COMP9020 Lecture 5
Session 2, 2015
Relations

Revision: 1.2

Relations and their representation

Relations are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are *related*. These objects may

- influence one another (each other for binary relations; self for unary?)
- share some common properties
- correspond to each other precisely when some constraints are satisfied

In general, relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.

Definition

An *n-ary relation* is a subset of the cartesian product of n sets.

$$R \subseteq S_1 \times S_2 \times \ldots \times S_n$$

$$x \in R \Rightarrow x = (x_1, x_2, \ldots x_n) \text{ where each } x_i \in S_i$$

If $n = 2$ we have a *binary* relation $\mathcal{R} \subseteq S \times T$.
(mostly we consider binary relations)
equivalent notations: $(x_1, x_2, \ldots x_n) \in R \iff R(x_1, x_2, \ldots x_n)$
for binary relations: $(x, y) \in R \iff R(x, y) \iff xRy$.

Database examples

Example (course enrolments)

S = set of CSE students
(S can be a subset of the set of all students)
C = set of CSE courses
(likewise)
E = enrolments = $\{ (s, c) \mid s \text{ takes } c \}$

$$E \subseteq S \times C$$

In practice, almost always there are various 'onto' (nonemptiness) and 1–1 (uniqueness) constraints on database relations.

Example (class schedule)

C = CSE courses
T = starting time (hour & day)
R = lecture rooms
S = schedule =

$$\{\,(c, t, r) \mid c \text{ is at } t \text{ in } r\,\} \subseteq C \times T \times R$$

Example (sport stats)

$$R \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$

n-ary relations

Relations can be defined linking $k \geq 1$ domains $D_1, \ldots, D_k$ simultaneously. In database situations one also allows for unary ($n = 1$) relations and even for nullary ($n = 0$) case.
A function $f : S \longrightarrow T$ can be identified with its *graph*, the set of pairs

$$\{ (s, f(s)) \mid s \in S \} \subseteq S \times T$$

Such a subset must satisfy certain conditions w.r.t. S and T (which?) Any subset satisfying these can be viewed as defining a function.
The language and notation of relations is used mostly for nonfunctional, binary situations; in practical applications it is extremely popular in database systems, in AI, semantics...

A binary relation, say $\mathcal{R} \subseteq S \times T$, can be presented as a matrix with rows enumerated by (the elements of) S and the columns by T; eg. for $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3, t_4\}$ we may have

$$\begin{bmatrix} \bullet & \circ & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \bullet & \bullet & \circ & \circ \end{bmatrix}$$

If relation is binary and links the domain with itself it can be presented as a graph on the elements of that domain.

Relations on a single domain

Particularly important are binary relationships between the elements of the same set. We say that '$\mathcal{R}$ is a relation on S' if

$$\mathcal{R} \subseteq S \times S$$

Special (trivial) relations

(all w.r.t. set S)

Identity (diagonal, equality) $E = \{ (x, x) \mid x \in S \}$

Empty $\emptyset$

Universal $U = S \times S$

Basic properties that $\mathcal{R} \subseteq S \times S$ may satisfy

(R) *reflexive* $(x, x) \in \mathcal{R}$ for all $x \in S$

(AR) *antireflexive* $(x, x) \notin \mathcal{R}$

(S) *symmetric* $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$

(AS) *antisymmetric* $(x, y), (y, x) \in \mathcal{R} \Rightarrow x = y$

(T) *transitive* $(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$

NB

An object, notion etc. is considered to satisfy a property if none of its instances violates any defining statement of that property.

Interaction of properties

A relation *can* be both symmetric and antisymmetric. It is when $\mathcal{R}$ consists only of some pairs $(x,x), x \in S$.

A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).

NB

$$\left.\begin{array}{r} \textit{nonreflexive} \\ \textit{nonsymmetric} \end{array}\right\} \textit{ is not the same as } \left\{\begin{array}{l} \textit{antireflexive/irreflexive} \\ \textit{antisymmetric} \end{array}\right.$$

Most important kinds of relations on S

- total order $\begin{pmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet \\ \circ & \circ & \bullet \end{pmatrix}$
- partial order $\begin{pmatrix} \bullet & \bullet & \bullet \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{pmatrix}$, $\begin{pmatrix} \bullet & \bullet & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{pmatrix}$
- quasi-orders (strict total/partial orders)
  $\begin{pmatrix} \circ & \bullet & \bullet \\ \circ & \circ & \bullet \\ \circ & \circ & \circ \end{pmatrix}$, $\begin{pmatrix} \circ & \bullet & \bullet \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{pmatrix}$
- equivalence $\begin{pmatrix} \bullet & \bullet & \circ \\ \bullet & \bullet & \circ \\ \circ & \circ & \bullet \end{pmatrix}$
- identity $\begin{pmatrix} \bullet & \circ & \circ \\ \circ & \bullet & \circ \\ \circ & \circ & \bullet \end{pmatrix}$

NB

Some of those are special cases of the others, eg. 'total order' of a 'partial order', 'identity' of an 'equivalence'.

Image and Inverse

Definition

Let $\mathcal{R} \subseteq S \times T$; let $A \subseteq S$; let $B \subseteq T$.
The *inverse* $\mathcal{R}^{-1}$ of $\mathcal{R}$ is $\mathcal{R}^{-1} = \{\, (t, s) \mid (s, t) \in \mathcal{R} \,\}$.
The *image* of A is $\mathcal{R}(A) \stackrel{\text{def}}{=} \{\, t \mid \exists s \in A\,((s, t) \in \mathcal{R}) \,\}$.

Of course we can also use the inverse to form images: $\mathcal{R}^{-1}(B) = \{\, s \mid \exists t \in B\left((t, s) \in \mathcal{R}^{-1}\right) \,\} = \{\, s \mid \exists t \in B\,((s, t) \in \mathcal{R}) \,\}$.

Corollary

$(\mathcal{R}^{-1})^{-1} = \mathcal{R}$

Viewed this way $\mathcal{R}$ becomes a function from $\mathrm{pow}(S)$ to $\mathrm{pow}(T)$. However, *not* every $g : \mathrm{pow}(S) \longrightarrow \mathrm{pow}(T)$ can be matched to a relation.
(Using a small domain like $S = \{a, b\}$ provide an example of a function $g : \mathrm{pow}(S) \longrightarrow \mathrm{pow}(S)$ which does not correspond to any relation on S. Can you do it with $S' = \{a\}$?)

NB

The order of axes — S and T — is important. For $\mathcal{R} \subseteq S \times S$, its inverse $\mathcal{R}^{-1}$ is usually quite different from $\mathcal{R}$.

Example: divisibility relation on $\mathbb{P}$

$$\begin{aligned} D &\stackrel{\text{def}}{=} \{ (p, q) \mid p|q \} = \{(1,1), (1,2), \ldots, (2,2), (2,4), \ldots\} \\ D^{-1} &= \{ (p, q) \mid p \in q\mathbb{P} \} \\ &= \{(1,1), (2,1), (2,2), (3,1), (3,3), (4,1), (4,2), \ldots\} \end{aligned}$$

For every $n \in \mathbb{P}$, $D(\{n\})$ is infinite, $D^{-1}(\{n\})$ is finite.

Question

Consider a function $f : S \longrightarrow T$ as a relation. Then f^{-1} is a relation; when is it a function?

Answer

When f is 1-1 and onto.

Question

Consider a function $f : S \longrightarrow T$ *as a relation. Then* f^{-1} *is a relation; when is it a function?*

Answer

When f *is 1-1 and onto.*

Which properties carry from individual relations $\mathcal{R}_1, \mathcal{R}_2 \subseteq S \times T$ to their union?

1. $\mathcal{R}_1, \mathcal{R}_2 \in (\mathrm{R}) \Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in (\mathrm{R})$
2. $\mathcal{R}_1, \mathcal{R}_2 \in (\mathrm{S}) \Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in (\mathrm{S})$
3. $\mathcal{R}_1, \mathcal{R}_2 \in (\mathrm{T}) \not\Rightarrow \mathcal{R}_1 \cup \mathcal{R}_2 \in (\mathrm{T})$
   E.g., $S = \{a, b, c\}$, $a\mathcal{R}_1 b$, $b\mathcal{R}_2 c$ and no other relationships

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Equivalence relations and partitions

Relation $\mathcal{R}$ is an *equivalence* relation if it satisfies (R), (S), and (T). Every equivalence relation $\mathcal{R}$ defines *equivalence classes* on its domain S.
The equivalence class $[s]$ (w.r.t. $\mathcal{R}$) of an element $s \in S$ is

$$[s] = \{ t \in S \mid t\mathcal{R}s \}$$

This notion is well defined only for $\mathcal{R}$ which is an equivalence relation. Collection of all equivalence classes $[S]_{\mathcal{R}} = \{ [s] \mid s \in S \}$ is a partition of S

$$S = \bigcup_{s \in S} [s]$$

Thus the equivalence classes are disjoint and jointly cover the entire domain. It means that every element belongs to one (and only one) equivalence class.

We call $s_1, s_2, \ldots$ *representatives* of (different) equivalence classes

For $s, t \in S$ either $[s] = [t]$, when $s\mathcal{R}t$, or $[s] \cap [t] = \emptyset$, when $s \not\mathcal{R} t$.

We commonly write $s \sim_{\mathcal{R}} t$ when s, t are in the same equivalence class.

In the opposite direction, a partition of a set defines the equivalence relation on that set. If $S = S_1 \dot\cup \ldots \dot\cup S_k$, then we specify $s \sim t$ exactly when s and t belong to the same S_i.

If the relation $\sim$ is an equivalence on S and $[S]$ the corresponding partition, then

$$\nu : S \longrightarrow [S], \quad \nu : s \mapsto [s] = \{ x \in S \mid x \sim s \}$$

is termed the *natural* map. It is always onto.

Question

When is ν also 1–1 ?

Answer

When $\sim$ is the identity on S.

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A function $f : S \longrightarrow T$ defines equivalence relation on S by

$$s_1 \sim s_2 \quad \text{iff} \quad f(s_1) = f(s_2)$$

These sets $f^{-1}(t)$, $t \in T$ that are nonempty form the corresponding partition

$$S = \bigcup_{t \in T} f^{-1}(t)$$

Question

When are all $f^{-1}(t) \neq \emptyset$?

Answer

When f is onto.

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Answer

When f is onto.

It is often necessary to define a function on $[S]$ by describing it on the individual representatives $t \in [s]$ for each equivalence class $[s]$. If $\phi : [S] \longrightarrow X$ is to be defined in this way, one must

- define $\phi(s)$ for all $s \in S$, making sure that $\phi(s) \in X$;
- make sure that $\phi(s) = \phi(r)$ whenever $r \sim s$,
  ie. when $[s] = [r]$;
- define $\phi([s]) \stackrel{\text{def}}{=} \phi(s)$.

The second condition is critical for ϕ to be well defined.

Example

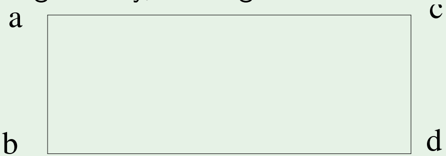
Partition of $\mathbb{Z}$ into classes of numbers with the same remainder $\pmod{p}$; it is particularly important for p prime

$$\mathbb{Z}(p) = \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

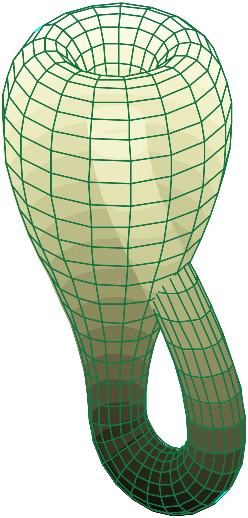
One can define all four arithmetic operations (with the usual properties) on $\mathbb{Z}_p$ for a prime p; division has to be restricted when p is not prime.

Example

In geometry, starting from the rectangle



and identifying points of $[a, b]$ and $[c, d]$ (in this direction) gives a cylinder, while identifying also $[a, c]$ and $[b, d]$ gives a torus. Points in the interior of the rectangle are not 'glued' together; thus the equivalence classes (for cylinder) have either one or two elements, while for torus there is also one class with four elements (which points of the original figure are in it?)
Identifying $[a, b]$ with $[d, c]$ (in that direction) gives a *one-sided* Moebius strip; furthermore, putting it together with identifying $[a, c]$ and $[b, d]$ produces a one-sided closed surface. That surface ("Klein bottle") cannot be represented in the 3-dim space without self-intersection.



Order relations

Total order $\leq$ *on* S
(R) $x \leq x$ for all $x \in S$
(AS) $x \leq y, y \leq x \Rightarrow x = y$
(T) $x \leq y, y \leq z \Rightarrow x \leq z$
(L) *Linearity* — any two elements are comparable
for all x, y either $x \leq y$ or $y \leq x$, and both if $x = y$

On a finite set all linear orders are isomorphic

$$x_1 \leq x_2 \leq \cdots \leq x_n$$

On an infinite set there is quite a variety of possibilities.

Examples

- discrete with the initial element $\mathbb{N} = \{0, 1, 2, \ldots\}$
- discrete without an initial element $\mathbb{Z} = \{\ldots, 0, 1, 2, \ldots\}$
- various dense/locally dense orders
  - rational numbers $\mathbb{Q}$: $\quad p < q \Rightarrow \exists r (p < r < q)$
  - $S = [a, b]$ — both least and greatest elements
  - $S = (a, b]$ — no least element
  - $S = [a, b)$ — no greatest element
  - other $[0, 1] \cup [2, 3] \cup [4, 5] \cup \ldots$

Partial order $\preceq$ on S

Satisfies (R), (AS), (T); need not be (L)

We call $(S, \preceq)$ a *poset* (for *partially ordered set*)

To each order/partial order one can associate a unique *quasi-order*

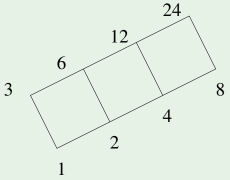
$$x \prec y \text{ iff } x \preceq y \text{ and } x \neq y$$

It satisfies (AS), (AR), and (T); it satisfies (L) if it corresponds to a total order (we could call it a total quasi-order).

Finite posets can be represented as lattice-like diagrams (Hasse diagrams).

Example

Hasse diagram for positive divisors of 24



Ordering concepts

- *Minimal* and *maximal* elements (they always exist in every finite poset).
- *Minimum* and *maximum*.
- *lub* (least upper bound) and *glb* (greatest lower bound) of a subset of elements
- *Cover* of an element in a poset
- *Lattice* — a poset where lub and glb exist for every pair of elements
  (by induction, they exist for every *finite* subset of elements)

Examples

- $\mathrm{pow}(\{a,b,c\})$ with the order $\subseteq$
  $\emptyset$ is minimum; $\{a,b,c\}$ is maximum
- $\mathrm{pow}(\{a,b,c\}) \setminus \{a,b,c\}$ (proper subsets of $\{a,b,c\}$)
  Two-element subsets $\{a,b\}, \{a,c\}, \{b,c\}$ are maximal.
- $\{1,2,3,4,6,8,12,24\}$ partially ordered by divisibility is a lattice
- $\{1,2,3\}$ partially ordered by divisibility is not a lattice
  - $\{2,3\}$ have no lub
- $\{2,3,6\}$ partially ordered by divisibility is not a lattice
  - $\{2,3\}$ have no glb
- $\{1,2,3,12,18,36\}$ partially ordered by divisibility is not a lattice
  - $\{2,3\}$ have no lub ($12, 18$ are minimal upper bounds)

NB

An infinite lattice need not have the lub (or the glb) for an arbitrary subset of its elements, in particular such a bound for all its elements.

Examples

- $\mathbb{Z}$ — neither lub nor glb;
- $\mathbb{F}(\mathbb{N})$ — all finite subsets, has no *arbitrary* lub property; glb exists — it is the intersection, hence always finite;
- co-$\mathbb{F}(\mathbb{N})$ — complements of finite subsets, may not have an arbitrary glb; lub, the union, is always infinite.

Consider the poset $(\mathbb{R}, \leq)$

1. It is a lattice.
2. subset with no upper bound: $\mathbb{R}_{>0} = \{\, r \in \mathbb{R} \mid r > 0 \,\}$
3. $\mathrm{lub}\{\, x \mid x < 73 \,\} = \mathrm{lub}\{\, x \mid x \leq 73 \,\} = 73$
4. $\mathrm{lub}\{\, x \mid x^2 < 73 \,\} = \sqrt{73}$
5. $\mathrm{glb}\{\, x \mid x^2 < 73 \,\} = -\sqrt{73}$

For $\omega_1, \omega_2 \in \Sigma^*$ define $\omega_1 \preceq \omega_2$ when $\omega_2 = \omega\omega_1\omega'$.
Relation $\preceq$ means being a substring; it is a partial order.
$\mathbb{F}(\mathbb{N})$ - collection of all *finite* subsets of $\mathbb{N}$

1. No maximal elements
2. $\emptyset$ is the minimum
3. $\text{lub}(A, B) = A \cup B$
4. $\text{glb}(A, B) = A \cap B$
5. $\mathbb{F}(\mathbb{N})$ is a lattice — is has *finite* union and intersection properties.

$\mathbb{I}(\mathbb{N}) = \mathrm{pow}(\mathbb{N}) \setminus \mathbb{F}(\mathbb{N})$ — collection of all *infinite* subsets of $\mathbb{N}$
(a) $\mathbb{N}$ is the maximum
(b) No minimum element ($\emptyset$ is not in $\mathbb{I}(\mathbb{N})$)
(c) lub$(A, B) = A \cup B$
(d) glb$(A, B) = A \cap B$ *if it exists*; it does not exist when $A \cap B$ is finite, eg. when empty.
(e) $\mathbb{I}(\mathbb{N})$ is not a lattice - it has finite union but not finite intersection property; eg. sets $2\mathbb{N}$ and $2\mathbb{N} + 1$ have the empty intersection.

Well-ordered set: every nonvoid subset has a least element.

NB

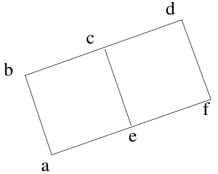
The greatest element is not required.

Examples

- $\mathbb{N} = \{0, 1, \ldots\}$
- $\mathbb{N}_1 \dot{\cup} \mathbb{N}_2 \dot{\cup} \mathbb{N}_3 \dot{\cup} \ldots$, where each $\mathbb{N}_i \sim \mathbb{N}$ and $\mathbb{N}_1 < \mathbb{N}_2 < \mathbb{N}_3 \cdots$
- The previous poset is isomorphic to the subset of $\mathbb{Q}$, consisting of all numbers of the form $n - \frac{1}{m}, n, m \geq 1$

Ordering of a poset — topological sort

For a poset $(S, \preceq)$ any linear order $\leq$ that is consistent with $\preceq$ is termed *topological sort*. Consistency means that if $a \preceq b$ then it remains $a \leq b$. Consider



Various possible
topological sortings

Various t-sorts (topological sortings)

$a \leq b \leq e \leq c \leq f \leq d$

$a \leq e \leq b \leq f \leq c \leq d$

$\ldots\ldots$

$a \leq e \leq f \leq b \leq c \leq d$

Combining orders

Product order - can combine any partial orders. In general, it is only a *partial order*, even if combining total orders.
For $s, s' \in S$ and $t, t' \in T$ define

$$(s, t) \preceq (s', t') \quad \text{if } s \preceq s' \text{ and } t \preceq t'$$

Practical orderings

They are, effectively, various *total* orders on the products of ordered sets.

- *Lexicographic order* - defined on all of Σ^*. It extends a total order already assumed to exist on Σ.
- *Lenlex* - the order on (potentially) the entire Σ^*, where the elements are ordered first by length.
  $\Sigma^{(1)} \prec \Sigma^{(2)} \prec \Sigma^{(3)} \prec \cdots$, then lexicographically within each $\Sigma^{(k)}$. In practice it is applied only to the finite subsets of Σ^*.
- *Filing order* - lexicographic order confined to the strings of the same length.
  It defines total orders on Σ^i, separately for each i.

Ordering of functions

T - arbitrary set (no order required)
S partially ordered set
$M = \{f : T \longrightarrow S\}$ - set of all functions from T to S
It has a natural partial order

$$f \preceq g \quad \text{iff} \quad \forall t \in T \, (f(t) \preceq g(t))$$

It is, in effect, a product order on $T^{|S|}$.
In most applications T has a linear ordering; however, it does not affect the order of the functions defined on T (only the order on S matters).

Let $A = \{1, 2, 3, 4\}$ and $S = A \times A$ with the product order. (a) A chain with seven elements

$(1, 1)\,(1, 2)\,(2, 2)\,(2, 3)\,(2, 4)\,(3, 4)\,(4, 4)$

(b) The above is a maximal chain; no chains of eight elements.

Take $(S, \preceq_1)$, $(T, \preceq_2)$ to be any posets (even chains) of more than one element. Then $S \times T$ with the product order is not a chain. for any $s_1 \prec s_2, t_1 \prec t_2$ the pair (s_1, t_2) and (s_2, t_1) are *not* comparable.

Let $\mathbb{B} = \{0, 1\}$ has the usual order. List the given elements of $\mathbb{B}^*$ in a certain order.

(a) Lexicographic order

$000, 0010, 010, 10, 1000, 101, 11$

(b) Len-lex order

$10, 11, 000, 010, 101, 0010, 1000$

When are *len* and *lenlex* on Σ^* the same?

Only if $|\Sigma| = 1$.

Take $(S, \preceq_1)$, $(T, \preceq_2)$ to be any posets (even chains) and define $\sqsubseteq$ on $S \times T$

$$(s,t) \sqsubseteq (s',t') \text{ if } s \preceq s' \text{ or } t \preceq t'$$

Not a partial order, as it need not satisfy (AS) nor (T).

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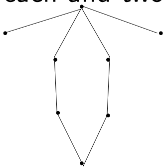
Not a partial order, as it need not satisfy (AS) nor (T).

Find the number of different maximal chains formed from the subsets of an n-element set.
(b) For $n = 5$ it is $5! = 120$ - why?
In general, the powerset $\mathrm{pow}(S)$ has $n!$ maximal chains
(Proof?)

Assume a poset with no chain of more than m elements.
(a) Chains of exactly m elements must be maximal.
(b) An example of a poset with two max chains of four elements each and two max chains of two elements each.



and many more

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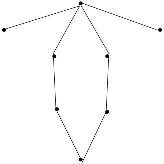


and many more

(Suppl)

(a) and (b) - True; this is the idea behind various lex-sorts

(c) Yes.

(d) Yes.

(e) False - consider a two-element set with the identity as p.o.

(f) True - due to the finiteness

(g) False, eg. $\mathbb{Z}$

Beginning of the lists:

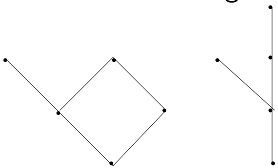
(a) Lenlex: λ, arc, bar, cab, car, ...

(b) lexicographic: λ, arc, bar, bare, bear, ...

Topological order for $\{1,2,3,4,5\} \times \{15,16,17\}$
Something simple, eg. filing order

$$(s,t) \preceq (m,n) \text{ if } s < m \text{ or } s = m, t < n$$

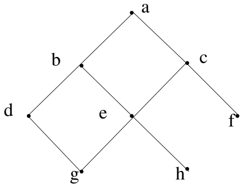
Some Hasse diagrams



and more ...

Existence of various glb's and lub's.

glb(b, f) does not exist, others are clear from the diagram



Properties of four relations, all defined on $\mathbb{P} = \{1, 2, \ldots\}$

- $\mathcal{R}_1$ if $m|n$
- $\mathcal{R}_2$ if $|m - n| \leq 2$
- $\mathcal{R}_3$ if $2|m + n$
- $\mathcal{R}_4$ if $3|m + n$

	$\mathcal{R}_1$	$\mathcal{R}_2$	$\mathcal{R}_3$	$\mathcal{R}_4$
(R)	Yes	Yes	Yes	
(S)		Yes	Yes	Yes
(AS)	Yes			
(T)	Yes		Yes	
Partial order	Yes			
Equivalence			Yes	