FUNDAMENTAL ALGEBRA & ANALYSIS

Contents

Chapter 1

Differential Calculus

1.1 Landau symbol

In this section, we fix a complete valued field $(K, |\cdot|)$ and a normed vector space $(V, \|\cdot\|)$ over K.

Definition 1.1.1 Let X be a set, $f : X \longrightarrow V$, $g : X \longrightarrow \mathbb{R}_{\geq 0}$ be mappings. Let $Y \subseteq X$ be a subset. We use the expression

$$f(x) = \mathcal{O}(g(x))$$

to denote the statement:

$$\exists C > 0, \forall x \in Y, \|f(x)\| \leq C \cdot g(x).$$

Let $\mathcal{F}$ be a filter on X, we use the expression

$$f(x) = \mathcal{O}\left(g(x)\right) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists C > 0, \exists A \in \mathcal{F}, \|f(x)\| \leq C \cdot g(x), \forall x \in A.$$

We use the expression

$$f(x) = o\left(g(x)\right) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists \varepsilon : X \longrightarrow \mathbb{R}_{\geq 0}, \exists A \in \mathcal{F}, \lim_{\mathcal{F}} \varepsilon = 0 \text{ and } \forall x \in A, \|f(x)\| \leq \varepsilon(x)g(x).$$

Proposition 1.1.2 Let X be a set and $\mathcal{F}$ be a filter on X.
(1) Let $f : X \longrightarrow V$, $g : X \longrightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f(x) = o(g(x))$ along $\mathcal{F}$, then $f(x) = \mathcal{O}(g(x))$ along $\mathcal{F}$.
(2)

1. Let $f_1 : X \longrightarrow V$, $f_2 : X \longrightarrow V$ and $g : X \longrightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f_1(x) = \mathcal{O}(g(x))$ and $f_2(x) = \mathcal{O}(g(x))$ along $\mathcal{F}$, then $f_1(x) + f_2(x) = \mathcal{O}(g(x))$ along $\mathcal{F}$.

2. Let $f_1 : X \longrightarrow V$, $f_2 : X \longrightarrow V$ and $g : X \longrightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f_1(x) = o(g(x))$ and $f_2(x) = o(g(x))$ along $\mathcal{F}$, then $f_1(x)+f_2(x) = o(g(x))$ along $\mathcal{F}$.

(3) Let $\lambda : X \longrightarrow K$, $f : X \longrightarrow V$, $g : X \longrightarrow \mathbb{R}_{\geq 0}$, $h : X \longrightarrow \mathbb{R}_{\geq 0}$ be mappings.

1. If $\lambda(x) = \mathcal{O}\left(g(x)\right)$ along $\mathcal{F}$, $f(x) = \mathcal{O}\left(h(x)\right)$ along $\mathcal{F}$, then
$$(\lambda f)(x) = \lambda(x)f(x) = \mathcal{O}\left(g(x)h(x)\right).$$

2. If $\lambda(x) = \mathcal{O}\left(g(x)\right)$ along $\mathcal{F}$, $f(x) = o\left(h(x)\right)$ along $\mathcal{F}$, or if $\lambda(x) = o\left(g(x)\right)$ along $\mathcal{F}$, $f(x) = \mathcal{O}\left(h(x)\right)$ along $\mathcal{F}$, then
$$\lambda(x)f(x) = o\left(g(x)h(x)\right).$$

Proof
(1) We have $\varepsilon : X \longrightarrow \mathbb{R}_{\geq 0}$, $A \in \mathcal{F}$ such that $\lim_{\mathcal{F}} \varepsilon = 0$ and $\forall x \in A$, $\|f(x)\| \leq \varepsilon(x)g(x)$. Since $\lim_{\mathcal{F}} \varepsilon = 0$, there exists $B \in \mathscr{T}$ such that $\forall x \in B$, $|\varepsilon(x)| < 1$, hence $\forall x \in A \cap B$, $\|f(x)\| \geq (x)$.
(2)

1. $A_1, A_2 \in \mathcal{F}, C_1, C_2 > 0, \forall x \in A_1, \|f_1(x)\| \leq C_1 g(x), \forall x \in A_2, \|f_2(x)\| \leq C_2 g(x)$. So $f_1(x) + f_2(x) = \mathcal{O}(g(x))$

2. Let $\varepsilon_1 : X \longrightarrow \mathbb{R}_{\geq 0}$, $\varepsilon_2 : X \longrightarrow \mathbb{R}_{\geq 0}$, $A \in \mathcal{F}$, $\lim_{\mathcal{F}} \varepsilon_1 = \lim_{\mathcal{F}} \varepsilon_2 = 0$. $\forall x \in A_1, \|f_1(x)\| \leq \varepsilon_1(x) \cdot g(x), \forall x \in A_2, \|f_2(x)\| \leq \varepsilon_2(x)g(x)$. So $\lim_{\mathcal{F}} \varepsilon_1 + \varepsilon_2 = 0$.
$$\forall x \in A_1 \cap A_2,\ \|f_1(x) + f_2(x)\| \leq \|f_1(x)\| + \|f_2(x)\| \leq \left(\varepsilon_1(x) + \varepsilon_2(x)\right) g(x).$$

(3)

1. There exists $(C_1, C_2) \in \mathbb{R}^2_{>0}$ and $(A_1, A_2) \in \mathcal{F}^2$ such that

$$\forall x \in A_1,\ |\lambda(x)| \leq C_1 g(x),\ \forall x \in A_2,\ \|f(x)\| \leq C_2 h(x).$$

   Hence,

$$\forall x \in A_1 \cap A_2,\ \|(\lambda(x) f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C_1 C_2 g(x) h(x).$$

2. We assume that

$$\lambda(x) = \mathcal{O}\left(g(x)\right) \text{ along } \mathcal{F},\ f(x) = o\left(h(x)\right) \text{ along } \mathcal{F}.$$

   There exists $(A_1, A_2) \in \mathcal{F} \times \mathcal{F}, C \in \mathbb{R}_{\geq 0}$ and a mapping $\varepsilon : X \longrightarrow \mathbb{R}_{\geq 0}$ such that

$$\forall x \in A_1,\ |\lambda(x)| \leq C \cdot g(x),\ \forall x \in A_2, \|f(x)\| \leq \varepsilon(x) h(x).$$

   Then one has

$$\lim_{\mathcal{F}} C\varepsilon(x) = 0$$

   and

$$\forall x \in A_1 \cap A_2,\ \|(\lambda(x) f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C \cdot g(x) \cdot \varepsilon(x) h(x)$$

   As required.

□

Example 1.1.3
(1) Let $I \subseteq \mathbb{N}$ infinite. Let $(V, \|\cdot\|)$ be a normed vector space over complete valued field $(K, |\cdot|)$. Let $\mathcal{F}$ be the filter on I. Let $(x_n)_{n\in I} \in V^I$, $(b_n)_{n\in I} \in \mathbb{R}^I_{\geq 0}$. We denote by

$$x_n = \mathcal{O}\left(b_n\right),\ n \in I,\ n \to +\infty$$

the statement $x_n = \mathcal{O}\left(b_n\right)$ along $\mathcal{F}$. Namely,

$$\exists N \in \mathbb{N},\ \exists C > 0,\ \forall n \in I_{\geq N},\ \|x_n\| \leq C \cdot b_n.$$

$$x_n = o\left(b_n\right),\ n \in I,\ n \to +\infty$$

denotes the statement $x_n = o\left(b_n\right)$ along $\mathcal{F}$. Namely,

$$\exists (\varepsilon_n)_{n\in I} \text{ such that } \lim_{n\to+\infty} \varepsilon_n = 0,\ \exists N \in \mathbb{N},\ \forall n \in I_{\geq N},\ \|x_n\| \leq \varepsilon_n \cdot b_n.$$

(2) Let $(X, \mathscr{T})$ be a topological space, $Y \subseteq X$, $y_0 \in \overline{Y}$. Let $f : Y \longrightarrow V$ and $g : Y \longrightarrow \mathbb{R}_{\geq 0}$ be mappings.

$$\mathcal{F} = \mathcal{V}_{y_0}(\mathscr{T})|_Y := \{U \cap Y \mid U \text{ is a neighborhood of } y_0\}$$

$f(y)\mathcal{O}(g(y))$, $y \in Y$, $y \to y_0$ denotes $f(y) = \mathcal{O}(g(y))$ along $\mathcal{F}$. Namely,

$$\exists C > 0,\ \exists U \in \mathcal{V}_{y_0}(\mathscr{T}),\ \forall y \in U \cap Y,\ \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)),\ y \in Y,\ y \to y_0$$

denotes $f(y) = o(g(y))$ along $\mathcal{F}$. Namely,

$$\exists \varepsilon : Y \longrightarrow \mathbb{R}_{\geq 0},\ \lim_{y \in Y, y \to y_0} \varepsilon(y) = 0,\ \exists U \in \mathcal{V}_{y_0}(\mathscr{T}),$$

$$\forall y \in U \cap Y,\ \|f(y)\| \leq \varepsilon(y)g(y).$$

(3) Let $\mathcal{F}$ be a filter on $\mathbb{R}$ generated by subsets of the form $[a, +\infty[$. Let $Y \subseteq \mathbb{R}$ not bounded from above. Let $f : Y \longrightarrow V$ and $g : Y \longrightarrow \mathbb{R}_{\geq 0}$ be mappings. Then

$$f(y) = \mathcal{O}(g(y)),\ y \in Y,\ y \to +\infty$$

denotes $f(y) = \mathcal{O}(g(y))$ along $\mathcal{F}|_Y$. Namely,

$$\exists C > 0,\ \exists a \in \mathbb{R},\ \forall y \in Y_{\geq a},\ \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)),\ y \in Y,\ y \to +\infty$$

denotes $f(y) = o(g(y))$ along $\mathcal{F}|_Y$. Namely,

$$\exists \varepsilon : Y \longrightarrow \mathbb{R}_{\geq 0},\ \lim_{y \to +\infty} \varepsilon(y) = 0,\ \exists a \in \mathbb{R},\ \forall y \in Y_{\geq a},\ \|f(y)\| \leq \varepsilon(y)g(y).$$

1.2 Differentiability

We fix a complete valued field $(K, |\cdot|)$. We suppose that there exists $a \in K^\times$, such that $|a| < 1$. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K.

$$\mathscr{L}(E, F) := \{\varphi \in \mathrm{Hom}_K(E, F) \mid \|\varphi\| < +\infty\}.$$

$(\mathscr{L}(E, F), \|\cdot\|)$ is a normed vector space over K.

Definition 1.2.1 Let $U \subseteq E$ be subset and $p \in U^\circ$. We say that a mapping $f : U \longrightarrow F$ is **differentiable** at p if there exists $\varphi \in \mathscr{L}(E, F)$ such that

$$f(p+h) - f(p) - \varphi(h) = o\left(\|h\|_E\right),\ h \to 0_E.$$

If $U = U^\circ$ and f is differentiable at every point of U, we say that f is **differentiable** on U.

Proposition 1.2.2 Assume that $f : U \longrightarrow F$ is differentiable at $p \in U^\circ$. There exists a unique $\varphi \in \mathscr{L}(E, F)$ such that

$$f(p+h) - f(p) - \varphi(h) = o\left(\|h\|_E\right),\ h \to 0_E.$$

Lemma 1.2.3 $\forall \eta \in \mathscr{L}(E, F), \forall r > 0.$

$$\|\eta\| = \sup_{x\in E, 0<\|x\|_E \leq r} \frac{\|\eta(x)\|_F}{\|x\|_E} = \sup_{\substack{x\in E \\ 0<\|x\|_E<r}} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

Proof (of Lemma) $\|\eta\| \geq \sup_{x\in E, 0<\|x\|_E<r} \frac{\|\eta(x)\|_F}{\|x\|_E}$. $\forall y \in E\backslash\{0\}$, $\|a^N y\|_E = |a|^N \|y\|_E < r$.

$$\frac{\|\eta(a^N y)\|_F}{\|a^N y\|_E} = \frac{|a|^N \cdot \|\eta(y)\|_F}{|a|^N \cdot \|y\|_E} = \frac{\|\eta(y)\|_F}{\|y\|_E} \leq \sup_{\substack{x\in E \\ 0<\|x\|_E<r}} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

□

Proof (of Proposition) Suppose $\varphi, \psi \in \mathscr{L}(E, F)$ are such that

$$f(p+h) - f(p) - \varphi(h) = o\left(\|h\|_E\right),\ h \to 0_E,$$

$$f(p+h) - f(p) - \psi(h) = o\left(\|h\|_E\right),\ h \to 0_E.$$

Then

$$\varphi(h) - \psi(h) = o\left(\|h\|_E\right),\ h \to 0_E.$$

$$\exists r > 0, \exists \varepsilon : \overline{B}(0_E, r) \longrightarrow \mathbb{R}_{\geq 0} \text{ such that } \lim_{h\to 0_E} \varepsilon(h) = 0.$$

$$\forall h \in \overline{B}(0_E, r),\ \|(\varphi - \psi)(h)\|_F = \varepsilon(h)\|h\|_E.$$

$$\|\varphi - \psi\| = \sup_{\substack{x\in E \\ 0<\|h\|_E<r'}} \frac{\|\varphi(h) - \psi(h)\|_F}{\|h\|_E} \leq \sup_{0<\|h\|_E<r'} \varepsilon(h).$$

Taking the limit when $r' \to 0$, by $\limsup_{h\to 0_E} \varepsilon(h) = 0$. We get $\|\varphi - \psi\| = 0$, hence $\varphi = \psi$. □

Definition 1.2.4 Let $U \subseteq E$ and $f : U \longrightarrow F$ be a mapping that is differentiable at $p \in U^\circ$. The unique $\varphi \in \mathscr{L}(E, F)$ such that

$$f(p+h) - f(p) - \varphi(h) = o\left(\|h\|_E\right), \ h \to 0_E$$

is called the **differential** of f at p and is denoted as

$$\mathrm{D}(f(p)).$$

Example 1.2.5
(1) $f : U \longrightarrow F$, $f(x) \equiv c$, $c \in F$.

$$f(x+h) - f(x) = 0_E = o\left(\|h\|_E\right).$$

So f is differentiable at every point of U and $\mathrm{D}(f(x)) = 0_F$.
(2) $\varphi \in \mathscr{L}(E, F)$.

$$\varphi(p+h) - \varphi(p) - \varphi(h) = 0_F = o\left(\|h\|_E\right).$$

So φ is differentiable at every point of E and $\mathrm{D}(\varphi(p)) = \varphi$.
(3) Let $(F_i, \|\cdot\|_i)$ be normed vector spaces over K, $i \in \{1, \ldots, n\}$, $n \in \mathbb{N}$. Suppose that $F = F_1 \oplus \cdots \oplus F_n$ and

$$\|\,(s_1, \ldots, s_n)\,\|_F = \max\{\|s_1\|_1, \ldots, \|s_n\|_n\}.$$

Let $U \subseteq E$ be an open subset, $f_i : U \longrightarrow F_i$ be a mapping.

$$f : U \longrightarrow F, \ f(x) = \left(f_1(x), \ldots, f_n(x)\right).$$

$$f(p+h) - f(p) = \left(f_1(p+h) - f_1(p), \ldots, f_n(p+h) - f_n(p)\right).$$

Suppose that each f_i is differentiable

$$\begin{aligned} & f(p+h) - f(p) - \left(\mathrm{D}f_1(p)(h), \ldots, \mathrm{D}f_n(p)(h)\right)|_F \\ = & \max_{i\in\{1,\ldots,n\}} \|f_i(p+h) - f_i(p) - \mathrm{D}f_i(p)(h)\|_{F_i} \\ = & o\left(\|h\|_E\right). \end{aligned}$$

So f is differentiable at p and

$$\mathrm{D}f(p)(h) = (\mathrm{D}f_1(p)(h), \dots, \mathrm{D}f_n(p)(h)) .$$

(4) Suppose that $E = K$. If $U \subseteq K$ is open and $f : U \longrightarrow F$ is differentiable at $p \in U$. We denote by $f'(p)$ the element $\mathrm{D}f(p)(1) \in F$.

$$f(p+h) - f(p) - \mathrm{D}f(p)(h) = o\left(\|h\|_E\right) .$$

So

$$f(p+h) - f(p) - hf'(p) = o\left(\|h\|_E\right) ,$$

$$\frac{f(p+h) - f(p)}{h} - f'(p) = o(1).$$

That is,

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h} = f'(p).$$

Theorem 1.2.6 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, $(G, \|\cdot\|_G)$ be normed vector spaces over a complete valued field $(K, |\cdot|)$. Let $U \subseteq E$ and $V \subseteq F$ be open subsets, $f : U \longrightarrow F$ and $g : V \longrightarrow G$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If f is differentiable at p and g is differentiable at $f(p)$, then $g \circ f : U \longrightarrow G$ is differentiable at p and

$$\mathrm{D}(g \circ f)(p)(h) = \mathrm{D}g(f(p))\left(\mathrm{D}f(p)(h)\right) .$$

Proof

$$f(p+h) - f(p) - \mathrm{D}f(p)(h) = o\left(\|h\|_E\right) ,$$

so,

$$f(p+h) - f(p) = \mathcal{O}\left(\|h\|_E\right) .$$

$$\begin{aligned} &g(f(p+h)) - g(f(p)) - \mathrm{D}g(f(p))\left(f(p+h) - f(p)\right) \\ =&o\left(\|f(p+h) - f(p)\|_F\right) = o\left(\mathcal{O}\|h\|_E\right) = o\left(\|h\|_E\right) . \end{aligned}$$

$$\begin{aligned} &\mathrm{D}g(f(p))\left(f(p+h) - f(p)\right) - \mathrm{D}g(f(p))\left(\mathrm{D}f(p)(h)\right) \\ =&\mathrm{D}g(f(p))\left(f(p+h) - f(p) - \mathrm{D}f(p)(h)\right) \\ =&\mathcal{O}\left(o\left(\|h\|_E\right)\right) = o\left(\|h\|_E\right) . \end{aligned}$$

So,

$$g(f(p+h)) - g(f(p)) - \mathrm{D}g(f(p))\left(\mathrm{D}f(p)(h)\right) = o\left(\|h\|_E\right) .$$

□

Remark 1.2.7 If $(E, \|\cdot\|_E) = (K, |\cdot|)$,

$$(g \circ f)'(p) = \mathrm{D}g(f(p))(f'(p)).$$

If $E = F = K$, $\|\cdot\|_E = \|\cdot\|_F = |\cdot|$.

$$(g \circ f)'(p) = g'(f(p)) \cdot f'(p).$$

Remark 1.2.8 Let $U \subseteq E$ be open. $f : U \longrightarrow F_1 \times \cdots \times F_n$. If f is differentiable at $p \in U$, for any $i \in \{1, \ldots, n\}$, the mapping

$$f_i := \pi_i \circ f : U \longrightarrow F_i$$

is differentiable at p and

$$\mathrm{D}(f_i)(p)(h) = \mathrm{D}\pi_i(f(p))\left(\mathrm{D}f(p)(h)\right) = \pi_i\left(\mathrm{D}f(p)(h)\right).$$

1.3 Multilineal Mappings

Definition 1.3.1 Let K be a commutative unitary ring. Let $E_1, \ldots, E_n; F$ be K-modules. We say that

$$\varphi : E_1 \times \ldots \times E_n \longrightarrow F$$

is n-linear if for any $i \in \{1, \ldots, n\}$ and any $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in E_1 \times \ldots \times E_{i-1} \times E_{i+1} \times \ldots \times E_n$, the mapping

$$E_i \longrightarrow F,\ x_i \mapsto \varphi(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

is a homomorphism of K-modules. (K-linear mapping)
If $n = 1$, 1-linear is also called linear.
If $n = 2$, 2-linear is also called bilinear.

Example 1.3.2
(1) $K \times K \longrightarrow K$ $(a, b) \longmapsto ab$ is bilinear.
(2) $K^n \times K^n \longrightarrow K$ $(x, y) \longmapsto x \cdot y = \sum_{i=1}^n x_i y_i$ is bilinear.
(3) $K \times \ldots \times K \longrightarrow K$ $(x_1, \ldots, x_n) \longmapsto x_1 \cdots x_n$ is n-linear.

Definition 1.3.3 We denote by $\mathrm{Hom}_K^{(n)}(E_1 \times \ldots \times E_n, F)$ the set of n-linear mappings from $E_1 \times \ldots \times E_n$ to F.

Definition 1.3.4 Let $(K, |\cdot|)$ be a complete valued field.
Let $(E_1, \|\cdot\|_{E_1}), \ldots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$ be normed vector spaces over K. For any $\varphi \in \mathrm{Hom}_K^{(n)}(E_1 \times \ldots \times E_n, F)$, we define

$$\|\varphi\| := \sup_{\substack{x_i \in E_i \setminus \{0\} \\ i \in \{1,\ldots,n\}}} \frac{\|\varphi(x_1, \ldots, x_n)\|_F}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_n}}.$$

We denote by $\mathscr{L}(E_1 \times \cdots \times E_n, F)$ the set

$$\{\varphi \in \mathrm{Hom}_K^{(n)}(E_1 \times \cdots \times E_n, F) \mid \|\varphi\| < +\infty\}.$$

$\mathscr{L}^{(n)}(E_1 \times \ldots \times E_n, F)$ is a normed vector space of $\mathrm{Hom}_K^{(n)}(E_1 \times \ldots \times E_n, F)$, and the norm is $\|\cdot\|$.

Theorem 1.3.5 Let $(E_1, \|\cdot\|_{E_1}), \ldots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$ be normed vector spaces over K. Let $\varphi \in \mathscr{L}^{(n)}(E_1 \times \ldots \times E_n, F)$. For any $p = (p_1, \ldots, p_n) \in E_1 \times \ldots \times E_n$, φ is differentiable at p and

$$\mathrm{D}\varphi(p)(h_1, \ldots, h_n) = \sum_{i=1}^n \varphi(p_1, \ldots, p_{i-1}, h_i, p_{i+1}, \ldots, p_n).$$

Proof

$$\begin{aligned}\varphi(p+h) - \varphi(p) = \sum_{i=1}^n & \varphi(p_1 + h_1, \ldots, p_{i-1} + h_{i-1}, p_i + h_i, p_{i+1}, \ldots, p_n) \\ & -\varphi(p_1 + h_1, \ldots, p_{i-1} + h_{i-1}, p_i, p_{i+1}, \ldots, p_n)\end{aligned}$$

$$\begin{aligned}&\varphi(p_1 + h_1, \ldots, p_{i-1} + h_{i-1}, h_i, p_{i+1}, \ldots, p_n) \\ &-\varphi(p_1, \ldots, p_{i-1}, h_i, p_{i+1}, \ldots, p_n) \\ &= \sum_{j=1}^{i-1} \varphi(p_1 + h_1, \ldots, p_{j-1} + h_{j-1}, h_j, p_{j+1}, \ldots, h_i, \ldots, p_n).\end{aligned}$$

$$
\begin{aligned}
&\|\varphi(p_1+h_1,\dots,p_{j-1}+h_{j-1},h_j,p_{j+1},\dots,h_i,\dots,p_n)\|_F\\
\leq&\|\varphi\|\cdot\prod_{k=1}^{j-1}\|p_k+h_k\|_{E_k}\cdot\|h_j\|_{E_j}\cdot\prod_{k=j+1}^{i-1}\|p_k\|_{E_k}\cdot\|h_i\|_{E_i}\cdot\prod_{k=i+1}^{n}\|p_k\|_{E_k}\\
=&\mathcal{O}\left(\|h\|^2\right)=o(h),\ h\to 0.
\end{aligned}
$$

□

Definition 1.3.6 Let K be a commutative unitary ring. $n\in\mathbb{N}_{\geq 1}$, E and F be K-modules. We say that

$$\varphi\in\mathrm{Hom}_K^{(n)}(E_1\times\ldots\times E_n,F)$$

is **symmetric** if

$$\forall\sigma\in\mathfrak{S}_{\{1,\dots,n\}},\ \forall(x_1,\dots,x_n)\in E^n,\ \varphi(x_{\sigma(1)},\dots,x_{\sigma(n)})=\varphi(x_1,\dots,x_n).$$

Let $P:E\longrightarrow F$ be a mapping. If there exists a symmetric $\varphi\in\mathrm{Hom}_K^{(n)}(E\times\ldots\times E,F)$ such that

$$\forall x\in E,\ P(x)=\varphi(x,\dots,x),$$

we say that P is a **homogeneous polynomial mapping of degree** n.
If $F=K$, P is called a **homogeneous polynomial** on E. The symmetric polynomial mapping φ is called the **polarization** of P.

Proposition 1.3.7 Let $(K,|\cdot|)$ be a complete valued field that is non-trivial. Let $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ be normed vector spaces over K. Assume that $P:E\longrightarrow F$ is a homogeneous polynomial mapping of degree n. Which admits a bounded polarization φ. Then P is differentiable on E and,

$$\forall(x,h)\in E\times E,\ \mathrm{D}P(x)(h)=n\varphi(x,\dots,x,h).$$

Proof Let

$$
\begin{array}{rccc}
\Delta: & E & \longrightarrow & E^n,\\
 & x & \longmapsto & (x,\dots,x).
\end{array}
$$

Then $P=\varphi\circ\Delta$. Since φ and Δ are differentiable, so it is P.

Moreover,

$$\begin{aligned}
\mathrm{D}P(x)(h) &= \mathrm{D}\varphi(\Delta(x))\,(\mathrm{D}\Delta(x)(h)) \\
&= \mathrm{D}\varphi(x,\dots,x)(h,\dots,h) \\
&= \sum_{i=1}^{n} \varphi(x,\dots,x,h,x,\dots,x) \\
&= n\varphi(x,\dots,x,h).
\end{aligned}$$

□

Remark 1.3.8 Assume that $E = K$. Let $P : K \longrightarrow F$ be a homogeneous polynomial mapping of degree n of form $P(x) = x^n s$, where $s \in F$. Its polarization is of the form

$$\varphi(a_1,\dots,a_n) = a_1 \cdots a_n s.$$

$$P'(x) = \mathrm{D}P(x)(1) = n\varphi(x,\dots,x,1) = nx^{n-1}s.$$

Proposition 1.3.9 Let n be a positive integer $n \geq 2$. Let $(E_1, \|\cdot\|_1), \dots, (E_n, \|\cdot\|_n), (F, \|\cdot\|_F)$ be normed vector spaces. For any $i \in \{1,\dots,n_1\}$, the mapping

$$\mathscr{L}^{(n)}(E_1,\dots,E_n;F) \xrightarrow{f} \mathscr{L}^{(i)}(E_1,\dots,E_i,\mathscr{L}^{(n-i)}(E_{i+1},\dots,E_n;F))$$

$$\varphi \longmapsto \left(\substack{E_1\times\ldots e_n \longrightarrow \mathscr{L}^{(i)}(E_{i+1},\ldots,E_n;F) \\ (x_1,\ldots,x_i)\longmapsto \left(\substack{(x_{i+1},\ldots,x_n)\longmapsto\varphi(x_1,\ldots,x_n) \\ E_{i+1}\times\ldots\times E_n \in F} \right)} \right)$$

is an isomorphism of vector spaces over K, and in the same time an isometry, ($\|f(\varphi)\| = \|\varphi\|$).

Remark 1.3.10
$\forall (x_1,\dots,x_n) \in E_1 \times \ldots \times E_n$, $f(\varphi)(x_1,\dots,x_i)(x_{i+1},\dots,x_n) = \varphi(x_1,\dots,x_n)$

Proof $\forall (x_1,\dots,x_n) \in E_1 \times \ldots \times E_n$,

$$\begin{aligned}
\varphi(x_1,\dots,x_n) :&E_{i+1} \times \ldots \times E_n \longrightarrow F \text{ is bounded} \\
&(x_{i+1},\dots,x_n) \longmapsto \varphi(x_1,\dots,x_n)
\end{aligned}$$

Since

$$\|\varphi(x_1,\dots,x_n)\|_F \leq (\|\varphi\| \cdot \|x_1\| \dots \|x_i\|)\,\|x_{i+1}\| \dots \|x_n\|.$$

$$\begin{aligned}\|f(\varphi)\| &= \sup_{x_j\in E_j\setminus\{0\},j=1,\ldots,i} \frac{\|\varphi(x_1,\ldots,x_i,\cdot)\|}{\|x_1\|_{E_1}\cdots\|x_n\|_{E_i}} \\ &= \sup_{x_j\in E_j\setminus\{0\},j=1,\ldots,i}\ \sup_{x_k\in E_k\setminus\{0\},k=i+1,\ldots,n} \frac{\|\varphi(x_1,\ldots,x_n)\|_F}{\|x_1\|_{E_1}\cdots\|x_n\|_{E_n}} \\ &= \|\varphi\|.\end{aligned}$$

Hence f is injective. ($\ker(f) = \{0\}$)
For any $\psi \in \mathscr{L}^{(i)}(E_1,\ldots,E_i,\mathscr{L}^{(n-i)}(E_{i+1},\ldots,E_n))$,

$$\begin{array}{rccl}\varphi: & E_1\times\ldots\times E_n & \longrightarrow & F \\ & (x_1,\ldots,x_n) & \longmapsto & \psi(x_1,\ldots,x_i)(x_{i+1},\ldots,x_n)\end{array}$$

belongs to $\mathscr{L}^{(n)}(E_1,\ldots,E_n;F)$ and $f(\varphi)=\psi$. So f is surjective. □

Corollary 1.3.11 If $E_1,\ldots,E_n$ are all finite dimensional, then

$$\mathscr{L}^{(n)}(E_1,\ldots,E_n;F) = \mathrm{Hom}_K^{(n)}(E_1\times\ldots\times E_n,F).$$

Proof If $n=1$, $\mathscr{L}(E_1,F)=\mathrm{Hom}_K(E_1,F)$.

$$\begin{aligned}\mathscr{L}^{(n)}(E_1,\ldots,E_n;F) &\cong \mathscr{L}(E_1,\mathscr{L}^{(n-1)}(E_2,\ldots,E_n;F)) \\ &= \mathrm{Hom}_K(E_1,\mathrm{Hom}_K^{(n-1)}(E_2\times\ldots\times E_n,F)) \\ &\cong \mathrm{Hom}_K^{(n)}(E_1\times\ldots\times E_n,F).\end{aligned}$$

□

Let $(K,|\cdot|)$ be a complete nontrivial valued field. Let $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ be normed vector spaces over K.

Definition 1.3.12 Let $U\subseteq E$ be an open subset of E, $f: U\longrightarrow F$ be a mapping.
If f is continuous on U, we say that f is **of class** $\mathcal{C}^0$ and we denote by

$$\mathrm{D}^0 f$$

the mapping $f: U\longrightarrow F$. Denote by

$$\mathcal{C}^0(U,F)$$

the set of mappings from U to F.

$$U \xrightarrow{(f,g)} K \times K \xrightarrow{\times} K$$

$$p \longmapsto (f(p), g(p)) \longmapsto f(p) \times g(p)$$

Let $p \in U$. If f is differentiable on an open neighborhood V of p such that $V \subseteq U$. Then

$$\begin{aligned} \mathrm{D}f : & V \longrightarrow \mathscr{L}(E, F) \\ & x \longmapsto \mathrm{D}f(x) \end{aligned}$$

is a mapping. If $\mathrm{D}f$ is $(n-1)$-times differentiable at p, we say that f is **of class $\mathcal{C}^n$** at p. If f is of class $\mathcal{C}^n$ at every point of U, we say that f is **n-times differentiable** at p. We denote by

$$\mathrm{D}^n f(p) \in \mathscr{L}^{(n)}(E, \dots, E, F)$$

the n-linear mapping that sends $(h_1, \dots, h_n) \in E^n$ to

$$\mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \dots, h_{n-1})(h_n) \in F.$$

Remark 1.3.13

$$\mathrm{D}^n f(p)(h_1, \dots, h_n) = \mathrm{D}^i(\mathrm{D}^{n-i} f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n).$$

1.4 Convexity

Definition 1.4.1 Let E be a vector space over a field K. $S \subseteq E$ be a non-empty subset.
We call affine combination of elements of S any element of E of the form

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n,$$

where $n \in \mathbb{N}_{\geq 1}$, $s_1, \ldots, s_n \in S$, $a_1, \ldots, a_n \in K$ such that

$$a_1 + a_2 + \cdots + a_n = 1.$$

We denote by $\mathrm{Aff}(S)$ the set of all affine combinations of elements of S. One has $S \subseteq \mathrm{Aff}(S)$. $\mathrm{Aff}(S)$ is called the affine hull of S.
If $S = \mathrm{Aff}(S)$, we say that S is an affine subspace of E.

Proposition 1.4.2
(1) If F is a vector subspace of E, $\forall p \in E$,

$$p + F = \{p + x \mid x \in F\}$$

is an affine subspace of E.
(2) If $A \subseteq E$ is an affine subspace of E. For any $p \in A$,

$$A - p := \{x - p \mid x \in A\}$$

is a vector subspace of E, which is not dependent on the choice of p. We call it the vector space **associated** with A.

Proof
(1) Let $(x_1, \ldots, x_n) \in F^n$, $(a_1, \ldots, a_n) \in K^n$, such that $\sum_{i=1}^n a_i = 1$. Then

$$\begin{aligned}\sum_{i=1}^n a_i(p + x_i) &= p \cdot \sum_{i=1}^n a_i + \sum_{i=1}^n a_i x_i \\ &= p + \sum_{i=1}^n a_i x_i \in p + F.\end{aligned}$$

(2) Let $(x_1, \ldots, x_n) \in A^n$, $(b_1, \ldots, b_n) \in K^n$.

$$\begin{aligned}\sum_{i=1}^{n} b_i(x_i - p) &= \sum_{i=1}^{n} b_i x_i - \left(\sum_{i=1}^{n} b_i\right) p \\ &= \left(\sum_{i=1}^{n} b_i x_i + \left(1 - \sum_{i=1}^{n} b_i\right) p\right) - p \\ &\in A - p.\end{aligned}$$

Let $q \in A$, $\forall x \in A$, $x - p = (x - q) + (q - p) \in A - q$. So $A - p \subseteq A - q$. By symmetry, $A - q \subseteq A - p$. Hence $A - p = A - q$. □

Example 1.4.3 Let A be an m by p matrix with coefficients in $\mathbb{R}$. Let $(b_1, \ldots, b_n) \in E^m$. Consider the linear equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

The solution set is

$$S := \{(x_1, \ldots, x_p) \in E^p \mid A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\}.$$

Claim: S is an affine subspace of E^p.

Proof Let $\underline{x}^{(1)}, \ldots, \underline{x}^{(n)}$ be elements of S, where $\underline{x}^{(i)} = (x_1^{(i)}, \ldots, x_p^{(i)})$. Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$, $\underline{x} = a_1 x^{(1)} + \cdots + a_n x^{(n)}$.

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = A \left(a_1 \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_p^{(1)} \end{pmatrix} + \cdots + a_n \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_p^{(n)} \end{pmatrix} \right).$$

$$a_1 A \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_p^{(1)} \end{pmatrix} + \ldots a_n A \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_p^{(n)} \end{pmatrix} = (a_1 + \cdots + a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

$$x_j = a_1 x_j^{(1)} + \cdots + a_n x_j^{(n)}.$$

□

Proposition 1.4.4 Let $S \subseteq E$. Then $\mathrm{Aff}(S)$ is the smallest affine subspace of E containing S.

Proof
Let $A \subseteq E$ be an affine subspace containing S. $\forall n \in \mathbb{N}_{\geq 1}$, $\forall (x_1, \ldots, x_n) \in S^n \subseteq A^n$, $(a_1, \ldots, a_n) \in \mathbb{R}$, $a_1 + \cdots + a_n = 1$, one has

$$\sum_{i=1}^{n} a_i x_i \in A.$$

So $\mathrm{Aff}(S) \subseteq A$.

To show that $\mathrm{Aff}(S)$ is an affine subspace containing S, it is sufficient to check that $\mathrm{Aff}(S)$ is an affine subspace.

If $S = \varnothing$, then $\mathrm{Aff}(S) = \varnothing$. It is an affine subspace.
Suppose that $S \neq \varnothing$, $p \in S$. We prove that $\mathrm{Aff}(S) - p$ is equal to $\mathrm{Span}_{\mathbb{R}}(S - p)$.
Let $y = a_1 x_1 + \cdots + a_n x_n \in \mathrm{Aff}(S)$.

$$y - p = a_1(x_1 - p) + \cdots + a_n(x_n - p) \in \mathrm{Span}_{\mathbb{R}}(S - p).$$

Let $(x_1, \ldots, x_n) \in S^n$, $(b_1, \ldots, b_n) \in \mathbb{R}^n$.

$$\sum_{i=1}^{n} b_i(x_i - p) = \left(\sum_{i=1}^{n} b_i x_i + \left(1 - \sum_{i=1}^{n} b_i \right) p \right) - p \in \mathrm{Aff}(S) - p.$$

□

Definition 1.4.5 Let $S \subseteq E$. We call **convex combination** of elements of S any element of E of the form

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n,$$

where $n \in \mathbb{N}_{\geq 1}$, $s_1, \ldots, s_n \in S$, $a_1, \ldots, a_n \in \mathbb{R}_{\geq 0}$ such that

$$a_1 + a_2 + \cdots + a_n = 1.$$

We denote by $\mathrm{Conv}(S)$ the set of all convex combinations of elements of S. $\mathrm{Conv}(S)$ is called the **convex hull** of S. One has $S \subseteq \mathrm{Conv}(S) \subseteq \mathrm{Aff}(S)$.

Proposition 1.4.6 Let E be a vector space over $\mathbb{R}$ and $C \subseteq E$. Then C is convex if and only if

$$\forall (x, y) \in C^2,\ \forall \lambda \in [0, 1],\ \lambda x + (1 - \lambda) y \in C.$$

Proof It is sufficient to check "$\Leftarrow$". We prove by induction on n that

$$\forall n \in \mathbb{N}_{\geq 1},\ \forall (x_1, \ldots, x_n) \in C^n,\ \forall (a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0},\ \sum_{i=1}^{n} a_i = 1,\ \sum_{i=1}^{n} a_i x_i \in C.$$

The case where $n = 1$ is trivial. The case where $n = 2$ comes from the hypothesis. Suppose $n \geq 3$ in assuming that the statement holds for any integer less than n. If $a_n = 1$, then $a_1 = \cdots = a_{n-1} = 0$, so $\sum_{i=1}^{n} a_i x_i = x_n \in C$. If $a_n < 1$, we have $a_1 + \cdots + a_{n-1} = 1 - a_n > 0$. By the induction hypothesis,

$$x := \sum_{i=0}^{n-1} \frac{a_i}{1 - a_n} x_i \in C.$$

Taking $y = x_n$, $t = 1 - a_n$,

$$C \ni tx + (1 - t) y = \sum_{i=1}^{n} a_i x_i.$$

$\square$

1.5 Mean Value Theorems

Theorem 1.5.1 (Mean Value Inequality) Let $(F, \|\cdot\|_F)$ be normed vector spaces over $\mathbb{R}$. Let $(a, b) \in \mathbb{R}^2$ such that $a < b$. Let $f : [a, b] \longrightarrow F$ be a continuous mapping that is differentiable on $]a, b[$. Then

$$\|f(b) - f(a)\|_F \leq (b - a) \cdot \sup_{t \in]a,b[} \|f'(t)\|_F.$$

Proof We may suppose that $\sup_{t \in]a,b[} \|f'(t)\|_F < +\infty$. Take

$$M > \sup_{t \in]a,b[} \|f'(t)\|_F.$$

Let $m = \frac{a+b}{2}$. Let

$$J = \{x \in [m, b] \mid \forall t \in [m, x], \|f(t) - f(m)\|_F \leq M(t - m)\}.$$

It is an interval containing m. So it is of the form

$$[m, c[\text{ or } [m, c]$$

$$\forall t \in [m, c[, \ \|f(t) - f(m)\|_F \leq M(t - m).$$

Taking the limit $t < c$, $t \to c$, we get $c \in J$. So $J = [m, c]$. We then check $c = b$.

If $c \neq b$, then $c \in]a, b[$, so f is differentiable at c. That is

$$\|f(c + h) - f(c)\|_F = \|f'(c)h + o(\|h\|)\|_F \leq \|f'(c)\|_F h + o(\|h\|), \ h \to 0.$$

Since $M > \|f'(c)\|_F$, $\exists h_0 > 0$ such that

$$\forall h \in]0, h_0], \ \|f(c + h) - f(c)\|_F \leq Mh.$$

$$\begin{aligned} \|f(c + h) - f(m)\| &\leq \|f(c + h) - f(c)\| + \|f(c) - f(m)\| \\ &\leq Mh + M(c - m) = M(c + h - m). \end{aligned}$$

So $[m, c + h_0] \subseteq J$, contradiction. Thus $b = c$. $\|f(b) - f(m)\|_F \leq M(b - m)$.

By the same reason, $\|f(m) - f(a)\|_F \leq M(m - a)$. So

$$\|f(b) - f(a)\|_F \leq \|f(b) - f(m)\|_F + \|f(m) - f(a)\|_F \leq M(b - a).$$

Taking the limit when $M \to \sup\limits_{t \in]a,b[} \|f'(t)\|_F$, we get the announced result. □

Corollary 1.5.2 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over $\mathbb{R}$. $U \subseteq E$ be an open subset, and $(x, y) \in U^2$ such that

$$[x, y] = \{tx + (1 - t)y \mid t \in [0, 1]\} \subseteq U.$$

Let $f : U \longrightarrow F$ be a differentiable mapping. Then

$$\|f(x) - f(y)\|_F \leq \left(\sup_{z \in]x,y[} \|\mathrm{D}f(z)\| \right) \cdot \|x - y\|_E.$$

Proof Let

$$\begin{aligned} g : \ [0, 1] &\longrightarrow U \\ t &\longmapsto tx + (1 - t)y. \end{aligned}$$

$$g(0) = x, \ g(1) = y, \ g'(t) = x - y.$$

Then,

$$(f \circ g)'(t) = \mathrm{D}f(g(t))(x-y),$$
$$\mathrm{D}(f \circ g)(t)(1) = \mathrm{D}f(g(t))(\mathrm{D}g(t)(1)).$$

By the theorem,

$$\begin{aligned}\|f(x)-f(y)\|_F &= \|f(g(1))-f(g(0))\|_F \\ &\leq \sup_{t\in]0,1[} \|\mathrm{D}f(g(t))(x-y)\|_F \\ &\leq \sup_{t\in]0,1[} |\mathrm{D}f(g(t))| \cdot \|x-y\|_E \\ &= \sup_{z\in]x,y[} \|\mathrm{D}f(z)\| \cdot \|x-y\|_E.\end{aligned}$$

□

Definition 1.5.3 Let $(X, \mathscr{T})$ be a topological space, $p \in X$. Let U be a neighborhood of p and $f : U \longrightarrow \mathbb{R}$ be a mapping. If there exists a neighborhood V of p such that $p \in V \subseteq U$ and

$$\forall x \in V,\ f(p) \geq f(x),$$

we say that p is a **local maximum point** of f on U.

If p is a local maximum point or a local minimum point, we say that p is a **local extremum** of f on U.

If $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are normed vector spaces. $U \subseteq E$ open, $f : U \to F$ is differentiable. If $p \in U$ is such that

$$\mathrm{D}f(p) = 0 \in \mathscr{L}(E, F),$$

we say that p is a **critical point** of f.

Theorem 1.5.4 Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$. $U \subseteq E$ be an open subset, $f : U \longrightarrow \mathbb{R}$ be a differentiable mapping. If $p \in U$ is a local extremum point of f, then it is a critical point $(\mathrm{D}f(p) = 0)$.

Proof There exists $r > 0$ such that $p + B(0, r) \subseteq U$ and

$$(h \in B(0,r)) \longmapsto f(p+h) - f(p) \in \mathbb{R}$$

does not change the sign.

$\forall h \in B(0,r),\ \forall \in [0,1],$

$$(f(p+th)-f(p))\,(f(p-th)-f(p)) \geq 0.$$

Taking the limit when $t \to 0$, $-\mathrm{D}f(p)(h)^2 \geq 0$. So $\mathrm{D}f(p)(h) = 0$. □

Theorem 1.5.5 (Rolle) Let $(a,b) \in \mathbb{R}^2$, $a < b$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous mapping that is differentiable on $]a,b[$. If $f(a) = f(b)$, then

$$\exists t \in \,]a,b[,\ f'(t) = 0.$$

Proof If there exists t which is in $]a,b[$ and is an extremum point of f, then $f'(t) = 0$. Since $[a,b]$ is compact and f is continuous, so f attains its maximum and minimum.

If the extremum points of f are in $\{a,b\}$. Since $f(a) = f(b)$, f is compact, so $f'(t) = 0$ on $]a,b[$. □

Theorem 1.5.6 (Gronwall inequality) Let $(F, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$, $(a,b) \in \mathbb{R}^2$, $a < b$. Let $f : [a,b] \longrightarrow F$ and $g : [a,b] \longrightarrow \mathbb{R}$ be differentiable mappings on $]a,b[$. If $\forall t \in \,]a,b[$, $\|f'(t)\| \leq g'(t)$, then

$$\|f(b)-f(a)\|_F \leq g(b)-g(a).$$

Proof Let $m \in \,]a,b[$. Let $\varepsilon > 0$,

$$J := \{t \in [m,b] \mid \forall s \in [m,t],\ \|f(s)-f(m)\|_F \leq g(s)-g(m)+\varepsilon(s-m)\}.$$

Since f and g are continuous, J is a closed interval of the form $[m,c]$.
If $c < b$,

$$\begin{aligned} f(c+h) &= f(c)+hf'(c)+o(h), \\ g(c+h) &= g(c)+hg'(c)+o(h), \quad h > 0,\ h \to 0. \end{aligned}$$

$\exists \delta > 0$, such that $[c, c+\delta] \subseteq [c,b]$ and $\forall h \in [0,\delta]$,

$$\|f(c+h)-f(c)\| \leq h\|f'(c)\| + \frac{\varepsilon}{2}h.$$

$$g(c+h)-g(c) \geq hg'(c) - \frac{\varepsilon}{2}h.$$

So,

$$\|f(c+h)-f(c)\| \leq g(c+h)-g(c)+\varepsilon h.$$

By the triangle inequality,

$$\|f(c+h)-f(m)\| \leq g(c+h)-g(m)+\varepsilon(c+h-m).$$

So $J \supseteq [m, c+\delta]$, contradiction.
Therefore $c=b$.

$$\|f(b)-f(m)\| \leq g(b)-g(m)+\varepsilon(b-m).$$

A similar argument shows that

$$\|f(m)-f(a)\| \leq g(m)-g(a)+\varepsilon(m-a).$$

Hence,

$$\|f(b)-f(a)\| \leq g(b)-g(a)+\varepsilon(b-a).$$

$$\|f(c+h)-f(c)+hf'(c)\| \leq \varphi(h)h,\ \lim_{h\to 0}\varphi(h)=0.$$

$$\exists\delta>0,\ \forall h>0,\ 0\leq h<\delta \Rightarrow |\varphi(h)| \leq \frac{\varepsilon}{2}.$$

□

Theorem 1.5.7 (Mean value theorem of Lagrange) Let $(a,b)\in\mathbb{R}^2$, $a<b$. Let $f:[a,b]\longrightarrow\mathbb{R}$ be a continuous mapping that is differentiable on $]a,b[$. Then

$$\exists\xi\in\,]a,b[,\ f(b)-f(a)=f'(\xi)(b-a).$$

Proof Let $g:[a,b]\longrightarrow\mathbb{R}$.

$$g(t):=f(b)-f(t)+C(b-t),\ \text{where } C=-\frac{f(b)-f(a)}{b-a}.$$

Then $g(a)=g(b)=0$, $g'(t)=-f'(t)-C$.

$$\exists\xi\in\,]a,b[,\ g'(\xi)=0,\ f'(\xi)=-C=\frac{f(b)-f(a)}{b-a}.$$

□

Theorem 1.5.8 (Darboux) Let I be an open interval in $\mathbb{R}$ and $f:I\longrightarrow\mathbb{R}$ be a differentiable mapping. Then $f'(I)$ is an interval.

Proof Let a, b be two elements in I such that $a < b$. Let

$$\begin{aligned} g: \ [a,b] &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \frac{f(t)-f(a)}{t-a}, \ t \neq a \\ f'(a), \ t = a \end{cases} \end{aligned}$$

g is continuous, and $g([a,b])$ is an interval. By the mean value theorem of Lagrange, $g([a,b]) \subseteq f'(I)$.
Let

$$\begin{aligned} h: \ [a,b] &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} \frac{f(t)-f(b)}{t-b}, \ t \neq b \\ f'(b), \ t = b \end{cases} \end{aligned}$$

$h([a,b])$ is an interval contained in $f'(I)$.
$h([a,b]) \cup g([a,b])$ is an interval since

$$\frac{f(b)-f(a)}{b-a} \in h([a,b]) \cap g([a,b]),$$

$$\{f'(a), f'(b)\} \subseteq h([a,b]) \cup g([a,b]).$$

So the interval liking $f'(a)$, $f'(b)$ is contained in $f'(I)$. Hence, $f'(I)$ is an interval.
□

1.6 Higher Differential

We fix a complete non-trivially valued field $(K, |\cdot|)$. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K.

Definition 1.6.1 Let $U \subseteq E$ be an open subset, $f : U \longrightarrow F$ be a mapping, $p \in U$.
(1) If f is continuous at p, we say that f is 0-time differentiable at p, and we let

$$\mathrm{D}^0 f(p) := f(p).$$

(2) If f is differentiable at p, we say that f is 1-time differentiable at p, and we let

$$\mathrm{D}^1 f(p) := \mathrm{D}f(p).$$

(3) Let $n \geq 2$. If exists open neighborhood V of p such that $V \subseteq U$ and f is differentiable on V and $\mathrm{D}f$ is $n-1$-time differentiable on V, we say that f is

n-time differentiable at p, and we let

$$\mathrm{D}^n f(p) \in \mathscr{L}(E, \ldots, E, F)$$

be the multilineal mapping sending $(h_1, \ldots, h_n) \in E^n$ to

$$\mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \ldots, h_{n-1})(h_n).$$

If $E = K$, $\mathrm{D}^n f(p)(1, \ldots, 1)$ is denoted as $f^{(n)}(p) \in F$. $f^{(0)}(p)$ is often denoted as $f(p)$.

Remark 1.6.2 $\forall i \in \{1, \ldots, n\}$,

$$\mathrm{D}^n f(p)(h_1, \ldots, h_n) = \mathrm{D}^i(\mathrm{D}^{n-i} f)(p)(h_1, \ldots, h_i)(h_{i+1}, \ldots, h_n).$$

If $E = K$,

$$f^{(n)}(p)(h_1, \ldots, h_n) = \mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \ldots, h_{n-1})(h_n).$$

Definition 1.6.3 Let X be a set, we denote by $\mathfrak{S}_X$ the element of all bijection from X to X. $(\mathfrak{S}_X, \circ)$ forms a group. The identity mapping Id_X is the neutral element of $(\mathfrak{S}_X, \circ)$. $(\mathfrak{S}_X, \circ)$ is called the symmetric group of X. The elements of $(\mathfrak{S}_X, \circ)$ are called permutations of X.

Let $n \in \mathbb{N}_{\geq 2}$, $x_1, \ldots, x_n$ be distinct elements of X. We denote by $(x_1\, x_2 \cdots x_n)$ the element of $\mathfrak{S}_X$ that sends x_i to x_{i+1}, $(i \in \{1, \ldots, n-1\})$, x_n to x_1, $y \in X \backslash \{x_1, \ldots, x_n\}$ to y itself. This element is called an n-cycle. A 2-cycle is also called a transposition.

Remark 1.6.4 $\mathfrak{S}_X$ acts on X.

$$\begin{aligned} \mathfrak{S}_X \times X &\longrightarrow X \\ (\sigma, x) &\longmapsto \sigma(x). \end{aligned}$$

If $\sigma \in \mathfrak{S}_X$, $x \in X$, we denote by $\mathrm{orb}_\sigma(x)$ the set $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$.

$$\langle \sigma \rangle := \{\sigma^n \mid n \in \mathbb{Z}\} \subseteq \mathfrak{S}_X$$

is a group. $\mathrm{orb}_\sigma(x)$ is the orbit of x under the action of $\langle \sigma \rangle$.

Proposition 1.6.5 If $\mathrm{orb}_\sigma(x)$ is finite of d elements, then $\sigma^d(x) = x$, and $\mathrm{orb}_\sigma(x) = \{x, \sigma(x), \ldots, \sigma^{d-1}(x)\}$. Moreover, the restriction of σ to $\mathrm{orb}_\sigma(x)$ identifies to the restriction of the cycle $(x, \sigma(x), \ldots, \sigma^{d-1}(x))$.

Proof Since $\mathrm{orb}_\sigma(x)$ is finite,

$$\{(n,m) \in \mathbb{Z}^2 \mid n < m,\ \sigma^n(x) = \sigma^m(x)\}$$

Let

$$l = \min\{m - n \mid (n,m) \in \mathbb{Z}^2\ n < m,\ \sigma^n(x) = \sigma^m(x)\}.$$

Then $x, \sigma(x), \ldots, \sigma^{l-1}(x)$ are distinct, and $\sigma^l(x) = x$. $\forall n \in \mathbb{Z}$, then n can be written as $n = lp + r$, where $p \in \mathbb{Z}$, $r \in \{0, \ldots, l-1\}$.

$$\sigma^n(x) = \sigma^r(\sigma^{lp}(x)) = \sigma^r((\sigma^l \circ \ldots \sigma^l)(x)) = \sigma^r(x).$$

So, $\mathrm{orb}_\sigma(x) = \{x, \sigma(x), \ldots, \sigma^{l-1}(x)\}$, ($l = d$). $\square$

Remark 1.6.6 If X is finite, then X can be written as a distinct union of orbits (under the action of $\langle\sigma\rangle$). Let $d_i = \#(\mathrm{orb}_\sigma(x_i))$, $i = 1, \ldots, n$, then

$$\sigma|_{\mathrm{orb}_\sigma(x^{(i)})} = \left(x^{(i)}, \sigma(x^{(i)}), \ldots, \sigma^{d_i-1}(x^{(i)})\right)|_{\mathrm{orb}_\sigma(x^{(i)})}.$$

So $\sigma = \tau_1 \circ \cdots \circ \tau_n$, where $\tau_i := (x^{(i)}, \sigma(x^{(i)}), \ldots, \sigma^{d_i-1}(x^{(i)}))$.

Corollary 1.6.7 Suppose that X is finite. Any $\sigma \in \mathfrak{S}_X$ can be written as a composition of transpositions.

Proof

$$(x_1\ \ldots\ x_n) = (x_1\ x_2) \circ (x_2 \ldots\ x_n),$$

So,

$$(x_1\ \ldots\ x_n) = (x_1\ x_2) \circ (x_2\ x_3) \circ \cdots \circ (x_{n-1}\ x_n).$$

$\square$

Definition 1.6.8 Denote by $\mathfrak{S}_n$ the symmetric group $\mathfrak{S}_{\{1,\ldots,n\}}$. A composition of the form $(i\ i+1)$, $i \in \{1, \ldots, n-1\}$ is called an adjacent transposition.

Corollary 1.6.9 Any $\sigma \in \mathfrak{S}_n$ can be written as a composition of adjacent transpositions.

Proof Let $(j, k) \in \{1, \dots, n\}^2$, $j < k$,

$$(j-1\ j) \circ (j\ k) \circ (j-1\ j) = (j-1\ k).$$

$$(j\ k) = (j\ j+1) \circ (j+1\ j+2) \circ \cdots \circ (k-1\ k) \circ \dots (j\ j+1).$$

□

Theorem 1.6.10 (Schwarz) Let $U \subseteq E$ be an open subset, $f : U \longrightarrow F$ be a mapping. $n \in \mathbb{N}_{\geq 1}$, $p \in U$. Assume that f is n-times differentiable at p. Then $\forall \sigma \in \mathfrak{S}_n$, $\forall (h_1, \dots h_n) \in E^n$,

$$\mathrm{D}^n f(p)(h_1, \dots, h_n) = \mathrm{D}^n f(p)(h_{\sigma(1)}, \dots, h_{\sigma(n)}).$$

Proof (By induction) The case where $n = 1$ is trivial. Case $n = 2$: Exists V open, $p \in V \subseteq U$. f is differentiable on V and $\mathrm{D}f$ is differentiable at p.

$$\mathrm{D}f(p+h)(\cdot) - \mathrm{D}f(p)(\cdot) - \mathrm{D}^2 f(p)(h, \cdot) = o\left(\|h\|_E\right).$$

Let $\varepsilon > 0$, $\exists \delta > 0$, $\forall h \in E$, $\|h\|_E \leq 2\delta \Rightarrow p + h \in V$ and

$$\|\mathrm{D}f(p+h)(\cdot) - \mathrm{D}f(p)(\cdot) - \mathrm{D}^2 f(p)(h, \cdot)\| \leq \varepsilon \|h\|_E.$$

Let $h \in E$ such that $\|h\|_E \leq \delta$. Define $g_h : B(0, \delta) \longrightarrow F$ as

$$g_h(k) = f(p+h+k) - f(p+h) - f(p+k) + f(p) - \mathrm{D}^2 f(p)(h, k).$$

Then,

$$\begin{aligned} \mathrm{D}g_h(k)(\cdot) =& \mathrm{D}f(p+h+k)(\cdot) - \mathrm{D}f(p+k)(\cdot) - \mathrm{D}^2 f(p)(h, \cdot) \\ =& \mathrm{D}f(p+h+k)(\cdot) - \mathrm{D}f(p)(\cdot) - \mathrm{D}^2 f(p)(h+k, \cdot) \\ &-(\mathrm{D}f(p+k)(\cdot) - \mathrm{D}f(p)(\cdot) - \mathrm{D}^2 f(p)(k, \cdot)) \end{aligned}$$

$$\|\mathrm{D}g_h(k)(\cdot)\| \leq \varepsilon \|h+k\|_E + \varepsilon \|k\|_E \leq 3\varepsilon \max\{\|k\|_E, \|h\|_E\}.$$

$g_h(0) = 0$. Therefore, $\|g_h(k)\| \leq 3\varepsilon \max \|k\|_E, \|h\|_E{}^2$ (mean value inequality).

$$\|g_h(k) - g_h(0)\| \leq \left(\sup_{t \in]0,1[} \|\mathrm{D}g_h(tk)\|\right) \cdot \|k\|.$$

Therefore,

$$f(p+h+k)-f(p+h)-f(p+k)+f(p)-\mathrm{D}^2 f(p)(h,k) = o\left(\max\{\|k\|_E, \|h\|_E\}^2\right).$$

By symmetry,

$$f(p+h+k)-f(p+h)-f(p+k)+f(p)-\mathrm{D}^2 f(p)(k,h) = o\left(\max\{\|k\|_E, \|h\|_E\}^2\right).$$

$$\mathrm{D}^2 f(p)(h,k) - \mathrm{D}^2 f(p)(k,h) = o\left(\max\{\|h\|_E, \|k\|_h\}^2\right).$$

$$\mathrm{D}^2 f(p)(th,tk) - \mathrm{D}^2 f(p)(tk,th) = o(|t|^2),\ t \to 0.$$

$$\mathrm{D}^2 f(p)(h,k) - \mathrm{D}^2 f(p)(k,h) = o(1),\ t \to 0.$$

Suppose $n \geq 3$.

$$\mathrm{D}^n f(p)(h_1, \ldots, h_n) = \mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \ldots, h_{n-1})(h_n).$$

If $\sigma = (j\ j+1)$, $j \leq 2$,

$$\mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_1, \ldots, h_{n-1})(h_n) = \mathrm{D}^{n-1}(\mathrm{D}f)(p)(h_{\sigma(1)}, \ldots, h_{\sigma(n-1)})(h_n)$$

by the induction hypothesis, if $\sigma = (n-1\ n)$,

$$\mathrm{D}^n f(p)(h_1, \ldots, h_n) = \mathrm{D}^2\left((\mathrm{D}^{n-2} f)(h_1, \ldots, h_{n-2})(h_{n-1}, h_n)\right)$$

$$\begin{aligned}
\mathrm{D}^n f(p)(h_{\sigma(1)}, \ldots, h_{\sigma(n)}) &= \mathrm{D}^n f(p)(h_1, \ldots, h_{n-1}) \\
&= \mathrm{D}^2\left((\mathrm{D}^{n-2} f)(h_1, \ldots, h_{n-2})(h_n, h_{n-1})\right) \\
&= \mathrm{D}^n f(p)(h_1, \ldots, h_n).
\end{aligned}$$

□

1.7 Taylor's Formula

Theorem 1.7.1 (Toylor-Young) Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over $\mathbb{R}$, $U \subseteq E$ open, $n \in \mathbb{N}$, $f : U \longrightarrow F$ be a mapping, $p \in U$. Suppose that f is n-times differentiable at p. Then

$$f(x) = f(p) + \sum_{k=1}^{n} \frac{1}{k!} \mathrm{D}^k f(p)(x-p, \dots, x-p) + o(\|x-p\|^n),\ x \to p.$$

Proof (By induction on n)
$n = 0$, $f(x) = f(p) + o(1)$ follows by continuity of f; $n = 1$ follows by the differentiability of f.

From $n-1$ to n. Let $g : U \longrightarrow F$

$$g(x) = f(x) - f(p) - \sum_{k=1}^{n} \frac{1}{k!} \mathrm{D}^k f(p)(x-p, \dots, x-p).$$

g is differentiable on an open neighborhood of p,

$$\mathrm{D}g(x)(h) = \mathrm{D}f(x)(h) - \sum_{k=1}^{n} \frac{1}{k!} k \mathrm{D}^k f(p)(x-p, \dots, x-p, h)$$

$$\mathrm{D}g(x) = \mathrm{D}f(x) - \sum_{l=0}^{n-1} \frac{1}{l!} \mathrm{D}^l(\mathrm{D}f)(x-p, \dots, x-p) \overset{\text{hyp.}}{=\!=} o(\|x-p\|^{n-1}), x \to p.$$

So $g(x) = o(\|x-p\|^n)$.

$$\forall \varepsilon > 0,\ \exists \delta > 0, \forall x \in B(p, \delta),\ \|\mathrm{D}g(x)\| \le \varepsilon \|x-p\|^{n-1}.$$

$g(p) = 0$, so

$$\|g(x) - g(p)\| \le \varepsilon \|x-p\|^{n-1} \cdot \|x-p\| = \varepsilon \|x-p\|^n.$$

□

Theorem 1.7.2 (Taylor-Lagrange) Let $(a, b) \in \mathbb{R}^2$, $a < b$. $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous mapping. Suppose that f is $(n+1)$-times differentiable on $]a, b[$ and $\forall k \in \{3, \dots, n\}$, $f^{(k)} :]a, b[\longrightarrow \mathbb{R}$ tends to a continuous mapping $[a, b] \longrightarrow \mathbb{R}$.

Then

$$\exists \xi \in]a, b[, f(b) - \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) = \frac{f^{(n+1)}(\xi)(b-a)^{n+1}}{(n+1)!}.$$

Proof Let $g : [a, b] \longrightarrow \mathbb{R}$.

$$g(t) := \sum_{k=0}^{n} \frac{(b-t)^k}{k!} f^{(k)}(t) - C\frac{(b-t)^{n+1}}{(n+1)!}.$$

Then $g(b) = f(b)$, $g(a) = \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) - C\frac{(b-a)^{n+1}}{(n+1)!}$.

$$\begin{aligned} g'(t) &= \sum_{k=0}^{n} \frac{(b-t)^k}{k!} f^{(k+1)}(t) - \sum_{k=1}^{n} \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(t) + C\frac{(b-t)^n}{n!} \\ &= \frac{(b-t)^n}{n!} f^{(n+1)}(t) + C\frac{(b-t)^n}{n!}. \end{aligned}$$

Take C such that $g(a) = g(b)$. Then by Rolle's theorem, $\exists \xi \in]a, b[, g'(\xi) = 0$, $C = -f^{(n+1)}(\xi)$. Then,

$$g(a) = \sum_{k=0}^{n} \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{f^{(n+1)}(\xi)}{(n+1)!} + \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1} = f(b) = g(b).$$

□

Theorem 1.7.3 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces over $\mathbb{R}$, $U \subseteq E$ be an open subset, and $f : U \longrightarrow F$ be a mapping that is $(n+1)$-times differentiable, where $n \in \mathbb{N}$. Let $p \in U$, $h \in E$ such that $\forall t \in [0, 1], p + th \in U$. Let

$$M = \sup_{t \in [0,1]} \|\mathrm{D}^{n+1} f(p + th)\|.$$

Then,

$$\|f(p+h) - \sum_{k=0}^{n} \frac{1}{k!} \mathrm{D}^k f(p)(h, \ldots, h)\|_F \leq \frac{M}{(n+1)!} \|h\|_E^{n+1}.$$

Proof We define $\phi : [0,1] \longrightarrow F$

$$\phi(t) = f(p+th) + \sum_{k=1}^{n} \frac{(1-t)^k}{k!} \mathrm{D}^k f(p+th)(h,\dots,h).$$

$$\phi(0) = \sum_{k=0}^{n} \frac{1}{k!} \mathrm{D}^k f(p)(h,\dots,h),\ \phi(1) = f(p+h).$$

$$\begin{aligned}
\phi'(t) =& \mathrm{D}f(p+h)(h) + \sum_{k=1}^{n} \frac{(1-t)^k}{k!} \mathrm{D}^{k+1} f(p+th)(h,\dots,h) \\
&- \sum_{k=1}^{n} \frac{(1-t)^{k-1}}{(k-1)!} \mathrm{D}^k f(p+th)(h,\dots,h) \\
=& \sum_{k=0}^{n} \frac{(1-t)^k}{k!} \mathrm{D}^{k+1} f(p+th)(h,\dots,h) \\
&- \sum_{l=0}^{n-1} \frac{(1-t)^l}{(l)!} \mathrm{D}^{l+1} f(p+th)(h,\dots,h) \\
=& \frac{(1-t)^n}{n!} \mathrm{D}^{n+1} f(p+th)(h,\dots,h).
\end{aligned}$$

So,

$$\|\phi'(t)\| \leq M \|h\|_E^{n+1} \frac{(1-t)^n}{n!},\ t \in [0,1].$$

By Gronwall's inequality,

$$\|\phi(1) - \phi(0)\|_F \leq M \cdot \|h\|^{n+1} \frac{1}{(n+1)!}.$$

□

1.8 Banach Space

Proposition 1.8.1 Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If

$$\sum_{n\in\mathbb{N}} \mathrm{d}(x_n, x_{n+1}) < +\infty,$$

then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof Let $N \in \mathbb{N}$. If $(n,m) \in \mathbb{N}^2_{\geq N}$, $n > m$, by the triangle inequality,

$$\mathrm{d}(x_m, x_n) \leq \sum_{k=m}^{n-1} \mathrm{d}(x_k, x_{k+1}) \leq \sum_{k \geq N} \mathrm{d}(x_k, x_{k+1}).$$

So,

$$0 \leq \sup_{(n,m)\in\mathbb{N}_{\geq N}} \mathrm{d}(x_n, x_m) \leq \sum_{k \geq N} \mathrm{d}(x_k, x_{k+1}).$$

Taking the limit when $N \to +\infty$, we get

$$\lim_{N\to+\infty} \sup_{(n,m)\in\mathbb{N}^2_{\geq N}} \mathrm{d}(x_n, x_m) = 0.$$

Let $(a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. If $(\sum_{k=0}^{n} a_k)_{n\in\mathbb{N}}$ converges to some l in $\mathbb{R}$. Then, $l - \sum_{k=0}^{N-1} a_k$ converges to 0. If $a_k \leq 0$ for any $k \in \mathbb{N}$, $l - \sum_{k=0}^{N-1} a_k = \sum_{k=N}^{+\infty} a_k$.

$$l - \sum_{k=0}^{N-1} a_k = \lim_{n\to+\infty} \left(\sum_{k=0}^{n} a_k - \sum_{k=0}^{N-1} a_k \right) = \lim_{n\to+\infty} \sum_{k=N}^{n} a_k.$$

□

Definition 1.8.2 Let $(K, |\cdot|)$ be a complete valued field and $(E, \|\cdot\|)$ be a normed vector space over K. If E equipped with the metric

$$\begin{array}{rcl} E \times E & \longrightarrow & \mathbb{R}_{\geq 0} \\ (x,y) & \longmapsto & \|x - y\|_E. \end{array}$$

is complete, we say that $(E, \|\cdot\|)$ is a **Banach space**.
Let $(E, \|\cdot\|)$ be a Banach space. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in E such that $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$, we say that $\sum_{n\in\mathbb{N}} x_n$ **converges absolutely**.

Remark 1.8.3 Suppose that $\sum_{n\in\mathbb{N}} x_n$ converges absolutely. Then $\left(\sum_{k=0}^{n} x_k\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, since

$$\|x_n\| = \|\sum_{k=0}^{n} x_k - \sum_{k=0}^{n} x_{k-1}\|.$$

So, $\sum_{n\in\mathbb{N}} x_n$ converges.

Theorem 1.8.4 (Root test of Cauchy) Let $(E, \|\cdot\|)$ be a Banach space and $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$. Let

$$r = \limsup_{n\to+\infty} \|x_n\|^{\frac{1}{n}} \in [0, +\infty]$$

If $r < 1$, then $\sum_{n\in\mathbb{N}} x_n$ converges absolutely.
If $r > 1$, then $\sum_{n\in\mathbb{N}} x_n$ diverges.

Lemma 1.8.5 If a series $\sum_{n\in\mathbb{N}} x_n$ converges, then $\lim_{n\to+\infty} \|x_n\| = 0$.

Proof (of lemma)

$$\|x_n\| = \|\sum_{k=0}^{n} x_k - \sum_{k=0}^{n-1} x_k\|.$$

Since $\sum_{k}^{n} x_k$ converges to some $l \in E$.

$$\lim_{n\to+\infty} \|x_n\| = \lim_{n\to+\infty} \|\sum_{k=0}^{n} x_k - \sum_{k=0}^{n-1} x_k\| = \|l - l\| = 0.$$

□

Proof (of theorem) If $r > 1$, $\exists \beta > 1$ such that $r > \beta$. Since $r = \limsup_{n\to+\infty} \|x_n\|^{\frac{1}{n}}$, $\exists I \subseteq \mathbb{N}$ infinite such that $\lim_{n\in I, n\to+\infty} \|x_n\|^{\frac{1}{n}} = r$ (Bolzano-Weierstrass).

$$\exists N \in \mathbb{N}, \forall n \in I_{\geq N}, \|x_n\|^{\frac{1}{n}} \geq \beta.$$

So, $\|x_n\| \geq \beta^n \geq 1$. So $\sum_{n\in\mathbb{N}} x_n$ diverges.
If $r < 1$, $\exists \alpha \in]0, 1[, r < \alpha$. Since $r = \limsup_{n\to+\infty} \|x_n\|^{\frac{1}{n}}$,

$$\exists N \in \mathbb{N},\ \forall n \geq N,\ \|x_n\|^{\frac{1}{n}} \leq \alpha, \|x_n\| \leq \alpha^n.$$

So,

$$\sum_{n\geq N} \|x_n\| \leq \sum_{n\geq N} \alpha^n = \frac{\alpha^N}{1-\alpha} < +\infty.$$

Therefore, $\sum_{n\in\mathbb{N}} x_n$ converges absolutely. □

Theorem 1.8.6 (Ratio test of D'Alembert) Let $(E, \|\cdot\|)$ be a Banach space and $(x_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$.
(1) If

$$\limsup_{n\to+\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < 1,$$

then $\sum_{n\in\mathbb{N}} x_n$ converges absolutely.
(2) If

$$\limsup_{n\to+\infty} \frac{\|x_{n+1}\|}{\|x_n\|} > 1,$$

then $\sum_{n\in\mathbb{N}} x_n$ diverges.

Proof
(1) Let $0 < \alpha < 1$ such that

$$\limsup_{n\to+\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < \alpha.$$

$$\exists N \in \mathbb{N}, \forall x \in \mathbb{N}_{\geq N}, \|x_{n+1}\| \leq \alpha\|x_n\| \leq \alpha^{n+1-N}\|x_N\|.$$

Thus,

$$\sum_{n\geq\mathbb{N}} \|x_n\| \leq \sum_{n\geq\mathbb{N}} \|x_N\|\alpha^{n-N} = \|x_N\|\frac{1}{1-\alpha} < +\infty.$$

(2) Let $\beta > 1$ such that

$$\liminf_{n\to+\infty} \frac{\|x_{n+1}\|}{\|x_n\|} > \beta.$$

$$\exists N \in \mathbb{N}, x_N \neq 0, \text{ and } \forall n \in \mathbb{N}_{\geq N}, \|x_{n+1}\| \geq \beta\|x_n\|$$

$$\forall n \geq N, \|x_n\| \geq \beta^{n-N}\|x_N\| \to +\infty\ (n \to +\infty)$$

So $\sum_{n\in\mathbb{N}} x_n$ diverges. □

Let $z \in \mathbb{C}$. The series $\sum_{n\in\mathbb{N}} \frac{z^n}{n!}$ converges absolutely since

$$\left|\frac{z^{n+1}/(n+1)!}{z^n/n!}\right| = \frac{|z|}{n+1} \to 0\ (n \to +\infty).$$

We denote by e^z this limit.

1.9 Local inversion

Definition 1.9.1 Let X be a topological space and $Y \subseteq X$. If $\overline{Y} = X$, we say that Y is dense.

Theorem 1.9.2 (Baire) Let (X, d) be a complete metric space. Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of dense open subset of X. Let $\Omega = \bigcap_{n\in\mathbb{N}} \Omega_n$, then Ω is dense in X.

Proof Suppose that Ω is not dense. Let $x_0 \in X \backslash \overline{\Omega}$, exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq X \backslash \overline{\Omega}$.
Let $r_0 = \varepsilon$. We construct in a recursive way sequence $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}} \in \mathbb{R}^n_{\geq 0}$ as follows.

Suppose that (x_n, r_n) is chosen. $B(x_n, r_n) \cap \Omega_n \neq \varnothing$. We pick $x_{n+1} \in X$ and $r_{n+1} \leq \frac{x_n}{2}$ such that $B(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n) \cap \Omega_n$, $\mathrm{d}(x_{n+1}, x_n) < r_n$. $\sum_{n\in\mathbb{N}} r_n < +\infty$ (ratio test).

Then the sequence converges to some l. For any $n \in \mathbb{N}$, $x_n \in B(x_0, \varepsilon)$. So $l \in \overline{B}(x_0, \varepsilon)$.

Moreover, $\forall n \in \mathbb{N}$, $l \in \overline{B}(x_{n+1}, r_{n+1}) \subseteq B_{x_n, r_n} \cap \Omega_n$. Thus $l \in \cap_{n\in\mathbb{N}} \Omega_n = \Omega$. Contradiction. □

Corollary 1.9.3 Let (X, d) be a non-empty complete metric space and $(Y_n)_{n\in\mathbb{N}}$ be a family of closed subsets of X such that $X = \bigcup_{n\in\mathbb{N}} Y_n$. Then exists $n \in \mathbb{N}$ such that $Y^\circ \neq \varnothing$.

Proof Let $\Omega_n = X \backslash Y_n$. Suppose that $\forall n \in \mathbb{N}$, $Y_n^\circ = \varnothing$. Then $\overline{\Omega}_n = X \backslash Y_n^\circ = X$. Thus $\Omega := \bigcap_{n\in\mathbb{N}} \Omega_n$ is dense in X. Namely, $X = \overline{\Omega}$. So

$$\varnothing = X \backslash \overline{\Omega} = (X \backslash \Omega)^\circ = \left(X \backslash \bigcap_{n\in\mathbb{N}} \Omega_n \right)^\circ = \left(\bigcup_{n\in\mathbb{N}} Y_n \right)^\circ = X^\circ = X.$$

Contradiction. □

Theorem 1.9.4 (Banach) Let $(K, |\cdot|)$ be a complete non-trivially valued filed, and E be a vector space over K. Let $\|\cdot\|_1$, $\|\cdot\|_2$ be two norms on E such that $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are both Banach spaces.

If $\exists C > 0$ such that $\|\cdot\|_2 \leq C\|\cdot\|_1$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. $(\exists C' > 0, \|\cdot\|_1 \leq C'\|\cdot\|_2)$

Proof For $x \in E$ and $r > 0$. Let

$$B_i(x,r) := \{y \in E \mid \|y - x\|_i < r\},\ i = 1,2$$

$\forall y \subseteq E$, let $\overline{Y}^{\|\cdot\|_2}$ be the closure of Y in $(E, \|\cdot\|_2)$.

$$E = \bigcup_{n\geq 1} B_1(0,n) = \bigcup_{n\geq 1} \overline{B_1(0,n)}^{\|\cdot\|_2}.$$

Hence, $\exists n_0 \geq 1$, $p \in E$, $r_0 > 0$ such that

$$B_2(p, r_0) \subseteq \overline{B_1(0,n_0)}^{\|\cdot\|_2}$$

or equivalently,

$$B_2(0,r_0) \subseteq \overline{B_1(-p,n_0)}^{\|\cdot\|_2} \subseteq \overline{B_1(0,n_0 + \|p\|_1)}^{\|\cdot\|_2}$$

since $\forall x \in B_1(-p, n_0)$

$$\|x\|_1 = \|x - p + p\|_1 \leq \|x - p\| + \|p\|_1 < n_0 + \|p\|_1.$$

Let $r_1 = n_0 + \|p\|_1$,

$$B_2(0,r_0) \subseteq \overline{B_1(0,r_1)}^{\|\cdot\|_2} \subseteq B_1(0,r_1) + B_2(0, r_0|a|)$$

where $a \in K$, $0 < |a| < \frac{1}{2}$.
In fact, $\forall x \in \overline{B_1(0,r_0)}^{\|\cdot\|_2}$, exists sequence $(x_n)_{n\in\mathbb{N}} \in B_1(0,r_1)^{\mathbb{N}}$, such that $x_n \to x$ in $(E, \|\cdot\|_2)$, $\exists n \in \mathbb{N}$, $\|x_n - x\|_2 < r_0|a|$

$$B_2(0, r_0|a|^n) \subseteq B_1(0, r_1|a|^n) + B_2(0, r_0|a|^{n+1})$$

Let $y \in B_2(0,r_0)$, we choose $(x_0,y_0) \in B_1(0,r_1) \times B_2(0,r_0|a|)$ such that $y = x_0 + y_0$. When (x_n, y_n) si chosen, let $(x_{n+1}, y_[n+1]) \in B_1(0, r_0|a|^{n+1}) \times B_2(0, r_0|a|^{n+2})$, $y_n = x_{n+1} + y_{n+1}$, $y = y_n + \sum_{k=0}^n x_k$. So $\sum_{n\in\mathbb{N}} x_n$ converges to y.
Moreover, $\sum_{n\in\mathbb{N}} \|x_n\| < +\infty$, so it converges in $(E, \|\cdot\|_1)$ to some x. Therefore, $x = y$ since $\|\cdot\|_2 \leq C\|\cdot\|_1$. So $\|y\|_1 \|x\|_1 \leq \sum_{n\in\mathbb{N}} \|x_n\|_1 \leq \frac{r_1}{1-|a|}$.

Therefore $B_2(0,r_0) \subseteq B_1(0, \frac{r}{1-|a|})$. So $\|\cdot\|_1$ is bounded by a constant times $\|\cdot\|_2$. $\square$

Proposition 1.9.5 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces over a complete non-trivially valued filed $(K, |\cdot|)$, and $f : E \longrightarrow F$ be a bounded mapping.
(1) If f is invertible, then f^{-1} is bounded.
(2) If f is surjective, for any $U \subseteq E$ open, $f(U)$ is open in F.

Proof
(1) We define a mapping

$$\begin{array}{rccc} \|\cdot\|'_E : & E & \longrightarrow & \mathbb{R}_{\geq 0} \\ & x & \longmapsto & \|f(x)\|_F. \end{array}$$

This is a norm on E. In fact, if $\|x\|'_E = \|f(x)\|_F = 0$, then $f(x) = 0_F$. So $x = 0_E$. Moreover,

$$\forall x \in E,\ \|x\|'_E = \|f(x)\|_F \leq \|f\|\|x\|_E.$$

So there exists $C > 0$ such that $\|\cdot\|_E \leq C\|\cdot\|'_E$. That is,

$$\forall y \in F,\ \|y\|_F = \|f(f^{-1}(y))\|_F = \|f^{-1}(y)\|'_E \geq C^{-1}\|f^{-1}(y)\|_E.$$

So, $\|f^{-1}\| \leq C$.
(2) Let

$$E_0 = \ker(f) = \{x \in E \mid f(x) = 0_F\}.$$

This is a closed vector subspace of E. $\|\cdot\|_E$ induces by passing to quotient a norm $\|\cdot\|_Q$ on $Q := E/E_0$. Let

$$\begin{array}{rccc} g : & Q & \longrightarrow & F \\ & [x] & \longmapsto & f(x). \end{array}$$

This is a K-linear bijection.

If $\alpha \in Q$,

$$\forall x \in \alpha,\ \|g(\alpha)\|_F = \|f(x)\|_F \leq \|f\| \cdot \|x\|_E.$$

Since $\|\alpha\|_Q := \inf_{x\in\alpha} \|x\|_E$, $\|g(\alpha)\|_F \leq \|f\| \cdot \|\alpha\|_Q$. So $\|g\| \leq \|f\|$. By (1), g^{-1} is bounded (hence is continuous).

If $V \subseteq Q$ is open, then $g(V) \subseteq F$ is open. Let $U \subseteq E$ be an open subset. Let

$$\begin{array}{rccc} \pi : & E & \longrightarrow & Q \\ & x & \longmapsto & [x]. \end{array}$$

Let $x \in U$, $r > 0$ such that $B(x, r) \subseteq U$. For any $\alpha \in Q$, if

$$\|\alpha - [x]\|_Q = \inf_{y\in\alpha} \|y - x\|_E < r,$$

then, exists $y \in \alpha$ such that $\|y - x\|_E < r$.
Therefore,

$$B([x], r) \subseteq \pi(B(x, r)) \subseteq \pi(U).$$

This means that $\pi(U)$ is open. So $f(U) = g(\pi(U))$ is open. $\square$

Definition 1.9.6 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector space over a complete non-trivially valued field $(K, |\cdot|)$, $U \subseteq E$ open, $f : U \longrightarrow F$. If $\forall p \in U$, f is n-times differentiable at p, and $\mathrm{D}^n f : U \longrightarrow \mathscr{L}^{(n)}(E, \ldots, E, F)$ is continuous, we say that f is of class $\mathcal{C}^n$.

If $\forall n \in \mathbb{N}$, f is n-times differentiable on U, we say that f is smooth, or of class $\mathcal{C}^\infty$. ($\forall n \in \mathbb{N}$, f is of class $\mathcal{C}^n$.)

Proposition 1.9.7 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ and $(G, \|\cdot\|_G)$ be normed vector space over a complete non-trivially valued field $(K, |\cdot|)$. $U \subseteq E$, $V \subseteq F$ be open subsets, $f : U \longrightarrow V$, $g : V \longrightarrow G$ be mappings. $n \in \mathbb{N}$.
(1) Let $p \in U$. If f is n-times differentiable at p and g is n-times differentiable at $f(p)$, then $g \circ f$ is n-times differentiable at p.
(2) If f is of class $\mathcal{C}^n$ on U and g is of class $\mathcal{C}^n$ on V, then $g \circ f$ is of class $\mathcal{C}^n$ on U.

Proof (induction on n)
$n = 0$, continuity composition.
$n = 1$, differentiability of composition.
$n \geq 2$,

$$\mathrm{D}(g \circ f)(p)(\cdot) = \mathrm{D}g(f(p))(\mathrm{D}f(p)(\cdot))$$

Let

$$\begin{array}{rccc} \Phi : & \mathscr{L}(F, G) \times \mathscr{L}(E, F) & \longrightarrow & \mathscr{L}(E, G) \\ & (\alpha, \beta) & \longmapsto & \alpha \circ \beta. \end{array}$$

This is a bounded bilinear mapping. $\|\alpha \circ \beta\| \leq \|\alpha\| \cdot \|\beta\|$.

$$(\|\alpha \circ \beta(h)\|_G = \|\alpha(\beta(h))\|_G \leq \|\alpha\| \cdot \|\beta(h)\|_F \leq \|\alpha\|\|\beta\|\|h\|_E)$$

Φ is of class $\mathcal{C}^\infty$.

$$\mathrm{D}(g \circ f) = \Phi(\mathrm{D}g \circ f, \mathrm{D}f).$$

(2) Since $\mathrm{D}g$ and $\mathrm{D}f$ is of class $\mathcal{C}^{n-1}$, we obtain that $\mathrm{D}(g \circ f)$ is of class $\mathcal{C}^{n-1}$, so $g \circ f$ is of class $\mathcal{C}^n$.
(1) If g is n-times differentiable at $f(p)$, $\mathrm{D}g$ is $(n-1)$-times differentiable at $f(p)$.

So $\mathrm{D}g \circ f$ is $(n-1)$-times differentiable at p. $\mathrm{D}f$ is $(n-1)$-times differentiable at p. So $\mathrm{D}(g \circ f)$ is $(n-1)$-times differentiable at p. □

Theorem 1.9.8 Let $(E, \|\cdot\|)$ be a Banach space over a complete non-trivially valued filed $(K, |\cdot|)$. Let

$$\mathrm{GL}(E) := \{\varphi \in \mathscr{L}(E, E) \mid \varphi \text{ is invertible}\}.$$

This set forms a group under $\circ$.
(1) $\forall \varphi \in \mathscr{L}(E, E)$, if $\|\varphi\| < 1$, then $\mathrm{Id}_E + \varphi \in \mathrm{GL}(E)$.
(2) $\mathrm{GL}(E) \subseteq \mathscr{L}(E, E)$ is open.
(3)

$$\begin{array}{rccc} \iota : & \mathrm{GL}(E) & \longrightarrow & \mathrm{GL}(E) \\ & \varphi & \longmapsto & \varphi^{-1} \end{array}$$

is of class $\mathcal{C}^\infty$.

Proof
(1) The series $\sum_{n\in\mathbb{N}}(-1)^n\varphi^n$ converges absolutely since $\|\varphi^n\| \leq \|\varphi\|^n$.
Let η be the limit of $\sum_{n\in\mathbb{N}}(-1)^n\varphi^n$.

$$(\mathrm{Id} + \varphi) \circ \sum_{k=0}^{n}(-1)^k\varphi^k = \mathrm{Id} + (-1)^n\varphi^{n+1}.$$

Taking the limit when $n \to +\infty$, we get $(\mathrm{Id}_E + \varphi) \circ \eta = \mathrm{Id}_\mathrm{E}$. For the same reason, $\eta \circ (\mathrm{Id}_E + \varphi) = \mathrm{Id}_E$.
(2) If $f \in \mathrm{GL}(E)$, $\forall \varphi \in \mathscr{L}(E, E)$ such that

$$\|\varphi\| < \frac{1}{\|f^{-1}\|},\ f + \varphi = f \circ \left(\mathrm{Id}_E + f^{-1} \circ \varphi\right),\ \|f^{-1} \circ \varphi\| \leq \|f^{-1}\| \cdot \|\varphi\| < 1.$$

So $\mathrm{Id}_E + f^{-1} \circ \varphi \in \mathrm{GL}(E)$. Hence $f + \varphi \in \mathrm{GL}(E)$.
(3) Let $f \in \mathrm{GL}(E)$, $\varphi \in \mathscr{L}(E, E)$. $\|\varphi\| \leq \frac{1}{\|f^{-1}\|}$.

$$\begin{aligned} \iota(f + \varphi) - \iota(f) &= (f + \varphi)^{-1} - f^{-1} \\ &= \left(f \circ \left(\mathrm{Id}_E + f^{-1} \circ \varphi\right)\right)^{-1} - f^{-1} \\ &= \left(\mathrm{Id}_E + f^{-1} \circ \varphi\right)^{-1} \circ f^{-1} - f^{-1} \\ &= \sum_{n\in\mathbb{N}}(-1)^n \left(f^{-1} \circ \varphi\right)^n \circ f^{-1} - f^{-1} \\ &= -f^{-1} \circ \varphi \circ f^{-1} + o(\|\varphi\|) \end{aligned}$$

since

$$\begin{aligned}
&\sum_{n\geq 2}\|(-1)^n\left(f^{-1}\circ\varphi\right)^n\circ f^{-1}\|\\
\leq&\sum_{n\geq 2}\|f^{-1}\|\cdot\left(\|f^{-1}\|\cdot\|\varphi\|\right)^n\\
=&\|\varphi\|^2\left(\|f\|^3\cdot\sum_{n\geq 2}\left(\|f^{-1}\|\cdot\|\varphi\|\right)^{n-2}\right)\\
=&o(\|\varphi\|).
\end{aligned}$$

Let

$$\begin{array}{rccc}\Phi: & \mathscr{L}(E,E)^3 & \longrightarrow & \mathscr{L}(E,E)\\ & (\alpha,\beta,\gamma) & \longmapsto & \alpha\circ\beta\circ\gamma.\end{array}$$

bounded 3-linear mapping.

$$\mathrm{D}\iota(f)(\cdot) = -\Phi\left(\iota(f),\cdot,\iota(f)\right).$$

By induction, we obtain that ι is of class $\mathcal{C}^n$ for any $n\in\mathbb{N}$. □

Definition 1.9.9 Let (X,d) be a metric space, $f : X \longrightarrow X$ be a mapping. If exists $\alpha\in\,]0,1[$, such that f is α-Lipschitzian, we say that f is a **contraction**.

Definition 1.9.10 Let $f : X \longrightarrow X$ be a mapping. If $x\in X$ is such that $f(x)=x$, we say that x is a **fixed point** of f.

Theorem 1.9.11 (Banach fixed point theorem) Let (X,d) be a non-empty complete metric space and $f : X\longrightarrow X$ be a contraction. Then f admits a unique fixed point.

Proof

" Uniqueness": Let $\alpha\in\,]0,1[$, such that f is α-Lipschitzian. If a and b are fixed point of f, then $\mathrm{d}(a,b)=\mathrm{d}(f(a),f(b))\leq\alpha\mathrm{d}(a,b)$. So $\mathrm{d}(a,b)=0$, $a=b$.
" Existence": Let $x_0\in X$. For any $n\in\mathbb{N}$, let $x_n=f^n(x_0)$. Then

$$\mathrm{d}(x_n,x_{n+1})=\mathrm{d}(f(x_{n-1}),f(x_n))\leq\alpha\mathrm{d}(x_{n-1},x_n)\leq\cdots\leq\alpha^n\mathrm{d}(x_0,x_1).$$

So

$$\sum_{n\in\mathbb{N}}\mathrm{d}(x_n,x_{n+1})\leq\sum_{n\in\mathbb{N}}\alpha^n\mathrm{d}(x_0,x_1)=\frac{1}{1-\alpha}\mathrm{d}(x_0,x_1)<+\infty.$$

Hence $(x_n)_{n\in\mathbb{N}}$ converges to some $a \in X$.

$$\mathrm{d}(a, f(a)) = \lim_{n\to+\infty} \mathrm{d}(x_n, f(x_n)) = \lim_{n\to+\infty} \mathrm{d}(x_n, x_{n+1}) = 0.$$

So $a = f(a)$. □

Definition 1.9.12 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over a complete value filed $(K, |\cdot|)$, $U \subseteq E$, $V \subseteq F$ be open subsets, $f : U \longrightarrow V$ be a bijection, $n \in \mathbb{N} \cup \{\infty\}$. If f and f^{-1} are both of class $\mathcal{C}^n$, we say that f is a $\mathcal{C}^n$-diffeomorphism.

Theorem 1.9.13 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces over $\mathbb{R}$, $U \subseteq E$ open and $f : U \longrightarrow F$ be a mapping of class $\mathcal{C}^n$ ($n \in \mathbb{N} \cup \{\infty\}$). Let $p \in U$. Suppose that $\mathrm{D}f(p) \in \mathscr{L}(E, F)$ is invertible. Then there exists a open neighborhood V of p contained in U, such that $f|_V : V \longrightarrow f(V)$ is a $\mathcal{C}^n$-homeomorphism. Moreover

$$\mathrm{D}f^{-1}(y) = \mathrm{D}f(f^{-1}(y))^{-1}.$$

Proof By replacing f by

$$\tilde{f} : x \longmapsto \mathrm{D}f(p)^{-1}\left(f(p + x) - f(p)\right).$$

We may assume that $E = F$, $p = f(p) = 0$, $\mathrm{D}f(p) = \mathrm{Id}_E$.

$$\mathrm{D}\tilde{f}(0)(h) = \mathrm{D}f(p)^{-1}(\mathrm{D}f(p)(h)) = h, \mathrm{D}\tilde{f}(0) = \mathrm{Id}_E.$$

Let $\mu : U \longrightarrow E$, $\mu(x) = f(x) - x$, $\mathrm{D}\mu(0) = 0$. Since $\mathrm{D}f$ is continuous, so is $\mathrm{D}\mu$.

$$\exists r > 0,\ \forall x \in \overline{B}(0_E, r),\ \|\mathrm{D}\mu(x)\| \leq \frac{1}{2}.$$

So μ is $\frac{1}{2}$-Lipschitzian on $\overline{B}(0_E, r)$ (mean value inequality).

$$\forall (x, y) \in \overline{B}(0_E, r)^2,\ \|f(x) - f(y)\| \geq \|x - y\| \|\mu(x) - \mu(y)\| \geq \frac{1}{2}\|x - y\|.$$

So f is injective on $\overline{B}(0_E, r)$. Let $a \in \overline{B}(0_E, \frac{r}{2})$.

$$\forall x \in \overline{B}(0_E, r),\ \|a - \mu(x)\| \leq \|a\| + \|\mu(x)\| \leq \frac{1}{2}r + \frac{1}{2}r = r.$$

Let

$$\begin{aligned} \nu : \overline{B}(0,r) &\longrightarrow \overline{B}(0,r) \\ x &\longmapsto a - \mu(x) \end{aligned}$$

ν is a contraction. By Banach's fixed point theorem,

$$\exists! g(a) \in \overline{B}(0,r),\ \nu(g(a)) = a - \mu(g(a)) = a - f(g(a)).$$

That is $f(g(a)) = a$. Let $W = B(0, \frac{r}{2})$, $V = f^{-1}(W) \cap B(0,r)$, $f|_V : V \longrightarrow W$ is a bijection.

$\forall z \in B(0,r)$, $\mathrm{D}f(z) = \mathrm{Id}_E + \mathrm{D}\mu(z) \in \mathrm{GL}(E)$.

$$\forall (x, x_0) \in V \times V, y = f(x), y_0 = f(x_0), y - y_0 = \mathrm{D}f(x_0)(x - x_0) + o\left(\|x - x_0\|\right).$$

$$\|x - x_0\| = \|y - y_0 - (\mu(x) - \mu(x_0))\| \le \|y - y_0\| + \frac{1}{2}\|x - x_0\|,$$

$$\frac{1}{2}\|f^{-1}(y) - f^{-1}(y_0)\| = \frac{1}{2}\|x - x_0\| \le \|y - y_0\|.$$

So,

$$\mathrm{D}f(x_0)(x - x_0) = y - y_0 + o(\|y - y_0\|),$$

$$\begin{aligned} f^{-1}(y) - f^{-1}(y_0) =& x - x_0 = \mathrm{D}f(x_0)^{-1}(y - y_0) + o(\|y - y_0\|) \\ =& \mathrm{D}f(f^{-1}(y_0))^{-1}(y - y_0) + o(\|y - y_0\|) \end{aligned}$$

Thus,

$$\mathrm{D}f^{-1} = \iota \circ \mathrm{D}f \circ f^{-1}.$$

□

Proposition 1.9.14 Let $n \in \mathbb{N}_{\ge 1}$. Let $(K, |\cdot|)$ be a complete valued field, $(E_i, \|\cdot\|_i)$, $i \in \{1, \ldots, n\}$ be normed vector spaces over K, $(F, \|\cdot\|_F)$ be a Banach space over K. Then, $(\mathscr{L}^{(n)}(E_1, \ldots, E_n, F), \|\cdot\|)$ is a Banach space.

Proof Let $(\varphi_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}^{(n)}(E_1, \ldots, E_n, F)$. For $N \in \mathbb{N}$, let

$$\varepsilon_N := \sup_{(i,j) \in \mathbb{N}_{\ge N}} \|\varphi_i - \varphi_j\|, \ \lim_{N \to +\infty} \varepsilon_N = 0.$$

For any $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$, and any $(i, j) \in \mathbb{N}^2_{\geq N}$,

$$\|\varphi_i(x_1, \ldots, x_n) - \varphi_j(x_1, \ldots, x_n)\| \leq \|\varphi_i - \varphi_j\| \cdot \prod_{l=1}^{n} \|x_l\|_l \leq \varepsilon_N \prod_{l=1}^{n} \|x_l\|_l.$$

So $(\varphi_i(x_1, \ldots, x_n))_{i \in \mathbb{N}}$ is a Cauchy sequence in F, hence it converges to some element of F, denoted as $\varphi(x_1, \ldots, x_n)$.

Note that φ is a point-wise limit of an n-linear mapping, so it is also n-linear.

$$\begin{aligned} \|\varphi(x_1, \ldots, x_n)\|_F &= \lim_{i \to +\infty} \|\varphi_i(x_1, \ldots, x_n)\|_F \\ &\leq \limsup_{i \to +\infty} \|\varphi_i\| \cdot \|x_1\|_{E_1} \cdots \|x_n\|_{E_n} \\ &\leq \left(\sup_{i \in \mathbb{N}} \|\varphi_i\|\right) \cdot \|x_1\|_{E_1} \cdots \|x_n\|_{E_n} \end{aligned}$$

So $\varphi \in \mathscr{L}(E_1, \ldots, E_n, F)$.

For fixed $N \in \mathbb{N}$, $\forall (x_1, \ldots, x_n) \in E_1 \times \ldots E_n$,

$$\begin{aligned} &\|\varphi(x_1, \ldots, x_n) - \varphi_N(x_1, \ldots, x_n)\|_F \\ =& \lim_{n \to +\infty} \|\varphi_n(x_1, \ldots, x_n) - \varphi_N(x_1, \ldots, x_n)\|_F \\ \leq& \varepsilon_N \|x_1\| \cdot \|x_n\|. \end{aligned}$$

So $0 \leq \|\varphi - \varphi_N\| \leq \varepsilon_N$. By squeeze theorem

$$\lim_{N \to +\infty} \|\varphi - \varphi_N\| = 0.$$

$\square$

1.10 Uniform Convergence

Definition 1.10.1 Let X be a set, $(Y, \mathscr{T})$ be a topological space, $(f_n)_{n\in\mathbb{N}}$ be a sequence of mappings from X to Y. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges **point-wise** to a mapping $f : X \longrightarrow Y$ if for every $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to $f(x)$.

Suppose that (Y, d) is a metric space. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ converges **uniformly** to a mapping $f : X \longrightarrow Y$ if

$$\lim_{n\to+\infty} \sup_{x\in X} \mathrm{d}(f_n(x), f(x)) = 0.$$

Remark 1.10.2 Let $f, g : X \longrightarrow Y$ be mappings.

$$\mathrm{d}(f, g) = \sup_{x\in X} \mathrm{d}(f(x), g(x))$$

is a metric. Uniform convergence can be seen as convergence of $(f_n)_{n\in\mathbb{N}}$ with respect to this metric.

Theorem 1.10.3 Let $(X, \mathscr{T}_X)$ be a topological space, (Y, d_Y) be a metric space, $(f_n)_{n\in\mathbb{N}}$ be a sequence of mappings from X to Y that converges uniformly to a mapping $f : X \longrightarrow Y$. If $\forall n \in \mathbb{N}$, f_n is continuous at $p \in X$, then f is continuous at p.

Proof We will prove that for any $\varepsilon > 0$, $f^{-1}(B(f(p), \varepsilon))$ is a neighborhood of p.
Let $n \in \mathbb{N}$ such that

$$\sup_{x\in X} \mathrm{d}(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

We claim that

$$f_n^{-1}(B(f_n(x), \frac{\varepsilon}{3})) \subseteq f^{-1}(B(f(x), \varepsilon)).$$

Let x be a element of X such that

$$\mathrm{d}(f_n(x), f_n(p)) < \frac{\varepsilon}{3}.$$

One has

$$\mathrm{d}(f(x), f(p)) \leq \mathrm{d}(f(x), f_n(x)) + \mathrm{d}(f_n(x), f_n(p)) + \mathrm{d}(f_n(p), f(p)) < \varepsilon.$$

$\square$

Theorem 1.10.4 Let (X, d_X) and (Y, d_Y) be two metric spaces, $(f_n)_{n\in\mathbb{N}}$ be sequence of uniformly continuous mappings from X to Y. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f : X \longrightarrow Y$. Then f is uniformly continuous.

Proof Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that

$$\sup_{x\in X} \mathrm{d}(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

f_n is uniformly continuous, so the exists $\delta > 0$

$$\forall (x, y) \in X \times X,\ \mathrm{d}(x, y) < \delta \Rightarrow \mathrm{d}(f_n(x), f_n(y)) < \frac{\varepsilon}{3}.$$

Therefore, for any $(x, y) \in X \times X$ such that $\mathrm{d}(x, y) < \delta$,

$$\mathrm{d}(f(x), f(y)) \le \mathrm{d}(f(x), f_n(x)) + \mathrm{d}(f_n(x), f_n(y)) + \mathrm{d}(f_n(y), f(y)) < \varepsilon.$$

So f is uniformly continuous. $\square$

Theorem 1.10.5 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be normed vector spaces over a complete non-trivially valued field $(K, |\cdot|)$. Let $U \subseteq E$ open, $(f_n)_{n\in\mathbb{N}}$ a sequence of differentiable mappings from U to F. Let $f : U \longrightarrow F$, $g : U \longrightarrow \mathscr{L}(E, F)$ be mappings, $p \in U$. Suppose that
(1) The sequence $(\mathrm{D}f_n)_{n\in\mathbb{N}}$ converges uniformly to g.
(2) $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.
(3) There exists $N \in \mathbb{N}$ and mapping $\delta : U \longrightarrow \mathbb{R}_{\ge 0}$ such that $\lim_{x\to p} \delta(x) = 0$ and for any $n \in \mathbb{N}_{\ge N}$, any $x \in U$,

$$\|f_n(x) - f_n(p) - \mathrm{D}f_n(p)(x - p)\|_F \le \delta(x)\|x - p\|_E.$$

Then f is differentiable and $\mathrm{D}f = g$.

Proof For any $n \in \mathbb{N}$, define

$$\varepsilon_n := \sup_{x\in U} \|\mathrm{D}f_n(x) - g(x)\|,\ \mathrm{d}_n := \sup_{x\in U} \|f_n(x) - f(x)\|.$$

One has

$$\begin{aligned}\|f(x)-f(p)-g(p)(x-p)\| \leq & \|(f(x)-f_n(x))-(f(p)-f_n(p))\| \\ & +\|f_n(x)-f_n(p)-\mathrm{D}f_n(p)(x-p)\| \\ & +\|\mathrm{D}f_n(p)(x-p)-g(p)(x-p)\| \\ \leq & 2\mathrm{d}_n+\delta(x)\|x-p\|_E+\varepsilon_n\|x-p\|_E.\end{aligned}$$

for sufficiently large n.
Therefore,

$$\limsup_{x\to p}\frac{\|f(x)-f(p)-g(p)(x-p)\|_F}{\|x-p\|_E}\leq 2\varepsilon_n.$$

Taking the limit when $n\to\infty$, we obtain

$$\limsup_{x\to p}\frac{\|f(x)-f(p)-g(p)(x-p)\|_F}{\|x-p\|_E}=0.$$

Namely,

$$f(x)-f(p)-g(p)(x-p)=o(\|x-p\|_E),\ x\to p.$$

□

Theorem 1.10.6 Let $(E,\|\cdot\|_E)$ and $(F,\|\cdot\|_F)$ be normed vector spaces over $\mathbb{R}$, $U\subseteq E$ open, $(f_n)_{n\in\mathbb{N}}$ be sequence of differentiable mappings from U to F, $g:U\longrightarrow\mathscr{L}(E,F)$. We suppose that
(1) $(\mathrm{D}f_n)_{n\in\mathbb{N}}$ converges uniformly to g.
(2) $(f_n)_{n\in\mathbb{N}}$ converges point-wise to $f:U\longrightarrow F$.
Then f is differentiable and $\mathrm{D}f=g$.

Proof Let $p\in U$, for any $(n,m)\in\mathbb{N}\times\mathbb{N}$, for any $n\in\mathbb{N}$,

$$c_{m,n}\coloneqq\sup_{x\in U}\|\mathrm{D}f_m(x)-\mathrm{D}f_n(x)\|,\ \varepsilon_n\coloneqq\sup_{x\in U}\|\mathrm{D}f_n(x)-g(x)\|.$$

For $r>0$, $B(p,r)\subseteq U$ by the mean value inequality,

$$\|(f_n(x)-f_m(x))-(f_n(p)-f_m(p))\|\leq c_{m,n}\|x-p\|,\ x\in B(p,r).$$

Passing to the limit when $m\to+\infty$, we obtain

$$\|(f_n(x)-f(x))-(f_n(p)-f(p))\|\leq\varepsilon_n\|x-p\|.$$

we have

$$\begin{aligned}\|f(x)-f(p)-g(p)(x-p)\| \leq &\|(f(x)-f_n(x))-(f(p)-f_n(p))\| \\ &+\|f_n(x)-f_n(p)-\mathrm{D}f_n(p)(x-p)\| \\ &+\|\mathrm{D}f_n(p)(x-p)-g(p)(x-p)\|.\end{aligned}$$

$$\limsup_{x\to p}\frac{\|f(x)-f(p)-g(p)(x-p)\|_F}{\|x-p\|_E}\leq 3\varepsilon_n.$$

Taking the limit $n\to+\infty$

$$\lim_{x\to p}\frac{\|f(x)-f(p)-g(p)(x-p)\|_F}{\|x-p\|_E}=0.$$

Namely,

$$f(x)-f(p)-g(p)(x-p)=o(\|x-p\|_E),\ x\to p.$$

□

Proposition 1.10.7 Let $(E,\|\cdot\|_E)$, $(F,\|\cdot\|_F)$ be normed vector spaces over $\mathbb{R}$. Assume that $(F,\|\cdot\|_F)$ is a Banach space, $U\subseteq E$ be a path connected open, $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable mappings from U to F. Suppose that
(1) $(\mathrm{D}f_n)_{n\in\mathbb{N}}$ converges uniformly to $g: U\longrightarrow \mathscr{L}(E,F)$.
(2) There exists $p\in U$ such that $(f_n(p))_{n\in\mathbb{N}}$ converges.
Then the sequence $(f_n)_{n\in\mathbb{N}}$ converges point-wise on U to a differentiable mapping $f: U\longrightarrow F$ such that $\mathrm{D}f=g$.

Proof We first treat the case where U is convex.
For any $(n,m)\in\mathbb{N}\times\mathbb{N}$, let

$$c_{m,n}:=\sup_{x\in U}\|\mathrm{D}f_m(x)-\mathrm{D}f_n(x)\|.$$

Let $x\in U$. By the mean value inequality,

$$\|(f_n(x)-f_m(x))-(f_n(p)-f_m(p))\|\leq c_{m,n}\|x-p\|,$$

which leads to

$$\|f_n(x)-f_m(x)\|_F\leq\|f_n(p)-f_m(p)\|_F+c_{m,n}\|x-p\|_E.$$

Therefore $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in F (Banach space), so $f_n(x)$ converges in F to some $f(x)$. Now it suffices to use the theorem 1.10.6.

We will now treat the general case. Let $x \in U$. There exists $\gamma : [0,1] \longrightarrow U$ continuous such that $\gamma(0) = p, \gamma(1) = x$. Let I be the set of $t \in [0,1]$ such that $f_n(\gamma(s))$ converges for all $s \in [0,t]$. By definition, I is an interval in $[0,1]$ and $0 \in I$. Therefore, it is of the form $[0,c]$ or $[0,c[$.

Let $B(\gamma(c),r) \subseteq U$. Since γ is continuous, $\gamma^{-1}(B(\gamma(c),r))$ is open in $[0,1]$ and $c \in \gamma^{-1}(B(\gamma(c),r))$. Assume by contraction that $I = [0,c[$, then $I \cap \gamma^{-1}(B(\gamma(c),r)) \neq \varnothing$. There exists $q \in \gamma^{-1}(B(\gamma(c),r)) \cap I$ such that $f_n(q)$ converges. So from the "convex U version" f converges point-wise on $B(\gamma(c),r)$. So $f_n(\gamma(c))$ converges. Contradiction. We deduce that $I = [0,c]$.

If $c \neq 1$, then c is an adherent point of $]c,1]$. $\gamma^{-1}(B(\gamma(c),r))$ open, so there exists $r' > 0$ such that $B(c,r') \subseteq \gamma^{-1}(B(\gamma(c),r))$. In particular, $B(c,r') \cap \,]c,1]$ is an open interval in $[0,1]$ that continuous. So $I \supseteq \,]0,c+r']$. Contradiction. Therefore $c = 1$. $\square$

Definition 1.10.8 Let U be a set and $(F, \|\cdot\|)$ be a Banach space over complete valued field $(K, |\cdot|)$. $(f_n)_{n\in\mathbb{N}} \in \left(F^U\right)^{\mathbb{N}}$ be a sequence of mappings from U to F. If

$$\sum_{n\in\mathbb{N}} \sup_{p\in U} \|f_n(p)\|_F < +\infty,$$

then we say that $\sum\limits_{n\in\mathbb{N}} f_n$ **converges normally**.

Proposition 1.10.9 If $\sum f_n$ converges normally, then it converges uniformly.

Proof For any $n \in \mathbb{N}$, let $g_n = \sum_{k=0}^{n} f_k$. We need to check that the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly. For any $x \in U$, $\sum_{n\in\mathbb{N}} \|f_n(x)\| < +\infty$. So $\sum_{n\in\mathbb{N}} f_n(x)$ converges absolutely. In particular, $(g_n(x))_{n\in\mathbb{N}}$ converges to some $g(x)$.

$$\begin{aligned}\|g_n(x) - g(x)\|_F &= \lim_{m\to+\infty} \|g_n(x) - g_m(x)\|_F \\ &\leq \lim_{m\to+\infty} \|f_{n+1}(x) + \cdots + f_m(x)\|_F \\ &\leq \limsup_{m\to+\infty} \sum_{k\geq n+1} \|f_k(x)\|_F \\ &\leq \varepsilon_n.\end{aligned}$$

Let

$$\varepsilon_n = \sum_{k\geq n+1} \sup_{p\in U} \|f_k(p)\|_F, \quad \lim_{n\to+\infty} \varepsilon_n = 0.$$

So,

$$\limsup_{n\to+\infty}\left(\sup_{x\in U}\|g_n(x)-g(x)\|_F\right)=0,$$

namely, $(g_n)_{n\in\mathbb{N}}$ converges to g. □

Proposition 1.10.10 Let $(K,|\cdot|)$ be a complete valued field which is non-trivially valued, $(E,\|\cdot\|)$ be a normed vector space and $(F,\|\cdot\|_F)$ be a Banach space over K. $U\subseteq E$ be an open subset, $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable mappings $U\longrightarrow F$ and $p\in U$. Assume that

(1) $\sum\limits_{n\in\mathbb{N}} f_n$ converges normally (uniformly suffices).

(2) $\sum\limits_{n\in\mathbb{N}} \mathrm{D}f_n$ converges normally (uniformly suffices).

(3) $\exists N\in\mathbb{N}$ and mappings $(\delta_n : U\longrightarrow\mathbb{R}_{\geq 0})_{n\in\mathbb{N}_{\geq N}}$ such that

1. $\forall n\in\mathbb{N}_{\geq N}$, $\lim_{x\to p}\delta_n(x)=\delta_n(p)=0$.
2. $\sum_{n\in\mathbb{N}}\delta_n$ converges normally (uniformly suffices).
3. $\forall n\in\mathbb{N}_{\geq N}$, $\forall x\in U$,

$$\|f_n(x)-f_n(p)-\mathrm{D}f_n(p)(x-p)\|_F\leq\delta_n(x)\|x-p\|_E.$$

Let f and g be limits of $\sum\limits_{n\in\mathbb{N}} f_n$ and $\sum\limits_{n\in\mathbb{N}}\mathrm{D}f_n$ respectively. Then f is differentiable at p and $\mathrm{D}f=g$.

Proposition 1.10.11 Let $(E,\|\cdot\|_E)$ be a normed vector space and $(F,\|\cdot\|_F)$ be a Banach space over $\mathbb{R}$. Let $U\subseteq E$ open, and $(f_n : U\longrightarrow F)_{n\in\mathbb{N}}$ be a sequence of mappings $U\longrightarrow F$. Suppose that

(1) $\sum_{n\in\mathbb{N}}\mathrm{D}f_n$ converges normally (uniformly suffices) to some $g : U\longrightarrow\mathscr{L}(E,F)$.

(2) $\sum_{n\in\mathbb{N}} f_n$ converges point-wise to some $f : U\longrightarrow F$.

Then f is differentiable on U and $\mathrm{D}f=g$.

Remark 1.10.12 If U is path connected, one can replace (2) by (2'): $\exists p\in U$, $\sum_{n\in\mathbb{N}} f_n(p)$ converges.

1.11 Power Series

We fix a complete non-trivially valued field $(K, |\cdot|)$, and let $(E, \|\cdot\|_E)$ be a Banach space over K.

Definition 1.11.1 Let $(S_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ and $b \in K$. We call power series **centered at** b with coefficients $(s_n)_{n\in\mathbb{N}}$ the sequence of polynomial mappings.

$$\left((z \in K) \longmapsto \sum_{l=0}^{n} (z-b)^l s_l\right)_{n\in\mathbb{N}}$$

denoted as

$$\sum_{l=0}^{n} (z-b)^l s_l.$$

If $S = \sum_{n\in\mathbb{N}} (z-b)^n s_n$, we denote by $R(S)$ the element

$$\left(\limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}}\right)^{-1} \in [0, +\infty]$$

called the **convergence radius** of S. ($0^+ := +\infty$, $(+\infty)^{-1} := 0$)

Proposition 1.11.2 Let $S = \sum_{n\in\mathbb{N}} (z-b)^n s_n$.
(1) $\forall a \in K$, if $|a-b| < R(S)$, then $S(a) := \sum_{n\in\mathbb{N}} (a-b)^n s_n$ converges absolutely.
(2) If $r > 0$ such that $(r^n \|s_n\|)_{n\in\mathbb{N}}$ is bounded, then $R(S) \geq r$.
(3) If $a \in K$ is such that $|a-b| > R(S)$, then $\sum_{n\in\mathbb{N}} (a-b)^n s_n$ diverges.

Proof
(1)

$$\|(a-b)^n s\|^{\frac{1}{n}} = (|(a-b)^n| \cdot \|s_n\|)^{\frac{1}{n}} = |a-b| \cdot \|s_n\|^{\frac{1}{n}}.$$

$$\limsup_{n\to+\infty} \|(a-b)^n s_n\|^{\frac{1}{n}} = |a-b| \cdot \limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}}.$$

If $|a-b| < R(S)$, then $|a-b| \cdot \limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}} < 1$. By the root test of Cauchy, $\sum_{n\in\mathbb{N}} (a-b)^n s_n$ converges absolutely.
(2)

$$\|s_n\|^{\frac{1}{n}} = \frac{1}{r} \left(r^n \|s_n\|\right)^{\frac{1}{n}}.$$

Since $(r^n \|s_n\|)_{n\in\mathbb{N}}$ is bounded,

$$\limsup_{n\to+\infty} \left(r^n \|s_n\|\right)^{\frac{1}{n}} \leq 1.$$

So $\limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}} \leq \frac{1}{r}$. So $R(S) \geq r$.
(3) If $|a-b| > R(S)$, then

$$\limsup_{n\to+\infty} \|(a-b)^n s_n\|^{\frac{1}{n}} = |a-b| \cdot \limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}} > 1.$$

So $\sum_{n\in\mathbb{N}}(a-b)^n s_n$ diverges. □

Proposition 1.11.3 Let $S = \sum_{n\in\mathbb{N}} (z-b)^n s_n$ be a power series.
(1) $\forall r \in \mathbb{R}_{\geq 0}$ such that $r < R(S)$. the series S converges normally on $\overline{B}(b,r)$.
(2) $(a \in B(b, R(S))) \longmapsto S(a) := \sum_{n\in\mathbb{N}} (a-b)^n S_n$ is continuous.

Proof
(1) $\forall a \in \overline{B}(b,r)$,

$$\sum_{n\in\mathbb{N}} \|(a-b)^n s_n\| \leq \sum_{n\in\mathbb{N}} r^n \|s_n\| < +\infty$$

since $\limsup_{n\to+\infty} r \cdot \|s_n\|^{\frac{1}{n}} < 1$.
(2) $a \longmapsto S(a)$ is continuous on any $B(b,r)$, $r < R(S)$. Since

$$B(b, R(S)) = \bigcup_{r<R(S)} B(b,r),$$

S is continuous on $B(b, R(S))$. □

Definition 1.11.4 Let $S = \sum_{n\in\mathbb{N}} (z-b)^n s_n$. We define the formal derivative of S as

$$\sum_{n\in\mathbb{N}_{\geq 1}} (z-b)^{n-1}(n s_n).$$

Proposition 1.11.5 Let $S = \sum_{n\in\mathbb{N}}(z-b)^n s_n$ be a formal power series. Let $P \in K[T]$. For any $n \in \mathbb{N}$, let $P(n) := P(n1_K) \in K$. Let

$$S_p := \sum_{n\in\mathbb{N}} (z-b)^n (P(n) s_n).$$

Then $R(S_p) \geq R(S)$.

Proof We assume that $P \neq 0$, $P(T)$ is of the form

$$C_d T^d + C_{d-1} T^{d-1} + \cdots + C_1 T + C_0,\ C_d \neq 0.$$

$$|P(n)| = \mathcal{O}(n^d) = o(r^n),\ \text{for any } r > 1.$$

Hence, $\exists N \in \mathbb{N}$ such that $|P(n)| \leq r^n$, $\forall n \in \mathbb{N}_{\geq N}$.

$$\limsup_{n\to+\infty} \|P(n)s_n\|^{\frac{1}{n}} \leq r \cdot \limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}}.$$

Taking the limit when $r \to 1$, get

$$\limsup_{n\to+\infty} \|P(n)s_n\|^{\frac{1}{n}} \leq \limsup_{n\to+\infty} \|s_n\|^{\frac{1}{n}}.$$

$$R(S_p) \geq R(S).$$

□

Lemma 1.11.6 Let $(z_0, z) \in K^2$, $n \in \mathbb{N}_{\geq 1}$.

$$z^n - z_0^n - nz_0^{n-1}(z - z_0) = (z - z_0)^2 \sum_{j=0}^{n-2} (n - j - 1) z^j z_0^{n-2-j}.$$

Proof

$$\begin{aligned}
z^n - z_0^n =& (z - z_0) \sum_{i=0}^{n-1} z^i z^{n-1-i}. \\
z^n - z_0^n - nz_0^{n-1}(z - z_0) =& (z - z_0) \sum_{i=0}^{n-1} (z^i z_0^{n-1-i} - z_0^{n-1}) \\
=& (z - z_0) \sum_{i=0}^{n-1} z_0^{n-i-1}(z^i - z_0^i) \\
=& (z - z_0)^2 \sum_{i=1}^{n-1} z_0^{n-1-i} \sum_{j=0}^{i-1} z^j z_0^{i-j-1} \\
=& (z - z_0)^2 \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} z^j z_0^{n-2-j} \\
=& (z - z_0)^2 \sum_{j=0}^{n-2} (n - j - 1) z^j z_0^{n-2-j}.
\end{aligned}$$

□

Theorem 1.11.7 Let $\sum_{n\in\mathbb{N}}(z-b)^n s_n$ be a power series and R be its convergence radius. For any $z \in B(b,R)$, let $S(z)$ be the limit of the series. Then the mapping $S : B(b,R) \longrightarrow E$ is differentiable, and its derivative is given by the limit of the power series

$$\sum_{n\in\mathbb{N}_{\geq 1}} (z-b)^{n-1}(ns_n).$$

Proof Let $r < R$, $(z, z_0) \in B(b,r)^2$.

$$\begin{aligned}
&\|(z-b)^n s_n - (z_0-b)^n s_n - (z-z_0)(z_0-b)^{n-1} n s_n\| \\
=&|z-z_0|^2 \cdot \| \sum_{j=0}^{n-2} (n-1-j)(z-b)^j (z_0-b)^{n-2-j} s_n\|.
\end{aligned}$$

$$\begin{aligned}
&\| \sum_{j=0}^{n-2} (n-1-j)(z-b)^j (z_0-b)^{n-2-j} s_n\| \\
\leq& \sum_{j=0}^{n-2} (n-1-j) r^{n-2}\|s_n\| = \frac{n(n-1)}{2} r^{n-2}\|s_n\|.
\end{aligned}$$

We know that

$$\sum_{n\in\mathbb{N}} \frac{n(n-1)}{2} r^{n-2}\|s_n\| < +\infty.$$

Therefore, the result follows from the proposition 1.10.10. □

Definition 1.11.8 Let $(a_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$. We call **Cauchy product** of the series $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} s_n$ as the series:

$$\sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} a_k s_{n-k} \right).$$

Theorem 1.11.9 (Merterns) Let $(a_n)_{n\in\mathbb{N}} \in K^{\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$. Suppose that $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} s_n$ converges to $b \in K$ and $t \in E$ respectively.
(1) If at least one of $\sum_{n\in\mathbb{N}} a_n$, $\sum_{n\in\mathbb{N}} s_n$ converges absolutely, then the their Cauchy product converges to bt.
(2) If both $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} s_n$ converge absolutely, then the Cauchy product also converges absolutely.

Proof

(1) Suppose that $\sum_{n\in\mathbb{N}} a_n$ converges absolutely. For any $n \in \mathbb{N}$, let

$$A_n := \sum_{k=0}^{n} a_k,\ S_n := \sum_{k=0}^{n} s_k.$$

For any $N \in \mathbb{N}$, let

$$t_N = \sum_{n=0}^{N} \left(\sum_{k=0}^{n} a_k s_{n-k} \right) = \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ k+l\leq N}} a_k s_l = \sum_{k=0}^{N} a_k S_{N-k} = A_n t + \sum_{k=0}^{N} a_k \left(S_{N-k} - t\right).$$

Then,

$$t_N - bt = (A_N - b)\, t + \sum_{k=0}^{N} a_k \left(S_{N-k} - t\right).$$

Let $\alpha := \sum_{n\in\mathbb{N}} |a_n|$ and for any $n \in \mathbb{N}$ let

$$\varepsilon_n := \sup_{m\in\mathbb{N},\ m\leq n} \|S_m - t\|.$$

For any $l \in \{0, \ldots, N\}$, one has

$$\begin{aligned} \|\sum_{k=0}^{N} a_k \left(S_{N-k} - t\right)\| &\leq \sum_{k=0}^{N-l} |a_k| \cdot \|S_{N-k} - t\| + \sum_{k=N-l+1}^{N} |a_k| \cdot \|S_{N-k} - t\| \\ &\leq \varepsilon_l \cdot \alpha + \max_{i\in\{0,\ldots,l-1\}} \|S_i - t\| \cdot \sum_{k=N-l+1}^{N} |a_k|. \end{aligned}$$

We get

$$\forall l \in \mathbb{N},\ \limsup_{N\to+\infty} \|\sum_{k=0}^{N} a_k \left(S_{N-k} - t\right)\| \leq \varepsilon_l \alpha.$$

Taking the infimum with respect to l, we get

$$\limsup_{N\to+\infty} \|\sum_{k=0}^{N} a_k \left(S_{N-k} - t\right)\| = 0.$$

We deduce therefore that

$$\lim_{N\to+\infty} t_N = bt.$$

(2) Let

$$\alpha = \sum_{n\in\mathbb{N}} |a_n|,\ \beta = \sum_{n\in\mathbb{N}} \|s_n\|.$$

For any $N \in \mathbb{N}$, one has

$$\sum_{n=0}^{N} \| \sum_{k=0}^{n} a_k s_{n-k} \| \leq \sum_{n=0}^{N} \sum_{k=0}^{n} |a_k| \cdot \|s_n\| \leq \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ k\leq N,\ l\leq N}} |a_k| \|s_l\| \leq \alpha \cdot \beta.$$

So the Cauchy product of $\sum_{n\in\mathbb{N}} a_n$ and $\sum_{n\in\mathbb{N}} s_n$ converges absolutely. □

Example 1.11.10 Consider

$$\mathrm{e}^z = \exp(z) := \sum_{n\in\mathbb{N}} \frac{z^n}{n!}, \ z \in \mathbb{C}.$$

By thw ratio test of D'Alembert, for any $r > 0$, $\sum_{n\in\mathbb{N}} \frac{r^n}{n!} < +\infty$. e^z is well defined.
Let $\alpha \in \mathbb{C}$,

$$\exp'(\alpha z) = \alpha \exp(\alpha z).$$

We define

$$\cos(z) := \frac{\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}}{2}, \ \sin(z) := \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}},$$
$$\cosh(z) := \frac{\mathrm{e}^{z} + \mathrm{e}^{-z}}{2}, \ \sinh(z) := \frac{\mathrm{e}^{z} - \mathrm{e}^{-z}}{2}.$$

Proposition 1.11.11 Let $(a, b, z) \in \mathbb{C}^3$, then

$$\exp((a + b)z) = \exp(az) \exp(bz).$$

Proof The Cauchy product of $\sum_{n\in\mathbb{N}} \frac{(az)^n}{n!}$ and $\sum_{n\in\mathbb{N}} \frac{(bz)^n}{n!}$ is $\sum_{n\in\mathbb{N}} \frac{(a+b)^n z^n}{n!}$. Use the theorem of Merterns. □

1.12 Directional Differential

Definition 1.12.1 Let $(K, |\cdot|)$ be a complete non-trivially valued field, and $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K. Let $U \subseteq K$ open,

$f : U \longrightarrow F$ be a mapping, $p \in U$, $h \in E$. If the limit

$$\lim_{t\to 0} \frac{f(p+th) - f(p)}{t}$$

exists, we say that f admits the **directional derivative** at p along h.

Notation 1.12.2

$$\partial_h f(p) = \lim_{t\to 0} \frac{f(p+th) - f(p)}{t}.$$

Definition 1.12.3 Let $(E_1, \|\cdot\|_1), \dots, (E_n, \|\cdot\|_n)$ be normed vector spaces, $E := E_1 \times \cdots \times E_n$,

$$\|(s_1, \dots, s_n)\| = \max_{i\in\{1,\dots,n\}} \|s_i\|.$$

If $f : U \longrightarrow F$. We say that f has the **i-th partial differential** at $p = (p_1, \dots, p_n) \in U$, if the mapping

$$x_i \longmapsto f(p_1, \dots, p_{i-1}, x_i, p_{i+1}, \dots, p_n)$$

is differentiable at p_i. We denote the existing differential at p_i by

$$\mathrm{D}_i f(p) \in \mathscr{L}(E_i, F).$$

In the case when $E_i = K$,

$$\mathrm{D}_i f(p)(1) := \partial_i f(p) \text{ or } \frac{\partial f}{\partial x_i}(p).$$

Note that

$$\partial_i f(p) = \partial_{\underset{i\text{-th}}{(0,\cdots,1,\cdots,0)}} f(p).$$

Remark 1.12.4 Let $(K, |\cdot|)$ be a complete non-trivially valued filed, $(E_i, \|\cdot\|_i)$, $i \in \{1, \cdots, n\}$, $(F, \|\cdot\|_F)$ be normed vector spaces. $E = E_1 \times \cdots \times E_n$, equipped with the norm $\|\cdot\|$ defined as

$$\|(x_1, \cdots, x_n)\| = \max_{i\in\{1,\cdots,n\}} \|x_i\|_i.$$

Let $U \subseteq E$ be an open subset, $p \in U$, $f : U \longrightarrow F$ be a mapping. If f is differentiable at p, then f has the i-th partial differential at p for $i \in \{1, \cdots, n\}$.

In fact,

$$f(p_1,\cdots,p_i+h_i,\cdots,p_n)=f(p)+\mathrm{D}f(p)(0,\cdots,h_i,\cdots,0)+o(\|h_i\|_i).$$

$$\mathrm{D}_if(p)(h_i)=\mathrm{D}f(p)(0,\cdots,h_i,\cdots,0).$$

$$\mathrm{D}f(p)(h)=\sum_{i=1}^n\mathrm{D}f(p)(0,\cdots,h_i,\cdots,0)=\sum_{i=1}^n\mathrm{D}_if(p)(h_i).$$

Proposition 1.12.5 Let $(E_i,\|\cdot\|_i)$, $i\in\{1,\cdots,n\}$, $(F,\|\cdot\|_F)$ be normed vector spaces over $\mathbb{R}$, with $\dim_{\mathbb{R}}(F)<+\infty$. Let $E=E_1\times\cdots\times E_n$, equipped with the norm $\|\cdot\|$ defined as

$$\|(x_1,\cdots,x_n)\|=\max_{i\in\{1,\cdots,n\}}\|x_i\|_i.$$

Let $U\subseteq E$ be an open subset, $f:U\longrightarrow F$ be a mapping. Suppose that, for any $i\in\{1,\cdots,n\}$, f has i^{th} partial differential on U, and $\mathrm{D}_if:U\longrightarrow\mathscr{L}(E_i,F)$ is continuous. Then f is differentiable on U, and

$$\forall p\in U,\ \mathrm{D}f(p)(h_1,\cdots,h_n)=\sum_{i=1}^n\mathrm{D}_if(p)(h_i).$$

Proof We first treat the case where $F=\mathbb{R}$. Let $p\in U$, and $r>0$ such that $B(p,r)\subseteq U$. Let $h=(h_1,\cdots,h_n)\in B(0,r)$.

$$\begin{aligned}f(p+h)-f(p)=\sum_{i=1}^n(&f(p_1+h_1,\cdots,p_i+h_i+\cdots,p_{i+1},\cdots,p_n)\\&-f(p_1+h_1,\cdots,p_i,\cdots,p_{i+1},\cdots,p_n)).\end{aligned}$$

By the mean value theorem of Lagrange,

$$\exists(t_1(h),\cdots,t_n(h))\in\,]0,1[^n$$

such that

$$f(p+h)-f(p)=\sum_{i=1}^n\mathrm{D}_if(p_1+h_1,\cdots,p_i+t_i(h)h_i,\cdots,p_{i+1},\cdots,p_n)(h_i).$$

$$
\begin{aligned}
& f(p+h)-f(p)-\sum_{i=1}^{n} \mathrm{D}_i f(p)(h) \\
=& \sum_{i=1}^{n} \mathrm{D}_i f(p_1+h_1,\cdots,p_i+t_i(h)h_i,\cdots,p_{i+1},\cdots,p_n)(h_i) \\
-& \sum_{i=1}^{n} \mathrm{D}_i f(p_1,\cdots,p_i,\cdots,p_{i+1},\cdots,p_n)(h_i) \\
=& o(\|h\|).
\end{aligned}
$$

$\square$

Chapter 2

Integral Calculus

2.1 Differential 1-form

Definition 2.1.1 Let $(K, |\cdot|)$ be a complete non-trivially valued field. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K. Let $U \subseteq E$ be an open subset. We call **1-form** *on U with coefficients in F* any mapping

$$\alpha : U \longrightarrow \mathscr{L}(E, F).$$

If there exists $f : U \longrightarrow F$ differentiable such that $\mathrm{D}f = \alpha$, we say that α is an **exact** 1-form. (Sometimes $\mathrm{D}f$ is also written as $\mathrm{d}f$.)

Definition 2.1.2 We call a complete valued filed **extension** of $(K, |\cdot|)$ any complete valued field $(K', |\cdot|')$ such that $K \subseteq K'$ and $|\cdot| = |\cdot|'|_K$.
Let $(F, \|\cdot\|)$ be a normed vector space over K. If $\alpha : U \longrightarrow \mathscr{L}(E, K')$ and $s : U \longrightarrow F$ be mappings, we denote by

$$\alpha \otimes s : U \longrightarrow \mathscr{L}(E, F)$$

be the mapping sending $p \in U$ to

$$(h \in E) \longmapsto \alpha(p)(h)s(p).$$

Note that

$$\|\alpha(p)(h)s(p)\|_F \leq |\alpha(p)(h)|_{K'} \cdot \|s(p)\|_F \leq \|\alpha(p)\| \cdot \|s(p)\|_F \cdot \|h\|_E.$$

If $(F, \|\cdot\|_F) = (K', |\cdot|')$, $\alpha \otimes s$ is also written as αs.

Example 2.1.3 $(K, |\cdot|) = (\mathbb{R}, |\cdot|)$, $K' = \mathbb{C}$, $|x + \mathrm{i}y|' := \sqrt{x^2 + y^2}$.

Example 2.1.4 Let $\varphi \in \mathscr{L}(E, F)$,

$$\begin{aligned} \mathrm{D}\varphi : \ E &\longrightarrow \mathscr{L}(E, F) \\ p &\longmapsto \varphi. \end{aligned}$$

is a constant mapping.
As a 1-form, it is often written as $\mathrm{d}\varphi$.

Example 2.1.5 $E = K^n$, $x_i : K^n \longrightarrow K$, $(p_1, \cdots, p_n) \longmapsto p_i$. $U \subseteq E$ open, $f : U \longrightarrow K$ differentiable.

$$\mathrm{d}f(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\mathrm{d}x_i.$$

Example 2.1.6 Let $w \in \mathbb{C}$, $f : \mathbb{R} \longrightarrow \mathbb{C}$, $t \longmapsto \exp(wt)$.

$$\mathrm{d}f(t) = f'(t)\mathrm{d}t = w \exp(wt)\mathrm{d}t.$$

Proposition 2.1.7 Let $(K', |\cdot|)$ be a complete valued extension of $(K, |\cdot|)$, and $(F, \|\cdot\|_F)$ be a normed vector space over K'. Let $(E, \|\cdot\|_E)$ be a normed vector space over K, $U \subseteq E$ be ann open subset. Let $f : U \longrightarrow K'$ and $g : U \longrightarrow F$ be two mappings that are differentiable, then

$$\mathrm{d}(fg) = f\mathrm{d}g + \mathrm{d}f \otimes g.$$

Proposition 2.1.8 Let $(K', |\cdot|')$ be a complete valued extension of $(K, |\cdot|)$. $(E, \|\cdot\|_E)$ be a normed vector space over K, $(F, \|\cdot\|_F)$ be a normed vector space over K'. Let $U \subseteq E$ be an open subset, and $V \subseteq K'$ be an open subset. $f : U \longrightarrow V$, $g : V \longrightarrow F$ be differentiable mappings, then

$$\mathrm{d}(g \circ f) = \mathrm{d}f \otimes (g' \circ f).$$

Proof For $p \in U$ and $h \in E$,

$$\begin{aligned} \mathrm{D}(g \circ f)(p)(h) &= \mathrm{D}g(f(p))(\mathrm{D}f(p)(h)) \\ &= \mathrm{D}f(p)(h) \cdot \mathrm{D}g(f(p))(1) \\ &= \mathrm{D}f(p)(h) \cdot g'(f(p)) \end{aligned}$$

□

2.2 Primitive Functions

Proposition 2.2.1 Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be normed vector spaces over $\mathbb{R}$ and $U \subseteq E$ be a path connected open subset. If $f : U \longrightarrow F$ is a mapping such that $\mathrm{d}f = 0$, then f is a constant mapping.

Proof Let p and q be elements of U. There exists $\gamma : [0, 1] \longrightarrow U$ continuous and differentiable on $]0, 1[$, such that $\gamma(0) = p$, $\gamma(1) = q$.

$$\|f(p) - f(q)\|_F = \|f(\gamma(0)) - f(\gamma(1))\|_F \leq \sup_{t \in]0,1[} \|\mathrm{D}f(\gamma(t))(\gamma'(t))\|_F = 0.$$

So $f(p) = f(q)$. □

Definition 2.2.2 Let $I \subseteq \mathbb{R}$ be an open interval and $\varphi : I \longrightarrow F$ be a mapping. If there exists $\Phi : I \longrightarrow F$ such that $\Phi' = \varphi$, we say that Φ is a primitive function of φ. We denote by

$$\int \varphi(t)\mathrm{d}t$$

an arbitrary primitive function of φ. By the previous proposition,

$$\int \varphi(t)\mathrm{d}t = \Phi(t) + C.$$

where C is a constant mapping.

Example 2.2.3 Let $w \in \mathbb{C}$,

$$\int \exp(wt)\mathrm{d}t = \begin{cases} \frac{\exp(wt)}{w} + C & , w \neq 0 \\ t + C & , w = 0. \end{cases}$$

Proposition 2.2.4 Let $I \subseteq \mathbb{R}$ be an open interval. Let $g : I \longrightarrow \mathbb{R}$ and $\varphi : I \longrightarrow F$ be mappings having $G : I \longrightarrow \mathbb{R}$ and $\Phi : I \longrightarrow F$ as primitive functions. Then

$$\int G(t)\mathrm{d}\Phi(t) + \int \mathrm{d}G(t) \otimes \Phi(t) = G(t)\Phi(t) + C.$$

or equivalently,

$$\int G(t)\mathrm{d}t \otimes \varphi(t) + \int g(t)\mathrm{d}t \otimes \Phi(t) = G(t)\Phi(t) + C.$$

If $F = \mathbb{R}$ or $F = \mathbb{C}$, the formula can be written as

$$\int G(t)\mathrm{d}\Phi(t) + \int \Phi(t)\mathrm{d}G(t) = G(t)\Phi(t) + C.$$

or

$$\int G(t)\varphi(t)\mathrm{d}t + \int \Phi(t)g(t)\mathrm{d}t = G(t)\Phi(t) + C.$$

Example 2.2.5

$$\int t\mathrm{e}^t\mathrm{d}t = \int t\mathrm{d}(\mathrm{e}^t) = t\mathrm{e}^t - \int \mathrm{e}^t\mathrm{d}t = t\mathrm{e}^t - \mathrm{e}^t + C.$$

Proposition 2.2.6 Let $U \subseteq \mathbb{R}$ be an open subset, $V \subseteq \mathbb{R}$ be an open subset, $f : U \longrightarrow V$ and $g : V \longrightarrow F$ differentiable mappings. One has

$$\int \mathrm{d}f(t) \otimes g'(f(t)) = g(f(t)) + C.$$

Example 2.2.7

$$\int \sin(t)\cos(t)\mathrm{d}t = \int \sin(t)\mathrm{d}(\sin(t)) = \frac{1}{2}\sin(t)^2 + C.$$

2.3 Riesz Space

We fix a set Ω. We equipped $\mathbb{R}^\Omega$ with the partial order $\leq$ as follows:

$$\forall (f,g) \in \mathbb{R}^\Omega \times \mathbb{R}^\Omega,\ f \leq g \Leftrightarrow \forall \omega \in \Omega,\ f(\omega) \leq g(\omega).$$

If $(f_1, \cdots, f_n) \in \left(\mathbb{R}^\Omega\right)^n$, $\inf\{f_1, \cdots, f_n\}$ and $\sup\{f_1, \cdots, f_n\}$ exists.

$$\forall \omega \in \Omega,\ \inf\{f_1, \cdots, f_n\}(\omega) = \min\{f_1(\omega), \cdots, f_n(\omega)\}$$

$$\forall \omega \in \Omega,\ \sup\{f_1, \cdots, f_n\}(\omega) = \max\{f_1(\omega), \cdots, f_n(\omega)\}$$

Definition 2.3.1 We call Riesz space on Ω any vector space S of $\mathbb{R}^\Omega$, such that

$$\forall (f, g) \in S \times S,\ \inf\{f, g\} \in S.$$

Remark 2.3.2 $\forall (f, g) \in S \times S$,

$$\sup\{f, g\} = f + g - \inf\{f, g\} \in S.$$

$$|f| = \sup\{f, 0\} - \inf\{f, 0\} \in S.$$

By induction, $\forall n \in \mathbb{N}_{\geq 1}$, $\forall (f_1, \cdots, f_n) \in S^n$,

$$\inf\{f_1, \cdots, f_n\}, \sup\{f_1, \cdots, f_n\} \subseteq S.$$

$$\forall \omega \in \Omega,\ \sup\{f, g\}(\omega) = \max\{f(\omega), g(\omega)\} = f(\omega) + g(\omega) - \min\{f(\omega), g(\omega)\}.$$

Definition 2.3.3 Let S be a Riesz space on Ω. We call **integral operator** on S any $\mathbb{R}$-linear mapping $I : S \longrightarrow \mathbb{R}$ such that
(1) $\forall (f, g) \in S \times S$, if $f \leq g$, then $I(f) \leq I(g)$.
(2) If $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence in S, that converges point-wise to constant zero mapping 0, one has

$$\lim_{n \to +\infty} I(f_n) = 0.$$

Example 2.3.4 Let $\Omega = \mathbb{R}$, $\forall A \subseteq \mathbb{R}$, let

$$\begin{array}{rccl} \mathbb{1}_A : & \mathbb{R} & \longrightarrow & \{0, 1\} \\ & x & \longmapsto & \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \end{array}$$

Let S be the vector space of $\mathbb{R}^\mathbb{R}$ generated by mappings of the form $\mathbb{1}_{]a,b]}$, $(a \leq b)$
Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a right continuous mapping,

$$\forall t \in \mathbb{R},\ \varphi(t) = \lim_{\varepsilon > 0, \varepsilon \to 0} \varphi(t + \varepsilon).$$

which is increasing. Then $I_\varphi : S \longrightarrow \mathbb{R}$,

$$I_\varphi\left(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i, b_i]}\right) := \sum_{i=1}^{n} \lambda_i \left(\varphi(b_i) - \varphi(a_i)\right)$$

is an integral operator.

Proposition 2.3.5 Let Ω be a set and S be a Riesz space on Ω. An $\mathbb{R}$-linear mapping $I : S \longrightarrow \mathbb{R}$ that satisfies $(f \leq g \Rightarrow I(f) \leq I(g))$ is an integral operator if and only if, for any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in S that converges point-wise to some $f \in S$, one has

$$\lim_{n\to+\infty} I(f_n) = I(f).$$

Proof
"$\Rightarrow$": $(f - f_n)_{n\in\mathbb{N}}$ is decreasing and converges to 0. So

$$\lim_{n\to+\infty} I(f - f_n) = 0.$$

So $\lim_{n\to+\infty} I(f_n) = I(f)$.
"$\Leftarrow$": Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence in S that converges point-wise to 0. Then $(-f_n)_{n\in\mathbb{N}}$ is increasing and converges point-wise to 0. So

$$\lim_{n\to+\infty} I(-f_n) = 0.$$

So, $\lim_{n\to+\infty} I(f_n) = 0$. □

Proposition 2.3.6 Let Ω be a set and S be a Riesz space on Ω and $I : S \longrightarrow \mathbb{R}$ be an integral operator. Let $g \in S$ and $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in S. If

$$\forall \omega \in \Omega,\ g(\omega) \leq \lim_{n\to+\infty} f_n(\omega),$$

then

$$I(g) \leq \lim_{n\to+\infty} I(f_n).$$

Proof $(\inf\{g, f_n\})_{n\in\mathbb{N}}$ is an increasing sequence in S. It converges to g. Hence,

$$I(g) = \lim_{n\to+\infty} I(\inf\{g, f_n\}) \leq \lim_{n\to+\infty} I(f_n).$$

□

Corollary 2.3.7 Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be increasing sequences in S. Suppose that

$$\forall \omega \in \Omega,\ \lim_{n\to+\infty} f_n(\omega) \leq \lim_{n\to+\infty} g_n(\omega).$$

Then,

$$\lim_{n\to+\infty} I(f_n) \leq \lim_{n\to+\infty} I(g_n).$$

Proof $\forall k \in \mathbb{N}, \forall \omega \in \Omega,$

$$f_k(\omega) \leq \lim_{n\to+\infty} f_n(\omega) \leq \lim_{n\to+\infty} g_n(\omega).$$

So $I(f_k) \leq \lim_{n\to+\infty} I(g_n)$. Taking the limit when $k \to +\infty$, we get

$$\lim_{k\to+\infty} I(f_k) \leq \lim_{n\to+\infty} I(g_n).$$

□

Definition 2.3.8 Let $S^{\uparrow}$ be the set of all mappings $f : \Omega \longrightarrow]-\infty, +\infty]$ that can be written as the point-wise limit of an increasing sequence in S.

Remark 2.3.9
(1) If $f \in S^{\uparrow}$, $\lambda > 0$, then $\lambda f \in S^{\uparrow}$.
(2) If $(f, g) \in S^{\uparrow} \times S^{\uparrow}$, then $f + g \in S^{\uparrow}$, $\inf\{f, g\} \in S^{\uparrow}$, $\sup\{f, g\} \in S^{\uparrow}$.
(3) If $I : S \longrightarrow \mathbb{R}$ is an integral operator, then for any $f \in S^{\uparrow}$ that is written as the point-wise limit of two increasing sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in S, then

$$\lim_{n\to+\infty} I(f_n) = \lim_{n\to+\infty} I(g_n).$$

We denote by $I(f)$ this limit.

Proposition 2.3.10 Let $(f_n)_{n\in\mathbb{N}} \in (S^{\uparrow})^{\mathbb{N}}$ be an increasing sequence, and f be its point-wise limit. Then $f \in S^{\uparrow}$, and $I(f) = \lim\limits_{n\to+\infty} I(f_n)$ for any operator I.

Proof For any $k \in \mathbb{N}$, let $(g_{k,m})_{m\in\mathbb{N}} \in S^{\mathbb{N}}$ be an increasing sequence in S that converges point-wise to f_k. For any $n \in \mathbb{N}$, let

$$h_n = \sup\{g_{0,n}, g_{1,n}, \cdots, g_{n,n}\} \in S.$$

$(h_n)_{n\in\mathbb{N}}$ is an increasing sequence in S.
$\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, k \leq n$, one has

$$f_n \geq f_k \geq g_{k,n}, \;\; f_n \geq h_n.$$

So,

$$f = \lim_{n\to+\infty} f_n \geq \lim_{n\to+\infty} h_n \geq \lim_{n\to+\infty} g_{k,n} = f_k.$$

This leads to

$$f = \lim_{n\to+\infty} h_n,\ f \in S^{\uparrow}.$$

One has

$$I(f) = \lim_{n\to+\infty} I(h_n) \leq \lim_{n\to+\infty} I(f_n).$$

Moreover, $\forall n \in \mathbb{N}$, $f \geq f_n$, so $I(f) \geq I(f_n)$. Thus leads to

$$I(f) \geq \lim_{n\to+\infty} I(f_n).$$

□

Definition 2.3.11 Let Ω be a set and S be a Riesz space on Ω. We denote by $S^{\downarrow}$ the set of all mappings $f : \Omega \longrightarrow [-\infty, +\infty[$ that can be written as the point-wise limit of a decreasing sequence in S.

Remark 2.3.12
(1) $f \in S^{\downarrow} \Leftrightarrow -f \in S^{\uparrow}$.
(2) If $f \in S^{\downarrow}$, $\lambda > 0$, then $\lambda f \in S^{\downarrow}$.
(3) If $(f, g) \in S^{\downarrow} \times S^{\downarrow}$, then $f + g \in S^{\downarrow}$, $-\inf\{f, g\} \in S^{\downarrow}$, $-\sup\{f, g\} \in S^{\downarrow}$.
(4) If $(f_n)_{n\in\mathbb{N}} \in (S^{\downarrow})^{\mathbb{N}}$ is a decreasing sequence, then

$$\lim_{n\to+\infty} f_n \in S^{\downarrow}.$$

(5) If $I : S \longrightarrow \mathbb{R}$ is an integral operator. For any $f \in S^{\downarrow}$, let

$$I(f) := -I(f).$$

1. If $(f, g) \in S^{\downarrow} \times S^{\downarrow}$ or $(f, g) \in S^{\uparrow} \times S^{\uparrow}$,

$$f \leq g \Rightarrow I(f) \leq I(g), I(f + g) = I(f) + I(g),$$

$$I(\lambda f) = \lambda I(f), \forall \lambda \in \mathbb{R}\backslash\{0\}.$$

2. If $(f_n)_{n\in\mathbb{N}} \in (S^{\downarrow})^{\mathbb{N}}$ is a decreasing sequence, then

$$I(\lim_{n\to+\infty} f_n) = \lim_{n\to+\infty} I(f_n).$$

Proposition 2.3.13 Let Ω be a set, S be a Riesz space on Ω and $I : S \longrightarrow \Omega$ be an integral operator. For any $(f, g) \in (S^{\uparrow} \cup S^{\downarrow})^2$, if $f \leq g$, then $I(f) \leq I(g)$.

Proof It is suffices to treat the case where $(f, g) \in S^{\uparrow} \times S^{\downarrow}$ or $(f, g) \in S^{\downarrow} \times S^{\uparrow}$.

If $(f, g) \in S^{\uparrow} \times S^{\downarrow}$, then $(-f, g) \in S^{\downarrow} \times S^{\downarrow}$, so $g - f \in S^{\downarrow}$. $I(g - f) = I(g) - I(f) \geq 0$. So $I(f) \leq I(g)$.

If $(f, g) \in S^{\downarrow} \times S^{\uparrow}$, then $(-f, g) \in S^{\uparrow} \times S^{\uparrow}$, so $g - f \in S^{\uparrow}$. $I(g - f) = I(g) - I(f) \leq 0$. So $I(f) \leq I(g)$. □

Definition 2.3.14 Let Ω be a set, S be a Riesz space on Ω, and $I : S \longrightarrow \mathbb{R}$ be an integral operator. Let $f : \Omega \longrightarrow \mathbb{R}$ be a mapping. If

$$\sup_{\substack{l \in S \\ l \leq f}} I(l) = \inf_{\substack{\mu \in S \\ \mu \geq f}} I(\mu).$$

We say that f is **Riemann integrable**.

Let

$$\underline{I}(f) := \sup_{\substack{l \in S^{\downarrow} \\ l \leq f}} I(l),$$

$$\overline{I}(f) := \inf_{\substack{\mu \in S^{\uparrow} \\ \mu \geq f}} I(\mu),$$

then,

$$\underline{I}(f) \leq I(f) \leq \overline{I}(f).$$

If $\underline{I}(f) = \overline{I}(f) \in \mathbb{R}$, we say that f is **Daniell integrable**, and we denote by $I(f)$ the real number $\underline{I}(f) = \overline{I}(f)$.

We denote by $\mathcal{L}^1(I)$ the set of all Daniell integrable mappings from Ω to $\mathbb{R}$. We got a mapping

$$I : \mathcal{L}^1(I) \longrightarrow \mathbb{R}.$$

Lemma 2.3.15 Let Ω be a set, S be a Riesz space on Ω, and $I : S \longrightarrow \mathbb{R}$ be an integral operator.

(1) For any mapping $f : \Omega \longrightarrow \mathbb{R}$,

$$\underline{I}(-f) = -\overline{I}(f), \ \overline{I}(-f) = -\underline{I}(f).$$

In particular,

$$f \in \mathcal{L}^1(I) \Leftrightarrow -f \in \mathcal{L}^1(I).$$

And in this case,

$$-I(f) = I(-f).$$

(2) For any $(f,g) \in \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega}$,

$$\underline{I}(f+g) \geq \underline{I}(f) + \underline{I}(g),\ \overline{I}(f+g) \leq \overline{I}(f) + \overline{I}(g).$$

In particular, if $(f,g) \in \mathcal{L}^1(I) \times \mathcal{L}^1(I)$, then $f+g \in \mathcal{L}^1(I)$, and $I(f+g) = I(f) + I(g)$.
(3) For any $f \in \mathbb{R}^{\Omega}$ and any $\lambda \in \mathbb{R}_{>0}$,

$$\underline{I}(\lambda f) = \lambda \underline{I}(f),\ \overline{I}(\lambda f) = \lambda \overline{I}(f).$$

In particular, if $f \in \mathcal{L}^1(I)$, then $\lambda f \in \mathcal{L}^1(I)$, and $I(\lambda f) = \lambda I(f)$.
(4) If $(f,g) \in \mathbb{R}^{\Omega} \times \mathbb{R}^{\Omega}$ such that $f \leq g$, then

$$\underline{I}(f) \leq \underline{I}(g),\ \overline{I}(f) \leq \overline{I}(g).$$

(5) If $(f : \Omega \longrightarrow \mathbb{R}) \in S^{\uparrow} \cup S^{\downarrow}$ such that $I(f) \in \mathbb{R}$, then $f \in \mathcal{L}^1(I)$.

Proof
(1) If $\mu \in S^{\uparrow}$, $\mu \geq f$, then $-\mu \in S^{\downarrow}$, $-\mu \leq -f$. So

$$-I(\mu) = I(-\mu) \leq \underline{I}(-f).$$

$$I(\mu) \geq -\underline{I}(-f).$$

Taking $\inf\limits_{\substack{\mu \in S^{\uparrow} \\ \mu \geq f}}$, we get

$$\overline{I}(f) \geq -\underline{I}(-f).$$

$\forall l \in S^{\downarrow}$, $l \leq f$, one has $-l \in S^{\uparrow}$, $-l \geq -f$. So

$$I(-l) \geq \overline{I}(-f),\ I(l) \leq -\overline{I}(-f).$$

Taking $\sup\limits_{\substack{l \in S^{\downarrow} \\ \leq f}}$, we get

$$\underline{I}(f) \leq -\overline{I}(-f).$$

Replacing f by $-f$, we get

$$\underline{I}(-f) \geq -\overline{I}(f),\ -\overline{I}(-f) \geq \underline{I}(f).$$

So $-\underline{I}(-f) = \overline{I}(f),\ -\overline{I}(-f) = \underline{I}(f)$.
(2) For any $(l_1, l_2) \in S^{\downarrow} \times S^{\downarrow}$, $l_1 \leq f$, $l_2 \leq g$. One has $l_1 + l_2 \leq f + g$, so

$$\sup_{\substack{(l_1,l_2)\in S^{\downarrow}\times S^{\downarrow}\\ l_1\leq f, l_2\leq g}} I(l_1 + l_2) \leq \underline{I}(f+g).$$

$$\overline{I}(f+g) = -\underline{I}(-f-g) \geq -(\underline{I}(-f) + \underline{I}(-g)) = \overline{I}(f) + \overline{I}(g).$$

If $\overline{I}(f) = \underline{I}(f)$, $\overline{I}(g) = \underline{I}(g)$, one has

$$\overline{I}(f) + \overline{I}(g) = \underline{I}(f) + \underline{I}(g) \leq \underline{I}(f+g) \leq \overline{I}(f+g).$$

$$\overline{I}(f+g) \leq \overline{I}(f) + \overline{I}(g) = I(f) + I(g).$$

(3)

$$\underline{I}(\lambda f) = \sup_{\substack{l\in S\\ l\leq\lambda f}} I(l) = \sup_{\substack{l\in S^{\downarrow}\\ l\leq f}} I(\lambda l) = \lambda\underline{I}(f).$$

$$\overline{I}(\lambda f) = -\underline{I}(\lambda(-f)) = -\lambda\underline{I}(-f) = \lambda\overline{I}(f).$$

(5) Let $f \in S^{\uparrow}$. By definition, $\overline{I}(f) = I(f)$. Moreover, there exists an increasing sequence $(f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}} \subseteq (S^{\uparrow})^{\mathbb{N}}$ such that

$$I(f) = \lim_{n\to\infty} I(f_n) \leq \underline{I}(f).$$

So,

$$\underline{I}(f) = I(f) = \overline{I}(f).$$

□

Theorem 2.3.16 (Beppo Levi) Let $(f_n)_{n\in\mathbb{N}}$ be a monotone sequence in $\mathcal{L}^1(I)$ such that converges point-wise to a mapping $f : \Omega \longrightarrow \mathbb{R}$. If $\lim\limits_{n\to+\infty} I(f_n) \in \mathbb{R}$, then

$$f \in \mathcal{L}^1(I),\ I(f) = \lim_{n\to+\infty} I(f_n).$$

Proof Suppose that $(f_n)_{n\in\mathbb{N}}$ is increasing. By replacing f_n by $f_n - f_0$ and f by $f - f_0$, we may assume $f_0 = 0$.
Let $\varepsilon > 0$. For any $n \in \mathbb{N}_{\geq 1}$, let $\mu_0 \in S^{\uparrow}$ such that $f_n - f_{n-1} \leq \mu_n$ and

$$I(f_n - f_{n-1}) \geq I(\mu_n) - \frac{\varepsilon}{2^n}.$$

$$f_n = \sum_{k=1}^{n} (f_k - f_{k-1}) \leq \mu_1 + \cdots + \mu_n,$$

and

$$I(f_n) = \sum_{k=1}^{n} I(f_k - f_{k-1}) \geq \sum_{k=1}^{n} \Big(I(\mu_k) - \frac{\varepsilon}{2^n} \Big) \geq I(\mu_1) + \cdots + I(\mu_n) - \varepsilon.$$

Let

$$\mu = \lim_{N \to +\infty} \sum_{k=1} \mu_k \in S^{\uparrow}.$$

One has $I(\mu) = \sum_{n \in \mathbb{N}} I(\mu_n)$, $\mu \geq \lim_{n \to +\infty} f_n = f$. Let $\alpha = \lim_{n \to +\infty} I(f_n)$, one has

$$\alpha \geq I(\mu) - \varepsilon \geq \overline{I}(f) - \varepsilon.$$

For any $n \in \mathbb{N}$, let $l_n \in S^{\downarrow}$ such that $l_n \leq f_n \leq f$ and $I(l_n) \geq I(f_n) - \varepsilon$. Then

$$\alpha - \varepsilon \liminf_{n \to +\infty} I(l_n) \leq \underline{I}(f).$$

Thus,

$$\alpha - \varepsilon \underline{I}(f) \leq \overline{I}(f) \leq \alpha + \varepsilon.$$

Since ε is arbitrary, we have

$$\underline{I}(f) = \overline{I}(f) = \alpha \lim_{n \to +\infty} I(f_n).$$

□

Theorem 2.3.17 (Daniell) $\mathcal{L}^1(I)$ forms a Riesz space on Ω, and $I : \mathcal{L}^(I) \longrightarrow \mathbb{R}$ is an integral operator extending $I : S \longrightarrow \mathbb{R}$.

Proof By the property of $\overline{I}$ and $\underline{I}$, $\mathcal{L}^1(I)$ is a vector subspace of $\mathbb{R}^{\Omega}$ and $I : \mathcal{L}^1(I) \longrightarrow \mathbb{R}$ is an $\mathbb{R}$-linear mapping. Moreover, if $f \leq g$, then $I(f) \leq I(g)$. Let $(f_1, f_2) \in \mathcal{L}^1(I)^2$, $\forall \varepsilon > 0$, $\exists (l_1, l_2) \in \big(S^{\downarrow}\big)^2$, $\exists (\mu_1, \mu_2) \in \big(S^{\uparrow}\big)^2$,

$$l_i \leq f_i \leq \mu_i,\ i \in \{1, 2\} \text{ and } I(\mu_i - l_i) \leq \frac{\varepsilon}{2}.$$

Then,

$$\inf\{l_1, l_2\} \leq \inf\{f_1, f_2\} \leq \inf\{\mu_1, \mu_2\},$$

and

$$\inf\{\mu_1, \mu_2\} - \inf\{l_1, l_2\} \leq (\mu_1 - l_1) + (\mu_2 - l_2).$$

Suppose that $\mu_1(\omega) \leq \mu_2(\omega)$, $l_2(\omega) \leq l_1(\omega)$. LFS= $\mu_1(\omega) - l_2(\omega)$, $\mu_2(\omega) \geq \mu_1(\omega) \geq l_1(\omega)$,

$$I\left(\inf\{\mu_1, \mu_2\} - \inf\{l_1, l_2\}\right) \leq I(\mu_1 - l_1) + I(\mu_2 - l_2) \leq \varepsilon.$$

By Beppo Levi's theorem, if $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathcal{L}^1(I)$ that converges to some $f \in \mathcal{L}^1(I)$. One has $I(f) = \lim_{n\to+\infty} I(f_n)$. □

Remark 2.3.18 If $f \in \mathcal{L}^1(I)$, then $|f| \in \mathcal{L}^1(I)$.

Theorem 2.3.19 (Fatou's lemma) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1(I)$. Assume that there exists $g \in \mathcal{L}^1(I)$ such that $\forall n \in \mathbb{N}$, $f_n \geq g$. Then

$$\liminf_{n\to+\infty} f_n \in \mathcal{L}^1(I),$$

and

$$I\left(\liminf_{n\to+\infty} f_n\right) \leq \liminf_{n\to+\infty} I(f_n).$$

Moreover, when $\liminf_{n\to+\infty} I(f_n) < +\infty$ and $\liminf_{n\to+\infty} f_n$ takes finite values, then

$$\liminf_{n\to+\infty} f_n \in \mathcal{L}^1(I).$$

Proof For any $n \in \mathbb{N}$, let g_n be

$$\inf_{k\in\mathbb{N}} f_{n+k} = \lim_{k\to+\infty} \inf\{f_n, f_{n+1}, \cdots, f_{n+k}\} \geq g.$$

$$I(f_n) \geq \lim_{k\to+\infty} I\left(\inf\{f_n, \cdot, f_{n+k}\}\right) \geq I(g).$$

By Beppo Levi's theorem, $g_n \in \mathcal{L}^1(I)$, and $I(g_n) \leq I(f_n)$. The sequence $(g_n)_{n\in\mathbb{N}}$ is increasing and converges point-wise to $\liminf_{n\to+\infty} f_n$. So $\liminf_{n\to+\infty} f_n \in \mathcal{L}^1(I)^{\uparrow}$, and

$$I\left(\liminf_{n\to+\infty} f_n\right) = \lim_{n\to+\infty} I(g_n) \leq \liminf_{n\to+\infty} I(f_n).$$

If $\liminf_{n\to+\infty} I(f_n) < +\infty$, then $I\left(\liminf_{n\to+\infty} f_n\right) < +\infty$. By Beppo Levi's theorem,

$$\liminf_{n\to+\infty} f_n \in \mathcal{L}^1(I).$$

□

Theorem 2.3.20 (Dominated convergence theorem, Lebesgue) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1(I)$ that convergence pointwise to a mapping $f : \Omega \longrightarrow \mathbb{R}$. Assume that there exists $g \in \mathcal{L}^1(I)$ such that

$$\forall n \in \mathbb{N}, |f_n| \leq g,$$

then,

$$f \in \mathcal{L}^1(I) \text{ and } I(f) = \lim_{n\to+\infty} I(f_n).$$

Proof

$$f_n \geq g_n, \ -f_n \geq -g_n, \forall n \in \mathbb{N}.$$

By Fatou's lemma,

$$I(\lim_{n\to+\infty} f_n) \leq \lim_{n\to+\infty} I(f_n),$$

$$I(\lim_{n\to+\infty}(-f_n)) \leq \liminf_{n\to+\infty} I(-f_n) = -\limsup_{n\to+\infty} I(f_n),$$

$$-I(g) \leq \limsup_{n\to+\infty} I(f_n) \leq I(\lim_{n\to+\infty} f_n) \leq \liminf_{n\to+\infty} I(f_n) \leq I(g).$$

So $(I(f_n))_{n\in\mathbb{N}}$ converges to $I\left(\lim_{n\to+\infty} f_n\right) \in \mathbb{R}$. Hence,

$$\lim_{n\to+\infty} f_n \in \mathcal{L}^1(I).$$

□

2.4 Convexity*

Definition 2.4.1 Let E be a vector space over $\mathbb{R}$, $U \subseteq E$ convex. We say that the mapping $f : U \longrightarrow \mathbb{R}$ is **convex** if the **epigraph**

$$\Gamma_+(f) := \{(x,a) \in U \times \mathbb{R} \mid f(x) \leq a\}$$

is convex in $E \times \mathbb{R}$.
We say that $f : U \longrightarrow \mathbb{R}$ is **concave** if its **hypergraph**

$$\Gamma_-(f) := \{(x,a) \in U \times \mathbb{R} \mid f(x) \geq a\}$$

is convex in $E \times \mathbb{R}$.

Proposition 2.4.2 Let E be a vector space over $\mathbb{R}$, $U \subseteq E$ convex, and $f : U \longrightarrow \mathbb{R}$ a mapping. Then the following conditions are equivalent:
(1) f is convex.
(2) For any $(x, y) \in U \times U$, and $t \in [0, 1]$,

$$f(tx + y(1-t)) \leq tf(x) + y(1-t)f(y).$$

Proof
(1)⇒(2): Note that $((x, f(x)), (y, f(y))) \in \Gamma_+^2(f)$, $(x, y) \in U^2$.

$$t(x, f(x)) + (1-t)(y, f(y)) = (tx + y(1-t), tf(x) + (1-t)f(y)) \in \Gamma_+(f).$$

Hence,

$$f(tx + y(1-t)) \leq tf(x) + (1-t)f(y).$$

(2)⇒(1): Let $((x, a), (y, b)) \in \Gamma_+^2(f)$, then $a \geq f(x)$, $b \geq f(y)$. Let $t \in [0, 1]$, then

$$ta + (1-t)b \geq tf(x) + (1-t)f(y) \geq f(tx + (1-t)y).$$

Hence,

$$(tx + (1-t)y, ta + (1-t)b) \in \Gamma_+(f).$$

□

Proposition 2.4.3 Let E be a vector space over $\mathbb{R}$, $U \subseteq E$ convex, and $f : U \longrightarrow \mathbb{R}$ a mapping. Then the following conditions are equivalent:
(1) f is concave.
(2) For any $(x, y) \in U \times U$, and $t \in [0, 1]$,

$$f(tx + y(1-t)) \geq tf(x) + y(1-t)f(y).$$

Proof
(1)⇒(2): Note that $((x, f(x)), (y, f(y))) \in \Gamma_-^2(f)$, $(x, y) \in U^2$.

$$t(x, f(x)) + (1-t)(y, f(y)) = (tx + y(1-t), tf(x) + (1-t)f(y)) \in \Gamma_-(f).$$

Hence,

$$f(tx + y(1-t)) \geq tf(x) + (1-t)f(y).$$

(2)⇒(1): Let $((x, a), (y, b)) \in \Gamma_-^2(f)$, then $a \leq f(x)$, $b \leq f(y)$. Let $t \in [0, 1]$, then

$$ta + (1-t)b \leq tf(x) + (1-t)f(y) \leq f(tx + (1-t)y).$$

Hence,
$$(tx + (1-t)y, ta + (1-t)b) \in \Gamma_-(f).$$
□

Proposition 2.4.4 Let E be a vector space over $\mathbb{R}$, $U \subseteq E$ convex, and $f : U \longrightarrow \mathbb{R}$ a mapping. $(f_i)_{i\in I}$ is a family of linear forms on U. ($f_i : E \longrightarrow \mathbb{R}$ linear.) $(c_i)_{i\in I}$ is a family of real numbers. If

$$\forall p \in U, f(p) = \sup_{i\in I}(f_i(p) + c_i),$$

then, f is convex.

Proof Let $(x, y) \in U^2$, $t \in [0, 1]$, then for any $i \in I$,

$$f_i(tx+(1-t)y)+c_i = t(f_i(x)+c_i)+(1-t)(f_i(y)+c_i) \leq tf(x)+(1-t)f(y).$$

Taking the supremum with respect to i, we obtain

$$f(tx + y(1-t)) \leq tf(x) + (1-t)f(y).$$

□

Proposition 2.4.5 Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$, $U \subseteq E$ be a convex open subset, $f : U \longrightarrow \mathbb{R}$ be a differentiable mapping. Then f is convex if and only if

$$\forall (p, x) \in U^2,\ f(x) \geq f(p) + \mathrm{D}f(p)(x-p).$$

Moreover, when f is convex, then

$$\forall x \in U,\ f(x) = \sup_{p\in U}\left(f(p) + \mathrm{D}f(p)(x-p)\right).$$

Proof For any $p \in U$, we define
$$\begin{array}{rccl} g_p: & U & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(p) + \mathrm{D}f(p)(x-p). \end{array}$$
We have that $f(p) = g_p(p)$.
$$\forall (p, x) \in U^2,\ f(x) \geq g_p(x) \Rightarrow f = \sup_{p\in U} g_p.$$

By proposition 2.4.4, f is convex.

Conversely, assume that f is convex, $(p, x) \in U^2$, $t \in [0, 1]$,

$$f(tx+(1-t)p) = f(p+t(x-p)) \leq tf(x)+(1-t)f(p) = f(p)+t(f(x)-f(p)).$$

f is differentiable at p,

$$f(p + t(x - p)) = f(p) + t\mathrm{D}f(p)(x - p) + o(|t|).$$

Taking the limit when $t \to 0$, we get

$$f(x) - f(p) \geq \mathrm{D}f(p)(x - p).$$

□

Definition 2.4.6 Let E be a vector space over $\mathbb{R}$. **Bilinear form** on E is a bilinear mapping from $E \times E$ to $\mathbb{R}$. Let $\varphi : E \times E \longrightarrow \mathbb{R}$ be a symmetric bilinear form.
If

$$\forall x \in E,\ \varphi(x, x) \geq 0,$$

we say that φ is **semipositive**.
If

$$\forall x \in E \backslash \{0\},\ \varphi(x, x) > 0,$$

we say that φ is **positive define**.

Example 2.4.7 Let $(x_1, \cdots, x_n)$ and $(y_1, \cdot, y_n)$ be elements of $\mathbb{R}^n$,

$$((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \longmapsto \sum_{i=1}^{n} x_i y_i$$

is a linear bilinear positive define form on $\mathbb{R}^n$.

Definition 2.4.8 Let E be a vector space over $\mathbb{R}$, $\varphi : E \times E \longrightarrow \mathbb{R}$ be a symmetric bilinear form.

$$\ker(\varphi) := \{x \in E \mid \forall y \in E,\ \varphi(x, y) = 0\}$$

is the intersection of $\ker(\varphi(\cdot, y))$ over all $y \in E$.

The **isotropic cone** of φ is the set of $x \in E$ such that $\varphi(x, x) = 0$. $\ker(\varphi)$ is contained in the isotropic cone of φ.

Proposition 2.4.9 Let E be a vector space over $\mathbb{R}$, $\varphi : E \times E \longrightarrow \mathbb{R}$ be a symmetric bilinear form. If φ is semipositive, then $\ker(\varphi)$ is equal to the isotropic cone of φ.

Proof It is suffices to show that any element y of the isotropic cone of φ is in $\ker(\varphi)$.

Let $x \in E$, $t \in \mathbb{R}$,

$$\varphi(x + ty, x + ty) = \varphi(x, x) + 2t\varphi(x, y) + t^2\varphi(y, y) \geq 0.$$

Since $\varphi(y, y) = 0$, we obtain

$$\forall t \in \mathbb{R},\ \varphi(x, x) + 2t\varphi(x, y) \geq 0,$$

$$\forall -t \in \mathbb{R},\ \varphi(x, x) - 2t\varphi(x, y) \geq 0.$$

Thus, for any $t \in \mathbb{R}$,

$$(\varphi(x, x) + 2t\varphi(x, y))\,(\varphi(x, x) - 2t\varphi(x, y)) = \varphi(x, x)^2 - 4t^2\varphi(x, y)^2 \geq 0.$$

Take the limit $|t| \to +\infty$, we obtain, $\varphi(x, y) = 0$. $\square$

Theorem 2.4.10 (Cauchy-Schwartz) Let E be a vector space over $\mathbb{R}$, $\varphi : E \times E \longrightarrow \mathbb{R}$ be a semipositive, bilinear form. For any $(x, y) \in E \times E$,

$$\varphi(x, y)^2 \leq \varphi(x, x)\varphi(y, y).$$

The equality holds if and only if $\varphi(y - x, h) = 0$ for any $h \in E$.

Proof First, we show that if $[x] = h[y]$ in $E/\ker(\varphi)$ then $\varphi(x, y)^2 = \varphi(x, x)\varphi(y, y)$.

We have

$$\{x - ah, y - bh\} \subseteq \ker\varphi.$$

$$\varphi(x, y) = \varphi((x - ah) + ah, (y - bh) + bh) = \varphi(ah, bh) = ab\varphi(h, h).$$

$$\varphi(x, x) = a^2\varphi(h, h),\ \varphi(y, y) = b^2\varphi(h, h).$$

Hence,

$$\varphi(x, y)^2\varphi(x, x)\varphi(y, y).$$

We know if $\varphi(y, y) = 0$, then $y \in \ker\varphi$. In this case, $[y] = 0$. So $[x], [y]$ are colinear in $E/\ker\varphi$.

Assume that $\varphi(y,y) \neq 0$, $t \in \mathbb{R}$,

$$\varphi(x+ty, x+ty) = t^2\varphi(y,y) + \varphi(x,x) + 2t\varphi(x,y) \geq 0.$$

Take $t = -\frac{\varphi(x,y)}{\varphi(y,y)}$, we obtain

$$\varphi(x,y)^2 \leq \varphi(x,x)\varphi(y,y).$$

If the equality holds, then $\varphi(x+ty, x+ty) = 0$, for $t = -\frac{\varphi(x,y)}{\varphi(y,y)}$ and hence $x + ty \in \ker\varphi$. $\square$

Theorem 2.4.11 Let $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$, $U \subseteq E$ be an open convex subset, $f : U \longrightarrow \mathbb{R}$ be a second-order differentiable mapping. If $\mathrm{D}^2 f(p)$ is semipositive for any p, then f is convex.

Proof Let $(p,x) \in U^2$, we define

$$\begin{array}{rccl} g: & [0,1] & \longrightarrow & \mathbb{R} \\ & t & \longmapsto & f(tx + (1-t)p). \end{array}$$

Then,

$$g'(t) = \mathrm{D}f(p + t(x-p))(x-p), \; g''(t) = \mathrm{D}^2 f(p + t(x-p))(x-p, x-p) \geq 0.$$

By Taylor-Lagrange, there exists $\xi \in [0,1]$,

$$g(1) - g(0) = g'(0) + \xi g''(\xi) \leq g'(0) = \mathrm{D}f(p)(x-p).$$

So $f(x) - f(p) \geq \mathrm{D}f(p)(x-p)$. So f is convex. $\square$

2.5 Semirings

Definition 2.5.1 Let Ω be a set. We call semiring on Ω any $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ that satisfies
(1) $\varnothing \in \mathcal{C}$.
(2) $\forall (A,B) \in \mathcal{C} \times \mathcal{C}$, $A \cap B \in \mathcal{C}$.
(3) $\forall (A,B) \in \mathcal{C} \times \mathcal{C}$, there exists a family $C_1, \cdots, C_n$ of pairwise disjoint sets in

$\mathcal{C}$ such that
$$B\backslash A = \bigcup_{i=1}^{n} C_i.$$

Example 2.5.2 $\Omega = \mathbb{R}$, $\mathcal{C} = \{]a,b] \mid (a,b) \in \mathbb{R}^2, a \leq b\}$.
(1) $\varnothing =]0,0] \in \mathcal{C}$.
(2) $]a,b] \cap]c,d] \neq \varnothing \Leftrightarrow c < b$ $(a \leq c)$. When $c < b$, $]a,b] \cap]c,d] =]c,b]$.
(3) $B\backslash A = B\backslash (A \cap B)$. We may assume $A \subseteq B$. If $A =]a,b]$, $B =]c,d]$, $A \subseteq B$ implies $c \leq a$, $b \leq d$.

Proposition 2.5.3 Let Ω be a set and $\mathcal{C}$ be a semiring on Ω.
(1) Let $B \in \mathcal{C}$. Let $A_1, \cdots, A_n$ be sets in $\mathcal{C}$. Then $B\backslash (A_1 \cup \cdots \cup A_n)$ can be written as the union of a finite family of pairwise disjoint sets in $\mathcal{C}$.
(2) Let Θ be a finite subset of $\mathcal{C}$. There exists a finite family Φ of pairwise disjoint sets in $\mathcal{C}$ such that each element of Θ ca be written as the union of some elements of Φ.
(3) Let $\mathcal{A}$ be the set
$$\{A \in \mathcal{P}(\Omega) \mid \exists n \in \mathbb{N}, \exists (A_1, \cdots, A_n) \in \mathcal{C}^n, A = A_1 \cup \cdots \cup A_n\}.$$
Then any $A \in \mathcal{A}$ can be written as the union of a finite family of pairwise disjoint sets in $\mathcal{C}$. In particular, $\forall (A, A') \in \mathcal{A}^2$, $\{A \cup A', A \cap A', A\backslash A'\} \subseteq \mathcal{A}$.

Proof
(1) We reason by induction on n. The case where $n = 0$ is trivial. Suppose that $B\backslash(A_1, \cup \cdots, \cup A_{n-1}) = \cup_{i=1}^{n} C_i$, where $C_1, \cdots, C_m$ are pairwise disjoint sets in $\mathcal{C}$. Then
$$B\backslash(A_1, \cup \cdots, \cup A_n) = \bigcup_{i=1}^{n} C_i \backslash A_n.$$
Each $C_i\backslash A_i$ is of the form $\bigcup_{j=1}^{d_i} D_{ij}$ with $D_{i,1}, \cdots, D_{i,d_i}$ in $\mathcal{C}$, pairwise disjoint. So
$$B\backslash(A_1, \cup \cdots, \cup A_n) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{d_i} D_{ij}.$$

(2) Suppose that $\Theta = \{B_1, \cdots, B_n\}$. For any $i \in \{1, \cdot, n\}$, one has
$$B_i = \bigcup_{i \in J \subseteq \{1, \cdots, n\}} \left(\bigcap_{j \in J} B_j \right) \backslash \left(\bigcup_{k \in \{1, \cdots, n\} \backslash J} B_k \right).$$

For any $J \subseteq \{1, \cdots, n\}$, $J \neq \varnothing$, we let

$$B_j := \left(\bigcap_{j \in J} B_j \right) \setminus \left(\bigcup_{k \in \{1, \cdots, n\} \setminus J} B_k \right) = \left(\bigcap_{j \in J} B_j \right) \bigcap \left(\bigcap_{k \in \{1, \cdots, n\} \setminus J} \complement_\Omega B_k \right).$$

$(B_J)_{\substack{J \subseteq \{1, \cdots, n\} \\ J \neq \varnothing}}$ are pairwise disjoint. By (1), each B_J is the union of fa finite family of pairwise disjoint elements $C_{J,1}, \cdots, C_{J,d_J}$ in $\mathcal{C}$. Let

$$\Phi = \{C_{J,l} \mid l \in \{1, \cdots, d_J\}, J \subseteq \{1, \cdots, n\}, J \neq \varnothing\}.$$

(3) By (2), there exists a finite subset Φ of pairwise disjoint elements of $\mathcal{C}$ such that each A_i is the union of some sets in Φ. Then

$$A = \bigcup_{\substack{C \in \Phi \\ C \subseteq A}} C.$$

□

Proposition 2.5.4 Let Ω be a set and $\mathcal{C}$ be a semiring on Ω. Let S be the vector subspace of $\mathbb{R}^\Omega$ generated by mappings of the form $\mathbb{1}_A$, $A \in \mathcal{C}$.
(1) Any pair $(f, g) \in S^2$ can be written as

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{C_i}, \; g = \sum_{i=1}^{n} b_i \mathbb{1}_{C_i},$$

where $n \in \mathbb{N}$, $(a_1, \cdots, a_n) \in \mathbb{R}^{\mathbb{N}}$, $(b_1, \cdots, b_n) \in \mathbb{R}^{\mathbb{N}}$, $(C_1, \cdots, C_n) \in \mathcal{C}^n$, pairwise disjoint.
(2) S is a Riesz space.

Proof
(1) By definition, f and g are of the form

$$f = \sum_{A \in \Theta_f} \lambda_A \mathbb{1}_A, \; g = \sum_{B \in \Theta_g} \mu_B \mathbb{1}_B,$$

where Θ_f and Θ_g are finite subsets of $\mathcal{C}$, λ_A and μ_B are real numbers. Let $\Theta = \Theta_f \cup \Theta_g$. There is a subset $\Phi \subseteq \mathcal{C}$ consisting of pairwise disjoint sets, such that element of Θ can be written as the union of some sets in Φ.

Suppose that $\Phi = \{C_1, \cdots, C_n\}$. Then

$$f = \sum_{i=1}^{n} \left(\sum_{\substack{A \in \Theta_f \\ A \cap C_i \neq \varnothing}} \lambda_A \right) \mathbb{1}_{C_i},\ g = \sum_{i=1}^{n} \left(\sum_{\substack{B \in \Theta_g \\ B \cap C_i \neq \varnothing}} \mu_B \right) \mathbb{1}_{C_i}.$$

(2) If

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{C_i},\ g = \sum_{i=1}^{n} b_i \mathbb{1}_{C_i},$$

with $C_1, \cdots, C_n$ in $\mathcal{C}$ pairwise disjoint, then

$$\inf\{f, g\} = \sum_{i=1}^{n} \min\{a_i, b_i\} \mathbb{1}_{C_i} \in S.$$

□

2.6 σ-additive Functions

Definition 2.6.1 Let Ω be a set, $\mathcal{C} \subseteq \mathcal{P}(\Omega)$, $\mu : \mathcal{C} \longrightarrow \mathbb{R}_{\geq 0}$ be a mapping. We say that μ is **additive** if for any finite family $(A_n)_{i=1}^{n}$ of pairwise disjoint sets in $\mathcal{C}$ such that $A_1 \cap \cdots \cap A_n \in \mathcal{C}$. One has

$$\mu(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \mu(A_i).$$

Remark 2.6.2 If $\varnothing \in \mathcal{C}$, then $\varnothing = \varnothing \cup \varnothing$. So $\mu(\varnothing) = 2\mu(\varnothing)$, that means $\mu(\varnothing) = 0$.

Example 2.6.3 Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ right continuous and increasing.

$$\mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}.$$

We define

$$\mu_\varphi : \mathcal{C} \longrightarrow \mathbb{R}_{\geq 0},\ \mu_\varphi(]a, b]) = \varphi(b) - \varphi(a).$$

If $a_0 \leq \cdots \leq a_n$ are real numbers.

$$\mu_\varphi\left(]a_0, a_n]\right) = \varphi(a_n) - \varphi(a_0) = \sum_{i=1}^{n} \varphi(a_i) - \varphi(a_{i-1}) = \sum_{i=1}^{n} \mu_\varphi\left(]a_{i-1}, a_i]\right).$$

Therefore, μ_φ is additive.

Proposition 2.6.4 Let Ω be a set, $\mathcal{C}$ be a semiring on Ω, $\mu : \mathcal{C} \longrightarrow \mathbb{R}_{\geq 0}$ be a additive mapping, S be the vector subspace of $\mathbb{R}^{\Omega}$ generated by $\mathbb{1}_A$, where $A \in \mathcal{C}$.
(1) There exists a unique $\mathbb{R}$-linear mapping $I : S \longrightarrow \mathbb{R}$ such that $I(\mathbb{1}_A) = \mu(A)$ for any $A \in \mathcal{C}$.
(2) Let

$$\mathcal{A} = \{A \in \mathcal{P}(\Omega) \mid \exists n \in \mathbb{N}, \exists (A_1, \cdots, A_n) \in \mathcal{C}^n, A = A_1 \cup \cdots \cup A_n\}.$$

Then μ extends in a unique way to an additive mapping from $\mathcal{A}$ to $\mathbb{R}_{\geq 0}$.

Proof
(1) If $I : S \longrightarrow \mathbb{R}$ exists, then it is unique since S is generated by $\mathbb{1}_A$, $A \in \mathcal{C}$. ($\forall f \in S$, f is of form $\sum_{i=1}^n a_i \mathbb{1}_{A_i}$, $A_i \in \mathcal{C}$, $a_i \in \mathbb{R}$. $I(f)$ should be $\sum_{i=1}^n a_i \mu(A_i)$.) It remains to check that such I is well defined. Suppose that $f \in S$ can be written as

$$f = \sum_{A \in \Theta} \lambda_A \mathbb{1}_A = \sum_{B \in \Theta'} \lambda'_B \mathbb{1}_{B'}.$$

We aim to check that

$$\sum_{A \in \Theta} \lambda_A \mu(A) = \sum_{B \in \Theta'} \lambda'_B \mu(B).$$

Take $C_1, \cdots, C_n \in \mathcal{C}$, pairwise disjoint, such that exch $A \in \Theta \cup \Theta'$ can be written as the union of some sets among $\{C_1, \cdots, C_n\}$.

$$f = \sum_{i=1}^n \left(\sum_{\substack{A \in \Theta \\ C_i \cap A \neq \varnothing}} \lambda_A \right) \mathbb{1}_{C_i} = \sum_{i=1}^n \left(\sum_{\substack{B \in \Theta' \\ C_i \cap B \neq \varnothing}} \lambda'_B \right) \mathbb{1}_{C_i}.$$

$$\forall i \in \{1, \cdots, n\}, \quad \sum_{\substack{A \in \Theta \\ C_i \cap A \neq \varnothing}} \lambda_A = \sum_{\substack{B \in \Theta' \\ C_i \cap B \neq \varnothing}} \lambda'_B.$$

$$\begin{aligned}\sum_{A\in\Theta}\lambda_A\mu(A) &= \sum_{A\in\Theta}\lambda_A\sum_{\substack{i\in\{1,\cdots,n\}\\ A\cap C_i\neq\varnothing}}\mu(C_i)\\ &= \sum_{i=1}^n\mu(C_i)\sum_{\substack{A\in\Theta\\ A\cap C_i\neq\varnothing}}\lambda_A\\ &= \sum_{i=1}^n\mu(C_i)\sum_{\substack{B\in\Theta'\\ B\cap C_i\neq\varnothing}}\lambda'_B\\ &= \sum_{B\in\Theta'}\lambda'_B\sum_{\substack{i\in\{1,\cdots,n\}\\ B\cap C_i\neq\varnothing}}\mu(C_i)\\ &= \sum_{B\in\Theta'}\lambda'_B\mu(B).\end{aligned}$$

(2) We take, for any $A\in\mathcal{A}$, $\mu(A)$ as $I(\mathbb{1}_A)$. If A is write as a disjoint union $B_1\cup\cdots\cup B_m$, with $B_i\in\mathcal{A}$.

$$I(\mathbb{1}_A)=I(\sum_{j=1}^m\mathbb{1}_{B_j})=\sum_{j=1}^m I(\mathbb{1}_{B_j}).$$

□

Definition 2.6.5 Let Ω be a set and $\mathcal{C}\subseteq\mathcal{P}(\Omega)$. We say that a mapping $\mu:\mathcal{C}\longrightarrow\mathbb{R}_{\geq 0}$ is σ-additive if, for any countable set Θ and any family $(C_i)_{i\in\Theta}$ of pairwise disjoint sets in $\mathcal{C}$, one has

$$\bigcup_{i\in\Theta}C_i\in\mathcal{C}\Rightarrow\mu\left(\bigcup_{i\in\Theta}C_i\right)=\sum_{i\in\Theta}\mu(C_i).$$

$$\left(\sum_{i\in\Theta}\mu(C_i)\right):=\sup_{\substack{\Theta'\subseteq\Theta\\ \Theta'\text{ finite}}}\sum_{i\in\Theta'}\mu(C_i).$$