Westlake University
Fundamental Algebra and Analysis I

Exercise sheet 8–4 : Complements on metric space and normed vector space

1. Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disc of the complex plane. For $u, v \in D$, we define $d(u,v) = |u - v|$ if $v = ru$ for some $r \in \mathbb{R}$, and $d(u,v) = |u| + |v|$ otherwise.

(1) Show that d is a metric on D.

(2) What is the topology induced by d on the unit circle $\mathbb{T}$?

2. Recall that $C^n([0,1])$ is the space of real-valued functions whose n-th derivatives are continuous. For $f, g \in C^n([0,1])$, we define

$$d_n(f,g) = \sup_{x \in [0,1]} |f^{(n)}(x) - g^{(n)}(x)|.$$

(1) Is $d_1(f,g)$ a metric on $C^1([0,1])$?

(2) Show that

$$d(f,g) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(f,g)}{1 + d_n(f,g)}$$

is a metric on $C^\infty([0,1])$.

3. Show that the following subspaces of $\mathbb{C}$ are homeomorphic :

$$A = \mathbb{C} \setminus \{0\},\ B = \{z \in \mathbb{C} \mid |z| > 1\},\ C = \{z \in \mathbb{C} \mid 0 < |z| < 1\},\ D = \mathbb{C} \setminus [0,\, 1]$$

4. The following examples show that completeness is not a topological notion.

(1) Let ϕ be the function on $\mathbb{R}$ given by $\phi(x) = \frac{x}{1+|x|}$. We define

$$d(x,\, y) = |\phi(x) - \phi(y)|, \quad x, y \in \mathbb{R}.$$

Show that d is a metric that induces the same topology as the usual one on $\mathbb{R}$. However, $\mathbb{R}$ is not complete under the metric d.

(2) More generally, let O be an open subset of a complete metric space (E, d). Consider the map $\phi : O \to E \times \mathbb{R}$ defined by

$$\phi(x) = \left(x,\ \frac{1}{d(x, O^c)}\right) \stackrel{\text{def}}{=} (x, \rho(x)).$$

Show that ϕ is a homeomorphism from O onto a closed subset of $E \times \mathbb{R}$. Deduce that there exists a complete metric on O that induces the topology given by d on O. (**Remark :** $(O, d\big|_O)$ is in general not complete.)

5. Show that a metric space $(E,\, d)$ is complete if and only if every sequence $\{x_n\}_{n=1}^{\infty} \subset E$ satisfying $d(x_n, x_{n+1}) \leq 2^{-n}$ for all $n \geq 1$ converges.

6. Let (E, d) be a metric space and let $\{x_n\}_{n=1}^{\infty} \subseteq E$ be a Cauchy sequence. Let $A \subset E$ be a subset such that the closure $\overline{A}$ is complete and that $\lim_{n\to\infty} d(x_n,\, A) = 0$. Prove that $\{x_n\}_{n=1}^{\infty}$ converges in E.

7. Let $(E,\, d)$ be a metric space and $\alpha > 0$. Assume that $A \subset E$ such that $d(x,\, y) \geq \alpha$ for any $x, y \in A$ with $x \neq y$. Show that A is complete.

8. Let $(E,\, d)$ be a metric space and $A \subset E$ be a subset. Assume that every Cauchy sequence of A converges in E. Show that the closure $\overline{A}$ of A is complete.

9. Let $(E,\, d)$ be a metric space and let $\{x_n\}_{n=1}^{\infty} \subseteq E$ be a divergent Cauchy sequence.

(1) Show that for any $x \in E$, the sequence $\{d(x, x_n)\}_{n=1}^{\infty}$ converges to a positive number $g(x)$.
(2) Show that the function $x \mapsto \frac{1}{g(x)}$ is continuous from E to $\mathbb{R}$.
(3) Show that the function above is unbounded.

10. Let $f : (E,\, d) \to (F,\, \delta)$ be a uniformly continuous bijection such that the inverse f^{-1} is uniformly continuous too. Show that for any $A \subset E$, $f(A)$ is complete if and only if A is complete.

11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a uniformly continuous function. Show that there exist two nonnegative constants a and b such that

$$|f(x)| \leq a\|x\| + b,$$

where $\|x\|$ is the Euclidean norm of x.

12. Let $f :\ E \to F$ be a continuous map between two metric spaces. Assume that f is uniformly continuous on every bounded subset of E.

(1) Show that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence for every Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subseteq E$.
(2) Let $E \subset E'$ with E dense in E' and assume that F is complete. Show that f admits a unique continuous extension from E' to F.

13. Show, by a counter-example, that the assumption that f is a contraction in the Fixed Point Theorem cannot be relaxed to the following : $d(f(x), f(y)) < d(x, y)$ for all $x, y \in E$ with $x \neq y$.
(**Hint :** Consider the function $f(x) = (x^2 + 1)^{1/2}$ on $[0,\, +\infty[$).

14. Let (E, d) be a complete metric space and let f be a map such that $f^n = f \circ \cdots \circ f$ (n times) is a contraction. Show that f has a unique fixed point. Show, by an example, that such an f is not necessarily continuous.

15. Let $I = (0,\, \infty)$ be equipped with its usual topology τ.

(1) Show that τ is defined by the following complete distance :

$$d(x, y) = |\log x - \log y|.$$

(2) Let $f \in C^1(I)$ such that for some $\lambda < 1$, $x|f'(x)| \leq \lambda f(x)$ for all $x \in I$. Show that f has a unique fixed point in I.

16. Let $E = \{a_n\}_{n=1}^{\infty}$ be a countable set. We define

$$d(a_m, a_n) = \begin{cases} 0 & m = n \\ 10 + 1/m + 1/n & m \neq n \end{cases}$$

(1) Show that d is a metric and that E is complete under this metric.
(2) Let $f : E \to E$ be defined by $f(a_n) = a_{n+1}$. Show that $d(f(a_m), f(a_n)) < d(a_m, a_n)$ for $m \neq n$ but f does not have any fixed point.

17. The purpose here is to give an alternate proof of the Fixed Point Theorem. Let (E, d) be a nonempty complete metric space and let $f : E \to E$ be a Lipschtzian function of constant $\lambda < 1$. For any $R \geq 0$, set $A_R = \{x \in E : d(x, f(x)) \leq R\}$.
(1) Show that $f(A_R) \subset A_{\lambda R}$.
(2) Show that A_R is a nonempty closed subset of E for any $R > 0$.
(3) Show that for any $x, y \in A_R$, it holds that $d(x, y) \leq 2R + d(f(x), f(y))$. Then deduce that $\operatorname{diam}(A_R) \leq 2R/(1 - k)$.
(4) Show that A_0 is nonempty.

18. Let (E, d) be a complete metric space. Let f and g be two contractions on E that commute (i.e., $f \circ g = g \circ f$). Show that f and g have a unique common fixed point.
Show, by a counter-example, that the conclusion is not true if the commuting assumption is removed.

19. The aim of this exercise is to show that the conclusion of the preceding one can be extended, in a certain sense, to non-commuting contractions.
Let (E, d) be a complete metric space. Given $A \subset E$, let d_A be the distance function associated to A : $d_A(x) = \inf\{d(x, a) \mid a \in A\}$. Let $\mathcal{C}$ denote the family of all compact subsets of E. For $A, B \in \mathcal{C}$, we define

$$h(A, B) = \sup_{x \in E} |d_A(x) - d_B(x)|.$$

(1) Show that h is a distance on $\mathcal{C}$.
(2) For $F \subset E$, we write $F_\varepsilon = \{\varepsilon \mid d_F(x) \leq \varepsilon\}$. Show that

$$h(A, B) = \inf\{\varepsilon \geq 0 \mid, A \subset A_\varepsilon \ B \subset B_\varepsilon\}.$$

(3) Show that $(\mathcal{C}, h)$ is complete.
(4) Now let $f_1, \cdots, f_n$ be n contractions on E. Define T on $(\mathcal{C}, h)$ by

$$T(A) = \bigcup_{k=1}^{n} f_k(A).$$

Show that T is a contraction and that there exists a unique compact set K such that $T(K) = K$

20. Let K and H be two compact subsets of a topology space. Is their intersection $K \cap H$ also compact ?

21. Let A, B be two disjoint closed subsets of a compact Hausdorff E. Show that there exist two disjoint open sets U, V such that $A \subset U$ and $B \subset V$. Show, by a counter-example, that the compactness cannot be removed.

22. (**Separable metric space**). A metric space X is said to be separable provided that there exists a countable subset of X that is dense in X. Show that

    (1) A compact metric space is separable.

    (2) A metric space X is separable if and only if there is a countable collection of open subsets $\{O_n\}_{n=1}^{\infty}$ such that each open subset of X is the union of a subcollection of $\{O_n\}_{n=1}^{\infty}$.

23. Let E, F be two metric spaces, and let A be a compact subset of E and B a compact subset of F. Assume that Ω is an open set in $E \times F$ such that $A \times B \subset \Omega$ Show that there exist open sets $U \subset E$ and $V \subset F$ such that $A \times B \subset U \times V \subset \Omega$.

24. (**Lebesgue's covering lemma**). Let $\{O_\lambda\}_{\lambda \in \Lambda}$ be an open cover of a compact metric space X. Show the existence of $\varepsilon > 0$ such that for each $x \in X$ the open ball $B(x, \varepsilon)$ is contained in some member of the open cover.

25. Let (E, d) be a compact metric space. Suppose that $f : E \to E$ satisfy $d(f(x), f(y)) < d(x, y)$ for all $x, y \in E$ with $x \neq y$. Show that f has a unique fixed point.

26. For $f \in C^0([0,1])$, we define $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$. Is the set

$$S = \{f \in C^0([0,1]) \mid \|f\|_\infty \leq 1\}$$

compact ?

27. Consider the set of functions

$$S = \left\{ \sum_{n \in \mathbb{Z}} c_n e^{inx} \mid |c_n| \leq (1 + |n|)^{-2} \right\}.$$

    (1) Show that every function in S lies in $C^0([0, 2\pi])$.

    (2) Show that S is compact in $C^0([0, 2\pi])$ (under the supremum norm $\|\cdot\|_\infty$).

28. Let (E, d) be a connected metric space which is unbounded (i.e., its diameter is infinite). Show that every sphere $S(x, r) = \{y \in E : d(y, x) = r\}$ with $x \in E$ and $r > 0$ is nonempty. Deduce that E is not separable.

29. Let $\mathbb{R}^n$ be equipped with its euclidean topology. Show that a connected open subsets of $\mathbb{R}^n$ with $n \geq 2$ is not homeomorphic to any subset of $\mathbb{R}$.

30. Let E be a topological space and $A \subset E$. Show that if a connected subset $C \subset E$ meets $\mathring{A}$ and $\mathring{A}^c$, it must meet ∂A.

31. (1) Show that the subset of $\mathbb{R}^2$ consisting of points which have at least one coordinate is irrational is connected.

    (2) Give an example of two homeomorphic subsets $A, B \subset \mathbb{R}^2$ such that A^c is connected but B^c is disconnected.

32. Let (E, d) be a metric space. Assume that $\overline{B(x, r)} = \overline{B}(x, r)$ for all $x \in E$ and $r > 0$. Show that E is connected.

33. Show that the following set $\Gamma \subseteq \mathbb{R}^2$ is connected

$$\Gamma = \Big(\bigcup_{r \in [0,\,1] \cap \mathbb{Q}} \{r\} \times [0,\,1] \Big) \cup \Big(\bigcup_{r \in [0,\,1] \setminus \mathbb{Q}} \{r\} \times [-1,\,0) \Big).$$

Is Γ path-connected?

34. Let A, B be two subsets of a topological space E.

(1) Show that if A and B are closed and both $A \cup B$, $A \cap B$ are connected, then A and B are connected.

(2) Show that if A and B are connected and $\overline{A} \cap B \neq \emptyset$, then $A \cup B$ is connected.

35. (**Furstenberg's topological proof of infinitely many primes**). Let $\mathbb{Z}$ and $\mathbb{Z}_+$ be the sets of integers and positive integers, respectively. We define arithmetic sequences

$$V(a, b) = \{an + b : \ n \in \mathbb{Z}\},$$

where $a \in \mathbb{Z}_+, b \in \mathbb{Z}$. Let τ be a family of subsets of $\mathbb{Z}$, whose members are $\emptyset$, $\mathbb{Z}$, and the sets which can be represented as the union of arithmetic sequences. Show that

(1) τ is a topology on $\mathbb{Z}$.

(2) $V(a, b)$ is both open and closed.

(3) Apply (a) and (b) to show that there are infinitely many primes.

36. (**Vitali's covering lemma**). We denote by $B(x, r_x)$ the closed ball of radius $r_x > 0$ and centered at x. Let $\mathcal{F} = \{B(x, r_x) : \ x \in S\}$ be a collection of closed balls in $\mathbb{R}^d$ with uniformly bounded diameters, i.e.,

$$\sup\{r_x : B(x, r_x) \in \mathcal{F}\} < \infty.$$

Show that there exists a countable family of disjoint balls $\mathcal{G} = \{B(y, r_y) : \ y \in T\} \subseteq \mathcal{F}$ such that

$$\cup_{x \in S} B(x, r_x) \subset \cup_{y \in T} B(y, 5r_y).$$

37. Suppose that X is a vector space.

(1) If $A, B \subset X$ are convex, show that $A + B$ and $A \cap B$ are convex. What about $A \cup B$?

(2) Show that $2A \subset A + A$. When is it true that $2A = A + A$?

38. Let $\mathbb{R}^d$ be endowed with the Euclidean metric. Let $K \subset \mathbb{R}^d$ be a symmetric, convex, compact set with $0 \in K^\circ$, that is, the origin is an interior point of K. Show that

$$\rho_K(x) = \inf\{r > 0 : r^{-1}x \in K\}$$

defines a norm on $\mathbb{R}^d$.

39. Let X be an infinite dimensional normed vector space. A sequence $\{e_n\}_{n=1}^{\infty} \subset X$ is called a **Schauder basis** for X if for every $x \in X$ there is a unique sequence of scalars $\{\alpha_n\}_{n=1}^{\infty}$ such that

$$\|x - (\alpha_1 e_1 + \cdots + \alpha_n e_n)\| \to 0 \text{ as } n \to \infty,$$

and we have the expansion

$$x = \sum_{n=1}^{\infty} \alpha_n e_n.$$

(1) Find a Schauder basis for ℓ^p for $1 \leq p < \infty$.

(2) Show that if a normed vector space has a Schauder basis, then it is separable. Is the converse true?

40. Let X be a normed vector space and let M be a nonempty subset. The anihilator of M is defined as $M^a = \{f \in X^* : f(x) = 0 \text{ for all } x \in M\}$. Show that M^a is a closed subspace of X^*. What are X^a and $\{0\}^a$?

41. Let X be a normed vector space over $\mathbb{C}$ and let $X_{\mathbb{R}}$ be the same space treated as a normed space over $\mathbb{R}$. Show that, for each $\mathbb{R}$-linear functional $\varphi : X_{\mathbb{R}} \to \mathbb{R}$, there exists a unique $\mathbb{C}$-linear functional $\tilde{\varphi} : X \to \mathbb{C}$ such that $\varphi(x) = \mathrm{Re}\tilde{\varphi}(x)$ for each $x \in X$ and it is give by

$$\tilde{\varphi}(x) = \varphi(x) - i\varphi(ix).$$

42. Let $\Lambda^\alpha(\mathbb{R})$ be the space of all bounded functions on $\mathbb{R}$ which satisfy a Hölder condition of exponent α with $0 < \alpha \leq 1$, that is

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

(1) Show that a function which satisfies a Hölder condition of exponent α with $\alpha > 1$ is a constant.

(2) Show that

$$\|f\|_{\Lambda^\alpha(\mathbb{R})} = \sup_{x \in \mathbb{R}} |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

defines a norm on $\Lambda^\alpha(\mathbb{R})$, and with this norm, $\Lambda^\alpha(\mathbb{R})$ is a Banach space.

(3) We define $C^{k,\alpha}(\mathbb{R})$ to be the class of functions f on $\mathbb{R}$ whose derivatives of order less than or equal to k belong to $\Lambda^\alpha(\mathbb{R})$. Show that

$$\|f\|_{C^{k,\alpha}(\mathbb{R})} = \sum_{\ell=0}^{k} \|f^{(\ell)}\|_{\Lambda^\alpha(\mathbb{R})}$$

defines a norm on $C^{k,\alpha}(\mathbb{R})$, and with the norm, $C^{k,\alpha}(\mathbb{R})$ is a Banach space.

43. A normed vector space X is said to be **uniformly convex** if for every $0 < \varepsilon \leq 2$, there exists $\delta > 0$ such that for any two $x, y \in X$ with $\|x\| = 1, \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, it holds that

$$\left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

Let X be a uniformly convex Banach space. Let $K \subset X$ be a closed convex set. Show that for any $x \in X$, there exists a unique $x^* \in K$ such that

$$\|x - x^*\| = \inf_{y \in K} \|x - y\|.$$

44. Let X be a normed vector space and suppose S and T are in the dual space X^*. If $S(x) = 0$ implies $T(x) = 0$, show that there is a constant c such that $S(x) = cT(x)$ for all $x \in X$.

45. Let (X, ρ) be a metric space. We denote by $\mathrm{Lip}_0(X)$ the set of real-valued Lipschitz functions f on X that vanish at $x_0 \in X$.

(1) Show that $\mathrm{Lip}_0(X)$ is a Banach space endowed with the following norm

$$\|f\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}.$$

(2) For each $x \in X$, we define the linear functional $T_x : f \to f(x)$ for $f \in \mathrm{Lip}_0(X)$. Show that $\|T_x - T_y\| = \rho(x, y)$. Thus X is isometric to a subset of the Banach space $\mathrm{Lip}_0(X)^*$. (Since any closed subset of a complete metric space is complete, this provides another proof of the existence of a completion for any metric space X. It also shows that any metric space is isometric to a subset of a normed linear space).

46. Let $\{\Lambda_n\}_{n=1}^{\infty}$ be a sequence of bounded linear operators from a normed vector space X to a Banach space Y with $\|\Lambda_n\| \leq M < \infty$ for all $n \geq 1$. Suppose that there is a dense set $E \subset X$ such that $\{\Lambda_n x\}_{n=1}^{\infty}$ converges for each $x \in E$. Show that $\{\Lambda_n x\}_{n=1}^{\infty}$ converges for each $x \in X$.

47. Let X and Y be normed vector spaces. Let T be a bounded linear operator from X to Y with kernel $\ker(T) = Z$. Show that there is a unique bound linear operator S from X/Z to Y such that $T = S \circ \varphi$, where $\varphi : X \to X/Z$ is the natural map. Moreover, show that $\|T\| = \|S\|$.

48. (1) Let X be a complex normed vector space and let $T : X \to \mathbb{C}$ be a linear map. Show that T is continuous if and only if the kernel $\ker(T)$ is closed.

(2) More generally, let X and Y be normed vector spaces and let $T : X \to Y$ be a linear operator. If X is finite dimensional, show that T is continuous. If Y is finite dimensional, show that T is continuous if and only if $\ker(T)$ is closed.

49. Let X be the normed space of sequences of complex numbers $x = \{x_i\}_{i=1}^{\infty}$ with only finitely many nonzero terms and normed by $\|x\| = \sup_i |x_i|$. Let $T : X \to X$ be defined by

$$Tx = \left\{x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots\right\}.$$

Show that T is a bounded linear operator, but T^{-1} is unbounded.

50. Show that a subset of a finite dimensional normed vector space is compact if and only if it is closed and bounded.

51. (1) Give an open cover of the closed unit ball of $C([0,1])$ (with the norm $\|f\| = \max_{x\in[0,1]} |f(x)|$) that has no finite subcover.

(2) For $1 \leq p \leq \infty$, give an open cover of the closed unit ball of the Banach space ℓ^p that has no finite subcover.

52. (**Another proof of Riesz's Theorem**). Let X be an infinite dimensional normed vector space. Let B be the closed unit ball in X, and let B_0 be the unit open ball in X. If B is compact, then the open cover $\{x + B_0/3\}_{x\in B}$ of B has a finite subcover $\{x_i + B_0/3\}_{i=1}^n$. Use Riesz's Lemma with $Y = \text{span}\{x_1, x_2, \cdots, x_n\}$ to derieve a contradiction.

Riesz's Lemma : Let Y be a closed proper subspace of a normed vector space X. Then for each $\varepsilon > 0$, there exists a unit vector $x_0 \in X$ such that

$$\|x_0 - y\| > 1 - \varepsilon \quad \text{for all } y \in Y.$$

53. Define $X = \{f \in C([0,1]) : f(0) = 0\}$ and $V = \{f \in X : \int_0^1 f(x)dx = 0\}$. Show that V is a closed subspace of X, yet there is no $x \in X$ with $\|x\| = 1$ such that $d(x, V) = 1$. (This assertion means that in general we can not avoid the $\varepsilon > 0$ in Riesz's lemma).

54. Let X be a normed vector space and let B be the open unit ball. Show that X is infinite dimensonal if and only if B contains an infinite collection of non-overlapping open balls of radius $1/4$.

55. Show that, for every infinite dimensional normed vector space X, there exists an infinite sequence $\{x_n\}_{n=1}^\infty$ of norm one such that $\|x_m - x_n\| > 1$ for all $m, n \in \mathbb{N}$ and $m \neq n$.

56. (1) Suppose X is a vector space. The algebraic dual of X is the set of all linear functionals on X, and is also a vector space. Suppose also that X is a normed vector space. Show that X has finite dimension if and only if the algebraic dual and the dual space X^* coincide.

(2) Let X be an infinite dimensional normed vector space over $\mathbb{R}$ or $\mathbb{C}$. Prove that there exists a non-continuous linear functional.

(3) For any discontinuous linear functional Λ on a normed vector space X, show that $\overline{\ker(\Lambda)} = X$, that is, the kernel of Λ is a dense subspace of X.

57. (1) Show that a Banach space $\mathcal{B}$ is separable if its dual space $\mathcal{B}^*$ is separable.

(2) Show that a normed vector space X is separable if and only if there is a compact subset $K \subset X$ such that $\overline{\text{span}(K)} = X$.

58. Let X be a normed vector space and let X^* be its dual space with the norm

$$\|f\| = \sup_{\|x\|\leq 1} |f(x)|.$$

(1) For each $x \in X$, show that the mapping $T_x : f \to f(x)$ is a bounded linear functional on X^*. Moreover, show that $\|T_x\| = \|x\|$. (This gives a natural embedding of X into X^{**}, the dual space of X^*).

(2) Let $\{x_n\}_{n=1}^\infty \subset X$ be a sequence such that $\{f(x_n)\}_{n=1}^\infty$ is bounded for every $f \in X^*$. Show that $\{\|x_n\|\}_{n=1}^\infty$ is bounded.

59. Let $C([a,b])$ be the family of continuous functions on $[a,b]$ with the maximum norm. We say that a sequence $\{f_n\}_{n=1}^{\infty}$ weakly converges to f, denoted by $f_n \xrightarrow{w} f$ if, for all $\Lambda \in C([a,b])^*$, we have $\Lambda(f_n) \to \Lambda(f)$.

(1) Suppose that $f_n \xrightarrow{w} f$. Show that $\{f_n\}_{n=1}^{\infty}$ is also pointwise convergent, that is, $f_n(x) \to f(x)$ for all $x \in [a,b]$.

(2) Show that a weakly convergent sequence in $C^1([a,b])$ is convergent in $C([a,b])$. Is this still true when $[a,b]$ is replaced by $\mathbb{R}$.

60. For a complex sequence $x = \{x_i\}_{i=1}^{\infty}$, we define its ℓ^1 and ℓ^∞ norms as

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|, \quad \|x\|_\infty = \sup_{i\geq 1} |x_i|.$$

The Banach spaces ℓ^1 and ℓ^∞ are defined as

$$\ell^1 = \{x : \|x\|_1 < \infty\}, \quad \ell^\infty = \{x : \|x\|_\infty < \infty\}$$

and c_0 is the subspace of ℓ^∞ consisting of all x such that $\lim_{i\to\infty} x_i = 0$.

(1) For each $y \in \ell^1$, show that the mapping $\Lambda_y(x) = \sum_{i=1}^{\infty} x_i y_i$ is a bounded linear functional on c_0 with norm $\|\Lambda_y\| = \|y\|_1$. Moreover, every member in c_0^* can be obtained in this way and hence $c_0^* = \ell^1$.

(2) Similarly, show that $(\ell^1)^* = \ell^\infty$.

(3) Show that every $y \in \ell^1$ induces a bounded linear functional on ℓ^∞ as in (a). However, this does not give all mappings of $(\ell^\infty)^*$.

(4) Show that c_0 and ℓ^1 are separable but ℓ^∞ is not.

61. Suppose that X is a Banach space, M and N are linear subspaces, an that $X = M \bigoplus N$, that is

$$X = M + N = \{m + n : m \in M, n \in N\}$$

and $M \cap N = \{0\}$. Let P be the projection of X on to M, that is, if $x = m+n$, then $P(x) = m$. Show that P is well defined and linear. Prove that P is bounded if and only if both M and N are closed.

62. Let X be a normed vector space. Given $A \subset X$ and $B \subset X^*$, we define

$$\begin{aligned} A^\perp &= \{f \in X^* : f(x) = 0 \text{ for all } x \in A\} \\ {}^\perp B &= \{x \in X : f(x) = 0 \text{ for all } f \in B\} \end{aligned}$$

(1) Show that $A^\perp$ ad ${}^\perp B$ are closed subspaces of X^* and X, respectively.

(2) Let $V \subset X$ be a subspace. Show that $\overline{V} =^\perp (V^\perp)$.

63. An operator T on a Banach space X is called compact if, for every bounded subset B, the immage $T(B)$ has compact closure. Alternatively, T is compact if and only if for every sequence $\{x_n\}_{n=1}^{\infty} \subset X$, $\{T(x_n)\}_{n=1}^{\infty} \subset X$ has a convergent subsequence.

(Fredholm alternative). Suppose that T is a compact operator on a Banach space X. Show that exactly one of the following holds

(1) $I - T$ is invertible with bounded inverse.

(2) There exists a nonzero $x \in X$ such that $Tx = x$ (i.e., 1 is an eigenvalue of T).

64. Prove that a subset A of a metric space (X, d) is bounded if and only if every countable subset of A is bounded.

65. Let X be a normed vector space and $M \subset X$. For $\varepsilon > 0$, we say that a set $A \subset M$ is an *ε-net* in M provided that, for every $x \in M$ there is $y \in A$ such that $\|x - y\| < \varepsilon$. We say that A is *ε-separated* if $\|x - y\| \geq \varepsilon$ for all $x, y \in A$ and $x \neq y$.

Suppose that X has dimension n. Show that every ε-net in the unit ball of X contains at least ε^{-n} points. On the other hand, show that there exists an ε-net in the unit ball with at most $(1 + 2/\varepsilon)^n$ points.