Spherical Functions

1 Legendre Polynomial

$$P_l(x) = \sum_{n=0}^{l} \frac{1}{(n!)^2} \frac{(l+n)!}{(l-n)!} \left(\frac{x-1}{2} \right)^n . \tag{1}$$

Where l is a natural number.

1.1 Rodrigues Formula:

$$P_l\left(x\right) = \frac{1}{2^l l!} \frac{\mathrm{d}^l}{\mathrm{d}x^l} \left(x^2 - 1\right)^l . \tag{2}$$

1.2 Differential Equation:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[\left(1 - x^2\right) \frac{\mathrm{d}P_l(x)}{\mathrm{d}x} \right] + l(l+1)P_l(x) = 0. \tag{3}$$

We can get

$$P_l(1) = 1 \tag{4}$$

via (1) and obtain

$$P_l(-x) = (-)^l P_l(x) \tag{5}$$

from (2).

1.3 Explicit Expression:

$$P_l(x) = \sum_{k=0}^{[l/2]} (-)^k \frac{(2l-2k)!}{2^l k! \, (l-k)! \, (l-2k)!} x^{l-2k}. \tag{6}$$

So,

$$P_{2l}(0) = (-)^l \frac{(2l)!}{\left(2^l l!\right)^2}, \quad P_{2l+1}(0) = 0. \tag{7}$$

1.4 Orthogonal Completeness

$$\int_{-1}^{1} P_k(x) P_l(x) \, \mathrm{d}x = \frac{2}{2l+1} \delta_{kl}. \tag{8}$$

1.5 Generating Function

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l, \quad |t| < \min \left| x \pm \sqrt{x^2 - 1} \right| . \tag{9}$$

1.6 Recursive Relation

Use (9) to get

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x), \tag{10}$$

and

$$P_l(x) = P'_{l+1}(x) - 2xP'_l(x) + P_{l-1}(x), \tag{11}$$

$$P'_{l+1}(x) = xP'_l(x) + (l+1)\,P_l(x), \tag{12}$$

2 Associated Legendre Function

2.1 Differential Equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_l^m(x)\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P_l^m(x) = 0, \tag{13}$$

where, $m \leq l$ are natural numbers. We have the solution

$$P_l^m(x) = (-)^m(1-x^2)^{\frac{m}{2}}\frac{\mathrm{d}^m}{\mathrm{d}x}P_l(x), \tag{14}$$

$$P_l^{-m}(x) = (-)^m\frac{(l-m)!}{(l+m)!}P_l^m(x). \tag{15}$$

2.2 Orthogonal Relation

$$\int_{-1}^{1} P_l^m(x)P_k^m(x)\,\mathrm{d}x = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{lk}, \tag{16}$$

3 Spherical Harmonics

Special Harmonic functions are the normalized associated Legendre functions.

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}. \tag{17}$$

3.1 Orthogonal Relation

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi} Y_{l'}^{m'}(\theta,\phi)\,Y_l^{m*}(\theta,\phi)\sin\theta\,\mathrm{d}\theta = \delta_{ll'}\delta_{mm'}, \tag{18}$$

3.2 Parity

Under the transformation of $\mathbf{r} \to -\mathbf{r}$,

$$Y_l^m(\pi-\theta,\phi+\pi) = (-1)^m\,Y_l^m(\theta,\phi). \tag{19}$$

3.3 Addition Theorem

$$P_l\left(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'\right) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^{m*}\left(\hat{\mathbf{n}}\right) Y_l^{m}\left(\hat{\mathbf{n}}'\right). \tag{20}$$