FAA Chapter 4. exercises

5. · Implicit condition: $x \geq -2$.

~~Suppose y is a solution to this inequality.~~

~~then $(y-1)^2 \leq (\sqrt{y+2})^2 = y+2 \rightsquigarrow y^2-2y+1 \leq y+2 \Rightarrow y^2-3y-1 \leq 0$.~~

We want ~~to~~ use ~~the~~ $\boxed{a \leq b \Rightarrow a^2 \leq b^2}$. but please notice ~~the~~ condition $\boxed{a, b \geq 0}$!

- If $x-1 \leq 0$, i.e $x<1$, ~~then the inequality~~ holds.
- If $x-1 \geq 0$ ~~then~~. $(x-1)^2 \leq (x+2) \rightsquigarrow x^2-2x+1-x-2 \leq 0 \rightsquigarrow x^2-3x-1 \leq 0$

$$\rightsquigarrow (x-\tfrac{3}{2})^2 \leq 1+\tfrac{9}{4} = \tfrac{13}{4}$$

$$\rightsquigarrow x \in \left[\tfrac{3}{2}-\tfrac{\sqrt{13}}{2}, \tfrac{3}{2}+\tfrac{\sqrt{13}}{2}\right]$$

the final answer:

$$[-2,+\infty[\ \cap \left( ]-\infty,1[\ \cup \left([1,+\infty[\ \cap \left[\tfrac{3}{2}-\tfrac{\sqrt{13}}{2}, \tfrac{3}{2}+\tfrac{\sqrt{13}}{2}\right] \right) \right) = \left[-2, \tfrac{3+\sqrt{13}}{2}\right]$$

9: ~~We have AM-GM inequality: $\frac{x+y}{2} \geq \sqrt{xy}$. $\forall x,y \geq 0$ ("=" holds iff~~ $x=y$)

~~$\rightsquigarrow \forall (a,b) \in \mathbb{R}^2_{>0}$,~~ $(\frac{a}{b}+\frac{b}{a})/2 \geq \sqrt{\frac{a}{b}\cdot\frac{b}{a}} = 1 \Rightarrow \frac{a}{b}+\frac{b}{a} \geq 2$

When ~~$\frac{a}{b} = \frac{b}{a}$, i.e. $a=b$, we have~~ $\frac{a}{b}+\frac{b}{a} = 2$.

~~$\rightsquigarrow \inf \left\{ \frac{a}{b}+\frac{b}{a} \mid (a,b) \in \mathbb{R}^2_{>0} \right\} = 2$.~~

8. upper bound of A: 3. upper bound of B: 3.
   lower bound of A: -3. lower bound of B: -3.

   $\sup A = \sqrt{2}$ $\sup B = 1$
   $\inf A = -\sqrt{2}$ $\inf B = 0$.

10. $(*) = \left(\sum_{k=1}^{n} x_k\right)\cdot\left(\sum_{k=1}^{n} x_k^{-1}\right) = \underbrace{\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)}_{\text{arithmetic } \cancel{\text{inequality}} \text{ average.}} \cdot \underbrace{\left(\frac{n}{\sum_{k=1}^{n} x_k^{-1}}\right)^{-1}}_{\text{harmonic average.}} \cdot n^2$

(2)

$\overset{\text{AM-HM}}{\geq} \left(\frac{n}{\sum_{k=1}^{n} x_k^{-1}}\right)\cdot\left(\frac{n}{\sum_{k=1}^{n} x_k^{-1}}\right)^{-1}\cdot n^2 = n^2.$

When $x_1 = x_2 = \cdots = x_n$, we have. $(*) = n^2$

$\rightsquigarrow \inf\{\ \_\_\ \} = n^2.$

12 (1). ~~take~~ $b \in X_{<a}$. ~~need to~~ verify: $\underbrace{\{x\in X \mid x<b\}}_{X_{<b}} \subseteq X_{<a} = \{x \in X \mid x<a\}$.

Pf: Since $b \in X_{<a}$, we ~~know that~~ $b<a$.

~~for~~ any $y \in X_{<b}$, we have $y<b<a \rightsquigarrow y \in X_{<a}$.

~~thus~~. $X_{<b} \subseteq X_{<a}$. #.

(2). Exists, Unique

- $I \neq X \Rightarrow X_I = \{x\in X \mid x \notin I\} \neq \emptyset$. let a be the least element of X_I.

Claim: $I = X_{<a}$.

- let $y \in X_{<a}$, i.e. $y<a$. If $y \notin I$, then $y \in X_I$. This contradicts to the minimality of a in X_I. $\rightarrow X_{<a} \subseteq I$.
- let $y \in I$. If $y \notin X_{<a}$, i.e. ~~y~~ $y \not< a$, ~~then~~ $a \leq y$, i.e. $a = y$ or $a<y$.
  - If $a=y$, then $y \in X_I \Rightarrow y \notin I$, contradiction.
  - If $a<y$, then $a \in X_{<y} \subseteq I$, ~~contradict to~~ $a \in X_I$.

  $\rightsquigarrow I \subseteq X_{<a}$.

~~this proves the~~ existance.

~~Suppose~~ $I = X_{<a} = X_{<b}$ for ~~certain~~. $a, b \in X$. ~~then the minimality.~~

~~then~~ If $a<b$, then. $a \in X_{<b} = X_{<a} \Rightarrow a<a$ X.

If $b<a$, ~~then~~ $b \in X_{<a} = X_{<b} \Rightarrow b<b$ X $\quad \Rightarrow a=b$.

~~this proves the~~ uniqueness.

(3). Let $a \in I$, then $a \in I_\lambda$ for some $\lambda \in \Lambda$

Since I_λ is a ~~initial~~ segment, $X_{<a} \subseteq I_\lambda \subseteq I$. #

$\Rightarrow$.

(4). Let $S := \{x \in X \mid \neg P(x)\}$.

If $S \neq \phi$, then let a be the least element of S.

$\forall\, b \in X_{<a}$, i.e. $b < a$: ~~the mini~~, $b \notin S \Rightarrow \neg(\neg P(b)) = P(b)$.

by (b), $P(a)$ holds this contradicts to $a \in S$. #

14 omitted

16. Step 1: Prove by ~~induction that~~. $x_n < \frac{1}{2} + \sqrt{5}$, $\forall n \in \mathbb{N}$.

Step 2: Use the fact: if $a \in [0, \frac{1}{2} + \sqrt{5})$, then $\sqrt{1+a} < a$

18. By induction (Show the case for $n = 0, 1$ by hand!)

20 Omitted.

22 (1) easy

(2) If $d \mid n$, then $\exists\, m \in \mathbb{N}$ s.t. $n = m \cdot d$. If $n \neq 0$, ~~then~~ $m \neq 0$ and $d \neq 0$ $\Rightarrow m \geq 1$ and $d \geq 1$

$\Rightarrow$ ~~$n \geq d$~~ $n = (m-1)d + d \geq d$.

(3) Claim: 1 is the least element. Pf: $\forall n \in \mathbb{N}$, $n = 1 \cdot n \Rightarrow 1 \mid n$ #

(4). Claim. 0 is the greatest element. Pf: ~~$0 = 1 \cdot 0 \Rightarrow$~~ $\forall n \in \mathbb{N}$, $0 = 0 \cdot n \Rightarrow n \mid 0$ #

(5). Easy to check that 0 is an upperbound.

Claim: $\sup_{(\mathbb{N}, \mid)} A = 0$. Pf: We show that 0 is the only upperbound of A, and ~~the result~~ follows. If there exists a positive upperbound m of A, then ~~since~~. $\forall d \in A$, $d \mid m$ by (2), we know that $m \geq d$. Since A is an infinite set, d can be arbitrarily large (otherwise A is bounded in the usual order, which must be finite). Thus m is arbitrarily large. This is impossible.

(6). (a) let $m := \prod_{d \in A} d$. Easy to check ~~that~~ $m \in M(A)$.

(b). If ~~there exists~~ $n \in M(A)$ s.t. $n_0 \nmid n$, then we can write

$n = d \cdot n_0 + r$, where ~~$d \in \mathbb{N}$~~ $d, r \in \mathbb{N}$, $0 < r < n_0$.

Claim $r \in M(A)$. Pf: take ~~d~~ $t \in A$. Since $n, n_0 \in M(A)$, $\exists\, m, m_0 \in \mathbb{N}$ s.t.

$m t = n$, $m_0 \cdot t = n_0$ $\Rightarrow$ $mt = d(m_0 \cdot t) + r$ $\Rightarrow$ $t(m - dm_0) = r$

Since ~~$t \geq 1$~~, $t > 0$, $r > 0$ $\Rightarrow (m - dm_0) \in \mathbb{N}_{\geq 1} \Rightarrow t \mid r$ #.

This is impossible, since we assumed ~~that~~ n_0 is minimal in $M(A)$.

Thus $n_0 \mid n$ #.

(c) $M(A)$ is the set of upperbounds of A w.r.t. $(\mathbb{N}, |)$. By (b), n_0 is the least element of $M(A)$ under $(\mathbb{N}, |)$. Thus $\sup_{(\mathbb{N}, |)} A = n_0$.

(7). (a) Easy

(b). Easy

(c). Easy.

(d). $d\mathbb{Z} \subseteq A\mathbb{Z}$ is trivial.

If $A\mathbb{Z} \nsubseteq d\mathbb{Z}$, then $\exists\, x = a_1 n_1 + \cdots + a_k n_k$. with $a_i \in A$, $n_i \in \mathbb{N}$ s.t. $d \nmid x$.

We can write $x = dm + r$, with $m, r \in \mathbb{N}$, $1 \leq r < d$.

$\Rightarrow r = x - dm = a_1 n_1 + \cdots + a_k n_k + (-d) \cdot m \in A\mathbb{Z}$.

This is impossible, since we assumed ~~that~~ d is minimal in $A\mathbb{Z}$.

(e). by (d), $d\mathbb{Z} = A\mathbb{Z}$; by (c) $A \subseteq A\mathbb{Z}$ $\Rightarrow d \mid a, \forall a \in A$ $\Rightarrow d$ is a lower bound of A under $(\mathbb{N}, |)$.

~~Take~~ d' be another lowerbound of A in $(\mathbb{N}, |)$, then $d' \mid a, \forall a \in A$.

$\Rightarrow d' \mid a_1 n_1 + \cdots + a_k n_k$, $\forall a_1 n_1 + \cdots + a_k n_k \in A\mathbb{Z} = d\mathbb{Z} \Rightarrow d' \mid d$.

This shows that d is the greatest lowerbound of A in $(\mathbb{N}, |)$, i.e. d is the infimum of A in $(\mathbb{N}, |)$.

(8). If A is empty, then it is easy to check $\gcd(A) = 0$, $\operatorname{lcm}(A) = 1$

Now we assume $A \neq \emptyset$. | If $A = \{0\}$, then it is easy to check $\gcd(A) = 0$, $\operatorname{lcd}(A) = 0$.

~~If A contains 0, then~~

If $A \neq \{0\}$, let $A' = \{a \in A \mid a \neq 0\}$. then $A' \neq \emptyset$.

By (7)/(e). A' has infimum d. Claim: d is also infimum of A $\rightarrow$ easy to check.

By (5) and ~~(c)~~ (6)/(c), A' has ~~sup~~ supremum D. ~~D~~. easy to check D is also the supremum of A.

5

(9) Let $A=\{a,b\}$. by (7)/(d), (e), $A\mathbb{Z}=d\mathbb{Z} \rightsquigarrow d\in A\mathbb{Z}$

$\rightsquigarrow \exists m,n\in\mathbb{Z}, \ d=an+bm$.

(10). $\frac{ab}{\gcd(a,b)} = a\cdot\left(\frac{b}{\gcd(a,b)}\right) = b\cdot\left(\frac{a}{\gcd(a,b)}\right)$ (*)

Since $\gcd(a,b)\mid a,b$, $\frac{b}{\gcd(a,b)}, \frac{a}{\gcd(a,b)} \in \mathbb{N}_{\geq 1}$

$\rightsquigarrow$ (*) shows ~~that~~ $\frac{ab}{\gcd(a,b)}$ is ~~divide~~ a upperbound of $A=\{a,b\}$ in $(\mathbb{N},|)$.

Since $\mathrm{lcm}(a,b)$ is ~~the~~ least upper bound, we ~~know that~~.

$\mathrm{lcm}(a,b) \mid \frac{ab}{\gcd(a,b)}$, i.e. $\gcd(a,b)\cdot\mathrm{lcm}(a,b) \mid ab$. (A)

On the other hand, $a=\left(\frac{a\cdot b}{\mathrm{lcm}(a,b)}\right)\cdot\frac{\mathrm{lcm}(a,b)}{b}$, $b=\left(\frac{b\cdot a}{\mathrm{lcm}(a,b)}\right)\cdot\frac{\mathrm{lcm}(a,b)}{a}$. (**)

Since $a,b\mid\mathrm{lcm}(a,b)$, we ~~know that~~ $\frac{\mathrm{lcm}(a,b)}{b}, \frac{\mathrm{lcm}(a,b)}{a} \in \mathbb{N}_{\geq 1}$.

Since ~~lcm~~ $a,b\mid a\cdot b$, we ~~know that~~ $a\cdot b$ is also an upperbound of $A=\{a,b\}$ in $(\mathbb{N},|)$.

$\rightsquigarrow \mathrm{lcm}(a,b)\mid a\cdot b$, i.e. $\frac{a\cdot b}{\mathrm{lcm}(a,b)}\in\mathbb{N}_{\geq 1}$.

$\rightsquigarrow$ (**) shows ~~that~~ $\frac{a\cdot b}{\mathrm{lcm}(a,b)}$ is a common divisor of a and b.
(a lower bound of $A=\{a,b\}$ under $(\mathbb{N},|)$).

$\rightsquigarrow \frac{a\cdot b}{\mathrm{lcm}(a,b)} \mid \gcd(a,b)$, i.e. $a\cdot b \mid \gcd(a,b)\cdot\mathrm{lcm}(a,b)$. (B).

(A)+(B) $\Rightarrow a\cdot b=\mathrm{lcm}(a,b)\cdot\gcd(a,b)$.

24:

(1) "$\Rightarrow$" If α is successor, then $\alpha = A\cup\{A\}$. for a ordinal A.

By 23/(4), $A = \bigcup_{x\in A\cup\{A\}} x = \bigcup_{x\in\alpha} x \Rightarrow \bigcup_{x\in\alpha} x \subseteq \alpha$

If $\bigcup_{x\in\alpha} x = \alpha$, then $A = \alpha = A\cup\{A\} \Rightarrow A\in A$. ~~this~~ is ruled out by axiom of regularity.

$\leadsto \bigcup_{x\in\alpha} x \subsetneq \alpha$.

"$\Leftarrow$" Let $U = \bigcup_{x\in\alpha} x$, claim $\alpha = U\cup\{U\}$.

- Take $y\in\alpha$, then $y\subseteq U$. $\begin{cases} \text{if } y=U, \text{ then } y\in\{U\} \subseteq U\cup\{U\} \\ \text{if } y\subsetneq U, \text{ then } y\in U \subseteq U\cup\{U\} \leadsto 23/(10). \end{cases} \Rightarrow \alpha\subseteq U\cup\{U\}$.

- ~~take $y\in U\cup\{U\}$~~ ~~Note U~~ Take $y\in U$, then $y\in x$ for some $x\in\alpha$. $\Rightarrow y\in x \subsetneq \alpha$.
$\Rightarrow U\subseteq\alpha$, ①

If $U=\alpha$, then $U = U\cup\{U\} \Rightarrow U\in U$, impossible. $\leadsto U\subsetneq\alpha$, i.e. $U\in\alpha$. ②

①+② $\Rightarrow U\cup\{U\}\subseteq\alpha$.

$\Rightarrow \alpha = U\cup\{U\}$.

(3) Need to verify: ~~none~~ $\forall x\in\emptyset\cup\{\emptyset\}$, x is not a limit ordinal.
$\Leftrightarrow$ $\emptyset$ is not a limit ordinal. $\leadsto$ by definition.

(4) ~~$\alpha=n$ is natural number $\Leftrightarrow \forall x\in\alpha\cup\{\alpha\}$, x is not limit. $(*)$.~~

~~N.T.P: $\alpha=n$ $\alpha+1$ is natural $\Leftrightarrow \forall x\in(\alpha\cup\{\alpha\})\cup\{\alpha\cup\{\alpha\}\}$, x is not limit~~

- ~~If $x\in\alpha\cup\{\alpha\}$, then $(*)$~~ $\Rightarrow$ x is not limit.

- ~~If $x\in x=\alpha\cup\{\alpha\}=\alpha+1$ since it is succesor of α, x is not~~

$\alpha=n$ is natural num $\Leftrightarrow \forall x\in\alpha\cup\{\alpha\}$, x is not limit $(*)$.

N.T.P: $\alpha+1$ is natural; i.e. $\forall x\in(\alpha\cup\{\alpha\})\cup\{\alpha\cup\{\alpha\}\}$, x is not limit.

- If $x\in\alpha\cup\{\alpha\}$, then $(*)$ $\Rightarrow$ x is not limit.
- If $x=\alpha\cup\{\alpha\}=\alpha+1$, then it is succesor of $\alpha \Rightarrow x$ is not limit #

(10). $\alpha = n$ is natural number. ~~N.T.P. $\forall x \in \alpha$,~~ Take $x \in \alpha$.

N.T.P: $\forall y \in x+1$, y is not limit.

Since α is natural, $\forall z \in \alpha+1$, z is not limit.

Since. $y \in x+1 \Leftrightarrow y \subsetneq x+1 \subsetneq \alpha+1 \Rightarrow y \in \alpha+1 \Rightarrow y$ is not limit. #.

try by yourself!

(9). $\boxed{\text{Increasing}} \Leftrightarrow \forall x_1 \le x_2 \in \mathbb{N}$, $f(x_1) \le f(x_2)$. $(*)$

Prove of induction. • Claim $f(0) = 0$.

Pf: If not, then $f(0) \ge 1$.

By $(*)$, $\forall n > 0$, $f(n) \ge f(0) \ge 1$. $\Rightarrow \forall n \in \mathbb{N}$, $f(n) \neq 0$

$\Rightarrow f$ is not surjective. ($0 \notin \operatorname{Im} f$) Impossible

• ~~If f~~ Claim: If $f(n) = n$ for $\forall n \le m$, then $f(m+1) = m+1$.

Pf: By $(*)$, $f(m+1) \ge f(m) = m$.

• If $f(m+1) = m = f(m)$, then f is not injective, impossible

• If $f(m+1) > m+1$, then $\forall i > m+1$, $f(i) \ge f(m+1) > m+1$

$\therefore$ ~~$m+1 \notin$~~ $\therefore \operatorname{Im} f \subseteq [0, m] \cup]m+1, +\infty) \Rightarrow m+1 \notin \operatorname{Im} f$

$\Rightarrow f$ is not surjective impossible

$\rightsquigarrow f(m+1) = m+1$ #

~~Chapter 5.~~
~~1.~~

Recall:

Thm (Jensen's inequality) (1) Let φ be a convex function on an interval I. Let $x_1, \dots, x_n \in I$, let $a_1, \dots a_n \in \mathbb{R}_{\geq 0}$, $\sum a_i = 1$. Then $\varphi(\sum_{i=1}^n a_i x_i) \leq \sum_{i=1}^n a_i \varphi(x_i)$.

Equality holds if and only if $x_1 = \dots = x_n$ or φ is linear.

(2) If φ is concave, then "$\geq$".

① Cauchy-Schwarz inequality.

• Form of 2 variables: Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^2$ (real plane), then $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\|$

General (discrete) form:

Thm (Cauchy, 1821). Let $\vec{u} := (u_1, u_2, \dots, u_n)$, $\vec{v} := (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, then $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\cdot\|\vec{v}\|$, i.e. $\left(\sum_{i=1}^n u_i v_i\right)^2 \leq \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{j=1}^n v_i^2\right)$. $(*)$

Pf: ~~Let φ:~~ Case 1: Suppose $v_i \neq 0$ for all $i = 1, \dots, n$.

Let $\varphi(x) = x^2$, then φ is convex. Let $a_i = \dfrac{v_i^2}{\sum_{k=1}^n v_k^2}$, $x_i = \dfrac{u_i}{v_i}$, then $\sum_{i=1}^n a_i = 1$.

By Jensen's inequality, $\left(\sum_{i=1}^n \dfrac{v_i^2}{\sum_{k=1}^n v_k^2}\cdot\dfrac{u_i}{v_i}\right)^2 \leq \sum_{i=1}^n \dfrac{v_i^2}{\sum_{k=1}^n v_k^2}\cdot\left(\dfrac{u_i}{v_i}\right)^2 \Rightarrow (*)$.

Case 2 $v_i = 0$ for some $i \in \{1, \dots, n\}$. WLOG, assume $v_1, \dots, v_s \neq 0$, $v_{s+1} = \dots = v_n = 0$.

then $\left(\sum_{i=1}^n u_i v_i\right)^2 = \left(\sum_{i=1}^s u_i v_i\right)^2 \underset{\text{Case 1}}{\leq} \left(\sum_{i=1}^s u_i^2\right)\left(\sum_{i=1}^s v_i^2\right) \leq \left(\sum_{i=1}^n u_i^2\right)\left(\sum_{i=1}^n v_i^2\right)$ #

② AM-GM-HM inequality (arithmetic, geometric, harmonic): $\forall x_1, x_2, \dots, x_n > 0$, $\underbrace{\frac{1}{n}\sum_{k=1}^n x_k}_{AM} \geq \underbrace{\left(\prod_{k=1}^n x_k\right)^{\frac{1}{n}}}_{GM} \geq \underbrace{n\cdot\left(\sum_{k=1}^n \frac{1}{x_k}\right)^{-1}}_{HM.}$ #

• AM-GM: $\ln(x)$ is concave.

$\Rightarrow \ln\left(\sum_{k=1}^n \frac{1}{n}\cdot x_k\right) \geq \sum_{k=1}^n \frac{1}{n}\ln(x_k) = \ln\left(\prod_{k=1}^n x_k^{\frac{1}{n}}\right)$.

↳ Jensen's inequality

• GM-HM:

$\frac{1}{n}\sum_{k=1}^n \frac{1}{x_k} \underset{\text{AM-GM}}{\geq} \left(\prod_{k=1}^n \frac{1}{x_k}\right)^{\frac{1}{n}} \Rightarrow \left(\prod_{k=1}^n x_k\right)^{\frac{1}{n}} \geq \left(\frac{1}{n}\sum_{k=1}^n \frac{1}{x_k}\right)^{-1} = n\left(\sum_{k=1}^n \frac{1}{x_k}\right)^{-1}$