QM HW11

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Problem 1
We need to solve the equation:

$$\begin{cases} i\frac{\partial}{\partial t}G(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,t) \\ G(x,0) = -i\delta(x). \end{cases} \tag{1.1}$$

Let the Laplace transform of $G(x,t)$ with respect to t be

$$\widetilde{G}(x,s) = \mathcal{L}\{G(x,t)\} = \int_0^\infty e^{-st}\, G(x,t)\,\mathrm{d}t\,, \tag{1.2}$$

where s is a complex parameter with $\mathrm{Re}(s) > 0$.

$$\mathcal{L}\left\{ i\frac{\partial G}{\partial t} \right\} = i\big[s\widetilde{G}(x,s) - G(x,0)\big] = is\widetilde{G}(x,s) - \delta(x). \tag{1.3}$$

The right-hand side transforms as:

$$\mathcal{L}\left\{ -\frac{\hbar^2}{2m}\frac{\partial^2 G}{\partial x^2} \right\} = -\frac{\hbar^2}{2m}\frac{\partial^2 \widetilde{G}(x,s)}{\partial x^2}. \tag{1.4}$$

Thus the transformed equation is:

$$\frac{\partial^2 \widetilde{G}}{\partial x^2} - k^2\widetilde{G} = \frac{2m}{\hbar^2}\,\delta(x). \tag{1.5}$$

where $k = \sqrt{\frac{2mis}{\hbar^2}}$. We choose the branch such that $\mathrm{Re}(k) > 0$.

The solution decaying as $|x| \to \infty$ is:

$$\widetilde{G}(x,s) = -\frac{m}{\hbar^2 k}e^{-k|x|} = -\sqrt{\frac{m}{2i\hbar^2 s}}\, e^{-\sqrt{\frac{2mis}{\hbar^2}}\,|x|}. \tag{1.6}$$

We recognize the Laplace transform pair (from tables):

$$\mathcal{L}^{-1}\left\{ \frac{e^{-a\sqrt{s}}}{\sqrt{s}} \right\} = \frac{1}{\sqrt{\pi t}}\, e^{-a^2/(4t)}, \quad a > 0. \tag{1.7}$$

So,

$$\boxed{G(x,t) = -e^{-i\pi/4}\sqrt{\frac{m}{2\pi\hbar^2 t}}\ \exp\left(\frac{imx^2}{2\hbar^2 t}\right)}. \tag{1.8}$$