The Electromagnetic Radiation

1 Preamble and Notation

We start from D'Alembert equation considering the flat metric diag$(-1, 1, 1, 1)$:

$$\partial^\mu \partial_\mu A^\alpha = -\mu_0 J^\alpha, \tag{1}$$

where

$$x^\alpha = \left(ct, x^1, x^2, x^3\right), \partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{2}$$

$$A^\alpha = \left(\frac{\phi}{c}, \mathbf{A}\right), J^\alpha = (\rho c, \mathbf{J}). \tag{3}$$

The solution of the above equation is (suppose a Green function G_R).

$$A^\mu(\mathbf{x}) = \mu_0 \int \mathrm{d}^4\mathbf{x}' \; G_R(\mathbf{x}, \mathbf{x}') J^\mu(\mathbf{x}'). \tag{4}$$

G_R is the solution of

$$\partial^\mu \partial_\mu G_R(\mathbf{x}, \mathbf{x}') = \delta^4(\mathbf{x} - \mathbf{x}'). \tag{5}$$

Since it should satisfies the law of causality, we pick[1]

$$G_R(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \Theta(t - t') \delta((x^\mu - x'^\mu)(x_\mu - x'_\mu)), \tag{6}$$

exactly,

$$G_R(\mathbf{x}, \mathbf{x}') = \frac{\delta(t - t' - \frac{|\mathbf{r} - \mathbf{r}'|}{c})}{4\pi |\mathbf{r} - \mathbf{r}'|}. \tag{7}$$

Let

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \; \tilde{t} = t - \frac{R}{c}. \tag{8}$$

We agree that any time dependent variable $f(t)$ with[2] $\tilde{}$ means replace t as $\tilde{t}$. Which means $\tilde{f}(t) = f\left(\tilde{t}\right)$. Or[3] we will write it into the form of $\mathcal{R}[f(t)](\mathbf{r}_1, \mathbf{r}_2) =$

[1]Another one is of the form: $t + r/c$, space and time is some sense dissymmetry since they take different sign, so when using Fourier transform to solve it, $1/(k^2 - \frac{\omega^2}{c^2})$ contribute a pair of pole. Heaviside step function $\Theta = \mathbb{1}_{\mathbb{R}_{\geq 0}}$ ensure the causality law valid.

[2]Some books or articles use $[\,\cdot\,]$ to denote the retarded term.

[3]Seldom do that, since this operator is not a linear operator in the whole space. But it will help us clarify the relation of variables when doing complex work.

$f(\tilde{t})$ to remind us that it is not a simple function. For convenience, if there won't be any confusion, we neglect $\mathbf{r}'$ that have appeared in the inner function $f(\mathbf{r}', t)$, and write the retarded term as $\mathcal{R}[f(\mathbf{r}', t)](\mathbf{r})$. We even neglect $\mathbf{r}$ that point out the position where we evaluate the retarded term. Also, sometimes we will use lower index "ret" to emphasize the expressions.

Deduce the term of time in (4), we obtain,

$$A^\mu(\mathbf{x}) = \mu_0 \int \frac{\tilde{J}^\mu(\mathbf{x}')}{4\pi R}\, \mathrm{d}\mathbf{r}' \,. \tag{9}$$

2 Properties of Retardation Notation

Suppose $J = J(\mathbf{x})$, then

$$\frac{\partial \mathcal{R}[J(\mathbf{r}, t)]}{\partial x^\mu} = \frac{\partial \mathbf{r}}{\partial x^\mu} \cdot \nabla J(\mathbf{r}, \tilde{t}) + \frac{\partial \tilde{t}}{\partial x^\mu} \cdot \frac{\partial J(\mathbf{r}, \tilde{t})}{\partial \tilde{t}}. \tag{10}$$

In particular,

$$\frac{\partial}{\partial t}\mathcal{R}[J(\mathbf{r}, t)] = \mathcal{R}\left[\frac{\partial}{\partial t} J(\mathbf{r}, t)\right]. \tag{11}$$

$$\nabla \mathcal{R}[J(\mathbf{r}, t)] = \mathcal{R}[\nabla J(\mathbf{r}, t)] - \mathcal{R}\left[\frac{1}{c}\frac{\partial J(\mathbf{r}, t)}{\partial t}\right] \nabla R. \tag{12}$$

Note that $\mathcal{R}$ is equipped with a position $\mathbf{r}_\mathcal{R}$, so we can define another derivative, which act on the position $\mathbf{r}_\mathcal{R}$.

$$\nabla_{r_\mathcal{R}} \mathcal{R}[J(\mathbf{r}, t)] = -\frac{1}{c}\frac{\partial J(\mathbf{r}, \tilde{t})}{\partial \tilde{t}} \cdot \nabla_{r_\mathcal{R}} R = -\mathcal{R}\left[\frac{1}{c}\frac{\partial J(\mathbf{r}, t)}{\partial t}\right] \nabla_{r_\mathcal{R}} R. \tag{13}$$

Since $\nabla R = -\nabla_{r_\mathcal{R}} R$, we have

$$\nabla \mathcal{R}[J(\mathbf{r}, t)] + \nabla_{r_\mathcal{R}} \mathcal{R}[J(\mathbf{r}, t)] = \mathcal{R}\left[\nabla J(\mathbf{r}, t)\right] \tag{14}$$

By symmetry of definition,

$$\nabla_1 \mathcal{R}[f](\mathbf{r}_1, \mathbf{r}_2) = -\nabla_2 \mathcal{R}[f](\mathbf{r}_1, \mathbf{r}_2). \tag{15}$$

Example: Verify that our solution (9) satisfies Lorentz gauge.

$$\frac{\partial \phi}{\partial t} = \int \frac{1}{4\pi\varepsilon_0 R} \cdot \frac{\partial \tilde{\rho}(\mathbf{r}', t)}{\partial t}\, \mathrm{d}V' = \mathcal{R}\left[\int \frac{1}{4\pi\varepsilon_0 R} \cdot \frac{\partial \rho(\mathbf{r}', t)}{\partial t}\, \mathrm{d}V'\right] \tag{16}$$

$$\nabla \cdot \mathbf{A} = \frac{\mu_0}{4\pi} \int \nabla \cdot \left(\frac{\tilde{\mathbf{J}}(\mathbf{r}', t)}{R}\right) \mathrm{d}V' \,. \tag{17}$$

Use the conservation of charge to make a connection of two expressions above. By (14),

$$-\mathcal{R}\left[\frac{\partial \rho(\mathbf{r}', t)}{\partial t}\right] = \mathcal{R}\left[\nabla' \cdot \mathbf{J}(\mathbf{r}', t)\right] = \nabla' \cdot \mathcal{R}\left[\mathbf{J}(\mathbf{r}', t)\right] + \nabla \cdot \mathcal{R}\left[\mathbf{J}(\mathbf{r}', t)\right]. \tag{18}$$

So,

$$
\begin{aligned}
\frac{1}{c^2}\frac{\partial\phi}{\partial t}+\nabla\cdot\mathbf{A} =&\frac{\mu_0}{4\pi}\int\left[-\frac{1}{R}\left(\nabla'\cdot\tilde{\mathbf{J}}+\nabla\cdot\tilde{\mathbf{J}}\right)+\left(\frac{\nabla\cdot\tilde{\mathbf{J}}}{R}-\frac{\nabla R}{R^2}\cdot\tilde{\mathbf{J}}\right)\right]\mathrm{d}V' \\
=&-\frac{\mu_0}{4\pi}\int\left[\frac{1}{R}\left(\nabla'\cdot\tilde{\mathbf{J}}\right)+\nabla\left(\frac{1}{R}\right)\cdot\tilde{\mathbf{J}}\right]\mathrm{d}V' \\
=&-\frac{\mu_0}{4\pi}\int\left[\frac{\nabla'\cdot\tilde{\mathbf{J}}}{R}-\nabla'\cdot\left(\frac{1}{R}\right)\cdot\tilde{\mathbf{J}}\right]\mathrm{d}V \\
=&-\frac{\mu_0}{4\pi}\int\nabla'\cdot\left(\frac{\tilde{\mathbf{J}}}{R}\right)\mathrm{d}V'=-\frac{\mu_0}{4\pi}\oint_S\frac{\tilde{J}\cdot\mathrm{d}\mathbf{S}}{R}=0.
\end{aligned}
\tag{19}
$$

3 Distribution of Electromagnetic Field

3.1 Precise Solution

By

$$
\mathbf{E}=-\nabla\phi-\frac{\partial\mathbf{A}}{\partial t},\ \mathbf{B}=\nabla\times\mathbf{A}, \tag{20}
$$

and (9)

$$
\mathbf{E}(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\int\left[\frac{\rho_{\mathrm{ret}}(\mathbf{r}',t)}{R^2}\nabla R+\frac{\nabla R}{cR}\frac{\partial\rho_{\mathrm{ret}}(\mathbf{r}',t)}{\partial t}-\frac{1}{c^2R}\frac{\partial\mathbf{J}_{\mathrm{ret}}}{\partial t}\right]\mathrm{d}V'\,, \tag{21}
$$

$$
\mathbf{B}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int\left[\mathbf{J}_{\mathrm{ret}}\times\frac{\nabla R}{R^2}-\frac{\nabla R}{cR}\times\frac{\partial\mathbf{J}_{\mathrm{ret}}}{\partial t}\right]\mathrm{d}V'\,. \tag{22}
$$

3.2 Time-harmonic Solution

It is difficult to do the integral if there's some complicated retarded term, we try to solve some simple case. Suppose all the current evolute in the form of $\mathbf{j}_0\mathrm{e}^{-\mathrm{i}\omega t}$, then

$$
\mathbf{A}=\frac{\mu_0\mathrm{e}^{-\mathrm{i}\omega t}}{4\pi}\int\frac{\mathbf{j}_0\mathrm{e}^{\mathrm{i}kR}}{R}\,\mathrm{d}V'\,. \tag{23}
$$

This is also difficult to calculate if the distribution is not simple. We can use different approximations in the following according to the specific case.

- Near-field approximation: $kR\ll 1$, retardation effect can be neglected.
- Far-field approximation: $R\approx r\gg r'$, $\mathbf{A}\approx\frac{\mu_0}{4\pi r}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\int\mathbf{j}_0\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\,\mathrm{d}V'$.
- Long-wavelength approximation: In addition, $\mathbf{k}\cdot\mathbf{r}'\ll 1$, $\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'}\approx 1-\mathrm{i}\mathbf{k}\cdot\mathbf{r}'$.

Example: A linear antenna of length d oscillates in a full-wave mode with angular frequency $\omega=\frac{2\pi c}{d}$. Find the radiated power per unit solid angle.

To calculate the energy flux, we only need to consider a large sphere, so we take the far-field approximation,

$$\begin{aligned}\mathbf{A} =&\frac{\mu_0 I_0}{4\pi r}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\int_{-\frac{\lambda}{2}}^{+\frac{\lambda}{2}}\sin(kz')\mathrm{e}^{-\mathrm{i}kz'\cos\theta}\,\mathrm{d}z'\,\hat{\mathbf{z}}\\ =&-\frac{\mathrm{i}\mu_0 I_0}{2\pi kr}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\frac{\sin\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\hat{\mathbf{z}}\end{aligned} \tag{24}$$

Retain the leading term of $\frac{1}{r}$ in $\mathbf{B}$ and $\mathbf{E}$, $\nabla\to\mathrm{i}\mathbf{k}$,

$$\mathbf{B} = -\frac{\mu_0 I_0}{2\pi r}\frac{\sin\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\hat{\boldsymbol{\varphi}}, \tag{25}$$

$$\mathbf{E} = \frac{\mu_0 I_0 c}{2\pi r}\frac{\sin\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\mathrm{e}^{\mathrm{i}(kr-\omega t)}\hat{\boldsymbol{\theta}}. \tag{26}$$

$$\bar{\mathbf{S}} = \frac{c\mu_0 I_0^2}{8\pi^2 r^2}\left[\frac{\sin\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2\hat{\mathbf{r}}. \tag{27}$$

$$\frac{\mathrm{d}\bar{P}}{\mathrm{d}\Omega} = \frac{c\mu_0 I_0^2}{8\pi^2}\left[\frac{\sin\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2. \tag{28}$$

3.3 Multiple Expansion

If we have the condition of long-wavelength approximation, we can do the multiple expansion of the vector potential. Also, we retain the term of $\frac{1}{r}$.

$$\mathbf{A}\approx\frac{\mu_0}{4\pi r}\int\tilde{\mathbf{J}}\,\mathrm{d}V'\,. \tag{29}$$

We first introduce two lemmas.

Lemma.

$$\int\mathbf{J}\,\mathrm{d}V = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}. \tag{30}$$

Proof.

$$\mathbf{J} = \mathbf{J}\cdot\nabla\mathbf{r} = \nabla\cdot(\mathbf{J}\mathbf{r}) - (\nabla\cdot\mathbf{J})\mathbf{r} = \nabla\cdot(\mathbf{J}\mathbf{r}) + \frac{\partial\rho}{\partial t}\mathbf{r}. \tag{31}$$

The following is easy. □

Lemma.

Let

$$\mathbf{m} = \frac{1}{2}\int\mathbf{r}\times\mathbf{J}\,\mathrm{d}V\,,\ \mathbf{D} = \int 3\rho\mathbf{r}\mathbf{r}\,\mathrm{d}V\,, \tag{32}$$

then,

$$\mathbf{F}\cdot\int \mathbf{r}\mathbf{J}\,\mathrm{d}V=\frac{1}{6}\mathbf{F}\cdot\dot{\mathbf{D}}-\mathbf{F}\times\mathbf{m}. \tag{33}$$

where $\mathbf{F}$ is an arbitrary vector.

Proof. We have

$$\nabla\cdot(\mathbf{J}\mathbf{r}\mathbf{r})=(\nabla\cdot\mathbf{J})\,\mathbf{r}\mathbf{r}+\mathbf{J}\mathbf{r}+\mathbf{r}\mathbf{J}, \tag{34}$$

$$\mathbf{F}\cdot(\mathbf{J}\mathbf{r}-\mathbf{r}\mathbf{J})=\mathbf{F}\times(\mathbf{r}\times\mathbf{J})\,, \tag{35}$$

So,

$$\mathbf{r}\mathbf{J}=\frac{\nabla\cdot(\mathbf{J}\mathbf{r}\mathbf{r})+\frac{\partial\rho}{\partial t}\mathbf{r}\mathbf{r}+(\mathbf{r}\mathbf{J}-\mathbf{J}\mathbf{r})}{2}. \tag{36}$$

The following is easy. □

Now we expand $\mathcal{R}[\mathbf{J}(\mathbf{r}',t)](\mathbf{r}''=\mathbf{r}',\mathbf{r})$ around $\mathbf{r}''=0$,

$$\mathcal{R}[\mathbf{J}(\mathbf{r}',t)](\mathbf{r})=\mathcal{R}[\mathbf{J}(\mathbf{r}',t)](0,\mathbf{r})-\mathbf{r}'\cdot\nabla\mathcal{R}[\mathbf{J}(\mathbf{r}',t)](0,\mathbf{r})+o(\mathbf{r}'). \tag{37}$$

The minus sign comes from exchanging the derivative of retardation positions. In this cases, it is convenient to use retard-time-transition operator $\mathcal{T}$ defined as $(\mathcal{T}_a f)\,(t)=f(t-a)$. Then

$$\mathcal{R}[\mathbf{J}(\mathbf{r}',t)](\mathbf{r})=\mathcal{T}_{\frac{r}{c}}\mathbf{J}(\mathbf{r}',t)-\mathbf{r}'\cdot\nabla\left(\mathcal{T}_{\frac{r}{c}}\mathbf{J}(\mathbf{r}',t)\right)+o(\mathbf{r}'). \tag{38}$$

By (13),

$$\mathcal{R}[\mathbf{J}]=\mathcal{T}\mathbf{J}+\mathbf{r}'\cdot\nabla r\frac{(\mathcal{T}\dot{\mathbf{J}})}{c}+o(\mathbf{r}'). \tag{39}$$

By two lemmas above, we have

$$\mathbf{A}\approx\frac{\mu_0}{4\pi r}\mathcal{T}_{\frac{r}{c}}\left[\dot{\mathbf{p}}-\frac{\hat{\mathbf{r}}}{c}\times\dot{\mathbf{m}}+\frac{1}{6}\frac{\hat{\mathbf{r}}}{c}\cdot\ddot{\mathbf{D}}\right]. \tag{40}$$

In the following part, we can use $\nabla\to-\frac{\hat{\mathbf{r}}}{c}\frac{\partial}{\partial t}$ to calculate $\mathbf{E}$ and $\mathbf{B}$ retaining the leading term of $\frac{1}{r}$. If we only consider the electric dipole term, the energy flow density is

$$\mathbf{S}(t)=\mathcal{T}\left[\frac{\mu_0(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})^2}{16\pi^2r^2c}\hat{\mathbf{r}}\right]. \tag{41}$$

4 Point Charge

Let the particle's worldline be parameterized by its proper time τ:

$$x_0^\mu(\tau) = (ct_0(\tau), \mathbf{r}_0(t(\tau)))\,. \tag{42}$$

The four-velocity is

$$U_0^\mu(\tau) = \frac{\mathrm{d}x_0^\mu}{\mathrm{d}\tau} = \gamma(\tau)\,(c, \mathbf{v}_0)\,, \tag{43}$$

with

$$\gamma(\tau) = \frac{\mathrm{d}t_0}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}. \tag{44}$$

The four-current density of a point charge is

$$J^\mu(\mathbf{x}') = qc\int_{-\infty}^{+\infty} U_0^\mu(\tau)\delta^{(4)}\left(\mathbf{x}' - \mathbf{x}_0(\tau)\right)\mathrm{d}\tau \tag{45}$$

Now we want to reduce the integral into 3D form, by the property of Dirac delta function,

$$\delta(ct' - ct_0) = \frac{1}{c}\delta(t' - t_0), \tag{46}$$

So,

$$\begin{aligned} J^\mu(\mathbf{x}') =& q\int_{-\infty}^{+\infty} U_0^\mu(t_0)\delta(t'-t_0)\delta^{(3)}(\mathbf{r}' - \mathbf{r_0})\frac{\mathrm{d}\tau}{\mathrm{d}t_0}\,\mathrm{d}t_0 \\ =& \frac{qU_0^\mu(t')}{\gamma(t')}\delta^{(3)}\left(\mathbf{r}' - \mathbf{r}_0(t')\right). \end{aligned} \tag{47}$$

Thus,

$$\begin{aligned} \mathbf{A}(\mathbf{x}) =& \frac{\mu_0 q}{2\pi}\int \frac{U_0^\mu(t')}{\gamma(t')}\Theta(t-t')\delta((x^\mu - x'^\mu)(x_\mu - x'_\mu))\delta^{(3)}(\mathbf{r}' - \mathbf{r}_0(t'))\,\mathrm{d}^4\mathbf{x}' \\ =& \frac{\mu_0 qc}{2\pi}\int \frac{U_0^\mu(t_0)}{\gamma(t_0)}\Theta(t-t_0)\delta((x^\mu - {x_0}^\mu)(x_\mu - {x_0}_\mu))\,\mathrm{d}t_0\,. \end{aligned} \tag{48}$$

(The integral variable should be t', but we substitute it as t_0 for convenience.)

If we denote $R^\mu = x^\mu - x_0^\mu$, and the solution of $R^\mu(t_0)R_\mu(t_0) = 0$, are $t_{\mathrm{ret}}(\mathbf{x}, \mathbf{r}_0)$ and $t_{\mathrm{adv}}(\mathbf{x}, \mathbf{r}_0)$, then

$$\delta\left(R^\mu R_\mu\right) = \sum_{i=\text{ ret, adv}} \frac{\delta(t_0 - t_i)}{\left|\frac{\mathrm{d}\tau}{\mathrm{d}t_0}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(R^\mu(\tau)R_\mu(\tau)\right)\right|}, \tag{49}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[R^\mu(\tau)R_\mu(\tau)\right] = -2R_\mu(\tau)U_0^\mu(\tau). \tag{50}$$

So,

$$\Theta(t-t_0)\delta(R^\mu R_\mu) = \frac{\gamma(t_0)\delta(t_0 - t_{\mathrm{ret}})}{2|R_\mu U_0^\mu(t_0)|}. \tag{51}$$

Hence,

$$
\begin{aligned}
A^\mu(\mathbf{x}) =& \frac{\mu_0 q c}{2\pi} \int \frac{U_0^\mu(t_0)}{\gamma(t_0)} \Theta(t-t_0)\delta(R^\mu R_\mu)\,\mathrm{d}t_0 \\
=& \frac{\mu_0 q c}{4\pi} \int \frac{U_0^\mu(t_0)\delta(t_0 - t_{\mathrm{ret}}(\mathbf{x},\mathbf{r}_0))}{|R_\sigma U_0^\sigma(t_0)|}\,\mathrm{d}t_0 \\
=& \frac{\mu_0 q c}{4\pi} \mathcal{R}\left[\frac{U_0^\mu(t)}{|R_\sigma U_0^\sigma(t)|}\right].
\end{aligned}
\tag{52}
$$

Now we try to simplify it to a three-dimensional form.

We have $R^\alpha = (c\Delta t, \mathbf{R})$, with $c\Delta t = R$, $R_\sigma U_0^\sigma = c\gamma\left[(\mathbf{R}\cdot\mathbf{v}_0) - R\right]$, so,

$$
\mathbf{A}^\alpha(\mathbf{x}) = \frac{\mu_0 q}{4\pi}\mathcal{R}\left[\frac{(c,\mathbf{v}_0)}{R - \mathbf{R}\cdot\mathbf{v}_0/c}\right]. \tag{53}
$$

This is called the Liénard-Wiechert potential.