FUNDAMENTAL ALGEBRA & ANALYSIS

Contents

Chapter 1

Topology

1.1 Topological spaces

Proposition 1.1.1 Let X be a set and for any $x \in X$, let $\mathcal{G}_x$ be a filter contained in the principal filter of $\{x\}$ $(\forall U \in \mathcal{G}_x,\ x \in U)$. Denote by $\mathscr{T}$ the set

$$\{U \in \wp(X) \mid \forall x \in U, U \in \mathcal{G}_x\}.$$

Then the following conditions are satisfied.
(1) $\{\varnothing, X\} \subseteq \mathscr{T}$.
(2) If $(U_1, U_2) \in \mathscr{T}^2$, then $U_1 \cap U_2 \in \mathscr{T}$.
(3) If I is a set and $(U_i)_{i\in I} \in \mathscr{T}^I$, then $\bigcup_{i\in I} U_i \in \mathscr{T}$.
Moreover, $\forall x \in X,\ \mathcal{B}_x = \{U \in \mathscr{T} \mid x \in U\}$ is a filter basis contained in $\mathcal{G}_x$. It generates $\mathcal{G}_x$ if the following condition is satisfied:

$$\forall U \in \mathcal{G}_x,\ \exists V \in \mathcal{G}_x,\ V \subseteq U \text{ and } \forall y \in V,\ V \in \mathcal{G}_y.$$

Proof
(1) $\varnothing \in \mathscr{T}, X \in \bigcap_{x\in X} \mathcal{G}_x$.
(2) $\forall x \in U_1 \cap U_2,\ U_1 \in \mathcal{G}_x,\ U_2 \in \mathcal{G}_x$, so $U_1 \cap U_2 \in \mathcal{G}_x$.
(3) Let $U = \bigcup_{i\in I} U_i$. $\forall x \in U,\ \exists i \in I,\ x \in U_i$, so $U_i \in \mathcal{G}_x$. Since $U \supseteq U_i$, so $U \in \mathcal{G}_x$.

$$\mathcal{B}_x := \{U \in \mathscr{T} \mid x \in U\}.$$

If $U \in \mathcal{B}_x$, then $x \in U$, so $U \in \mathcal{G}_x$. Hence $\mathcal{B}_x \subseteq \mathcal{G}_x$. If $(U, V) \in \mathcal{B}_x^2$, then $U \cap V \in \mathscr{T}$, and $x \in U \cap V$. So $U \cap V \in \mathcal{B}_x$. So $\mathcal{B}_x$ is a filter basis. Suppose the condition is satisfied. For any $U \in \mathcal{G}_x$, $\exists V \in \mathcal{G}_x \cap \mathscr{T}$, such that $V \subseteq U$. Note that $V \in \mathcal{B}_x$, so $\mathcal{G}_x$ is generated by $\mathcal{B}_x$. $\square$

Definition 1.1.2 Let X be a set. We call **topology** on X any subset $\mathscr{T}$ of $\wp(X)$ that satisfies the following conditions:
(1) $\{\varnothing, X\} \subseteq \mathscr{T}$.
(2) If $(U_1, U_2) \in \mathscr{T}^2$, then $U_1 \cap U_2 \in \mathscr{T}$.
(3) For any set I, $\forall\, (U_i)_{i\in I} \in \mathscr{T}^I$, then $\bigcup\limits_{i\in I} U_i \in \mathscr{T}$.
$(X, \mathscr{T})$ is called a **topological space**.
If $\forall x \in X,\ \mathcal{B}_x$ is a filter basis of X contained in the principal filter of $\{x\}$, then

$$\mathscr{T} = \{U \in \wp(X) \mid \forall x \in U,\ \exists V_x \in \mathcal{B}_x,\ V_x \subseteq U\}$$

is a topology on X, called the **topology generated by** $(\mathcal{B}_x)_{x\in X}$. More generally, if $\forall x \in X,\ S_x$ is a subset of the principal filter of $\{x\}$ and $\mathcal{G}_x$ is the filter generated by S_x, then we say that

$$\mathscr{T} = \{U \in \wp(X) \mid \forall x \in U,\ U \in \mathcal{G}_x\}$$

is the topology generated by $(S_x)_{x\in X}$.

Example 1.1.3
(1) Let $\mathcal{G}_x = \{X\}$. The topology by $(\mathcal{G}_x)_{x\in X}$ is $\{\varnothing, X\}$, called the **trivial topology** on X.
(2) Let $\mathcal{G}_x = \mathcal{F}_{\{x\}}$ be the principal filter. The topology generated by $(\mathcal{G}_x)_{x\in X}$ is $\wp(X)$. This topology is called the **discrete topology** on X.
(3) Let (X, d) be a semimetric space.

$$\mathrm{d} : X \times X \longrightarrow \mathbb{R}_{\geq 0},\ \mathrm{d}(x,y) = \mathrm{d}(y,x),\ \mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z),\ \mathrm{d}(x,x) = 0.$$

$\forall \varepsilon > 0,\ \forall x \in X$, let $B(x,\varepsilon) = \{y \in X \mid \mathrm{d}(x,y) < \varepsilon\}$, $\{B(x,\varepsilon) \mid \varepsilon \in \mathbb{R}_{>0}\} =: \mathcal{B}_x$ is a filter basis on X, $\mathcal{B}_x$ is contained in the principal filter of $\{x\}$. The topology

$$\mathscr{T} = \{U \in \wp(X) \mid \forall x \in U,\ \exists \varepsilon \in \mathbb{R}_{>0},\ B(x,\varepsilon) \subseteq U\}$$

is called the **topology induced by the semimetric** d.
(4) Let $(G, \leq)$ be a totally ordered set $\forall x \in G$, let $S_x = \{G_{>a} \mid a < x\} \cup \{G_{<b} \mid x < b\}$

Proposition 1.1.4 $\forall x \in X,\ \forall \varepsilon \in \mathbb{R}_{>0}, B(x,\varepsilon) \in \mathscr{T}$.

Proof $\forall y \in B(x,\varepsilon),\ \mathrm{d}(x,y) < \varepsilon$. Let $r = \varepsilon - \mathrm{d}(x,y) > 0$, we claim that $B(y,r) \subseteq B(x,\varepsilon)$. Let $z \in B(y,r),\ \mathrm{d}(y,z) < r$. Hence,

$$\mathrm{d}(x,z) \leq \mathrm{d}(x,y) + \mathrm{d}(y,z) < \mathrm{d}(x,y) + r = \mathrm{d}(x,y) + \varepsilon - \mathrm{d}(x,y) = \varepsilon.$$

□

Remark 1.1.5 On $\mathbb{R}$, one has a metric

$$\mathrm{d} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0},$$

$$(a,b) \longmapsto |a-b|.$$

$$B(x,\varepsilon) = \,]x-\varepsilon, x+\varepsilon[.$$

$$\mathcal{B}_x = \{]x-\varepsilon, x+\varepsilon[\ \mid \varepsilon \in \mathbb{R}_{>0}\}.$$

Let $\mathscr{T}_{\mathrm{d}}$ be the topology generated by $(\mathcal{B}_x)_{x\in\mathbb{R}}$. Let $\mathscr{T}$ be the order topology generated by $(S_x)_{x\in\mathbb{R}}$, where

$$S_x := \{\mathbb{R}_{>a} \mid a < x\} \cup \{\mathbb{R}_{<b} \mid x < b\}.$$

Proposition 1.1.6 For any $x \in \mathbb{R}$, $\mathcal{F}(\mathcal{B}_x) = \mathcal{F}(S_x)$.

Proof $\forall \varepsilon > 0,\ ]x-\varepsilon, x+\varepsilon[= \mathbb{R}_{<x+\varepsilon} \cap \mathbb{R}_{>x-\varepsilon} \in \mathcal{F}(S_x)$. So $\mathcal{F}(\mathcal{B}_x) \subseteq \mathcal{F}(S_x)$.

$\forall a \in \mathbb{R},\ a < x,\ \mathbb{R}_{>a} \supseteq \,]a, 2x-a[= \,]x-(x-a), x+(x-a)[,\ \mathbb{R}_{>a} \in \mathcal{F}(\mathcal{B}_x)$.

$$\forall b \in \mathbb{R}, b > x, \mathbb{R}_{<b} \supseteq \,]2x-b, b[= \,]x+(b-x), x+(b-x)[,$$

So, $\mathbb{R}_{<b} \subseteq \mathcal{F}(\mathcal{B}_x)$. Hence $S_x \subseteq \mathcal{F}(\mathcal{B}_x)$, which leads to $\mathcal{F}(S_x) \subseteq \mathcal{F}(\mathcal{B}_x)$. □

Definition 1.1.7 Let $(X, \mathscr{T})$ be a topological space. For any $x \in X$ and any $V \in \wp(X)$, if there exists $U \in \mathscr{T}$ such that $x \in U \subseteq V$, then we say that V is a **neighborhood of** x. We call **open subset** of X any subset of X that belongs to $\mathscr{T}$. If $U \in \mathscr{T}$, such that $x \in U$, we say that U is an **open neighborhood of** x. We denote by $\mathcal{V}_x(\mathscr{T})$ the set of all neighborhoods of x.

Proposition 1.1.8 $\mathcal{V}_x(\mathscr{T})$ is a filter on X contained in the principal filter of $\{x\}$. Moreover, the topology generated by $(\mathcal{V}_x(\mathscr{T}))_{x\in X}$ identifies with $\mathscr{T}$.

Proof
(1) If $(V_1, V_2) \in \mathcal{V}_x(\mathscr{T})^2$, $\exists (U_1, U_2) \in \mathscr{T}^2$, such that $x \in U_1 \subseteq V_1$, $x \in U_2 \subseteq V_2$. Hence, $x \in U_1 \cap U_2 \subseteq V_1 \cap V_2$, so $V_1 \cap V_2 \in \mathcal{V}_x(\mathscr{T})$.
(2) If $V \in \mathcal{V}_x(\mathscr{T})$, $W \in \wp(X)$, $V \subseteq W$. $\exists U \in \mathscr{T}$, $x \in U \subseteq V \subseteq W$, so $W \in \mathcal{V}_x(\mathscr{T})$. Let $\mathscr{T}'$ be the topology generated by $(\mathcal{V}_x(\mathscr{T}))_{x\in X}$. By definition,

$$\mathscr{T}' = \{U \subseteq X \mid \forall x \in U, U \in \mathcal{V}_x(\mathscr{T})\}.$$

For any $U \in \mathscr{T}$, $\forall x \in U$, U is a open neighborhood of x, so $U \in \mathscr{T}'$. Let $U \in \mathscr{T}'$, $\forall x \in U$, $\exists V_2 \in \mathscr{T}$, $x \in V_x \subseteq U$.

$$U = \bigcup_{x\in U} \{x\} \subseteq \bigcup_{x\in U} V_x \subseteq U.$$

$$U = \bigcup_{x\in U} V_x \in \mathscr{T}.$$

□

Proposition 1.1.9 Let X be a set, $(\mathscr{T}_i)_{i\in I}$ be a family of topologies on X. Then

$$\mathscr{T} = \bigcap_{i\in I} \mathscr{T}_i$$

is a topology on X.

Proof
(1) $\forall i \in I$, $\{\varnothing, X\} \subseteq \mathscr{T}_i$, so $\{\varnothing, X\} \subseteq \mathscr{T}$.
(2) If $(U_1, U_2) \in \mathscr{T}^2$, then for any $i \in I$, $U_1 \cap U_2 \in \mathscr{T}_i$, so $U_1 \cap U_2 \in \bigcap_{i\in I} \mathscr{T}_i$.
(3) For any set J and any $(U_j)_{j\in J} \in \mathscr{T}^J$, one has $\forall i \in I$, $\forall j \in J$, $U_j \in \mathscr{T}_i$, so

$$\bigcup_{j\in J} U_j \in \mathscr{T}_i.$$

Therefore,

$$\bigcup_{j\in J} U_j \in \bigcap_{i\in I} \mathscr{T}_i.$$

□

Definition 1.1.10 Let S be a subset of $\wp(X)$, we denote by $\mathscr{T}_S$ the intersection of all topologies containing S, we call it the topology generated by S.

Definition 1.1.11
Let $\mathcal{B}$ be a subset of $\wp(X)$, we say that $\mathcal{B}$ is a topological basis if:
(1) $X = \bigcup_{V \in \mathcal{B}} V$.
(2) $\forall (U,V) \in \mathcal{B} \times \mathcal{B},\ \forall x \in U \cap V,\ \exists W_x \in \mathcal{B},\ x \in W_x \subseteq U \cap V$.

Definition 1.1.12 Let X be a set and $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on X. If $\mathscr{T}_1 \subseteq \mathscr{T}_2$, we say that $\mathscr{T}_1$ is coarser than $\mathscr{T}_2$ and $\mathscr{T}_2$ is finer than $\mathscr{T}_1$.
If $S \subseteq \wp(X)$, we denote by $\mathscr{T}_S$ the intersection of all topology containing S. It is the coarsest topology containing S.
Let $\mathcal{B} \subseteq \wp(X)$. If $X = \bigcup_{V \in \mathcal{B}} V$ and $\forall (U,V) \in \mathcal{B} \times \mathcal{B},\ \forall x \in U \cap V,\ \exists W_x \in \mathcal{B}$, such that $x \in W_x \subseteq U \cap V$, we say that $\mathcal{B}$ is a **topological basis** on X.

Proposition 1.1.13 Let S be s subset of $\wp(X)$. Let

$$\mathcal{B}_S := \{X\} \cup \left\{ \bigcap_{i=1}^{n} A_i \mid n \in \mathbb{N}_{\geq 1},\ (A_1, \dots, A_n) \in S^n \right\},$$

then, $\mathcal{B}_S$ is a topological basis on X. Moreover, $\mathscr{T}_S = \mathscr{T}_{\mathcal{B}_S}$.

Proof Since $X \in \mathcal{B}_S$, $\bigcup_{V \in \mathcal{B}_S} V = X$. Let $(U,V) \in \mathcal{B}_S \times \mathcal{B}_S$. If $U = X$, then $U \cap V = V \in \mathcal{B}_S$. Similarly, if $V = X$, then $U \cap V = U \in \mathcal{B}_S$.
If $U = A_1 \cap \cdots \cap A_n$, $V = B_1 \cap \cdots \cap B_m$, then $\{A_1, \dots, A_n, B_1, \dots, B_m\} \subseteq \mathcal{B}_S$.

$$U \cap V = A_1 \cap \cdots \cap A_n \cap B_1 \cap \cdots \cap B_m \in \mathcal{B}_S.$$

Since $S \subseteq \mathcal{B}_S \subseteq \mathscr{T}_{\mathcal{B}_S}$, so $\mathscr{T}_S \subseteq \mathscr{T}_{\mathcal{B}_S}$, $X \in \mathscr{T}_S$. If $(A_1, \dots, A_n) \in S^n$, then $(A_1, \dots, A_n) \in \mathscr{T}_{\mathcal{B}_S}$. So $A_1 \cap \cdots \cap A_n \in \mathscr{T}_S$. Hence $\mathcal{B}_S \subseteq \mathscr{T}_S$. Therefore, $\mathscr{T}_{\mathcal{B}_S} \subseteq \mathscr{T}_S$, so $\mathscr{T}_{\mathcal{B}_S} = \mathscr{T}_S$. $\square$

Proposition 1.1.14 Let $\mathcal{B}$ be a topological basis on a set X. Then

$$\mathscr{T}_{\mathcal{B}} = \left\{ U \in \wp(X) \mid \exists \text{ a set } I \text{ and } (V_i)_{i \in I} \in \mathcal{B}^I, U = \bigcup_{i \in I} V_i \right\}.$$

Proof We denote by $\mathscr{T}$ te set

$$\{U \in \wp(X) \mid U \text{ can be written as as the union of a family sets in } \mathcal{B}\}.$$

By definition, $\mathcal{B} \subseteq \mathscr{T} \subseteq \mathscr{T}_{\mathcal{B}}$. It remains to check that $\mathscr{T}$ is a topology.
By definition, $X \in \mathscr{T}$, $\varnothing \in \mathscr{T}$. Moreover, the union of a family of elements of $\mathscr{T}$ remains in $\mathscr{T}$. Let $U = \bigcup\limits_{i \in I} U_i$ and $V = \bigcup\limits_{j \in J} V_j$ be elements of $\mathscr{T}$, where, $U_i \in \mathcal{B}, V_j \in \mathcal{B}$. Then

$$U \cap V = \bigcup_{(i,j) \in I \times J} (U_i \cap V_j)\,.$$

For any $x \in U_i \cap V_j$, $\exists W_x^{(i,j)} \in \mathcal{B},\ x \in W_x^{(i,j)} \subseteq U_i \cap V_j$. $U_i \cap V_j = \bigcup\limits_{x \in U_i \cap V_j} W_x^{(i,j)}$, so

$$U \cap V = \bigcup_{(i,j) \in I \times J} \bigcup_{x \in U_i \cap V_j} W_x^{(i,j)}.$$

□

1.2 Convergence

We fix a topology space $(E, \mathscr{T})$, $l \in E$ and $S \subseteq \wp(E)$ that generates the filter $\mathcal{V}_l(\mathscr{T})$ of all neighborhood of l.

Definition 1.2.1 Let $f : X \longrightarrow Y$ be a mapping. If $\mathcal{F}$ is a filter on X, we denote by $f_*(\mathcal{F})$ the set $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{F}\}$.

Proposition 1.2.2 $f_*(\mathcal{F})$ is a filter on Y.

Proof Let $(B_1, B_2) \in f_*(\mathcal{F})$,

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \in \mathcal{F}.$$

Let $B \in f_*(\mathcal{F}),\ C \supseteq B.\ f^{-1}(C) \supseteq f^{-1}(B) \in \mathcal{F}$, so $f^{-1}(C) \in \mathcal{F}$. □

Proposition 1.2.3 Let $f : X \longrightarrow Y,\ g : Y \longrightarrow Z$, be mappings. $\mathcal{F}$ be a filter on X. Then

$$(g \circ f)_* (\mathcal{F}) = g_*(f_*(\mathcal{F})).$$

Proof

$$\begin{aligned}(g\circ f)_*(\mathcal{F}) =&\{C\subseteq Z \mid (g\circ f)^{-1}(C)\in\mathcal{F}\}\\ =&\{C\subseteq Z\mid f^{-1}(g^{-1}(C))\in\mathcal{F}\}\\ =&\{C\subseteq Z\mid g^{-1}(C)\in f_*(\mathcal{F})\}\\ =&g_*(f_*(\mathcal{F})).\end{aligned}$$

□

Proposition 1.2.4 Let $\mathcal{B}$ be a filter basis in X, $f: X\longrightarrow Y$ be a mapping and $\mathcal{F}$ be the filter generated by $\mathcal{B}$. Then $f(\mathcal{B}):\{f(U)\mid U\in\mathcal{B}\}$ is a filter basis on Y and $f_*(\mathcal{F})$ is the filter generated by $f(\mathcal{B})$.

Proof Let U and V be elements of $\mathcal{B}$. Then $\exists W\in\mathcal{B}$, $W\subseteq U\cap V$. Hence $f(W)\subseteq f(U\cap V)\subseteq f(U)\cap f(V)$. Moreover, for any $U\in\mathcal{B}$, $U\subseteq f^{-1}(f(U))$. So $f^{-1}(f(U))\in\mathcal{F}$. Therefore, $f(\mathcal{B})\subseteq f_*(\mathcal{F})$. Let $A\in f_*(\mathcal{F})$. Then, $f^{-1}(A)\in\mathcal{F}$. So $\exists V\in\mathcal{B}$, $V\subseteq f^{-1}(A)$. Hence $f(V)\subseteq A$. Therefore, $f_*(\mathcal{F})$ is a filter basis generated by $f_*(\mathcal{B})$ □

Definition 1.2.5 Let $f: X\longrightarrow E$ be a mapping, $\mathcal{F}$ be a non-degenerate filter on X. If $f_*(\mathcal{F})\supseteq\mathcal{V}_l(\mathscr{T})$, we say that f **converges** to l along $\mathcal{F}$.

$$\lim_{\mathcal{F}} f = l$$

denotes "f converges to l along $\mathcal{F}$".
This condition is equivalent to

$$\forall V\in S_l,\ f^{-1}(V)\in\mathcal{F}.$$

If $\mathcal{B}$ is a filter basis which generates $\mathcal{F}$. This condition is also

$$\forall V\in S_l,\ \exists U\in\mathcal{B},\ f(U)\subseteq V.$$

Example 1.2.6
(1) Let $I\subseteq\mathbb{N}$ be an infinite subset and $x=(x_n)_{n\in I}\in\mathbb{N}^I$. Let $\mathcal{F}$ be the Fréchet filter on I. If x converges to l along $\mathcal{F}$, or equivalently,

$$\forall V\in S_l,\ \exists N\in\mathbb{N},\ \forall n\in I_{\geq\mathbb{N}},\ x_n\in V.$$

We say that the sequence $(x_n)_{n\in I}$ **converges to** l when n tends to the infinity, denote as

$$\lim_{n\to+\infty} x_n = l.$$

(2) Let $(X, \mathscr{T}_X)$ be a topological space. Let $Y \subseteq X$ be a subset of X, $p \in X$. Let

$$\mathcal{F} = \mathcal{V}_p(\mathscr{T}_X)|_Y = \{V \cap Y\} \mid V \in \mathcal{V}_p(\mathscr{T}_X).$$

Assume that $\mathcal{F}$ is non-degenerate. Let $f : X \longrightarrow E$ be a mapping. If f converges to l along $\mathcal{F}$, we day that $f(x)$ converges to l when $x \in Y$ tends to p, denoted as

$$\lim_{x\in Y,\ x\to p} f(x) = l.$$

This condition is equivalent to:

$$\forall V \in S_l,\ \exists U \text{ a open neighborhood of } p,\ \forall x \in U \cap Y,\ f(x) \in V.$$

In general, if $g : Y \longrightarrow E$ is a mapping such that

$$\forall V \in S_l,\ \exists U \text{ a open neighborhood of } p,\ \forall x \in U \cap Y,\ g(x) \in V,$$

then we say $g(x)$ converges to l when $x \in Y$ tends to p, denoted as

$$\lim_{x\in Y,\ x\to p} g(x) = l.$$

Remark 1.2.7 If (E, d) is a semimetric space, $\mathscr{T}$ is the semimetric topology. Then condition in (1) becomes:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \in I_{\geq N},\ \mathrm{d}(x_n, l) < \varepsilon.$$

The conditions in (2) becomes:

$$\forall \varepsilon > 0,\ \exists U \text{ a open neighborhood of } p,\ \forall x \in U \cap Y,\ \mathrm{d}(g(x), l) < \varepsilon.$$

If furthermore, $(X, \mathscr{T}_X)$ is a semimetric space with semimetric d_X. The condition becomes

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in Y,\ \mathrm{d}_X(x, p) < \delta \Rightarrow \mathrm{d}(g(x), l) < \varepsilon.$$

Example 1.2.8
(3) Consider $X = \mathbb{R}$. Let $Y \subseteq X$. Consider the filter $\mathcal{F}$ generated by $\{\mathbb{R}_{>M},\ M \in \mathbb{R}\}$. Suppose that $\mathcal{F}|_Y$ is non-degenerate. Let $g : Y \longrightarrow E$. If $\lim\limits_{\mathcal{F}|_Y} g = l$, we say that $g(x)$ converges to l when x tends to $+\infty$, denoted as

$$\lim_{x \in Y,\ x \to +\infty} g(x) = l.$$

This condition is

$$\forall V \in S_l,\ \exists M \in \mathbb{R}_{>0},\ \forall x \in Y,\ x > M \Rightarrow g(x) \in V.$$

If (E, d) is a metric space, it becomes:

$$\forall \varepsilon > 0,\ \exists M \in \mathbb{R}_{>0},\ \forall x \in Y,\ x > M \Rightarrow \mathrm{d}(g(x), l) < \varepsilon.$$

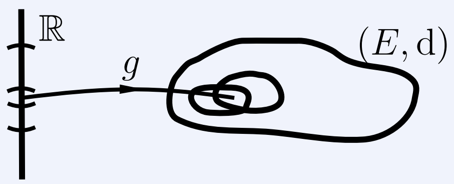

Figure 1.2.1: Example on $\mathbb{R}$

Example 1.2.9 Let $(G, \leq)$ be a totally ordered set, $\mathcal{F}$ be the ordered topology on G. It is generated by $\{G_{>a} \mid a \in g\} \cup \{G_{<b} \mid b \in G\}$. If $l \in G$, then $\mathcal{V}_x(\mathscr{T})$ is generated by

$$S_l := \{G_{>a} \mid a < l\} \cup \{G_{<b} \mid l < b\}.$$

Assume that $(G, \leq)$ is order complete. Let $f : X \longrightarrow G$ be a mapping and $\mathcal{F}$ be a non-degenerate filter on X.
(1) Assume that f converges to l along $\mathcal{F}$. $\forall a < l$, $U_a := f^{-1}(G_{>a}) \in \mathcal{F}$. $\forall x \in U_a, f(x) > a$. So $\liminf_{\mathcal{F}} f \geq a$. If $\sup(G_{<}l) = l$, then $\liminf_{\mathcal{F}} f \geq l$. If $\sup(G_{<l}) < l$, we denote $a = \sup(G_{<l})$. $\forall a \in U_a, f(x) \geq l$. So $\liminf_{\mathcal{F}} f \geq l$. Similarly, $\liminf_{\mathcal{F}} f \leq l$. So f admits l as its limit.
(2) Assume that $\limsup_{\mathcal{F}} f = \liminf_{\mathcal{F}} f = l$.

$$\liminf_{\mathcal{F}} f = l \Rightarrow \sup_{U \in \mathcal{F}} f^i(U) = l, \forall a < l, \exists U \in \mathcal{F}, f^i(U) > a, f^{-1}(G_{>a}) \in \mathcal{F}.$$

$$\limsup_{\mathcal{F}} f = l \Rightarrow \forall b > l, f^{-1}(G_{<b}) \in \mathcal{F}.$$

Therefore, f converges to l along $\mathcal{F}$.

1.3 Continuity

Definition 1.3.1 Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces, f be a function from X to Y, and $p \in \mathrm{Dom}(f)$. If for any neighborhood U of $f(p)$, there exists a neighborhood V of p such that

$$f(V) \subseteq U,$$

then we say that the function f is **continuous** at the point p.
If f is continuous at any $p \in \mathrm{Dom}(f)$, then we say that f is **continuous**.

Remark 1.3.2
(1) The continuity of f at p is equivalent to:

$$\lim_{\substack{x \in \mathrm{Dom}(f) \\ x \to p}} f(x) = f(p),$$

namely, f converges to $f(p)$ when x tends to p.
(2) Let $\mathcal{V}_{f(p)}(\mathscr{T}_Y)$ and $\mathcal{V}_p(\mathscr{T}_X)$ be filters of neighborhoods of $f(p)$ and p respectively. Let $\mathcal{B}_p$ be a filter basis that generates $\mathcal{V}_p(\mathscr{T}_X)$. Let $S_{f(p)}$ be a subset of $\mathcal{V}_{f(p)}(\mathscr{T}_Y)$ that generates $\mathcal{V}_{f(p)}(\mathscr{T}_Y)$. Then the continuity of f at p is equivalent to:

$$\forall U \in S_{f(p)},\ \exists V \in \mathcal{B}_p,\ f(V) \subseteq U.$$

In the case where X and Y are metric spaces, this condition becomes:

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathrm{Dom}(f),\ \mathrm{d}(x,p) < \delta \Rightarrow \mathrm{d}(f(x), f(p)) < \varepsilon.$$

Proposition 1.3.3 Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$, $(Z, \mathscr{T}_Z)$ be topological spaces. $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be functions. $p \in \mathrm{Dom}(g \circ f)$. Assume that f is continuous at p and g is continuous at $f(p)$. Then $g \circ f$ is continuous at p.

Proof Let U be a neighborhood of $g(f(p))$. Since g is continuous at $f(p)$, there exists a neighborhood V of $f(p)$ such that $g(V) \subseteq U$. Since f is continuous at p, there exists a neighborhood W of p, such that $f(W) \subseteq V$. Hence $g(f(W)) \subseteq g(V) \subseteq U$. So $g \circ f$ is continuous at p. □

Example 1.3.4 Let $(X, \mathscr{T}_X)$ be a topological space. Then Id_X and constant mapping are continuous.

Theorem 1.3.5 Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$ be topological spaces, $f : X \longrightarrow Y$ a function, $p \in \mathrm{Dom}(f)$. Consider the following conditions:
(1) f is continuous at p.
(2) For any $(x_n)_{n\in\mathbb{N}} \in \mathrm{Dom}(f)^{\mathbb{N}}$, if $\lim_{n\to\infty} x_n = p$, then

$$\lim_{n\to\infty} f(x_n) = f(p).$$

One has (1)⇒(2). If p has a countable basis of neighborhoods, (2)⇒(1).

Proof Let $(x_n)_{n\in\mathbb{N}} \in \mathrm{Dom}(f)^{\mathbb{N}}$ such that $\lim_{x\to\infty} x_n = p$. f is continuous at p, so for any neighborhood U of $f(p)$, there exists a neighborhood V of p, such that $f(V) \subseteq U$. Since $\lim_{n\to\infty} x_n = p$, there exists $N \in \mathbb{N}$, so that for any $n \in \mathbb{N}_{>N}$, $x_n \in V$. Hence for any $n \in \mathbb{N}_{>N}$, $f(x_n) \in U$. Hence $\lim_{n\to\infty} f(x_n) = f(p)$.
Assume that p has a countable basis of neighborhood. Let $(W_n)_{n\in\mathbb{N}}$ be a sequence of neighborhood of p, such that $\{W_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis. For $n \in \mathbb{N}$,

$$V_n := \bigcap_{i\in\mathbb{N}_{\leq n}} W_i.$$

V_n is a neighborhood of p. If f is not continuous at p, then there exists an neighborhood of p, U, such that

$$\forall n \in \mathbb{N},\ f(V_n) \nsubseteq U.$$

We pick $x_n \in V_n$ but $f(x_n) \notin U$. For any neighborhood V of p, there exists $N \in \mathbb{N}$ such that $V_N \subseteq V$, so that $x_n \in V$ for any $n \in \mathbb{N}_{>N}$. Hence, x_n converges to p. But $f(x_n)$ cannot converge to $f(p)$. □

Lemma 1.3.6 Let $(X, \mathscr{T}_X)$ be a topological space. $V \subseteq X$. If $\forall p \in V$, V is a neighborhood of p, then $V \in \mathscr{T}_X$. In fact $\forall p \in V$, there exists $W_p \in \mathscr{T}_X$, $p \in W_p \subseteq V$. Hence

$$V = \bigcup_{p\in V} \{p\} \subseteq \bigcup_{p\in V} W_p \subseteq V.$$

Proposition 1.3.7 Let $(X, \mathscr{T}_X)$, $(Y, \mathscr{T}_Y)$ be topological spaces, $\mathcal{S} \subseteq \mathscr{T}_Y$, $\mathcal{S}$ generates $\mathscr{T}_Y$. The following statements are equivalent:
(1) f is continuous.
(2) For any $U \in \mathscr{T}_Y$, $f^{-1}(U) \in \mathscr{T}_X$.
(3) For any $U \in \mathcal{S}$, $f^{-1}(U) \in \mathscr{T}_X$.

Proof
(1)⇒(2): For any $p \in f^{-1}(U)$, one has $f(p) \in U$. Hence, there is a neighborhood V_p of p, $f(V_p) \subseteq U$, or equivalently $V_p \subseteq f^{-1}(U)$. Therefore, $f^{-1}(U) \in \mathscr{T}_X$.
(3)⇒(2):

$$\mathscr{T}'_Y = \left\{U \in \wp(Y) \mid f^{-1}(U) \in \mathscr{T}_X\right\}$$

By definition, $\{\varnothing, Y\} \subseteq \mathscr{T}'_Y$. If $(U_1, U_2) \in \mathscr{T}'_Y \times \mathscr{T}'_Y$, then $f^{-1}(U_1 \cap U_2) = f^{-1}(U_1) \cap f^{-1}(U_2) \in \mathscr{T}_X$. So $U_1 \cap U_2 \in \mathscr{T}'_Y$. $(U_i)_{i\in I} \in (\mathscr{T}'_Y)^I$, then

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i) \in \mathscr{T}_X.$$

So $\mathscr{T}'_Y$ is a topology, by (3), $\mathcal{S} \subseteq \mathscr{T}'_Y \Rightarrow \mathscr{T}_Y \subseteq \mathscr{T}'_Y$. □

1.4 Initial Topology

Definition 1.4.1 Let X be a set, $((Y_i, \mathscr{T}_i))_{i\in I}$ a family of topological spaces, $(f_i : X \longrightarrow Y_i)_{i\in I}$ a family of mappings. We call **initial topology** on X induced by $(f_i)_{i\in I}$ the topology generated by

$$\bigcup_{i\in I} \left\{f_i^{-1}(U_i) \mid U_i \in \mathscr{T}_i\right\}.$$

It is the coarsest topology on X making all f_i continuous.

Proposition 1.4.2 Let $\mathscr{T}$ be the initial topology on X induced by $(f_i)_{i\in I}$.
(1)

$$\mathcal{B} = \left\{\bigcap_{j\in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, \ (U_j)_{j\in J} \in \prod_{j\in J} \mathscr{T}_j\right\}$$

is topological basis that generates $\mathscr{T}$.
(2) Let $(Z, \mathscr{T}_Z)$ be topological space, $h : Z \longrightarrow X$ be a function and $p \in \mathrm{Dom}(f)$. Then h is continuous at p if and only if $\forall i \in I, \ f_i \circ h$ is continuous at p.

Proof
(1) Let

$$S = \bigcup_{i\in I} \left\{f_i^{-1}(U_i) \mid U_i \in \mathscr{T}_i\right\},$$

$\mathcal{B}'$ be the set of the intersections of all finitely elements of S (We have proved that $\mathcal{B}'$ is a basis of $\mathscr{T}$). $\mathcal{B} \subseteq \mathcal{B}'$. Let $i_1, \ldots, i_n$ elements of I, $U_{i_k} \in \mathscr{T}_{i_k}$, $J = \{i_1, \ldots, i_n\}$, $j \in J$, $A_j = \{k \in \{1, \ldots, n\} \mid i_k = j\}$, $W_j = \bigcap_{k \in A_j} U_{i_k}$.

$$\bigcap_{k=1}^{n} f_{i_k}^{-1}(U_{i_k}) = \bigcap_{j \in J} f_j^{-1}(W_j) \in \mathcal{B}.$$

(2) Since f_i is continuous at p, if h is continuous then $\forall i \in I$, $f_i \circ h$ is continuous. Assume $\forall i \in I,\ f_i \circ h$ is continuous, then

$$\forall i \in I,\ \forall U_i \in \mathscr{T}_i,\ (f_i \circ h)^{-1}(U_i) = h^{-1}\left(f_i^{-1}(U_i)\right).$$

Therefore, for any $V \in S$, $h^{-1}(V) \in \mathscr{T}_Z$. Hence h is continuous. $\square$

Example 1.4.3 Let $(X_i, \mathscr{T}_i)$ be topological spaces, $X = \prod_{i \in I} X_i$, $\pi_i : X \longrightarrow X_i$ be a projection. The initial topology on X induced by $(\pi_i)_{i \in I}$ is called the **product topology**.

1.5 Uniform Continuity

Definition 1.5.1 Let (X, d_X), (Y, d_Y) be semimetric spaces, $f : X \longrightarrow Y$ be a function, $\alpha \in \mathbb{R}_{\geq 0}$. If for any $(x_1, x_2) \in \mathrm{Dom}(f)^2$, $\mathrm{d}\left(f(x_1), f(x_2)\right) \leq \alpha \cdot \mathrm{d}(x_1, x_2)$, then we say that f is α-Lipschitzian. If there exists $\alpha \in \mathbb{R}_{\geq 0}$ such that f is **α-Lipschitzian**, then we say that f is **Lipschitzian**.
If

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall (x_1, x_2) \in \mathrm{Dom}(f)^2,\ \mathrm{d}_X(x_1, x_2) < \delta \Rightarrow \mathrm{d}(f(x_1), f(x_2)) < \varepsilon,$$

then we say that f is **uniformly continuous**.

Proposition 1.5.2 Let (X, d_X), (Y, d_Y) be semimetric spaces, $f : X \longrightarrow Y$ be a function. If f is uniformly continuous, then f continuous.

Proof Let $p \in \mathrm{Dom}(f)$. For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall (x_1, x_2) \in \mathrm{Dom}(f)^2,\ \mathrm{d}_X(x_1, x_2) < \delta \Rightarrow \mathrm{d}(f(x_1), f(x_2)) < \varepsilon.$$

In particular,

$$\forall x \in \mathrm{Dom}(f),\ \exists \delta > 0,\ \mathrm{d}_X(x,p) < \delta \Rightarrow \mathrm{d}_Y(f(x), f(p)) < \varepsilon.$$

□

Definition 1.5.3 Let K be a field, we call absolute value on K any mapping,

$$|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0},$$

(1) $\forall a \in K,\ a = 0_K$ if and only if $|a| = 0$.
(2) $\forall (a,b) \in K \times K,\ |ab| = |a|\,|b|$.
(3) $\forall (a,b) \in K \times K,\ |a+b| \leq |a| + |b|$.
The pair $(K, |\cdot|)$ is called a **valued field**.

Example 1.5.4 Let $(K, \leq)$ be a totally ordered field, then $|a| = \max\{-a, a\}$ is an absolute value on K.

Example 1.5.5 Let p be a prime number. Any non-zero rational number α can be written in the form

$$\alpha = p^{\mathrm{ord}_p(\alpha)} \cdot \frac{m}{n},$$

where, $\mathrm{ord}_p(\alpha) \in \mathbb{Z},\ p \nmid mn$. If $\alpha = 0$, we set (by convention) $\mathrm{ord}_p(\alpha) = +\infty$.
Properties:
(1) $\mathrm{ord}_p(\alpha\beta) = \mathrm{ord}_p(\alpha) + \mathrm{ord}_p(\beta)$.
(2) $\alpha = p^{\mathrm{ord}_p(\alpha)}\frac{m}{n},\ \beta = p^{\mathrm{ord}_p(\beta)}\frac{u}{v}$, $\mathrm{ord}_p(\alpha) > \mathrm{ord}_p(\beta)$, $p \nmid nvu$.

$$\alpha + \beta = p^{\mathrm{ord}_p(\beta)}\frac{p^{\mathrm{ord}(\alpha)-\mathrm{ord}(\beta)}mv + nu}{nv}.$$

(3) If $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$, then $\mathrm{ord}_p(\alpha+\beta) \geq \mathrm{ord}_p(\alpha) = \mathrm{ord}_p(\beta)$.

$$\alpha + \beta = p^{\mathrm{ord}_p(\alpha)}\frac{mv + nu}{nv}.$$

Proposition 1.5.6 The mapping

$$|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0},$$

$$\begin{cases} |\alpha|_p = p^{-\mathrm{ord}(\alpha)}, & \text{if } \alpha \neq 0 \\ |\alpha|_p = 0, & \text{if } \alpha = 0 \end{cases}$$

is an absolute value on $\mathbb{Q}$.

Proof If $\alpha = 0$, then $|\alpha|_p > 0$. If $(\alpha,\beta) \in \mathbb{Q}^2$, when $0 \in \{\alpha,\beta\}$, then $\alpha\beta = 0$ and $0 = |\alpha\beta|_p = |\alpha|_p|\beta|_p$. When $0 \notin \{\alpha,\beta\}$,

$$|\alpha\beta|_p = p^{-\mathrm{ord}_p(\alpha\beta)} = p^{-\mathrm{ord}_p(\alpha)-\mathrm{ord}_p(\beta)} = |\alpha|_p|\beta|_p.$$

If $\alpha = 0$, $|\alpha\beta|_p = |\beta|_p$. If $\beta = 0$, $|\alpha\beta|_p = |\alpha|_p$., if $0 \notin \{\alpha,\beta\}$,

$$|\alpha+\beta|_p = p^{-\mathrm{ord}_p(\alpha+\beta)} \le p^{\max\{\mathrm{ord}_p(\alpha),\mathrm{ord}_p(\beta)\}} \le \max\{|\alpha|_p, |\beta|_p\} \le |\alpha|_p + |\beta|_p.$$

□

Remark 1.5.7 Let $(K, |\cdot|)$ be a valued field. If for any $(\alpha,\beta) \in K^2$ satisfies $|\alpha+\beta| \le \max\{|\alpha|,|\beta|\}$, we say that $(K,|\cdot|)$ is **non-archimedean**, otherwise, we say that $(K,|\cdot|)$ is **archimedean**. $(\mathbb{R},|\cdot|)$ and $(\mathbb{Q},|\cdot|)$ are archimedean.

Definition 1.5.8 Let $(K,|\cdot|)$ be a valued filed, V a vector spaced over K. We call **seminorm** on V any mapping

$$\|\cdot\| : V \longrightarrow \mathbb{R}_{\ge 0}$$

that satisfies the following conditions:
(1) $\forall (a,x) \in K \times V,\ \|ax\| = |a| \cdot \|x\|$.
(2) $\forall (x,y) \in V \times V,\ \|x+y\| \le \|x\| + \|y\|$.
Note that (1) implies that $\|0_V\| = |0_K| \cdot \|0_V\| = 0$.
The pair $(V,\|\cdot\|)$ is called **seminormed vector space** *over* $(K,|\cdot|)$. If $\forall (x,y) \in V \times V,\ \|x+y\| \le \max\{\|x\|,\|y\|\}$, then we say that $\|\cdot\|$ is **ultrametric**. If $\forall x \in V \cdot \{0\}$, $\|x\| > 0$, then we say that $\|\cdot\|$ is a **norm** and $(V,\|\cdot\|)$ is a **normed vector space** *over* $(K,|\cdot|)$.

Example 1.5.9 $\mathrm{d} : V \times V \longrightarrow \mathbb{R}_{\ge 0}$, $\mathrm{d}(x,y) := \|x-y\|$ is a semi-metric.

Example 1.5.10 Let $(K,|\cdot|)$ be a valued field.
(1) $(K,|\cdot|)$ is a normed vector space over $(K,|\cdot|)$. ($\mathrm{d}(x,y) = |x-y|$ is a metric.)
(2) Let $(V_1,\|\cdot\|_1),\ldots,(V_n,\|\cdot\|_n)$ be seminormed vector spaces over $(K,|\cdot|)$,

$V = V_1 \oplus \cdots \oplus V_n$.

$$\| \cdot \|_{l^\infty} : V \longrightarrow \mathbb{R}_{\geq 0},\ (x_1, \ldots, x_n) \longmapsto \max_{i \in \{1,\ldots,n\}} \|x_i\|_i,\ x_i \in V_i,\ i \in \{1, \ldots, n\},$$

$$\| \cdot \|_{l^1} : V \longrightarrow \mathbb{R}_{\geq 0},\ (x_1, \ldots, x_n) \longmapsto \sum_{i \in \{1,\ldots,n\}} \|x_i\|_i,\ x_i \in V_i,\ i \in \{1, \ldots, n\}.$$

$\forall \lambda \in K,\ \forall (x_1, \ldots, x_n) \in V,$

$$\begin{aligned} \|\lambda(x_1, \ldots, x_n)\|_{l^\infty} &= \|(\lambda \cdot x_1, \ldots, \lambda x_n)\|_{l^\infty} \\ &= \max_{i \in \{1,\ldots,n\}} \|\lambda x_i\|_i \\ &= \max_{i \in \{1,\ldots,n\}} |\lambda|\, \|x_i\|_i \\ &= |\lambda| \max_{i \in \{1,\ldots,n\}} \|x_i\|_i \\ &= |\lambda| \cdot \|(x_1, \ldots, x_n)\|_{l^\infty}. \end{aligned}$$

$$\begin{aligned} \|\lambda(x_1, \ldots, x_n)\|_{l^1} &= \|(\lambda \cdot x_1, \ldots, \lambda x_n)\|_{l^1} \\ &= \sum_{i \in \{1,\ldots,n\}} \|\lambda x_i\|_i \\ &= \sum_{i \in \{1,\ldots,n\}} |\lambda|\, \|x_i\|_i \\ &= |\lambda| \sum_{i \in \{1,\ldots,n\}} \|x_i\|_i \\ &= |\lambda| \cdot \|(x_1, \ldots, x_n)\|_{l^1}. \end{aligned}$$

$\forall x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n) \in V,$

$$\begin{aligned} \|x + y\|_{l^\infty} = \|(x_1 + y_1, \ldots, x_n + y_n)\|_{l^\infty} &= \max_{i \in \{1,\ldots,n\}} \|x_i + y_i\|_i \\ \leq \max_{i \in \{1,\ldots,n\}} \|x_i\|_i + \|y_i\|_i &\leq \|x\|_{l^\infty} + \|y\|_{l^\infty}. \end{aligned}$$

$$\begin{aligned} \|x + y\|_{l^1} &= \sum_{i \in \{1,\ldots,n\}} \|x_i + y_i\|_i \\ &\leq \sum_{i \in \{1,\ldots,n\}} \|x_i\|_i + \|y_i\|_i = \|x\|_{l^1} + \|y\|_{l^1}. \end{aligned}$$

(3) Let $(V, \|\cdot\|)$ be a seminormed vector space over K, $f : W \longrightarrow V$ be a K-linear mapping. We denote by $\| \cdot \|_f$ the mapping $W \longrightarrow \mathbb{R}_{\geq 0}$ define as

$$\forall x \in W,\ \|x\|_f := \|f(x)\|.$$

$\forall(\lambda, x) \in K \times W$,

$$\|\lambda x\|_f = \|f(\lambda x)\| = \|\lambda f(x)\| = |\lambda| \cdot \|f(x)\| = \lambda\|x\|_f.$$

$\forall(x, y) \in W \times W$,

$$\|x + y\|_f = \|f(x + y)\| = \|f(x)\| + \|f(y)\| \leq \|x\|_f + \|y\|_f.$$

Therefore, $\|\cdot\|_f$ is a seminorm on W, called the **seminorm** *induced by (the K-linear) mapping f.*
(4) Let $(V, |\cdot|)$ be a seminormed vector space over K, let $\pi : V \longrightarrow E$ be a surjective K-linear mapping. We denote by $\|\cdot\|_\pi$ the mapping

$$E \longrightarrow \mathbb{R}_{\geq 0},$$

$$\alpha \longmapsto \inf_{x \in \pi^{-1}(\alpha)} \|x\|.$$

If $(\lambda, \alpha) \in K \times E$,

$$\|\lambda\alpha\|_\pi = \inf_{x \in \pi^{-1}(\lambda\alpha)} \|x\| = \inf_{x \in \pi^{-1}(\alpha)} |\lambda|\,\|x\| = |\lambda|\,\|\alpha\|_\pi.$$

If $(\alpha, \beta) \in E \times E$,

$$\begin{aligned}\|\alpha + \beta\|_\pi = \inf_{z \in \pi^{-1}(\alpha+\beta)} \|z\| &= \inf_{(x,y) \in \pi^{-1}(\alpha) \times \pi^{-1}(\beta)} \|x + y\| \\ &\leq \inf_{(x,y) \in \pi^{-1}(\alpha) \times \pi^{-1}(\beta)} \|x\| + \|y\| \\ &= \inf_{x \in \pi^{-1}(\alpha)} \|x\| + \inf_{y \in \pi^{-1}(\beta)} \|y\|.\end{aligned}$$

Hence $\|\cdot\|_\pi$ is a seminorm on E called the **quotient seminorm** *of $\|\cdot\|$ induced by π.*

Proposition 1.5.11 Let $(V, \|\cdot\|)$ be a seminormed vector space over a valued field $(K, |\cdot|)$.
(1) For any $a \in V$, the mapping $\tau_a : V \longrightarrow V,\ \tau_a(x) = x + a$ is 1-Lipschitzian.
(2) For any $\lambda \in K$, the mapping $m_\lambda : V \longrightarrow V,\ m_\lambda(x) := \lambda \cdot x$ is λ-Lipschitzian.
(3) The mapping $\|\cdot\| : V \longrightarrow \mathbb{R}$ is 1-Lipschitzian.

Proof
(1) $\forall(x, y) \in V \times V,\ \|\tau_a(x) - \tau_a(y)\| = \|(x + a) - (y + a)\| = \|x - y\|.$

(2) $\forall (x,y) \in V \times V$, $\|m_\lambda(x) - m_\lambda(y)\| = \|\lambda x - \lambda y\| = |\lambda|\,\|x-y\|$.
(3) $\forall (x,y) \in V \times V$, $\|x\| = \|(x-y)+y\| \leq \|y\| + \|x-y\|$. So $\|x\| - \|y\| \leq \|x-y\|$. Similarly, $\|y\| - \|x\| \leq \|y-x\|$. Hence,

$$|\|x\| - \|y\|| \leq \|x-y\|.$$

□

Definition 1.5.12 $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two seminormed vector spaces over a valued field $(K, |\cdot|)$, and φ a K-linear mapping from E to F. We define $\|\varphi\| \in [0, +\infty]$ as

$$\|\varphi\| := \sup_{x\in E,\ \|x\|_E \neq 0} \frac{\|\varphi(x)\|_F}{\|x\|_E}.$$

In the case where $\|x_E\| = 0$, for any $x \in E$, by convention, $\|\varphi(x)\|$ is defined to be 0. If $\|\varphi\| < +\infty$, we say that φ is bounded. We denote by $\mathscr{L}(E,F)$ the set of all bounded K-linear mappings from E to F.

Remark 1.5.13 In the case when $(E, \|\cdot\|_E) = (K, \|\cdot\|)$,

$$\|\varphi\| = \sup_{x\in\mathcal{B}(0,1)} \|\varphi(x)\|_F.$$

Proposition 1.5.14
(1) For any $\varphi \in \mathscr{L}(E,F)$ be the mapping φ is $\|\varphi\|$-Lipschitzian. In particular, φ is continuous.
(2) Suppose that there exists $\lambda \in K$, such that $|\lambda| > 1$. If $\varphi : E \longrightarrow F$ is continuous at 0_E, then $\varphi \in \mathscr{L}(E,F)$.

Proof For any $(x,y) \in E \times E$:
(1) $\|\varphi(x) - \varphi(y)\|_F = \|\varphi(x-y)\|_F \leq \|\varphi\|\|x-y\|_E$.
(2) $\mathcal{B}(0_F, 1) := \{\alpha \in F \mid \|\alpha\|_F < 1\}$ is a neighborhood of 0_F. There exists $\varepsilon > 0$ such that

$$\varphi(\overline{\mathcal{B}}(0_E, \varepsilon)) \subseteq \mathcal{B}(0_F, 1)$$

where

$$\overline{\mathcal{B}}(0_E, \varepsilon) := \{x \in E \mid \|x\|_E < \varepsilon\}.$$

Let $x \in E\backslash\{0\}$, there exists $n \in \mathbb{Z}$, such that $\|\lambda^n x\|_E = |\lambda|^n \|x\|_E < \varepsilon$ and

$\|\lambda^{n+1}x\|_E = |\lambda|^{n+1}\,\|x\|_E \geq \varepsilon$. Thus,

$$\|\varphi(x)\|_F = \|\lambda^{-n}\varphi(\lambda^n x)\|_F = |\lambda|^{-n}\,\|\varphi(\lambda^n x)\|_F \leq |\lambda|^{-n} \leq \frac{|\lambda|}{\varepsilon}\|x\|_E.$$

Therefore, $\|\varphi\| \leq \frac{\lambda}{\varepsilon}$. □

Proposition 1.5.15 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be two seminormed vector spaces over a valued field $(K, |\cdot|)$. Then $\mathscr{L}(E, F)$ is a vector subspace of F^E, and $\|\cdot\|$ is a seminorm on $\mathscr{L}(E, F)$, called the operator seminorm.

Proof Let φ, ψ be to K-linear mappings from E to F. For any $x \in E$, such that $\|x\|_E \neq 0$.

$$\begin{aligned}\|(\varphi+\psi)(x)\|_F =& \|\varphi(x)+\|\psi(x)\|_F \leq \|\varphi(x)\|_F + \|\psi(x)\|_F \\ \leq& \|\varphi\|\|x\|_E + \|\psi\|\|x\|_E = (\|\varphi\| + \|\psi\|)\,\|x\|_E.\end{aligned}$$

$$\|\varphi+\psi\| \leq \|\varphi\| + \|\psi\|.$$

So, $\varphi, \psi \in \mathscr{L}(E, F) \Rightarrow \varphi + \psi \in \mathscr{L}(E, F)$. Let $\lambda \in K^\times$ and $\varphi \in \mathscr{L}(E, F)$, for any $x \in E$, $\|x\|_E \neq 0$. One has

$$\|(\lambda\varphi)(x)\|_F = \|\lambda \cdot \varphi(x)\|_F = |\lambda|\,\|\varphi(x)\|_F \leq |\lambda|\,\|\varphi\|\|x\|_E.$$

So, $\|\lambda\| \leq |\lambda|\,\|\varphi\|$, $\lambda\varphi \in \mathscr{L}(E, F)$. So $\mathscr{L}(E, F)$ is a vector subspace of F^E. Note that we can apply to λ^{-1} and $\lambda\varphi$ and get

$$\|\lambda^{-1}\cdot\lambda\varphi\| = \|\varphi\| \leq \left|\lambda^{-1}\right|\|\lambda\varphi\| = |\lambda|^{-1}\,\|\lambda\varphi\|,\ |\lambda| \leq \|\lambda\varphi\|.$$

Hence, $|\lambda|\,\|\varphi\| = \|\lambda\varphi\|$ and therefore $\|\cdot\|$ is a seminorm on $\mathscr{L}(E, F)$. □

Definition 1.5.16 Let E be a vector space over K, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be seminorms on E. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, if there exists $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$\forall x \in E,\ c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1.$$

Proposition 1.5.17 Let E be a vector space over K, and $\|\cdot\|_1$ and $\|\cdot\|_2$ be seminorms on E. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then they define the same topology on E.

Proof Let $\mathscr{T}_1$ and $\mathscr{T}_2$ to be the topologies defined by $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then

$$\mathrm{Id}_E : (E, \mathscr{T}_1) \longrightarrow (E, \mathscr{T}_2)$$

is bounded. So it is continuous, so $\mathscr{T}_2 \subseteq \mathscr{T}_1$. Similarly, $\mathscr{T}_1 \subseteq \mathscr{T}_2$. □

Proposition 1.5.18 Let $n \in \mathbb{N}_{\geq 1}$, and (X_i, d_i), $i \in \{1, 2, \ldots, n\}$ be n seminormed vector spaces over K. Let $X = \prod_{i=1}^{n} X_i$ and

$$\mathrm{d} : X \times X \longrightarrow \mathbb{R}_{\geq 0},$$

$$\mathrm{d}\left((\mathrm{x}_1, \ldots, \mathrm{x}_\mathrm{n}), (\mathrm{y}_1, \ldots, \mathrm{y}_\mathrm{n})\right) \longmapsto \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, y_i).$$

Then d is a semimetric on X, and the topology induced by d is the product topology of $\mathscr{T}_{\mathrm{d}_i}$ (topology induced on X_i by d_i) $i \in \{1, \ldots, n\}$.

Proof
(1) $\mathrm{d}\left((x_1, \ldots, x_n), (x_1, \ldots, x_n)\right) = \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, x_i) = 0$.
(2)

$$\mathrm{d}\left((x_1, \ldots, x_n), (y_1, \ldots, y_n)\right) = \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, y_i)$$
$$= \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(y_i, x_i) = \mathrm{d}\left((y_1, \ldots, y_n), (x_1, \ldots, x_n)\right).$$

(3)

$$\begin{aligned}
&\mathrm{d}((x_1, \ldots, x_n), (z_1, \ldots, z_n)) \\
=& \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, z_i) \\
\leq& \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, y_i) + \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(y_i, z_i) \\
=&\mathrm{d}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) + \mathrm{d}((y_1, \ldots, y_n), (z_1, \ldots, z_n)).
\end{aligned}$$

So d is a semimetric on X. For $i \in \{1, \ldots, n\}$, $\mathscr{T}_i := \mathscr{T}_{\mathrm{d}_i}$, where $\mathscr{T}_\mathrm{d}$ is the topology induced by d. Let $\pi_i : X \longrightarrow X_i$ be the project mapping (continuous with the product topology on X.) For any $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ in X,

$$\mathrm{d}(x, y) = \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, y_i) = \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(\pi_i(x), \pi_i(y)),$$

have $\forall i \in \{1, \ldots, n\}$, $\mathrm{d}_i(x_i, y_i) \leq \mathrm{d}(x, y)$ which implies that

$$\mathrm{Id}_X : (X, \mathscr{T}_\mathrm{d}) \longrightarrow (X, \mathscr{T})$$

is continuous. So $\mathscr{T} \subseteq \mathscr{T}_{\mathrm{d}}$.

$$\begin{aligned}\mathcal{B}((p_1,\dots,p_n),\varepsilon) =& \{(x_1,\dots,x_n) \mid \mathrm{d}((p_1,\dots,p_n),(x_1,\dots,x_n)) < \varepsilon\} \\ =& \prod_{i=1}^{n} \mathcal{B}(p_i,\varepsilon) \in \mathscr{T}.\end{aligned}$$

□

1.6 Closed Subsets

Example 1.6.1 (Review of open subsets) In $\mathbb{R}$, an interval of the form $]a,b[$ is open, since $]a,b[= \mathcal{B}(\frac{a+b}{2}, \frac{b-a}{2})$. An interval of the form $]a,+\infty[$ is open, since $]a,+\infty[= \bigcup\limits_{n\in\mathbb{N}_{\geq 1}}]a,a+n[$.

Definition 1.6.2 Let $(X,\mathscr{T})$ be a topological space. We say a subset Y of X is **closed** if $X \setminus Y$ is open.

Remark 1.6.3
(1) $\varnothing$, X are closed.
(2) If F_1, F_2 are closed subset of X, then $F_1 \cup F_2$ is closed.
(3) If $(F_i)_{i\in I}$ is a non-empty family of closed subsets of X, then $\bigcap\limits_{i\in I} F_i$ is closed.

Example 1.6.4 Let $(a,b) \in \mathbb{R}^2$, $a < b$, then $[a,b] \subseteq \mathbb{R}^2$ is closed. Moreover, $]-\infty,a]$ is closed.

Proposition 1.6.5 Let $(X,\mathscr{T}_X)$ and $(Y,\mathscr{T}_Y)$ be topological spaces, and $f : X \longrightarrow Y$ be a mapping, then the following statements are equivalent:
(1) f is continuous.
(2) For any closed subset F of Y, $f^{-1}(F)$ is a closed subset of X.

Proof
(1) $\Leftrightarrow$ (2): f is continuous if and only if, for any open subset U of Y, $f^{-1}(U) \in \mathscr{T}_X$. Let $F \subseteq Y$ be closed, then $Y \backslash F$ is open. So $f^{-1}(Y\backslash F) = X \backslash f^{-1}(F)$ is open, so $f^{-1}(F)$ is closed.
(2) $\Leftrightarrow$ (1): Let $U \in \mathscr{T}_Y$, then $F = Y\backslash U$ is closed, so $f^{-1}(F) = X\backslash f^{-1}(U)$ is

closed. So $f^{-1}(U) \in \mathscr{T}_Y$. $\square$

Example 1.6.6
In $\mathbb{R}^2$, $\{(x, y) \in \mathbb{R}^2 \mid x \geq 1\}$ is closed. Since $\mathbb{R}^2 \backslash \{(x, y) \mid x \geq 1\} =]-\infty, 0[\times \mathbb{R}$. Since $f(x, y) = x + y$ is continuous, then $f^{-1}([0, +\infty[) = \{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0\}$ is closed.

Example 1.6.7 Let (X, d) be a semimetric space. Let $Y \subseteq X$, $Y \neq \varnothing$. We define, for any $x \in X$,

$$\mathrm{d}(x, Y) = \inf_{y \in Y} \mathrm{d}(x, y) \in \mathbb{R}_{\geq 0}.$$

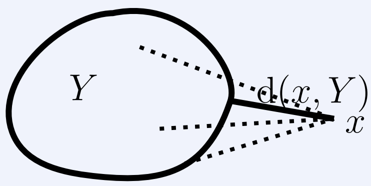

Figure 1.6.1: Definition of $\mathrm{d}(\cdot, Y)$.

The mapping $\mathrm{d}(\cdot, Y) : X \longrightarrow \mathbb{R}$, $x \longmapsto \mathrm{d}(x, Y)$ is 1-Lipschitzian.
Let $(x, x') \in X \times X$, $\forall y \in Y$,

$$\mathrm{d}(x, y) - \mathrm{d}(x', y) \leq \mathrm{d}(x, x').$$

Taking the infimum, we get

$$\mathrm{d}(x, Y) - \mathrm{d}(x', Y) \leq \mathrm{d}(x, x').$$

By symmetry between x and x', $\mathrm{d}(x', Y) - \mathrm{d}(x, Y) \leq \mathrm{d}(x, x')$. So

$$|\mathrm{d}(x, Y) - \mathrm{d}(x', Y)| \leq |\mathrm{d}(x, x')|.$$

For any $r > 0$, define

$$B(Y, r) := \{x \in X \mid \mathrm{d}(x, Y) < r\},$$

$$\overline{B}(Y, r) := \{x \in X \mid \mathrm{d}(x, Y) \leq r\}$$

$B(Y, r)$ is open, $\overline{B}(Y, r)$ is closed. If $Y = \{y\}$ is a one point set, $B(Y, r)$ and $\overline{B}(Y, r)$ are defined as $B(y, r)$ and $\overline{B}(y, r)$ respectively.

Definition 1.6.8 Let $(X, \mathscr{T})$ be a topological space.
(1) Let $\mathcal{F}$ be a non-degenerate filter, an element $p \in X$ is called **adherent point** *of* $\mathcal{F}$ if $\mathcal{F} \cup \mathcal{V}_p(\mathscr{T})$ generates a non-degenerate filter. ($\forall U \in \mathcal{F}, \forall V \in \mathcal{V}_p(\mathscr{T}), U \cap V \neq \varnothing$.)
(2) Let $Y \subseteq X$. We say that $p \in X$ is an adherent point of Y if it is an adherent point of the principal filter $\mathcal{F}_Y = \{U \subseteq X \mid Y \subseteq U\}$. (For any neighborhood V of p, $Y \cap V \neq \varnothing$.) We denote by $\overline{Y}$ the set of all adherent points of Y called the **closure** of Y. Clearly, $Y \subseteq \overline{Y}$.

Proposition 1.6.9 Let $(X, \mathscr{T}_X)$ be a topological space, $Y \subseteq X$. Then $\overline{Y}$ is the smallest closed subset containing Y. Namely,

$$\overline{Y} = \bigcap_{\substack{F \subseteq X \text{ closed}, \\ Y \subseteq F}} F.$$

Proof Let $p \in \overline{Y}$. If there exists a closed subset F containing Y such that $p \notin F$. So $p \in X \backslash F$. Hence $X \backslash F \in \mathcal{V}_p(\mathscr{T})$. So $\varnothing = (X \backslash Y) \cap Y \supseteq (X \backslash F) \cap Y \neq \varnothing$. Contradiction. Therefore,

$$\overline{Y} \subseteq \bigcap_{\substack{F \subseteq X \text{ closed}, \\ Y \subseteq F}} F.$$

Suppose that $x \in X \backslash \overline{Y}$. There exists an open neighborhood U of x such that $U \cap Y = \varnothing$. So $x \notin F := X \backslash U$.

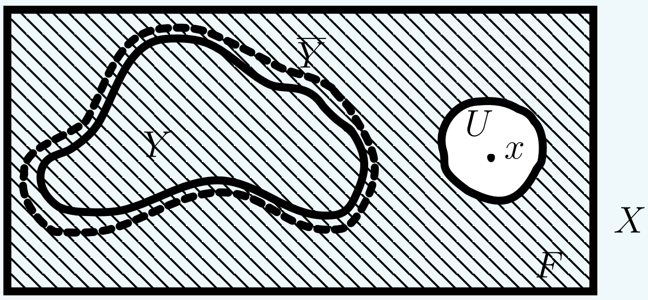

Figure 1.6.2: Closure

Note that F is closed and $F \supseteq Y$. Therefore,

$$X \backslash \overline{Y} \subseteq \bigcup_{\substack{F \subseteq X \text{ closed}, \\ Y \subseteq F}} (X \backslash F).$$

which leads to

$$\overline{Y} \supseteq X \backslash \left(\bigcup_{\substack{F \subseteq X \text{ closed}, \\ Y \subseteq F}} (X \backslash F) \right) = \bigcap_{\substack{F \subseteq X \text{ closed}, \\ Y \subseteq F}} F.$$

□

Definition 1.6.10 Let $(X, \mathscr{T})$ be a topological space and $Y \subseteq X$. We denote by Y° the set of $p \in Y$ such that Y is a neighborhood of p.

Proposition 1.6.11 Y° is the least[a] open subset of X such that is contained in Y. Moreover,

$$X \backslash Y^\circ = \overline{X \backslash Y}.$$

[a] largest

Proof $\forall y \in Y^\circ$, there exists $U_y \in \mathscr{T}$ such that $y \in U_y \subseteq Y$. Therefore, $\forall x \in U_y$, Y is a neighborhood of x, hence, $U_y \subseteq Y^\circ$. We thus obtain

$$Y^\circ = \bigcup_{y \in Y^\circ} \{y\} \subseteq \bigcup_{y \in Y^\circ} U_y \subseteq Y^\circ.$$

Hence Y° is open.
If $U \subseteq Y$ is open, then $\forall x \in U$, Y is a neighborhood of x. So $U \subseteq Y^\circ$. Therefore, Y° is the largest open subset that is contained in Y.

$$X \backslash Y^\circ = X \backslash \bigcup_{\substack{U \in \mathscr{T} \\ U \subseteq Y}} U = \bigcap_{\substack{U \in \mathscr{T} \\ U \subseteq Y}} X \backslash U \overset{F = X \backslash U}{=} \bigcap_{\substack{F \text{ closed} \\ X \backslash Y \subseteq F}} F = \overline{X \backslash Y}.$$

□

Definition 1.6.12 Let $(X, \mathscr{T})$ be a topological space. We equip $X \times X$ with the product topology, (a topological basis is given by $\{U \times V \mid (U, V) \in \mathscr{T}^2\}$) Let

$$\Delta_X := \{(x, x) \mid x \in X\} \subseteq X \times X.$$

If Δ_X is closed, we say that $(X, \mathscr{T})$ is a **Hausdorff space**. (Or $(X, \mathscr{T})$ is separated.)

Proposition 1.6.13 $(X, \mathscr{T})$ is a Hausdorff space if and only if $\forall (x,y) \in X \times X$, $x \neq y$, there exists $(U, V) \in \mathcal{V}_x(\mathscr{T}) \times \mathcal{V}_y(\mathscr{T})$, such that $U \cap V = \varnothing$.

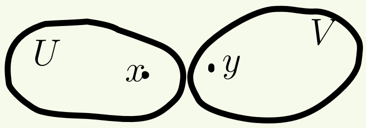

Figure 1.6.3: Hausdorff space

Proof
"$\Rightarrow$": If $(x, y) \in X \times X$, $x \neq y$, then $(x, y) \in (X \times X)\backslash\Delta_X$. There exists $(U, V) \subseteq (X \times X)\backslash\Delta_X$, such that $(x, y) \in U \times V$, so $(U \times V) \cap \Delta_X = \varnothing$. Thus $U \cap V = \varnothing$. (If $p \in U \cap V$, then $(p, p) \in (U \times V) \cap \Delta_X$.)
"$\Leftarrow$": For any $(x, y) \in X \times X$, $x \neq y$, $\exists U \in \mathscr{T}, \forall V \in \mathscr{T}, x \in U, y \in V, U \cap V = \varnothing$. Then $(x, y) \in U \times V$ and $(U \times V) \cap \Delta_X = \varnothing$. So Δ_X is closed. $\square$

Proposition 1.6.14 Let $(X, \mathscr{T})$ be a Hausdorff space. Let $\mathcal{F}$ be a non-degenerate filter on X. If $\mathcal{F}$ has a limit point[a], then its limit point is unique.

[a] Let $(X, \mathscr{T})$ be a topological space and $\mathcal{F}$ be a filter on X. If $p \in X$ is such that $\mathcal{V}_p(\mathscr{T}) \subseteq \mathcal{F}$, then we say that p is a limit point of $\mathcal{F}$.

Proof (By contradiction) Suppose that x and y are limit points of $\mathcal{F}$, $x \neq y$. Since X is Hausdorff, $\exists (U, V) \in \mathscr{T}^2, x \in U, y \in V, U \cap V = \varnothing$. Since x and y are limit points of $\mathcal{F}$, $U \in \mathcal{F}, V \in \mathcal{F}$. This contradicts the hypothesis that $\mathcal{F}$ is non-degenerate. $\square$

Example 1.6.15 Any metric space is Hausdorff.
Let (X, d) be a metric space, $\forall (x, y) \in X \times X, x \neq y, \mathrm{d}(x, y) > 0$. Let $\varepsilon = \frac{\mathrm{d}(x,y)}{2}$. $B(x, \varepsilon) \cap B(y, \varepsilon) = \varnothing$. In fact, if $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$,

$$\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y) < 2\varepsilon = \mathrm{d}(x, y).$$

Proposition 1.6.16 Let $(X, \mathscr{T})$ be a topological space, Y be a subset of X and $p \in X$.
(1) Let Z be a set and $f : Z \longrightarrow X$ be a mapping. Let $\mathcal{F}$ be a non-degenerate filter on Z. If p is a limit of f along $\mathcal{F}$, and if $f(Z) \subseteq Y$, then $p \in \overline{Y}$.
(2) Suppose that p has a countable neighborhood basis. If $p \in \overline{Y}$, then there exists a sequence $(y_n)_{n \in \mathbb{N}}$ in Y that converges to p.

Proof
(1) p is a limit of f along $\mathcal{F}$ if and only if $\mathcal{V}_p(\mathscr{T}) \subseteq f_*(\mathcal{F})$, or equivalently

$$\forall U \in \mathcal{V}_p(\mathscr{T}), f^{-1}(U) \in \mathcal{F}.$$

$f(f^{-1}(U)) \subseteq U \cap Y$, since $f(X) \subseteq Y$. Hence $U \cap Y \neq \varnothing$. So $p \in \overline{Y}$.
(2) Since p has a countable neighborhood basis, there exists a decreasing sequence $V_0 \supseteq V_1 \supseteq \dots$ of neighborhood of p such that $\{V_n \mid n \in \mathbb{N}\}$ forms a filter basis of $\mathcal{V}_p(\mathscr{T})$. For any $n \in \mathbb{N}$, $V_n \cap Y = \varnothing$, we take $y_n \in V_n \cap Y$. The sequence $(y_n)_{n\in\mathbb{N}}$ converges to p since $\forall n \in \mathbb{N}$, $\{y_k \mid k \in \mathbb{N}, k \geq n\} \subseteq V_n$. $\square$

Example 1.6.17 Let (X, d) be a semimetric space. $Y \subseteq X$, $\varepsilon > 0$. If $(y_n)_{n\in\mathbb{N}}$ is a sequence in $B(Y, \varepsilon)$, that converges to some $p \in X$, then

$$\lim_{n\to\infty} \mathrm{d}(y_n, Y) = \mathrm{d}(p, Y).$$

Therefore, $\overline{B(Y,\varepsilon)} \subseteq \overline{B}(Y, \varepsilon) := \{x \in X \mid \mathrm{d}(x, Y) \leq \varepsilon\}$.

Proposition 1.6.18 Let (X, d) be a semimetric space, $Y \subseteq X$ be a closed subset. $\forall x \in X\backslash Y$, $\mathrm{d}(x, Y) > 0$.

Proof $X\backslash Y$ is open, so $\exists\varepsilon > 0$ such that $B(X, \varepsilon) \subseteq X\backslash Y$. So $\forall y \in Y$, $\mathrm{d}(x, y) \geq \varepsilon$. Hence, $\mathrm{d}(x, y) \geq \varepsilon$. $\square$

Corollary 1.6.19 Let $(V, \|\cdot\|)$ be a semimetric space, W be a closed vector subspace of V. $Q = V/W$. Then the quotient seminorm

$$\|\cdot\|_Q : Q \longrightarrow R,$$

$$\alpha \longmapsto \inf_{x\in V, [x]=\alpha} \|x\|$$

is a norm.

Proof Let $\alpha \in Q\backslash\alpha$ and $x \in V$ such that $\alpha = [x]$. Since $\alpha \neq 0, x \notin W$.

$$0 < \mathrm{d}(x, W) := \inf_{y\in W} \|x - y\| = \inf_{\substack{x'\in V\\ [x']=\alpha}} \|x'\| = \|\alpha\|_Q.$$

$\square$

Proposition 1.6.20 If $(X, \mathscr{T})$ is a Hausdorff space, then, $\forall x \in X$, $\{x\}$ is closed.

Proof $\forall y \in X \backslash \{x\}, y \neq x$. So $\exists (U, V) \in \mathscr{T} \times \mathscr{T}, x \in U, y \in V$. $U \cap V = \varnothing$. So $V \subseteq X \backslash \{x\}$. Hence $X \backslash \{x\}$ is a neighborhood of y. $\square$

Remark 1.6.21 Let $(V, \|\cdot\|)$ be a seminorm space. $W \subseteq V$, $Q = V/W$ and $\|\cdot\|_Q$ is the quotient seminorm. The mapping $\pi : V \longrightarrow Q, x \longmapsto [x]$ is continuous since $\|[x]\|_Q \leq \|x\|$. If $\|\cdot\|_Q$ is a norm then $\{0_Q\}$ is closed (since Q is Hausdorff). So $W = \pi^{-1}(\{0_Q\}) = \ker(\pi)$ is closed. This shows that $\|\cdot\|_Q$ is a norm $\Leftrightarrow$ W is closed.

1.7 Completeness

Definition 1.7.1 Let (X, d) be a semimetric space, $Y \subseteq X$, we define the diameter of Y as $\mathrm{diam}(Y) := \sup_{(x,y) \in Y^2} \mathrm{d}(x, y)$. If $\mathrm{diam}(Y) < +\infty$, we say that Y is **bounded**.

Remark 1.7.2 Let (E, d) be a semimetric space.
(1) If A and B are subsets of E, then

$$A \subseteq B \Rightarrow \mathrm{diam}(A) \leq \mathrm{diam}(B).$$

(2) If $A \subseteq B \subseteq E$ and B is bounded, then A is bounded.
(3) If $A = \{x_1, \ldots, x_n\} \subseteq E$, $n \in \mathbb{N}_{\geq 1}$. Then

$$\mathrm{diam}(A) = \max_{(i,j) \in \{1,\ldots,n\}^2} \mathrm{d}(x_i, x_j) < +\infty.$$

So A is bounded.
(4) $\forall p \in X$, $\mathrm{diam}(\bar{B}(p, r)) \leq 2r$, $\forall r \in \mathbb{R}_{>0}$. In fact, $\forall (x, y) \in \bar{B}(p, r)^2$, $\mathrm{d}(x, y) \leq \mathrm{d}(x, p) + \mathrm{d}(p, y) \leq 2r$.

Proposition 1.7.3 Let (E, d) be a semimetric space, $A \subseteq E$. Suppose that A is bounded. Let $r = \mathrm{diam}(A)$. For any $p \in A$, $A \subseteq \bar{B}(p, r)$.

Proof $\forall x \in A$, $\mathrm{d}(p, x) \leq \mathrm{diam}(A) = r$. $\square$

Proposition 1.7.4 Let (E, d) be a semimetric space, $A \subseteq E$, $B \subseteq E$ and $(x_0, y_0) \in A \times B$. Then

$$\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B) + \mathrm{d}(x_0, y_0).$$

Proof Let $(x, y) \in (A \cup B)^2$.
Case 1. $\{x, y\} \in A$, $\mathrm{d}(x, y) \leq \operatorname{diam}(A)$.
Case 2. $\{x, y\} \in B$, $\mathrm{d}(x, y) \leq \operatorname{diam}(B)$.
Case 3. $x \in A,\ y \in B$,

$$\mathrm{d}(x, y) \leq \mathrm{d}(x, x_0) + \mathrm{d}(x_0, y_0) + \mathrm{d}(y_0, y) \leq \operatorname{diam}(A) + \operatorname{diam}(B) + \mathrm{d}(x_0, y_0).$$

Case 4. $x \in B,\ y \in A$. Same as case 3.

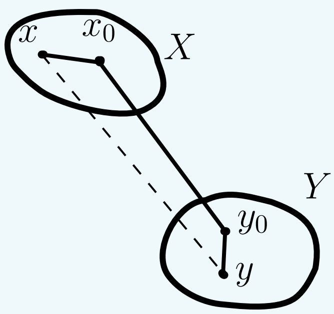

Figure 1.7.1: Prop. 1.7.4

□

Corollary 1.7.5 If $A \cap B \neq \varnothing$, then

$$\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + \operatorname{diam}(B).$$

Definition 1.7.6 Let (E, d) be a semimetric space, and $\mathcal{F}$ be a non-degenerate filter on E. If $\inf_{A \in \mathcal{F}} \operatorname{diam}(A) = 0$, we say that $\mathcal{F}$ is a **Cauchy filter**.

Example 1.7.7 $E =]0, 1]$. If $\mathcal{F}$ is generated by $]0, \varepsilon[$, $\varepsilon \leq 1$, then $\mathcal{F}$ is a Cauchy filter.

Proposition 1.7.8 Let (E, d) be a semimetric space and $\mathcal{F}$ be a non-degenerate filter on E. If $\mathcal{F}$ has a limit point p, then $\mathcal{F}$ is a Cauchy filter.

Proof $\forall \varepsilon > 0$, $\bar{B}(p, \frac{\varepsilon}{2}) \in \mathcal{F}$, $\operatorname{diam}(\bar{B}(p, \frac{\varepsilon}{2})) \leq \varepsilon$. So $\inf_{A \in \mathcal{F}} \operatorname{diam}(A) = 0$. □

Proposition 1.7.9 Let (E, d) be a semimetric space, and $\mathcal{F}$ be a Cauchy filter on E. Any adherent point of $\mathcal{F}$ is a limit point of $\mathcal{F}$.

Proof Let $\varepsilon > 0$. Let $A \in \mathcal{F}$ such that $\operatorname{diam}(A) < \frac{\varepsilon}{2}$. Note that $B(p, \frac{\varepsilon}{4}) \cap A \neq \varnothing$. So

$$\operatorname{diam}(B(p, \frac{\varepsilon}{4}) \cup A) \leq \operatorname{diam}(B(p, \frac{\varepsilon}{4})) + \operatorname{diam}(A) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

So $A \subseteq B(p, \varepsilon)$. □

Definition 1.7.10 Let (E, d) be a semimetric space. $I \subseteq \mathbb{N}$ be an infinite subset, $\mathcal{F}$ be the Fréchet filter on I. We say that a sequence $x := (x_n)_{n \in I} \in E^I$ is a Cauchy sequence if $x_*(\mathcal{F}) := \{U \subseteq E \mid x^{-1}(U) \in \mathcal{F}\}$ is a Cauchy filter. Or equivalently,

$$\forall \varepsilon > 0, \; \exists N \in I, \; \forall n, m \in I^2_{\geq N}, \; \mathrm{d}(x_n, x_m) \leq \varepsilon.$$

Proof "$\Rightarrow$" $\forall \varepsilon > 0, \; \exists U \in x_*(\mathcal{F}), \; \operatorname{diam}(U) \leq \varepsilon$. Since $x^{-1}(U) \in \mathcal{F}$, $\exists N \in \mathbb{N}$ such that $I_{\geq N} \subseteq x^{-1}(U)$. So $\{x_n \mid n \in I_{\geq N}\} \subseteq U$, which leads to

$$\forall (n, m) \in I^2_{\geq N}, \; \mathrm{d}(x_n, x_m) \leq \varepsilon.$$

"$\Leftarrow$" $\forall \varepsilon > 0, \; \exists N \in \mathbb{N}$, $\operatorname{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \varepsilon$. □

Proposition 1.7.11 Let (E, d) be a semimetric space, and $(x_n)_{n \in I}$ be a sequence in E.
(1) If $(x_n)_{n \in I}$ is convergent, then it is a Cauchy sequence.
(2) If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\{x_n \mid n \in I\}$ is bounded.
(3) If $(x_n)_{n \in I}$ is a Cauchy sequence, then any of its subsequence is a Cauchy sequence.
(4) If $(x_n)_{n \in I}$ is a Cauchy sequence, and if there exists a subsequence $(x_n)_{n \in J}$ that converges to some $p \in E$, then $(x_n)_{n \in I}$ converges to p.

Proof Let $\mathcal{F}_I$ be a Fréchet filter on I.
(1) $x_*(\mathcal{F}_I)$ has a limit point. So it is a Cauchy filter.

(2) $\exists N \in \mathbb{N}$ such that $\operatorname{diam}(\{x_n \mid n \in I_{\geq \mathbb{N}}\}) \leq 1 < +\infty$. So $\operatorname{diam}(\{x_n \mid n \in I\}) < +\infty$ since

$$\{x_n \mid n \in I\} = \{x_n \mid n \in I_{<N}\} \cup \{x_n \mid n \in I_{\geq N}\}.$$

(3) Let J be an infinite subset of I, $\lambda : J \longrightarrow I$ be the inclusion mapping, and $\mathcal{F}_J$ be the Fréchet filter. Then $\lambda_*(\mathcal{F}_J) \supseteq \mathcal{F}_I$.

$$(x \circ \lambda)_*(\mathcal{F}_J) = x_*(\lambda_*(\mathcal{F}_J)) \supseteq x_*(\mathcal{F}_I.)$$

Since $x_*(\mathcal{F}_I)$ is a Cauchy filter, $(x \circ \lambda)_*(\mathcal{F}_J)$ is a Cauchy filter.
(4) We keep the notation introduced in the proof of (3). If p is a limit point of $x \circ \lambda_*(\mathcal{F}_J)$. Then p is an adherent point of $x_*(\mathcal{F}_I)$. Since $x_*(\mathcal{F}_I)$ is a Cauchy filter, p is a limit point of $x_*(\mathcal{F}_I)$. □

Definition 1.7.12 Let (X, d) be a semimetric space. If any Cauchy filter on X has a limit point, then we say that (X, d) is complete.

Proposition 1.7.13 Let (X, d) be a metric space and Y be a subset of X. If (Y, d) is complete, then Y is a closed subset of X.

Proof Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in Y that converges in X to some $p \in X$. So $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, thus it converges to some $q \in Y$. Since X is Hausdorff, $p = q$. Then $\bar{Y} = Y$ (Since $\bar{Y}$ is the set of limits of all sequence in Y). So Y is closed. □

Example 1.7.14 $(\mathbb{R}, |\cdot|)$ is complete.
Let $(x_n)_{n\in I}$ be a Cauchy sequence in $\mathbb{R}$. Let $M > 0$ such that $\forall n \in I, |x_n| \leq M$. Hence,

$$-M \leq \liminf_{n\to+\infty} x_n \leq \limsup_{n\to+\infty} x_n \leq M.$$

By Bolzano-Weierstrass, $\exists J \subseteq I$ infinite such that $(x_n)_{n\in I}$ converges to $\limsup_{n\to+\infty} x_n \in \mathbb{R}$. So $(x_n)_{n\in I}$ converges.

Proposition 1.7.15 Let (X, d) be a semimetric space, (X, d) is complete if and only if any Cauchy sequence in X is convergent.

Proof It is suffices to prove "$\Leftarrow$". Suppose that all Cauchy sequence in X converges. Let $\mathcal{F}$ be a Cauchy filter on X. $\forall n \in \mathbb{N}$, let $A_n \in \mathcal{F}$, $\operatorname{diam}(A) < \frac{1}{n+1}$. $\forall n \in \mathbb{N}$, let $B_n = A_0 \cap \ldots \cap A_n \in \mathcal{F}$, $\operatorname{diam}(B) \leq \frac{1}{n+1}$. Take $x_n \in B_n$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Hence $(x_n)_{n\in\mathbb{N}}$ converges to $p \in X$.

We claim that p is a limit point of $\mathcal{F}$. If V is a neighborhood of p, then $\exists N \in \mathbb{N}$ such that $\{x_n \mid n \in \mathbb{N}_{\geq N}\} \subseteq V$. So $V \cap B_n \neq \varnothing$, $\forall n \in \mathbb{N}_{\geq N}$.

Let $A \in \mathcal{F}$, $A \cap B_n \neq \varnothing$. Take $y_n \in A \cap B_n$, $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. $\mathrm{d}(y_n, x_n) \leq \frac{1}{1+n}$, so $(y_n)_{n\in\mathbb{N}}$ converges to p. Thus $V \cap A \neq \varnothing$. $\square$

Proposition 1.7.16 Let n be a positive integer. For any $i \in \{1, \ldots, n\}$, let (X_i, d_i) be a semimetric space. Let $X = X_1 \times \ldots \times X_n$ and

$$\pi_i : X \longrightarrow X_i$$

$$(x_1, \ldots, x_n) \longmapsto x_i$$

be the projection mapping. We equip X with the product semimetric d

$$\mathrm{d}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \max_{i\in\{1,\ldots,n\}} \mathrm{d}_i(x_i, y_i).$$

Let $\mathcal{F}$ be a non-degenerate filter on X. For any $i \in \{1, \ldots, n\}$, let

$$\mathcal{F}_i = (\pi_i)_* (\mathcal{F}).$$

(1) The filter $\mathcal{F}$ is a Cauchy filter if and only if $\mathcal{F}_i$ is a Cauchy filter for any $i \in \{1, \ldots, n\}$.
(2) The filter $\mathcal{F}$ has a limit point if and only if $\mathcal{F}_i$ has a limit point for any $i \in \{1, \ldots, n\}$. If p_i is the limit point of $\mathcal{F}_i$, then $p = (p_1, \ldots, p_n)$ is a limit point of $\mathcal{F}$.
(3) If $(X_1, \mathrm{d}_1), \ldots, (X_n, \mathrm{d}_n)$ are complete, then (X, d) is complete.

Proof
(1) Since π_i are Lipschitzian, if $\mathcal{F}$ is a Cauchy filter, then $\mathcal{F}_i$ are all Cauchy filters. Conversely, let $\mathcal{F}$ be a non-degenerate filter such that $\mathcal{F}_i$ is a Cauchy filter for all $i \in \{1, ., n\}$. For any $\varepsilon > 0$, any $i \in \{1, \ldots, n\}$, there exists $A_i \in \mathcal{F}_i$ such that $\operatorname{diam}(A_i) < \varepsilon$. We define

$$A := A_1 \times \ldots \times A_n = \bigcap_{i=1}^{n} \pi_i^{-1}(A_i) \in \mathcal{F}.$$

For any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in A$, $\mathrm{d}(x,y) = \max_{i \in \{1,\ldots,n\}} \mathrm{d}_i(x_i, y_i) < \varepsilon$.
(2) π_i is Lipschitzian so also continuous $i \in \{1, \ldots, n\}$, therefore, if $p = (p_1, \ldots, p_n)$ a limit point of $\mathcal{F}$, then $\pi_i(p) = p_i$ is a limit point of $\mathcal{F}_i$.

Conversely, Suppose that p_i is a limit point of $\mathcal{F}_i$. Let U be a neighborhood of p. There exists $U_i \in \mathcal{F}_i$ neighborhood of p_i such that $U_1 \times \ldots \times U_n \subseteq U$.

$$U \supseteq \bigcap_{i=1}^{n} \pi_i^{-1}(U_i) \in \mathcal{F},$$

so $U \in \mathcal{F}$. □

Proposition 1.7.17 Let (X, d_X), (Y, d_Y) be two semimetric spaces, $f : X \longrightarrow Y$ be uniformly continuous. For any Cauchy filter $\mathcal{F}$ on X, $f_*(\mathcal{F})$ is also a Cauchy filter on Y.

Proof $\forall \varepsilon > 0,\ \exists \delta > 0$ such that

$$\forall (x,y) \in X \times X,\ \mathrm{d}_X(x,y) < \delta \Rightarrow \mathrm{d}_Y(f(x), f(y)) < \varepsilon.$$

Let $A \in \mathcal{F}$ such that $\operatorname{diam}(A) < \delta$, then $f(A) \in f_*(\mathcal{F})$ and $\operatorname{diam}(f(A)) < \varepsilon$. □

Proposition 1.7.18 Let (X, d), (X', d') be two semimetric spaces, $f : X \longrightarrow X'$ injective mapping. Assume that there exists positive constants C_1 and C_2 such that $\forall (x,y) \in X \times X$, $C_1 \cdot \mathrm{d}(x,y) \le \mathrm{d}'(f(x), f(y)) \le C_2 \cdot \mathrm{d}(x,y)$.
(1) For any sequence $(x_i)_{i \in I}$ in X, the sequence $(x_i)_{i \in I}$ is a Cauchy sequence if and only if the sequence $(f(x_i))_{i \in I}$ is a Cauchy sequence.
(2) For any $(x_i)_{i \in I}$ in X, $(x_i)_{i \in I}$ is a convergent if and only if $(f(x_i))_{i \in I}$ is convergent.
(3) If (X, d) is complete, then $(f(X), \mathrm{d}')$ is complete. $f(X)$ is closed in X' if in addition we assume that d' is a metric.

Proof Note that $f : X \longrightarrow f(X)$, $f^{-1} : f(X) \longrightarrow X$ are Lipschitzian. Apply the previous proposition. □

Theorem 1.7.19 Let $(K, |\cdot|)$ be a valued filed such that $(K, |\cdot|)$ is complete. Let V be a finite dimensional vector space over K.
(1) All possible norms on K are equivalent.
(2) For any norm $\|\cdot\|$ on V, the normed space $(V, \|\cdot\|)$ is complete.

(3) If we equip V with the topology induced by an arbitrary norm $\|\cdot\|$. For any normed vector space $(V, \|\cdot\|)$, any K-linear mapping $f : V \longrightarrow V'$ is bounded and $f(V)$ is closed in V'.

Proof Let $e := (e_i)_{i=1}^n$ be a basis of V. Then, the mapping

$$\|\cdot\|_e : V \longrightarrow \mathbb{R}_{>0}, \ \|a_1e_1 + \ldots + a_ne_n\| = \max_{i\in\{1,\ldots,n\}} |e_i|$$

is a norm on V. For $f : V \longrightarrow V'$,

$$\begin{aligned} \|f(a_1e_1 + \ldots + a_ne_n)\|' &= \|a_1f(e_1) + \ldots + a_nf(e_n)\|' && (1.7.1)\\ &\le |a_1| \|f(e_1)\|' + \ldots + |a_n| \|f(e_n)\|' && (1.7.2)\\ &\le \max_{i\in\{1,\ldots,n\}} |a_i| \sum_{i=1}^{n} \|f(e_i)\|'. && (1.7.3) \end{aligned}$$

Therefore the K-linear mapping $f : (V, \|\cdot\|_e) \longrightarrow (V', \|\cdot\|')$ is bounded. In particular, for any $\|\cdot\|$ on V, $\mathrm{Id} : (V, \|\cdot\|) \longrightarrow (V, \|\cdot\|)$ is bounded. So there exists $C > 0$, $\|\cdot\| \le C\|\cdot\|_e$.

To prove the theorem, we reason by induction with respect to $n = \dim(V)$.

In case when $n = 0$, $V = \{0\}$ has the unique norm $\|\cdot\|_0$ (constant mapping with the value $0 \in \mathbb{R}$). Any sequence in $\{0_V\}$ is constant, and hence convergent $((\{0\}, \|\cdot\|_0)$ complete). If $f : (\{0_V\}, \|\cdot\|) \longrightarrow (V, \|\cdot\|)$ is K-linear, then it is bounded. The unique $f(\{0_V\})$ is a one-point set, which is closed, since $(V, \|\cdot\|)$ is Hausdorff.

In case when $n = 1$, let e_1 be the basis of V, and $\|\cdot\|$ be an arbitrary norm on V.

$$\|a_1e_1\| = |a_1| \|e_1\| = \|a_1e_1\|_{e_1} \cdot \|e_1\|,$$

so $\|\cdot\|$ is equivalent to $\|\cdot\|_{e_1}$.
(2) Since $(K, |\cdot|)$ is complete, so $(V, \|\cdot\|_{e_1})$ is also complete, since

$$(K, |\cdot|) \longrightarrow (V, \|\cdot\|_{e_1})$$

$$a \longmapsto ae_1$$

is an isomorphism.
(3) We have seen that the mapping $f : (V, \|\cdot\|_{e_1}) \longrightarrow (V', \|\cdot\|')$ is bounded. So $f : (V, \|\cdot\|) \longrightarrow (V', \|\cdot\|')$ is also bounded. $f(V) = \{0\}$, or $(f(V), \|\cdot\|')$ is of dimension 1, so $(f(V), \|\cdot\|')$ is complete, so $f(V)$ is closed.
Suppose that the theorem holds for normed vector space of dimension $< n$. Then the case of dimension n:

Let $e = (e_i)_{i=1}^n$ be a basis of V. Let $W = \operatorname{span}_K(\{e_1, \ldots, e_n\})$. If $\|\cdot\|$ is a norm on V, then, by the induction hypothesis,
(i) $\|\cdot\|$ and $\|\cdot\|_{\underline{e}}$ are equivalent on W, that is, $\exists A > 0$ such that $\forall (a_1, \ldots, a_{n-1})$,

$$\max\{|a_1|, \ldots, |a_n|\} = \|a_1e_1 + \ldots + a_{n-1}e_{n-1}\|_{\underline{e}} \leq A\|a_1e_1 + \ldots + a_{n-1}e_{n-1}\|.$$

(ii) $(W, \|\cdot\|)$ is complete.
(iii) W is a closed subset of V.
Let $Q = V/W$, and $\|\cdot\|_Q$ be the quotient norm on Q. Let $(b_1, \ldots, b_n) \in K^n$,

$$s = b_1e_1 + \ldots + b_ne_n \in V,\ t = b_1e_1 + \ldots + b_ne_n + w \in V,\ \alpha = [s] = b_n[e_n] \in Q.$$

$$\|t\| = \|s - b_ne_n\| \leq \|s\| + |b_n|\,\|e_n\|.$$

Take $B = \frac{\|e_n\|}{\|[e_n]\|_Q} \in \mathbb{R}$, $\|s\| \geq \|\alpha\|_Q = |b_n|\,\|[e_n]\|_Q = B^{-1}|b_n|\|e_n\|$.

$$B^{-1}\|s\| \geq (\|t\| - |b_n| \cdot \|e_n\|)\, B^{-1}$$

$$\|s\| \geq B^{-1} \cdot |b_n| \cdot \|e_n\|.$$

$$\left(B^{-1} + 1\right)\|s\| \geq B^{-1}\|t\| \geq B^{-1}A^{-1}\max\{|b_1|, \ldots, |b_n|\}$$

Take $C = \min\left\{\frac{B^{-1}A^{-1}}{B^{-1}+1}, B^{-1} \cdot \|e_n\|\right\}$. Then $\|s\| \geq C\max\{|b_1|, \ldots, |b_n|\}$. So $\|\cdot\|$ is equivalent to $\|\cdot\|_{\underline{e}}$.
(2) Since $(V, \|\cdot\|)_e$ is complete and $\|\cdot\|$ is equivalent to $\|\cdot\|_e$, we obtain that $(V, \|\cdot\|)$ is also complete.
(3) Since $f : (V, \|\cdot\|_e) \longrightarrow (V', \|\cdot\|')$ is bounded, $f : (V, \|\cdot\|) \longrightarrow (V', \|\cdot\|')$ is also bounded.

If f is not injective, $\dim(f(V)) < n$, so $(f(V), \|\cdot\|')$ is complete, $f(V)$ is thus closed.

If f is injective, then $\|f(\cdot)\|'$ and $\|\cdot\|$ are equivalent norms on V. So $(f(V), \|\cdot\|')$ is complete. Hence $f(V)$ is closed.

□

Proposition 1.7.20 Let (X, d) be a complete semimetric space. Let $Y \subseteq X$ be a closed subset. Then (Y, d) is complete.

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. It is also a Cauchy sequence in X, so converges to some $l \in X$. Since Y is closed, one has $l \in Y$. So (Y, d) is complete. □

1.8 Compactness

Definition 1.8.1 Let X be a set and $\mathcal{F}$ be a non-degenerate filter on X. If there does not exists any non-degenerate filter $\mathcal{G}$ on X such that $\mathcal{F} \subsetneq \mathcal{G}$, then we say that $\mathcal{F}$ is an ultrafilter.

Proposition 1.8.2 For any non-degenerate filter $\mathcal{F}$ on X, there exists an ultrafilter on X containing $\mathcal{F}$.

Proof Let Θ be a set of non-degenerate filters containing $\mathcal{F}$, equipped with $\subseteq$. Let Θ_0 be a non-empty totally ordered subset. Let $\mathcal{F}' = \bigcup\limits_{\mathcal{H} \in \Theta_0} \mathcal{H}$.
We prove that $\mathcal{F}'$ is a filter.
(1) Let $(V_1, V_2) \in \mathcal{F}' \times \mathcal{F}'$, $\exists \mathcal{H}_1, \mathcal{H}_2$ in Θ_0, $V_1 \in \mathcal{H}_1,\ V_2 \in \mathcal{H}_2$. Since Θ_0 is totally ordered, either $\mathcal{H}_1 \subseteq \mathcal{H}_2$ or $\mathcal{H}_2 \subseteq \mathcal{H}_1$.
If $\mathcal{H}_1 \subseteq \mathcal{H}_2, V_1 \cap V_2 \in \mathcal{H}_2 \subseteq \mathcal{F}'$. If $\mathcal{H}_2 \subseteq \mathcal{H}_1, V_1 \cap V_2 \in \mathcal{H}_1 \subseteq \mathcal{F}'$.
(2) Let $V \in \mathcal{F}'$, let $\mathcal{H} \in \Theta_0$ such that $V \in \mathcal{H}$. $\forall U \in \wp(X), U \supseteq V$, one has $U \in \mathcal{H} \subseteq \mathcal{F}'$. So $\mathcal{F}' \in \Theta$. It is an upper bound of Θ_0. By Zorn's lemma, there exists maximal $\mathcal{G} \in \Theta$, it is an ultrafilter containing $\mathcal{F}$.

□

Proposition 1.8.3 Let X be a set and $\mathcal{F}$ be a non-degenerate filter on X. The following conditions are equivalent.
(1) $\mathcal{F}$ is an ultrafilter.
(2) $\forall A \in \wp(X)$, either $A \in \mathcal{F}$ or $X \backslash A \in \mathcal{F}$.
(3) $\forall (A, B) \in \wp(X)^2$, if $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Proof
(1)$\Rightarrow$(2): Suppose that $A \in \wp(X)$ such that $A \notin \mathcal{F}$ and $X \backslash A \notin \mathcal{F}$. Let $B \in \mathcal{F}$. If $B \cap A = \varnothing$, then $B \subseteq X \backslash A$. So $\mathcal{F} \cup \{A\}$ generates a non-degenerate filter $\mathcal{F}' \supsetneq \mathcal{F}$, contradiction.
(2)$\Rightarrow$(3): Suppose that $B \notin \mathcal{F}$. Then $X \backslash B \in \mathcal{F}$. So $(A \cup B) \cap (X \backslash B) \in \mathcal{F}$. So $A \in \mathcal{F}$.
(3)$\Rightarrow$(1): Let $\mathcal{F}'$ be a non-degenerate filter such that $\mathcal{F} \subsetneq \mathcal{F}'$. Take $A \in \mathcal{F}' \backslash \mathcal{F}$. Then $X = A \cup (X \backslash A)$. Since $A \notin \mathcal{F}, X \backslash A \in \mathcal{F} \subseteq \mathcal{F}'$. So $\varnothing = A \cap (X \backslash A) \in \mathcal{F}'$. Contradiction. □

Corollary 1.8.4 Let $f : X \longrightarrow Y$ be mapping of sets. If $\mathcal{F}$ is an ultrafilter on X, then $f_*(\mathcal{F})$ is an ultrafilter on Y.

Proof Let A and B be subsets of Y such that

$$A \cup B \in f_*(\mathcal{F}) := \{C \subseteq Y \mid f^{-1}(C) \in \mathcal{F}\}$$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \in \mathcal{F}.$$

Since $\mathcal{F}$ is an ultrafilter, $f^{-1}(A) \in \mathcal{F}$ or $f^{-1}(B) \in \mathcal{F}$. Namely, $A \in f_*(\mathcal{F})$ or $B \in f_*(\mathcal{F})$. □

Definition 1.8.5 Let $(X, \mathscr{T})$ be a topological space, $Y \subseteq X$ and $(U_i)_{i \in I}$ be a family of subset of X.
(1) If $Y \subseteq \bigcup_{i \in I} U_i$, we say that $(U_i)_{i \in I}$ is a **cover** of Y.
(2) If $\exists J \in I$ such that $Y \subseteq \bigcup_{j \in J} U_j$, we say that $(U_j)_{j \in J}$ is a **subcover** of $(U_i)_{i \in I}$ of Y.
(3) If $(U_i)_{i \in I} \in \mathscr{T}^I$ is a cover of Y, we say that ti is an **open cover** of Y.
(4) If I is a finite set and $(U_i)_{i \in I}$ is a cover of Y, we say that $(U_i)_{i \in I}$ is a **finite open cover**.

Proposition 1.8.6 Let $(X, \mathscr{T})$ be a topological space and $Y \subseteq X$. The following conditions are equivalent:
(1) For any ultrafilter $\mathcal{G}$ on X such that $Y \in \mathcal{G}$, $\mathcal{G}$ has a limit point in Y.
(2) For any non-degenerate filter $\mathcal{F}$ on X, such that $Y \in \mathcal{F}$, $\mathcal{F}$ has an adherent point in Y.
(3) Any open cover of Y has a finite subcover.

Proof
(1)⇒(2) Let $\mathcal{G}$ be an ultrafilter containing $\mathcal{F}$ and $x \in Y$ be a limit point of $\mathcal{G}$. For any $U \in \mathcal{V}_x(\mathcal{T})$, $U \in \mathcal{G}$, so $\forall A \in \mathcal{F}, U \cap A \neq \varnothing$.
(2)⇒(1) Let $\mathcal{G}$ be an ultrafilter, and let $x \in Y$ be an adherent point of $\mathcal{G}$. Then the filter $\mathcal{G}'$ generated by $\mathcal{G} \cup \mathcal{V}_x(\mathscr{T})$ is non-degenerate and contains $\mathcal{G}$. So $\mathcal{G}' = \mathcal{G}$. This means $\mathcal{V}_x(\mathscr{T}) \subseteq \mathcal{G}$.
(2)⇒(3) Let $(\mathcal{U}_i)_{i \in I}$ be an open cover of Y. Suppose that it does not have any finite subcover. For any $i \in I$, let $F_i = X \backslash U_i$. For any finite subset J of I,

$Y \subseteq \bigcup_{j\in\mathcal{F}} U_j$. So

$$Y \bigcap \left(X \setminus \bigcup_{j\in J} U_j \right) = Y \bigcup \bigcap_{j\in J} F_j \neq \varnothing.$$

So $\{F_i \mid i \in I\} \cup \{Y\}$ generates a non-degenerate filter $\mathcal{F}$. It has an adherent point of $x \in Y$. Since $Y \subseteq \bigcup_{i\in I} U_i$, $\exists i_0 \in I$, $x \in U_{i_0}$. So $U_{i_0} \in \mathcal{V}_x(\mathcal{T})$. This is impossible since $U_{i_0} \cap F_{i_0} = \varnothing$.

(3)⇒(2) Let $\mathcal{F}$ be a non-degenerate filter such that $Y \in \mathcal{F}$. Suppose that $\mathcal{F}$ does not have any adherent point in Y.

For any $y \in Y$, there exists open neighborhood U_y of y and $A_y \in \mathcal{F}$, such that $U_y \cap A_y = \varnothing$. Since $Y \subseteq \bigcup_{y\in Y} U_y$, $\exists\{y_1, \dots, y_n\} \subseteq Y$ such that $Y \subseteq \bigcup_{i=1}^{n} U_{y_i}$. Take $A = \left(\bigcap_{i=1}^{n} A_{y_i} \right) \cap Y \in \mathcal{F}$, $A \neq \varnothing$, $A \subseteq Y$.

$$\begin{aligned} A = A \cap Y \subseteq A \bigcap \bigcup_{i=1}^{n} U_{y_i} \\ = \bigcup_{i=1}^{n} (A \cap U_{y_i}) \\ \subseteq \bigcup_{i=1}^{n} (A_{y_i} \cap U_{y_i}) \\ = \varnothing. \end{aligned}$$

Contradiction. □

Definition 1.8.7 Let $(X, \mathscr{T})$ eb a topological space. If $Y \subseteq X$ satisfies the equivalent conditions described in the previous proposition, we say that Y is **compact**.

Proposition 1.8.8 Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be two topological spaces, $f : X \longrightarrow Y$ be a continuous mapping. If $F \subseteq X$ is compact, then $f(F)$ is also compact.

Proof Let $(V_i)_{i\in I}$ be a open cover of $f(F)$. Then

$$F \subseteq f^{-1}(f(F)) \subseteq \bigcup_{i\in I} f^{-1}(V_i).$$

Since F is closed, $\exists J \subseteq I$ finite such that $F \subseteq \bigcup_{j\in J} f^{-1}(V_j)$. Hence $f(F) \subseteq \bigcup_{j\in J} V_j$. □

Proposition 1.8.9 Let $(X, \mathscr{T})$ be a topological space, A be a compact subset of X, F is a closed subset of X, then $A \cap F$ is a compact subset of X.

Proof Let $(U_i)_{i\in I}$ be an open cover of $A \cap F$. Then

$$A \subseteq \left(\bigcup_{i\in I} U_i\right) \cup (X\backslash F).$$

So $\exists J \subseteq I$ finite, $A \subseteq (\bigcup_{j\in J} U_j) \cup (X\backslash F)$. Hence

$$A \cap F \subseteq \left(\bigcup_{j\in J} U_j\right) \cup ((X\backslash F) \cap F) = \bigcup_{j\in J} U_j.$$

□

Proposition 1.8.10 Let $(X, \mathscr{T})$ be a topological space, A be a compact subset of X.
(1) For any $x \in X\backslash A$, there exists open subsets U and V of X, such that $A \subseteq U$, $x \in V$ and $U \cap V = \varnothing$.
(2) A is closed.

Proof (1)⇒(2): $\forall x \in X\backslash A$, $\exists (U, V) \in \mathscr{T}^2$, $A \subseteq U$, $x \in V$, $U \cap V = \varnothing$. Hence, $V \subseteq X\backslash A$. So $X\backslash A$ is a neighborhood of x.
Proof of (1): $\forall y \in A$, $\exists U_y \in \mathscr{T}$, $V_y \in \mathscr{T}$ such that $y \in U_y$, $x \in V_y$, $U_y \cap V_y = \varnothing$. (Hausdorff) Since $A \subseteq \bigcup_{y\in A}$ and A is compact. $\exists \{y_1, \ldots, y_n\} \subseteq A$, $A \subseteq \bigcup_{i=1}^{n} U_{y_i}$.
Let $U = \bigcup_{i=1}^{n} U_{y_i}$, $V = \bigcap_{i=1}^{n} V_{y_i}$. These are open subsets of X, and $A \subseteq U$, $x \in V$.

$$U \cap V = \bigcup_{i=1}^{n} U_{y_i} \cap V \subseteq \bigcup_{i=1}^{n} U_{y_i} \cap V_{y_i} = \varnothing.$$

□

Corollary 1.8.11 Let $(X, \mathscr{T})$ be a topological space, and A and B be disjoint compact subset of X. There exist $(U, V) \in \mathscr{T}^2$, such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$.

Proof By (1) of the previous proposition.

$$\forall x \in B,\ \exists (U_x, V_x) \in \mathscr{T}^2,\ A \subseteq U_x,\ x \in V_x,\ U_x \cap V_x = \varnothing.$$

$B \subseteq \bigcup_{x \in B} V_x$. So, $\exists \{x_1, \dots, x_n\} \subseteq B$, such that $B \subseteq \bigcup_{i=1}^{n} V_{x_i}$. Let

$$U = \bigcap_{i=1}^{n} U_{x_i} \in \mathscr{T},\ V = \bigcup_{i=1}^{n} V_{x_i} \in \mathscr{T}.$$

Then, $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. $\square$