left(2 - \frac{r}{a_0}\right) e^{-r/(2a_0)} \tag{1.6}$$

$$\psi_{210}(r,\theta,\phi) = \frac{1}{\sqrt{32\pi a_0^3}} \frac{r}{a_0} e^{-r/(2a_0)} \cos\theta \tag{1.7}$$

$$\psi_{21\pm1}(r,\theta,\phi) = \frac{1}{\sqrt{64\pi a_0^3}} \frac{r}{a_0} e^{-r/(2a_0)} \sin\theta\, e^{\pm i\phi} \tag{1.8}$$

$$\psi_{300}(r,\theta,\phi) = \frac{1}{81\sqrt{3\pi a_0^3}} \left(27 - 18\frac{r}{a_0} + 2\frac{r^2}{a_0^2}\right) e^{-r/(3a_0)} \tag{1.9}$$

$$\psi_{310}(r,\theta,\phi) = \frac{\sqrt{2}}{81\sqrt{\pi a_0^3}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/(3a_0)} \cos\theta \tag{1.10}$$

$$\psi_{31\pm1}(r,\theta,\phi) = \frac{1}{81\sqrt{\pi a_0^3}} \left(6 - \frac{r}{a_0}\right) \frac{r}{a_0} e^{-r/(3a_0)} \sin\theta\, e^{\pm i\phi} \tag{1.11}$$

$$\psi_{320}(r,\theta,\phi) = \frac{1}{81\sqrt{6\pi a_0^3}} \frac{r^2}{a_0^2} e^{-r/(3a_0)} (3\cos^2\theta - 1) \tag{1.12}$$

$$\psi_{32\pm1}(r,\theta,\phi) = \frac{1}{81\sqrt{\pi a_0^3}}\frac{r^2}{a_0^2}e^{-r/(3a_0)}\sin\theta\cos\theta\, e^{\pm i\phi} \tag{1.13}$$

$$\psi_{32\pm2}(r,\theta,\phi) = \frac{1}{162\sqrt{\pi a_0^3}}\frac{r^2}{a_0^2}e^{-r/(3a_0)}\sin^2\theta\, e^{\pm 2i\phi} \tag{1.14}$$

When $l > 0$, r^l tend to 0 with r tend to 0. When $l = 0$, value at $r = 0$ is not 0 with a non-zero derivative. So cusps appear when $l = 0$.

Problem 2 (Gaussian orbital approximation)
(1)

$$\psi_{1s}(\mathbf{r}) = \frac{1}{\sqrt{\pi a^3}}\mathrm{e}^{-\frac{r}{a}},\ \psi_{1s}^G(\mathbf{r}) = \sqrt{\frac{2\sqrt{2}}{\pi^{\frac{3}{2}}\lambda^3}}\mathrm{e}^{-\frac{r^2}{\lambda^2}}. \tag{2.1}$$

$$\begin{aligned}\mathrm{Err}(\lambda) &= 4\pi\int_0^{+\infty}\left|\frac{1}{\sqrt{\pi a^3}}\mathrm{e}^{-\frac{r}{a}} - \sqrt{\frac{2\sqrt{2}}{\pi^{\frac{3}{2}}\lambda^3}}\mathrm{e}^{-\frac{r^2}{\lambda^2}}\right|^2 r^2\,\mathrm{d}r \\ &= 2 - \sqrt{\frac{128\sqrt{2}}{\sqrt{\pi}a^3\lambda^3}}\mathrm{e}^{\left(\frac{\lambda}{2a}\right)^2}\int_0^{+\infty}\mathrm{e}^{-\left(\frac{r}{\lambda}+\frac{\lambda}{2a}\right)^2}r^2\,\mathrm{d}r\,.\end{aligned} \tag{2.2}$$

Let $\lambda = ka$, then

$$\mathrm{Err} = 2 - 8\sqrt{2}\left(\frac{2}{\pi}\right)^{1/4}k^{\frac{3}{2}}\mathrm{e}^{\left(\frac{k}{2}\right)^2}\int_0^{+\infty}\mathrm{e}^{-\left(x+\frac{k}{2}\right)^2}x^2\,\mathrm{d}x\,. \tag{2.3}$$

We require

$$\frac{\partial\ln\mathrm{Err}}{\partial k} = 0, \tag{2.4}$$

$$\frac{3}{2k} + \frac{k}{2} - \frac{\int_0^{+\infty}\left(x+\frac{k}{2}\right)\mathrm{e}^{-\left(x+\frac{k}{2}\right)^2}x^2\,\mathrm{d}x}{\int_0^{+\infty}\mathrm{e}^{-\left(x+\frac{k}{2}\right)^2}x^2\,\mathrm{d}x} = 0. \tag{2.5}$$

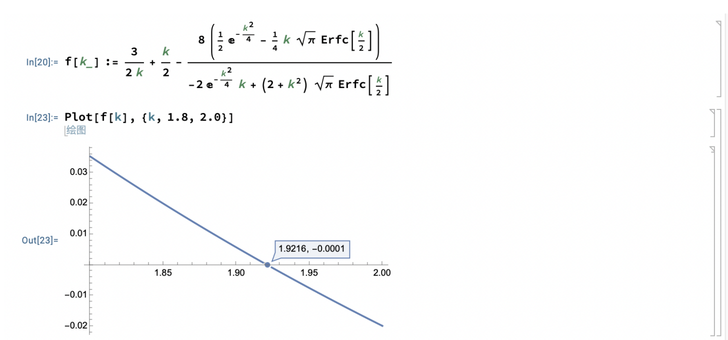

Figure 1: The relation between k and the LHS of the equation of 1s orbit.

We can get the value of k is about 1.92. (But AI said it is about 1.34, and $k = 1.92$ let $\mathrm{Err} < 0$.)

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(2)

$$\psi_{2p_z} = \frac{1}{\sqrt{32\pi a^5}} r \cos\theta \mathrm{e}^{-\frac{r}{2a}}. \tag{2.6}$$

$$\psi_{2p_z}^G = \sqrt{\frac{8\sqrt{2}}{\pi^{\frac{3}{2}}\lambda^5}} r \cos\theta \mathrm{e}^{-\frac{r^2}{\lambda^2}}. \tag{2.7}$$

The exponent part is similar to (1). So we want to find the extreme point of

$$\left(\frac{k}{4}\right)^2 + \frac{5}{2}\ln k + \ln \int_0^{+\infty} \mathrm{e}^{-(x+k/4)^2} x^4 \,\mathrm{d}x\,. \tag{2.8}$$

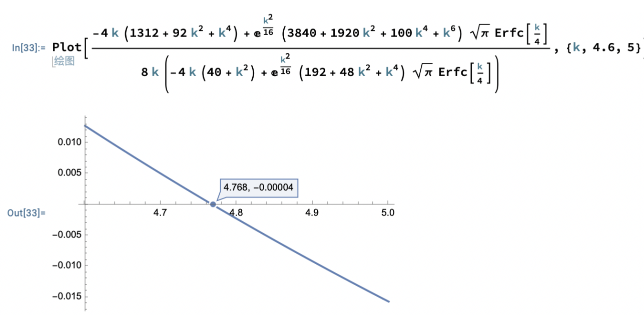
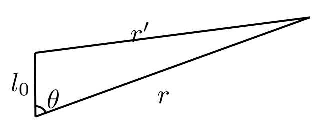


Figure 2: The relation between k and the LHS of the equation of 2p orbit.

$k \approx 4.77$

(3)



$$r'^2 = r^2 - 2rl_0\cos\theta + l_0^2. \tag{2.9}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}\, \psi_{1s}^{G*}(\mathbf{r})\psi_{1s}^G(\mathbf{r} - l_0\hat{\mathbf{z}}) \\ =&\frac{4\sqrt{2}}{\sqrt{\pi}\lambda^3}\int_{(r,\theta)\in[0,+\infty]\times[0,\pi]} \exp\left[-\frac{2r^2 - 2rl_0\cos\theta + l_0^2}{\lambda^2}\right] r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta \\ =&\frac{4\sqrt{2}}{\sqrt{\pi}\lambda^3}\int_0^{+\infty} \left[\mathrm{e}^{-2(r-l_0/2)^2/\lambda^2} - \mathrm{e}^{-2(r+l_0/2)^2/\lambda^2}\right] \frac{\lambda^2 r}{2l_0}\,\mathrm{d}r\,.\end{aligned} \tag{2.10}$$

(4)

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}\, \psi^{G*}_{2p_z}(\mathbf{r})\psi^{G}_{2p_z}(\mathbf{r}-l_0\hat{\mathbf{z}})\\ =&\frac{16\sqrt{2}}{\sqrt{\pi}\lambda^5}\int_{(r,\theta)\in[0,+\infty]\times[0,\pi]}\exp\left[-\frac{2r^2-2rl_0\cos\theta+l_0^2}{\lambda^2}\right]r^4\cos^2\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}r\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\int \mathrm{d}\mathbf{r}\, \psi^{G*}_{2p_z}(\mathbf{r})\psi^{G}_{2p_z}(\mathbf{r}-l_0\hat{\mathbf{x}})\\ =&\frac{8\sqrt{2}}{\pi^{\frac{3}{2}}\lambda^5}\int_{(r,\theta,\phi)\in D}\exp\left[-\frac{2r^2-2rl_0\sin\theta\cos\phi+l_0^2}{\lambda^2}\right]r^4\cos^2\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi\,.\end{aligned} \tag{2.12}$$

σ-bounding is stronger.

Problem 3 (2D hydrogen atom)
Let $\psi = R(r)\mathrm{e}^{\mathrm{i}n\phi}$, then the Schrödinger equation is

$$R'' + \frac{1}{\rho}R' - \frac{n^2}{\rho^2}R + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)R = 0, \tag{3.1}$$

where,

$$\kappa^2 = -\frac{2mE}{\hbar^2},\ \rho = 2\kappa r,\ \lambda = \frac{me^2}{\hbar^2\kappa}. \tag{3.2}$$

Considering the tendency at $\rho \to \infty$ and $\rho \to 0$, we have the form of R as

$$R(\rho) = \rho^{|n|}e^{-\frac{\rho}{2}}w(\rho). \tag{3.3}$$

Then, $w(\rho)$ satisfies confluent hypergeometric equation:

$$\rho w'' + (2|n| + 1 - \rho)\, w' + \left(\lambda - |n| - \frac{1}{2}\right) w = 0. \tag{3.4}$$

When

$$\lambda = n_r + |n| + \frac{1}{2},\ \text{with } n_r \text{ a natural number} \tag{3.5}$$

the solution is a polynomial. So

$$E_n = -\frac{me^4}{2\hbar^2\left(N + \frac{1}{2}\right)^2}. \tag{3.6}$$

where $N = n_r + |n|$. The degeneracy is $2N + 1$. In 3D case, the energy is related to a integer with power of -2, and has a degeneracy of n^2.

Problem 4 (Edge spectrum of the edge state of QHE)
Suppose there's no boundary, then the wave function of ground state is

$$\psi_{0,m} \sim w^m \mathrm{e}^{-\frac{|w|^2}{4l_B^2}}, \tag{4.1}$$

where $w = x + \mathrm{i}y$, and $l_B = \sqrt{\frac{\hbar}{eB}}$.

If there's a disk boundary, then $\psi(R)$ should be 0. $|\psi_{0,m}|$ take its maximum when $r = \sqrt{2m}l_B$. So the leading term of extra effect$\sim w^m$. This will lead to an energy$\sim \psi'' \sim m^2$. Hence, spectrum is about

$$\hbar\omega_c(n + 1/2) + km^2. \tag{4.2}$$

Where, k is a constant and $m \sim \left(\frac{R_{\mathrm{disk}}}{l_B}\right)^2$.

Problem 5 (Schwinger boson representation of angular momentum)
(1)

$$[J_\mu, J_\nu] = \frac{1}{4}\sigma^\mu_{\alpha\beta}\sigma^\nu_{\rho\lambda}[a^\dagger_\alpha a_\beta, a^\dagger_\rho a_\lambda]. \tag{5.1}$$

$$[a\alpha^\dagger a_\beta, a^\dagger_\rho a_\lambda] = a^\dagger_\alpha a_\lambda \delta_{\beta\rho} - a^\dagger_\beta a_\rho \delta_{\alpha\lambda}. \tag{5.2}$$

So,

$$[J_\mu, J_\nu] = \frac{1}{4}a^\dagger_\alpha a_\beta[\sigma^\mu, \sigma^\nu]_{\alpha\beta} = \mathrm{i}\epsilon_{\mu\nu\lambda}\frac{1}{2}a^\dagger_\alpha \sigma^\lambda_{\alpha\beta} a_\beta = \mathrm{i}\epsilon_{\mu\nu\lambda}J_\lambda. \tag{5.3}$$

(2)

$$\sigma^\mu_{\alpha\beta}\sigma^\mu_{\rho\lambda} = 2\delta_{\alpha\lambda}\delta_{\beta\rho} - \delta_{\alpha\beta}\delta_{\rho\lambda}. \tag{5.4}$$

Thus,

$$\begin{aligned} J^\mu J_\mu =& \frac{1}{2}a^\dagger_\alpha a^\beta a^\dagger_\beta a^\alpha - \frac{1}{4}a^\dagger_\alpha a^\alpha a^\dagger_\beta a^\beta \\ =& \frac{1}{2}a^\dagger_\alpha a^\beta(a^\alpha a^\dagger_\beta + [a^\alpha, a^\dagger_\beta]) - J^2 \\ =& \frac{1}{2}a^\dagger_\alpha a^\alpha(a^\dagger_\beta a^\beta + [a^\beta, a^\dagger_\beta]) - J - J^2 \\ =& J(J+1). \end{aligned} \tag{5.5}$$

$[a^\beta, a^\dagger_\beta] = 2$, $!^T\omega^T!$
(3)

$$J_z = \frac{1}{2}(a^\dagger_1 a_1 - a^\dagger_2 a_2). \tag{5.6}$$

$$a^\dagger_1 a_1 (a^\dagger_1)^{j+m} = (a^\dagger_1)^{j+m+1}a_1 + a^\dagger_1[a_1, (a^\dagger_1)^{j+m}] = (a^\dagger_1)^{j+m+1}a_1 + (j+m)(a^\dagger_1)^{j+m}. \tag{5.7}$$

Deal with a_2 similarly, we have

$$J_z|jm\rangle = \frac{(j+m) - (j-m)}{2}|jm\rangle = m|jm\rangle. \tag{5.8}$$

$$J_x = \frac{1}{2}(a^\dagger_1 a_2 + a^\dagger_2 a_1),\ \mathrm{i}J_y = \frac{1}{2}(a^\dagger_1 a_2 - a^\dagger_2 a_1). \tag{5.9}$$

So,

$$J_+ = a^\dagger_1 a_2,\ J_- = a^\dagger_2 a_1. \tag{5.10}$$

Thus,

$$\begin{aligned} J_+ |j,m\rangle &= \frac{(a_1^\dagger)^{j+(m+1)}}{\sqrt{j+(m+1)!}}\sqrt{j+m+1}\cdot\frac{(a_2^\dagger)^{j-(m+1)}}{\sqrt{(j-m+1)!}}\sqrt{j-m}\,|00\rangle \\ &= \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle\,. \end{aligned} \tag{5.11}$$

Similarly,

$$J_- |j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle\,. \tag{5.12}$$

Problem 6 (CG coefficients)
Since

$$(J_z - J_{1z} - J_{2z})\,|j_1 j_2; jm\rangle = 0, \tag{6.1}$$

we have CG coefficients vanish unless $m = m_1 + m_2$. By recursion relations:

$$\begin{aligned} &\sqrt{(j\mp m)(j\pm m+1)}\,\langle j_1 j_2; j, m\pm 1|j_1 j_2; m_1 m_2\rangle \\ =&\sqrt{(j_1\mp m_1+1)(j_1\pm m_1)}\,\langle j_1 j_2; j, m|j_1 j_2; m_1\mp 1, m_2\rangle \\ +&\sqrt{(j_2\mp m_2+1)(j_2\pm m_2)}\,\langle j_1 j_2; j, m|j_1 j_2; m_1, m_2\mp 1\rangle\,. \end{aligned} \tag{6.2}$$

We obtain

j, m	(m_1, m_2)	CG coefficients
$2, 2$	$(1, 1)$	1
$2, 1$	$(1, 0), (0, 1)$	$1/\sqrt{2}$
$2, 0$	$(1, -1), (0, 0), (-1, 1)$	$1/\sqrt{6},\ 2/\sqrt{6},\ 1/\sqrt{6}$
$2, -1$	$(0, -1), (-1, 0)$	$1/\sqrt{2}$
$2, -2$	$(-1, -1)$	1
$1, 1$	$(1, 0), (0, 1)$	$1/\sqrt{2},\ -1/\sqrt{2}$
$1, 0$	$(1, -1), (-1, 1)$	$1/\sqrt{2},\ -1/\sqrt{2}$
$1, -1$	$(0, -1), (-1, 0)$	$1/\sqrt{2},\ -1/\sqrt{2}$
$0, 0$	$(1, -1), (0, 0), (-1, 1)$	$1/\sqrt{3},\ -1/\sqrt{3},\ 1/\sqrt{3}$

Table 1: CG coefficients