Quantum Mechanics 2025 HW5

Due 10/21 in Class

October 14, 2025

Problem 1. Coherent states

Use the fact that $a|\alpha\rangle = \alpha|\alpha\rangle$. Prove that for the coherent state $|\alpha\rangle$, define $\overline{\Delta x^2} = \langle\alpha|x^2|\alpha\rangle - (\langle\alpha|x^2|\alpha\rangle)$, and $\overline{\Delta p^2} = \langle\alpha|p^2|\alpha\rangle - (\langle\alpha|p^2|\alpha\rangle)$. Prove that $\sqrt{\overline{\Delta x^2}}\sqrt{\overline{\Delta p^2}} = \hbar/2$. Explain why coherent states are call the most classical quantum state.

Problem 2. Wavefunctions of Harmonic Oscillator

In class we have derived that the ground state wavefunction of a harmonic oscillator satisfies

$$a\psi_0(x) = 0. \tag{1}$$

Start from here, please derive the normalized expression of $\psi_0(x)$. Then according to $|n\rangle = (a^\dagger)^n/\sqrt{n!}|0\rangle$, derive the normalized expression of the n-th excited wavefunction $\psi_n(x)$ for $n = 1, 2$.

Problem 3. High dimensional Oscillator

1) Consider the D-dimensional harmonic oscillators

$$H = -\sum_{i=1}^{D} \tfrac{\hbar^2\nabla_i^2}{2m} + \tfrac{1}{2}m\omega^2 \sum_{i=1}^{D} x_i^2. \tag{2}$$

For each dimension, we define the creation and annihilation operators a_i and $a_i^\dagger$. Define $a = (a_1, a_2, ..., a_D)^T$. Show that for an arbitrary special unitary transformation U satisfying $U^\dagger U = 1$ and $\det U = 1$, the Hamiltonian is invariant under the transformation $a' = Ua$.

2) This transformation is called the $SU(D)$ transformation. It can be viewed as a rotation in the D-dimensional complex space. If the rotation angle is small, we can write down $U = 1 - iA\theta$ where A is a $D \times D$ Hermitian matrix and $\mathrm{tr}A = 0$. Prove that $Q_{ij} = a_i^\dagger A_{ij} a_j$ communtes with H.

2) For the case $D = 2$, we have found the three bases for A, which are the Pauli matrices. Write down the corresponding conserved quantities. Please find the bases for A for $D = 3$, and write down the corresponding conserved quantities.

Problem 4. Quantum Virial Theorem

1) Consider a particle in the 3D whose Hamiltonian is given by

$$H = p^2/2m + V(x). \tag{3}$$

By calculating $[\mathbf{x} \cdot \mathbf{p}, H]$, prove that

$$\tfrac{d}{dt}\langle \mathbf{x} \cdot \mathbf{p} \rangle = \langle p^2/m \rangle - \langle \mathbf{x} \cdot \nabla V \rangle. \tag{4}$$

To identify the relation with the classic Virial theorem, we eed the left-hand-side vanish. Under what condition does it vanish?

2) Feynman-Hellman theorem: Consider a Hamiltonian $H(\lambda)$ with a parameter λ, whose normalized eigen-state $\psi_{n,\lambda}$ satisfies

$$H(\lambda)\psi_{n,\lambda} = E_n(\lambda)\psi_{n,\lambda}. \tag{5}$$

Prove that

$$\tfrac{\partial E_n(\lambda)}{\partial \lambda} = \langle \psi_{n\lambda} | \tfrac{\partial H}{\partial \lambda} | \psi_{n\lambda} \rangle. \tag{6}$$

3) Another way to prove Virial theorem is by Feynman-Hellman theorem. Define $H(\lambda)$ as the result of performing the a scaling transformation $x \to x' = \lambda x$ to $H(x)$. Write down the form of $H(\lambda)$. Then use the Feynman-Hellman theorem to prove the Virial theorem.

4) Apply the quantum Viral theorem to the harmonic oscillator. What is your conclusion?

Problem 5. Operator normal product

Consider $A = \alpha a + \alpha' a^\dagger$, and $B = \beta a + \beta' a^\dagger$. Define the normal product of e^A as

$$: e^A :=: e^{\alpha a + \alpha' a^\dagger} := e^{\alpha' a^\dagger} e^{\alpha a}. \tag{7}$$

Prove that $\langle 0| : e^A : |0\rangle = 1$ where $|0\rangle$ is the ground state associated with $H = \omega a^\dagger a$. Prove that

$$: e^A :: e^B :=: e^{A+B} : e^{\langle 0|AB|0\rangle} \tag{8}$$

$$e^A e^B =: e^{A+B} : e^{\langle 0|AB + \frac{A^2}{2} + \frac{B^2}{2}|0\rangle}. \tag{9}$$

Then calculate $\langle 0| e^A e^B |0\rangle$.