

QM HW1

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Problem 1 (Canonical Transformation)

(1)

$$d\Phi(q, P, t) = pdq + QdP + (H' - H)dt \quad (1.1)$$

$$\frac{\partial \Phi}{\partial q} = p, \frac{\partial \Phi}{\partial P} = Q, \frac{\partial \Phi}{\partial t} + H = H'. \quad (1.2)$$

(2) We have proved:

$$\frac{\partial S}{\partial q_t} = -p_t, \frac{\partial S}{\partial q_{t+\tau}} = p_{t+\tau}, \quad (1.3)$$

First, we can check:

$$\frac{\partial \Psi}{\partial q_t} = -\frac{\partial S}{\partial q_t} = p_t \quad (1.4)$$

Second, by chain rule, we obtain

$$\frac{\partial \Psi}{\partial p_{t+\tau}} = q_{t+\tau} + \left(p_{t+\tau} - \frac{\partial S}{\partial p_{t+\tau}} \right) \frac{\partial q_{t+\tau}}{\partial p_{t+\tau}} = q_{t+\tau} \quad (1.5)$$

(3)

$$dp_t \wedge dq_t = dp_{t+\tau} \wedge dq_{t+\tau}, \frac{\partial(q_{t+\tau}, p_{t+\tau})}{\partial(q_t, p_t)} = 1 \quad (1.6)$$

Problem 2 (Hamilton-Jacobi equation)

(1)

$$\frac{\partial S}{\partial t} = \frac{dS}{dt} - \frac{\partial S}{\partial q} \dot{q} = L - p\dot{q} = -H \quad (2.1)$$

(2)

$$\beta = \frac{\partial S}{\partial \alpha} \quad (2.2)$$

$$H' = H + \frac{\partial S}{\partial t} = 0 \quad (2.3)$$

Problem 3 (Harmonic Oscillator)

(1) easy to check:

$$\frac{\partial S}{\partial x} = p, \frac{\partial S}{\partial t} = -E \quad (3.1)$$

(2)

$$\frac{\partial S}{\partial t_f} = -E \quad (3.2)$$

$$\frac{\partial S}{\partial x_f} = p = \pm m\omega \sqrt{A^2 - x^2} \quad (3.3)$$

(3)

$$S = \pm m\omega \int \sqrt{\frac{2E}{m\omega^2} - x^2} dx - Et + \text{const.} \quad (3.4)$$

$$\frac{\partial S}{\partial E} = \pm \frac{\arcsin(\sqrt{\frac{m\omega^2}{2E}} x)}{\omega} - t \quad (3.5)$$

New Hamitonian:

$$H' = H + \frac{\partial S}{\partial t} = 0 \quad (3.6)$$

By Hamilton equation

$$\dot{\beta} = \frac{\partial H'}{\partial E} = 0 \quad (3.7)$$

Therefore β is a constant. It means the initial phase of oscillator.**Problem 4** (Planck's derivation of black body radiation)(1) $\omega \rightarrow +\infty$:

$$\ln u = -\frac{\hbar\omega}{k_B T} + \text{const.} \quad (4.1)$$

$$\frac{\partial(\ln u)}{\partial(\frac{1}{T})} = -\frac{\hbar\omega}{k_B} \quad (4.2)$$

 $\omega \rightarrow 0$:

$$\frac{1}{u} \frac{\partial u}{\partial(\frac{1}{T})} = -T = -\frac{u}{k_B} \quad (4.3)$$

We assume that

$$\frac{\partial(\ln u)}{\partial(\frac{1}{T})} = -\frac{\hbar\omega + u}{k_B} \quad (4.4)$$

That leads to

$$\frac{u}{\hbar\omega + u} = C e^{-\frac{\hbar\omega}{k_B T}} \quad (4.5)$$

By the expression at low frequency, we can deduce that $C = 1$.

$$u = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} \quad (4.6)$$

Problem 5 (Heisenberg's magic)

Quantization condition:

$$\oint p dq = \int_0^T m \dot{x}^2 dt = nh \quad (5.1)$$

$$x_{n\leftarrow m}(t) = x_{n\leftarrow m} e^{i\omega_{n\leftarrow m} t} \quad (5.2)$$

$$\dot{x}_{n\leftarrow m}^2(t) = \sum_{k=-\infty}^{+\infty} x_{n\leftarrow k} x_{k\leftarrow m} \omega_{n\leftarrow k} \omega_{k\leftarrow m} e^{i\omega_{n\leftarrow m} t} \quad (5.3)$$

Let $m = n + l$, we obtain f-sum rule

$$\sum_{l=0}^{+\infty} \left(\omega_{n \rightarrow n+l} |x_{n+l,n}|^2 - \omega_{n-l \rightarrow n} |x_{n,n-l}|^2 \right) = \frac{\hbar}{2m} \quad (5.4)$$

In classic mechanics ,we have

$$\ddot{x} + \omega_0^2 x = 0 \quad (5.5)$$

Plug in $x_{n\pm l\leftarrow n}(t) = x_{n\pm l\leftarrow n} e^{i\omega_{n\pm l\leftarrow n} t}$

$$(\omega_0^2 - \omega_{n\pm l\leftarrow n}^2) x_{n\pm l\leftarrow n} = 0 \quad (5.6)$$

Only when $l = \pm 1, \omega_0^2 - \omega_{n\pm l\leftarrow n}^2 = 0, x_{n\pm l\leftarrow n} \neq 0$

$$\frac{\hbar}{2m} = \omega_0 (|x_{n+1,n}|^2 - |x_{n,n-1}|^2) \quad (5.7)$$

There must exists a lower bound ,or $|x_{n,n-1}|^2 \geq 0$ can not hold. Thus,

$$x_{n+1,n} = \sqrt{\frac{\hbar}{2m\omega_0}} (n+1) \quad (5.8)$$

Therefore,

$$(\dot{x})_{nn}^2 = (n+1/2) \frac{\hbar\omega_0}{m} \quad (5.9)$$