Family name: ____________ Given name: ____________ Student ID: ____________

Westlake University
Fundamental Algebra and Analysis I

Test of November 15th 2025

Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. If X' is a subset of X, we equip X' with the induced topology, which is defined as

$$\{X' \cap U \,|\, U \in \mathscr{T}_X\}.$$

We assume that $(X, \mathscr{T}_X)$ is Hausdorff.

Let Z be a subset of X and $p \in \overline{Z} \setminus Z$. Let $f : Z \to Y$ be a continuous mapping and $\ell \in Y$. Assume that

$$\lim_{z \in Z,\, z \to p} f(z) = \ell.$$

(1) Prove that there exists a continuous mapping

$$F : Z \cup \{p\} \longrightarrow Y$$

such that

$$\forall\, z \in Z,\ F(z) = f(z).$$

The mapping is called a *continuous extension* of f to $Z \cup \{p\}$

(2) Assume that $(Y, \mathscr{T}_Y)$ is Hausdorff. Prove that the continuous extension of f to $Z \cup \{p\}$ is unique. One can assume that F_1 and F_2 are two continuous extensions of f and consider the mapping $F : Z \cup \{p\} \to Y \times Y$ that sends z to $(F_1(z), F_2(z))$.

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Answer. (1) We extend f to a mapping $F : Z \cup \{p\} \to Y$ such that $F(z) = f(z)$ for $z \in Z$ and $F(p) = \ell$. By definition, for any neighbourhood U of ℓ, there exists a neighbourhood V of p such that

$$\forall\, z \in V \cap Z,\ f(z) \in U.$$

Therefore

$$\forall\, z \in V \cap (Z \cup \{p\}),\ F(z) \in U,$$

which shows the continuity of F at p.

Let $x \in Z$. Since f is continuous at x, for any neighbourhood W of $f(x)$, there exists a neighbourhood A of x such that

$$\forall\, z \in A \cap Z,\ f(z) \in W.$$

Since X is Hausdorff, $A \setminus \{p\}$ is also a neighbourhood of x. Note that

$$\forall\, z \in (A \setminus \{p\}) \cap (Z \cup \{p\}),\ F(z) = f(z) \in W.$$

Hence F is continuous at x.

(2) Assume that F_1 and F_2 are two continuous extensions of f to $Z \cup \{p\}$. Then

$$F : Z \cup \{p\} \longrightarrow Y \times Y,\ F(z) := (F_1(z), F_2(z))$$

is continuous. Since Y is Hausdorff, $F^{-1}(\Delta_Y)$ is closed. Since it contains Z, it also contains $Z \cup \{p\}$. Hence $F_1 = F_2$.