GP1 HW6

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Problem 1 (Kepler's problem)
(1)

$$c = \frac{b^2}{a}. \tag{1.1}$$

$$E = \frac{1}{2}mv_1^2 - \frac{\gamma}{a + \sqrt{a^2 - b^2}} = \frac{1}{2}mv_2 - \frac{\gamma}{a - \sqrt{a^2 - b^2}}. \tag{1.2}$$

$$l = mv_1 \left(a + \sqrt{a^2 - b^2}\right) = mv_2 \left(a - \sqrt{a^2 - b^2}\right). \tag{1.3}$$

So,

$$E = -\frac{\gamma}{2a}. \tag{1.4}$$

$$l = m\sqrt{\gamma c}. \tag{1.5}$$

(2) We find that a is only depends on E, and c is only depends on l.
(3) If E is fixed, then a is fixed. l reaches its maximum when $b = a$. So it is a circle. If l is fixed, c is fixed. Also when $b = a$, E reaches its minimum. It is also a circle.
(4) The area of the ellipse is

$$S = \pi ab. \tag{1.6}$$

$$\frac{\Delta S}{\Delta t} = \frac{l}{2m}. \tag{1.7}$$

Thus,

$$T = \frac{2m\pi ab}{l}. \tag{1.8}$$

Hence,

$$\frac{T^2}{a^3} = \frac{4m\pi^2}{\gamma}. \tag{1.9}$$

Problem 2 (Cosmic velocities)
Let r be the radius of the Earth and R_e is the distance between the Earth and the Sun. M_e is the mass of the Earth, and M_s is the mass of the Sun.
(1)

$$m\frac{v_1^2}{r} = \frac{GM_e m}{r^2},\ g = \frac{GM_e}{r^2} \tag{2.1}$$

So,

$$v_1 = \sqrt{gr}. \tag{2.2}$$

(2) The energy can't less than 0, if it can escape.

$$E = \frac{1}{2}mv_2^2 - \frac{GM_e m}{r} = 0. \tag{2.3}$$

So,

$$v_2 = \sqrt{2gr} \approx 11.2\text{km/s}. \tag{2.4}$$

(3) To escape from the Sun, the velocity after escaping the Earth but around the Earth should be

$$v_{\text{esc}} = \sqrt{\frac{2GM_s}{R_e}}. \tag{2.5}$$

The velocity of the Earth is

$$v_e = \sqrt{\frac{GM_s}{R_e}}. \tag{2.6}$$

The relative velocity can least be

$$v_r = (\sqrt{2} - 1)\sqrt{\frac{GM_s}{R_e}}. \tag{2.7}$$

By energy conservation in the Earth system,

$$\frac{1}{2}mv_3^2 - \frac{GM_e m}{r} = \frac{1}{2}mv_r^2. \tag{2.8}$$

Hence,

$$v_3 = \sqrt{(3 - 2\sqrt{2})\frac{GM_s}{R_e} + \frac{2GM_e}{r}} \approx 16.7\text{km/s}. \tag{2.9}$$

(4) Here we use the notation in the lecture note,

Initial energy (rocket + Earth system):

$$E_1 = \frac{1}{2}(m + M)v_0^2 + E_{\text{ch}} - \frac{GMm}{R},$$

where E_{ch} is chemical energy.

After burning chemical fuel, rocket has velocity $v_0 + v_3$ near Earth's surface:

$$E_2 = \frac{m}{2}(v_0 + v_3)^2 + \frac{M}{2}(v_0 + \Delta v)^2 - \frac{GMm}{R},$$

where Δv is Earth's recoil.

Momentum conservation:

$$(m + M)v_0 = m(v_0 + v_3) + M(v_0 + \Delta v) \Rightarrow mv_3 + M\Delta v = 0.$$

Substitute into E_2:

$$E_2 = \frac{1}{2}(m+M)v_0^2 + \frac{m}{2}v_3^2 + \frac{M}{2}\Delta v^2 - \frac{GMm}{R}.$$

Energy conservation $E_1 = E_2$ gives:

$$E_{\text{ch}} = \frac{m}{2}\left(1 + \frac{m}{M}\right)v_3^2 \approx \frac{m}{2}v_3^2.$$

$$E_3 = \frac{1}{2}m(\sqrt{2}v_0)^2 + \frac{M}{2}(v_0 + \Delta v')^2,$$

where $\Delta v'$ is Earth's final recoil.

Momentum conservation:

$$(m+M)v_0 = m\sqrt{2}v_0 + M(v_0 + \Delta v') \Rightarrow M\Delta v' = -m(\sqrt{2}-1)v_0.$$

Thus:

$$E_3 = \frac{m}{2}(\sqrt{2}v_0)^2 + \frac{M}{2}v_0^2 - m(\sqrt{2}-1)v_0^2.$$

Energy conservation $E_1 = E_3$ yields:

$$v_3^2 = v_0^2(\sqrt{2}-1)^2 + v_2^2.$$

With $v_0 = 30\,\text{km/s}$ and $v_2 = 11.2\,\text{km/s}$:

$$v_3 = \sqrt{30^2 \times 0.414^2 + 11.2^2} = 16.7\,\text{km/s}.$$

(5) v_3 is the escape velocity to Earth, for planets at different place in the solar system, they have different escape velocity. So not anything can't get out from the solar system.

Problem 3 (Shallow impact - Double Asteroid Redirection Test)

(1) Since $m1 \gg m2$, we suppose m_1 is fixed,

$$m_2\left(\frac{2\pi}{T}\right)^2 R = \frac{Gm_1m_2}{R^2}. \tag{3.1}$$

So,

$$R = \sqrt[3]{\frac{Gm_1T^2}{4\pi^2}} \approx 1180\text{m}. \tag{3.2}$$

$$v = \frac{2\pi}{T}R = \sqrt[3]{\frac{2\pi Gm_1}{T}} \approx 0.17\text{m/s}. \tag{3.3}$$

(2) By Kepler's Third Law, we want to shorten the half-major axis a, that is also means to make the total energy get smaller, so we will make the velocity after crashing reaches the least. So we will choose to have a face-to-face collision.

$$m_2v - m_0v_0 = (m_2 + m_0)v'. \tag{3.4}$$

$$\delta v \approx -7.8 \times 10^{-4}\text{m/s}. \tag{3.5}$$

$$T = \sqrt{\frac{4\pi^2 a^3}{Gm_1}} = 2\pi G m_1 \left(-2E\right)^{-\frac{3}{2}}. \tag{3.6}$$

So,

$$-2\delta E = -\frac{2}{3}(2\pi G m_1 m_2)^{\frac{2}{3}}\sqrt{m_2}T^{-\frac{5}{3}}\delta T. \tag{3.7}$$

$$\delta E = m_2 v \delta v \approx -6.48 \times 10^5 \text{kg} \cdot \text{m}^2/\text{s}^2. \tag{3.8}$$

$$\delta T = -14\text{s}. \tag{3.9}$$

Maybe it is not a completely inelastic collision.