Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 9–5 : Differentiability : series and uniform convergence

1. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent. Show that $\sum_{n=1}^{\infty} \sqrt{a_n}/n^{\alpha}$ is also convergent when $\alpha > 1/2$. Is the statement valid when $\alpha = 1/2$?

2. Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers. Sow that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ either converge or diverge at the same time.

3. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(a) If $\{a_n\}_{n=1}^{\infty}$ is decreasing, show that $\lim_{n\to\infty} na_n = 0$.

(b) Give a counter example of (a) if the monotonicity assumption is removed.

4. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive reals. Denote $S_n = \sum_{k=1}^{n} a_k$ and $R_n = \sum_{k=n}^{\infty} a_k$.

(a) Suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Show that $\sum_{n=1}^{\infty} a_n/S_n^{\alpha}$ converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.

(b) Suppose that $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} a_n/R_n^{\alpha}$ diverges if $\alpha \geq 1$ and converges if $\alpha < 1$.

5. Let $\sum_{n=1}^{\infty} a_n$ be a positive convergent series. Denote $b_n = \left(\prod_{k=1}^{n} a_k\right)^{1/n}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent.

6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive reals. Suppose that $\sum_{n=1}^{\infty} 1/a_n$ is convergent. Show the convergence of the following series

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + \cdots + a_n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^2 a_n}{(a_1 + \cdots + a_n)^2}\,.$$

7. Cesáro summation. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Define $S_n = \sum_{k=1}^{n} a_k$. We say that $\{a_n\}_{n=1}^{\infty}$ is Cesáro summable, with Cesáro sum A, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} S_k = A.$$

(a) Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. Show that $\{a_n\}_{n=1}^{\infty}$ is Cesáro summable and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} S_k = \sum_{n=1}^{\infty} a_n.$$

(b) Is the sequence $1, -1, 1, -1, \cdots$ Cesáro summable?

8. Let $\sum_{n=1}^{\infty} a_n$ be a convergent positive series. Show that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} k a_k = 0.$$

9. Let $\{a_n\}_{n=1}^{\infty}$ be a real sequence such that $a_n \searrow 0$. Show that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ either converge or diverge at the same time. If they converge, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n(a_n - a_{n+1}).$$

10. Let $\{b_n\}_{n=1}^{\infty}$ be a decreasing sequence and $\lim_{n\to\infty} b_n = 0$. Suppose that $\sum_{n=1}^{\infty} a_n b_n$ is convergent. Define $S_n = \sum_{k=1}^{n} a_k$. Show that

(a) $\lim_{n\to\infty} S_n \cdot b_n = 0$.

(b) $\sum_{n=1}^{\infty} S_n(b_n - b_{n+1})$ is convergent and

$$\sum_{n=1}^{\infty} S_n(b_n - b_{n+1}) = \sum_{n=1}^{\infty} a_n b_n.$$

11. Suppose that $\sum_{n=1}^{\infty} a_n$ converges and that $\sum_{n=1}^{\infty}(b_{n+1}-b_n)$ is absolutely convergent. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

12. **(Riemann rearrangement theorem)** Let $\sum_{n=1}^{\infty} u_n$ be a conditional convergent series. We denote by v_n (resp. w_n) the n-th nonnegative (resp. negative) term of $\{u_n\}_{n=1}^{\infty}$. We define $V_n = \sum_{k=1}^{n} v_k$ and $W_n = \sum_{k=1}^{n} w_k$.

(a) Show that $\lim\limits_{n\to\infty} V_n = +\infty$ and $\lim\limits_{n\to\infty} W_n = -\infty$.

(b) Let $\ell \in \mathbb{R}$. Show, by induction, that there exist two strictly increasing sequences of integers $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=0}^{\infty}$ such that $\ell < V_{m_0}$ and for all $k \geq 0$ that

$$\begin{aligned} W_{n_k} + V_{m_{k+1}} - v_{m_{k+1}} \leq \ell < W_{n_k} + V_{m_{k+1}}, \\ W_{n_k} + V_{m_k} < \ell \leq W_{n_k} + V_{m_k} - w_{n_k}. \end{aligned}$$

Deduce that there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} u_{\sigma(n)} = \ell$.

(c) Show, by induction, that there exist two strictly increasing sequences of integers $\{p_k\}_{k=0}^{\infty}$ and $\{q_k\}_{k=0}^{\infty}$ such that $V_{p_0} > 0$ and for all $k \geq 1$ that

$$\begin{aligned} W_{q_k} + V_{p_{k+1}} - v_{p_{k+1}} \leq k < W_{q_k} + V_{p_{k+1}}, \\ W_{q_k} + V_{p_k} < k - 1 \leq W_{q_k} + V_{p_k} - w_{p_k}. \end{aligned}$$

Deduce that there exists a bijection $\nu : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} u_{\nu(n)}$ diverges.

(d) Let $z_n = \text{ß}^n/n$ for $n \geq 1$. Show that the complex series $\sum_{n=1}^{\infty} z_n$ is conditional convergent. Show that for any complex number $z \in \mathbb{C}$ there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} z_{\varphi(n)}$ converges to z.

13. By rearranging the terms of the series

$$S_{1,1} := \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots = 1 - \log 2$$

we obtain

$$S_{p,q} := \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p} - \frac{1}{3} - \frac{1}{5} - \cdots - \frac{1}{2q+1} + \frac{1}{2p+2} + \cdots$$

in which blocks of p positive terms alternate with blocks of q negative terms. Show that

$$S_{p,q} - S_{1,1} = \frac{1}{2} \log \frac{p}{q}.$$

14. Let $\{a_n\}_{n=1}^{\infty}$ be positive sequence such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n$ diverges. Denote $S_n = \sum_{k=1}^{n} a_k$. Show that $\{S_n - \lfloor S_n \rfloor\}_{n=1}^{\infty}$ is dense in $[0, 1]$.

15. Let $\sum_{n=1}^{\infty} u_n$ be a convergent series of positive terms such that

$$u_n \leq \sum_{k=n+1}^{\infty} u_k, \quad \forall\, n \in \mathbb{N}.$$

Put

$$r_n = \sum_{k=n}^{\infty} u_k.$$

Show that for any $x \in (0, r_0]$ there exists a strictly increasing sequence $\{n_k\}_{k=0}^{\infty} \subset \mathbb{N}$ such that

$$u_{n_0} + u_{n_1} + \cdots + u_{n_{k-1}} + r_{n_k} \geq x > u_{n_0} + u_{n_1} + \cdots + u_{n_{k-1}} + r_{n_k+1}.$$

Deduce that

$$\sum_{k=0}^{\infty} u_{n_k} = x.$$

16. Let $\sum_{n=1}^{\infty} u_n$ be a convergent series of nonnegative terms. Let $R_n = \sum_{k=n}^{\infty} u_k$ be the remainder of order n. Assume that there exists a constant $C > 0$ such that $R_n \leq C u_n$ for all $n \in \mathbb{N}$. Show that there exist $a \in (0,1)$ and $M > 0$ such that $u_n \leq M a^n$ for all $n \in \mathbb{N}$.

17. The **Betrand series** is defined as follows

$$\sum_{n=2}^{\infty} \frac{1}{n^{\alpha} \log^{\beta} n}.$$

When is this series convergent ?

18. For any $k \in \mathbb{Z}_+$ and $x > 0$, we define $\log^{(k)} x = \log(|\log^{(k-1)} x|)$ with $\log^{(1)} x = \log x$. Show that the following series is convergent

$$\sum_{n=2}^{\infty} \frac{1}{n \log^{(1)} n \cdots \log^{(k-1)} n (\log^{(k)} n)^2}.$$

19. **Kummer's test :** Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two positive sequences.

(a) Show that $\sum_{n=1}^{\infty} a_n$ is convergent if

$$\liminf_{n\to\infty} \left(b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} \right) > 0.$$

(b) Suppose that $\sum_{n=1}^{\infty} 1/b_n$ diverges and

$$\limsup_{n\to\infty} \left(b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} \right) < 0.$$

Show that $\sum_{n=1}^{\infty} a_n$ diverges.

20. **Raabe's test :** Let $\{u_n\}_{n=1}^{\infty}$ be a positive sequence. Put

$$\alpha_n = n\left(1 - \frac{u_{n+1}}{u_n}\right), \quad n \geq 1.$$

We propose to show the following

(a) If $\ell = \liminf \alpha_n > 1$, then $\sum_{n=1}^{\infty} u_n$ converges.

(b) If $L = \limsup \alpha_n < 1$, then $\sum_{n=1}^{\infty} u_n$ diverges.

(a) Let $\beta \in \mathbb{R}$ and $v_n = n^{-\beta}$. Show that

$$\frac{v_{n+1}}{v_n} = 1 - \frac{\beta}{n} + o\left(\frac{1}{n}\right) \quad \text{as} \quad n \to \infty.$$

(b) Show that if $\beta < \ell$, there exists a positive constant A such that

$$u_n \leq \frac{A}{n^\beta}, \quad n \geq 1;$$

and if $\beta > L$, there exists a positive constant B such that

$$u_n \geq \frac{B}{n^\beta}, \quad n \geq 1.$$

(c) Deduce the desired results.

(d) Application : Let $a, b, \lambda \in \mathbb{R}$ with a, b not negative integers. Determine the nature of the series of general term given by

$$u_n = \left[\frac{(a+1)(a+2)\cdots(a+n)}{(a+1)(b+2)\cdots(b+n)}\right]^\lambda .$$

21. Betrand's test : Let $\{u_n\}_{n=1}^\infty$ be a positive sequence. Assume that

$$\frac{u_{n+1}}{u_n} = 1 - \frac{1}{n} - \frac{\alpha_n}{n\log n}.$$

Show that

(a) If $\liminf \alpha_n > 1$, the series $\sum_{n=1}^\infty u_n$ converges ;

(b) If $\limsup \alpha_n < 1$, the series $\sum_{n=1}^\infty u_n$ diverges.

State a similar criterion when

$$\frac{u_{n+1}}{u_n} = 1 - \frac{1}{n} - \frac{1}{n\log n} - \frac{\alpha_n}{n\log\log n}.$$

Extend the results.

22. Let $\sum_{n=1}^\infty a_n$ be a complex series. Suppose that

$$\frac{a_{n+1}}{a_n} = 1 + \frac{A}{n} + \frac{B_n}{n^2},$$

where $\mathrm{Re}(A) < -1$ and B_n is bounded. Show that $\sum_{n=1}^\infty |a_n|$ is convergent.

23. Let $\sum_{n=1}^\infty a_n$ be a real series such that $\lim_{n\to\infty} a_n = 0$ and

$$-1 \leq \frac{a_{n+1}}{a_n} < r, \quad \text{where } r \in (0,1).$$

Show that $\sum_{n=1}^\infty a_n$ is convergent.

24. Let $\{u_n\}_{n=1}^\infty$ be a positive sequence. Show that

(a) $\limsup_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)^n < \frac{1}{e}$, the series $\sum_{n=1}^{\infty} u_n$ converges;

(b) $\liminf_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right)^n > \frac{1}{e}$, the series $\sum_{n=1}^{\infty} u_n$ diverges.

25. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers.

(a) Suppose there is $\alpha \in (0,1)$ such that $a_n^{1/n} \le 1 - n^{-\alpha}$ for all $n \in \mathbb{Z}_+$. Is $\sum_{n=1}^{\infty} a_n$ convergent?

(b) Suppose there is $\beta > 1$ such that $n(a_n/a_{n+1} - 1) \ge \beta$ for all $n \in \mathbb{Z}_+$. Is $\sum_{n=1}^{\infty} a_n$ convergent?

26. Let $\{a_n\}_{n=1}^{\infty}$ be a positive sequence such that

$$a_n^{1/n} < 1 - \frac{1}{n} - \frac{\log\log n}{\log n}, \quad n \ge N.$$

Is $\sum_{n=1}^{\infty} a_n$ convergent?

27. Let $\{a_n\}_{n=1}^{\infty}$ be a positive sequence such that $\lim_{n\to\infty} na_n = 0$ and

$$\frac{a_{n+1}}{a_n} = \frac{n+a}{n+b}, \quad a, b \in \mathbb{R}.$$

(a) Show that $\sum_{n=1}^{\infty} a_n$ is convergent and determine its value.

(b) Determine the value of $\sum_{n=1}^{\infty} a_n$, where

$$a_n = \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot \cdots \cdot 2n \cdot (2n+2)}$$

28. Let $\sum_{n=1}^{\infty} a_n$ be a complex series. Suppose that

$$\frac{a_{n+1}}{a_n} = e^{i\theta}\left(1 - \frac{b}{n} + O\left(\frac{1}{n^2}\right)\right), \quad \theta, b \in \mathbb{R},$$

Show that $\sum_{n=1}^{\infty} a_n$ diverges if $b < 0$, and converges if $b > 0$ and $\theta \ne 2k\pi, k \in \mathbb{Z}$.

29. Let $\alpha \in (0, 1)$.

(a) Show that the series $\sum_{n\ge 1} \frac{|\sin(n^\alpha)|}{n}$ diverges.

(b) Show that there exists a positive constant A such that

$$\left|\int_0^x \sin(t^\alpha)dt\right| \le Ax^{1-\alpha}, \quad \left|\int_0^n \sin(t^\alpha)dt - \sum_{k=0}^{n} \sin(k^\alpha)\right| \le An^\alpha, \quad x > 0, n \in \mathbb{N}.$$

(c) Deduce that if $\beta > \max(\alpha, 1-\alpha)$, the series $\displaystyle\sum_{n\geq 1} \frac{\sin(n^\alpha)}{n^\beta}$ converges.

30. Let $\sum_{n=1}^{\infty} a_n$ be a positive series. The sequence $\{s_n\}_{n=1}^{\infty}$ is defind via the following equation

$$2s_{n+1} = s_n + \sqrt{s_n^2 + a_n} \text{ with } s_1 = 1.$$

Show that $\lim_{n\to\infty} s_n$ exists provided that $\sum_{n=1}^{\infty} a_n$ converges. Is the converse statement true?

31. Let $f : \mathbb{R}_+ \to (0, \infty)$ be a decreasing function. Suppose that

$$\lim_{x\to\infty} \frac{e^x f(e^x)}{f(x)} = \lambda.$$

Show that $\sum_{n=1}^{\infty} f(n)$ converges if $\lambda < 1$ and diverges if $\lambda > 1$.

32. Let $f : [1, +\infty) \to \mathbb{R}_+$ be a decreasing function such that the integral $\int_1^{+\infty} f(x)dx$ diverges. Let $u_n = f(n)$. Show that

$$\sum_{k=1}^{n} u_k \sim \int_1^n f(x)dx \quad \text{as} \quad n \to +\infty.$$

33. Let $f : \mathbb{R}_+ \to (0, \infty)$ be a continuously differentiable function such that $\lim_{x\to\infty} \frac{f'(x)}{f(x)} = \lambda \neq 0$.

(a) If $\int_0^{\infty} f(x)dx$ converges, show that

$$\sum_{k=n+1}^{\infty} f(k) \sim \frac{\lambda}{1 - e^{-\lambda}} \int_n^{\infty} f(x)dx \text{ as } n \to \infty.$$

(b) If $\int_0^{\infty} f(x)dx$ diverges, show that

$$\sum_{k=n+1}^{\infty} f(k) \sim \frac{\lambda}{1 - e^{-\lambda}} \int_0^{n} f(x)dx \text{ as } n \to \infty.$$

34. Let $\{u_n\}_{n=1}^{\infty}$ be a positive sequence. Assume that there exist $\alpha > 1$ and $\lambda > 0$ such that $u_{n+1} - u_n \sim -\lambda u_n^\alpha$ as $n \to \infty$.

(a) Show that starting from a certain rank $\{u_n\}_{n=1}^{\infty}$ is decreasing and $\lim_{n\to\infty} u_n = 0$.

(b) Show that

$$\lim_{n\to\infty} \left(\frac{1}{u_{n+1}^{\alpha-1}} - \frac{1}{u_n^{\alpha-1}} \right) = \lambda(\alpha - 1).$$

(c) Deduce that
$$u_n \sim \left(\frac{1}{\lambda(\alpha-1)n}\right)^{\frac{1}{\alpha-1}}.$$
(d) Determine the nature of the series $\sum_{n=1}^{\infty} u_n^{\beta}$ according to the positive values of β.
(e) Application : the sequence $\{u_n\}_{n=1}^{\infty}$ is defined by $u_{n+1} = \sin u_n$.

35. We aim to study the harmonic series $\sum_{n=1}^{\infty} 1/n$ and the Euler constant. Let
$$\sigma_n = \sum_{k=1}^{n} \frac{1}{k} - \log n \text{ and } u_1 = \sigma_1, \quad u_n = \sigma_n - \sigma_{n-1}, \quad n \geq 2.$$
(a) Show that
$$u_n \sim \frac{-1}{2n^2} \text{ as } n \to \infty.$$
(b) Deduce that the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges; its limit is called the Euler constant γ.
(c) Show that the remainder R_n of order n of $\sum u_n$ is equivalent to $\frac{-1}{2n}$ as $n \to \infty$.
(d) Deduce that
$$\sigma_n = \gamma + \frac{1}{2n} + o\left(\frac{1}{n}\right).$$

36. Let $\sum_{n=0}^{\infty} z_n$ be a conditional convergent complex series. Let $z_n = x_n + ßy_n$ with $x_n, y_n \in \mathbb{R}$.
(a) Show that there exists at most one $\lambda \in \mathbb{R}$ such that the series $\sum_{n=0}^{\infty}(\lambda x_n + y_n)$ absolutely converges.
(b) Assume that there exists one $\lambda \in \mathbb{R}$ such that $\sum_{n=0}^{\infty}(\lambda x_n + y_n)$ absolutely converges. Show that the set
$$\left\{\sum_{n=0}^{\infty} z_{\sigma(n)} : \sigma : \mathbb{N} \to \mathbb{N} \text{ is a bijection s.t. } \sum_{n=0}^{\infty} z_{\sigma(n)} \text{ converges}\right\}$$
is a line in the complex plane.

37. Let $\sum_{n=0}^{\infty} u_n$, $\sum_{n=0}^{\infty} v_n$ be two convergent series of sums U, V respectively, and let $\sum_{n=0}^{\infty} w_n$ be their product series. Assume that $\sum_{n=0}^{\infty} w_n$ converges to W. Let U_n, V_n, W_n be the respective n-th partial sums.
(a) Show that $W_0 + W_1 + \cdots + W_n = U_0V_n + U_1V_{n-1} + \cdots + U_nV_0$.
(b) Deduce that $W = UV$.

38. Discuss the convergence of the following series.

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(2\pi e n!)}{n^{\alpha}}$

(2) $\displaystyle\sum_{n=1}^{\infty} \sin(\pi e n!)$

(3) $\displaystyle\sum_{n=1}^{\infty} \cos\left(\pi n^2 \log \frac{n}{n-1}\right)$

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^k(n\theta)}{n^{\alpha}}$, $(\alpha > 0, \theta \in \mathbb{R}, k \in \mathbb{N})$

(5) $\displaystyle\sum_{n=1}^{\infty} (-1)^n n^{\alpha}\, a^{n^{\beta}}$, $(\alpha, \beta, a > 0)$

(6) $\displaystyle\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k}\right) \frac{\cos(n\theta)}{n + (-1)^n}$, $(\theta \in \mathbb{R})$

(7) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\alpha} + \sin(n\theta)}$, $(\alpha > 0, \theta \in \mathbb{R})$

(8) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^{\alpha} \sin(n^{-\alpha})}{n^{\beta} + (-1)^n}$, $(\alpha, \beta > 0)$

(9) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{\alpha \log n}{n}\right)^n$

(10) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{p+1/n}}$

(11) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sqrt{n} \sin \frac{1}{n}$

(12) $\displaystyle\sum_{n=1}^{\infty} \log n \log\left(1 + \frac{(-1)^n}{n}\right)$

(13) $\displaystyle\sum_{n=1}^{\infty} \sin(\pi\sqrt{n^2+1})$

(14) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{\sqrt{n} + \cos n}$

(15) $\displaystyle\sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n}\right)\right)^{n^{\alpha}}$, $(\alpha \in \mathbb{R})$

(16) $\displaystyle\sum_{n=1}^{\infty} \left|\sin \frac{\pi\sqrt{n}}{n^{\alpha}}\right|$, $(\alpha \in \mathbb{R})$

(17) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdots 2n-1}{2 \cdot 4 \cdots 2n}\right)^{\alpha}$, $(\alpha \in \mathbb{R})$

(18) $\displaystyle\sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)}{n!c(c+1)\cdots(c+n-1)}$, $(a, b, c \in \mathbb{R})$

39. Discuss the convergence of the following infinite products.

(1) $\displaystyle\prod_{n=1}^{\infty} \big(\cos \frac{1}{n^{\alpha}}\big)^n$, $\alpha > 0$ (2) $\displaystyle\prod_{n=0}^{\infty} (1 + z^{2^n})$, $z \in \mathbb{C}$ (value in case of convergence)

(3) $\displaystyle\prod_{n=1}^{\infty} \big(1 + \frac{ß}{n}\big)$, $\displaystyle\prod_{n=1}^{\infty} \big|1 + \frac{ß}{n}\big|$ (4) $\displaystyle\prod_{n=0}^{\infty} \cos \frac{a}{2^n}$, $a \in \mathbb{R}$, $\displaystyle\prod_{n=0}^{\infty} \mathrm{ch} \frac{a}{2^n}$ (value)

40. Let $(u_n) \subset \mathbb{C} \setminus \{-1\}$ such that the series $\sum u_n$ absolutely converges. Show that for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$

$$\prod_{n=0}^{\infty} (1 + u_n) = \prod_{n=0}^{\infty} (1 + u_{\sigma(n)}).$$

41. (**More on Cesáro summation**) Let $(s_n)_{n\geq 0}$ be a sequence of complex numbers, and define

$$\sigma_n = \frac{s_0 + \cdots + s_n}{1+n} \qquad (n = 0, 1, 2, \ldots).$$

(a) Can it happen that $\lim_{n\to\infty} s_n = +\infty$ but $\lim_{n\to\infty} \sigma_n = 0$?

(b) For $n \geq 1$, put $a_n = s_n - s_{n-1}$. Show that

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} k a_k.$$

Assume that $\lim_{n\to\infty} n a_n = 0$ and that (σ_n) converges, show that (s_n) converges to the same limit.

(c) We aim to show that the last conclusion still holds under the weak assumption that there exists $M > 0$, such that $|n a_n| \leq M$ for all n.

i. For $m < n$, show that

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i).$$

ii. For $i = m+1, \ldots, n$, show that

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

iii. Fix an arbitrary $\varepsilon > 0$. For any $i \leq n$, choose m to be the unique integer such that

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Show that

$$\frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$$

and that

$$|s_n - s_i| \leq M\varepsilon.$$

iv. Conclude, i.e. assume $M > 0$, $|n a_n| \leq M$ for all n, and (σ_n) converges to σ, then (s_n) also converges to σ.

42. (Iterated series.) Consider an infinite array of numbers $(a_{ij})_{i,j\geq 0}$ defined by

$$a_{ij} = \begin{cases} 0 & \text{if } i < j; \\ -1 & \text{if } i = j; \\ 2^{j-i} & \text{if } i > j. \end{cases}$$

(a) Compute $\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij} \right)$ and $\sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ij} \right)$.

(b) Show that if $(a_{ij})_{i,j\geq 0}$ is an infinite array of *non-negative* numbers, then

$$\sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{ij} \right) = \sum_{j=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ij} \right).$$

Here, we allow $\infty = \infty$.

43. (There are many prime numbers.) It is known that Riemann's ζ-function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(a) Show that

$$\zeta(s) = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}},$$

where $\mathcal{P}$ is the set of all primes.

(b) Deduce that the series

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s}$$

converges if $s > 1$, and diverges if $0 < s \leq 1$. Particularly, it implies that there are infinitely many primes.