

FUNDAMENTAL ALGEBRA & ANALYSIS

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Chapter 1

Differential Calculus

1.1 Landau symbol

In this section, we fix a complete valued field $(K, |\cdot|)$ and a normed vector space $(V, \|\cdot\|)$ over K .

Definition 1.1.1 Let X be a set, $f : X \longrightarrow V$, $g : X \longrightarrow \mathbb{R}_{\geq 0}$ be mappings. Let $Y \subseteq X$ be a subset. We use the expression

$$f(x) = \mathcal{O}(g(x))$$

to denote the statement:

$$\exists C > 0, \forall x \in Y, \|f(x)\| \leq C \cdot g(x).$$

Let \mathcal{F} be a filter on X , we use the expression

$$f(x) = \mathcal{O}(g(x)) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists C > 0, \exists A \in \mathcal{F}, \|f(x)\| \leq C \cdot g(x), \forall x \in A.$$

We use the expression

$$f(x) = o(g(x)) \text{ along } \mathcal{F}$$

to denote the statement:

$$\exists \varepsilon : X \longrightarrow \mathbb{R}_{\geq 0}, \exists A \in \mathcal{F}, \lim_{\mathcal{F}} \varepsilon = 0 \text{ and } \forall x \in A, \|f(x)\| \leq \varepsilon(x)g(x).$$

Proposition 1.1.2 Let X be a set and \mathcal{F} be a filter on X .

(1) Let $f : X \rightarrow V$, $g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f(x) = o(g(x))$ along \mathcal{F} , then $f(x) = \mathcal{O}(g(x))$ along \mathcal{F} .

(2)

1. Let $f_1 : X \rightarrow V$, $f_2 : X \rightarrow V$ and $g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f_1(x) = \mathcal{O}(g(x))$ and $f_2(x) = \mathcal{O}(g(x))$ along \mathcal{F} , then $f_1(x) + f_2(x) = \mathcal{O}(g(x))$ along \mathcal{F} .

2. Let $f_1 : X \rightarrow V$, $f_2 : X \rightarrow V$ and $g : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings. If $f_1(x) = o(g(x))$ and $f_2(x) = o(g(x))$ along \mathcal{F} , then $f_1(x) + f_2(x) = o(g(x))$ along \mathcal{F} .

(3) Let $\lambda : X \rightarrow K$, $f : X \rightarrow V$, $g : X \rightarrow \mathbb{R}_{\geq 0}$, $h : X \rightarrow \mathbb{R}_{\geq 0}$ be mappings.

1. If $\lambda(x) = \mathcal{O}(g(x))$ along \mathcal{F} , $f(x) = \mathcal{O}(h(x))$ along \mathcal{F} , then

$$(\lambda f)(x) = \lambda(x)f(x) = \mathcal{O}(g(x)h(x)).$$

2. If $\lambda(x) = \mathcal{O}(g(x))$ along \mathcal{F} , $f(x) = o(h(x))$ along \mathcal{F} , or if $\lambda(x) = o(g(x))$ along \mathcal{F} , $f(x) = \mathcal{O}(h(x))$ along \mathcal{F} , then

$$\lambda(x)f(x) = o(g(x)h(x)).$$

Proof

(1) We have $\varepsilon : X \rightarrow \mathbb{R}_{\geq 0}$, $A \in \mathcal{F}$ such that $\lim_{\mathcal{F}} \varepsilon = 0$ and $\forall x \in A$, $\|f(x)\| \leq \varepsilon(x)g(x)$. Since $\lim_{\mathcal{F}} \varepsilon = 0$, there exists $B \in \mathcal{F}$ such that $\forall x \in B$, $|\varepsilon(x)| < 1$, hence $\forall x \in A \cap B$, $\|f(x)\| \leq g(x)$.

(2)

1. $A_1, A_2 \in \mathcal{F}$, $C_1, C_2 > 0$, $\forall x \in A_1$, $\|f_1(x)\| \leq C_1g(x)$, $\forall x \in A_2$, $\|f_2(x)\| \leq C_2g(x)$. So $f_1(x) + f_2(x) = \mathcal{O}(g(x))$

2. Let $\varepsilon_1 : X \rightarrow \mathbb{R}_{\geq 0}$, $\varepsilon_2 : X \rightarrow \mathbb{R}_{\geq 0}$, $A \in \mathcal{F}$, $\lim_{\mathcal{F}} \varepsilon_1 = \lim_{\mathcal{F}} \varepsilon_2 = 0$. $\forall x \in A_1$, $\|f_1(x)\| \leq \varepsilon_1(x) \cdot g(x)$, $\forall x \in A_2$, $\|f_2(x)\| \leq \varepsilon_2(x)g(x)$. So $\lim_{\mathcal{F}} \varepsilon_1 + \varepsilon_2 = 0$.

$$\forall x \in A_1 \cap A_2, \|f_1(x) + f_2(x)\| \leq \|f_1(x)\| + \|f_2(x)\| \leq (\varepsilon_1(x) + \varepsilon_2(x))g(x).$$

(3)

1. There exists $(C_1, C_2) \in \mathbb{R}_{>0}^2$ and $(A_1, A_2) \in \mathcal{F}^2$ such that

$$\forall x \in A_1, |\lambda(x)| \leq C_1 g(x), \forall x \in A_2, \|f(x)\| \leq C_2 h(x).$$

Hence,

$$\forall x \in A_1 \cap A_2, \|(\lambda(x)f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C_1 C_2 g(x) h(x).$$

2. We assume that

$$\lambda(x) = \mathcal{O}(g(x)) \text{ along } \mathcal{F}, f(x) = o(h(x)) \text{ along } \mathcal{F}.$$

There exists $(A_1, A_2) \in \mathcal{F} \times \mathcal{F}, C \in \mathbb{R}_{\geq 0}$ and a mapping $\varepsilon : X \longrightarrow \mathbb{R}_{\geq 0}$ such that

$$\forall x \in A_1, |\lambda(x)| \leq C \cdot g(x), \forall x \in A_2, \|f(x)\| \leq \varepsilon(x) h(x).$$

Then one has

$$\lim_{\mathcal{F}} C\varepsilon(x) = 0$$

and

$$\forall x \in A_1 \cap A_2, \|(\lambda(x)f(x))\| \leq |\lambda(x)| \cdot \|f(x)\| \leq C \cdot g(x) \cdot \varepsilon(x) h(x)$$

As required. □

Example 1.1.3

(1) Let $I \subseteq \mathbb{N}$ infinite. Let $(V, \|\cdot\|)$ be a normed vector space over complete valued field $(K, |\cdot|)$. Let \mathcal{F} be the filter on I . Let $(x_n)_{n \in I} \in V^I, (b_n)_{n \in I} \in \mathbb{R}_{\geq 0}^I$. We denote by

$$x_n = \mathcal{O}(b_n), n \in I, n \rightarrow +\infty$$

the statement $x_n = \mathcal{O}(b_n)$ along \mathcal{F} . Namely,

$$\exists N \in \mathbb{N}, \exists C > 0, \forall n \in I_{\geq N}, \|x_n\| \leq C \cdot b_n.$$

$$x_n = o(b_n), n \in I, n \rightarrow +\infty$$

denotes the statement $x_n = o(b_n)$ along \mathcal{F} . Namely,

$$\exists (\varepsilon_n)_{n \in I} \text{ such that } \lim_{n \rightarrow +\infty} \varepsilon_n = 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, \|x_n\| \leq \varepsilon_n \cdot b_n.$$

(2) Let (X, \mathcal{T}) be a topological space, $Y \subseteq X$, $y_0 \in \bar{Y}$. Let $f : Y \rightarrow V$ and $g : Y \rightarrow \mathbb{R}_{\geq 0}$ be mappings.

$$\mathcal{F} = \mathcal{V}_{y_0}(\mathcal{T})|_Y := \{U \cap Y \mid U \text{ is a neighborhood of } y_0\}$$

$f(y)\mathcal{O}(g(y))$, $y \in Y$, $y \rightarrow y_0$ denotes $f(y) = \mathcal{O}(g(y))$ along \mathcal{F} . Namely,

$$\exists C > 0, \exists U \in \mathcal{V}_{y_0}(\mathcal{T}), \forall y \in U \cap Y, \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)), y \in Y, y \rightarrow y_0$$

denotes $f(y) = o(g(y))$ along \mathcal{F} . Namely,

$$\exists \varepsilon : Y \rightarrow \mathbb{R}_{\geq 0}, \lim_{y \in Y, y \rightarrow y_0} \varepsilon(y) = 0, \exists U \in \mathcal{V}_{y_0}(\mathcal{T}),$$

$$\forall y \in U \cap Y, \|f(y)\| \leq \varepsilon(y)g(y).$$

(3) Let \mathcal{F} be a filter on \mathbb{R} generated by subsets of the form $[a, +\infty[$. Let $Y \subseteq \mathbb{R}$ not bounded from above. Let $f : Y \rightarrow V$ and $g : Y \rightarrow \mathbb{R}_{\geq 0}$ be mappings. Then

$$f(y) = \mathcal{O}(g(y)), y \in Y, y \rightarrow +\infty$$

denotes $f(y) = \mathcal{O}(g(y))$ along $\mathcal{F}|_Y$. Namely,

$$\exists C > 0, \exists a \in \mathbb{R}, \forall y \in Y_{\geq a}, \|f(y)\| \leq C \cdot g(y).$$

$$f(y) = o(g(y)), y \in Y, y \rightarrow +\infty$$

denotes $f(y) = o(g(y))$ along $\mathcal{F}|_Y$. Namely,

$$\exists \varepsilon : Y \rightarrow \mathbb{R}_{\geq 0}, \lim_{y \rightarrow +\infty} \varepsilon(y) = 0, \exists a \in \mathbb{R}, \forall y \in Y_{\geq a}, \|f(y)\| \leq \varepsilon(y)g(y).$$

1.2 Differentiability

We fix a complete valued field $(K, |\cdot|)$. We suppose that there exists $a \in K^\times$, such that $|a| < 1$. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K .

$$\mathcal{L}(E, F) := \{\varphi \in \text{Hom}_K(E, F) \mid \|\varphi\| < +\infty\}.$$

$(\mathcal{L}(E, F), \|\cdot\|)$ is a normed vector space over K .

Definition 1.2.1 Let $U \subseteq E$ be subset and $p \in U^\circ$. We say that a mapping $f : U \rightarrow F$ is **differentiable** at p if there exists $\varphi \in \mathcal{L}(E, F)$ such that

$$f(p+h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

If $U = U^\circ$ and f is differentiable at every point of U , we say that f is **differentiable** on U .

Proposition 1.2.2 Assume that $f : U \rightarrow F$ is differentiable at $p \in U^\circ$. There exists a unique $\varphi \in \mathcal{L}(E, F)$ such that

$$f(p+h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

Lemma 1.2.3 $\forall \eta \in \mathcal{L}(E, F), \forall r > 0$.

$$\|\eta\| = \sup_{x \in E, 0 < \|x\|_E \leq r} \frac{\|\eta(x)\|_F}{\|x\|_E} = \sup_{\substack{x \in E \\ 0 < \|x\|_E < r}} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

Proof (of Lemma) $\|\eta\| \geq \sup_{x \in E, 0 < \|x\|_E < r} \frac{\|\eta(x)\|_F}{\|x\|_E}$. $\forall y \in E \setminus \{0\}, \|a^N y\|_E = |a|^N \|y\|_E < r$.

$$\frac{\|\eta(a^N y)\|_F}{\|a^N y\|_E} = \frac{|a|^N \cdot \|\eta(y)\|_F}{|a|^N \cdot \|y\|_E} = \frac{\|\eta(y)\|_F}{\|y\|_E} \leq \sup_{\substack{x \in E \\ 0 < \|x\|_E < r}} \frac{\|\eta(x)\|_F}{\|x\|_E}.$$

□

Proof (of Proposition) Suppose $\varphi, \psi \in \mathcal{L}(E, F)$ are such that

$$f(p+h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E,$$

$$f(p+h) - f(p) - \psi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

Then

$$\varphi(h) - \psi(h) = o(\|h\|_E), \quad h \rightarrow 0_E.$$

$$\exists r > 0, \exists \varepsilon : \overline{B}(0_E, r) \rightarrow \mathbb{R}_{\geq 0} \text{ such that } \lim_{h \rightarrow 0_E} \varepsilon(h) = 0.$$

$$\forall h \in \overline{B}(0_E, r), \|(\varphi - \psi)(h)\|_F = \varepsilon(h)\|h\|_E.$$

$$\|\varphi - \psi\| = \sup_{\substack{x \in E \\ 0 < \|h\|_E < r'}} \frac{\|\varphi(h) - \psi(h)\|_F}{\|h\|_E} \leq \sup_{0 < \|h\|_E < r'} \varepsilon(h).$$

Taking the limit when $r' \rightarrow 0$, by $\limsup_{h \rightarrow 0_E} \varepsilon(h) = 0$. We get $\|\varphi - \psi\| = 0$, hence $\varphi = \psi$. \square

Definition 1.2.4 Let $U \subseteq E$ and $f : U \rightarrow F$ be a mapping that is differentiable at $p \in U^\circ$. The unique $\varphi \in \mathcal{L}(E, F)$ such that

$$f(p + h) - f(p) - \varphi(h) = o(\|h\|_E), \quad h \rightarrow 0_E$$

is called the **differential** of f at p and is denoted as

$$D(f(p)).$$

Example 1.2.5

(1) $f : U \rightarrow F, f(x) \equiv c, c \in F$.

$$f(x + h) - f(x) = 0_E = o(\|h\|_E).$$

So f is differentiable at every point of U and $D(f(x)) = 0_F$.

(2) $\varphi \in \mathcal{L}(E, F)$.

$$\varphi(p + h) - \varphi(p) - \varphi(h) = 0_F = o(\|h\|_E).$$

So φ is differentiable at every point of E and $D(\varphi(p)) = \varphi$.

(3) Let $(F_i, \|\cdot\|_i)$ be normed vector spaces over $K, i \in \{1, \dots, n\}, n \in \mathbb{N}$. Suppose that $F = F_1 \oplus \dots \oplus F_n$ and

$$\|(s_1, \dots, s_n)\|_F = \max\{\|s_1\|_1, \dots, \|s_n\|_n\}.$$

Let $U \subseteq E$ be an open subset, $f_i : U \rightarrow F_i$ be a mapping.

$$f : U \rightarrow F, f(x) = (f_1(x), \dots, f_n(x)).$$

$$f(p + h) - f(p) = (f_1(p + h) - f_1(p), \dots, f_n(p + h) - f_n(p)).$$

Suppose that each f_i is differentiable

$$\begin{aligned} & f(p + h) - f(p) - (Df_1(p)(h), \dots, Df_n(p)(h))|_F \\ &= \max_{i \in \{1, \dots, n\}} \|f_i(p + h) - f_i(p) - Df_i(p)(h)\|_{F_i} \\ &= o(\|h\|_E). \end{aligned}$$

So f is differentiable at p and

$$Df(p)(h) = (Df_1(p)(h), \dots, Df_n(p)(h)).$$

(4) Suppose that $E = K$. If $U \subseteq K$ is open and $f : U \rightarrow F$ is differentiable at $p \in U$. We denote by $f'(p)$ the element $Df(p)(1) \in F$.

$$f(p+h) - f(p) - Df(p)(h) = o(\|h\|_E).$$

So

$$f(p+h) - f(p) - hf'(p) = o(\|h\|_E),$$

$$\frac{f(p+h) - f(p)}{h} - f'(p) = o(1).$$

That is,

$$\lim_{h \rightarrow 0} \frac{f(p+h) - f(p)}{h} = f'(p).$$

Theorem 1.2.6 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$, $(G, \|\cdot\|_G)$ be normed vector spaces over a complete valued field $(K, |\cdot|)$. Let $U \subseteq E$ and $V \subseteq F$ be open subsets, $f : U \rightarrow F$ and $g : V \rightarrow G$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If f is differentiable at p and g is differentiable at $f(p)$, then $g \circ f : U \rightarrow G$ is differentiable at p and

$$D(g \circ f)(p)(h) = Dg(f(p))(Df(p)(h)).$$

Proof

$$f(p+h) - f(p) - Df(p)(h) = o(\|h\|_E),$$

so,

$$f(p+h) - f(p) = \mathcal{O}(\|h\|_E).$$

$$\begin{aligned} & g(f(p+h)) - g(f(p)) - Dg(f(p))(f(p+h) - f(p)) \\ &= o(\|f(p+h) - f(p)\|_F) = o(\mathcal{O}\|h\|_E) = o(\|h\|_E). \end{aligned}$$

$$\begin{aligned} & Dg(f(p))(f(p+h) - f(p)) - Dg(f(p))(Df(p)(h)) \\ &= Dg(f(p))(f(p+h) - f(p) - Df(p)(h)) \\ &= \mathcal{O}(o(\|h\|_E)) = o(\|h\|_E). \end{aligned}$$

So,

$$g(f(p+h)) - g(f(p)) - Dg(f(p))(Df(p)(h)) = o(\|h\|_E).$$



Remark 1.2.7 If $(E, \|\cdot\|_E) = (K, |\cdot|)$,

$$(g \circ f)'(p) = Dg(f(p))(f'(p)).$$

If $E = F = K$, $\|\cdot\|_E = \|\cdot\|_F = |\cdot|$.

$$(g \circ f)'(p) = g'(f(p)) \cdot f'(p).$$

Remark 1.2.8 Let $U \subseteq E$ be open. $f : U \longrightarrow F_1 \times \cdots \times F_n$. If f is differentiable at $p \in U$, for any $i \in \{1, \dots, n\}$, the mapping

$$f_i := \pi_i \circ f : U \longrightarrow F_i$$

is differentiable at p and

$$D(f_i)(p)(h) = D\pi_i(f(p)) (Df(p)(h)) = \pi_i (Df(p)(h)).$$

1.3 Multilinear Mappings

Definition 1.3.1 Let K be a commutative unitary ring. Let $E_1, \dots, E_n; F$ be K -modules. We say that

$$\varphi : E_1 \times \cdots \times E_n \longrightarrow F$$

is n -linear if for any $i \in \{1, \dots, n\}$ and any $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in E_1 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_n$, the mapping

$$E_i \longrightarrow F, x_i \mapsto \varphi(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)$$

is a homomorphism of K -modules. (K -linear mapping)

If $n = 1$, 1-linear is also called linear.

If $n = 2$, 2-linear is also called bilinear.

Example 1.3.2

(1) $K \times K \longrightarrow K$ $(a, b) \mapsto ab$ is bilinear.

(2) $K^n \times K^n \longrightarrow K$ $(x, y) \mapsto x \cdot y = \sum_{i=1}^n x_i y_i$ is bilinear.

(3) $K \times \cdots \times K \longrightarrow K$ $(x_1, \dots, x_n) \mapsto x_1 \cdots x_n$ is n -linear.

Definition 1.3.3 We denote by $\text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$ the set of n -linear mappings from $E_1 \times \dots \times E_n$ to F .

Definition 1.3.4 Let $(K, |\cdot|)$ be a complete valued field. Let $(E_1, \|\cdot\|_{E_1}), \dots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$ be normed vector spaces over K . For any $\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$, we define

$$\|\varphi\| := \sup_{\substack{x_i \in E_i \setminus \{0\} \\ i \in \{1, \dots, n\}}} \frac{\|\varphi(x_1, \dots, x_n)\|_F}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_n}}.$$

We denote by $\mathcal{L}(E_1 \times \dots \times E_n, F)$ the set

$$\{\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F) \mid \|\varphi\| < +\infty\}.$$

$\mathcal{L}^{(n)}(E_1 \times \dots \times E_n, F)$ is a normed vector space of $\text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$, and the norm is $\|\cdot\|$.

Theorem 1.3.5 Let $(E_1, \|\cdot\|_{E_1}), \dots, (E_n, \|\cdot\|_{E_n}), (F, \|\cdot\|_F)$ be normed vector spaces over K . Let $\varphi \in \mathcal{L}^{(n)}(E_1 \times \dots \times E_n, F)$. For any $p = (p_1, \dots, p_n) \in E_1 \times \dots \times E_n$, φ is differentiable at p and

$$D\varphi(p)(h_1, \dots, h_n) = \sum_{i=1}^n \varphi(p_1, \dots, p_{i-1}, h_i, p_{i+1}, \dots, p_n).$$

Proof

$$\begin{aligned} \varphi(p+h) - \varphi(p) &= \sum_{i=1}^n \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, p_i + h_i, p_{i+1}, \dots, p_n) \\ &\quad - \varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, p_i, p_{i+1}, \dots, p_n) \end{aligned}$$

$$\begin{aligned} &\varphi(p_1 + h_1, \dots, p_{i-1} + h_{i-1}, h_i, p_{i+1}, \dots, p_n) \\ &- \varphi(p_1, \dots, p_{i-1}, h_i, p_{i+1}, \dots, p_n) \\ &= \sum_{j=1}^{i-1} \varphi(p_1 + h_1, \dots, p_{j-1} + h_{j-1}, h_j, p_{j+1}, \dots, h_i, \dots, p_n). \end{aligned}$$

$$\begin{aligned}
& \|\varphi(p_1 + h_1, \dots, p_{j-1} + h_{j-1}, h_j, p_{j+1}, \dots, h_i, \dots, p_n)\|_F \\
& \leq \|\varphi\| \cdot \prod_{k=1}^{j-1} \|p_k + h_k\|_{E_k} \cdot \|h_j\|_{E_j} \cdot \prod_{k=j+1}^{i-1} \|p_k\|_{E_k} \cdot \|h_i\|_{E_i} \cdot \prod_{k=i+1}^n \|p_k\|_{E_k} \\
& = \mathcal{O}(\|h\|^2) = o(h), \quad h \rightarrow 0.
\end{aligned}$$

□

Definition 1.3.6 Let K be a commutative unitary ring. $n \in \mathbb{N}_{\geq 1}$, E and F be K -modules. We say that

$$\varphi \in \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F)$$

is **symmetric** if

$$\forall \sigma \in \mathfrak{S}_{\{1, \dots, n\}}, \forall (x_1, \dots, x_n) \in E^n, \varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varphi(x_1, \dots, x_n).$$

Let $P : E \longrightarrow F$ be a mapping. If there exists a symmetric $\varphi \in \text{Hom}_K^{(n)}(E \times \dots \times E, F)$ such that

$$\forall x \in E, P(x) = \varphi(x, \dots, x),$$

we say that P is a **homogeneous polynomial mapping of degree n** .

If $F = K$, P is called a **homogeneous polynomial** on E . The symmetric polynomial mapping φ is called the **polarization** of P .

Proposition 1.3.7 Let $(K, |\cdot|)$ be a complete valued field that is non-trivial. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K . Assume that $P : E \longrightarrow F$ is a homogeneous polynomial mapping of degree n . Which admits a bounded polarization φ . Then P is differentiable on E and,

$$\forall (x, h) \in E \times E, DP(x)(h) = n\varphi(x, \dots, x, h).$$

Proof Let

$$\begin{aligned}
\Delta : E &\longrightarrow E^n, \\
x &\longmapsto (x, \dots, x).
\end{aligned}$$

Then $P = \varphi \circ \Delta$. Since φ and Δ are differentiable, so it is P .

Moreover,

$$\begin{aligned}
 DP(x)(h) &= D\varphi(\Delta(x)) (D\Delta(x)(h)) \\
 &= D\varphi(x, \dots, x)(h, \dots, h) \\
 &= \sum_{i=1}^n \varphi(x, \dots, x, h, x, \dots, x) \\
 &= n\varphi(x, \dots, x, h).
 \end{aligned}$$

□

Remark 1.3.8 Assume that $E = K$. Let $P : K \rightarrow F$ be a homogeneous polynomial mapping of degree n of form $P(x) = x^n s$, where $s \in F$. Its polarization is of the form

$$\begin{aligned}
 \varphi(a_1, \dots, a_n) &= a_1 \cdots a_n s. \\
 P'(x) = DP(x)(1) &= n\varphi(x, \dots, x, 1) = nx^{n-1}s.
 \end{aligned}$$

Proposition 1.3.9 Let n be a positive integer $n \geq 2$. Let $(E_1, \|\cdot\|_1), \dots, (E_n, \|\cdot\|_n), (F, \|\cdot\|_F)$ be normed vector spaces. For any $i \in \{1, \dots, n\}$, the mapping

$$\begin{aligned}
 \mathcal{L}^{(n)}(E_1, \dots, E_n; F) &\xrightarrow{f} \mathcal{L}^{(i)}(E_1, \dots, E_i, \mathcal{L}^{(n-i)}(E_{i+1}, \dots, E_n; F)) \\
 \varphi &\longmapsto \left(\begin{array}{c} E_1 \times \dots \times E_n \longrightarrow \mathcal{L}^{(i)}(E_{i+1}, \dots, E_n; F) \\ (x_1, \dots, x_i) \longmapsto \left(\begin{array}{c} (x_{i+1}, \dots, x_n) \longmapsto \varphi(x_1, \dots, x_n) \\ E_{i+1} \times \dots \times E_n \in F \end{array} \right) \end{array} \right)
 \end{aligned}$$

is an isomorphism of vector spaces over K , and in the same time an isometry, $(\|f(\varphi)\| = \|\varphi\|)$.

Remark 1.3.10

$\forall (x_1, \dots, x_n) \in E_1 \times \dots \times E_n, f(\varphi)(x_1, \dots, x_i)(x_{i+1}, \dots, x_n) = \varphi(x_1, \dots, x_n)$

Proof $\forall (x_1, \dots, x_n) \in E_1 \times \dots \times E_n,$

$$\begin{aligned}
 \varphi(x_1, \dots, x_n) : E_{i+1} \times \dots \times E_n &\longrightarrow F \text{ is bounded} \\
 (x_{i+1}, \dots, x_n) &\longmapsto \varphi(x_1, \dots, x_n)
 \end{aligned}$$

Since

$$\|\varphi(x_1, \dots, x_n)\|_F \leq (\|\varphi\| \cdot \|x_1\| \cdots \|x_i\|) \|x_{i+1}\| \cdots \|x_n\|.$$

$$\begin{aligned}
\|f(\varphi)\| &= \sup_{x_j \in E_j \setminus \{0\}, j=1, \dots, i} \frac{\|\varphi(x_1, \dots, x_i, \cdot)\|}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_i}} \\
&= \sup_{x_j \in E_j \setminus \{0\}, j=1, \dots, i} \sup_{x_k \in E_k \setminus \{0\}, k=i+1, \dots, n} \frac{\|\varphi(x_1, \dots, x_n)\|_F}{\|x_1\|_{E_1} \cdots \|x_n\|_{E_n}} \\
&= \|\varphi\|.
\end{aligned}$$

Hence f is injective. ($\ker(f) = \{0\}$)

For any $\psi \in \mathcal{L}^{(i)}(E_1, \dots, E_i, \mathcal{L}^{(n-i)}(E_{i+1}, \dots, E_n)),$

$$\begin{aligned}
\varphi : E_1 \times \dots \times E_n &\longrightarrow F \\
(x_1, \dots, x_n) &\longmapsto \psi(x_1, \dots, x_i)(x_{i+1}, \dots, x_n)
\end{aligned}$$

belongs to $\mathcal{L}^{(n)}(E_1, \dots, E_n; F)$ and $f(\varphi) = \psi$. So f is surjective. \square

Corollary 1.3.11 If E_1, \dots, E_n are all finite dimensional, then

$$\mathcal{L}^{(n)}(E_1, \dots, E_n; F) = \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F).$$

Proof If $n = 1$, $\mathcal{L}(E_1, F) = \text{Hom}_K(E_1, F)$.

$$\begin{aligned}
\mathcal{L}^{(n)}(E_1, \dots, E_n; F) &\cong \mathcal{L}(E_1, \mathcal{L}^{(n-1)}(E_2, \dots, E_n; F)) \\
&= \text{Hom}_K(E_1, \text{Hom}_K^{(n-1)}(E_2 \times \dots \times E_n, F)) \\
&\cong \text{Hom}_K^{(n)}(E_1 \times \dots \times E_n, F).
\end{aligned}$$

\square

Let $(K, |\cdot|)$ be a complete nontrivial valued field. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K .

Definition 1.3.12 Let $U \subseteq E$ be an open subset of E , $f : U \longrightarrow F$ be a mapping.

If f is continuous on U , we say that f is **of class** \mathcal{C}^0 and we denote by

$$D^0 f$$

the mapping $f : U \longrightarrow F$. Denote by

$$\mathcal{C}^0(U, F)$$

the set of mappings from U to F .

$$U \xrightarrow{(f,g)} K \times K \xrightarrow{\times} K$$

$$p \longmapsto (f(p), g(p)) \longmapsto f(p) \times g(p)$$

Let $p \in U$. If f is differentiable on an open neighborhood V of p such that $V \subseteq U$. Then

$$\begin{aligned} Df : V &\longrightarrow \mathcal{L}(E, F) \\ x &\longmapsto Df(x) \end{aligned}$$

is a mapping. If Df is $(n-1)$ -times differentiable at p , we say that f is **of class \mathcal{C}^n** at p . If f is of class \mathcal{C}^n at every point of U , we say that f is **n -times differentiable** at p . We denote by

$$D^n f(p) \in \mathcal{L}^{(n)}(E, \dots, E, F)$$

the n -linear mapping that sends $(h_1, \dots, h_n) \in E^n$ to

$$D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n) \in F.$$

Remark 1.3.13

$$D^n f(p)(h_1, \dots, h_n) = D^i(D^{n-i} f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n).$$

1.4 Convexity

Definition 1.4.1 Let E be a vector space over a field K . $S \subseteq E$ be a non-empty subset.

We call affine combination of elements of S any element of E of the form

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n,$$

where $n \in \mathbb{N}_{\geq 1}$, $s_1, \dots, s_n \in S$, $a_1, \dots, a_n \in K$ such that

$$a_1 + a_2 + \cdots + a_n = 1.$$

We denote by $\text{Aff}(S)$ the set of all affine combinations of elements of S . One has $S \subseteq \text{Aff}(S)$. $\text{Aff}(S)$ is called the affine hull of S .

If $S = \text{Aff}(S)$, we say that S is an affine subspace of E .

Proposition 1.4.2

(1) If F is a vector subspace of E , $\forall p \in E$,

$$p + F = \{p + x \mid x \in F\}$$

is an affine subspace of E .

(2) If $A \subseteq E$ is an affine subspace of E . For any $p \in A$,

$$A - p := \{x - p \mid x \in A\}$$

is a vector subspace of E , which is not dependent on the choice of p . We call it the vector space **associated** with A .

Proof

(1) Let $(x_1, \dots, x_n) \in F^n$, $(a_1, \dots, a_n) \in K^n$, such that $\sum_{i=1}^n a_i = 1$. Then

$$\begin{aligned} \sum_{i=1}^n a_i(p + x_i) &= p \cdot \sum_{i=1}^n a_i + \sum_{i=1}^n a_i x_i \\ &= p + \sum_{i=1}^n a_i x_i \in p + F. \end{aligned}$$

(2) Let $(x_1, \dots, x_n) \in A^n$, $(b_1, \dots, b_n) \in K^n$.

$$\begin{aligned} \sum_{i=1}^n b_i(x_i - p) &= \sum_{i=1}^n b_i x_i - \left(\sum_{i=1}^n b_i \right) p \\ &= \left(\sum_{i=1}^n b_i x_i + \left(1 - \sum_{i=1}^n b_i \right) p \right) - p \\ &\in A - p. \end{aligned}$$

Let $q \in A$, $\forall x \in A$, $x - p = (x - q) + (q - p) \in A - q$. So $A - p \subseteq A - q$. By symmetry, $A - q \subseteq A - p$. Hence $A - p = A - q$. \square

Example 1.4.3 Let A be an m by p matrix with coefficients in \mathbb{R} . Let $(b_1, \dots, b_m) \in E^m$. Consider the linear equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

The solution set is

$$S := \{(x_1, \dots, x_p) \in E^p \mid A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}\}.$$

Claim: S is an affine subspace of E^p .

Proof Let $\underline{x}^{(1)}, \dots, \underline{x}^{(n)}$ be elements of S , where $\underline{x}^{(i)} = (x_1^{(i)}, \dots, x_p^{(i)})$. Let $(a_1, \dots, a_n) \in \mathbb{R}^n$, $\underline{x} = a_1 \underline{x}^{(1)} + \dots + a_n \underline{x}^{(n)}$.

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} = A \left(a_1 \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_p^{(1)} \end{pmatrix} + \dots + a_n \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_p^{(n)} \end{pmatrix} \right).$$

$$a_1 A \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_p^{(1)} \end{pmatrix} + \dots + a_n A \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_p^{(n)} \end{pmatrix} = (a_1 + \dots + a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

$$x_j = a_1 x_j^{(1)} + \dots + a_n x_j^{(n)}.$$

\square

Proposition 1.4.4 Let $S \subseteq E$. Then $\text{Aff}(S)$ is the smallest affine subspace of E containing S .

Proof

Let $A \subseteq E$ be an affine subspace containing S . $\forall n \in \mathbb{N}_{\geq 1}, \forall (x_1, \dots, x_n) \in S^n \subseteq A^n, (a_1, \dots, a_n) \in \mathbb{R}, a_1 + \dots + a_n = 1$, one has

$$\sum_{i=1}^n a_i x_i \in A.$$

So $\text{Aff}(S) \subseteq A$.

To show that $\text{Aff}(S)$ is an affine subspace containing S , it is sufficient to check that $\text{Aff}(S)$ is an affine subspace.

If $S = \emptyset$, then $\text{Aff}(S) = \emptyset$. It is an affine subspace.

Suppose that $S \neq \emptyset, p \in S$. We prove that $\text{Aff}(S) - p$ is equal to $\text{Span}_{\mathbb{R}}(S - p)$. Let $y = a_1 x_1 + \dots + a_n x_n \in \text{Aff}(S)$.

$$y - p = a_1(x_1 - p) + \dots + a_n(x_n - p) \in \text{Span}_{\mathbb{R}}(S - p).$$

Let $(x_1, \dots, x_n) \in S^n, (b_1, \dots, b_n) \in \mathbb{R}^n$.

$$\sum_{i=1}^n b_i(x_i - p) = \left(\sum_{i=1}^n b_i x_i + \left(1 - \sum_{i=1}^n b_i\right) p \right) - p \in \text{Aff}(S) - p.$$

□

Definition 1.4.5 Let $S \subseteq E$. We call **convex combination** of elements of S any element of E of the form

$$a_1 s_1 + a_2 s_2 + \dots + a_n s_n,$$

where $n \in \mathbb{N}_{\geq 1}, s_1, \dots, s_n \in S, a_1, \dots, a_n \in \mathbb{R}_{\geq 0}$ such that

$$a_1 + a_2 + \dots + a_n = 1.$$

We denote by $\text{Conv}(S)$ the set of all convex combinations of elements of S . $\text{Conv}(S)$ is called the **convex hull** of S . One has $S \subseteq \text{Conv}(S) \subseteq \text{Aff}(S)$.

Proposition 1.4.6 Let E be a vector space over \mathbb{R} and $C \subseteq E$. Then C is convex

if and only if

$$\forall (x, y) \in C^2, \forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in C.$$

Proof It is sufficient to check “ \Leftarrow ”. We prove by induction on n that

$$\forall n \in \mathbb{N}_{\geq 1}, \forall (x_1, \dots, x_n) \in C^n, \forall (a_1, \dots, a_n) \in \mathbb{R}_{\geq 0}^n, \sum_{i=1}^n a_i = 1, \sum_{i=1}^n a_i x_i \in C.$$

The case where $n = 1$ is trivial. The case where $n = 2$ comes from the hypothesis. Suppose $n \geq 3$ in assuming that the statement holds for any integer less than n . If $a_n = 1$, then $a_1 = \dots = a_{n-1} = 0$, so $\sum_{i=1}^n a_i x_i = x_n \in C$. If $a_n < 1$, we have $a_1 + \dots + a_{n-1} = 1 - a_n > 0$. By the induction hypothesis,

$$x := \sum_{i=1}^{n-1} \frac{a_i}{1 - a_n} x_i \in C.$$

Taking $y = x_n$, $t = 1 - a_n$,

$$C \ni tx + (1 - t)y = \sum_{i=1}^n a_i x_i.$$

□

1.5 Mean Value Theorems

Theorem 1.5.1 (Mean Value Inequality) Let $(F, \|\cdot\|_F)$ be normed vector spaces over \mathbb{R} . Let $(a, b) \in \mathbb{R}^2$ such that $a < b$. Let $f : [a, b] \rightarrow F$ be a continuous mapping that is differentiable on $]a, b[$. Then

$$\|f(b) - f(a)\|_F \leq (b - a) \cdot \sup_{t \in]a, b[} \|f'(t)\|_F.$$

Proof We may suppose that $\sup_{t \in]a, b[} \|f'(t)\|_F < +\infty$. Take

$$M > \sup_{t \in]a, b[} \|f'(t)\|_F.$$

Let $m = \frac{a+b}{2}$. Let

$$J = \{x \in [m, b] \mid \forall t \in [m, x], \|f(t) - f(m)\|_F \leq M(t - m)\}.$$

It is an interval containing m . So it is of the form

$$[m, c[\text{ or } [m, c]$$

$$\forall t \in [m, c[, \|f(t) - f(m)\|_F \leq M(t - m).$$

Taking the limit $t < c, t \rightarrow c$, we get $c \in J$. So $J = [m, c]$. We then check $c = b$.

If $c \neq b$, then $c \in]a, b[$, so f is differentiable at c . That is

$$\|f(c + h) - f(c)\|_F = \|f'(c)h + o(\|h\|)\|_F \leq \|f'(c)\|_F h + o(\|h\|), \quad h \rightarrow 0.$$

Since $M > \|f'(c)\|_F$, $\exists h_0 > 0$ such that

$$\forall h \in]0, h_0], \|f(c + h) - f(c)\|_F \leq Mh.$$

$$\begin{aligned} \|f(c + h) - f(m)\| &\leq \|f(c + h) - f(c)\| + \|f(c) - f(m)\| \\ &\leq Mh + M(c - m) = M(c + h - m). \end{aligned}$$

So $[m, c + h_0] \subseteq J$, contradiction. Thus $b = c$. $\|f(b) - f(m)\|_F \leq M(b - m)$.

By the same reason, $\|f(m) - f(a)\|_F \leq M(m - a)$. So

$$\|f(b) - f(a)\|_F \leq \|f(b) - f(m)\|_F + \|f(m) - f(a)\|_F \leq M(b - a).$$

Taking the limit when $M \rightarrow \sup_{t \in]a, b[} \|f'(t)\|_F$, we get the announced result. \square

Corollary 1.5.2 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over \mathbb{R} . $U \subseteq E$ be an open subset, and $(x, y) \in U^2$ such that

$$[x, y] = \{tx + (1 - t)y \mid t \in [0, 1]\} \subseteq U.$$

Let $f : U \rightarrow F$ be a differentiable mapping. Then

$$\|f(x) - f(y)\|_F \leq \left(\sup_{z \in]x, y[} \|Df(z)\| \right) \cdot \|x - y\|_E.$$

Proof Let

$$\begin{aligned} g : [0, 1] &\longrightarrow U \\ t &\longmapsto tx + (1 - t)y. \end{aligned}$$

$$g(0) = x, \quad g(1) = y, \quad g'(t) = x - y.$$

Then,

$$\begin{aligned}(f \circ g)'(t) &= Df(g(t))(x - y), \\ D(f \circ g)(t)(1) &= Df(g(t))(Dg(t)(1)).\end{aligned}$$

By the theorem,

$$\begin{aligned}\|f(x) - f(y)\|_F &= \|f(g(1)) - f(g(0))\|_F \\ &\leq \sup_{t \in]0,1[} \|Df(g(t))(x - y)\|_F \\ &\leq \sup_{t \in]0,1[} |Df(g(t))| \cdot \|x - y\|_E \\ &= \sup_{z \in]x,y[} \|Df(z)\| \cdot \|x - y\|_E.\end{aligned}$$

□

Definition 1.5.3 Let (X, \mathcal{T}) be a topological space, $p \in X$. Let U be a neighborhood of p and $f : U \rightarrow \mathbb{R}$ be a mapping. If there exists a neighborhood V of p such that $p \in V \subseteq U$ and

$$\forall x \in V, f(p) \geq f(x),$$

we say that p is a **local maximum point** of f on U .

If p is a local maximum point or a local minimum point, we say that p is a **local extremum** of f on U .

If $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are normed vector spaces. $U \subseteq E$ open, $f : U \rightarrow F$ is differentiable. If $p \in U$ is such that

$$Df(p) = 0 \in \mathcal{L}(E, F),$$

we say that p is a **critical point** of f .

Theorem 1.5.4 Let $(E, \|\cdot\|)$ be a normed vector space over \mathbb{R} . $U \subseteq E$ be an open subset, $f : U \rightarrow \mathbb{R}$ be a differentiable mapping. If $p \in U$ is a local extremum point of f , then it is a critical point ($Df(p) = 0$).

Proof There exists $r > 0$ such that $p + B(0, r) \subseteq U$ and

$$(h \in B(0, r)) \mapsto f(p + h) - f(p) \in \mathbb{R}$$

does not change the sign.

$$\forall h \in B(0, r), \forall \epsilon \in [0, 1],$$

$$(f(p + th) - f(p))(f(p - th) - f(p)) \geq 0.$$

Taking the limit when $t \rightarrow 0$, $-Df(p)(h)^2 \geq 0$. So $Df(p)(h) = 0$. \square

Theorem 1.5.5 (Rolle) Let $(a, b) \in \mathbb{R}^2$, $a < b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$. If $f(a) = f(b)$, then

$$\exists t \in]a, b[, f'(t) = 0.$$

Proof If there exists t which is in $]a, b[$ and is an extremum point of f , then $f'(t) = 0$. Since $[a, b]$ is compact and f is continuous, so f attains its maximum and minimum.

If the extremum points of f are in $\{a, b\}$. Since $f(a) = f(b)$, f is constant, so $f'(t) = 0$ on $]a, b[$. \square

Theorem 1.5.6 (Gronwall inequality) Let $(F, \|\cdot\|)$ be a normed vector space over \mathbb{R} , $(a, b) \in \mathbb{R}^2$, $a < b$. Let $f : [a, b] \rightarrow F$ and $g : [a, b] \rightarrow \mathbb{R}$ be differentiable mappings on $]a, b[$. If $\forall t \in]a, b[, \|f'(t)\| \leq g'(t)$, then

$$\|f(b) - f(a)\|_F \leq g(b) - g(a).$$

Proof Let $m \in]a, b[$. Let $\epsilon > 0$,

$$J := \{t \in [m, b] \mid \forall s \in [m, t], \|f(s) - f(m)\|_F \leq g(s) - g(m) + \epsilon(s - m)\}.$$

Since f and g are continuous, J is a closed interval of the form $[m, c]$.

If $c < b$,

$$\begin{aligned} f(c + h) &= f(c) + hf'(c) + o(h), \\ g(c + h) &= g(c) + hg'(c) + o(h), \quad h > 0, h \rightarrow 0. \end{aligned}$$

$\exists \delta > 0$, such that $[c, c + \delta] \subseteq [c, b]$ and $\forall h \in [0, \delta]$,

$$\|f(c + h) - f(c)\| \leq h\|f'(c)\| + \frac{\epsilon}{2}h.$$

$$g(c + h) - g(c) \geq hg'(c) - \frac{\epsilon}{2}h.$$

So,

$$\|f(c + h) - f(c)\| \leq g(c + h) - g(c) + \epsilon h.$$

By the triangle inequality,

$$\|f(c+h) - f(m)\| \leq g(c+h) - g(m) + \varepsilon(c+h-m).$$

So $J \supseteq [m, c+\delta]$, contradiction.

Therefore $c = b$.

$$\|f(b) - f(m)\| \leq g(b) - g(m) + \varepsilon(b-m).$$

A similar argument shows that

$$\|f(m) - f(a)\| \leq g(m) - g(a) + \varepsilon(m-a).$$

Hence,

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \varepsilon(b-a).$$

$$\|f(c+h) - f(c) + hf'(c)\| \leq \varphi(h)h, \lim_{h \rightarrow 0} \varphi(h) = 0.$$

$$\exists \delta > 0, \forall h > 0, 0 \leq h < \delta \Rightarrow |\varphi(h)| \leq \frac{\varepsilon}{2}.$$

□

Theorem 1.5.7 (Mean value theorem of Lagrange) Let $(a, b) \in \mathbb{R}^2$, $a < b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$. Then

$$\exists \xi \in]a, b[, f(b) - f(a) = f'(\xi)(b-a).$$

Proof Let $g : [a, b] \rightarrow \mathbb{R}$.

$$g(t) := f(b) - f(t) + C(b-t), \text{ where } C = -\frac{f(b) - f(a)}{b-a}.$$

Then $g(a) = g(b) = 0$, $g'(t) = -f'(t) - C$.

$$\exists \xi \in]a, b[, g'(\xi) = 0, f'(\xi) = -C = \frac{f(b) - f(a)}{b-a}.$$

□

Theorem 1.5.8 (Darboux) Let I be an open interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ be a differentiable mapping. Then $f'(I)$ is an interval.

Proof Let a, b be two elements in I such that $a < b$. Let

$$g : [a, b] \longrightarrow \mathbb{R}$$

$$t \longmapsto \begin{cases} \frac{f(t)-f(a)}{t-a}, & t \neq a \\ f'(a), & t = a \end{cases}$$

g is continuous, and $g([a, b])$ is an interval. By the mean value theorem of Lagrange, $g([a, b]) \subseteq f'(I)$.

Let

$$h : [a, b] \longrightarrow \mathbb{R}$$

$$t \longmapsto \begin{cases} \frac{f(t)-f(b)}{t-b}, & t \neq b \\ f'(b), & t = b \end{cases}$$

$h([a, b])$ is an interval contained in $f'(I)$.

$h([a, b]) \cup g([a, b])$ is an interval since

$$\frac{f(b) - f(a)}{b - a} \in h([a, b]) \cap g([a, b]),$$

$$\{f'(a), f'(b)\} \subseteq h([a, b]) \cup g([a, b]).$$

So the interval linking $f'(a), f'(b)$ is contained in $f'(I)$. Hence, $f'(I)$ is an interval.

□

1.6 Higher Differential

We fix a complete non-trivially valued field $(K, |\cdot|)$. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over K .

Definition 1.6.1 Let $U \subseteq E$ be an open subset, $f : U \longrightarrow F$ be a mapping, $p \in U$.

(1) If f is continuous at p , we say that f is 0-time differentiable at p , and we let

$$D^0 f(p) := f(p).$$

(2) If f is differentiable at p , we say that f is 1-time differentiable at p , and we let

$$D^1 f(p) := Df(p).$$

(3) Let $n \geq 2$. If exists open neighborhood V of p such that $V \subseteq U$ and f is differentiable on V and Df is $n - 1$ -time differentiable on V , we say that f is

n -time differentiable at p , and we let

$$D^n f(p) \in \mathcal{L}(E, \dots, E, F)$$

be the multilinear mapping sending $(h_1, \dots, h_n) \in E^n$ to

$$D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n).$$

If $E = K$, $D^n f(p)(1, \dots, 1)$ is denoted as $f^{(n)}(p) \in F$. $f^{(0)}(p)$ is often denoted as $f(p)$.

Remark 1.6.2 $\forall i \in \{1, \dots, n\}$,

$$D^n f(p)(h_1, \dots, h_n) = D^i(D^{n-i}f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n).$$

If $E = K$,

$$f^{(n)}(p)(h_1, \dots, h_n) = D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n).$$

Definition 1.6.3 Let X be a set, we denote by \mathfrak{S}_X the element of all bijection from X to X . (\mathfrak{S}_X, \circ) forms a group. The identity mapping Id_X is the neutral element of (\mathfrak{S}_X, \circ) . (\mathfrak{S}_X, \circ) is called the symmetric group of X . The elements of (\mathfrak{S}_X, \circ) are called permutations of X .

Let $n \in \mathbb{N}_{\geq 2}$, x_1, \dots, x_n be distinct elements of X . We denote by $(x_1 x_2 \cdots x_n)$ the element of \mathfrak{S}_X that sends x_i to x_{i+1} , ($i \in \{1, \dots, n-1\}$), x_n to x_1 , $y \in X \setminus \{x_1, \dots, x_n\}$ to y itself. This element is called an n -cycle. A 2-cycle is also called a transposition.

Remark 1.6.4 \mathfrak{S}_X acts on X .

$$\begin{aligned} \mathfrak{S}_X \times X &\longrightarrow X \\ (\sigma, x) &\longmapsto \sigma(x). \end{aligned}$$

If $\sigma \in \mathfrak{S}_X$, $x \in X$, we denote by $\text{orb}_\sigma(x)$ the set $\{\sigma^n(x) \mid n \in \mathbb{Z}\}$.

$$\langle \sigma \rangle := \{\sigma^n \mid n \in \mathbb{Z}\} \subseteq \mathfrak{S}_X$$

is a group. $\text{orb}_\sigma(x)$ is the orbit of x under the action of $\langle \sigma \rangle$.

Proposition 1.6.5 If $\text{orb}_\sigma(x)$ is finite of d elements, then $\sigma^d(x) = x$, and $\text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{d-1}(x)\}$. Moreover, the restriction of σ to $\text{orb}_\sigma(x)$ identifies to the restriction of the cycle $(x, \sigma(x), \dots, \sigma^{d-1}(x))$.

Proof Since $\text{orb}_\sigma(x)$ is finite,

$$\{(n, m) \in \mathbb{Z}^2 \mid n < m, \sigma^n(x) = \sigma^m(x)\}$$

Let

$$l = \min\{m - n \mid (n, m) \in \mathbb{Z}^2, n < m, \sigma^n(x) = \sigma^m(x)\}.$$

Then $x, \sigma(x), \dots, \sigma^{l-1}(x)$ are distinct, and $\sigma^l(x) = x$. $\forall n \in \mathbb{Z}$, then n can be written as $n = lp + r$, where $p \in \mathbb{Z}, r \in \{0, \dots, l-1\}$.

$$\sigma^n(x) = \sigma^r(\sigma^{lp}(x)) = \sigma^r((\sigma^l \circ \dots \sigma^l)(x)) = \sigma^r(x).$$

So, $\text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{l-1}(x)\}$, ($l = d$). □

Remark 1.6.6 If X is finite, then X can be written as a distinct union of orbits (under the action of $\langle \sigma \rangle$). Let $d_i = \#(\text{orb}_\sigma(x_i))$, $i = 1, \dots, n$, then

$$\sigma|_{\text{orb}_\sigma(x^{(i)})} = (x^{(i)}, \sigma(x^{(i)}), \dots, \sigma^{d_i-1}(x^{(i)}))|_{\text{orb}_\sigma(x^{(i)})}.$$

So $\sigma = \tau_1 \circ \dots \circ \tau_n$, where $\tau_i := (x^{(i)}, \sigma(x^{(i)}), \dots, \sigma^{d_i-1}(x^{(i)}))$.

Corollary 1.6.7 Suppose that X is finite. Any $\sigma \in \mathfrak{S}_X$ can be written as a composition of transpositions.

Proof

$$(x_1 \dots x_n) = (x_1 x_2) \circ (x_2 \dots x_n),$$

So,

$$(x_1 \dots x_n) = (x_1 x_2) \circ (x_2 x_3) \circ \dots \circ (x_{n-1} x_n).$$

□

Definition 1.6.8 Denote by \mathfrak{S}_n the symmetric group $\mathfrak{S}_{\{1, \dots, n\}}$. A composition of the form $(i \ i+1)$, $i \in \{1, \dots, n-1\}$ is called an adjacent transposition.

Corollary 1.6.9 Any $\sigma \in \mathfrak{S}_n$ can be written as a composition of adjacent transpositions.

Proof Let $(j, k) \in \{1, \dots, n\}^2, j < k$,

$$(j-1\ j) \circ (j\ k) \circ (j-1\ j) = (j-1\ k).$$

$$(j\ k) = (j\ j+1) \circ (j+1\ j+2) \circ \dots \circ (k-1\ k) \circ \dots (j\ j+1).$$

□

Theorem 1.6.10 (Schwarz) Let $U \subseteq E$ be an open subset, $f : U \rightarrow F$ be a mapping. $n \in \mathbb{N}_{\geq 1}, p \in U$. Assume that f is n -times differentiable at p . Then $\forall \sigma \in \mathfrak{S}_n, \forall (h_1, \dots, h_n) \in E^n$,

$$D^n f(p)(h_1, \dots, h_n) = D^n f(p)(h_{\sigma(1)}, \dots, h_{\sigma(n)}).$$

Proof (By induction) The case where $n = 1$ is trivial. Case $n = 2$: Exists V open, $p \in V \subseteq U$. f is differentiable on V and Df is differentiable at p .

$$Df(p+h)(\cdot) - Df(p)(\cdot) - D^2f(p)(h, \cdot) = o(\|h\|_E).$$

Let $\varepsilon > 0, \exists \delta > 0, \forall h \in E, \|h\|_E \leq 2\delta \Rightarrow p+h \in V$ and

$$\|Df(p+h)(\cdot) - Df(p)(\cdot) - D^2f(p)(h, \cdot)\| \leq \varepsilon \|h\|_E.$$

Let $h \in E$ such that $\|h\|_E \leq \delta$. Define $g_h : B(0, \delta) \rightarrow F$ as

$$g_h(k) = f(p+h+k) - f(p+h) - f(p+k) + f(p) - D^2f(p)(h, k).$$

Then,

$$\begin{aligned} Dg_h(k)(\cdot) &= Df(p+h+k)(\cdot) - Df(p+k)(\cdot) - D^2f(p)(h, \cdot) \\ &= Df(p+h+k)(\cdot) - Df(p)(\cdot) - D^2f(p)(h+k, \cdot) \\ &\quad - (Df(p+k)(\cdot) - Df(p)(\cdot) - D^2f(p)(k, \cdot)) \end{aligned}$$

$$\|Dg_h(k)(\cdot)\| \leq \varepsilon \|h+k\|_E + \varepsilon \|k\|_E \leq 3\varepsilon \max\{\|k\|_E, \|h\|_E\}.$$

$g_h(0) = 0$. Therefore, $\|g_h(k)\| \leq 3\varepsilon \max\{\|k\|_E, \|h\|_E\}^2$ (mean value inequality).

$$\|g_h(k) - g_h(0)\| \leq \left(\sup_{t \in [0,1]} \|Dg_h(tk)\| \right) \cdot \|k\|.$$

Therefore,

$$f(p+h+k) - f(p+h) - f(p+k) + f(p) - D^2 f(p)(h, k) = o(\max\{\|k\|_E, \|h\|_E\}^2).$$

By symmetry,

$$f(p+h+k) - f(p+h) - f(p+k) + f(p) - D^2 f(p)(k, h) = o(\max\{\|k\|_E, \|h\|_E\}^2).$$

$$D^2 f(p)(h, k) - D^2 f(p)(k, h) = o(\max\{\|h\|_E, \|k\|_E\}^2).$$

$$D^2 f(p)(th, tk) - D^2 f(p)(tk, th) = o(|t|^2), \quad t \rightarrow 0.$$

$$D^2 f(p)(h, k) - D^2 f(p)(k, h) = o(1), \quad t \rightarrow 0.$$

Suppose $n \geq 3$.

$$D^n f(p)(h_1, \dots, h_n) = D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n).$$

If $\sigma = (j \ j+1)$, $j \leq 2$,

$$D^{n-1}(Df)(p)(h_1, \dots, h_{n-1})(h_n) = D^{n-1}(Df)(p)(h_{\sigma(1)}, \dots, h_{\sigma(n-1)})(h_n)$$

by the induction hypothesis, if $\sigma = (n-1 \ n)$,

$$D^n f(p)(h_1, \dots, h_n) = D^2 \left((D^{n-2} f)(h_1, \dots, h_{n-2})(h_{n-1}, h_n) \right)$$

$$\begin{aligned} D^n f(p)(h_{\sigma(1)}, \dots, h_{\sigma(n)}) &= D^n f(p)(h_1, \dots, h_{n-1}) \\ &= D^2 \left((D^{n-2} f)(h_1, \dots, h_{n-2})(h_n, h_{n-1}) \right) \\ &= D^n f(p)(h_1, \dots, h_n). \end{aligned}$$

□

1.7 Taylor's Formula

Theorem 1.7.1 (Taylor-Young) Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over \mathbb{R} , $U \subseteq E$ open, $n \in \mathbb{N}$, $f : U \rightarrow F$ be a mapping, $p \in U$. Suppose that f is n -times differentiable at p . Then

$$f(x) = f(p) + \sum_{k=1}^n \frac{1}{k!} D^k f(p)(x - p, \dots, x - p) + o(\|x - p\|^n), \quad x \rightarrow p.$$

Proof (By induction on n)

$n = 0$, $f(x) = f(p) + o(1)$ follows by continuity of f ; $n = 1$ follows by the differentiability of f .

From $n - 1$ to n . Let $g : U \rightarrow F$

$$g(x) = f(x) - f(p) - \sum_{k=1}^n \frac{1}{k!} D^k f(p)(x - p, \dots, x - p).$$

g is differentiable on an open neighborhood of p ,

$$Dg(x)(h) = Df(x)(h) - \sum_{k=1}^n \frac{1}{k!} k D^k f(p)(x - p, \dots, x - p, h)$$

$$Dg(x) = Df(x) - \sum_{l=0}^{n-1} \frac{1}{l!} D^l(Df)(x - p, \dots, x - p) \stackrel{\text{hyp.}}{=} o(\|x - p\|^{n-1}), \quad x \rightarrow p.$$

So $g(x) = o(\|x - p\|^n)$.

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in B(p, \delta), \|Dg(x)\| \leq \varepsilon \|x - p\|^{n-1}.$$

$g(p) = 0$, so

$$\|g(x) - g(p)\| \leq \varepsilon \|x - p\|^{n-1} \cdot \|x - p\| = \varepsilon \|x - p\|^n.$$

□

Theorem 1.7.2 (Taylor-Lagrange) Let $(a, b) \in \mathbb{R}^2$, $a < b$. $f : [a, b] \rightarrow \mathbb{R}$ be a continuous mapping. Suppose that f is $(n + 1)$ -times differentiable on $]a, b[$ and $\forall k \in \{3, \dots, n\}$, $f^{(k)} :]a, b[\rightarrow \mathbb{R}$ tends to a continuous mapping $[a, b] \rightarrow \mathbb{R}$.

Then

$$\exists \xi \in]a, b[, f(b) - \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) = \frac{f^{(n+1)}(\xi)(b-a)^{n+1}}{(n+1)!}.$$

Proof Let $g : [a, b] \rightarrow \mathbb{R}$.

$$g(t) := \sum_{k=0}^n \frac{(b-t)^k}{k!} f^{(k)}(t) - C \frac{(b-t)^{n+1}}{(n+1)!}.$$

$$\text{Then } g(b) = f(b), g(a) = \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) - C \frac{(b-a)^{n+1}}{(n+1)!}.$$

$$\begin{aligned} g'(t) &= \sum_{k=0}^n \frac{(b-t)^k}{k!} f^{(k+1)}(t) - \sum_{k=1}^n \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(t) + C \frac{(b-t)^n}{n!} \\ &= \frac{(b-t)^n}{n!} f^{(n+1)}(t) + C \frac{(b-t)^n}{n!}. \end{aligned}$$

Take C such that $g(a) = g(b)$. Then by Rolle's theorem, $\exists \xi \in]a, b[, g'(\xi) = 0$, $C = -f^{(n+1)}(\xi)$. Then,

$$g(a) = \sum_{k=0}^n \frac{(b-a)^k}{k!} f^{(k)}(a) + \frac{f^{(n+1)}(\xi)}{(n+1)!} + \frac{f^{(n+1)}(\xi)}{(n+1)!} (b-a)^{n+1} = f(b) = g(b).$$

□

Theorem 1.7.3 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed vector spaces over \mathbb{R} , $U \subseteq E$ be an open subset, and $f : U \rightarrow F$ be a mapping that is $(n+1)$ -times differentiable, where $n \in \mathbb{N}$. Let $p \in U$, $h \in E$ such that $\forall t \in [0, 1], p + th \in U$. Let

$$M = \sup_{t \in [0, 1]} \|D^{n+1}f(p + th)\|.$$

Then,

$$\|f(p+h) - \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h)\|_F \leq \frac{M}{(n+1)!} \|h\|_E^{n+1}.$$

Proof We define $\phi : [0, 1] \longrightarrow F$

$$\phi(t) = f(p + th) + \sum_{k=1}^n \frac{(1-t)^k}{k!} D^k f(p + th)(h, \dots, h).$$

$$\phi(0) = \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h), \quad \phi(1) = f(p + h).$$

$$\begin{aligned} \phi'(t) &= Df(p + h)(h) + \sum_{k=1}^n \frac{(1-t)^k}{k!} D^{k+1} f(p + th)(h, \dots, h) \\ &\quad - \sum_{k=1}^n \frac{(1-t)^{k-1}}{(k-1)!} D^k f(p + th)(h, \dots, h) \\ &= \sum_{k=0}^n \frac{(1-t)^k}{k!} D^{k+1} f(p + th)(h, \dots, h) \\ &\quad - \sum_{l=0}^{n-1} \frac{(1-t)^l}{(l)!} D^{l+1} f(p + th)(h, \dots, h) \\ &= \frac{(1-t)^n}{n!} D^{n+1} f(p + th)(h, \dots, h). \end{aligned}$$

So,

$$\|\phi'(t)\| \leq M \|h\|_E^{n+1} \frac{(1-t)^n}{n!}, \quad t \in [0, 1].$$

By Gronwall's inequality,

$$\|\phi(1) - \phi(0)\|_F \leq M \cdot \|h\|^{n+1} \frac{1}{(n+1)!}.$$

□

1.8 Banach Space

Proposition 1.8.1 Let (X, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . If

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) < +\infty,$$

then $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

Proof Let $N \in \mathbb{N}$. If $(n, m) \in \mathbb{N}^2$, $n > m$, by the triangle inequality,

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k \geq N} d(x_k, x_{k+1}).$$

So,

$$0 \leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \leq \sum_{k \geq N} d(x_k, x_{k+1}).$$

Taking the limit when $N \rightarrow +\infty$, we get

$$\lim_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0.$$

Let $(a_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$. If $\left(\sum_{k=0}^n a_k \right)_{n \in \mathbb{N}}$ converges to some l in \mathbb{R} . Then, $l - \sum_{k=0}^{N-1} a_k$ converges to 0. If $a_k \leq 0$ for any $k \in \mathbb{N}$, $l - \sum_{k=0}^{N-1} a_k = \sum_{k=N}^{+\infty} a_k$.

$$l - \sum_{k=0}^{N-1} a_k = \lim_{n \rightarrow +\infty} \left(\sum_{k=0}^n a_k - \sum_{k=0}^{N-1} a_k \right) = \lim_{n \rightarrow +\infty} \sum_{k=N}^n a_k.$$

□

Definition 1.8.2 Let $(K, |\cdot|)$ be a complete valued field and $(E, \|\cdot\|)$ be a normed vector space over K . If E equipped with the metric

$$\begin{aligned} E \times E &\longrightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\longmapsto \|x - y\|_E. \end{aligned}$$

is complete, we say that $(E, \|\cdot\|)$ is a **Banach space**.

Let $(E, \|\cdot\|)$ be a Banach space. If $(x_n)_{n \in \mathbb{N}}$ is a sequence in E such that $\sum_{n \in \mathbb{N}} \|x_n\| < +\infty$, we say that $\sum_{n \in \mathbb{N}} x_n$ **converges absolutely**.

Remark 1.8.3 Suppose that $\sum_{n \in \mathbb{N}} x_n$ converges absolutely. Then $\left(\sum_{k=0}^n x_k \right)_{n \in \mathbb{N}}$ is a Cauchy sequence, since

$$\|x_n\| = \left\| \sum_{k=0}^n x_k - \sum_{k=0}^{n-1} x_k \right\|.$$

So, $\sum_{n \in \mathbb{N}} x_n$ converges.

Theorem 1.8.4 (Root test of Cauchy) Let $(E, \|\cdot\|)$ be a Banach space and $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$. Let

$$r = \limsup_{n \rightarrow \infty} \|x_n\|^{\frac{1}{n}} \in [0, +\infty]$$

If $r < 1$, then $\sum_{n \in \mathbb{N}} x_n$ converges absolutely.

If $r > 1$, then $\sum_{n \in \mathbb{N}} x_n$ diverges.

Lemma 1.8.5 If a series $\sum_{n \in \mathbb{N}} x_n$ converges, then $\lim_{n \rightarrow +\infty} \|x_n\| = 0$.

Proof (of lemma)

$$\|x_n\| = \left\| \sum_{k=0}^n x_k - \sum_{k=0}^{n-1} x_k \right\|.$$

Since $\sum_k^n x_k$ converges to some $l \in E$.

$$\lim_{n \rightarrow +\infty} \|x_n\| = \lim_{n \rightarrow +\infty} \left\| \sum_{k=0}^n x_k - \sum_{k=0}^{n-1} x_k \right\| = \|l - l\| = 0.$$

□

Proof (of theorem) If $r > 1$, $\exists \beta > 1$ such that $r > \beta$. Since $r = \limsup_{n \rightarrow +\infty} \|x_n\|^{\frac{1}{n}}$, $\exists I \subseteq \mathbb{N}$ infinite such that $\lim_{n \in I, n \rightarrow +\infty} \|x_n\|^{\frac{1}{n}} = r$ (Bolzano-Weierstrass).

$$\exists N \in \mathbb{N}, \forall n \in I_{\geq N}, \|x_n\|^{\frac{1}{n}} \geq \beta.$$

So, $\|x_n\| \geq \beta^n \geq 1$. So $\sum_{n \in \mathbb{N}} x_n$ diverges.

If $r < 1$, $\exists \alpha \in]0, 1[$, $r < \alpha$. Since $r = \limsup_{n \rightarrow +\infty} \|x_n\|^{\frac{1}{n}}$,

$$\exists N \in \mathbb{N}, \forall n \geq N, \|x_n\|^{\frac{1}{n}} \leq \alpha, \|x_n\| \leq \alpha^n.$$

So,

$$\sum_{n \geq N} \|x_n\| \leq \sum_{n \geq N} \alpha^n = \frac{\alpha^N}{1 - \alpha} < +\infty.$$

Therefore, $\sum_{n \in \mathbb{N}} x_n$ converges absolutely.

□

Theorem 1.8.6 (Ratio test of D'Alembert) Let $(E, \|\cdot\|)$ be a Banach space and $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$.

(1) If

$$\limsup_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < 1,$$

then $\sum_{n \in \mathbb{N}} x_n$ converges absolutely.

(2) If

$$\limsup_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} > 1,$$

then $\sum_{n \in \mathbb{N}} x_n$ diverges.

Proof

(1) Let $0 < \alpha < 1$ such that

$$\limsup_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} < \alpha.$$

$$\exists N \in \mathbb{N}, \forall n \in \mathbb{N}_N, \|x_{n+1}\| \leq \alpha \|x_n\| \leq \alpha^{n+1-N} \|x_N\|.$$

Thus,

$$\sum_{n \geq N} \|x_n\| \leq \sum_{n \geq N} \|x_N\| \alpha^{n-N} = \|x_N\| \frac{1}{1-\alpha} < +\infty.$$

(2) Let $\beta > 1$ such that

$$\liminf_{n \rightarrow +\infty} \frac{\|x_{n+1}\|}{\|x_n\|} > \beta.$$

$$\exists N \in \mathbb{N}, x_N \neq 0, \text{ and } \forall n \in \mathbb{N}_{\geq N}, \|x_{n+1}\| \geq \beta \|x_n\|$$

$$\forall n \geq N, \|x_n\| \geq \beta^{n-N} \|x_N\| \rightarrow +\infty (n \rightarrow +\infty)$$

So $\sum_{n \in \mathbb{N}} x_n$ diverges. □

Let $z \in \mathbb{C}$. The series $\sum_{n \in \mathbb{N}} \frac{z^n}{n!}$ converges absolutely since

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \rightarrow 0 (n \rightarrow +\infty).$$

We denote by e^z this limit.

1.9 Local inversion

Definition 1.9.1 Let X be a topological space and $Y \subseteq X$. If $\overline{Y} = X$, we say that Y is dense.

Theorem 1.9.2 (Baire) Let (X, d) be a complete metric space. Let $(\Omega_n)_{n \in \mathbb{N}}$ be a sequence of dense open subset of X . Let $\Omega = \bigcap_{n \in \mathbb{N}} \Omega_n$, then Ω is dense in X .

Proof Suppose that Ω is not dense. Let $x_0 \in X \setminus \overline{\Omega}$, exists $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subseteq X \setminus \overline{\Omega}$.

Let $r_0 = \varepsilon$. We construct in a recursive way sequence $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ and $(r_n)_{n \in \mathbb{N}} \in \mathbb{R}_{\geq 0}^{\mathbb{N}}$ as follows.

Suppose that (x_n, r_n) is chosen. $B(x_n, r_n) \cap \Omega_n \neq \emptyset$. We pick $x_{n+1} \in X$ and $r_{n+1} \leq \frac{r_n}{2}$ such that $B(x_{n+1}, r_{n+1}) \subseteq B(x_n, r_n) \cap \Omega_n$, $d(x_{n+1}, x_n) < r_n$. $\sum_{n \in \mathbb{N}} r_n < +\infty$ (ratio test).

Then the sequence converges to some l . For any $n \in \mathbb{N}$, $x_n \in B(x_0, \varepsilon)$. So $l \in \overline{B(x_0, \varepsilon)}$.

Moreover, $\forall n \in \mathbb{N}$, $l \in \overline{B(x_{n+1}, r_{n+1})} \subseteq B_{x_n, r_n} \cap \Omega_n$. Thus $l \in \bigcap_{n \in \mathbb{N}} \Omega_n = \Omega$. Contradiction. \square

Corollary 1.9.3 Let (X, d) be a non-empty complete metric space and $(Y_n)_{n \in \mathbb{N}}$ be a family of closed subsets of X such that $X = \bigcup_{n \in \mathbb{N}} Y_n$. Then exists $n \in \mathbb{N}$ such that $Y^\circ \neq \emptyset$.

Proof Let $\Omega_n = X \setminus Y_n$. Suppose that $\forall n \in \mathbb{N}$, $Y_n^\circ = \emptyset$. Then $\overline{\Omega_n} = X \setminus Y_n^\circ = X$. Thus $\Omega := \bigcap_{n \in \mathbb{N}} \Omega_n$ is dense in X . Namely, $X = \Omega$. So

$$\emptyset = X \setminus \overline{\Omega} = (X \setminus \Omega)^\circ = \left(X \setminus \bigcap_{n \in \mathbb{N}} \Omega_n \right)^\circ = \left(\bigcup_{n \in \mathbb{N}} Y_n \right)^\circ = X^\circ = X.$$

Contradiction. \square

Theorem 1.9.4 (Banach) Let $(K, |\cdot|)$ be a complete non-trivially valued field, and E be a vector space over K . Let $\|\cdot\|_1, \|\cdot\|_2$ be two norms on E such that $(E, \|\cdot\|_1)$ and $(E, \|\cdot\|_2)$ are both Banach spaces.

If $\exists C > 0$ such that $\|\cdot\|_2 \leq C \|\cdot\|_1$. Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent. ($\exists C' > 0, \|\cdot\|_1 \leq C' \|\cdot\|_2$)

Proof For $x \in E$ and $r > 0$. Let

$$B_i(x, r) := \{y \in E \mid \|y - x\|_i < r\}, \quad i = 1, 2$$

$\forall y \subseteq E$, let $\overline{Y}^{\|\cdot\|_2}$ be the closure of Y in $(E, \|\cdot\|_2)$.

$$E = \bigcup_{n \geq 1} B_1(0, n) = \bigcup_{n \geq 1} \overline{B_1(0, n)}^{\|\cdot\|_2}.$$

Hence, $\exists n_0 \geq 1, p \in E, r_0 > 0$ such that

$$B_2(p, r_0) \subseteq \overline{B_1(0, n_0)}^{\|\cdot\|_2}$$

or equivalently,

$$B_2(0, r_0) \subseteq \overline{B_1(-p, n_0)}^{\|\cdot\|_2} \subseteq \overline{B_1(0, n_0 + \|p\|_1)}^{\|\cdot\|_2}$$

since $\forall x \in B_1(-p, n_0)$

$$\|x\|_1 = \|x - p + p\|_1 \leq \|x - p\| + \|p\|_1 < n_0 + \|p\|_1.$$

Let $r_1 = n_0 + \|p\|_1$,

$$B_2(0, r_0) \subseteq \overline{B_1(0, r_1)}^{\|\cdot\|_2} \subseteq B_1(0, r_1) + B_2(0, r_0|a|)$$

where $a \in K, 0 < |a| < \frac{1}{2}$.

In fact, $\forall x \in \overline{B_1(0, r_0)}^{\|\cdot\|_2}$, exists sequence $(x_n)_{n \in \mathbb{N}} \in B_1(0, r_1)^{\mathbb{N}}$, such that $x_n \rightarrow x$ in $(E, \|\cdot\|_2)$, $\exists n \in \mathbb{N}, \|x_n - x\|_2 < r_0|a|$

$$B_2(0, r_0|a|^n) \subseteq B_1(0, r_1|a|^n) + B_2(0, r_0|a|^{n+1})$$

Let $y \in B_2(0, r_0)$, we choose $(x_0, y_0) \in B_1(0, r_1) \times B_2(0, r_0|a|)$ such that $y = x_0 + y_0$. When (x_n, y_n) is chosen, let $(x_{n+1}, y_{n+1}) \in B_1(0, r_1|a|^{n+1}) \times B_2(0, r_0|a|^{n+2})$, $y_n = x_{n+1} + y_{n+1}$, $y = y_n + \sum_{k=0}^n x_k$. So $\sum_{n \in \mathbb{N}} x_n$ converges to y .

Moreover, $\sum_{n \in \mathbb{N}} \|x_n\| < +\infty$, so it converges in $(E, \|\cdot\|_1)$ to some x . Therefore, $x = y$ since $\|\cdot\|_2 \leq C\|\cdot\|_1$. So $\|y\|_1 \|x\|_1 \leq \sum_{n \in \mathbb{N}} \|x_n\|_1 \leq \frac{r_1}{1-|a|}$.

Therefore $B_2(0, r_0) \subseteq B_1(0, \frac{r}{1-|a|})$. So $\|\cdot\|_1$ is bounded by a constant times $\|\cdot\|_2$. \square

Proposition 1.9.5 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces over a complete non-trivially valued field $(K, |\cdot|)$, and $f : E \longrightarrow F$ be a bounded mapping.

(1) If f is invertible, then f^{-1} is bounded.

(2) If f is surjective, for any $U \subseteq E$ open, $f(U)$ is open in F .

Proof

(1) We define a mapping

$$\begin{aligned} \|\cdot\|'_E : E &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \|f(x)\|_F. \end{aligned}$$

This is a norm on E . In fact, if $\|x\|'_E = \|f(x)\|_F = 0$, then $f(x) = 0_F$. So $x = 0_E$. Moreover,

$$\forall x \in E, \quad \|x\|'_E = \|f(x)\|_F \leq \|f\| \|x\|_E.$$

So there exists $C > 0$ such that $\|\cdot\|_E \leq C \|\cdot\|'_E$. That is,

$$\forall y \in F, \quad \|y\|_F = \|f(f^{-1}(y))\|_F = \|f^{-1}(y)\|'_E \geq C^{-1} \|f^{-1}(y)\|_E.$$

So, $\|f^{-1}\| \leq C$.

(2) Let

$$E_0 = \ker(f) = \{x \in E \mid f(x) = 0_F\}.$$

This is a closed vector subspace of E . $\|\cdot\|_E$ induces by passing to quotient a norm $\|\cdot\|_Q$ on $Q := E/E_0$. Let

$$\begin{aligned} g : Q &\longrightarrow F \\ [x] &\longmapsto f(x). \end{aligned}$$

This is a K -linear bijection.

If $\alpha \in Q$,

$$\forall x \in \alpha, \quad \|g(\alpha)\|_F = \|f(x)\|_F \leq \|f\| \cdot \|x\|_E.$$

Since $\|\alpha\|_Q := \inf_{x \in \alpha} \|x\|_E$, $\|g(\alpha)\|_F \leq \|f\| \cdot \|\alpha\|_Q$. So $\|g\| \leq \|f\|$. By (1), g^{-1} is bounded (hence is continuous).

If $V \subseteq Q$ is open, then $g(V) \subseteq F$ is open. Let $U \subseteq E$ be an open subset. Let

$$\begin{aligned} \pi : E &\longrightarrow Q \\ x &\longmapsto [x]. \end{aligned}$$

Let $x \in U$, $r > 0$ such that $B(x, r) \subseteq U$. For any $\alpha \in Q$, if

$$\|\alpha - [x]\|_Q = \inf_{y \in \alpha} \|y - x\|_E < r,$$

then, exists $y \in \alpha$ such that $\|y - x\|_E < r$.

Therefore,

$$B([x], r) \subseteq \pi(B(x, r)) \subseteq \pi(U).$$

This means that $\pi(U)$ is open. So $f(U) = g(\pi(U))$ is open. \square

Definition 1.9.6 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector space over a complete non-trivially valued field $(K, |\cdot|)$, $U \subseteq E$ open, $f : U \rightarrow F$. If $\forall p \in U$, f is n -times differentiable at p , and $D^n f : U \rightarrow \mathcal{L}^{(n)}(E, \dots, E, F)$ is continuous, we say that f is of class \mathcal{C}^n .

If $\forall n \in \mathbb{N}$, f is n -times differentiable on U , we say that f is smooth, or of class \mathcal{C}^∞ . ($\forall n \in \mathbb{N}$, f is of class \mathcal{C}^n .)

Proposition 1.9.7 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ and $(G, \|\cdot\|_G)$ be normed vector space over a complete non-trivially valued field $(K, |\cdot|)$. $U \subseteq E$, $V \subseteq F$ be open subsets, $f : U \rightarrow V$, $g : V \rightarrow G$ be mappings. $n \in \mathbb{N}$.

(1) Let $p \in U$. If f is n -times differentiable at p and g is n -times differentiable at $f(p)$, then $g \circ f$ is n -times differentiable at p .

(2) If f is of class \mathcal{C}^n on U and g is of class \mathcal{C}^n on V , then $g \circ f$ is of class \mathcal{C}^n on U .

Proof (induction on n)

$n = 0$, continuity composition.

$n = 1$, differentiability of composition.

$n \geq 2$,

$$D(g \circ f)(p)(\cdot) = Dg(f(p))(Df(p)(\cdot))$$

Let

$$\begin{aligned} \Phi : \mathcal{L}(F, G) \times \mathcal{L}(E, F) &\longrightarrow \mathcal{L}(E, G) \\ (\alpha, \beta) &\longmapsto \alpha \circ \beta. \end{aligned}$$

This is a bounded bilinear mapping. $\|\alpha \circ \beta\| \leq \|\alpha\| \cdot \|\beta\|$.

$$(\|\alpha \circ \beta(h)\|_G = \|\alpha(\beta(h))\|_G \leq \|\alpha\| \cdot \|\beta(h)\|_F \leq \|\alpha\| \|\beta\| \|h\|_E)$$

Φ is of class \mathcal{C}^∞ .

$$D(g \circ f) = \Phi(Dg \circ f, Df).$$

(2) Since Dg and Df is of class \mathcal{C}^{n-1} , we obtain that $D(g \circ f)$ is of class \mathcal{C}^{n-1} , so $g \circ f$ is of class \mathcal{C}^n .

(1) If g is n -times differentiable at $f(p)$, Dg is $(n-1)$ -times differentiable at $f(p)$.

So $Dg \circ f$ is $(n-1)$ -times differentiable at p . Df is $(n-1)$ -times differentiable at p . So $D(g \circ f)$ is $(n-1)$ -times differentiable at p . \square

Theorem 1.9.8 Let $(E, \|\cdot\|)$ be a Banach space over a complete non-trivially valued field $(K, |\cdot|)$. Let

$$\mathrm{GL}(E) := \{\varphi \in \mathcal{L}(E, E) \mid \varphi \text{ is invertible}\}.$$

This set forms a group under \circ .

(1) $\forall \varphi \in \mathcal{L}(E, E)$, if $\|\varphi\| < 1$, then $\mathrm{Id}_E + \varphi \in \mathrm{GL}(E)$.

(2) $\mathrm{GL}(E) \subseteq \mathcal{L}(E, E)$ is open.

(3)

$$\begin{aligned} \iota : \mathrm{GL}(E) &\longrightarrow \mathrm{GL}(E) \\ \varphi &\longmapsto \varphi^{-1} \end{aligned}$$

is of class \mathcal{C}^∞ .

Proof

(1) The series $\sum_{n \in \mathbb{N}} (-1)^n \varphi^n$ converges absolutely since $\|\varphi^n\| \leq \|\varphi\|^n$. Let η be the limit of $\sum_{n \in \mathbb{N}} (-1)^n \varphi^n$.

$$(\mathrm{Id} + \varphi) \circ \sum_{k=0}^n (-1)^k \varphi^k = \mathrm{Id} + (-1)^n \varphi^{n+1}.$$

Taking the limit when $n \rightarrow +\infty$, we get $(\mathrm{Id}_E + \varphi) \circ \eta = \mathrm{Id}_E$. For the same reason, $\eta \circ (\mathrm{Id}_E + \varphi) = \mathrm{Id}_E$.

(2) If $f \in \mathrm{GL}(E)$, $\forall \varphi \in \mathcal{L}(E, E)$ such that

$$\|\varphi\| < \frac{1}{\|f^{-1}\|}, \quad f + \varphi = f \circ (\mathrm{Id}_E + f^{-1} \circ \varphi), \quad \|f^{-1} \circ \varphi\| \leq \|f^{-1}\| \cdot \|\varphi\| < 1.$$

So $\mathrm{Id}_E + f^{-1} \circ \varphi \in \mathrm{GL}(E)$. Hence $f + \varphi \in \mathrm{GL}(E)$.

(3) Let $f \in \mathrm{GL}(E)$, $\varphi \in \mathcal{L}(E, E)$. $\|\varphi\| \leq \frac{1}{\|f^{-1}\|}$.

$$\begin{aligned} \iota(f + \varphi) - \iota(f) &= (f + \varphi)^{-1} - f^{-1} \\ &= (f \circ (\mathrm{Id}_E + f^{-1} \circ \varphi))^{-1} - f^{-1} \\ &= (\mathrm{Id}_E + f^{-1} \circ \varphi)^{-1} \circ f^{-1} - f^{-1} \\ &= \sum_{n \in \mathbb{N}} (-1)^n (f^{-1} \circ \varphi)^n \circ f^{-1} - f^{-1} \\ &= -f^{-1} \circ \varphi \circ f^{-1} + o(\|\varphi\|) \end{aligned}$$

since

$$\begin{aligned}
 & \sum_{n \geq 2} \|(-1)^n (f^{-1} \circ \varphi)^n \circ f^{-1}\| \\
 & \leq \sum_{n \geq 2} \|f^{-1}\| \cdot (\|f^{-1}\| \cdot \|\varphi\|)^n \\
 & = \|\varphi\|^2 \left(\|f\|^3 \cdot \sum_{n \geq 2} (\|f^{-1}\| \cdot \|\varphi\|)^{n-2} \right) \\
 & = o(\|\varphi\|).
 \end{aligned}$$

Let

$$\begin{aligned}
 \Phi : \mathcal{L}(E, E)^3 & \longrightarrow \mathcal{L}(E, E) \\
 (\alpha, \beta, \gamma) & \longmapsto \alpha \circ \beta \circ \gamma.
 \end{aligned}$$

bounded 3-linear mapping.

$$D\iota(f)(\cdot) = -\Phi(\iota(f), \cdot, \iota(f)).$$

By induction, we obtain that ι is of class \mathcal{C}^n for any $n \in \mathbb{N}$. □

Definition 1.9.9 Let (X, d) be a metric space, $f : X \longrightarrow X$ be a mapping. If exists $\alpha \in]0, 1[$, such that f is α -Lipschitzian, we say that f is a **contraction**.

Definition 1.9.10 Let $f : X \longrightarrow X$ be a mapping. If $x \in X$ id such that $f(x) = x$, we say that x is a **fixed point** of f .

Theorem 1.9.11 (Banach fixed point theorem) Let (X, d) be a non-empty complete metric space and $f : X \longrightarrow X$ be a contraction. Then f admits a unique fixed point.

Proof

“Uniqueness”: Let $\alpha \in]0, 1[$, such that f is α -Lipschitzian. If a and b are fixed point of f , then $d(a, b) = d(f(a), f(b)) \leq \alpha d(a, b)$. So $d(a, b) = 0$, $a = b$.

“Existence”: Let $x_0 \in X$. For any $n \in \mathbb{N}$, let $x_n = f^n(x_0)$. Then

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \leq \alpha d(x_{n-1}, x_n) \leq \cdots \leq \alpha^n d(x_0, x_1).$$

So

$$\sum_{n \in \mathbb{N}} d(x_n, x_{n+1}) \leq \sum_{n \in \mathbb{N}} \alpha^n d(x_0, x_1) = \frac{1}{1 - \alpha} d(x_0, x_1) < +\infty.$$

Hence $(x_n)_{n \in \mathbb{N}}$ converges to some $a \in X$.

$$d(a, f(a)) = \lim_{n \rightarrow +\infty} d(x_n, f(x_n)) = \lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0.$$

So $a = f(a)$. □

Definition 1.9.12 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over a complete value field $(K, |\cdot|)$, $U \subseteq E$, $V \subseteq F$ be open subsets, $f : U \rightarrow V$ be a bijection, $n \in \mathbb{N} \cup \{\infty\}$. If f and f^{-1} are both of class \mathcal{C}^n , we say that f is a \mathcal{C}^n -diffeomorphism.

Theorem 1.9.13 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces over \mathbb{R} , $U \subseteq E$ open and $f : U \rightarrow F$ be a mapping of class \mathcal{C}^n ($n \in \mathbb{N} \cup \{\infty\}$). Let $p \in U$. Suppose that $Df(p) \in \mathcal{L}(E, F)$ is invertible. Then there exists a open neighborhood V of p contained in U , such that $f|_V : V \rightarrow f(V)$ is a \mathcal{C}^n -homeomorphism. Moreover

$$Df^{-1}(y) = Df(f^{-1}(y))^{-1}.$$

Proof By replacing f by

$$\tilde{f} : x \mapsto Df(p)^{-1} (f(p+x) - f(p)).$$

We may assume that $E = F$, $p = f(p) = 0$, $Df(p) = \text{Id}_E$.

$$D\tilde{f}(0)(h) = Df(p)^{-1}(Df(p)(h)) = h, D\tilde{f}(0) = \text{Id}_E.$$

Let $\mu : U \rightarrow E$, $\mu(x) = f(x) - x$, $D\mu(0) = 0$. Since Df is continuous, so is $D\mu$.

$$\exists r > 0, \forall x \in \overline{B}(0_E, r), \|D\mu(x)\| \leq \frac{1}{2}.$$

So μ is $\frac{1}{2}$ -Lipschitzian on $\overline{B}(0_E, r)$ (mean value inequality).

$$\forall (x, y) \in \overline{B}(0_E, r)^2, \|f(x) - f(y)\| \geq \|x - y\| \|\mu(x) - \mu(y)\| \geq \frac{1}{2} \|x - y\|.$$

So f is injective on $\overline{B}(0_E, r)$. Let $a \in \overline{B}(0_E, \frac{r}{2})$.

$$\forall x \in \overline{B}(0_E, r), \|a - \mu(x)\| \leq \|a\| + \|\mu(x)\| \leq \frac{1}{2}r + \frac{1}{2}r = r.$$

Let

$$\begin{aligned}\nu : \overline{B}(0, r) &\longrightarrow \overline{B}(0, r) \\ x &\longmapsto a - \mu(x)\end{aligned}$$

ν is a contraction. By Banach's fixed point theorem,

$$\exists! g(a) \in \overline{B}(0, r), \nu(g(a)) = a - \mu(g(a)) = a - f(g(a)).$$

That is $f(g(a)) = a$. Let $W = B(0, \frac{r}{2})$, $V = f^{-1}(W) \cap B(0, r)$, $f|_V : V \longrightarrow W$ is a bijection.

$$\forall z \in B(0, r), Df(z) = \text{Id}_E + D\mu(z) \in \text{GL}(E).$$

$$\forall (x, x_0) \in V \times V, y = f(x), y_0 = f(x_0), y - y_0 = Df(x_0)(x - x_0) + o(\|x - x_0\|).$$

$$\|x - x_0\| = \|y - y_0 - (\mu(x) - \mu(x_0))\| \leq \|y - y_0\| + \frac{1}{2}\|x - x_0\|,$$

$$\frac{1}{2}\|f^{-1}(y) - f^{-1}(y_0)\| = \frac{1}{2}\|x - x_0\| \leq \|y - y_0\|.$$

So,

$$Df(x_0)(x - x_0) = y - y_0 + o(\|y - y_0\|),$$

$$\begin{aligned}f^{-1}(y) - f^{-1}(y_0) &= x - x_0 = Df(x_0)^{-1}(y - y_0) + o(\|y - y_0\|) \\ &= Df(f^{-1}(y_0))^{-1}(y - y_0) + o(\|y - y_0\|)\end{aligned}$$

Thus,

$$Df^{-1} = \iota \circ Df \circ f^{-1}.$$

□

Proposition 1.9.14 Let $n \in \mathbb{N}_{\geq 1}$. Let $(K, |\cdot|)$ be a complete valued field, $(E_i, \|\cdot\|_i)$, $i \in \{1, \dots, n\}$ be normed vector spaces over K , $(F, \|\cdot\|_F)$ be a Banach space over K . Then, $(\mathcal{L}^{(n)}(E_1, \dots, E_n, F), \|\cdot\|)$ is a Banach space.

Proof Let $(\varphi_i)_{i \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^{(n)}(E_1, \dots, E_n, F)$. For $N \in \mathbb{N}$, let

$$\varepsilon_N := \sup_{(i,j) \in \mathbb{N}_{\geq N}} \|\varphi_i - \varphi_j\|, \quad \lim_{N \rightarrow +\infty} \varepsilon_N = 0.$$

For any $(x_1, \dots, x_n) \in E_1 \times \dots \times E_n$, and any $(i, j) \in \mathbb{N}_{\geq N}^2$,

$$\|\varphi_i(x_1, \dots, x_n) - \varphi_j(x_1, \dots, x_n)\| \leq \|\varphi_i - \varphi_j\| \cdot \prod_{l=1}^n \|x_l\| \leq \varepsilon_N \prod_{l=1}^n \|x_l\|.$$

So $(\varphi_i(x_1, \dots, x_n))_{i \in \mathbb{N}}$ is a Cauchy sequence in F , hence it converges to some element of F , denoted as $\varphi(x_1, \dots, x_n)$.

Note that φ is a point-wise limit of an n -linear mapping, so it is also n -linear.

$$\begin{aligned} \|\varphi(x_1, \dots, x_n)\|_F &= \lim_{i \rightarrow +\infty} \|\varphi_i(x_1, \dots, x_n)\|_F \\ &\leq \limsup_{i \rightarrow +\infty} \|\varphi_i\| \cdot \|x_1\|_{E_1} \cdots \|x_n\|_{E_n} \\ &\leq \left(\sup_{i \in \mathbb{N}} \|\varphi_i\| \right) \cdot \|x_1\|_{E_1} \cdots \|x_n\|_{E_n} \end{aligned}$$

So $\varphi \in \mathcal{L}(E_1, \dots, E_n, F)$.

For fixed $N \in \mathbb{N}$, $\forall (x_1, \dots, x_n) \in E_1 \times \dots \times E_n$,

$$\begin{aligned} &\|\varphi(x_1, \dots, x_n) - \varphi_N(x_1, \dots, x_n)\|_F \\ &= \lim_{n \rightarrow +\infty} \|\varphi_n(x_1, \dots, x_n) - \varphi_N(x_1, \dots, x_n)\|_F \\ &\leq \varepsilon_N \|x_1\| \cdots \|x_n\|. \end{aligned}$$

So $0 \leq \|\varphi - \varphi_N\| \leq \varepsilon_N$. By squeeze theorem

$$\lim_{N \rightarrow +\infty} \|\varphi - \varphi_N\| = 0.$$

□

1.10 Uniform Convergence

Definition 1.10.1 Let X be a set, (Y, \mathcal{T}) be a topological space, $(f_n)_{n \in \mathbb{N}}$ be a sequence of mappings from X to Y . We say that the sequence $(f_n)_{n \in \mathbb{N}}$ converges **point-wise** to a mapping $f : X \rightarrow Y$ if for every $x \in X$ the sequence $(f_n(x))_{n \in \mathbb{N}}$ converges to $f(x)$.

Suppose that (Y, d) is a metric space. We say that the sequence $(f_n)_{n \in \mathbb{N}}$ converges **uniformly** to a mapping $f : X \rightarrow Y$ if

$$\lim_{n \rightarrow +\infty} \sup_{x \in X} d(f_n(x), f(x)) = 0.$$

Remark 1.10.2 Let $f, g : X \rightarrow Y$ be mappings.

$$d(f, g) = \sup_{x \in X} d(f(x), g(x))$$

is a metric. Uniform convergence can be seen as convergence of $(f_n)_{n \in \mathbb{N}}$ with respect to this metric.

Theorem 1.10.3 Let (X, \mathcal{T}_X) be a topological space, (Y, d_Y) be a metric space, $(f_n)_{n \in \mathbb{N}}$ be a sequence of mappings from X to Y that converges uniformly to a mapping $f : X \rightarrow Y$. If $\forall n \in \mathbb{N}$, f_n is continuous at $p \in X$, then f is continuous at p .

Proof We will prove that for any $\varepsilon > 0$, $f^{-1}(B(f(p), \varepsilon))$ is a neighborhood of p .

Let $n \in \mathbb{N}$ such that

$$\sup_{x \in X} d(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

We claim that

$$f_n^{-1}(B(f_n(x), \frac{\varepsilon}{3})) \subseteq f^{-1}(B(f(x), \varepsilon)).$$

Let x be a element of X such that

$$d(f_n(x), f_n(p)) < \frac{\varepsilon}{3}.$$

One has

$$d(f(x), f(p)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) < \varepsilon.$$

□

Theorem 1.10.4 Let (X, d_X) and (Y, d_Y) be two metric spaces, $(f_n)_{n \in \mathbb{N}}$ sequence of uniformly continuous mappings from X to Y . Suppose that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to $f : X \rightarrow Y$. Then f is uniformly continuous.

Proof Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that

$$\sup_{x \in X} d(f_n(x), f(x)) < \frac{\varepsilon}{3}.$$

f_n is uniformly continuous, so there exists $\delta > 0$

$$\forall (x, y) \in X \times X, d(x, y) < \delta \Rightarrow d(f_n(x), f_n(y)) < \frac{\varepsilon}{3}.$$

Therefore, for any $(x, y) \in X \times X$ such that $d(x, y) < \delta$,

$$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) < \varepsilon.$$

So f is uniformly continuous. □

Theorem 1.10.5 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be normed vector spaces over a complete non-trivially valued field $(K, |\cdot|)$. Let $U \subseteq E$ open, $(f_n)_{n \in \mathbb{N}}$ a sequence of differentiable mappings from U to F . Let $f : U \rightarrow F$, $g : U \rightarrow \mathcal{L}(E, F)$ be mappings, $p \in U$. Suppose that

- (1) The sequence $(Df_n)_{n \in \mathbb{N}}$ converges uniformly to g .
- (2) $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f .
- (3) There exists $N \in \mathbb{N}$ and mapping $\delta : U \rightarrow \mathbb{R}_{\geq 0}$ such that $\lim_{x \rightarrow p} \delta(x) = 0$ and for any $n \in \mathbb{N}_{\geq N}$, any $x \in U$,

$$\|f_n(x) - f_n(p) - Df_n(p)(x - p)\|_F \leq \delta(x)\|x - p\|_E.$$

Then f is differentiable and $Df = g$.

Proof For any $n \in \mathbb{N}$, define

$$\varepsilon_n := \sup_{x \in U} \|Df_n(x) - g(x)\|, \quad d_n := \sup_{x \in U} \|f_n(x) - f(x)\|.$$

One has

$$\begin{aligned} \|f(x) - f(p) - g(p)(x - p)\| &\leq \|(f(x) - f_n(x)) - (f(p) - f_n(p))\| \\ &\quad + \|f_n(x) - f_n(p) - Df_n(p)(x - p)\| \\ &\quad + \|Df_n(p)(x - p) - g(p)(x - p)\| \\ &\leq 2d_n + \delta(x)\|x - p\|_E + \varepsilon_n\|x - p\|_E. \end{aligned}$$

for sufficiently large n .

Therefore,

$$\limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x - p)\|_F}{\|x - p\|_E} \leq 2\varepsilon_n.$$

Taking the limit when $n \rightarrow \infty$, we obtain

$$\limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x - p)\|_F}{\|x - p\|_E} = 0.$$

Namely,

$$f(x) - f(p) - g(p)(x - p) = o(\|x - p\|_E), \quad x \rightarrow p.$$

□

Theorem 1.10.6 Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces over \mathbb{R} , $U \subseteq E$ open, $(f_n)_{n \in \mathbb{N}}$ be sequence of differentiable mappings from U to F , $g : U \rightarrow \mathcal{L}(E, F)$. We suppose that

- (1) $(Df_n)_{n \in \mathbb{N}}$ converges uniformly to g .
- (2) $(f_n)_{n \in \mathbb{N}}$ converges point-wise to $f : U \rightarrow F$.

Then f is differentiable and $Df = g$.

Proof Let $p \in U$, for any $(n, m) \in \mathbb{N} \times \mathbb{N}$, for any $n \in \mathbb{N}$,

$$c_{m,n} := \sup_{x \in U} \|Df_m(x) - Df_n(x)\|, \quad \varepsilon_n := \sup_{x \in U} \|Df_n(x) - g(x)\|.$$

For $r > 0$, $B(p, r) \subseteq U$ by the mean value inequality,

$$\|(f_n(x) - f_m(x)) - (f_n(p) - f_m(p))\| \leq c_{m,n}\|x - p\|, \quad x \in B(p, r).$$

Passing to the limit when $m \rightarrow +\infty$, we obtain

$$\|(f_n(x) - f(x)) - (f_n(p) - f(p))\| \leq \varepsilon_n\|x - p\|.$$

we have

$$\begin{aligned} \|f(x) - f(p) - g(p)(x - p)\| &\leq \|(f(x) - f_n(x)) - (f(p) - f_n(p))\| \\ &\quad + \|f_n(x) - f_n(p) - Df_n(p)(x - p)\| \\ &\quad + \|Df_n(p)(x - p) - g(p)(x - p)\|. \end{aligned}$$

$$\limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x - p)\|_F}{\|x - p\|_E} \leq 3\varepsilon_n.$$

Taking the limit $n \rightarrow +\infty$

$$\lim_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x - p)\|_F}{\|x - p\|_E} = 0.$$

Namely,

$$f(x) - f(p) - g(p)(x - p) = o(\|x - p\|_E), \quad x \rightarrow p.$$

□

Proposition 1.10.7 Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be normed vector spaces over \mathbb{R} . Assume that $(F, \|\cdot\|_F)$ is a Banach space, $U \subseteq E$ be a path connected open, $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable mappings from U to F . Suppose that

- (1) $(Df_n)_{n \in \mathbb{N}}$ converges uniformly to $g : U \rightarrow \mathcal{L}(E, F)$.
- (2) There exists $p \in U$ such that $(f_n(p))_{n \in \mathbb{N}}$ converges.

Then the sequence $(f_n)_{n \in \mathbb{N}}$ converges point-wise on U to a differentiable mapping $f : U \rightarrow F$ such that $Df = g$.

Proof We first treat the case where U is convex.

For any $(n, m) \in \mathbb{N} \times \mathbb{N}$, let

$$c_{m,n} := \sup_{x \in U} \|Df_m(x) - Df_n(x)\|.$$

Let $x \in U$. By the mean value inequality,

$$\|(f_n(x) - f_m(x)) - (f_n(p) - f_m(p))\| \leq c_{m,n}\|x - p\|,$$

which leads to

$$\|f_n(x) - f_m(x)\|_F \leq \|f_n(p) - f_m(p)\|_F + c_{m,n}\|x - p\|_E.$$

Therefore $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in F (Banach space), so $f_n(x)$ converges in F to some $f(x)$. Now it suffices to use the theorem 1.10.6.

We will now treat the general case. Let $x \in U$. There exists $\gamma : [0, 1] \rightarrow U$ continuous such that $\gamma(0) = p, \gamma(1) = x$. Let I be the set of $t \in [0, 1]$ such that $f_n(\gamma(s))$ converges for all $s \in [0, t]$. By definition, I is an interval in $[0, 1]$ and $0 \in I$. Therefore, it is of the form $[0, c]$ or $[0, c[$.

Let $B(\gamma(c), r) \subseteq U$. Since γ is continuous, $\gamma^{-1}(B(\gamma(c), r))$ is open in $[0, 1]$ and $c \in \gamma^{-1}(B(\gamma(c), r))$. Assume by contradiction that $I = [0, c[$, then $I \cap \gamma^{-1}(B(\gamma(c), r)) \neq \emptyset$. There exists $q \in \gamma^{-1}(B(\gamma(c), r)) \cap I$ such that $f_n(q)$ converges. So from the “convex U version” f converges point-wise on $B(\gamma(c), r)$. So $f_n(\gamma(c))$ converges. Contradiction. We deduce that $I = [0, c]$.

If $c \neq 1$, then c is an adherent point of $]c, 1]$. $\gamma^{-1}(B(\gamma(c), r))$ open, so there exists $r' > 0$ such that $B(c, r') \subseteq \gamma^{-1}(B(\gamma(c), r))$. In particular, $B(c, r') \cap]c, 1]$ is an open interval in $[0, 1]$ that continuous. So $I \supseteq]0, c + r']$. Contradiction. Therefore $c = 1$. \square

Definition 1.10.8 Let U be a set and $(F, \|\cdot\|)$ be a Banach space over complete valued field $(K, |\cdot|)$. $(f_n)_{n \in \mathbb{N}} \in (F^U)^{\mathbb{N}}$ be a sequence of mappings from U to F . If

$$\sum_{n \in \mathbb{N}} \sup_{p \in U} \|f_n(p)\|_F < +\infty,$$

then we say that $\sum_{n \in \mathbb{N}} f_n$ **converges normally**.

Proposition 1.10.9 If $\sum f_n$ converges normally, then it converges uniformly.

Proof For any $n \in \mathbb{N}$, let $g_n = \sum_{k=0}^n f_k$. We need to check that the sequence $(g_n)_{n \in \mathbb{N}}$ converges uniformly. For any $x \in U$, $\sum_{n \in \mathbb{N}} \|f_n(x)\| < +\infty$. So $\sum_{n \in \mathbb{N}} f_n(x)$ converges absolutely. In particular, $(g_n(x))_{n \in \mathbb{N}}$ converges to some $g(x)$.

$$\begin{aligned} \|g_n(x) - g(x)\|_F &= \lim_{m \rightarrow +\infty} \|g_n(x) - g_m(x)\|_F \\ &\leq \lim_{m \rightarrow +\infty} \|f_{n+1}(x) + \cdots + f_m(x)\|_F \\ &\leq \limsup_{m \rightarrow +\infty} \sum_{k \geq n+1} \|f_k(x)\|_F \\ &\leq \varepsilon_n. \end{aligned}$$

Let

$$\varepsilon_n = \sum_{k \geq n+1} \sup_{p \in U} \|f_k(p)\|_F, \quad \lim_{n \rightarrow +\infty} \varepsilon_n = 0.$$

So,

$$\limsup_{n \rightarrow +\infty} \left(\sup_{x \in U} \|g_n(x) - g(x)\|_F \right) = 0,$$

namely, $(g_n)_{n \in \mathbb{N}}$ converges to g . □

Proposition 1.10.10 Let $(K, |\cdot|)$ be a complete valued field which is non-trivially valued, $(E, \|\cdot\|)$ be a normed vector space and $(F, \|\cdot\|_F)$ be a Banach space over K . $U \subseteq E$ be an open subset, $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable mappings $U \rightarrow F$ and $p \in U$. Assume that

- (1) $\sum_{n \in \mathbb{N}} f_n$ converges normally (uniformly suffices).
- (2) $\sum_{n \in \mathbb{N}} Df_n$ converges normally (uniformly suffices).
- (3) $\exists N \in \mathbb{N}$ and mappings $(\delta_n : U \rightarrow \mathbb{R}_{\geq 0})_{n \in \mathbb{N}_{\geq N}}$ such that

- 1. $\forall n \in \mathbb{N}_{\geq N}, \lim_{x \rightarrow p} \delta_n(x) = \delta_n(p) = 0$.
- 2. $\sum_{n \in \mathbb{N}} \delta_n$ converges normally (uniformly suffices).
- 3. $\forall n \in \mathbb{N}_{\geq N}, \forall x \in U$,

$$\|f_n(x) - f_n(p) - Df_n(p)(x - p)\|_F \leq \delta_n(x) \|x - p\|_E.$$

Let f and g be limits of $\sum_{n \in \mathbb{N}} f_n$ and $\sum_{n \in \mathbb{N}} Df_n$ respectively. Then f is differentiable at p and $Df = g$.

Proposition 1.10.11 Let $(E, \|\cdot\|_E)$ be a normed vector space and $(F, \|\cdot\|_F)$ be a Banach space over \mathbb{R} . Let $U \subseteq E$ open, and $(f_n : U \rightarrow F)_{n \in \mathbb{N}}$ be a sequence of mappings $U \rightarrow F$. Suppose that

- (1) $\sum_{n \in \mathbb{N}} Df_n$ converges normally (uniformly suffices) to some $g : U \rightarrow \mathcal{L}(E, F)$.
- (2) $\sum_{n \in \mathbb{N}} f_n$ converges point-wise to some $f : U \rightarrow F$.

Then f is differentiable on U and $Df = g$.

Remark 1.10.12 If U is path connected, one can replace (2) by (2'): $\exists p \in U$, $\sum_{n \in \mathbb{N}} f_n(p)$ converges.

1.11 Power Series

We fix a complete non-trivially valued field $(K, |\cdot|)$, and let $(E, \|\cdot\|_E)$ be a Banach space over K .

Definition 1.11.1 Let $(S_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$ and $b \in K$. We call power series **centered at b** with coefficients $(s_n)_{n \in \mathbb{N}}$ the sequence of polynomial mappings.

$$\left((z \in K) \mapsto \sum_{l=0}^n (z - b)^l s_l \right)_{n \in \mathbb{N}}$$

denoted as

$$\sum_{l=0}^n (z - b)^l s_l.$$

If $S = \sum_{n \in \mathbb{N}} (z - b)^n s_n$, we denote by $R(S)$ the element

$$\left(\limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}} \right)^{-1} \in [0, +\infty]$$

called the **convergence radius** of S . ($0^+ := +\infty$, $(+\infty)^{-1} := 0$)

Proposition 1.11.2 Let $S = \sum_{n \in \mathbb{N}} (z - b)^n s_n$.

- (1) $\forall a \in K$, if $|a - b| < R(S)$, then $S(a) := \sum_{n \in \mathbb{N}} (a - b)^n s_n$ converges absolutely.
- (2) If $r > 0$ such that $(r^n \|s_n\|)_{n \in \mathbb{N}}$ is bounded, then $R(S) \geq r$.
- (3) If $a \in K$ is such that $|a - b| > R(S)$, then $\sum_{n \in \mathbb{N}} (a - b)^n s_n$ diverges.

Proof

(1)

$$\|(a - b)^n s\|^{\frac{1}{n}} = (|(a - b)^n| \cdot \|s_n\|)^{\frac{1}{n}} = |a - b| \cdot \|s_n\|^{\frac{1}{n}}.$$

$$\limsup_{n \rightarrow +\infty} \|(a - b)^n s_n\|^{\frac{1}{n}} = |a - b| \cdot \limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}}.$$

If $|a - b| < R(S)$, then $|a - b| \cdot \limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}} < 1$. By the root test of Cauchy, $\sum_{n \in \mathbb{N}} (a - b)^n s_n$ converges absolutely.

(2)

$$\|s_n\|^{\frac{1}{n}} = \frac{1}{r} (r^n \|s_n\|)^{\frac{1}{n}}.$$

Since $(r^n \|s_n\|)_{n \in \mathbb{N}}$ is bounded,

$$\limsup_{n \rightarrow +\infty} (r^n \|s_n\|)^{\frac{1}{n}} \leq 1.$$

So $\limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}} \leq \frac{1}{r}$. So $R(S) \geq r$.

(3) If $|a - b| > R(S)$, then

$$\limsup_{n \rightarrow +\infty} \|(a - b)^n s_n\|^{\frac{1}{n}} = |a - b| \cdot \limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}} > 1.$$

So $\sum_{n \in \mathbb{N}} (a - b)^n s_n$ diverges. □

Proposition 1.11.3 Let $S = \sum_{n \in \mathbb{N}} (z - b)^n s_n$ be a power series.

(1) $\forall r \in \mathbb{R}_{\geq 0}$ such that $r < R(S)$. the series S converges normally on $\overline{B}(b, r)$.

(2) $(a \in B(b, R(S))) \mapsto S(a) := \sum_{n \in \mathbb{N}} (a - b)^n s_n$ is continuous.

Proof

(1) $\forall a \in \overline{B}(b, r)$,

$$\sum_{n \in \mathbb{N}} \|(a - b)^n s_n\| \leq \sum_{n \in \mathbb{N}} r^n \|s_n\| < +\infty$$

since $\limsup_{n \rightarrow +\infty} r \cdot \|s_n\|^{\frac{1}{n}} < 1$.

(2) $a \mapsto S(a)$ is continuous on any $B(b, r)$, $r < R(S)$. Since

$$B(b, R(S)) = \bigcup_{r < R(S)} B(b, r),$$

S is continuous on $B(b, R(S))$. □

Definition 1.11.4 Let $S = \sum_{n \in \mathbb{N}} (z - b)^n s_n$. We define the formal derivative of S as

$$\sum_{n \in \mathbb{N}_{\geq 1}} (z - b)^{n-1} (n s_n).$$

Proposition 1.11.5 Let $S = \sum_{n \in \mathbb{N}} (z - b)^n s_n$ be a formal power series. Let $P \in K[T]$. For any $n \in \mathbb{N}$, let $P(n) := P(n1_K) \in K$. Let

$$S_p := \sum_{n \in \mathbb{N}} (z - b)^n (P(n) s_n).$$

Then $R(S_p) \geq R(S)$.

Proof We assume that $P \neq 0$, $P(T)$ is of the form

$$C_d T^d + C_{d-1} T^{d-1} + \cdots + C_1 T + C_0, \quad c_d \neq 0.$$

$$|P(n)| = \mathcal{O}(n^d) = o(r^n), \text{ for any } r > 1.$$

Hence, $\exists N \in \mathbb{N}$ such that $|P(n)| \leq r^n, \forall n \in \mathbb{N}_{\geq N}$.

$$\limsup_{n \rightarrow +\infty} \|P(n)s_n\|^{\frac{1}{n}} \leq r \cdot \limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}}.$$

Taking the limit when $r \rightarrow 1$, get

$$\limsup_{n \rightarrow +\infty} \|P(n)s_n\|^{\frac{1}{n}} \leq \limsup_{n \rightarrow +\infty} \|s_n\|^{\frac{1}{n}}.$$

$$R(S_p) \geq R(S).$$

□

Lemma 1.11.6 Let $(z_0, z) \in K^2, n \in \mathbb{N}_{\geq 1}$.

$$z^n - z_0^n - n z_0^{n-1}(z - z_0) = (z - z_0)^2 \sum_{j=0}^{n-2} (n - j - 1) z^j z_0^{n-2-j}.$$

Proof

$$\begin{aligned} z^n - z_0^n &= (z - z_0) \sum_{i=0}^{n-1} z^i z_0^{n-1-i}. \\ z^n - z_0^n - n z_0^{n-1}(z - z_0) &= (z - z_0) \sum_{i=0}^{n-1} (z^i z_0^{n-1-i} - z_0^{n-1}) \\ &= (z - z_0) \sum_{i=0}^{n-1} z_0^{n-i-1} (z^i - z_0^i) \\ &= (z - z_0)^2 \sum_{i=1}^{n-1} z_0^{n-1-i} \sum_{j=0}^{i-1} z^j z_0^{i-j-1} \\ &= (z - z_0)^2 \sum_{j=0}^{n-2} \sum_{i=j+1}^{n-1} z^j z_0^{n-2-j} \\ &= (z - z_0)^2 \sum_{j=0}^{n-2} (n - j - 1) z^j z_0^{n-2-j}. \end{aligned}$$

□